# A study of singularities with the theory of mixed Hodge modules

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# Contents

CHAPTER 1

# Introduction

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with $f(0) = 0$. If some partial derivatives $\partial f/\partial x_i(0)$ are not 0, the hypersurface $V := f^{-1}(0) \subset \mathbb{C}^n$ is a submanifold of $\mathbb{C}^n$ in a neighborhood of $0 \in V$ and the point 0 is called a smooth point. If all the partial derivatives $\partial f/\partial x_i(0)$ are 0, the hypersurface $V$ may not be a submanifold and the point 0 is called a singular point. In this thesis, we study singular points of hypersurfaces in $\mathbb{C}^n$ defined by polynomials. We consider three topics associated to a given singular point $0 \in V = f^{-1}(0)$:

(i) (Chapter 3) the stalk at the singular point $0 \in V$ of the intersection cohomology complex $\mathrm{IC}_V$,
(ii) (Chapter 4) the monodromy of the family of smooth hypersurfaces around the singular fiber $V$, and
(iii) (Chapter 5) the Milnor fibration of 0 at the singular point $0 \in V$.

Throughout this thesis, we use the theory of mixed Hodge structures, in particular some techniques of the theory of mixed Hodge modules. We give a brief review of them.

**(i) (Chapter 3)** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with $f(0) = 0$ and $V = f^{-1}(0) \subset \mathbb{C}^n$ the hypersurface defined by $f$. The intersection cohomology complex $\mathrm{IC}_V \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V)$ of $V$ is a complex of sheaves on $V$ whose hypercohomology groups are the intersection cohomology groups of $V$. Moreover, by Morihiko Saito's theory of mixed Hodge modules, it is the underlying complex of the mixed Hodge module $\mathrm{IC}_V^H$. Therefore, the stalk $(\mathrm{IC}_V)_0$ at the origin $0 \in V$ is a complex of mixed Hodge structures and its cohomology groups $H^j((\mathrm{IC}_V)_0)$ carries a natural mixed Hodge structure. If $0 \in V$ is smooth, $\mathrm{IC}_V$ is isomorphic to the shifted constant sheaf $\mathbb{Q}_V[n-1]$ in a neighborhood of 0 and the mixed Hodge structure of $H^j((\mathrm{IC}_V)_0)$ is the trivial one. However, if $0 \in V$ is a singular point, we know almost nothing about the mixed Hodge structure of $H^j((\mathrm{IC}_V)_0)$ until now. We study how the complexity of the singularity reflects the complexity of the mixed Hodge structure of $H^j((\mathrm{IC}_V)_0)$. In Chapter 3, we will describe the dimension $\dim \mathrm{Gr}_r^W H^j((\mathrm{IC}_V)_0)$ of the graded pieces of $H^j((\mathrm{IC}_V)_0)$ in terms of the Milnor monodromy of $f$ at the singular point 0 (see Theorem 3.5.1).

**(ii) (Chapter 4)** We consider a polynomial $f(t, x) \in \mathbb{C}[t, x_1, \ldots, x_n]$. We set $Y := f^{-1}(0) \subset \mathbb{C}^*_t \times \mathbb{C}^n$. The first projection $Y \to \mathbb{C}^*_t$ defines a family of hypersurfaces of $\mathbb{C}^n$ on $\mathbb{C}^*_t$. Its fiber $Y_{t_0}$ at $t_0 \in \mathbb{C}^*_t$ is a hypersurface of $\mathbb{C}^n$ defined by $f(t_0, x) \in \mathbb{C}[x_1, \ldots, x_n]$. Considering a path around the origin $0 \in \mathbb{C}$ in $\mathbb{C}^*_t$, we get a monodromy automorphism $\Phi_j \colon H^j(Y_{t_0}; \mathbb{C}) \xrightarrow{\sim} H^j(Y_{t_0}; \mathbb{C})$ of the cohomology group $H^j(Y_{t_0}; \mathbb{C})$. To compute it, we consider a mixed Hodge structure of $H^j(Y_{t_0}; \mathbb{Q})$ called the limit mixed Hodge structure. This mixed Hodge structure is very related to $\Phi_j$; the dimensions of the graded pieces of $H^j(Y_{t_0}; \mathbb{C})$ with respect to the weight filtration recover the Jordan normal form of $\Phi_j$. Assume that $f$ is schön (see Definition 4.3.1). Then by Denef-Loeser's motivic Milnor fiber, Stapledon [**44**] described the $E_\lambda$-polynomial ($\lambda \in \mathbb{C}$):

$$E_\lambda(Y_{t_0}; u, v) := \sum_{p,q\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{dim}\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^j(Y_{t_0}; \mathbb{C})_\lambda u^p v^q \in \mathbb{Z}[u, v]$$

in terms of the polyhedron $\mathrm{UH}_f := \mathrm{NP}(f) + (\mathbb{R}_{\geq 0} \times \mathbb{R}^n) \subset \mathbb{R}^{n+1}$, where $\mathrm{NP}(f)$ is the Newton polytope. In Chapter 4, we will reprove his formula in a different way. However, in general we can not compute each mixed Hodge number $\mathrm{dim}\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^{n-1}(Y_{t_0}; \mathbb{C})_\lambda$ from the $E_\lambda$-polynomial because $H^j(Y_{t_0}; \mathbb{C})_\lambda$ are not 0 for several $j$. In order to overcome this difficulty, we define a "set of bad eigenvalues" $R_f$ of $\Phi_j$ by $\mathrm{UH}_f$ (see (27)) and prove the concentration

$$H^j(Y_{t_0}; \mathbb{C})_\lambda = 0 \quad (j \neq n-1),$$

for $\lambda \notin R_f$ (see Corollary 4.4.7). Then for $\lambda \notin R_f$ we can compute the mixed Hodge numbers $\mathrm{dim}\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^{n-1}(Y_{t_0}; \mathbb{C})_\lambda$ from the formula for the $E_\lambda$-polynomial. In this way, we describe the Jordan normal form of $\Phi_{n-1}$ for an eigenvalue $\lambda \notin R_f$ in terms of $\mathrm{UH}_f$.

**(iii) (Chapter 5)** We consider the Milnor fibration of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ with $f(0) = 0$ at a singular point $0 \in V = f^{-1}(0)$. For a sufficiently small ball $B(0, \epsilon) \subset \mathbb{C}^n$ and a punctured disc $B(0, \eta)^* \subset \mathbb{C}$, the restriction $f|\colon f^{-1}(B(0, \eta)^*) \cap B(0, \epsilon) \longrightarrow B(0, \eta)^*$ of $f$ is a locally trivial fibration called the Milnor fibration of $f$ at 0 and its generic fiber $F_{f,0}$ of $f$ is called the Milnor fiber. Moreover, the monodromy automorphism $\Phi_{j,0}$ of $H^j(F_{f,0}; \mathbb{C})$ associated to the fibration is called the $j$-th Milnor monodromy automorphism. By the theory of mixed Hodge modules, the cohomology groups $H^j(F_{f,0}; \mathbb{Q})$ are endowed with mixed Hodge structures. If 0 is an isolated singular point, they have some nice properties: the cohomology group $H^j(F_{f,0}; \mathbb{Q})$ vanishes except for $j = 0, n-1$ and the weight filtration on $H^{n-1}(F_{f,0}; \mathbb{C})$ is the monodromy weight filtration. However, if 0 is a non-isolated singular point, we do not have such nice properties. By Varchenko [**51**], if $f$ is non-degenerate at 0 with respect to the Newton polyhedron $\Gamma_+(f)$, we can describe the monodromy zeta function

$$\zeta_{f,0}(z) := \prod_{j\in\mathbb{Z}} \det(\mathrm{Id} - z\Phi_{j,0})^{(-1)^j} \in \mathbb{C}(z),$$

in terms of the Newton polyhedron $\Gamma_+(f)$. If $0 \in V$ is an isolated singular point, the non-trivial factor is only $\det(\mathrm{Id} - z\Phi_{n-1,0})$ and thus we can compute the characteristic polynomial of $\Phi_{n-1,0}$ in terms of $\Gamma_+(f)$. We can not apply this argument to non-isolated singular points. On the other hand, as seen in [**25**] we can describe the $E_\lambda$-polynomial ($\lambda \in \mathbb{C}$):

$$E_\lambda(F_{f,0}; u, v) = \sum_{p,q\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{dim}\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^j(F_{f,0}; \mathbb{C})_\lambda u^p v^q \in \mathbb{Z}[u, v],$$

in terms of the Newton polyhedron $\Gamma_+(f)$, where $H^j(F_{f,0}; \mathbb{C})_\lambda$ is the generalized eigenspace of $\Phi_{j,0}$ for the eigenvalue $\lambda$ (see Corollary 5.3.5). If $0 \in V$ is an isolated singular point, similarly to the above argument for $\zeta_{f,0}$, we can compute $\mathrm{dim}\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^{n-1}(F_{f,0}; \mathbb{C})_\lambda$ from the computation of $E_\lambda(F_{f,0}; u, v)$. Moreover, since the weight filtration of $H^{n-1}(F_{f,0}; \mathbb{Q})$ coincides with the monodromy weight filtration in this case, eventually we can describe the Jordan normal form of $\Phi_{n-1,0}$ in terms of the Newton polyhedron $\Gamma_+(f)$. However, if $0 \in V$ is a non-isolated singular point, we can not argue in this way. In Chapter 5, we overcome this difficulty as follows. Similarly to the way in Chapter 4, we define "a set of bad eigenvalues" $R_f$ by $\Gamma_+(f)$ (see Definition 5.3.7) and prove the concentration

$$H^j(F_{f,0};\mathbb{C})_\lambda = 0 \quad (j \neq n-1)$$

(see Theorem 5.4.1). Then, for $\lambda \notin R_f$, we can compute the characteristic polynomial of $\Phi_{n-1,0}$ for the eigenvalue $\lambda$ from Varchenko's formula for $\zeta_{f,0}$ and the mixed Hodge numbers $\dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ from the formula for $E_\lambda$. Moreover, we prove that for $\lambda \notin R_f$ the weight filtration on $H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ coincides with the monodromy weight filtration of $\Phi_{n-1,0}$ (see Theorem 5.4.6). In this way, for $\lambda \notin R_f$ we describe the Jordan normal form of the Milnor monodromy $\Phi_{n-1,0}$ (see Corollary 5.5.1).

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CHAPTER 2

# Preliminary

In this section, we recall some basic notations and recall the theory of mixed Hodge structures and modules. Basically, we follow the notations of [**18**] and [**16**] (see also [**9**] and [**41**]).

Let $X$ be an algebraic variety over $\mathbb{C}$ and $\mathbb{K}$ a field. We denote by $\mathbb{K}_X$ the constant sheaf on $X$ with stalk $\mathbb{K}$ and by $\mathrm{D^b}(\mathbb{K}_X)$ or $\mathrm{D^b}(X)$ the bounded derived category of sheaves of $\mathbb{K}_X$-modules on $X$. For $F \in \mathrm{D^b}(X)$ and an integer $d \in \mathbb{Z}$, we denote by $F[d]$ the shifted complex of $F$ by the degree $d$. Moreover, we denote by $\tau^{\geq d}F, \tau^{\leq d}F$ the truncated complexes of $F$. We denote by $\mathrm{D^b_c}(X)$ the full triangulated subcategory of $\mathrm{D^b}(X)$ consisting of complexes whose cohomology sheaves are (algebraically) constructible. For a morphism $f\colon X \to Y$ of algebraic varieties, one can define Grothendieck's six operations $\mathrm{R}f_*$, $\mathrm{R}f_!$, $f^{-1}$, $f^!$, $\overset{\mathrm{L}}{\otimes}$ and $\mathrm{R}\mathcal{H}om$ as functors among $\mathrm{D^b_c}(X)$, $\mathrm{D^b_c}(Y)$ and $\mathrm{D^b_c}(X)\times\mathrm{D^b_c}(X)$ (see [**41**, Theorem 4.0.2]). Let $g\colon X \to \mathbb{C}$ be a morphism of algebraic varieties. Denote by $\widetilde{\mathbb{C}^*}$ the universal covering of $\mathbb{C}^*$ and by $p\colon \widetilde{\mathbb{C}^*} \simeq \mathbb{C} \to \mathbb{C}^*, x \mapsto \exp(2\pi\sqrt{-1}x)$ the covering map. Consider the diagram:

$$\begin{array}{ccccccc} & & & & (X\setminus g^{-1}(0))\times_{\mathbb{C}^*}\widetilde{\mathbb{C}^*} & \xrightarrow{\pi'} & \widetilde{\mathbb{C}^*} \\ & & & & \downarrow{\scriptstyle\pi} & \square & \downarrow{\scriptstyle p} \\ g^{-1}(0) & \xhookrightarrow[i]{} & X & \xhookleftarrow[j]{} & X\setminus g^{-1}(0) & \xrightarrow[g]{} & \mathbb{C}^* \end{array},$$

where the maps $i$ and $j$ are the inclusion maps and the square $\square$ on the right is Cartesian. For $F \in \mathrm{D^b_c}(X)$, we define $\psi_g(F) \in \mathrm{D^b_c}(g^{-1}(0))$ as $i^{-1}\mathrm{R}(j\circ\pi)_*(j\circ\pi)^{-1}F \in \mathrm{D^b_c}(g^{-1}(0))$. Then we obtain the functor $\psi_g : \mathrm{D^b_c}(X) \to \mathrm{D^b_c}(g^{-1}(0))$ called the *nearby cycle functor*. Moreover, we can define the functor $\phi_g : \mathrm{D^b_c}(X) \to \mathrm{D^b_c}(g^{-1}(0))$ which fits into a distinguished triangle $i^{-1} \longrightarrow \psi_g \longrightarrow \phi_g \xrightarrow{+1}$ . The functor $\phi_g$ is called the *vanishing cycle functor*.

We denote by SHM the abelian category of mixed Hodge structures, by $\mathrm{SHM}^p$ the full subcategory of SHM consisting of *graded-polarizable* mixed Hodge structures. Recall the Deligne's fundamental theorem on the theory of mixed Hodge structures.

Theorem 2.0.1 (Deligne [**5**]). *Let $X$ be an arbitrary algebraic variety over $\mathbb{C}$. Then $H^j(X;\mathbb{Q})$ and $H^j_c(X;\mathbb{Q})$ have canonical mixed Hodge structures for any $j \in \mathbb{Z}$.*

Note that if $X$ is a smooth projective variety, these mixed Hodge structures coincide with the usual pure Hodge structures. We call the mixed Hodge structures in Theorem 2.0.1 *Deligne's mixed Hodge structures.*

Let $X$ be an algebraic variety over $\mathbb{C}$. We denote by $\mathrm{MHM}(X)$ the abelian category of *mixed Hodge modules* on $X$, and by $\mathrm{D^b\,MHM}(X)$ the bounded derived category of mixed Hodge modules. Moreover, we denote by $\mathrm{Perv}(X)(= \mathrm{Perv}(\mathbb{K}_X))$

the category of perverse sheaves over the field $\mathbb{K}$ on $X$. This is a full abelian subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ given by the heart of the self-dual perverse $t$-structure, with cohomology functors ${}^{p}\mathcal{H}^{j}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X) \longrightarrow \mathrm{Perv}(X)$. Denote by $\mathrm{rat}\colon \mathrm{MHM}(X) \to \mathrm{Perv}(X)(= \mathrm{Perv}(\mathbb{Q}_X))$ the forgetful functor, which assigns a mixed Hodge module to its underlying perverse sheaf. It induces a functor from $\mathrm{D}^{\mathrm{b}}\,\mathrm{MHM}(X)$ to $\mathrm{D}^{\mathrm{b}}(\mathrm{Perv}(X)) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, which we denote by the same symbol rat. We denote by $\mathrm{MF}_{\mathrm{rh}}(D_X, \mathbb{Q})$ the category of triples $(\mathcal{M}, F^{\bullet}, K)$ consisting of a regular holonomic $D_X$-module $\mathcal{M}$, a good filtration $F^{\bullet}$ of $\mathcal{M}$ and a perverse sheaf $K \in \mathrm{Perv}(X)$ such that the image of $\mathcal{M}$ by the de Rham functor is $\mathbb{C} \otimes_{\mathbb{Q}} K$. We write $\mathrm{MF}_{\mathrm{rh}}\mathrm{W}(D_X, \mathbb{Q})$ for the category of quadruples $(\mathcal{M}, F^{\bullet}, K, W_{\bullet})$ of $(\mathcal{M}, F^{\bullet}, K) \in \mathrm{MF}_{\mathrm{rh}}(D_X, \mathbb{Q})$ and its finite increasing filtration $W_{\bullet}$. Recall that if $X$ is smooth, $\mathrm{MHM}(X)$ is an abelian subcategory of $\mathrm{MF}_{\mathrm{rh}}\mathrm{W}(D_X, \mathbb{Q})$. Even if $X$ is singular, $\mathrm{MHM}(X)$ can be defined by using a local embedding of $X$ into a smooth variety. The category of mixed Hodge modules on the one point variety pt is equivalent to that of graded-polarizable mixed Hodge structures $\mathrm{SHM}^{p}$ (see [**37**, (4.2.12)]). Let $f\colon X \to Y$ and $g\colon X \to \mathbb{C}$ be morphisms of algebraic varieties. Then we have functors between derived categories of mixed Hodge modules, $f_*$, $f^*$, $f_!$, $f^!$, $\psi_g$, $\phi_g$, $D$, $\otimes$ and $\mathcal{H}om$, and all of them are compatible with the corresponding functors for constructible sheaves via the functor rat. For example, we have $\mathrm{rat} \circ f_* = \mathrm{R}f_* \circ \mathrm{rat}$, $\mathrm{rat} \circ \psi_g = \psi_g[-1] \circ \mathrm{rat}$ and $\mathrm{rat} \circ \phi_g = \phi_g[-1] \circ \mathrm{rat}$.

The following definition is important.

Definition 2.0.2. Let $\mathcal{V} \in \mathrm{D}^{\mathrm{b}}\,\mathrm{MHM}(X)$. Then for any $w \in \mathbb{Z}$, we say that $\mathcal{V}$ *has mixed weights* $\leq w$ (resp. $\geq w$) if $\mathrm{Gr}^W_r H^j(\mathcal{V}) = 0$ for any $r, j \in \mathbb{Z}$ with $r > j+w$ (resp. $r < j+w$), where $\mathrm{Gr}^W_r H^j(\mathcal{V})$ is the $r$-th graded piece of $H^j(\mathcal{V})$ with respect to its weight filtration. Moreover we say that $\mathcal{V}$ *has a pure weight* $w$ if $\mathrm{Gr}^W_r H^j(\mathcal{V}) = 0$ for any $r \neq j + w$.

We have the following relation between weights and functors of derived categories of mixed Hodge modules by [**37**, (4.5.2)].

Proposition 2.0.3 (M. Saito [**37**]). *Let $X$, $Y$ and $Z$ be algebraic varieties and $f\colon X \to Y$ and $g\colon Z \to X$ be morphisms of algebraic varieties. Let $\mathcal{V} \in \mathrm{D}^{\mathrm{b}}\,\mathrm{MHM}(X)$. Then we have the following:*

(i) *If $\mathcal{V}$ has mixed weights $\leq w$, then $f_!\mathcal{V}$ and $g^*\mathcal{V}$ have mixed weights $\leq w$.*
(ii) *If $\mathcal{V}$ has mixed weights $\geq w$, then $f_*\mathcal{V}$ and $g^!\mathcal{V}$ have mixed weights $\geq w$.*

In particular, if $f$ is a proper morphism, the functor $f_* = f_!$ preserves the purity of $\mathcal{V}$.

We denote by pt the one point variety. Let $X$ be an algebraic variety and $a_X\colon X \to \mathrm{pt}$ a morphism from $X$ to pt. We denote by $\mathbb{Q}^H_{\mathrm{pt}}$ the trivial Hodge module on pt, and set $\mathbb{Q}^H_X := a_X^*\mathbb{Q}^H_{\mathrm{pt}}$. Since the image of $a_{X*}\mathbb{Q}^H_X$ by rat is $\mathrm{R}\Gamma(X; \mathbb{Q}_X)$, $H^j(X; \mathbb{Q})$ has a mixed Hodge structure for any $j \in \mathbb{Z}$. This mixed Hodge structure coincides with the one in Theorem 2.0.1 (see [**39**, Theorem 0.2 and Corollary 4.3]). Similarly, by considering $a_{X!}\mathbb{Q}^H_X$, $H^j_c(X; \mathbb{Q})$ has a mixed Hodge structure, which coincides with the one in Theorem 2.0.1 for any $j \in \mathbb{Z}$.

CHAPTER 3

# On the mixed Hodge structures of the intersection cohomology stalks of complex hypersurfaces

## 3.1. Introduction for Chapter 3

In this chapter, we reveal a new relationship between the mixed Hodge structures of the stalks of the intersection cohomology complexes (we call them IC stalks for short) and the Milnor monodromies, by using the results on motivic Milnor fibers shown by Matsui-Takeuchi [**25**] and on motivic Milnor fibers by Stapledon [**44**].

For a natural number $n \geq 2$, let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial of $n$ variables with coefficients in $\mathbb{C}$ such that $f(0) = 0$. Assume that $f$ is convenient and non-degenerate at 0 (see Definitions 3.4.7 and 3.4.9). We denote by $V$ the hypersurface $\{x \in \mathbb{C}^n \mid f(x) = 0\}$ in $\mathbb{C}^n$. Then, it is well-known that $0 \in V$ is a smooth or isolated singular point of $V$. We denote by $\mathrm{IC}_V := \mathrm{IC}_V(\mathbb{Q})$ the intersection cohomology complex of $V$ with rational coefficients. This is the underlying perverse sheaf of the mixed Hodge module $\mathrm{IC}_V^H$. By Morihiko Saito's theory, the stalk $(\mathrm{IC}_V)_0$ of $\mathrm{IC}_V$ at 0 is a complex of mixed Hodge structures. However, to the best of our knowledge, the mixed Hodge structure of $(\mathrm{IC}_V)_0$ has not been fully studied yet. The mixed Hodge module $\mathrm{IC}_V^H$ has a pure weight $n$ (see Definition 2.0.2). Nevertheless, in general, the stalk of $\mathrm{IC}_V^H$ does not have a pure weight. The purity of the weights of IC stalks is important. Kazhdan-Lustzig computed the Kazhdan-Lustzig polynomials by using the purity of IC stalks of Schubert varieties in flag varieties in [**21**]. Denef-Loeser proved that if $f$ is quasi-homogeneous, $(\mathrm{IC}_V)_0[-(n-1)](=: (\widetilde{\mathrm{IC}_V})_0)$ has a pure weight 0 in [**6**] (see Proposition 3.4.15). By using this result, they computed the dimensions of the intersection cohomology groups of complete toric varieties.

In general, $(\widetilde{\mathrm{IC}_V})_0$ has mixed weights $\leq 0$, that is $\mathrm{Gr}_r^W H^j((\widetilde{\mathrm{IC}_V})_0) = 0$ for $r > j$. In this chapter, we will describe the dimensions of $\mathrm{Gr}_r^W H^j((\widetilde{\mathrm{IC}_V})_0)$ very explicitly. We denote by $N_0$ the dimension of the invariant subspace of the $(n-1)$-st Milnor monodromy $\Phi_{n-1,0}$ of $f$ at 0. Assuming that $n \geq 3$, for any $j \in \mathbb{Z}$, we have

$$\mathrm{dim} H^j((\widetilde{\mathrm{IC}_V})_0) = \begin{cases} 1 & \text{if } \ j = 0, \\ N_0 & \text{if } \ j = n-2, \ \text{and} \\ 0 & \text{otherwise} \end{cases}$$

(see Proposition 3.4.13). Thus, if $H^{n-2}((\widetilde{\mathrm{IC}_V})_0)$ does not have a pure weight, the dimension $N_0$ is decomposed into those of the graded pieces $\mathrm{Gr}_r^W H^{n-2}((\widetilde{\mathrm{IC}_V})_0)$. We shall describe $\mathrm{dim}\mathrm{Gr}_r^W H^{n-2}((\widetilde{\mathrm{IC}_V})_0)$ in terms of the numbers of the Jordan blocks for the eigenvalue 1 in the $(n-1)$-st Milnor monodromy $\Phi_{n-1,0}$. For a natural number $m \in \mathbb{Z}_{\geq 0}$, we denote by $J_m^1$ the number of the Jordan blocks in $\Phi_{n-1,0}$ for the eigenvalue 1 with size $m$. Then our main result is the following.

THEOREM 3.1.1 (see Theorem 3.5.1). *Assume that $n \geq 3$ and $f$ is convenient and non-degenerate at $0$. Then for any $r \in \mathbb{Z}$, we have*

$$\dim \mathrm{Gr}_r^W H^0((\widetilde{\mathrm{IC}_V})_0) = \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{if } r \neq 0, \end{cases}$$

*and*

$$\dim \mathrm{Gr}_r^W H^{n-2}((\widetilde{\mathrm{IC}_V})_0) = J^1_{n-r-1}.$$

For the case where $n = 2$, see Theorem 3.5.10. In particular, we obtain the following result on the purity of the IC stalk $(\widetilde{\mathrm{IC}_V})_0$.

COROLLARY 3.1.2 (see Corollary 3.5.2). *In the situation of Theorem 3.1.1, the following conditions are equivalent.*

(i) *The IC stalk $(\widetilde{\mathrm{IC}_V})_0$ has a pure weight $0$.*
(ii) *There is no Jordan block for the eigenvalue $1$ with size $> 1$ of the $(n-1)$-st Milnor monodromy $\Phi_{n-1,0}$.*

Therefore, the result on the purity of the IC stalks for quasi-homogeneous polynomials by Denef-Loeser (Proposition 3.4.15) is a special case of our result.

Moreover, as a corollary of the above theorem, we obtain a result on the mixed Hodge structures of the cohomology groups of the link of the isolated singular point $0$ in $V$. We denote by $L$ the link of $0$ in $V$, that is, the intersection of $V$ and a sufficiently small sphere centered at 0. Then $L$ is a $(2n-3)$-dimensional orientable compact real manifold and each cohomology group $H^j(L;\mathbb{Q})$ of $L$ has a canonical mixed Hodge structure, which was studied in Durfee-Saito [**11**]. For any $j \leq n-2$, the $j$-th cohomology group $H^j(L;\mathbb{Q})$ is isomorphic (as mixed Hodge structures) to $H^j((\widetilde{\mathrm{IC}_V})_0)$. Assuming that $n \geq 3$, by the Poincaré duality, we have $H^j(L;\mathbb{Q}) = 0$ for any $j \neq 0, n-2, n-1, 2n-3$. Then we obtain the following result.

COROLLARY 3.1.3 (see Corollary 3.5.5). *In the situation of Theorem 3.1.1, for any $r \in \mathbb{Z}$, we have*

$$\dim \mathrm{Gr}_r^W H^0(L;\mathbb{Q}) = \dim \mathrm{Gr}^W_{2(n-1)-r} H^{2n-3}(L;\mathbb{Q}) = \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{if } r \neq 0, \end{cases}$$

*and*

$$\dim \mathrm{Gr}_r^W H^{n-2}(L;\mathbb{Q}) = \dim \mathrm{Gr}^W_{2(n-1)-r} H^{n-1}(L;\mathbb{Q}) = J^1_{n-r-1}.$$

We prove Theorem 3.1.1 by combining the results of Stapledon [**44**] with that of Matsui-Takeuchi [**25**]. For this purpose, we deform the hypersurface $V$ to a suitable one which can be compactified in $\mathbb{P}^n$ nicely (see the proof of Theorem 3.5.1).

## 3.2. Intersection cohomology

For a $\mathbb{K}$-vector space $H$, we denote by $H^*$ the dual $\mathbb{K}$-vector space of $H$. If $X$ is a purely $n$-dimensional smooth variety, there exist canonical isomorphisms

$$H^j(X;\mathbb{K}) \simeq (H_c^{2n-j}(X;\mathbb{K}))^*$$

for any $j \in \mathbb{Z}$ by the Poincaré duality. However, if $X$ has some singular points, we can not expect such a nice symmetry in its cohomology groups $H^j(X;\mathbb{K})$. Intersection cohomology theory was invented in order to overcome this problem. Nowadays this theory is formulated in terms of perverse sheaves. Let $U$ be the regular part $X_{\mathrm{reg}}$ of $X$. Then $\mathbb{K}_U[n]$ is a perverse sheaf on $U$. Let $j\colon U \hookrightarrow X$ be the inclusion map. Then there exists a natural morphism $j_!\mathbb{K}_U[n] \to \mathrm{R}j_*\mathbb{K}_U[n]$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. Recall the notation ${}^p j_! := {}^p\mathcal{H}^0 \circ j_!$ and ${}^p\mathrm{R}j_* := {}^p\mathcal{H}^0 \circ \mathrm{R}j_*$. Taking the 0-th perverse cohomology groups of both sides of the above morphism, we obtain a morphism ${}^p j_!\mathbb{K}_U[n] \to {}^p\mathrm{R}j_*\mathbb{K}_U[n]$ in $\mathrm{Perv}(X)$.

Definition 3.2.1. We define $\mathrm{IC}_X(\mathbb{K})(=: \mathrm{IC}_X)$ as the image of the morphism

$$ {}^p j_!\mathbb{K}_U[n] \to {}^p\mathrm{R}j_*\mathbb{K}_U[n] $$

in the abelian category $\mathrm{Perv}(X)$. We call $\mathrm{IC}_X$ the *intersection cohomology complex* of $X$ with coefficients in $\mathbb{K}$.

Definition 3.2.2. We set

$$ \begin{aligned} IH^j(X;\mathbb{K}) &:= H^j(\mathrm{R}\Gamma(X;\mathrm{IC}_X(\mathbb{K})[-n])), \text{ and} \\ IH^j_c(X;\mathbb{K}) &:= H^j(\mathrm{R}\Gamma_c(X;\mathrm{IC}_X(\mathbb{K})[-n])) \end{aligned} $$

for any $j \in \mathbb{Z}$. We call $IH^j(X;\mathbb{K})$ (resp. $IH^j_c(X;\mathbb{K})$) the *$j$-th intersection cohomology group* of $X$ (resp. the *$j$-th intersection cohomology group with compact supports* of $X$).

Proposition 3.2.3. *In the situation as above, for any $j \in \mathbb{Z}$, we have the generalized Poincaré duality isomorphism*

$$ IH^j(X;\mathbb{K}) \simeq (IH^{2n-j}_c(X;\mathbb{K}))^*. $$

We can express $\mathrm{IC}_X$ more concretely as follows. Take a Whitnety stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of $X$ and set $X_d := \bigsqcup_{\dim X_\alpha \leq d} X_\alpha$. Then we have a sequence of closed subvarieties

$$ X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset $$

of $X$. Set $U_d := X \setminus X_{d-1}$. Then we have a sequence of inclusion maps

$$ U_n \overset{j_n}{\hookrightarrow} U_{n-1} \overset{j_{n-1}}{\hookrightarrow} \ldots \overset{j_2}{\hookrightarrow} U_1 \overset{j_1}{\hookrightarrow} U_0 = X. $$

Proposition 3.2.4. *There is an isomorphism*

$$ \mathrm{IC}_X \simeq (\tau^{\leq -1}\mathrm{R}j_{1*}) \circ (\tau^{\leq -2}\mathrm{R}j_{2*}) \circ \cdots \circ (\tau^{\leq -n}\mathrm{R}j_{n*})(\mathbb{K}_{U_n}[n]) $$

*in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$.

Remark 3.2.5. Note that one can weaken the assumption that the stratification is a Whitney stratification. For example, for a stratification $X = X_d \sqcup (X \setminus X_d)$ where $X_d$ is a pure $d$-dimensional closed smooth subvariety of $X$ and the inclusion $j\colon U := X \setminus X_d \hookrightarrow X$, if the restrictions of the cohomology sheaves of $\mathrm{R}j_*\mathbb{K}_U[n]$ to $X_d$ are locally constant, then we have

$$ \mathrm{IC}_X \simeq \tau^{\leq -d-1}(\mathrm{R}j_*\mathbb{K}_U[n]). $$

In particular, we have

Corollary 3.2.6. *Let $V$ be an algebraic variety over $\mathbb{C}$ of pure dimension $n$, and assume that $V$ has only one singular point at $p \in V$. Then we have an isomorphism*

$$ \mathrm{IC}_V \simeq \tau^{\leq -1}(\mathrm{R}j_*\mathbb{K}_{V_{\mathrm{reg}}}[n]), $$

*where $j : V_{\mathrm{reg}} = V \setminus \{p\} \hookrightarrow V$ is the inclusion map.*

In this chapter, we mainly consider the shifted complex $\widetilde{\mathrm{IC}_V} := \mathrm{IC}_V[-n]$ of $\mathrm{IC}_V$. By Corollary 3.2.6, we have

$$\widetilde{\mathrm{IC}_V} \simeq \tau^{\leq n-1}\mathrm{R}j_*\mathbb{K}_{V_{\mathrm{reg}}}.$$

Let $X$ be a $n$-dimensional algebraic variety over $\mathbb{C}$. Then there is an object in $\mathrm{MHM}(X)$ whose underlying perverse sheaf is $\mathrm{IC}_X$. We denote it by $\mathrm{IC}^H_X$. Note that $a_{X*}\mathrm{IC}^H_X[-n]$ is a complex of mixed Hodge structures whose image by the functor rat is $\mathrm{R}\Gamma(X;\mathrm{IC}_X[-n])$. Therefore, $IH^j(X;\mathbb{Q})$ has a natural mixed Hodge structure. The same thing holds also for $IH^j_c(X;\mathbb{Q})$. By the construction, the mixed Hodge module $\mathrm{IC}^H_X$ has a pure weight $n$. Hence $\mathrm{IC}^H_X[-n]$ has a pure weight 0. According to Proposition 2.0.3, $\mathrm{R}\Gamma(X;\mathrm{IC}_X[-n])$ has mixed weights $\geq 0$, and we have $\mathrm{Gr}^W_r IH^j(X;\mathbb{Q}) = 0$ for $r < j$. If $X$ is complete, we have $a_{X*} = a_{X!}$, and hence $\mathrm{R}\Gamma(X;\mathrm{IC}_X[-n])$ has a pure weight 0 by Proposition 2.0.3. Therefore, we have $\mathrm{Gr}^W_r IH^j(X;\mathbb{Q}) = 0$ for $r \neq j$, and $IH^j(X;\mathbb{Q})$ has a pure Hodge structure of weight $j$.

Finally, we define *virtual Poincaré polynomials*.

DEFINITION 3.2.7. Let $C$ be a bounded complex of mixed Hodge structures. Then we define a Laurent polynomial $\mathrm{P}(C)(T) \in \mathbb{Z}[T^{\pm}]$ with coefficients in $\mathbb{Z}$ by

$$\mathrm{P}(C)(T) := \sum_{r\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}}(-1)^j \dim \mathrm{Gr}^W_r H^j(C)\right)T^r,$$

where $\mathrm{Gr}^W_r H^j(C)$ stands for the $r$-th graded piece with respect to the weight filtration $W_\bullet$ of the mixed Hodge structure of $H^j(C)$. We call it the *virtual Poincaré polynomial* of $C$.

Virtual Poincaré polynomials satisfy the following property.

PROPOSITION 3.2.8. (i) *Let $C \to D \to E \xrightarrow{+1}$ be a distinguished triangle in $\mathrm{D^b(SHM)}$. Then we have*

$$\mathrm{P}(C) + \mathrm{P}(E) = \mathrm{P}(D).$$

(ii) *Let $C$ and $D$ be elements of $\mathrm{D^b(SHM)}$. Then we have*

$$\mathrm{P}(C\otimes D) = \mathrm{P}(C)\mathrm{P}(D).$$

Proposition 3.2.8 can be restated as the fact that the virtual Poincaré polynomial induces a ring homomorphism on Grothendieck groups

$$\mathrm{P}\colon (\mathrm{K}_0(\mathrm{D^b(SHM)}),\otimes) = (\mathrm{K}_0(\mathrm{SHM}),\otimes) \to \mathbb{Z}[T^{\pm}].$$

Then, the virtual Poincaré polynomial of $\mathrm{R}\Gamma(X;\mathrm{IC}_X[-n])$ for $X$ complete is explicitly written as follows:

$$\mathrm{P}(\mathrm{R}\Gamma(X;\mathrm{IC}_X[-n]))(T) = \sum_{j\in\mathbb{Z}}(-1)^j \dim IH^j(X;\mathbb{Q})T^j.$$

Fix a point $p$ in $X$. We denote by $i_p\colon \{p\} \hookrightarrow X$ the inclusion map. Then the image of $i_p^*\mathrm{IC}^H_X[-n]$ by the functor rat is $(\widetilde{\mathrm{IC}_X})_p$, and has a mixed Hodge structure. By Proposition 2.0.3, it has mixed weights $\leq 0$. This property will play an important role in the proof of Theorem 3.5.1.

### 3.3. Limit mixed Hodge structures

We denote by $B(0,\eta)$ the open disc in $\mathbb{C}$ centered at 0 with radius $\eta$ and by $B(0,\eta)^*$ the punctured open disc $B(0,\eta)\setminus\{0\}$. Let $f\colon \mathbb{C}^n\to\mathbb{C}$ be a non-constant polynomial map satisfying that $f(0)=0$. Then for a sufficiently small $\eta>0$, the restriction of $f$ to $f^{-1}(B(0,\eta)^*)$ is a locally trivial fibration. Considering a lift of a path along a small circle around 0 in $B(0,\eta)^*$, for any $\epsilon\in B(0,\eta)^*$ we can define a *monodromy automorphism*

$$\Psi_j\colon H^j_c(f^{-1}(\epsilon);\mathbb{C})\xrightarrow{\sim} H^j_c(f^{-1}(\epsilon);\mathbb{C})$$

for any $j\in\mathbb{Z}$. For any $\epsilon\in B(0,\eta)^*$, the $\mathbb{Q}$-vector space $H^j_c(f^{-1}(\epsilon);\mathbb{Q})$ has a canonical mixed Hodge structure by Theorem 2.0.1. We denote by $F^\bullet$ and $W_\bullet$ its Hodge and weight filtrations. The monodromy automorphism $\Psi_j$ is compatible with the weight filtration $W_\bullet$, since $H^j_c(f^{-1}(\epsilon);\mathbb{Q})$ is the stalk of the (geometric) variation of mixed Hodge structures $\mathrm{R}f_!\mathbb{Q}_{\mathbb{C}^n}$ on $B(0,\eta)^*$, i.e. $W_\bullet$ comes from a sub-local system of these locally constant direct image sheaves. On the other hand, we can endow $H^j_c(f^{-1}(\epsilon);\mathbb{C})$ and $H^j_c(f^{-1}(\epsilon);\mathbb{Q})$ with another filtrations $F^\bullet_\infty$ and $M_\bullet$, which are called the *limit Hodge filtration* and the *relative monodromy filtration*, respectively. Moreover, Steenbrink-Zucker proved the following theorem.

Theorem 3.3.1 (Steenbrink-Zucker [**47**]). *For any $j\in\mathbb{Z}$, the triple*

$$(H^j_c(f^{-1}(\epsilon);\mathbb{Q}),F^\bullet_\infty,M_\bullet)$$

*defines a mixed Hodge structure.*

This mixed Hodge structure in Theorem 3.3.1 is called the *limit mixed Hodge structure*. The relative monodromy filtration $M_\bullet$ satisfies the following properties. The monodromy automorphims $\Psi_j$ are decomposed as $\Psi_j=\Psi^s_j\Psi^u_j$, where $\Psi^s_j$ is the semisimple part of $\Psi_j$ and $\Psi^u_j$ is the unipotent part of $\Psi_j$. We denote by $N$ the logarithm operator $\log\Psi^u_j$ of $\Psi^u_j$ on $H^j_c(f^{-1}(\epsilon);\mathbb{C})$ and for $r\in\mathbb{Z}$ by $N(r)$ the nilpotent operator on $\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})$ induced by $N$. We denote by $M(r)_\bullet$ the filtration of $\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})$ induced by $M_\bullet$. Then $M(r)_\bullet$ satisfies the following properties:

(i) $M(r)_{2r}=\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})$,
(ii) $N(M(r)_l)\subset M(r)_{l-2}$ for any $l\geq 0$, and
(iii) the map

$$N^l\colon \mathrm{Gr}^{M(r)}_{r+l}\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})\to \mathrm{Gr}^{M(r)}_{r-l}\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})$$

is an isomorphism.

For an eigenvalue $\lambda\in\mathbb{C}$ of $\Psi_j$, we denote by $H^j_c(f^{-1}(\epsilon);\mathbb{C})_\lambda$ the generalized eigenspace of $\Psi_j$ for the eigenvalue $\lambda$. Then the number of the Jordan blocks of the linear operator on $\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})$ induced by $\Psi_j$ for the eigenvalue $\lambda$ with size $m$ is equal to

$$\dim\mathrm{Gr}^M_{r+1-m}\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})_\lambda-\dim\mathrm{Gr}^M_{r-1-s}\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})_\lambda.$$

Eventually, $H^j_c(f^{-1}(\epsilon);\mathbb{Q})$ has three filtrations $F^\bullet_\infty$, $M_\bullet$ and $W_\bullet$. We set

$$h^{p,q,r}_\lambda(H^j_c(f^{-1}(\epsilon);\mathbb{C})):=\dim\mathrm{Gr}^p_{F_\infty}\mathrm{Gr}^M_{p+q}\mathrm{Gr}^W_r H^j_c(f^{-1}(\epsilon);\mathbb{C})_\lambda.$$

Then we have the following symmetry:

$$h^{p,q,r}_\lambda(H^j_c(f^{-1}(\epsilon);\mathbb{C}))=h^{r-q,r-p,r}_\lambda(H^j_c(f^{-1}(\epsilon);\mathbb{C})).$$

The limit mixed Hodge structure encodes the data of the monodromy $\Psi_j$ only partially. In general, the limit mixed Hodge structure does not determine the Jordan normal form of the monodromy $\Psi_j$. However, in some good situation as in the following proposition, it completely recovers the Jordan normal form of the monodromy.

PROPOSITION 3.3.2 (Stapledon [**44**, Example 6.7]). *Assume that $f$ is convenient (see Definition 3.4.7) and non-degenerate (see Definition 3.4.8). Then we have $H_c^j(f^{-1}(\epsilon);\mathbb{C})=0$ for any $j\in\mathbb{Z}$ with $j\neq n-1, 2(n-1)$ and the monodromy action on $H_c^{2(n-1)}(f^{-1}(\epsilon);\mathbb{C})\simeq\mathbb{C}$ is trivial. Moreover, the monodromy automorphism $\Psi_{n-1}$ on the graded piece $\mathrm{Gr}_r^W H_c^{n-1}(f^{-1}(\epsilon);\mathbb{C})$ is trivial for $r\neq n-1$. Therefore, by the properties of $M_\bullet$, for any $r\neq n-1$ and $q\neq r$, we have*

$$\mathrm{Gr}_q^M \mathrm{Gr}_r^W H_c^{n-1}(f^{-1}(\epsilon);\mathbb{C})=0.$$

*In particular, for any eigenvalue $\lambda\neq 1$ and any $r\neq n-1$, we have*

$$\mathrm{Gr}_r^W H_c^{n-1}(f^{-1}(\epsilon);\mathbb{C})_\lambda=0,$$

*and*

$$H_c^{n-1}(f^{-1}(\epsilon);\mathbb{C})_\lambda\simeq \mathrm{Gr}_{n-1}^W H_c^{n-1}(f^{-1}(\epsilon);\mathbb{C})_\lambda.$$

This result follows from a deep argument combining the computation of the motivic Milnor fiber and some combinatorial results (see Stapledon [**44**], Guibert-Loeser-Merle [**15**] and Section 4). This proposition implies that for an eigenvalue $\lambda\neq 1$, the filtration $M_\bullet$ is equal to $M(n-1)_\bullet$ on $H_c^{n-1}(f^{-1}(\epsilon);\mathbb{C})_\lambda$. Thus, the Jordan normal form of the monodromy automorphism $\Psi_{n-1}$ for the eigenvalue $\lambda\neq 1$ can be completely determined by the dimensions of the graded pieces $\mathrm{Gr}_q^M H_c^{n-1}(f^{-1}(\epsilon);\mathbb{C})_\lambda$ with respect to the monodromy filtration $M_\bullet$.

We can explain the limit mixed Hodge structure in terms of mixed Hodge modules. As an object in $\mathrm{D^b}(\mathrm{Mod}(\mathbb{Q}))$, the complex $\psi_t \mathrm{R}f_! \mathbb{Q}_{\mathbb{C}^n}$ is isomorphic to $\mathrm{R}\Gamma_c(f^{-1}(\epsilon);\mathbb{Q})$ for a sufficiently small $\epsilon\in B(0,\eta)^*$, where $\psi_t$ is the nearby cycle functor of the coordinate $t$ of $\mathbb{C}$. Note that $\psi_t\mathrm{R}f_!\mathbb{Q}_{\mathbb{C}^n}$ is the underlying constructible sheaf of the complex of mixed Hodge modules $\psi_t\mathrm{R}f_!\mathbb{Q}^H_{\mathbb{C}^n}$. Thus, each cohomology group $H_c^j(f^{-1}(\epsilon);\mathbb{Q})$ of $\mathrm{R}\Gamma_c(f^{-1}(\epsilon);\mathbb{Q})(\simeq \psi_t\mathrm{R}f_!\mathbb{Q}_{\mathbb{C}^n})$ has a mixed Hodge structure. Moreover these mixed Hodge structures of $H_c^j(f^{-1}(\epsilon);\mathbb{Q})$ coincide with the limit mixed Hodge structures introduced in the above (For the case where the monodromy action is unipotent, see [**38**, p49]. If the monodromy action is quasi-unipotent, we can reduce the claim to the unipotent case via a suitable base change $t\mapsto t^m$ (see [**17**, Lemma 1.9.1]) as in [**36**, Remark 3.4].).

## 3.4. Milnor monodromies and IC stalks

For a natural number $n\geq 2$, let $f\in\mathbb{C}[x_1,\dots,x_n]$ be a non-constant polynomial of $n$ variables with coefficients in $\mathbb{C}$ such that $f(0)=0$. For any positive real number $r>0$ and any natural number $m\geq 1$, we denote by $B(0,r)$ the open ball in $\mathbb{C}^m$ centered at 0 with radius $r$.

THEOREM 3.4.1 (Milnor [**27**]). *There exists $\epsilon>0$ such that for a sufficiently small $(\epsilon\gg)\eta>0$, the restriction of $f$*

$$f\colon B(0,\epsilon)\cap f^{-1}(B(0,\eta)^*)\to B(0,\eta)^*$$

*is a locally trivial fibration. Moreover, if* 0 *is an isolated singular point of* $f^{-1}(0)$*, its fiber is homotopic to a bouquet of some* $(n-1)$*-spheres.*

This fibration is called the *Milnor fibration* of $f$ at 0 and its general fiber $F_{f,0}$ is called the *Milnor fiber of* $f$ *at* 0. When $0 \in f^{-1}(0)$ is an isolated singular point, we denote by $\mu$ the number of $(n-1)$-spheres in the bouquet which is homotopic to $F_{f,0}$ and call it the *Milnor number*. In this case, for any $j \in \mathbb{Z}$, we have

$$H^j(F_{f,0};\mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \text{if } j=0, \\ \mathbb{Q}^{\mu} & \text{if } j=n-1, \text{ and} \\ 0 & \text{if } j \neq 0, n-1. \end{cases}$$

Considering a path along a small circle around the origin in $\mathbb{C}^*$, we obtain an automorphism of $F_{f,0}$ called the *geometric Milnor monodromy*. It induces an automorphism

$$\Phi_{j,0} \colon H^j(F_{f,0};\mathbb{C}) \xrightarrow{\sim} H^j(F_{f,0};\mathbb{C})$$

for any $k \in \mathbb{Z}$. This automorphism is called the *k-th Milnor monodromy* of $f$ at 0. In this chapter, when 0 is an isolated singular point of $f^{-1}(0)$, we call $\Phi_{n-1,0}$ the *Milnor monodromy* of $f$, for short. The following well-known fact for $\Phi_{n-1,0}$ is called the *monodromy theorem*.

THEOREM 3.4.2 (van Doorn-Steenbrink [**50**]). *Assume that* 0 *is an isolated singular point of* $f^{-1}(0)$*. Then any eigenvalue of* $\Phi_{n-1,0}$ *is a root of unity. Moreover, the maximal size of the Jordan blocks in* $\Phi_{n-1,0}$ *for the eigenvalue* $\lambda \neq 1$ (resp. 1) *is bounded by* $n$ (resp. $n-1$).

In what follows, we assume that 0 is an isolated singular point of $f^{-1}(0)$. Then by a theorem of Steenbrink [**45**], $H^{n-1}(F_{f,0};\mathbb{Q})$ has a canonical mixed Hodge structure. Recall that we defined the nearby cycle functor $\psi_f$ in Chapter 2. We denote by $i_0 \colon \{0\} \hookrightarrow f^{-1}(0)$ the inclusion map. One can show that for any integer $j \in \mathbb{Z}$, the $k$-th cohomology group of the complex of sheaves $i_0^{-1}\psi_f(\mathbb{Q}_{\mathbb{C}^n})$ is isomorphic to $H^j(F_{f,0};\mathbb{Q})$. Since $i_0^{-1}\psi_f(\mathbb{Q}_{\mathbb{C}^n})$ is the underlying complex of that of mixed Hodge structures $i_0^*\psi_f(\mathbb{Q}^H_{\mathbb{C}^n})$, the $\mathbb{Q}$-vector space $H^j(i_0^*\psi_f(\mathbb{Q}^H_{\mathbb{C}^n})) \simeq H^j(F_{f,0};\mathbb{Q})$ has a canonical mixed Hodge structure for any $j \in \mathbb{Z}$. These mixed Hodge structures coincide with Steenbrink's mixed Hodge structures (see [**40**]). We denote by $F^{\bullet}$ its Hodge filtration and $W_{\bullet}$ its weight filtration. For $\lambda \in \mathbb{C}$, let $H^{n-1}(F_{f,0};\mathbb{C})_{\lambda} \subset H^{n-1}(F_{f,0};\mathbb{C})$ be the generalized eigenspace of $\Phi_{n-1,0}$ for the eigenvalue $\lambda$ and set

$$h^{p,q}_{\lambda}(H^{n-1}(F_{f,0};\mathbb{C})) := \dim \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H^{n-1}(F_{f,0};\mathbb{C})_{\lambda}.$$

For these numbers, the following results are well-known (see Steenbrink [**45**]).

PROPOSITION 3.4.3. *(i) For* $\lambda \in \mathbb{C}^* \setminus \{1\}$ *and* $(p,q) \notin [0,n-1] \times [0,n-1]$ *we have* $h^{p,q}_{\lambda}(H^{n-1}(F_{f,0};\mathbb{C})) = 0$*. For* $(p,q) \in [0,n-1] \times [0,n-1]$ *we have*

$$h^{p,q}_{\lambda}(H^{n-1}(F_{f,0};\mathbb{C})) = h^{n-1-q,n-1-p}_{\lambda}(H^{n-1}(F_{f,0};\mathbb{C})).$$

*(ii) For* $(p,q) \notin [1,n-1] \times [1,n-1]$ *we have* $h^{p,q}_1(H^{n-1}(F_{f,0};\mathbb{C})) = 0$*. For* $(p,q) \in [1,n-1] \times [1,n-1]$ *we have*

$$h^{p,q}_1(H^{n-1}(F_{f,0};\mathbb{C})) = h^{n-q,n-p}_1(H^{n-1}(F_{f,0};\mathbb{C})).$$

Proposition 3.4.4. *(i) For $\lambda \in \mathbb{C}^* \setminus \{1\}$, $s \geq 1$, the number of the Jordan blocks in $\Phi_{n-1,0}$ with size $\geq s$ for the eigenvalue $\lambda$ is equal to*

$$\sum_{p+q=n-2+s,n-1+s} h_\lambda^{p,q}(H^{n-1}(F_{f,0};\mathbb{C})).$$

*(ii) For $s \geq 1$, the number of the Jordan blocks in $\Phi_{n-1,0}$ with size $\geq s$ for the eigenvalue 1 is equal to*

$$\sum_{p+q=n-1+s,n+s} h_1^{p,q}(H^{n-1}(F_{f,0};\mathbb{C})).$$

In what follows, for any $\lambda \in \mathbb{C}^*$ and $m \in \mathbb{Z}_{\geq}$, we denote by $J_m^\lambda$ the number of the Jordan blocks in $\Phi_{n-1,0}$ with size $m$ for the eigenvalue $\lambda$. By the monodrmy theorem, if $\lambda \neq 1$ (resp. $\lambda = 1$) we have $J_m^\lambda = 0$ for any $m \geq n+1$ (resp. $m \geq n$).

By Propositions 3.4.3 and 3.4.4, we can describe the dimension of each graded piece $\mathrm{Gr}_r^W H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ of $H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ with respect to the weight filtration as follows.

Proposition 3.4.5. *(i) For $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $r < 0$ or $2(n-1) < r$, we have $\mathrm{Gr}_r^W H^{n-1}(F_{f,0};\mathbb{C})_\lambda = 0$. For $0 \leq r \leq 2(n-1)$, we have the symmetry*

$$\dim\mathrm{Gr}_r^W H^{n-1}(F_{f,0};\mathbb{C})_\lambda = \dim\mathrm{Gr}_{2(n-1)-r}^W H^{n-1}(F_{f,0};\mathbb{C})_\lambda,$$

*centered at degree $n-1$. Moreover, for $0 \leq r \leq n-1$, we have*

$$\dim\mathrm{Gr}_r^W H^{n-1}(F_{f,0};\mathbb{C})_\lambda = \sum_{s\geq 0} J_{n-r+2s}^\lambda.$$

*(ii) For $r < 2$ or $2(n-1) < r$, we have $\mathrm{Gr}_r^W H^{n-1}(F_{f,0};\mathbb{C})_1 = 0$. For $2 \leq r \leq 2(n-1)$, we have the symmetry*

$$\dim_r^W H^{n-1}(F_{f,0};\mathbb{C})_1 = \dim\mathrm{Gr}_{2n-r}^W H^{n-1}(F_{f,0};\mathbb{C})_1,$$

*centered at degree $n$. Moreover, for $2 \leq r \leq n$, we have*

$$\dim\mathrm{Gr}_r^W H^{n-1}(F_{f,0};\mathbb{C})_1 = \sum_{s\geq 0} J_{n+1-r+2s}^1.$$

By Proposition 3.4.5 we have the following table for the dimensions of the graded pieces of $H^{n-1}(F_{f,0};\mathbb{C})_\lambda$.

| $r$ | $0$ | $1$ | $2$ | $3$ | $\cdots$ |
|---|---|---|---|---|---|
| $\dim\mathrm{Gr}_r^W H^{n-1}(F_{f,0})_\lambda$ | $J_n^\lambda$ | $J_{n-1}^\lambda$ | $J_{n-2}^\lambda + J_n^\lambda$ | $J_{n-3}^\lambda + J_{n-1}^\lambda$ | $\cdots$ |

| $n-2$ | $n-1$ | $n$ | $\cdots$ | $2(n-1)$ |
|---|---|---|---|---|
| $J_2^\lambda + J_4^\lambda + \cdots$ | $J_1^\lambda + J_3^\lambda + \cdots$ | $J_2^\lambda + J_4^\lambda + \cdots$ | $\cdots$ | $J_n^\lambda$ |

Table 1. The case $\lambda \neq 1$

Next, in order to introduce some results for the numbers of the Jordan blocks in $\Phi_{n-1,0}$, we recall some familiar notions about polynomials.

Definition 3.4.6. Let $f = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha x^\alpha \in \mathbb{C}[x_1^\pm,\ldots,x_n^\pm]$ be a Laurent polynomial with coefficients in $\mathbb{C}$. The convex hull in $\mathbb{R}^n$ of $\mathrm{supp}(f) = \{\alpha \in \mathbb{Z}_{\geq 0}^n \mid a_\alpha \neq 0\}$ is called the *Newton polytope* of $f$ and we denote it by $\mathrm{NP}(f)$.

| $r$ | 2 | 3 | 4 | 5 | $\cdots$ | $n-1$ | $n$ | $n+1$ | $\cdots$ | $2(n-1)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\dim \mathrm{Gr}_r^W H^{n-1}(F_{f,0})_1$ | $J^1_{n-1}$ | $J^1_{n-2}$ | $J^1_{n-3}+J^1_{n-1}$ | $J^1_{n-4}+J^1_{n-2}$ | $\cdots$ | $J^1_2+J^1_4+\cdots$ | $J^1_1+J^1_3+\cdots$ | $J^1_2+J^1_4+\cdots$ | $\cdots$ | $J^1_{n-1}$ |

TABLE 2. The case $\lambda = 1$

DEFINITION 3.4.7. Let $f = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with coefficients in $\mathbb{C}$ such that $f(0) = 0$.

(i) The convex hull of $\bigcup_{\alpha \in \mathrm{supp}\,(f)} \{\alpha + \mathbb{R}^n_{\geq 0}\}$ is called the *Newton polyhedron* of $f$ at the origin $0 \in \mathbb{C}^n$ and we denote it by $\Gamma_+(f)$.
(ii) The union of all bounded face of $\Gamma_+(f)$ is called the *Newton boundary* of $f$ and we denote it by $\Gamma_f$.
(iii) The polynomial $f$ is called *convenient* if $\Gamma_+(f)$ intersects the positive part of each coordinate axis of $\mathbb{R}^n$.

For a Laurent polynomial $f = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha x^\alpha$ and a polytope $F$ in $\mathbb{R}^n$, set $f_F := \sum_{\alpha \in F} a_\alpha x^\alpha$.

DEFINITION 3.4.8. Let $f = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha x^\alpha \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ be a Laurent polynomial with coefficients in $\mathbb{C}$. We say that $f$ is *non-degenerate* if for any face $F$ of $\mathrm{NP}(f)$ the hypersurface $\{x \in (\mathbb{C}^*)^n \mid f_F(x) = 0\}$ of $(\mathbb{C}^*)^n$ is smooth and reduced.

DEFINITION 3.4.9. Let $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with coefficients in $\mathbb{C}$ such that $f(0) = 0$. We say that $f$ is *non-degenerate at* $0 \in \mathbb{C}^n$ if for any face $F$ of $\Gamma_f$, the hypersurface $\{x \in (\mathbb{C}^*)^n \mid f_F(x) = 0\}$ of $(\mathbb{C}^*)^n$ is smooth and reduced.

Let us recall a well-known condition ofor the origin $0 \in V$ to be an isolated singular point.

PROPOSITION 3.4.10. *If $f$ is convenient and non-degenerate at $0$, then the origin $0$ is a smooth or an isolated singular point of $V$.*

Let $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with coefficients in $\mathbb{C}$ such that $f(0) = 0$. Let $q_1, ..., q_l \in \mathbb{Z}^n$ be the vertices in $\Gamma_f \cap \mathrm{int}(\mathbb{R}^n_{\geq 0})$. We denote by $d_i \in \mathbb{Z}_{>0}$ the lattice distance between $q_i$ and $0 \in \mathbb{R}^n$, and by $\Pi_f$ the number of lattice points in the union of one dimensional faces of $\Gamma_f \cap \mathrm{int}(\mathbb{R}^n_{\geq 0})$.

THEOREM 3.4.11 ([**50**], [**25**]). *Assume that $f$ is convenient and non-degenerate at $0 \in \mathbb{C}^n$. Then we have*

(i) *for $\lambda \in \mathbb{C}^* \setminus \{1\}$, $J^\lambda_n = \#\{q_i \mid \lambda^{d_i} = 1\}$, and*
(ii) $J^1_{n-1} = \Pi_f$.

For a related result, see Raibaut [**31**].
The next theorem was proved by Matsui-Takeuchi [**25**].

THEOREM 3.4.12 ([**25**]). *If $f$ is convenient and non-degenerate at $0$, all the numbers $J^\lambda_m$ are determined from the Newton boundary $\Gamma_f$ of $f$.*

**3.4.1. IC stalks and Milnor monodromies.** As we mentioned in the introduction, there exists a relationship between IC stalks and Milnor monodromies. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial of $n$ variables with coefficients in $\mathbb{C}$ such that $f(0) = 0$. We assume that 0 is an isolated singular point of $V = f^{-1}(0) \subset \mathbb{C}^n$. We denote by $N_0$ the dimension of the invariant subspace of $\Phi_{n-1,0}$ in $H^{n-1}(F_{f,0}; \mathbb{Q})$.

PROPOSITION 3.4.13 (see e.g. [**28**, Lemma 4.3]). *If $n \geq 3$, then for any $j \in \mathbb{Z}$, we have*

$$\dim H^j((\widetilde{\mathrm{IC}_V})_0) = \begin{cases} 1 & \text{if } j = 0, \\ N_0 & \text{if } j = n-2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

*If $n = 2$, then for any $j \in \mathbb{Z}$, we have*

$$\dim H^j((\widetilde{\mathrm{IC}_V})_0) = \begin{cases} N_0 + 1 & \text{if } j = 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

As mentioned in Section 3.2, $(\widetilde{\mathrm{IC}_V})_0$ is a complex of mixed Hodge structures having mixed weights $\leq 0$ (see Proposition 2.0.3). Then for any $r \in \mathbb{Z}$, we have

$$\mathrm{Gr}_r^W H^0((\widetilde{\mathrm{IC}_V})_0) = 0 \quad \text{if } r > 0, \text{ and}$$
$$\mathrm{Gr}_r^W H^{n-2}((\widetilde{\mathrm{IC}_V})_0) = 0 \quad \text{if } r > n-2.$$

DEFINITION 3.4.14. Let $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with coefficients in $\mathbb{C}$ such that $f(0) = 0$. We say that $f$ is *quasi-homogeneous* if there exists a natural number $C \in \mathbb{Z}_{>0}$ and $v = (v_1, \ldots, v_n) \in (\mathbb{Z}_{>0})^n$ such that for any $\alpha$ with $a_\alpha \neq 0$, $v \cdot \alpha = C$.

For a strictly convex rational polyhedral cone $\sigma$ of $\mathbb{R}^n$ with $0 \in \sigma$ and the associated affine toric variety $X(\sigma)$, Denef-Loeser [**6**] showed that the stalk of $\mathrm{IC}_{X(\sigma)}$ at the fixed point under the $\mathbb{C}^*$-action on $X(\sigma)$ has a pure weight to compute the dimensions of the intersection cohomology groups of complete toric varieties. In fact, their proof of Theorem 6.2 of [**6**] applies also to varieties with $\mathbb{C}^*$-actions. If $f$ is quasi-homogeneous, $V$ is equipped with a natural $\mathbb{C}^*$-action. Hence we can obtain the following result.

THEOREM 3.4.15 (Denef-Loeser [**6**, (6.2.3) in Theorem 6.2]). *Assume that the polynomial $f$ is quasi-homogeneous. Then, the IC stalk $(\widetilde{\mathrm{IC}_V})_0$ has a pure weight* 0.

We introduce the new notion which is a generalization of quasi-homogeneousness.

DEFINITION 3.4.16. Let $f$ be a polynomial with coefficients in $\mathbb{C}$ such that $f(0) = 0$. We say that $\Gamma_+(f)$ *is flat* if the affine space spanned by $\Gamma_f$ is a hyperplane in $\mathbb{R}^n$ (see Figures 1 and 2).

For example, the Newton polyhedron $\Gamma_+(f)$ of a quasi-homogeneous polynomial $f$ is flat.

Under the assumption that $f$ is convenient and non-degenerate at 0, we will show that the Newton polyhedron $\Gamma_+(f)$ completely determines the mixed Hodge numbers of the mixed Hodge structures of $H^j((\widetilde{\mathrm{IC}_V})_0)$, in Proposition 3.5.8.

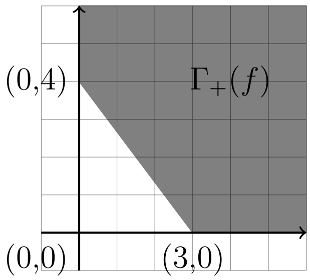


FIGURE 1. $\Gamma_+(f)$ is flat

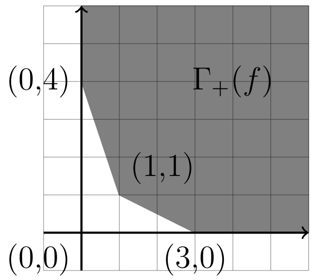


FIGURE 2. $\Gamma_+(f)$ is not flat

## 3.5. Main theorem

Let $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha x^\alpha \in \mathbb{C}[x_1, ..., x_n]$ be a non-constant polynomial of $n$ variables with coefficients in $\mathbb{C}$ such that $f(0) = 0$. Assume that $f$ is convenient (see Definition 3.4.7) and non-degenerate at 0 (see Definition 3.4.9). Set $V := \{x \in \mathbb{C}^n \mid f(x) = 0\} \subset \mathbb{C}^n$. Then $0 \in V$ is a smooth or isolated singular point of $V$ (Proposition 3.4.10). In what follows, we assume that $n \geq 3$. For the case where $n = 2$, see Theorem 3.5.10.

THEOREM 3.5.1. *Under the above conditions, for any $r \in \mathbb{Z}$, we have*

$$\dim \mathrm{Gr}^W_r H^0((\widetilde{\mathrm{IC}_V})_0) = \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{if } r \neq 0, \end{cases}$$

*and*

$$\dim \mathrm{Gr}^W_r H^{n-2}((\widetilde{\mathrm{IC}_V})_0) = J^1_{n-r-1},$$

*where $J^1_m$ is the number of the Jordan blocks in $\Phi_{n-1,0}$ with size $m$ for the eigenvalue 1.*

COROLLARY 3.5.2. *In the situation of Theorem 3.5.1, the following three conditions are equivalent.*

(i) *The IC stalk $(\widetilde{\mathrm{IC}_V})_0$ has a pure weight 0.*
(ii) *There is no Jordan block of $\Phi_{n-1,0}$ with size $> 1$ for the eigenvalue 1.*
(iii) *The Newton polyhedron $\Gamma_+(f)$ of $f$ is flat (see Definition 3.4.16).*

PROOF. The equivalence (i) $\iff$ (ii) follows from Theorem 3.5.1. Since $n \geq 3$, (ii) $\Rightarrow$ (iii) follows from Theorem 3.4.11. Suppose that $\Gamma_+(f)$ is flat. We define a quasi-homogeneous polynomial $f' \in \mathbb{C}[x_1, \ldots, x_n]$ as $f' := \sum_{\alpha \in \Gamma_f} a_\alpha x^\alpha$, and a hypersurface $V' \subset \mathbb{C}^n$ as the zero set of $f'$ in $\mathbb{C}^n$. Note that $\Gamma_+(f') = \Gamma_+(f)$ and $f'$ is non-degenerate at 0. By Proposition 3.5.8 below, we will show that the mixed Hodge numbers of $H^j((\widetilde{\mathrm{IC}_V})_0)$ are the same of the mixed Hodge numbers of $H^j((\widetilde{\mathrm{IC}_{V'}})_0)$. The Milnor monodromy of a quasi-homogeneous polynomial is semisimple. Therefore we have (iii) $\Rightarrow$ (ii). $\square$

Remark 3.5.3. We can rewrite Theorem 3.5.1 in terms of the virtual Poincaré polynomial as follows:

$$\begin{aligned}\mathrm{P}((\widetilde{\mathrm{IC}_V})_0)(T) &= 1+(-1)^{n-2}\sum_{i=0}^{n-2} J^1_{n-i-1}T^i \\ &= 1+(-1)^{n-2}(J^1_{n-1}+J^1_{n-2}T+J^1_{n-3}T^2+\cdots+J^1_1T^{n-2}).\end{aligned}$$

Remark 3.5.4. By Theorem 3.4.2, there is no Jordan block of $\Phi_{n-1,0}$ with size $\geq n$ for the eigenvalue 1. Hence the dimension $N_0$ of the invariant subspace of $\Phi_{n-1,0}$ in $H^{n-1}(F_{f,0};\mathbb{Q})$ is equal to $J^1_1+\cdots+J^1_{n-1}$. On the other hand, by Proposition 3.4.13 and Theorem 3.5.1 we have $N_0 = \dim H^{n-2}((\widetilde{\mathrm{IC}_V})_0) = \sum_{r\in\mathbb{Z}} \dim \mathrm{Gr}^W_r H^{n-2}((\widetilde{\mathrm{IC}_V})_0)$. Namely Theorem 3.5.1 means that these two decompositions of $N_0$ are the same.

As a corollary of Theorem 3.5.1, we get the following result of the mixed Hodge structures of the cohomology groups of the link of the isolated singular point 0 in $V$. The intersection of $V$ and a small sphere centered at 0 is called the *link of* 0 *in* $V$ and we denote it by $L$. It is known that $L$ is a $(2n-3)$-dimensional orientable compact real manifold. We denote by $i$ the inclusion map $V\setminus\{0\}\hookrightarrow V$ and by $i_0$ the inclusion map $\{0\}\hookrightarrow V$. Then for any $j\in\mathbb{Z}$, the cohomology group $H^j(L;\mathbb{Q})$ of the link can be expressed as $H^j(i_0^{-1}\mathrm{R}j_*\mathbb{Q}_{V\setminus\{0\}})$. Note that $H^j(i_0^{-1}\mathrm{R}j_*\mathbb{Q}_{V\setminus\{0\}})$ has a canonical mixed Hodge structure by using the same argument for $H^j((\widetilde{\mathrm{IC}_V})_0)$, and hence we can endow $H^k(L;\mathbb{Q})$ with a canonical mixed Hodge structure. It is clear that for $j\leq n-2$, $H^j((\widetilde{\mathrm{IC}_V})_0)$ is isomorphic to $H^j(L;\mathbb{Q})$ as mixed Hodge structures. Moreover by [**11**, Proposition 3.3], for any $j\in\mathbb{Z}$, we have the Poincaré duality isomorphism as mixed Hodge structures:

$$H^j(L;\mathbb{Q})\simeq (H^{2n-3-j}(L;\mathbb{Q})(n-1))^*,$$

where $H^{2n-3-j}(L;\mathbb{Q})(n-1)$ stands for the $(n-1)$-Tate twist of $H^{2n-3-j}(L;\mathbb{Q})$. Therefore, $H^0(L;\mathbb{Q})$, $H^{n-2}(L;\mathbb{Q})$, $H^{n-1}(L;\mathbb{Q})$ and $H^{2n-3}(L;\mathbb{Q})$ are the only non-trivial cohomology groups and for any $r,j\in\mathbb{Z}$, we have

$$\dim \mathrm{Gr}^W_r H^j(L;\mathbb{Q}) = \dim \mathrm{Gr}^W_{2(n-1)-r} H^{2n-3-j}(L;\mathbb{Q}).$$

We thus obtain the following.

Corollary 3.5.5. *In the situation of Theorem 3.5.1, for any $r\in\mathbb{Z}$, we have*

$$\dim \mathrm{Gr}^W_r H^0(L;\mathbb{Q}) = \dim \mathrm{Gr}^W_{2(n-1)-r} H^{2n-3}(L;\mathbb{Q}) = \begin{cases} 1 & \text{if } r=0,\\ 0 & \text{if } r\neq 0,\end{cases}$$

*and*

$$\dim \mathrm{Gr}^W_r H^{n-2}(L;\mathbb{Q}) = \dim \mathrm{Gr}^W_{2(n-1)-r} H^{n-1}(L;\mathbb{Q}) = J^1_{n-r-1}.$$

First, we show Theorem 3.5.1 in the following very special case.

Lemma 3.5.6. *Assume that $f$ satisfies the conditions of Theorem 3.5.1. Moreover, suppose that $f$ has no linear term, its degree $m$ part is $\sum_{1\leq i\leq n} x_i^m$ and $f$ is non-degenerate (see Definition 3.4.8). Then, Theorem 3.5.1 holds.*

Proof. Let $g\in\mathbb{C}[x_0,\ldots,x_n]$ be the projectivization of $f$. Namely we set $g = x_0^m f(x_1/x_0,\ldots,x_n/x_0)$. Define a hypersurface $X\subset\mathbb{P}^n$ of $\mathbb{P}^n$ by $X := \{x\in\mathbb{P}^n \mid g(x)=0\}$. Then under the identification $\{[x_0:\cdots:x_n]\in\mathbb{P}^n \mid x_0\neq 0\}\simeq\mathbb{C}^n$, we have $X \cap \mathbb{C}^n = V$. We decompose $\mathbb{P}^n$ into a disjoint union of small tori as $\mathbb{P}^n = \bigsqcup_{\emptyset \neq J \subset \{0,\dots,n\}} T_J$, where we set

$$T_J := \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_i = 0 \ (i \notin J) \ x_i \neq 0 \ (i \in J)\} \simeq (\mathbb{C}^*)^{|J|-1},$$

for $\emptyset \neq J \subset \{0,\dots,n\}$. By the non-degeneracy of $f$, for any $J \neq \emptyset$, the torus $T_J$ intersects $X$ transversally. Thus there is only one singular point $p = [1 : 0 : \cdots : 0]$ in the hypersurface $X \subset \mathbb{P}^n$.

Consider the following distinguished triangle in $\mathrm{D^b}(\mathrm{SHM}^p)$:

$$\mathrm{R}\Gamma_c(X \setminus \{p\}; \widetilde{\mathrm{IC}_X}) \to \mathrm{R}\Gamma_c(X; \widetilde{\mathrm{IC}_X}) \to \mathrm{R}\Gamma_c(\{p\}; \widetilde{\mathrm{IC}_X}) \xrightarrow{+1} . \tag{1}$$

By $(\widetilde{\mathrm{IC}_X})|_{X\setminus\{p\}} = \mathbb{Q}_{X\setminus\{p\}}$, we have

$$\mathrm{R}\Gamma_c(X \setminus \{p\}; \widetilde{\mathrm{IC}_X}) = \mathrm{R}\Gamma_c(X \setminus \{p\}; \mathbb{Q}_{X\setminus\{p\}}).$$

Moreover, obviously we have

$$\mathrm{R}\Gamma_c(\{p\}; \widetilde{\mathrm{IC}_X}) = (\widetilde{\mathrm{IC}_V})_0.$$

Taking the virtual Poincaré polynomials of the terms in the distinguished triangle (1), we obtain

$$\begin{aligned} \mathrm{P}(\mathrm{R}\Gamma_c(X; \widetilde{\mathrm{IC}_X}))(T) &= \mathrm{P}(\mathrm{R}\Gamma_c(X \setminus \{p\}; \mathbb{Q}))(T) + \mathrm{P}((\widetilde{\mathrm{IC}_V})_0)(T) \\ &= \mathrm{P}(\mathrm{R}\Gamma_c(X; \mathbb{Q}))(T) - 1 + \mathrm{P}((\widetilde{\mathrm{IC}_V})_0)(T). \end{aligned} \tag{2}$$

Consider the following distinguished triangle in $\mathrm{D^b}(\mathrm{SHM}^p)$:

$$(\mathrm{R}f_!\mathbb{Q}^H_{\mathbb{C}^n})_0 \to \psi_t(\mathrm{R}f_!\mathbb{Q}^H_{\mathbb{C}^n}) \to \phi_t(\mathrm{R}f_!\mathbb{Q}^H_{\mathbb{C}^n}) \xrightarrow{+1}, \tag{3}$$

where $t : \mathbb{C} \to \mathbb{C}$ is the identity map. As objects in $\mathrm{D^b}(\mathrm{Mod}(\mathbb{Q}))$, the first term is isomorphic to $\mathrm{R}\Gamma_c(V; \mathbb{Q})$, and the second one is isomorphic to $\mathrm{R}\Gamma_c(V_\epsilon; \mathbb{Q})$, where we set $V_\epsilon := f^{-1}(\epsilon)$ for a sufficiently small $\epsilon > 0$. By the next lemma, the third one is isomorphic to $H^{n-1}(F_{f,0}; \mathbb{Q})[-(n-1)]$, where $F_{f,0}$ is the Milnor fiber of $f$ at 0.

Lemma 3.5.7 (cf. [**24**]). *We have an isomorphism in* $\mathrm{D^b}(\mathrm{MHM}(\mathrm{pt})) = \mathrm{D^b}(\mathrm{SHM}^p)$*:*

$$\phi_t(f_!\mathbb{Q}^H_{\mathbb{C}^n}) \simeq \phi_f(\mathbb{Q}^H_{\mathbb{C}^n})_0.$$

Proof. Let $\Gamma_\infty(f)$ be the convex hull of the union of $\{0\}$ and $\mathrm{NP}(f)$ in $\mathbb{R}^n$. Let $\Sigma_0$ be the fan formed by all faces of $(\mathbb{R}_{\geq 0})^n$ in $\mathbb{R}^n$, $\Sigma_1$ the normal fan of $\Gamma_\infty(f)$. Since $f$ is convenient, $\Sigma_0$ is a subfan of $\Sigma_1$. There exists a smooth subdivision $\Sigma$ of $\Sigma_1$ which contains $\Sigma_0$ as a subfan. We denote by $X_\Sigma$ the toric variety associated to $\Sigma$. We regard $f$ as an element of the function field of $X_\Sigma$, and eliminate its indeterminacy by blowing up $X_\Sigma$ (see [**24**, the proof of Theorem 3.6]). Then we obtain a smooth compact variety $\widetilde{X_\Sigma}$ and a proper morphism $\pi \colon \widetilde{X_\Sigma} \to X_\Sigma$ such that $g := f \circ \pi$ has no point of indeterminacy. Then $g$ can be considered as a proper map from $\widetilde{X_\Sigma}$ to $\mathbb{P}^1$. The restriction $g^{-1}(\mathbb{C}) \to \mathbb{C}$ of $g$ is also proper, and we use the same symbol $g$ for this restriction map. We thus obtain a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xhookrightarrow{\iota} & g^{-1}(\mathbb{C}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{C} & \overset{t}{=\!=\!=} & \mathbb{C}\ , \end{array}$$

where $\iota$ is the inclusion map. By [**9**, Proposition 4.2.11] and the properness of $g$, we obtain the following isomorphism:

$$\begin{aligned}\mathrm{R}\Gamma(g^{-1}(0);\phi_g(\iota_!\mathbb{Q}_{\mathbb{C}^n})) &\simeq \phi_t(\mathrm{R}g_*\iota_!\mathbb{Q}_{\mathbb{C}^n})\\ &\simeq \phi_t(\mathrm{R}f_!\mathbb{Q}_{\mathbb{C}^n}).\end{aligned}$$

It follows from the construction of $\widetilde{X_\Sigma}$ that the support of $\phi_g(\iota_!\mathbb{Q}_{\mathbb{C}^n})$ does not intersect $\widetilde{X_\Sigma}\setminus\mathbb{C}^n$ (see [**24**, the proof of Theorem 3.6]). Moreover, since $V=f^{-1}(0)\subset\mathbb{C}^n$ has only one singular point 0, the support of $\phi_g(\iota_!\mathbb{Q}_{\mathbb{C}^n})$ is $\{0\}$. Furthermore, obviously we have $\phi_g(\iota_!\mathbb{Q}_{\mathbb{C}^n})_0\simeq\phi_f(\mathbb{Q}_{\mathbb{C}^n})_0$. Combining these results, we finally obtain

$$\phi_t(\mathrm{R}f_!\mathbb{Q}_{\mathbb{C}^n})\simeq\mathrm{R}\Gamma(g^{-1}(0);\phi_g(\iota_!\mathbb{Q}_{\mathbb{C}^n}))\simeq\phi_g(\iota_!\mathbb{Q}_{\mathbb{C}^n})_0\simeq\phi_f(\mathbb{Q}_{\mathbb{C}^n})_0.$$

Since the forgetful functors from the categories of mixed Hodge modules to the categories of perverse sheaves are faithful, we have the above isomorphisms not only in $\mathrm{D^b}(\mathrm{Mod}(\mathbb{Q}))$ but also in $\mathrm{D^b}(\mathrm{MHM}(\mathrm{pt}))=\mathrm{D^b}(\mathrm{SHM}^p)$.

□

The $k$-th cohomology group $H^k_c(V;\mathbb{Q})$ of $\mathrm{R}\Gamma_c(V;\mathbb{Q})$ has Deligne's mixed Hodge structure (see Theorem 2.0.1). The $k$-th cohomology group $H^k_c(V_\epsilon;\mathbb{Q})$ of $\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})$ has the limit mixed Hodge structure which encodes the information of the monodromy action (see Theorem 3.3.1). Moreover the $(n-1)$-st cohomology group $H^{n-1}(F_{f,0};\mathbb{Q})$ of the Milnor fiber $F_{f,0}$ of $f$ at 0 has Steenbrink's mixed Hodge structure which encodes some informations of the Milnor monodromy $\Phi_{n-1,0}$ (see Section 3.4). On the other hand, the cohomology group of $\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})$ is endowed with also Deligne's mixed Hodge structure. In what follows, we use the symbol $\mathrm{R}\Gamma_c(f^{-1}(\epsilon);\mathbb{Q})_{\mathrm{Del}}$ to express $\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})$ with Deligne's mixed Hodge structure, and the symbol $\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})_{\lim}$ to express $\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})$ with the limit mixed Hodge structure.

Taking the virtual Poincaré polynomials of the terms in the distinguished triangle (3), we obtain

$$\mathrm{P}(\mathrm{R}\Gamma_c(V,\mathbb{Q})(T)=\mathrm{P}(\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})_{\lim})(T)-\mathrm{P}(H^{n-1}(F_{f,0};\mathbb{Q})[-(n-1)])(T). \quad (4)$$

Decompose $\mathrm{P}(\mathrm{R}\Gamma_c(X;\mathbb{Q}))(T)$ into $\mathrm{P}(\mathrm{R}\Gamma_c(V;\mathbb{Q}))(T)$ and $\mathrm{P}(\mathrm{R}\Gamma_c(X\setminus V;\mathbb{Q}))(T)$ and by using (4), we have

$$\begin{aligned}\mathrm{P}(\mathrm{R}\Gamma_c(X;\mathbb{Q}))(T)&=\mathrm{P}(\mathrm{R}\Gamma_c(V;\mathbb{Q}))(T)+\mathrm{P}(\mathrm{R}\Gamma_c(X\setminus V;\mathbb{Q}))(T)\\ &=\mathrm{P}(\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})_{\lim})(T)-\mathrm{P}(H^{n-1}(F_{f,0};\mathbb{Q})[-(n-1)])(T)\\ &\quad+\mathrm{P}(\mathrm{R}\Gamma_c(X\setminus V;\mathbb{Q}))(T).\end{aligned}$$

Putting this into (2), we obtain

$$\begin{aligned}\mathrm{P}(\mathrm{R}\Gamma_c(X;\widetilde{\mathrm{IC}_X}))(T)=\mathrm{P}(\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})_{\lim})(T)-\mathrm{P}(H^{n-1}(F_{f,0};\mathbb{Q})[-(n-1)])(T) \quad (5)\\ +\mathrm{P}(\mathrm{R}\Gamma_c(X\setminus V;\mathbb{Q}))(T)-1+\mathrm{P}((\widetilde{\mathrm{IC}_V})_0)(T).\end{aligned}$$

First, we examine the first term in this equation. We denote by $W_\bullet$ Deligne's weight filtration of $H_c^{n-1}(V_\epsilon;\mathbb{C})$, and by $M_\bullet$ the relative monodromy filtartion, respectively (see Section 3.3). Then for $q \neq n-1$, we have

$$\begin{aligned}
\dim\mathrm{Gr}_q^M H_c^{n-1}(V_\epsilon;\mathbb{C}) &= \sum_{r\in\mathbb{Z}} \dim\mathrm{Gr}_q^M \mathrm{Gr}_r^W H_c^{n-1}(V_\epsilon;\mathbb{C}) \\
&= \dim\mathrm{Gr}_q^M \mathrm{Gr}_q^W H_c^{n-1}(V_\epsilon;\mathbb{C}) \\
&\quad + \dim\mathrm{Gr}_q^M \mathrm{Gr}_{n-1}^W H_c^{n-1}(V_\epsilon;\mathbb{C}) \\
&= \dim\mathrm{Gr}_q^W H_c^{n-1}(V_\epsilon;\mathbb{C}) \\
&\quad + \dim\mathrm{Gr}_q^M \mathrm{Gr}_{n-1}^W H_c^{n-1}(V_\epsilon;\mathbb{C})
\end{aligned}$$

Here, the second and third equalities follow from Proposition 3.3.2. According to the weak Lefschetz type theorem (see [**4**, Proposition 3.9]), we have $H_c^k(V_\epsilon;\mathbb{C}) = 0$ for any $k \neq n-1, 2(n-1)$. The monodromy action on $H_c^{2(n-1)}(V_\epsilon;\mathbb{C}) \simeq \mathbb{C}$ is trivial, and hence $\dim\mathrm{Gr}_{2(n-1)}^M H_c^{2(n-1)}(V_\epsilon;\mathbb{Q}) = 1$ and $\dim\mathrm{Gr}_q^M H_c^{2(n-1)}(V_\epsilon;\mathbb{Q}) = 0$ for any $q \neq 2(n-1)$. We thus obtain

$$\begin{aligned}
\mathrm{P}(\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})_{\lim})(T) &= \sum_{i\in\mathbb{Z}} (-1)^{n-1}\dim\mathrm{Gr}_i^M H_c^{n-1}(V_\epsilon;\mathbb{Q})T^i + T^{2(n-1)} \qquad (6)\\
&= \sum_{n-1\neq i\in\mathbb{Z}} (-1)^{n-1}(\dim\mathrm{Gr}_i^W H_c^{n-1}(V_\epsilon;\mathbb{Q}) \\
&\qquad + \dim\mathrm{Gr}_i^M \mathrm{Gr}_{n-1}^W H_c^{n-1}(V_\epsilon;\mathbb{Q}))T^i \\
&\quad + (-1)^{n-1}\dim\mathrm{Gr}_{n-1}^M H_c^{n-1}(V_\epsilon;\mathbb{Q})T^{n-1} + T^{2(n-1)} \\
&= \sum_{i\in\mathbb{Z}} (-1)^{n-1}\dim\mathrm{Gr}_i^W H_c^{n-1}(V_\epsilon;\mathbb{Q})T^i \\
&\quad + T^{2(n-1)} + Q_1(T) \\
&= \mathrm{P}(\mathrm{R}\Gamma_c(f^{-1}(\epsilon);\mathbb{Q})_{\mathrm{Del}})(T) + Q_1(T),
\end{aligned}$$

where we set

$$\begin{aligned}
Q_1(T) :=& (-1)^{n-1} \sum_{n-1\neq i\in\mathbb{Z}} \dim\mathrm{Gr}_i^M \mathrm{Gr}_{n-1}^W H_c^{n-1}(V_\epsilon;\mathbb{C})T^i \\
&+ (-1)^{n-1}\dim\mathrm{Gr}_{n-1}^M H_c^{n-1}(V_\epsilon;\mathbb{Q})T^{n-1} \\
&- (-1)^{n-1}\dim\mathrm{Gr}_{n-1}^W H_c^{n-1}(V_\epsilon;\mathbb{Q})T^{n-1}.
\end{aligned}$$

Since the polynomial $\sum_{i\neq n-1} \dim\mathrm{Gr}_i^M \mathrm{Gr}_{n-1}^W H_c^{n-1}(V_\epsilon;\mathbb{C})T^i$ has a symmetry centered at the degree $n-1$ (see Section 3.3), $Q_1(T)$ is a symmetric polynomial centered at the degree $n-1$. The projectivization of $f - \epsilon \in \mathbb{C}[x_1, ..., x_n]$ is a homogeneous polynomial $g - \epsilon x_0^m \in \mathbb{C}[x_0, ..., x_n]$. We denote by $X_\epsilon$ the hypersurface in $\mathbb{P}^n$ defined by this polynomial. Then we have $X_\epsilon \cap \mathbb{C}^n = V_\epsilon$. Recall that we decomposed $\mathbb{P}^n$ into small tori. Since the intersection of each torus and $X_\epsilon$ is smooth, $X_\epsilon$ is smooth in $\mathbb{P}^n$. We define the hyperplane $\mathbb{L}$ of $\mathbb{P}^n$ by $\mathbb{L} := \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_0 = 0\} \simeq \mathbb{P}^{n-1}$. Then we have $\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{L}$ and $X_\epsilon \cap \mathbb{L} = X \cap \mathbb{L}$. By $f^{-1}(\epsilon) \sqcup (X \cap \mathbb{L}) = X_\epsilon$, we have

$$\mathrm{P}(\mathrm{R}\Gamma_c(f^{-1}(\epsilon);\mathbb{Q})_{\mathrm{Del}})(T) + \mathrm{P}(\mathrm{R}\Gamma_c(X\cap\mathbb{L};\mathbb{Q}))(T) = \mathrm{P}(\mathrm{R}\Gamma_c(X_\epsilon;\mathbb{Q}))(T).$$

We set $Q_2(T) := \mathrm{P}(\mathrm{R}\Gamma_c(X_\epsilon;\mathbb{Q}))(T)$. Since $X_\epsilon$ is a smooth hypersurface in $\mathbb{P}^n$, the polynomial $Q_2(T)$ has a symmetry centered at the degree $n-1$ by the Poincaré duality. Note that $X \cap \mathbb{L} = X \setminus V$. Thus the sum of the first and third terms of the right-hand side of the equation (5) is calculated as:

$$\begin{aligned}&\mathrm{P}(\mathrm{R}\Gamma_c(V_\epsilon;\mathbb{Q})_{\mathrm{lim}})(T) + \mathrm{P}(\mathrm{R}\Gamma_c(X\cap\mathbb{L};\mathbb{Q}))(T)\\ &= \mathrm{P}(\mathrm{R}\Gamma_c(X_\epsilon;\mathbb{Q}))(T) + Q_1(T)\\ &= Q_2(T) + Q_1(T).\end{aligned}$$

Next, we examine the second term of the right hand side of the equation (5). For a Laurent polynomial $\mathrm{P}(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathbb{Z}[T^{\pm}]$ of one variable with coefficients in $\mathbb{Z}$ and any integer $i_0 \in \mathbb{Z}$, we define the two polynomials:

$$\mathrm{trun}_{\leq i_0} \mathrm{P}(T) := \sum_{i\leq i_0} a_i T^i \quad \text{and} \quad \mathrm{trun}_{\geq i_0} \mathrm{P}(T) := \sum_{i\geq i_0} a_i T^i.$$

Let $Q_3(T) \in \mathbb{Z}[T]$ be the symmetric polynomial centered at $n-1$ satisfying the condition

$$\mathrm{trun}_{\leq n-1} Q_3(T) = (-1)^{n-1} \sum_{\lambda\neq 1} (\sum_{i=0}^{n-1} (\sum_{s\geq 0} J^{\lambda}_{n-i+2s}) T^i).$$

Moreover we also define the symmetric polynomial $Q_4(T) \in \mathbb{Z}[T]$ centered at $n$ by the condition

$$\mathrm{trun}_{\leq n} Q_4(T) = (-1)^{n-1} \sum_{i=2}^{n} (\sum_{s\geq 0} J^{1}_{n+1-i+2s}) T^i.$$

By Proposition 3.4.5, we have

$$\mathrm{P}(H^{n-1}(F_{f,0};\mathbb{Q})[-(n-1)])(T) = Q_3(T) + Q_4(T).$$

Then we can rewrite the equation (5) as,

$$\begin{aligned}\mathrm{P}(\mathrm{R}\Gamma_c(X;\widetilde{\mathrm{IC}_X}))(T) &= Q_1(T) + Q_2(T) - (Q_3(T) + Q_4(T))\\ &\quad - 1 + \mathrm{P}((\widetilde{\mathrm{IC}_V})_0)(T).\end{aligned} \tag{7}$$

As we saw in Section 3.4.1, the polynomial $\mathrm{P}((\widetilde{\mathrm{IC}_V})_0)(T)$ has a degree $\leq n-2$. Thus we have

$$\begin{aligned}&\mathrm{trun}_{\geq n-1} \mathrm{P}(\mathrm{R}\Gamma_c(X;\widetilde{\mathrm{IC}_X}))(T)\\ &= \mathrm{trun}_{\geq n-1}(Q_1(T) + Q_2(T) - Q_3(T) - Q_4(T)).\end{aligned}$$

Since $\mathrm{P}(\mathrm{R}\Gamma_c(X;\widetilde{\mathrm{IC}_X}))(T)$ has a symmetry centered at the degree $n-1$ by the generalized Poincaré duality, we also have

$$\begin{aligned}&\mathrm{trun}_{\leq n-1} \mathrm{P}(\mathrm{R}\Gamma_c(X;\widetilde{\mathrm{IC}_X}))(T)\\ &= T^{2(n-1)} \mathrm{trun}_{\geq n-1}(Q_1(T') + Q_2(T') - Q_3(T') - Q_4(T'))|_{T'=T^{-1}}.\end{aligned} \tag{8}$$

Then by taking the truncations $\mathrm{trun}_{\le n-1}(*)$ of the both sides of (7), we obtain

$$\begin{aligned} &\mathrm{P}((\widetilde{\mathrm{IC}_V})_0)(T) \\ &\quad = 1 + T^{2(n-1)}\,\mathrm{trun}_{\ge n-1}(Q_1(T') + Q_2(T') - Q_3(T') - Q_4(T'))|_{T'=T^{-1}} \\ &\qquad - \mathrm{trun}_{\le n-1}(Q_1(T) + Q_2(T) - Q_3(T) - Q_4(T)) \\ &\quad = 1 + \mathrm{trun}_{\ge n-1}(-Q_4(T'))|_{T'=T^{-1}} - \mathrm{trun}_{\le n-1}(-Q_4(T)) \\ &\quad = 1 + (-1)^{n-2} \sum_{s=0}^{n-2} J^1_{n-s+1} T^s, \end{aligned}$$

where the second equality follows from the symmetries centered at the degree $n-1$ of $Q_1(T), Q_2(T), Q_3(T)$. This completes the proof of Lemma 3.5.6. □

As we will see in the following Proposition 3.5.8, the coefficients of $f$ in $\Gamma_+(f)\backslash\Gamma_f$ do not affect the mixed Hodge numbers of the IC stalk. Fix a polytope $P \subset \Gamma_+(f)$ containing the Newton polytope of $f$ such that the union of bounded faces of $\mathrm{Conv}(\bigcup_{\alpha \in P\cap\mathbb{Z}^n}(\alpha + \mathbb{R}^n_{\ge 0}))$ is equal to $\Gamma_f$. Set $l := \#\{P \cap \mathbb{Z}^n\}$ and identify $\mathbb{C}^l$ with $\mathbb{C}^{P\cap\mathbb{Z}^n}$. For $y = (y_\alpha) \in \mathbb{C}^{P\cap\mathbb{Z}^n}$, set $\sigma_y f := \sum_\alpha (a_\alpha + y_\alpha)x^\alpha$ (when $\alpha \notin P$, we set $y_\alpha = 0$) and denote by $V_y$ the zero set of $\sigma_y f$ in $\mathbb{C}^n$. Let $\Omega \subset \mathbb{C}^l$ be the subset of $\mathbb{C}^l$ consisting $y \in \mathbb{C}^l$ such that $\Gamma_{\sigma_y f}$ is equal to $\Gamma_f$ and $\sigma_y f$ is non-degenerate at $0 \in \mathbb{C}^n$. Note that $\Omega$ is a Zariski open and hence path-connected subset of $\mathbb{C}^l$. Under this setting, we have the following result.

PROPOSITION 3.5.8. *For any $y \in \Omega$ and $j \in \mathbb{Z}$, the mixed Hodge numbers of $H^j((\mathrm{IC}_{V_y})_0)$ is the same as those of $H^j((\mathrm{IC}_V)_0)$.*

PROOF. We denote by $\Sigma_0$ the normal fan of $\Gamma_+(f)$ in $\mathbb{R}^n$. Take a smooth subdivision $\Sigma$ of the fan $\Sigma_0$ without subdividing the cones in the boundary of $(\mathbb{R}_{\ge 0})^n$ in $\mathbb{R}^n$. Let $\Sigma'$ be the fan formed by all faces of $(\mathbb{R}_{\ge 0})^n$ in $\mathbb{R}^n$. We denote by $X_\Sigma$ and $X_{\Sigma'}$ the smooth toric varieties associated to $\Sigma$ and $\Sigma'$ respectively. $X_{\Sigma'}$ is isomorphic to $\mathbb{C}^n$. The identity map $\mathbb{R}^n \to \mathbb{R}^n$ induces a morphism of fans from $\Sigma$ to $\Sigma'$, and we obtain a proper morphism $\pi\colon X_\Sigma \to \mathbb{C}^n$. Then $\pi$ induces an isomorphism $X_\Sigma \setminus \pi^{-1}(0) \simeq \mathbb{C}^n \setminus \{0\}$. Set $\widetilde{\pi} := \pi \times \mathrm{id}_\Omega\colon X_\Sigma \times \Omega \to \mathbb{C}^n \times \Omega$. Define a polynomial $\widetilde{f} \in \mathbb{C}[x_1, \dots, x_n, (y_\alpha)_{\alpha \in P\cap\mathbb{Z}^n}]$ of $n+l$ variables by $\widetilde{f}(x_1, \dots, x_n, (y_\alpha)_{\alpha\in P\cap\mathbb{Z}^n}) := \sum_{\alpha \in P\cap\mathbb{Z}^n}(a_\alpha + y_\alpha)x^\alpha$. By the definition of $\Omega$, the hypersurface in $X_\Sigma \times \Omega$ defined by $\widetilde{f} \circ \widetilde{\pi}$ is a normal crossing divisor in a sufficiently small open neighborhood $W'$ of $\pi^{-1}(0) \times \Omega$. Since $X_\Sigma \setminus \pi^{-1}(0) \simeq \mathbb{C}^n \setminus \{0\}$, $W := \widetilde{\pi}(W')$ is also an open neighborhood of $\{0\} \times \Omega$ with $W' = \widetilde{\pi}^{-1}(W)$. Let $\widetilde{V}$ be the hypersurface in $W$ defined by $\widetilde{f}$ and $\widetilde{V}'$ be that in $W'$ defined by $\widetilde{f} \circ \widetilde{\pi}$. Note that $D := \{0\} \times \Omega \subset \widetilde{V}$, $D' := \pi^{-1}(0) \times \Omega \subset \widetilde{V}'$ and $(\widetilde{\pi})^{-1}(\widetilde{V}) = \widetilde{V}'$. For $y \in \Omega$, we set $\widetilde{V}_y := \widetilde{V} \cap (\mathbb{C}^n \times \{y\}) \subset \mathbb{C}^n \times \{y\} (\simeq \mathbb{C}^n)$ and $\widetilde{V}'_y := \widetilde{V} \cap (X_\Sigma \times \{y\}) \subset X_\Sigma \times \{y\} (\simeq X_\Sigma)$. Then, $\widetilde{V}_y$ is equal to $V_y$ in a small neighborhood of 0 and we thus obtain $(\mathrm{IC}_{\widetilde{V}_y})_0 \simeq (\mathrm{IC}_{V_y})_0$. Set $\widetilde{U} := \widetilde{V} \setminus D$, $\widetilde{U}' := \widetilde{V}' \setminus D'$,

$\widetilde{U}_y := \widetilde{V}_y \setminus \{0\}$ and $\widetilde{U}'_y := \widetilde{V}'_y \setminus D'$. We consider the Cartesian diagrams:

$$\begin{array}{ccccc}
D' & \overset{i'}{\hookrightarrow} & \widetilde{V}' & \overset{j'}{\hookleftarrow} & \widetilde{U}' \\
{\scriptstyle \widetilde{\pi}'_0}\downarrow & & \downarrow{\scriptstyle \widetilde{\pi}'} & & \downarrow\wr \\
D & \overset{i}{\hookrightarrow} & \widetilde{V} & \overset{j}{\hookleftarrow} & \widetilde{U} \\
{\scriptstyle 0_y}\uparrow & & {\scriptstyle s_y}\uparrow & & \uparrow{\scriptstyle t_y} \\
\{0\} & \overset{i_y}{\hookrightarrow} & \widetilde{V}_y & \overset{j_y}{\hookleftarrow} & \widetilde{U}_y,
\end{array}$$

where $i$, $i'$, $i_y$, $j$, $j'$, $j_y$, $s_y$ and $t_y$ are the natural inclusions, $\widetilde{\pi}'$ and $\widetilde{\pi}'_0$ are the restrictions of $\widetilde{\pi}$ and $0_y$ is the map which sends $0$ to $(0, y)$.

LEMMA 3.5.9. *The restriction $i^{-1}\mathrm{R}j_*\mathbb{Q}_{\widetilde{U}}$ of $\mathrm{R}j_*\mathbb{Q}_{\widetilde{U}}$ is cohomologically constant i.e. all cohomology sheaves $H^k(i^{-1}\mathrm{R}j_*\mathbb{Q}_{\widetilde{U}})$ $(k \in \mathbb{Z})$ are locally constant on $\{0\}\times\Omega(\simeq \Omega)$. In particular, the mixed Hodge module $H^k(i^*j_*\mathbb{Q}^H_{\widetilde{U}})$ is a variation of mixed Hodge structure on $\Omega$, and thus for fixed $p, q \in \mathbb{Z}$ and $k \in \mathbb{Z}$ the $(p, q)$-mixed Hodge numbers of all the stalks of the variation of mixed Hodge structure $H^k(i^*j_*\mathbb{Q}^H_{\widetilde{U}})$ on $\Omega$ are equal.*

PROOF. We have

$$\begin{aligned}
i^{-1}\mathrm{R}j_*\mathbb{Q}_{\widetilde{U}} \simeq & i^{-1}\mathrm{R}\widetilde{\pi}'_*\mathrm{R}j'_*\mathbb{Q}_{\widetilde{U}'} \\
\simeq & \mathrm{R}(\widetilde{\pi}'_0)_* i'^{-1}\mathrm{R}j'_*\mathbb{Q}_{\widetilde{U}'}.
\end{aligned}$$

Since $\widetilde{V}'$ is a normal crossing divisor and $D'$ is a union of some irreducible components of it, by Proposition 4.2.1 of [**41**] $i'^{-1}\mathrm{R}j'_*\mathbb{Q}_{\widetilde{U}'}$ is constructible with respect to the natural stratification of the normal crossing divisor $D'$ (in particular, any stratum is locally of the form $A \times \Omega$ for a stratum $A \subset \pi^{-1}(0)$). Therefore, $\mathrm{R}(\widetilde{\pi}'_0)_* i'^{-1}\mathrm{R}j'_*\mathbb{Q}_{\widetilde{U}'}$ is cohomologically locally constant on $\Omega$ (by Proposition 5.4.4 and 8.4.1 in [**18**] or Thom's isotopy lemma). □

Let us continue the proof of Proposition 3.5.8. We consider the Cartesian diagrams:

$$\begin{array}{ccccc}
\widetilde{V} & \overset{\widetilde{\pi}'}{\longleftarrow} & \widetilde{V}' & \overset{j'}{\hookleftarrow} & \widetilde{U}' \\
{\scriptstyle s_y}\uparrow & & {\scriptstyle s'_y}\uparrow & & \uparrow{\scriptstyle t'_y} \\
\widetilde{V}_y & \underset{\widetilde{\pi}'_y}{\longleftarrow} & \widetilde{V}'_y & \underset{j'_y}{\hookleftarrow} & \widetilde{U}'_y,
\end{array}$$

where $j'_y$, $s'_y$ and $t'_y$ are the natural inclusions and $\widetilde{\pi}'_y$ is the restriction of $\widetilde{\pi}$. As stated in the proof of Lemma 3.5.9, $\widetilde{V}'$ is a normal crossing divisor and $D'$ is a union of some irreducible components of it, and $\mathrm{R}j'_*\mathbb{Q}_{\widetilde{U}'}$ is a cohomologically constructible with respect to the natural stratification of $\widetilde{V'}$. Therefore by Proposition 4.3.1 of [**41**] (or Proposition 5.4.13 of [**18**] and the proper base change) the natural morphism

$$(s'_y)^{-1}\mathrm{R}j'_*\mathbb{Q}_{\widetilde{U}'} \to \mathrm{R}(j'_y)_*(t'_y)^{-1}\mathbb{Q}_{\widetilde{U}'}$$

is an isomorphism. Then we have

$$\begin{aligned}(0_y)^{-1}i^{-1}\mathrm{R}j_*\mathbb{Q}_{\widetilde{U}} &\simeq i_y^{-1}s_y^{-1}\mathrm{R}j_*\mathbb{Q}_{\widetilde{U}}\\ &\simeq i_y^{-1}s_y^{-1}\mathrm{R}\widetilde{\pi}'_*\mathrm{R}j'_*\mathbb{Q}_{\widetilde{U}'}\\ &\simeq i_y^{-1}\mathrm{R}(\widetilde{\pi}'_y)_*(s'_y)^{-1}\mathrm{R}j'_*\mathbb{Q}_{\widetilde{U}'}\\ &\simeq i_y^{-1}\mathrm{R}(\widetilde{\pi}'_y)_*\mathrm{R}(j'_y)_*(t'_y)^{-1}\mathbb{Q}_{\widetilde{U}'}\\ &\simeq i_y^{-1}\mathrm{R}(\widetilde{\pi}'_y)_*\mathrm{R}(j'_y)_*\mathbb{Q}_{\widetilde{U}'_y}\\ &\simeq i_y^{-1}\mathrm{R}(j_y)_*\mathbb{Q}_{\widetilde{U}_y}.\end{aligned}$$

Therefore, by the isomorphism $i_y^{-1}H^k(\mathrm{R}(j_y)_*\mathbb{Q}_{\widetilde{U}_y}) \simeq i_y^{-1}H^k(\widetilde{\mathrm{IC}_{\widetilde{V_y}}})(\simeq i_y^{-1}H^k(\widetilde{\mathrm{IC}_{V_y}}))$ $(k \leq n-2)$, we obtain

$$(0_y)^{-1}H^k(i^{-1}\mathrm{R}j_*\mathbb{Q}_{\widetilde{U}}) \simeq i_y^{-1}H^k(\widetilde{\mathrm{IC}_{V_y}}) \quad (k \leq n-2). \tag{9}$$

Since the forgetful functors from the categories of mixed Hodge modules to the categories of perverse sheaves are faithful, the isomorphism (9) also holds in the context of the corresponding mixed Hodge modules. Therefore, by the latter part of Lemma 3.5.9 the $(p,q)$-mixed Hodge numbers of $H^k((\mathrm{IC}_{V_y})_0)$ do not depend on $y \in \Omega$.

□

By Proposition 3.5.8, the proof of Theorem 3.5.1 is reduced to the case in Lemma 3.5.6 as follows.

Proof of Theorem 3.5.1. If the polynomial $f$ has some linear terms, $V$ is non-singular at $0 \in V$ and $\Phi_{n-1,0} = 0$. Thus, our assertion is trivial. Therefore, in what follows, we assume that the polynomial $f$ has no linear term. Since the non-degeneracy condition is a generic condition for each fixed Newton polytope, there exist a polytope $P \subset \mathbb{R}^n_{\geq 0}$ and $y \in \Omega$ for which $\sigma_y f$ satisfies the conditions of Lemma 3.5.6. By Proposition 3.5.8, the mixed Hodge numbers of $\widetilde{(\mathrm{IC}_{V_y})_0}$ and $(\widetilde{\mathrm{IC}_V})_0$ are the same. On the other hand, since both $\sigma_y f$ and $f$ are non-degenerate at 0, convenient and satisfy the condition $\Gamma_{\sigma_y f} = \Gamma_f$, the Jordan normal forms of the Milnor monodromies of $\sigma_y f$ and $f$ are the same (see Theorem 3.4.12). Then the assertion immediately follows from Lemma 3.5.6. □

For $n = 2$, similarly we obtain the following theorem.

Theorem 3.5.10. *Assume that $n = 2$ and $f$ is convenient and non-degenerate at 0. Then for any $r \in \mathbb{Z}$, we have*

$$\dim \mathrm{Gr}_r^W H^0((\widetilde{\mathrm{IC}_V})_0) = \begin{cases} N_0 + 1 & \text{if } r = 0, \text{ and} \\ 0 & \text{if } r \neq 0. \end{cases}$$

CHAPTER 4

# On the monodromies and the limit mixed Hodge structures of families of algebraic varieties (joint work with Kiyoshi Takeuchi)

## 4.1. Introduction for Chapter 4

Families of algebraic varieties are basic objects in algebraic geometry. Here we are interested in the special but fundamental case where such one $Y$ is smooth and defined over the punctured disk $B(0;\varepsilon)^* = \{t \in \mathbb{C} \mid 0 < |t| < \varepsilon\} \subset \mathbb{C}$ $(0 < \varepsilon \ll 1)$ in $\mathbb{C}$. For this family $\pi_Y : Y \longrightarrow B(0;\varepsilon)^*$ let us consider its fibers $Y_t = \pi_Y^{-1}(t) \subset Y$ $(0 < |t| < \varepsilon)$ and their cohomology groups $H^j(Y_t;\mathbb{C})$ $(j \in \mathbb{Z})$. Then it is our primary interest to know $H^j(Y_t;\mathbb{C})$ themselves and the monodromy operators acting on them. Moreover, in addition to the classical mixed Hodge structure of Deligne, each cohomology group $H^j(Y_t;\mathbb{C})$ is endowed with the limit mixed Hodge structure which encodes some information of the monodromy (see El Zein [**12**] and Steenbrink-Zucker [**47**] etc.). However in general, it is very hard to compute the monodromies and the limit mixed Hodge numbers explicitly. Very recently, based on our previous works [**13**], [**24**], [**25**] and some new results in [**19**], [**20**], Stapledon [**44**] succeeded in computing the latter ones for families $Y \subset B(0;\varepsilon)^* \times \mathbb{C}^n$ of schön hypersurfaces in $\mathbb{C}^n$. Here the schönness is a very weak condition which is almost always satisfied. More precisely, in [**44**] the author expressed the motivic nearby fiber $\psi_t([Y])$ of such $Y \longrightarrow B(0;\varepsilon) = \{t \in \mathbb{C} \mid |t| < \varepsilon\} = B(0;\varepsilon)^* \sqcup \{0\}$ by the function $t = \mathrm{id}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$ in terms of the tropical variety associated to the defining Laurent polynomial $f(t,x) \in \mathbb{C}(t)[x_1,\dots,x_n]$ of the hypersurface $Y \subset B(0;\varepsilon)^* \times \mathbb{C}^n$ and obtained a complete description of the limit mixed Hodge numbers of $H^j(Y_t;\mathbb{C})$ $(0 < |t| < \varepsilon)$. His idea is to subdivide $\mathbb{C}^n$ into some algebraic tori $(\mathbb{C}^*)^k$ $(0 \leq k \leq n)$ by the additivity of $\psi_t$ and apply the arguments of Batyrev-Borisov [**1**] and Borisov-Mavlyutov [**2**] etc. in mirror symmetry to each piece $(\mathbb{C}^*)^k$ by using the new special polynomials introduced in Katz-Stapledon [**19**]. In particular, he effectively used the purity and the generalized Poincaré duality for the intersection cohomology groups of some singular hypersurfaces in the toric compactifications of $(\mathbb{C}^*)^k$ to obtain these remarkable results. Thanks to them, in Section 3, we obtained some new results on the weight filtrations of the stalks of intersection cohomology complexes. However the motivic nearby fiber $\psi_t([Y])$ used in the paper [**44**] is the one introduced by Steenbrink [**46**] very recently with the help of the semi-stable reduction theorem and it is not clear for us if it coincides with the classical (and more standard) one of [**7**], [**8**] and [**15**] etc. (see also Raibaut [**31**] for a nice introduction to this subject). Moreover the arguments in [**44**] heavily depend on some deep technical results in recent tropical geometry. It is therefore desirable to use the classical motivic nearby fibers and describe them without using the tropical geometry. The aim of this chapter is to simplify Stapledon's arguments and extend his results to families of schön complete intersection subvarieties in $\mathbb{C}^n$. Moreover, by extending our previous results in [**13**], [**24**], [**48**] etc., as in [**48**] we show that some parts of the Jordan normal forms of the monodromies on $H^j(Y_t;\mathbb{C})$ $(0<|t|<\varepsilon)$ can be described very explicitly. In order to explain these results more precisely, let us introduce our geometric situation and notations. Let $\mathbb{K}=\mathbb{C}(t)$ be the field of rational functions of $t$ and $f(t,x)=\sum_{v\in\mathbb{Z}^n} a_v(t)x^v \in \mathbb{K}[x_1^{\pm},\dots,x_n^{\pm}]$ $(a_v(t)\in\mathbb{K})$ a Laurent polynomial of $x=(x_1,\dots,x_n)$ with coefficients in $\mathbb{K}$. Then we define a family $Y\subset B(0;\varepsilon)^*\times(\mathbb{C}^*)^n$ of hypersurfaces in the algebraic torus $(\mathbb{C}^*)^n$ over the punctured disk $B(0;\varepsilon)^*$ by $Y=f^{-1}(0)\subset B(0;\varepsilon)^*\times(\mathbb{C}^*)^n$. If $f$ has a very special form $f(t,x)=1-tg(x)$ for some Laurent polynomial $g(x)\in\mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}]$ the monodromy of $Y$ around the origin $0\in\mathbb{C}$ is nothing but the monodromy at infinity of the polynomial map $g:(\mathbb{C}^*)^n\longrightarrow\mathbb{C}$. In this sense, our setting is a vast generalization of the classical ones of [**3**], [**13**], [**14**], [**22**], [**23**], [**24**], [**26**], [**31**], [**33**], [**34**], [**42**], [**49**] etc. For $v\in\mathbb{Z}^n$ by the Laurent expansion $a_v(t)=\sum_{j\in\mathbb{Z}} a_{v,j}t^j$ $(a_{v,j}\in\mathbb{C})$ of the rational function $a_v(t)$ we set

$$o(v):=\operatorname{ord}_t a_v(t)=\min\{j \mid a_{v,j}\neq 0\}.$$

If $a_v(t)\equiv 0$ we set $o(v)=+\infty$. Then we define an (unbounded) polyhedron $\mathrm{UH}_f$ in $\mathbb{R}^{n+1}$ by

$$\mathrm{UH}_f=\operatorname{Conv}\Big[\bigcup_{v\in\mathbb{Z}^n}\{(v,s)\in\mathbb{R}^{n+1} \mid s\geq o(v)\}\Big]\subset\mathbb{R}^{n+1},$$

where Conv $(\,\cdot\,)$ stands for the convex hull. We call it the upper-half polyhedron of $f$. Throughout this chapter we assume that the dimension of $\mathrm{UH}_f$ is $n+1$. Then by the projection $p:\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R}^1\twoheadrightarrow\mathbb{R}^n$ we obtain an $n$-dimensional polytope $P:=p(\mathrm{UH}_f)\subset\mathbb{R}^n$ which we call the Newton polytope of $f$. Let $\nu_f:P\to\mathbb{R}$ be the function defining the bottom part of the boundary $\partial\mathrm{UH}_f$ of $\mathrm{UH}_f$ and $\mathcal{S}$ the subdivision of $P$ by the lattice polytopes $p(\widetilde{F})\subset\mathbb{R}^n$ $(\widetilde{F}\prec\mathrm{UH}_f)$. We call such $F=p(\widetilde{F})\in\mathcal{S}$ a cell in $\mathcal{S}$. We denote by rel.int$F$ its relative interior i.e. its interior in the affine span $\mathrm{Aff}(F)\simeq\mathbb{R}^{\dim F}$ of $F$. Then by defining a hypersurface $V_F$ of the algebraic torus $T_F=\mathrm{Spec}(\mathbb{C}[\mathrm{Aff}(F)\cap\mathbb{Z}^n])\simeq(\mathbb{C}^*)^{\dim F}$ for each cell $F\in\mathcal{S}$ and the schönness of the family $Y$ etc. (see Section 4.3 for the details), we reobtain the following beautiful result of Stapledon [**44**]. For the family $Y$ over the punctured disk, denote by $\psi_t([Y])\in\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ its motivic nearby fiber by the function $t=\mathrm{id}_{\mathbb{C}}:\mathbb{C}\longrightarrow\mathbb{C}$ (see Section 4.2 for the details).

THEOREM 4.1.1. *Assume that the family $Y$ of hypersurfaces in $(\mathbb{C}^*)^n$ is schön. Then we have an equality*

$$\psi_t([Y])=\sum_{\mathrm{rel.int}F\subset\mathrm{Int}\,P}[V_F\circlearrowleft\hat{\mu}]\cdot(1-\mathbb{L})^{n-\dim F}$$

*in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.*

We prove this theorem by using only the classical toric geometry and a result of Guibert-Loeser-Merle [**15**] (see Theorem 4.2.1). Then in Section 4.4 we apply it to families $Y\subset B(0;\varepsilon)^*\times\mathbb{C}^n$ of schön hypersurfaces in $\mathbb{C}^n$ and describe some parts of the Jordan normal forms of the monodromies on $H^j(Y_t;\mathbb{C})$ $(0<|t|<\varepsilon)$ explicity. More precisely, as in [**48**] we define a finite subset $R_f$ of $\mathbb{C}$ by $\mathrm{UH}_f$ and describe the Jordan normal forms for the eigenvalues $\lambda\notin R_f$. For this purpose, first we prove the following concentration theorem. For $j\in\mathbb{Z}$ and $\lambda\in\mathbb{C}$ let

$$H^j(Y_t;\mathbb{C})_\lambda\subset H^j(Y_t;\mathbb{C}) \qquad (\text{resp. } H_c^j(Y_t;\mathbb{C})_\lambda\subset H_c^j(Y_t;\mathbb{C}))$$

be the generalized eigenspace of the monodromy automorphism $\Phi_j : H^j(Y_t;\mathbb{C}) \xrightarrow{\sim} H^j(Y_t;\mathbb{C})$ (resp. $\Phi_j : H^j_c(Y_t;\mathbb{C}) \xrightarrow{\sim} H^j_c(Y_t;\mathbb{C})$) for $0 < |t| < \varepsilon$.

THEOREM 4.1.2. *(see Theorem 4.4.6, Corollary 4.4.7 and Remark 4.4.8) Assume that the family $Y$ of hypersurfaces in $\mathbb{C}^n$ is schön. Then for any $\lambda \notin R_f$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ we have isomorphisms*

$$H^j_c(Y_t;\mathbb{C})_\lambda \simeq H^j(Y_t;\mathbb{C})_\lambda \qquad (j \in \mathbb{Z})$$

*and the concentration*

$$H^j(Y_t;\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1).$$

By the proofs of this theorem (see Theorem 4.4.6) and Sabbah's one [**34**, Theorem 13.1] we see also that for $\lambda \notin R_f$ the filtration on the only non-trivial cohomology group $H^{n-1}(Y_t;\mathbb{C})_\lambda$ induced by Deligne's weight filtration on $H^{n-1}(Y_t;\mathbb{C})$ is concentrated in degree $n-1$. Since $R_f$ is just a small part of the set of the eigenvalues of the monodromies of $Y$, Theorem 4.1.2 asserts that the geometric complexity of the family $Y$ is concentrated in the middle dimension $n-1 = \dim Y_t$. This enables us to describe the Jordan normal forms of the middle-dimensional monodromies

$$\Phi_{n-1} : H^{n-1}(Y_t;\mathbb{C})_\lambda \xrightarrow{\sim} H^{n-1}(Y_t;\mathbb{C})_\lambda \qquad (0 < |t| < \varepsilon)$$

for the eigenvalues $\lambda \notin R_f$ as follows, in terms of the equivariant limit mixed Hodge polynomials obtained by Stapledon's results in [**44**].

THEOREM 4.1.3. *Assume that the family $Y$ of hypersurfaces in $\mathbb{C}^n$ is schön. For $\lambda \in \mathbb{C}$ and $m \geq 1$ denote by $J_{\lambda,m}$ the number of the Jordan blocks in the monodromy*

$$\Phi_{n-1} : H^{n-1}(Y_t;\mathbb{C})_\lambda \xrightarrow{\sim} H^{n-1}(Y_t;\mathbb{C})_\lambda \qquad (0 < |t| < \varepsilon)$$

*for the eigenvalue $\lambda$ with size $m$. Then for $\lambda \notin R_f$ we have*

$$\sum_{m=0}^{n-1} J_{\lambda,n-m} s^{m+2} = \sum_{F \in \mathcal{S}} s^{\dim F+1} l^*_\lambda(F, \nu_f|_F; 1) \cdot \widetilde{l}_P(\mathcal{S}, F; s^2)$$

*(for the definitions of the polynomials $l^*_\lambda(F, \nu_f|_F; u) \in \mathbb{Z}[u]$ and $\widetilde{l}_P(\mathcal{S}, F; t) \in \mathbb{Z}[t]$ see Sections 4.2 and 4.3).*

The proof of Theorem 4.1.2 relies on Morihiko Saito's theorem on the primitive decompositions of the mixed Hodge modules over nearby cycle preverse sheaves associated to normal crossing divisors and we apply it to nearby cycle preverse sheaves on some smooth toric varieties. For the proof, we are also indepted to [**44**, Theorem 5.7] which is proved by using [**24**, Section 2] and some deep results on combinatorics obtained by Stanley [**43**]. Theorem 4.1.2 holds true without any assumption on the shape of the Newton polytope $P = p(\mathrm{UH}_f) \subset \mathbb{R}^n$. In particular, we do not require here that $P$ is convenient as in [**44**]. The convenience of $P$ of [**44**] is stronger than the usual one (see Definition 4.4.1) and we cannot expect it in general. For the treatment of the non-convenient case, we have to prove the topological concentration

$$H^j(Y_t;\mathbb{C})_\lambda \simeq 0 \qquad (\lambda \notin R_f, j \neq n-1)$$

in Theorem 4.1.2 which does not follow from the results in Danilov-Khovanskii [**4**] and Stapledon [**44**]. Moreover in Section 4.5, we also extend these results to families of schön complete intersection subvarieties in $\mathbb{C}^n$ and obtain a formula for the Jordan normal forms of their monodromies. To our surprise, the results that we obtain in this generalized situation are completely parallel to the ones for families of hypersurfaces in $\mathbb{C}^n$. See Section 4.5 for the details.

## 4.2. Preliminary notions and results

In this section, we introduce some preliminary notions and results which will be used in this chapter.

**4.2.1. Motivic Milnor fibers.** Throughtout this chapter we consider only varieties over the field $\mathbb{C}$ of complex numbers. From now we shall introduce the theory of motivic nearby fibers of Denef-Loeser [**7**], [**8**] and Guibert-Loeser-Merle [**15**] in this special case (see also Raibaut [**31**]). For a variety $S$ denote by $\mathrm{K}_0(\mathrm{Var}_S)$ the Grothendieck ring of varieties over $S$. Recall that the ring structure is defined by the fiber products over $S$. Moreover we denote by $\mathcal{M}_S$ the ring obtained from it by inverting the Lefschetz motive $\mathbb{L} \simeq \mathbb{C} \times S \in \mathrm{K}_0(\mathrm{Var}_S)$. If $S = \mathrm{Spec}(\mathbb{C})$ we denote $\mathrm{K}_0(\mathrm{Var}_S)$ and $\mathcal{M}_S$ simply by $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ and $\mathcal{M}_{\mathbb{C}}$ respectively. Note that $\mathcal{M}_S$ has a natural structure of an $\mathcal{M}_{\mathbb{C}}$-module. For $d \in \mathbb{Z}_{>0}$, let $\mu_d = \{\zeta \in \mathbb{C} \mid \zeta^d = 1\} \simeq \mathbb{Z}/\mathbb{Z}d$ be the multiplicative group consisting of $d$ roots of unity in $\mathbb{C}$. We denote by $\hat{\mu}$ the projective limit $\varprojlim_d \mu_d$ of the projective system $\{\mu_i\}_{i\geq 1}$ with morphisms $\mu_{id} \longrightarrow \mu_i$ given by $t \longmapsto t^d$. Then we define the Grothendieck ring $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_S)$ of varieties over $S$ with good $\hat{\mu}$-actions and its localization $\mathcal{M}_S^{\hat{\mu}}$ as in [**8**, Section 2.4]. Note that $\mathcal{M}_S^{\hat{\mu}}$ is naturally a $\mathcal{M}_{\mathbb{C}}$-module. Recall also that for a morphism $\pi : S \longrightarrow S'$ of varieties we have a group homomorphism

$$\pi_! : \mathcal{M}_S^{\hat{\mu}} \longrightarrow \mathcal{M}_{S'}^{\hat{\mu}} \tag{10}$$

obtained by the composition with $\pi$. Now let $Z$ be a smooth variety and $U \subset Z$ its Zariski open subset such that $D = Z \setminus U$ is a normal crossing divisor in $Z$. Moreover let $f : Z \longrightarrow \mathbb{C}$ be a regular function on $Z$ such that $f^{-1}(0) \subset Z$ is contained in $D$. Denote by $D' \subset D$ the union of irreducible components of $D$ which are not contained in $f^{-1}(0)$ and set $\Omega = Z \setminus D'$. Then we have $U \subset \Omega$. Let $E_1, E_2, \ldots, E_k$ be the irreducible components of the normal crossing divisor $\Omega \cap f^{-1}(0)$ in $\Omega \subset Z$. For each $1 \leq i \leq k$, let $b_i > 0$ be the order of the zero of $f$ along $E_i$. For a non-empty subset $I \subset \{1, 2, \ldots, k\}$, let us set

$$E_I = \bigcap_{i \in I} E_i, E_I^{\circ} = E_I \setminus \bigcup_{i \notin I} E_i \tag{11}$$

and $d_I = \gcd(b_i)_{i \in I} > 0$. Then, as in [**8**, Section 3.3], we can construct an unramified Galois covering $\widetilde{E_I^{\circ}} \longrightarrow E_I^{\circ}$ of $E_I^{\circ}$ as follows. First, for a point $p \in E_I^{\circ}$ we take an affine open neighborhood $W \subset \Omega \setminus (\cup_{i \notin I} E_i)$ of $p$ on which there exist regular functions $\xi_i$ $(i \in I)$ such that $E_i \cap W = \{\xi_i = 0\}$ for any $i \in I$. Then on $W$ we have $f = f_{1,W}(f_{2,W})^{d_I}$, where we set $f_{1,W} = f \prod_{i \in I} \xi_i^{-b_i}$ and $f_{2,W} = \prod_{i \in I} \xi_i^{\frac{b_i}{d_I}}$. Note that $f_{1,W}$ is a unit on $W$ and $f_{2,W} \colon W \longrightarrow \mathbb{C}$ is a regular function. It is easy to see that $E_I^{\circ}$ is covered by such affine open subsets $W$ of $\Omega \setminus (\cup_{i \notin I} E_i)$. Then as in [**8**, Section 3.3] by gluing the varieties

$$\widetilde{E_{I,W}^{\circ}} = \{(t, z) \in \mathbb{C}^* \times (E_I^{\circ} \cap W) \mid t^{d_I} = (f_{1,W})^{-1}(z)\} \tag{12}$$

together in an obvious way, we obtain the variety $\widetilde{E_I^\circ}$ over $E_I^\circ$. This unramified Galois covering $\widetilde{E_I^\circ}$ of $E_I^\circ$ admits a natural $\mu_{d_I}$-action defined by assigning the automorphism $(t,z) \longmapsto (\zeta_{d_I} t, z)$ of $\widetilde{E_I^\circ}$ to the generator $\zeta_{d_I} := \exp(2\pi\sqrt{-1}/d_I) \in \mu_{d_I}$. Namely the variety $\widetilde{E_I^\circ}$ is endowed with a good $\hat{\mu}$-action in the sense of [**8**, Section 2.4] and defines an element $[\widetilde{E_I^\circ}]$ of $\mathcal{M}_{f^{-1}(0)}^{\hat{\mu}}$. Finally we set

$$\mathcal{S}_{f,U} = \sum_{I \neq \emptyset} (1-\mathbb{L})^{|I|-1} \cdot [\widetilde{E_I^\circ}] \in \mathcal{M}_{f^{-1}(0)}^{\hat{\mu}}. \tag{13}$$

Then we have the following result.

THEOREM 4.2.1 ([**15**, Theorem 3.9]). *Let $X$ be a variety and $g : X \longrightarrow \mathbb{C}$ a regular function on it. Then there exists a morphism of $\mathcal{M}_{\mathbb{C}}$-modules*

$$\psi_g : \mathcal{M}_X \longrightarrow \mathcal{M}_{g^{-1}(0)}^{\hat{\mu}} \tag{14}$$

*such that for any proper morphism $\pi : Z \longrightarrow X$ from a smooth variety $Z$ and its Zariski open subset $U \subset Z$ whose complement $D = Z \setminus U$ is a normal crossing divisor in $Z$ containing $(g \circ \pi)^{-1}(0)$ we have the equality*

$$\psi_g([U \longrightarrow X]) = (\pi|_{(g\circ\pi)^{-1}(0)})_!(\mathcal{S}_{g\circ\pi,U}) \tag{15}$$

*in $\mathcal{M}_{g^{-1}(0)}^{\hat{\mu}}$.*

DEFINITION 4.2.2. In the situation of Theorem 4.2.1, for a variety $[Y \longrightarrow X] \in \mathcal{M}_X$ over $X$ we call $\psi_g([Y \longrightarrow X]) \in \mathcal{M}_{g^{-1}(0)}^{\hat{\mu}}$ the motivic nearby fiber of $Y$ by $g$ and denote it simply by $\psi_g([Y])$.

Following the notations in [**8**, Sections 3.1.2 and 3.1.3], we denote by $\mathrm{HS}^{\mathrm{mon}}$ the abelian category of Hodge structures with a quasi-unipotent endomorphism. Let

$$\chi_h \colon \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \longrightarrow \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{16}$$

be the Hodge characteristic morphism defined in [**8**] which associates to a variety $Z$ with a good $\mu_d$-action the Hodge structure

$$\chi_h([Z]) = \sum_{j \in \mathbb{Z}} (-1)^j [H_c^j(Z;\mathbb{Q})] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{17}$$

with the actions induced by the one $z \longmapsto \exp(2\pi\sqrt{-1}/d)z$ $(z \in Z)$ on $Z$. We can generalize this construction as follows. For a variety $X$ let $\mathrm{MHM}_X$ be the abelian category of mixed Hodge modules on $X$ (see [**16**, Section 8.3] etc.) and $\mathrm{K}_0(\mathrm{MHM}_X)$ its Grothendieck ring. Then there exists a group homomorphism

$$H_X \colon \mathcal{M}_X \longrightarrow \mathrm{K}_0(\mathrm{MHM}_X) \tag{18}$$

such that for any morphism $\pi \colon Z \longrightarrow X$ from a smooth variety $Z$ and the trivial Hodge module $\mathbb{Q}_Z^H$ on it we have

$$H_X([Z \longrightarrow X]) = \sum_{j \in \mathbb{Z}} (-1)^j [H^j R\pi_!(\mathbb{Q}_Z^H)]. \tag{19}$$

Here the Grothendieck ring $\mathrm{K}_0(\mathrm{MHM}_X)$ has a natural $\mathcal{M}_{\mathbb{C}}$-module structure defined by the Hodge realization map $H \colon \mathcal{M}_{\mathbb{C}} \longrightarrow \mathrm{K}_0(\mathrm{MHM}_{\mathrm{Spec}(\mathbb{C})})$ and $H_X$ is moreover $\mathcal{M}_{\mathbb{C}}$-linear. By using the abelian category $\mathrm{MHM}_X^{\mathrm{mon}}$ of mixed Hodge modules on $X$

with a finite order automorphism and its Grothendieck ring $\mathrm{K}_0(\mathrm{MHM}_X^{\mathrm{mon}})$ we have also a group homomorphism

$$H_X^{\mathrm{mon}} \colon \mathcal{M}_X^{\hat{\mu}} \longrightarrow \mathrm{K}_0(\mathrm{MHM}_X^{\mathrm{mon}}) \tag{20}$$

(see [**15**] and [**31**] for the details).

PROPOSITION 4.2.3 ([**15**, Proposition 3.17]). *In the situation of Theorem 4.2.1 there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_X & \xrightarrow{\psi_g} & \mathcal{M}^{\hat{\mu}}_{g^{-1}(0)} \\ {\scriptstyle H_X}\downarrow & & \downarrow{\scriptstyle H^{\mathrm{mon}}_{g^{-1}(0)}} \\ \mathrm{K}_0(\mathrm{MHM}_X) & \xrightarrow[\Psi_g]{} & \mathrm{K}_0(\mathrm{MHM}^{\mathrm{mon}}_{-}g^{-1}(0)), \end{array}$$

*where $\Psi_g$ is induced by the nearby cycles of mixed Hodge modules.*

**4.2.2. Equivariant Ehrhart theory of Katz and Stapledon.** In this section, we introduce some polynomials in the Equivariant Ehrhart theory in Katz-Stapledon [**19**] and Stapledon [**44**]. Throughout this chapter, we regard the empty set $\emptyset$ as a $(-1)$-dimensional polytope, and as a face of any polytope. Let $P$ be a polytope. If a subset $F \subset P$ is a face of $P$, we write $F \prec P$. For a pair of faces $F \prec F' \prec P$ of $P$, we denote by $[F, F']$ the face poset $\{F'' \prec P \mid F \prec F'' \prec F'\}$, and by $[F, F']^*$ a poset which is equal to $[F, F']$ as a set with the reversed order.

DEFINITION 4.2.4. Let $B$ be a poset $[F, F']$ or $[F, F']^*$. We define a polynomial $g(B, t)$ of degree $\leq (\mathrm{dim}F' - \mathrm{dim}F)/2$ as follows. If $F = F'$, we set $g(B; t) = 1$. If $F \neq F'$ and $B = [F, F']$ (resp. $B = [F, F']^*$), we define $g(B; t)$ inductively by

$$t^{\mathrm{dim}F' - \mathrm{dim}F} g(B; t^{-1}) = \sum_{F'' \in [F, F']} (t-1)^{\mathrm{dim}F' - \mathrm{dim}F''} g([F, F'']; t).$$

$$(\text{resp. } t^{\mathrm{dim}F' - \mathrm{dim}F} g(B; t^{-1}) = \sum_{F'' \in [F, F']^*} (t-1)^{\mathrm{dim}F'' - \mathrm{dim}F} g([F'', F']^*; t).)$$

In what follows, we assume that $P$ is a lattice polytope in $\mathbb{R}^n$. Let $S$ be a subset of $P \cap \mathbb{Z}^n$ containing the vertices of $P$, and $\omega \colon S \to \mathbb{Z}$ be a function. We denote by $\mathrm{UH}_\omega$ the convex hull in $\mathbb{R}^n \times \mathbb{R}$ of the set $\{(v, s) \in \mathbb{R}^n \times \mathbb{R} \mid v \in S, s \geq \omega(v)\}$. Then, the set of all the projections of the bounded faces of $\mathrm{UH}_\omega$ to $\mathbb{R}^n$ defines a lattice polyhedral subdivision $\mathcal{S}$ of $P$. Here a lattice polyhedral subdivision $\mathcal{S}$ of a polytope $P$ is a set of some polytopes in $P$ such that the intersection of any two polytopes in $\mathcal{S}$ is a face of both and all vertices of any polytope in $\mathcal{S}$ are in $\mathbb{Z}^n$. Moreover, the set of all the bounded faces of $\mathrm{UH}_\omega$ defines a piecewise $\mathbb{Q}$-affine convex function $\nu \colon P \to \mathbb{R}$. For a cell $F \in \mathcal{S}$, we denote by $\sigma(F)$ the smallest face of $P$ containing $F$, and $\mathrm{lk}_{\mathcal{S}}(F)$ the set of all cells of $\mathcal{S}$ containing $F$. We call $\mathrm{lk}_{\mathcal{S}}(F)$ the link of $F$ in $\mathcal{S}$. Note that $\sigma(\emptyset) = \emptyset$ and $\mathrm{lk}_{\mathcal{S}}(\emptyset) = \mathcal{S}$.

DEFINITION 4.2.5. For a cell $F \in \mathcal{S}$, the $h$-polynomial $h(\mathrm{lk}_{\mathcal{S}}(F); t)$ of the link $\mathrm{lk}_{\mathcal{S}}(F)$ of $F$ is defined by

$$t^{\mathrm{dim}P - \mathrm{dim}F} h(\mathrm{lk}_{\mathcal{S}}(F); t^{-1}) = \sum_{F' \in \mathrm{lk}_{\mathcal{S}}(F)} g([F, F']; t)(t-1)^{\mathrm{dim}P - \mathrm{dim}F'}.$$

The local $h$-polynomial $l_P(\mathcal{S}, F; t)$ of $F$ in $\mathcal{S}$ is defined by

$$l_P(\mathcal{S}, F; t) = \sum_{\sigma(F) \prec Q \prec P} (-1)^{\dim P - \dim Q} h(\mathrm{lk}_{\mathcal{S}|_Q}(F); t) \cdot g([Q, P]^*; t).$$

For $\lambda \in \mathbb{C}$ and $v \in mP \cap \mathbb{Z}^n$ ($m \in \mathbb{Z}_+ := \mathbb{Z}_{\geq 0}$) we set

$$w_\lambda(v) = \begin{cases} 1 & \left(\exp\left(2\pi\sqrt{-1} \cdot m\nu(\frac{v}{m})\right) = \lambda\right) \\ 0 & \text{(otherwise)}. \end{cases}$$

We define the $\lambda$-weighted Ehrhart polynomial $f_\lambda(P, \nu; m) \in \mathbb{Z}[m]$ of $P$ with respect to $\nu \colon P \to \mathbb{R}$ by

$$f_\lambda(P, \nu; m) := \sum_{v \in mP \cap \mathbb{Z}^n} w_\lambda(v).$$

Then $f_\lambda(P, \nu; m)$ is a polynomial in $m$ with coefficients $\mathbb{Z}$ whose degree is $\leq \dim P$ (see [**44**]).

Definition 4.2.6 ([**44**]). (i) We define the $\lambda$-weighted $h^*$-polynomial $h^*_\lambda(P, \nu; u) \in \mathbb{Z}[u]$ by

$$\sum_{m \geq 0} f_\lambda(P, \nu; m) u^m = \frac{h^*_\lambda(P, \nu; u)}{(1-u)^{\dim P + 1}}.$$

If $P$ is the empty polytope, we set $h^*_1(P, \nu; u) = 1$ and $h^*_\lambda(P, \nu; u) = 0$ ($\lambda \neq 1$).

(ii) We define the $\lambda$-local weighted $h^*$-polynomial $l^*_\lambda(P, \nu; u) \in \mathbb{Z}[u]$ by

$$l^*_\lambda(P, \nu; u) = \sum_{Q \prec P} (-1)^{\dim P - \dim Q} h^*_\lambda(Q, \nu|_Q; u) \cdot g([Q, P]^*; u).$$

If $P$ is the empty polytope, we set $l^*_1(P, \nu; u) = 1$ and $l^*_\lambda(P, \nu; u) = 0$ ($\lambda \neq 1$).

Definition 4.2.7 ([**44**]). (i) We define the $\lambda$-weighted limit mixed $h^*$-polynomial $h^*_\lambda(P, \nu; u, v) \in \mathbb{Z}[u, v]$ by

$$h^*_\lambda(P, \nu; u, v) := \sum_{F \in \mathcal{S}} v^{\dim F + 1} l^*_\lambda(F, \nu|_F; uv^{-1}) \cdot h(\mathrm{lk}_{\mathcal{S}}(F); uv).$$

(ii) We define the $\lambda$-local weighted limit mixed $h^*$-polynomial $l^*_\lambda(P, \nu; u, v) \in \mathbb{Z}[u, v]$ by

$$l^*_\lambda(P, \nu; u, v) := \sum_{F \in \mathcal{S}} v^{\dim F + 1} l^*_\lambda(F, \nu|_F; uv^{-1}) \cdot l_P(\mathcal{S}, F; uv).$$

(iii) We define $\lambda$-weighted refined limit mixed $h^*$-polynomial $h^*_\lambda(P, \nu; u, v, w) \in \mathbb{Z}[u, v, w]$ by

$$h^*_\lambda(P, \nu; u, v, w) := \sum_{Q \prec P} w^{\dim Q + 1} l^*_\lambda(Q, \nu|_Q; u, v) \cdot g([Q, P]; uvw^2).$$

Proposition 4.2.8 (see Theorem 4.21 of Stapledon [**44**]). *In the situation as above, the following holds.*

(i) *We have*

$$l^*_\lambda(P,\nu;u,1) = l^*_\lambda(P,\nu;u),$$
$$h^*_\lambda(P,\nu;u,1) = h^*_\lambda(P,\nu;u) \quad \textit{and}$$
$$h^*_\lambda(P,\nu;u,v,1) = h^*_\lambda(P,\nu;u,v).$$

(ii) *We have*

$$l^*_\lambda(P,\nu;u,v) = l^*_{\overline{\lambda}}(P,\nu;v,u) = (uv)^{\dim P+1} l^*_\lambda(P,\nu;v^{-1},u^{-1}) \quad \textit{and}$$
$$h^*_\lambda(P,\nu;u,v,w) = h^*_{\overline{\lambda}}(P,\nu;v,u,w) = h^*_\lambda(P,\nu;v^{-1},u^{-1},uvw).$$

(iii) *We have*

$$h^*_\lambda(P,\nu;u,v) = \sum_{\substack{F\in\mathcal{S}_\nu\\ \sigma(F)=P}} h^*_\lambda(F,\nu|_F;u,v)(uv-1)^{\dim P-\dim F}.$$

For the practical computation of the $h^*$-polynomials, the following example is useful.

Example 4.2.9 (Example 4.23 of Stapledon [**44**]). Assume that $P$ is an $n$-dimensional simplex in $\mathbb{R}^n$ and $\nu$ is a $\mathbb{Q}$-affine function on it. For a face $(\emptyset \neq)Q \prec P$, we denote by $C_Q \subset \mathbb{R}^n \times \mathbb{R}$ the rational polyhedral cone in $\mathbb{R}^n \times \mathbb{R}$ generated by the vectors $\{(v,1)\}_{v\in Q}$. For the empty face $Q=\emptyset$, we set $C_Q = \{0\} \subset \mathbb{R}^n \times \mathbb{R}$. Let $v_0, \ldots, v_n \in P \times \{1\}$ be the ray generators of edges of $C_P$. We define a finite subset Box in $\mathbb{Z}^n \times \mathbb{Z}$ by

$$\mathrm{Box} := \{v \in C_P \cap (\mathbb{Z}^n \times \mathbb{Z}) \mid v = \sum_{i=0}^n a_i v_i, 0 \leq a_i < 1\}.$$

We denote by $\mathrm{pr}\colon \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ the projection. Then we have

$$h^*(P,\nu;u,v,w) = \sum_{Q\prec P} \sum_{v\in \mathrm{Box}\cap \mathrm{rel.int}\,(C_Q)} w_\lambda(v) u^{\mathrm{pr}(v)} v^{\dim Q+1-\mathrm{pr}(v)} w^{\dim Q+1} \quad \text{and}$$

$$l^*(P,\nu;u,v) = \sum_{v\in \mathrm{Box}\cap \mathrm{rel.int}\,(C_P)} w_\lambda(v) u^{\mathrm{pr}(v)} v^{\dim P+1-\mathrm{pr}(v)}.$$

## 4.3. Monodromies and limit mixed Hodge structures of families of hypersurfaces in algebraic tori $((\mathbb{C}^*))^n$

Let $\mathbb{K} = \mathbb{C}(t)$ be the field of rational functions of $t$ and $f(t,x) = \sum_{v\in\mathbb{Z}^n} a_v(t) x^v \in \mathbb{K}[x_1^\pm, \ldots, x_n^\pm]$ ($a_v(t) \in \mathbb{K}$) a Laurent polynomial of $x = (x_1, \ldots, x_n)$ with coefficients in $\mathbb{K}$. For $v \in \mathbb{Z}^n$ by the Laurent expansion $a_v(t) = \sum_{j\in\mathbb{Z}} a_{v,j} t^j$ ($a_{v,j} \in \mathbb{C}$) of the rational function $a_v(t)$ we set

$$o(v) := \mathrm{ord}_t\, a_v(t) = \min\{j \mid a_{v,j} \neq 0\}.$$

If $a_v(t) \equiv 0$ we set $o(v) = +\infty$. Then we define an (unbounded) polyhedron $\mathrm{UH}_f$ in $\mathbb{R}^{n+1}$ by

$$\mathrm{UH}_f = \mathrm{Conv}\left[\bigcup_{v\in\mathbb{Z}^n} \{(v,s) \in \mathbb{R}^{n+1} \mid s \geq o(v)\}\right] \subset \mathbb{R}^{n+1},$$

where $\mathrm{Conv}\,(\,\cdot\,)$ stands for the convex hull. Throughout this chapter we assume that the dimension of $\mathrm{UH}_f$ is $n+1$. Then by the projection $p: \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1 \twoheadrightarrow \mathbb{R}^n$ we obtain an $n$-dimensional polytope $P := p(\mathrm{UH}_f) \subset \mathbb{R}^n$. We call it the Newton polytope of $f$. Set $\mathbb{R}_+ := \mathbb{R}_{\geq 0} \subset \mathbb{R}$. Let $\Sigma_0$ be the dual fan of $\mathrm{UH}_f$ in $\mathbb{R}^n \times \mathbb{R}^1_+ \subset$

$\mathbb{R}^{n+1}$. We call its subfan $\Xi_0$ in $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$ consisting of cones $\sigma \in \Sigma_0$ contained in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ the recession fan of $\mathrm{UH}_f$. Let $\nu_f : P \to \mathbb{R}$ be the function defining the bottom part of the boundary $\partial \mathrm{UH}_f$ of $\mathrm{UH}_f$ and $\mathcal{S}$ the subdivision of $P$ by the lattice polytopes $p(\widetilde{F}) \subset \mathbb{R}^n$ ($\widetilde{F} \prec \mathrm{UH}_f$). Then for each cell $F$ in $\mathcal{S}$ the restriction $\nu_f|_F$ of $\nu_f$ to $F \subset P$ is an affine $\mathbb{Q}$-linear function taking integral values on the vertices of $F$. Let us identify the affine subspace $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$ of $\mathbb{R}^{n+1}$ with $\mathbb{R}^n$ by the projection. Then by using the cones $\sigma \in \Sigma_0$ such that $\dim\sigma < n+1$ we define a polyhedral hypersurface $\mathrm{Trop}(Y)$ in $\mathbb{R}^n \times \{1\} \simeq \mathbb{R}^n$ by

$$\mathrm{Trop}(Y) = \bigcup_{\dim\sigma < n+1} \Big\{\sigma \cap (\mathbb{R}^n \times \{1\})\Big\} \subset \mathbb{R}^n \times \{1\} \simeq \mathbb{R}^n.$$

We call it the tropical variety of $Y$ (see [**32**] etc.). It has a decomposition

$$\mathrm{Trop}(Y) = \bigsqcup_{\dim\sigma < n+1} \Big\{\mathrm{rel.int}\sigma \cap (\mathbb{R}^n \times \{1\})\Big\}$$

into the (locally closed) cells $\mathrm{rel.int}\sigma \cap (\mathbb{R}^n \times \{1\}) \subset \mathrm{Trop}(Y)$. It is clear that there exists a one to one correspondence between the cells $F$ in $\mathcal{S}$ such that $\dim F > 0$ and those in $\mathrm{Trop}(Y)$. In [**44**] the author used the cell decomposition of $\mathrm{Trop}(Y)$ to express the motivic nearby fiber $\psi_t([Y])$. However in this chapter, we use only the subdivision $\mathcal{S}$ of $P$. For a cell $F$ in $\mathcal{S}$ by taking the (unique) compact face $\widetilde{F} \prec \mathrm{UH}_f$ of $\mathrm{UH}_f$ such that $F = p(\widetilde{F})$ we define the initial Laurent polynomial $I_f^F(x) \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ by

$$I_f^F(x) = \sum_{(v,j) \in \widetilde{F}} a_{v,j} x^v \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}].$$

By identifying $\mathrm{Aff}(F) \cap \mathbb{Z}^n$ with $\mathbb{Z}^{\dim F}$ we may consider $I_f^F(x)$ as a Laurent polynomial on the algebraic torus $T_F = \mathrm{Spec}(\mathbb{C}[\mathrm{Aff}(F) \cap \mathbb{Z}^n]) \simeq ((\mathbb{C}^*))^{\dim F}$. We denote by $V_F \subset T_F$ the hypersurface defined by $I_f^F(x)$ in $T_F$. By the affine $\mathbb{Q}$-linear extension $\nu_F : \mathrm{Aff}(F) \to \mathbb{R}$ of $\nu_f|_F$ to $\mathrm{Aff}(F) \simeq \mathbb{R}^{\dim F}$ we define an element $e_F$ of the algebraic torus $T_F = \mathrm{Spec}(\mathbb{C}[\mathrm{Aff}(F) \cap \mathbb{Z}^n]) \simeq \mathrm{Hom}_{\mathrm{group}}(\mathrm{Aff}(F) \cap \mathbb{Z}^n, (\mathbb{C}^*))$ by

$$e_F(v) = \exp\Big(-2\pi\sqrt{-1}\nu_F(v)\Big) \qquad (v \in \mathrm{Aff}(F) \cap \mathbb{Z}^n).$$

Then by the multiplication $\ell_{e_F} : T_F \xrightarrow{\sim} T_F$ by $e_F \in T_F$ we have $\ell_{e_F}(V_F) = V_F$. We thus obtain an element $[V_F \circlearrowleft \hat{\mu}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. The hypersurface $f^{-1}(0) \subset T = (\mathbb{C}^*)_t \times ((\mathbb{C}^*))_x^n$ defines a family $Y$ of hypersurfaces of $T_0 = ((\mathbb{C}^*))_x^n$ over a small punctured disk $B(0;\varepsilon)^* = \{t \in \mathbb{C} \mid 0 < |t| < \varepsilon\}$ ($0 < \varepsilon \ll 1$). By the projection $\pi : T = (\mathbb{C}^*)_t \times ((\mathbb{C}^*))_x^n \twoheadrightarrow (\mathbb{C}^*)_t$, for $t \in \mathbb{C}$ such that $0 < |t| < \varepsilon$ we set $Y_t := \pi^{-1}(t) \cap Y \subset \{t\} \times T_0 \simeq T_0 \simeq ((\mathbb{C}^*))_x^n$.

Definition 4.3.1. We say that the family $Y$ of hypersurfaces $\{Y_t\}_{0<|t|<\varepsilon}$ of $T_0 \simeq ((\mathbb{C}^*))_x^n$ is schön if for any cell $F$ in $\mathcal{S}$ the hypersurface $V_F \subset T_F$ of $T_F$ is smooth and reduced.

For the family $Y$ over the punctured disk, denote by $\psi_t([Y]) \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ its motivic nearby fiber by the function $t = \mathrm{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ (see Section 4.2). Then the following beautiful result was first obtained by Stapledon [**44**].

THEOREM 4.3.2. *Assume that the family* $Y$ *is schön. Then we have an equality*

$$\psi_t([Y]) = \sum_{\mathrm{rel.int}F \subset \mathrm{Int}\, P} [V_F \circlearrowleft \hat{\mu}] \cdot (1-\mathbb{L})^{n-\dim F}$$

*in* $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

PROOF. Let $\Sigma$ be a smooth subdivision of the dual fan $\Sigma_0$ of $\mathrm{UH}_f$ and $\Xi$ its subfan in $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$ consisting of cones $\sigma \in \Sigma$ contained in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. We denote by $\Lambda$ the fan $\{\{0\}, \mathbb{R}^1_+\}$ in $\mathbb{R}^1$ formed by the faces of the closed half line $\mathbb{R}^1_+$ in $\mathbb{R}^1$. Let $X_\Sigma$ (resp. $X_\Xi$) be the toric variety associated to $\Sigma$ (resp. $\Xi$). Recall that the algebraic torus $T = (\mathbb{C}^*)^{n+1}$ acts on $X_\Sigma$. For a cone $\sigma \in \Sigma$ denote by $T_\sigma \simeq (\mathbb{C}^*)^{n+1-\dim\sigma}$ the $T$-orbit in $X_\Sigma$ associated to it. Then by the morphism $\Sigma \to \Lambda$ of fans induced by the projection $\mathbb{R}^n \times \mathbb{R}^1_+ \twoheadrightarrow \mathbb{R}^1_+$ we obtain a morphism

$$\pi_\Sigma : X_\Sigma \longrightarrow \mathbb{C}$$

of toric varieties. Restricting it to $(\mathbb{C}^*) \subset \mathbb{C}$, we obtain the projection $(\mathbb{C}^*) \times X_\Xi \twoheadrightarrow (\mathbb{C}^*)$ which extends naturally the previous one $\pi : T = (\mathbb{C}^*) \times T_0 \twoheadrightarrow (\mathbb{C}^*)$. Let $\rho_1, \dots, \rho_N \in \Sigma$ be the rays i.e. the 1-dimensional cones in $\Sigma$. We may assume that for some $r \leq N$ we have $\rho_i \cap (\mathbb{R}^n \times \{0\}) = \{0\} \iff 1 \leq i \leq r$. For $1 \leq i \leq r$ the $T$-orbit $T_i = T_{\rho_i} \subset X_\Sigma$ associated to $\rho_i$ satisfies the condition $\pi_\Sigma(T_i) = \{0\} \subset \mathbb{C}$. We can easily see that for their closures $\overline{T_i} \subset X_\Sigma$ in $X_\Sigma$ we have

$$\pi_\Sigma^{-1}(\{0\}) = \bigcup_{i=1}^{r} \overline{T_i}.$$

For $1 \leq i \leq N$ let $\alpha_i \in \rho_i \cap (\mathbb{Z}^{n+1} \setminus \{0\})$ be the primitive vector on the ray $\rho_i$. We define a non-negative integer $b_i \geq 0$ by $b_i = q(\alpha_i)$, where $q : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1 \twoheadrightarrow \mathbb{R}^1$ is the projection. Then it is easy to see that for $1 \leq i \leq r$ the order of the zeros of the function $t \circ \pi_\Sigma : X_\Sigma \to \mathbb{C}$ along the toric divisor $\overline{T_i} \subset X_\Sigma$ is equal to $b_i > 0$. For a cone $\sigma \in \Sigma$ such that $\mathrm{rel.int}\sigma \subset \mathrm{Int}(\mathbb{R}^n \times \mathbb{R}^1_+)$ we denote by $\widetilde{F_\sigma} \prec \mathrm{UH}_f$ its supporting face in $\mathrm{UH}_f$. Then by the schönness of $Y$, the closure $\overline{Y}$ of $Y \subset T \subset X_\Sigma$ in $X_\Sigma$ intersects the $T$-orbit $T_\sigma \simeq ((\mathbb{C}^*))^{n+1-\dim\sigma} \subset X_\Sigma$ transversally. This in particular implies that $\overline{Y}$ is smooth on a neighborhood of $\pi_\Sigma^{-1}(\{0\})$ in $X_\Sigma$. The resulting smooth hypersurface $W_\sigma := \overline{Y} \cap T_\sigma \subset T_\sigma$ of $T_\sigma$ is defined by the $\widetilde{F_\sigma}$-part $f_{\widetilde{F_\sigma}}(t,x)$ of $f(t,x)$. We can also easily see that for any cone $\sigma \in \Xi$ the hypersurface $\overline{Y}$ intersects $T_\sigma$ transversally on a neighborhood of $\pi_\Sigma^{-1}(\{0\})$ in $X_\Sigma$. Moreover the divisor $D = \overline{Y} \setminus Y = \overline{Y} \cap (X_\Sigma \setminus T)$ in $\overline{Y}$ is normal crossing. Let $\pi_{\overline{Y}} = \pi_\Sigma|_{\overline{Y}} : \overline{Y} \to \mathbb{C}$ be the restriction of $\pi_\Sigma$ to $\overline{Y}$. Then by Theorem 4.2.1 and $t \circ \pi_{\overline{Y}} = \pi_{\overline{Y}}$ we have

$$\psi_t([Y]) = (\pi_{\overline{Y}}|_{\pi_{\overline{Y}}^{-1}(0)})_!(\mathcal{S}_{\pi_{\overline{Y}},Y}).$$

Moreover the morphism $(\pi_{\overline{Y}}|_{\pi_{\overline{Y}}^{-1}(0)})_!$ sends $\mathcal{S}_{\pi_{\overline{Y}},Y}$ to its underlying variety over the point $\{0\} \simeq \mathrm{Spec}(\mathbb{C})$, which we still denote by $\mathcal{S}_{\pi_{\overline{Y}},Y}$ for short. Define an open subset $\Omega$ of $\overline{Y}$ containing $Y = \overline{Y} \cap T$ by

$$\Omega = \overline{Y} \cap \big(X_\Sigma \setminus \bigcup_{\sigma \in \Xi \setminus \{0\}} \overline{T_\sigma}\big) \subset \overline{Y}.$$

Let $\Sigma' \subset \Sigma$ be the subset of $\Sigma$ consisting of cones $\sigma$ satisfying the condition $\sigma \cap (\mathbb{R}^n \times \{0\}) = \{0\}$. Then we have

$$\Omega = \overline{Y} \cap \Big( \bigsqcup_{\sigma \in \Sigma'} T_\sigma \Big)$$

and $\mathcal{S}_{\pi_{\overline{Y}}, Y}$ is described by some varieties over the normal crossing divisor

$$\Omega \cap \pi_{\overline{Y}}^{-1}(0) = \overline{Y} \cap \Big( \bigsqcup_{\sigma \in \Sigma' \setminus \{0\}} T_\sigma \Big) = \Omega \setminus Y.$$

For a cone $\sigma \in \Sigma' \setminus \{0\}$ let $\widetilde{W_\sigma}$ be the unramified Galois covering of $W_\sigma = \overline{Y} \cap T_\sigma$. From now, we shall prove that there exists an isomorphism

$$[\widetilde{W_\sigma}] = [V_{F_\sigma} \circlearrowleft \hat{\mu}] \cdot (\mathbb{L} - 1)^{n+1-\mathrm{dim}\sigma - \mathrm{dim}F_\sigma}, \tag{21}$$

where we set $F_\sigma = p(\widetilde{F_\sigma}) \in \mathcal{S}$. Assume first that $\mathrm{dim}\sigma + \mathrm{dim}F_\sigma = n+1$. Let $\tau \in \Sigma$ be an $(n+1)$-dimensional cone such that $\sigma \prec \tau$ and set $l = \mathrm{dim}\sigma$. After reordering the rays $\rho_i$ $(1 \leq i \leq N)$ we may assume that

$$\sigma = \mathbb{R}_+\alpha_1 + \cdots + \mathbb{R}_+\alpha_l, \qquad \tau = \mathbb{R}_+\alpha_1 + \cdots + \mathbb{R}_+\alpha_{n+1}.$$

Let us set $\sigma' = \mathbb{R}_+\alpha_{l+1} + \cdots + \mathbb{R}_+\alpha_{n+1} \prec \tau$ and

$$I = \{1, 2, \dots, l\}, \qquad I' = \{l+1, l+2, \dots, n+1\}.$$

Moreover set $d_I = \gcd\{b_i \mid 1 \leq i \leq l\} \geq 1$ and let $d_{I'} \geq 0$ be the generator of the subgroup $\mathbb{Z}b_{l+1} + \mathbb{Z}b_{l+2} + \cdots + \mathbb{Z}b_{n+1} \subset \mathbb{Z}$. We may define $\gcd\{b_i \mid l+1 \leq i \leq n+1\}$ to be $d_{I'}$. Then by the smoothness of the cone $\tau$ similarly we have $\gcd(d_I, d_{I'}) = \gcd\{b_i \mid 1 \leq i \leq n+1\} = 1$. We shall prove the equality (21) only in the case $d_{I'} \geq 1$. In the case $d_{I'} = 0$ ($\iff \sigma' \subset \mathbb{R}^n \times \{0\}$) we have $d_I = 1$ and the proof is much easier. Assume that $d_{I'} \geq 1$. Let $\alpha_1^*, \dots, \alpha_{n+1}^* \in (\mathbb{Z}^{n+1})^* \simeq \mathbb{Z}^{n+1}$ be the dual basis of $\alpha_1, \dots, \alpha_{n+1} \in \mathbb{Z}^{n+1}$ and $\tau^\vee \subset (\mathbb{R}^{n+1})^* \simeq \mathbb{R}^{n+1}$ the dual cone of $\tau \subset \mathbb{R}^{n+1}$. Then the affine open subset $\mathbb{C}^{n+1}(\tau) (\simeq \mathbb{C}^{n+1})$ of $X_\Sigma$ is defined by

$$\mathbb{C}^{n+1}(\tau) = \mathrm{Spec}(\mathbb{C}[\tau^\vee \cap \mathbb{Z}^{n+1}]) \simeq \mathrm{Spec}(\mathbb{C}[\mathbb{Z}_+\alpha_1^* + \cdots + \mathbb{Z}_+\alpha_{n+1}^*])$$
$$\simeq \mathrm{Spec}(\mathbb{C}[\xi_1, \dots, \xi_{n+1}]) \simeq \mathbb{C}_\xi^{n+1} \quad (\alpha_i^* \longleftrightarrow \xi_i).$$

By the coordinates $\xi = (\xi_1, \dots, \xi_{n+1})$ of $\mathbb{C}^{n+1}(\tau) \subset X_\Sigma$ its subset $T_\sigma = \mathrm{Spec}(\mathbb{C}[\mathrm{Aff}(\sigma)^\perp \cap \mathbb{Z}^{n+1}]) \simeq (\mathbb{C}^*)^{n+1-l}$ is explicitly given by

$$T_\sigma = \{\xi \in \mathbb{C}^{n+1}(\tau) \mid \xi_i = 0 \ (1 \leq i \leq l), \ \xi_i \neq 0 \ (l+1 \leq i \leq n+1)\}.$$

Moreover the restriction of the function $t \circ \pi_\Sigma \colon X_\Sigma \to \mathbb{C}$ to $\mathbb{C}^{n+1}(\tau)$ is equal to $\xi_1^{b_1} \dots \xi_{n+1}^{b_{n+1}} \in \mathbb{C}[\xi_1, \dots, \xi_{n+1}]$ which corresponds to the element $b_1\alpha_1^* + \cdots + b_{n+1}\alpha_{n+1}^*$ in the group ring $\mathbb{C}[\tau^\vee \cap \mathbb{Z}^{n+1}]$. From this we see that the unramified Galois covering $\widetilde{W_\sigma}$ of $W_\sigma = \overline{Y} \cap T_\sigma$ is given by

$$\widetilde{W_\sigma} = \{(t, (\xi_{l+1}, \dots, \xi_{n+1})) \in \mathbb{C}^* \times T_\sigma \mid t^{d_I} = \xi_{l+1}^{-b_{l+1}} \dots \xi_{n+1}^{-b_{n+1}}, \ f_{\widetilde{F_\sigma}}(\xi_{l+1}, \dots, \xi_{n+1}) = 0\}.$$

Recall that the action of the group $\hat{\mu}$ on $[\widetilde{W_\sigma}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ is defined by the multiplication of $\zeta_{d_I} = \exp(2\pi\sqrt{-1}/d_I) \in \mathbb{C}$ to the coordinate $t$. In order to rewrite $[\widetilde{W_\sigma}]$ we shall introduce a new basis of the lattice $\mathbb{Z}^{n+1}$. First by our assumption on $\sigma$, the affine span $\mathrm{Aff}(\sigma) \simeq \mathbb{R}^l$ of $\sigma$ intersects the hyperplane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ transversally in $\mathbb{R}^{n+1}$ and there exists a basis $\beta_1, \dots, \beta_l$ of the lattice $\mathrm{Aff}(\sigma) \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^l$ such that $\beta_1, \dots, \beta_{l-1} \subset \mathbb{Z}^n \times \{0\}$ ($\iff q(\beta_1) = \cdots = q(\beta_{l-1}) = 0$) and $q(\beta_l) = d_I$. By the condition $\sigma' \not\subset \mathbb{R}^n \times \{0\}$ there exists also a basis $\beta_{l+1}, \dots, \beta_{n+1}$ of the lattice $\mathrm{Aff}(\sigma') \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{n+1-l}$ such that $\beta_{l+1}, \dots, \beta_n \in \mathbb{Z}^n \times \{0\}$ and $q(\beta_{n+1}) = d_{I'}$. By $\mathbb{R}^{n+1} = \mathrm{Aff}(\sigma) \oplus \mathrm{Aff}(\sigma')$ we thus obtain a basis $\beta_1, \dots, \beta_l, \beta_{l+1}, \dots, \beta_{n+1}$ of the lattice $\mathbb{Z}^{n+1} = \{\mathrm{Aff}(\sigma) \cap \mathbb{Z}^{n+1}\} \oplus \{\mathrm{Aff}(\sigma') \cap \mathbb{Z}^{n+1}\}$. For the dual basis $\alpha_1^*, \dots, \alpha_{n+1}^*$ we have the decomposition

$$w_1 + w_2 = b_1 \alpha_1^* + \cdots + b_{n+1} \alpha_{n+1}^* = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where we set $w_1 = b_1\alpha_1^* + \cdots + b_l \alpha_l^* \subset \mathrm{Aff}(\sigma')^\perp \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^l$ and $w_2 = b_{l+1}\alpha_{l+1}^* + \cdots + b_{n+1}\alpha_{n+1}^* \in \mathrm{Aff}(\sigma)^\perp \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{n+1-l}$. Moreover by the construction of $\beta_1, \dots, \beta_{n+1}$, for the dual basis $\beta_1^*, \dots, \beta_{n+1}^*$ of it we have also

$$d_I \beta_l^* + d_{I'} \beta_{n+1}^* = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{22}$$

We thus obtain $w_1 = d_I \beta_l^*, w_2 = d_{I'} \beta_{n+1}^*$. By the condition $\sigma' \not\subset \mathbb{R}^n \times \{0\}$ we have $w_2 \neq 0$. It follows also from our assumption on $\sigma$ that the restriction of the projection $p\colon \mathbb{R}^{n+1} \twoheadrightarrow \mathbb{R}^n$ to $\mathrm{Aff}(\sigma)^\perp \simeq \mathbb{R}^{n+1-l}$ is injective. Hence we get $p(w_2) \neq 0$ and the non-vanishing $p(w_1) \neq 0$ follows from $p(w_1) + p(w_2) = 0$. Since the two vectors $p(w_1)$ and $p(w_2)$ in $\mathbb{Z}^n \subset \mathbb{R}^n$ are divisible by $d_I$ and $d_{I'}$ respectively and $\gcd(d_I, d_{I'}) = 1$, they are divisible also by $d_I d_{I'}$. Let us show that the lattice vector

$$\frac{1}{d_I d_{I'}} p(w_2) = \frac{1}{d_I} p(\beta_{n+1}^*) \in \mathbb{Z}^n$$

thus obtained is primitive. Suppose that it is divisible again by an integer $d \geq 2$. Then we have

$$0 = \langle \beta_l, w_2 \rangle = \langle p(\beta_l), p(w_2) \rangle + q(\beta_l) \cdot q(w_2) = \langle p(\beta_l), p(w_2) \rangle + d_I d_{I'} q(\beta_{n+1}^*).$$

This implies that

$$q(\beta_{n+1}^*) = -\left\langle p(\beta_l), \ \frac{1}{d_I d_{I'}} p(w_2) \right\rangle$$

is divisible by $d$. So $\beta_{n+1}^*$ is also divisible by $d \geq 2$, which contradicts the fact that $\beta_{n+1}^*$ is primitive. It follows also from

$$1 = \langle \beta_i, \beta_i^* \rangle = \langle p(\beta_i), p(\beta_i^*) \rangle \quad (l+1 \leq i \leq n)$$

that the vectors $p(\beta_i^*) \in \mathbb{Z}^n \subset \mathbb{R}^n$ $(l+1 \leq i \leq n)$ are primitive. Note that $\beta_{l+1}^*, \dots, \beta_{n+1}^*$ form a basis of the lattice $\mathrm{Aff}(\sigma)^\perp \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{n+1-l}$. For their projections by $p\colon \mathbb{R}^{n+1} \twoheadrightarrow \mathbb{R}^n$ we have the following result.

Proposition 4.3.3. *The vectors* $p(\beta_{l+1}^*), \dots, p(\beta_n^*), \frac{1}{d_I} p(\beta_{n+1}^*) \in p(\mathrm{Aff}(\sigma)^\perp) \cap \mathbb{Z}^n \simeq p(\mathrm{Aff}(\widetilde{F_\sigma})) \cap \mathbb{Z}^n = \mathrm{Aff}(F_\sigma) \cap \mathbb{Z}^n$ *form a basis of the lattice* $p(\mathrm{Aff}(\sigma)^\perp) \cap \mathbb{Z}^n \simeq \mathbb{Z}^{n+1-l}$.

PROOF. We have seen that $p(\beta^*_{l+1}), \dots, p(\beta^*_n), \frac{1}{d_I}p(\beta^*_{n+1})$ are primitive. First let us show that they are linearly independent over $\mathbb{R}$. Suppose that we have

$$\lambda_{l+1}p(\beta^*_{l+1}) + \cdots + \lambda_n p(\beta^*_n) + \lambda_{n+1}\frac{p(\beta^*_{n+1})}{d_I} = 0$$

for some $\lambda_i \in \mathbb{R}$ $(l+1 \leq i \leq n+1)$. Then by taking the pairings with $\beta_i$ $(l+1 \leq i \leq n)$ we obtain $\lambda_i = 0$ $(l+1 \leq i \leq n)$ and hence $\lambda_{n+1} = 0$. Next we show that they generate the lattice $p(\mathrm{Aff}(\sigma)^\perp) \cap \mathbb{Z}^n$ over $\mathbb{Z}$. For this we use the following result.

LEMMA 4.3.4. *The vectors $p(\beta^*_{l+1}), \dots, p(\beta^*_n)$ form a basis of the lattice $\{\mathbb{R}p(\beta^*_{l+1}) \oplus \cdots \oplus \mathbb{R}p(\beta^*_n)\} \cap \mathbb{Z}^n$.*

PROOF. Suppose that they do not generate the lattice over $\mathbb{Z}$. Then there exist integers $d \geq 2$ and $\lambda_{l+1}, \dots, \lambda_n \in \mathbb{Z}$ such that

$$\frac{1}{d}\left\{\lambda_{l+1}p(\beta^*_{l+1}) + \cdots + \lambda_n p(\beta^*_n)\right\} \in \mathbb{Z}^n$$

and $\frac{\lambda_{i_0}}{d} \in \mathbb{Q} \setminus \mathbb{Z}$ for some $l+1 \leq i_0 \leq n$. By taking the pairing with $p(\beta_{i_0})$ we obtain $\frac{\lambda_{i_0}}{d} \in \mathbb{Z}$, which is a contradiction. □

Let us continue the proof of Proposition 4.3.3. By Lemma 4.3.4, if the vectors $p(\beta^*_{l+1}), \dots, p(\beta^*_n), \frac{1}{d_I}p(\beta^*_{n+1})$ do not generate the lattice $p(\mathrm{Aff}(\sigma)^\perp) \cap \mathbb{Z}^n$, then there exist integers $d \geq 2$ and $\lambda_{l+1}, \dots, \lambda_n \in \mathbb{Z}$ such that

$$\frac{1}{d}\left\{\frac{1}{d_I}p(\beta^*_{n+1}) - \lambda_{l+1}p(\beta^*_{l+1}) - \cdots - \lambda_n p(\beta^*_n)\right\} \in \mathbb{Z}^n.$$

Set $\gamma^* = \beta^*_{n+1} - \lambda_{l+1}d_I\beta^*_{l+1} - \cdots - \lambda_n d_I \beta^*_n$. Then we obtain a new basis $\beta^*_{l+1}, \dots, \beta^*_n, \gamma^*$ of the lattice $\mathrm{Aff}(\sigma)^\perp \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{n+1-l}$. By taking the pairing with $\beta_l \in \mathrm{Aff}(\sigma) \cap \mathbb{Z}^{n+1}$ we get

$$0 = \langle \beta_l, \gamma^* \rangle = \langle p(\beta_l), p(\gamma^*) \rangle + d_I \cdot q(\gamma^*).$$

Since the lattice vector $p(\gamma^*) \in \mathbb{Z}^n$ is divisible by $dd_I$, the integer $q(\gamma^*) \in \mathbb{Z}$ and hence $\gamma^* \in \mathbb{Z}^{n+1}$ itself is so. This contradicts the fact that $\gamma^*$ is primitive. □

Now we return to the proof of Theorem 4.3.2. By the new basis $\beta^*_{l+1}, \dots, \beta^*_{n+1}$ of the lattice $\mathrm{Aff}(\sigma)^\perp \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{n+1-l}$ we have an isomorphism

$$\begin{aligned} T_\sigma &= \mathrm{Spec}(\mathbb{C}[\mathrm{Aff}(\sigma)^\perp \cap \mathbb{Z}^{n+1}]) \simeq \mathrm{Spec}(\mathbb{C}[\mathbb{Z}\beta^*_{l+1} + \cdots + \mathbb{Z}\beta^*_{n+1}]) \\ &\simeq \mathrm{Spec}(\mathbb{C}[z_{l+1}, \dots, z_{n+1}]) \quad (\beta^*_i \longleftrightarrow z_i). \end{aligned}$$

By the new coordinates $z = (z_{l+1}, \dots, z_{n+1})$ of $T_\sigma \simeq (\mathbb{C}^*)^{n+1-l}$ we have

$$\widetilde{W_\sigma} = \{(t, (z_{l+1}, \dots, z_{n+1})) \in \mathbb{C}^* \times T_\sigma \mid t^{d_I} = z_{n+1}^{-d_{I'}},\ f_{\widetilde{F_\sigma}}(z) = 0\}.$$

On the other hand, by taking the projection $q\colon \mathbb{R}^{n+1} \twoheadrightarrow \mathbb{R}^1$ of the both sides of the decomposition (22), we obtain

$$d_I \cdot q(\beta^*_l) + d_{I'} \cdot q(\beta^*_{n+1}) = 1. \tag{23}$$

Note that for the basis $p(\beta^*_{l+1}), \dots, p(\beta^*_n), \frac{1}{d_I}p(\beta^*_{n+1})$ of the lattice $\mathrm{Aff}(F_\sigma) \cap \mathbb{Z}^n$ constructed in Proposition 4.3.3 we have $\nu_f(p(\beta^*_{l+1})), \dots, \nu_f(p(\beta^*_n)) \in \mathbb{Z}$ and $\nu_f(\frac{1}{d_I}p(\beta^*_{n+1})) \equiv \frac{q(\beta^*_{n+1})}{d_I} \mod \mathbb{Z}$. By Proposition 4.3.3, we obtain an isomorphism

$$V_{F_\sigma} \simeq \{(s, (z_{l+1}, \dots, z_{n+1})) \in \mathbb{C}^* \times T_\sigma \mid s^{d_I} = z_{n+1},\ f_{\widetilde{F_\sigma}}(z) = 0\}.$$

Moreover the action of the group $\hat{\mu}$ on $[V_{F_\sigma} \circlearrowleft \hat{\mu}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ corresponds to the multiplication of $\exp(-2\pi\sqrt{-1}q(\beta_{n+1}^*)/d_I) \in \mathbb{C}$ to the coordinate $s$. By the equality (23) we have

$$\exp\left(-\frac{2\pi\sqrt{-1}q(\beta_{n+1}^*)}{d_I}\right)^{-d_{I'}} = \exp\left(\frac{2\pi\sqrt{-1}}{d_I}\right) = \zeta_{d_I}. \tag{24}$$

Furthermore by $\gcd(d_I, d_{I'}) = 1$ the morphism $V_{F_\sigma} \to \widetilde{W_\sigma}$ defined by $(s, z) \longmapsto (s^{-d_{I'}}, z)$ is an isomorphism. It is compatible with the actions of $\hat{\mu}$ on the both sides by the equality (24). We thus obtained the required isomorphism $[\widetilde{W_\sigma}] \simeq [V_{F_\sigma} \circlearrowleft \hat{\mu}]$ in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. If $\mathrm{dim}\sigma + \mathrm{dim}F_\sigma < n+1$, similarly we can prove an isomorphism

$$\widetilde{W_\sigma} \simeq V_{F_\sigma} \times (\mathbb{C}^*)^{n+1-\mathrm{dim}\sigma-\mathrm{dim}F_\sigma},$$

but the action of $\hat{\mu}$ on the second factor $(\mathbb{C}^*)^{n+1-\mathrm{dim}\sigma-\mathrm{dim}F_\sigma}$ of the right hand side might be non-trivial. Nevertheless by [**44**, Example 2.2] (witch follows essentially from the definition of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$) we obtain an isomorphism

$$[\widetilde{W_\sigma}] \simeq [V_{F_\sigma} \circlearrowleft \hat{\mu}] \cdot (\mathbb{L}-1)^{n+1-\mathrm{dim}\sigma-\mathrm{dim}F_\sigma}$$

in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. Then by Theorem 4.2.1 we have

$$\begin{aligned}\psi_t([Y]) &= \sum_{\sigma\in\Sigma'\setminus\{0\}} [\widetilde{W_\sigma}] \cdot (1-\mathbb{L})^{\mathrm{dim}\sigma-1} \\ &= \sum_{\sigma\in\Sigma'\setminus\{0\}} (-1)^{n+1-\mathrm{dim}\sigma-\mathrm{dim}F_\sigma}[V_{F_\sigma} \circlearrowleft \hat{\mu}] \cdot (1-\mathbb{L})^{n-\mathrm{dim}F_\sigma}.\end{aligned}$$

For a cell $F \in \mathcal{S}$ denote by $\widetilde{F} \prec \mathrm{UH}_f$ the unique compact face of $\mathrm{UH}_f$ such that $p(\widetilde{F}) = F$ and let $F^\circ \in \Sigma_0$ be the cone which corresponds to it in the dual fan $\Sigma_0$. Then we can easily show

$$\begin{aligned}&\sum_{\substack{\sigma\in\Sigma'\setminus\{0\} \\ \mathrm{rel.int}\sigma\subset\mathrm{rel.int}F^\circ}} (-1)^{n+1-\mathrm{dim}\sigma-\mathrm{dim}F_\sigma}[V_{F_\sigma} \circlearrowleft \hat{\mu}] \cdot (1-\mathbb{L})^{n-\mathrm{dim}F_\sigma} \\ &= \begin{cases} [V_F \circlearrowleft \hat{\mu}] \cdot (1-\mathbb{L})^{n-\mathrm{dim}F} & (\mathrm{rel.int}F \subset \mathrm{Int}\, P) \\ 0 \quad (\text{otherwise}) \end{cases}\end{aligned}$$

(cf. the proof of Matsui-Takeuchi [**24**, Theorem 5.7 and Proposition 5.5]) . We thus obtain the desired formula

$$\psi_t([Y]) = \sum_{\mathrm{rel.int}F\subset\mathrm{Int}\, P} [V_F \circlearrowleft \hat{\mu}] \cdot (1-\mathbb{L})^{n-\mathrm{dim}F}.$$

This completes the proof. □

REMARK 4.3.5. As is clear from the above proof of Theorem 4.3.2, it can be immediately generalized to any schön family of subvarieties of $T_0 \simeq ((\mathbb{C}^*))^n$ as in [**44**, Theorem 3.2 and Corollary 5.3]. However in this chapter, we do not use such a generalization.

By the proof of the above theorem we obtain the following result.

LEMMA 4.3.6. *Assume that the family $Y$ is schön. Then there exists small $\varepsilon > 0$ such that the hypersurface $Y_t = Y \cap \pi^{-1}(t) \subset T_0 = ((\mathbb{C}^*))_x^n$ is Newton non-degenerate (see [**30**]) for any $t \in (\mathbb{C}^*)$ satisfying the condition $0 < |t| < \varepsilon$.*

Recall that for a constructible sheaf $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C})$ on $\mathbb{C}$ its nearby cycle sheaf $\psi_t(F) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{0\})$ by the function $t$ has a direct sum decomposition

$$\psi_t(F) = \bigoplus_{\lambda \in \mathbb{C}} \psi_{t,\lambda}(F)$$

with respect to the generalized eigenspaces $\psi_{t,\lambda}(F) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{0\})$ for eigenvalues $\lambda \in \mathbb{C}$ (see Section 2). Let $j : B(0;\varepsilon)^* \hookrightarrow B(0;\varepsilon) = B(0;\varepsilon)^* \sqcup \{0\}$ be the inclusion map. By the rotation on the punctured disk we obtain the monodromy automorphisms

$$\Phi_j : H^j_c(Y_t;\mathbb{C}) \xrightarrow{\sim} H^j_c(Y_t;\mathbb{C}) \qquad (j \in \mathbb{Z})$$

for $0 < |t| < \varepsilon$. For $\lambda \in \mathbb{C}$ let

$$H^j_c(Y_t;\mathbb{C})_\lambda \subset H^j_c(Y_t;\mathbb{C})$$

be the generalized eigenspace of $\Phi_j$ for the eigenvalue $\lambda$. Then we have an isomorphism

$$H^j\psi_{t,\lambda}(j_!\mathrm{R}\pi_!\mathbb{C}_Y) \simeq H^j_c(Y_t;\mathbb{C})_\lambda$$

for any $j \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$.

PROPOSITION 4.3.7. *Assume that the family $Y$ is schön. Then for $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ we have*

$$H^j_c(Y_t;\mathbb{C}) \simeq 0 \qquad (j < n-1)$$

*and the Gysin map*

$$H^j_c(Y_t;\mathbb{C}) \longrightarrow H^{j+2}_c(T_0;\mathbb{C})$$

*associated to the inclusion map $Y_t \hookrightarrow T_0$ is an isomorphism (resp. surjective) for $j > n-1$ (resp. $j = n-1$). Moreover the monodromy $\Phi_j : H^j_c(Y_t;\mathbb{C}) \xrightarrow{\sim} H^j_c(Y_t;\mathbb{C})$ is identity for any $j > n-1$. In particular, for any $\lambda \neq 1$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ we have the concentration*

$$H^j_c(Y_t;\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1).$$

PROOF. Since $Y_t \subset T_0$ is smooth and affine, the first assertion follows from the (generalized) Poincaré duality theorem

$$H^j_c(Y_t;\mathbb{C}) \simeq [H^{2n-2-j}(Y_t;\mathbb{C})]^* \qquad (j \in \mathbb{Z}).$$

Moreover by the above lemma, the second assertion follows from the weak Lefschetz theorem (see Danilov-Khovanskii [**4**, Proposition 3.9]) for the Newton non-degenerate hypersurface $Y_t \subset T_0 = ((\mathbb{C}^*))^n_x$. Note also that the morphism

$$\psi_t(j_!\mathrm{R}\pi_!\mathbb{C}_T) \longrightarrow \psi_t(j_!\mathrm{R}\pi_!\mathbb{C}_Y)$$

induced by the one $\mathbb{C}_T \to \mathbb{C}_Y$ is compatible with the monodromy automorphisms on $\psi_t(j_!\mathrm{R}\pi_!\mathbb{C}_T)$ and $\psi_t(j_!\mathrm{R}\pi_!\mathbb{C}_Y)$. Then the remaining assertion follows immediately from the previous ones. □

Now let $X$ be a general variety and for some $\varepsilon > 0$ consider a family $Y \subset B(0;\varepsilon)^* \times X$ of subvarieties of $X$ over the punctured disk $B(0;\varepsilon)^* \subset \mathbb{C}$. Let $\pi : B(0;\varepsilon)^* \times X \to B(0;\varepsilon)^*$ be the projection and for $t \in \mathbb{C}$ such that $0 < |t| < \varepsilon$ set $Y_t = Y \cap \pi^{-1}(t) \subset X$. Then we obtain the monodromy automorphisms

$$\Phi_j : H^j_c(Y_t;\mathbb{C}) \xrightarrow{\sim} H^j_c(Y_t;\mathbb{C}) \qquad (j \in \mathbb{Z})$$

for $0 < |t| < \varepsilon$. For $\lambda \in \mathbb{C}$ let

$$H_c^j(Y_t; \mathbb{C})_\lambda \subset H_c^j(Y_t; \mathbb{C})$$

be the generalized eigenspace of $\Phi_j$ for the eigenvalue $\lambda$. Then we have an isomorphism

$$H^j \psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_Y) \simeq H_c^j(Y_t; \mathbb{C})_\lambda$$

for any $j \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$. As a generalization of Theorem 3.3.1, considering the mixed Hodge module over the left hand side $H^j \psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_Y)$ we can endow the right hand side $H_c^j(Y_t; \mathbb{C})_\lambda$ with a mixed Hodge structure which coincides with the classical limit mixed Hodge structure (see El Zein [**12**] and Steenbrink-Zucker [**47**] etc.). The weight filtration $M_\bullet$ on it is the "relative" monodromy filtration with respect to its Deligne's weight filtration $W_\bullet$ in the following sence. Let $\Phi_j^u$ be the unipotent part of the monodromy $\Phi_j$ and set $N = \log \Phi_j^u : H_c^j(Y_t; \mathbb{C}) \to H_c^j(Y_t; \mathbb{C})$. Then for any $r \in \mathbb{Z}$ the filtration $M(r)_\bullet$ on the graded piece $V_r = \mathrm{Gr}_r^W H_c^j(Y_t; \mathbb{C})$ induced by $M_\bullet$ and the morphism $N(r) : V_r \to V_r$ induced by $N$ give rise to isomorphisms

$$N(r)^k : \mathrm{Gr}_{r+k}^{M(r)} V_r \xrightarrow{\sim} \mathrm{Gr}_{r-k}^{M(r)} V_r \qquad (k \geq 0).$$

Namely the filtration $M(r)_\bullet$ on $V_r$ is the monodromy filtration of the automorphism $\mathrm{Gr}_r^W(\Phi_j) : V_r \xrightarrow{\sim} V_r$ centered at $r$. Deligne proved that such a filtration $M_\bullet$ on $H_c^j(Y_t; \mathbb{C})$ is unique (if it exists). For $\lambda \in \mathbb{C}$ and $p, q, r \in \mathbb{Z}$ let $h^{p,q}(\mathrm{Gr}_r^W H_c^j(Y_t; \mathbb{C})_\lambda) \geq 0$ be the dimension of the $(p,q)$-part of the above limit mixed Hodge structure on the graded piece $\mathrm{Gr}_r^W H_c^j(Y_t; \mathbb{C})_\lambda$ defined by the weight filtration $M(r)_\bullet$.

Definition 4.3.8 (Stapledon [**44**]). For $\lambda \in \mathbb{C}$ we define the equivariant refined limit mixed Hodge polynomial (resp. the equivariant limit mixed Hodge polynomial) $E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w]$ (resp. $E_\lambda(Y_t; u, v) \in \mathbb{Z}[u, v]$) for the eigenvalue $\lambda \in \mathbb{C}$ by

$$E_\lambda(Y_t; u, v, w) = \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}(\mathrm{Gr}_r^W H_c^j(Y_t; \mathbb{C})_\lambda) u^p v^q w^r,$$

$$E_\lambda(Y_t; u, v) = \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}(H_c^j(Y_t; \mathbb{C})_\lambda) u^p v^q.$$

By this definition, obviously we have $E_\lambda(Y_t; u, v, 1) = E_\lambda(Y_t; u, v)$ for any $\lambda \in \mathbb{C}$.

Lemma 4.3.9. *Let $Z \subset Y \subset B(0;\varepsilon)^* \times X$ be a subfamily of $Y$. Then for any $\lambda \in \mathbb{C}$ we have*

$$E_\lambda(Y_t; u, v, w) = E_\lambda(Z_t; u, v, w) + E_\lambda((Y \setminus Z)_t; u, v, w) \qquad (0 < |t| < \varepsilon).$$

Proof. There exists a long exact sequence

$$\cdots \to H^j \mathrm{R}\pi_! \mathbb{Q}_{Y \setminus Z}^H \to H^j \mathrm{R}\pi_! \mathbb{Q}_Y^H \to H^j \mathrm{R}\pi_! \mathbb{Q}_Z^H \to H^{j+1} \mathrm{R}\pi_! \mathbb{Q}_{Y \setminus Z}^H \to \cdots$$

of mixed Hodge modules. For any $r \in \mathbb{Z}$ by taking the $r$-th graded piece $\mathrm{Gr}_r^W(\cdot)$ of each term in it, we obtain again a long exact sequence. Then the assertion follows by applying the (exact) nearby cycle functor $\psi_t(\cdot)$ of mixed Hodge modules to them. □

Now let us return to the family $Y \subset B(0;\varepsilon)^* \times (\mathbb{C}^*)^n$ of hypersurfaces of $T_0 = (\mathbb{C}^*)^n$ over the punctured disk $B(0;\varepsilon)^*$. Then by Proposition 4.2.3 and Theorem 4.3.2 we obtain the following corollary.

COROLLARY 4.3.10. *Assume that the family* $Y \subset B(0;\varepsilon)^* \times (\mathbb{C}^*)^n$ *is schön. Then we have*

$$E_\lambda(Y_t; u, v) = \sum_{\mathrm{rel.int} F \subset \mathrm{Int}\, P} E_\lambda(V_F \circlearrowleft \hat{\mu}; u, v) \cdot (1 - uv)^{n - \dim F},$$

*for any* $\lambda \in \mathbb{C}$. *Here the equivariant mixed Hodge polynomials* $E_\lambda(V_F \circlearrowleft \hat{\mu}; u, v) \in \mathbb{Z}[u, v]$ *are defined by Deligne's mixed Hodge structure of the variety* $V_F$ *and the semisimple action on its cohomology groups as*

$$E_\lambda(V_F \circlearrowleft \hat{\mu}; u, v) = \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}(H_c^j(V_F; \mathbb{C})_\lambda) u^p v^q.$$

The following fundamental result was obtained by Stapledon in [**44**]. For $\lambda \in \mathbb{C}$ set

$$\varepsilon(\lambda) = \begin{cases} 1 & (\lambda = 1) \\ 0 & (\lambda \neq 1) \end{cases}$$

and recall that we have

$$h^*_\lambda(P, \nu_f; u, v, w) = \sum_{Q \prec P} w^{\dim Q + 1} l^*_\lambda(Q, \nu_f|_Q; u, v) \cdot g([Q, P]; uvw^2). \tag{25}$$

THEOREM 4.3.11 ([**44**, Theorem 5.7]). *Assume that the family* $Y$ *is schön. Then for any* $t \in (\mathbb{C}^*)$ *such that* $0 < |t| \ll 1$ *and* $\lambda \in \mathbb{C}$ *we have*

$$uvw^2 E_\lambda(Y_t; u, v, w) = \varepsilon(\lambda) \cdot (uvw^2 - 1)^n + (-1)^{n-1} h^*_\lambda(P, \nu_f; u, v, w). \tag{26}$$

In [**19**] and [**44**] Katz and Stapledon proved this theorem from Corollary 4.3.10 by using [**24**, Section 2] and some deep results on combinatorics obtained by Stanley [**43**]. For a cell $F \in \mathcal{S}$ by using the affine linear extension $\nu_F \colon \mathrm{Aff}(F) \simeq \mathbb{R}^{\dim F} \longrightarrow \mathbb{R}$ of $\nu|_F \colon F \longrightarrow \mathbb{R}$ we define a positive integer $m_F$ to be the minimal one $m$ for which $m \cdot \nu_F$ takes only integer values on $\mathrm{Aff}(F) \cap \mathbb{Z}^n$. Then we define a finite subset $R_f \subset \mathbb{C}$ by

$$R_f = \bigcup_{F \subset \partial P} \{\lambda \in \mathbb{C} \mid \lambda^{m_F} = 1\} \subset \mathbb{C}.$$

Note that we have $1 \in R_f$. Then by Theorem 4.3.11 and Proposition 4.3.7 we immediately obtain the following result. Note that by the definition of the integers $m_F$ the condition $\lambda \notin R_f$ implies the vanishing $l^*_\lambda(Q, \nu_f|_Q; u, v) = 0$ for any proper face $Q \neq P$ of $P$.

THEOREM 4.3.12. *Assume that the family* $Y$ *is schön. Then for any* $\lambda \notin R_f$ *and* $t \in (\mathbb{C}^*)$ *such that* $0 < |t| \ll 1$ *the equivariant mixed Hodge polynomial* $E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w]$ *for the eigenvalue* $\lambda$ *is concentrated in degree* $n-1$ *in the variable* $w$ *and given by*

$$\begin{aligned} E_\lambda(Y_t; u, v, w) &= (-1)^{n-1} w^{n-1} \sum_{p,q} h^{p,q}(H_c^{n-1}(Y_t; \mathbb{C})_\lambda) u^p v^q \\ &= (-1)^{n-1} \frac{w^{n-1}}{uv} l^*_\lambda(P, \nu_f; u, v) \\ &= (-1)^{n-1} \frac{w^{n-1}}{uv} \sum_{F \in \mathcal{S}} v^{\dim F + 1} l^*_\lambda(F, \nu_f|_F; uv^{-1}) \cdot l_P(\mathcal{S}, F; uv). \end{aligned}$$

*In particular, by setting $u = v = s$ and $w = 1$ we have*

$$E_\lambda(Y_t; s, s) = (-1)^{n-1} \sum_{k\geq 0} \Big( \sum_{p+q=k} h^{p,q}(H_c^{n-1}(Y_t;\mathbb{C})_\lambda) \Big) s^k$$
$$= (-1)^{n-1} \frac{1}{s^2} \sum_{F\in\mathcal{S}} s^{\dim F+1} l_\lambda^*(F, \nu_f|_F; 1) \cdot l_P(\mathcal{S}, F; s^2).$$

At the end of this section, by using mixed Hodge modules we will give a geometric proof to this concentration in degree $n-1$. We also obtain the following corollary. Note that for $\lambda \notin R_f$ by Theorem 4.3.12 and the construction of the weight filtration of the limit mixed Hodge structure of $H_c^{n-1}(Y_t;\mathbb{C})$ the filtration on $H_c^{n-1}(Y_t;\mathbb{C})_\lambda$ induced by it is equal to the monodromy filtration for the monodromy $\Phi_{n-1} : H_c^{n-1}(Y_t;\mathbb{C})_\lambda \xrightarrow{\sim} H_c^{n-1}(Y_t;\mathbb{C})_\lambda$ centered at $n-1$.

Corollary 4.3.13. *Assume that the family $Y \subset B(0;\varepsilon)^* \times (\mathbb{C}^*)^n$ is schön. Then for any $\lambda \notin R_f$ and $t \in \mathbb{C}$ such that $0 < |t| \ll 1$ we have the symmetry*

$$\sum_{p+q=n-1+k} h^{p,q}(H_c^{n-1}(Y_t;\mathbb{C})_\lambda) = \sum_{p+q=n-1-k} h^{p,q}(H_c^{n-1}(Y_t;\mathbb{C})_\lambda)$$

*for any $k \geq 0$.*

By Theorem 4.3.12 and Corollary 4.3.13, for any $\lambda \notin R_f$ the Jordan normal form of the middle-dimensional monodromy

$$\Phi_{n-1} : H_c^{n-1}(Y_t;\mathbb{C})_\lambda \xrightarrow{\sim} H_c^{n-1}(Y_t;\mathbb{C})_\lambda$$

on $H_c^{n-1}(Y_t;\mathbb{C})_\lambda$ can be recovered from the polynomial $E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w]$ as follows. By Theorem 4.3.12 for the polynomial $E_\lambda(Y_t; u, v) = E_\lambda(Y_t; u, v, 1) \in \mathbb{Z}[u, v]$ we have

$$E_\lambda(Y_t; u, v, w) = E_\lambda(Y_t; u, v) \cdot w^{n-1}.$$

Moreover the polynomial $\widetilde{E_\lambda}(Y_t; s) := (-1)^{n-1} E_\lambda(Y_t; s, s) \in \mathbb{Z}[s]$ has only non-negative coefficients and the symmetry centered at $n-1$. By the Lefschetz decomposition of $H_c^{n-1}(Y_t;\mathbb{C})_\lambda$ there exist non-negative integers $q_{\lambda,i} \geq 0$ $(0 \leq i \leq n-1)$ such that

$$\begin{aligned}\widetilde{E_\lambda}(Y_t; s) =& q_{\lambda,0}(1 + s^2 + \cdots + s^{2n-4} + s^{2n-2}) \\ &+ q_{\lambda,1}(s + s^3 + \cdots + s^{2n-3}) \\ &+ q_{\lambda,2}(s^2 + \cdots + s^{2n-4}) \\ &+ \cdots\cdots \\ &+ q_{\lambda,n-1} s^{n-1}.\end{aligned}$$

For $\lambda \in \mathbb{C}$ and $m \geq 1$ denote by $J_{\lambda,m}$ the number of the Jordan blocks in the monodromy automorphism

$$\Phi_{n-1} : H_c^{n-1}(Y_t;\mathbb{C}) \xrightarrow{\sim} H_c^{n-1}(Y_t;\mathbb{C}) \qquad (0 < |t| \ll 1)$$

for the eigenvalue $\lambda$ with size $m$.

Proposition 4.3.14. *Assume that the family $Y$ is schön. Then for any $\lambda \notin R_f$ we have*

$$J_{\lambda,m} = q_{\lambda,n-m} \qquad (1 \leq m \leq n).$$

Recall that for a cell $F \in \mathcal{S}$ the local $h$-polynomial $l_P(\mathcal{S}, F; t) \in \mathbb{Z}[t]$ has non-negative coefficients and the symmetry

$$l_P(\mathcal{S}, F; t) = t^{n-\dim F} l_P(\mathcal{S}, F; t^{-1})$$

(see [**44**, Remark 4.9]). Moreover it is unimodal. Hence there exist non-negative integers $l_{F,i}$ $(0 \leq i \leq \lfloor \frac{n-\dim F}{2} \rfloor)$ such that

$$\begin{aligned} l_P(\mathcal{S}, F; t) = \ & l_{F,0}(1 + t + t^2 + \cdots + t^{n-\dim F}) \\ & + l_{F,1}(t + t^2 + \cdots + t^{n-\dim F-1}) \\ & + l_{F,2}(t^2 + \cdots + t^{n-\dim F-2}) \\ & + \cdots\cdots . \end{aligned}$$

We set

$$\widetilde{l}_P(\mathcal{S}, F; t) = \sum_{i=0}^{\lfloor \frac{n-\dim F}{2} \rfloor} l_{F,i} t^i.$$

Then by Theorem 4.3.12 and Proposition 4.3.14 we obtain the following result.

THEOREM 4.3.15. *Assume that the family $Y$ is schön. Then for $\lambda \notin R_f$ we have*

$$\sum_{m=0}^{n-1} J_{\lambda,n-m} s^{m+2} = \sum_{F \in \mathcal{S}} s^{\dim F+1} l^*_\lambda(F, \nu_f|_F; 1) \cdot \widetilde{l}_P(\mathcal{S}, F; s^2).$$

*In particular, we have*

$$J_{\lambda,n} = \sum_{F \in \mathcal{S},\ \dim F = 1} l^*_\lambda(F, \nu_f|_F; 1) \cdot l_{F,0}.$$

The multiplicities of the eigenvalues $\lambda \neq 1$ in the middle-dimensional monodromy $\Phi_{n-1}$ are described more simply as follows.

THEOREM 4.3.16. *Assume that the family $Y$ is schön. Then for $\lambda \neq 1$ the multiplicity of the factor $t - \lambda$ in the characteristic polynomial of the monodromy*

$$\Phi_{n-1} : H^{n-1}_c(Y_t; \mathbb{C}) \xrightarrow{\sim} H^{n-1}_c(Y_t; \mathbb{C}) \qquad (0 < |t| \ll 1)$$

*is equal to that in*

$$\prod_{\mathrm{rel.int} F \subset \mathrm{Int}\, P,\ \dim F = n} (t^{m_F} - 1)^{\mathrm{Vol}_{\mathbb{Z}}(\widetilde{F})},$$

*where $\mathrm{Vol}_{\mathbb{Z}}(\widetilde{F}) \in \mathbb{Z}_{>0}$ is the normalized volume i.e. the $n!$ times usual volume $\mathrm{Vol}(\widetilde{F})$ of $\widetilde{F}$ with respect to the lattice $\mathrm{Aff}(\widetilde{F}) \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^n$ in $\mathrm{Aff}(\widetilde{F}) \simeq \mathbb{R}^n$.*

PROOF. By Proposition 4.3.7 and the proof of Theorem 4.3.2, the assertion can be proved by calculating monodromy zeta functions as in [**23**]. We can obtain it also just by taking the Euler characteristics of the both sides of the equality in Theorem 4.3.2. □

Now let us give a geometric proof to the concentration in Theorem 4.3.12.

THEOREM 4.3.17. *Assume that the family $Y$ is schön. Then for any $\lambda \notin R_f$ the morphism*

$$\psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_Y) \longrightarrow \psi_{t,\lambda}(j_! \mathrm{R}\pi_* \mathbb{C}_Y)$$

*induced by the one $\mathrm{R}\pi_! \mathbb{C}_Y \to \mathrm{R}\pi_* \mathbb{C}_Y$ is an isomorphism.*

Proof. The proof is similar to that of [**48**, Theorem 4.5]. We shall use the notations $\Sigma$, $\Xi$, $X_\Sigma$, $\pi_\Sigma : X_\Sigma \to \mathbb{C}$, $\overline{Y}$, $\pi_{\overline{Y}} : \overline{Y} \to \mathbb{C}$ etc. in the proof of Theorem 4.3.2. By the schönness of $Y$ the hypersurface $\overline{Y} \subset X_\Sigma$ intersects $T$-orbits in $\pi_\Sigma^{-1}(\{0\})$ transversally. Recall that for any cone $\sigma \in \Xi$ the hypersurface $\overline{Y}$ intersects the $T$-orbit $T_\sigma$ associated to it transversally on a neighborhood of $\pi_\Sigma^{-1}(\{0\})$ in $X_\Sigma$. Recall also that the divisor $D = \overline{Y} \setminus Y = \overline{Y} \cap (X_\Sigma \setminus T)$ in $\overline{Y}$ is normal crossing there. Let $i_D : D \hookrightarrow \overline{Y}$ and $j_Y : Y \hookrightarrow \overline{Y}$ be the inclusion maps. Then there exists a commutative diagram

$$\begin{array}{ccc} Y & \overset{j_Y}{\hookrightarrow} & \overline{Y} \\ {\scriptstyle \pi_Y = \pi|_Y}\downarrow & & \downarrow{\scriptstyle \pi_{\overline{Y}}} \\ (\mathbb{C}^*) & \overset{j}{\hookrightarrow} & \mathbb{C}. \end{array}$$

Since $\pi_{\overline{Y}} : \overline{Y} \to \mathbb{C}$ is proper, we have isomorphisms

$$\begin{cases} \psi_t(j_! \mathrm{R}\pi_! \mathbb{C}_Y) \simeq \psi_t(\mathrm{R}(\pi_{\overline{Y}})_*(j_Y)_! \mathbb{C}_Y), \\ \psi_t(j_! \mathrm{R}\pi_* \mathbb{C}_Y) \simeq \psi_t(\mathrm{R}j_* \mathrm{R}(\pi_Y)_* \mathbb{C}_Y) \simeq \psi_t(\mathrm{R}(\pi_{\overline{Y}})_* \mathrm{R}(j_Y)_* \mathbb{C}_Y). \end{cases}$$

Therefore, by applying the functor $\psi_t \circ \mathrm{R}(\pi_{\overline{Y}})_* : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\overline{Y}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{0\})$ to the distinguished triangle

$$(j_Y)_! \mathbb{C}_Y \to \mathrm{R}(j_Y)_* \mathbb{C}_Y \to (i_D)_*(i_D)^{-1} \mathrm{R}(j_Y)_* \mathbb{C}_Y \overset{+1}{\longrightarrow}$$

we obtain the new one

$$\psi_t(j_! \mathrm{R}\pi_! \mathbb{C}_Y) \to \psi_t(j_! \mathrm{R}\pi_* \mathbb{C}_Y) \to \psi_t(\mathrm{R}(\pi_{\overline{Y}})_*(i_D)_*(i_D)^{-1} \mathrm{R}(j_Y)_* \mathbb{C}_Y) \overset{+1}{\longrightarrow} .$$

This implies that for the proof of the theorem it suffices to prove the vanishing

$$\psi_{t,\lambda}(\mathrm{R}(\pi_{\overline{Y}})_*(i_D)_*(i_D)^{-1} \mathrm{R}(j_Y)_* \mathbb{C}_Y) \simeq 0$$

for any $\lambda \notin R_f$. Since $\pi_{\overline{Y}} : \overline{Y} \to \mathbb{C}$ is proper, by [**9**, Proposition 4.2.11] and [**18**, Exercise VIII.15] for any $\lambda \in \mathbb{C}$ we have an isomorphism

$$\psi_{t,\lambda}(\mathrm{R}(\pi_{\overline{Y}})_*(i_D)_*(i_D)^{-1} \mathrm{R}(j_Y)_* \mathbb{C}_Y) \simeq \mathrm{R}\Gamma(\pi_{\overline{Y}}^{-1}(\{0\}); \psi_{\pi_{\overline{Y}},\lambda}((i_D)_*(i_D)^{-1} \mathrm{R}(j_Y)_* \mathbb{C}_Y)).$$

Now let us set

$$D' = \overline{D \setminus \pi_{\overline{Y}}^{-1}(\{0\})} = \bigcup_{\sigma \in \Xi} (\overline{Y} \cap \overline{T_\sigma}) \subset D$$

and $Y' = \overline{Y} \setminus D' \supset Y$. Then $D'$ is a normal crossing divisor of $\overline{Y}$ on a neighborhood of $\pi_{\overline{Y}}^{-1}(\{0\})$ and for the inclusion maps $i_{D'} : D' \hookrightarrow \overline{Y}$ and $j_{Y'} : Y' \hookrightarrow \overline{Y}$ there exists an isomorphism

$$\psi_{\pi_{\overline{Y}},\lambda}((i_D)_*(i_D)^{-1} \mathrm{R}(j_Y)_* \mathbb{C}_Y) \simeq \psi_{\pi_{\overline{Y}},\lambda}((i_{D'})_*(i_{D'})^{-1} \mathrm{R}(j_{Y'})_* \mathbb{C}_{Y'}).$$

Note that for the natural stratification of the normal crossing divisor $D' \subset \overline{Y}$ by the strata

$$D'_\sigma := \overline{Y} \cap \big(\overline{T_\sigma} \setminus \bigcup_{\tau \in \Xi,\ \sigma \subsetneqq \tau} \overline{T_\tau}\big) \subset D' \qquad (\sigma \in \Xi \setminus \{0\})$$

the cohomology sheaves $H^j(i_{D'})^{-1} \mathrm{R}(j_{Y'})_* \mathbb{C}_{Y'}$ ($j \in \mathbb{Z}$) are constructible. Moreover their restrictions to each stratum $D'_\sigma \subset D'$ are constant. Then by cutting the support of the complex $(i_{D'})^{-1} \mathrm{R}(j_{Y'})_* \mathbb{C}_{Y'} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(D')$ by the stratification and truncating each of the resulting complexes, it suffices to prove the vanishing

$$\mathrm{R}\Gamma(\pi_{\overline{Y}}^{-1}(\{0\}); \psi_{\pi_{\overline{Y}},\lambda}(\mathbb{C}_{\overline{Y} \cap \overline{T_\sigma}})) \simeq 0.$$

for any $\lambda \notin R_f$ and any cone $\sigma \in \Xi \setminus \{0\}$. Fixing such $\lambda$ and $\sigma$ we shall prove the vanishing from now. Set $\pi_\sigma = \pi_{\overline{Y}}|_{\overline{Y}\cap\overline{T_\sigma}} : \overline{Y} \cap \overline{T_\sigma} \longrightarrow \mathbb{C}$ and $\mathcal{F}_\sigma = \mathbb{C}_{\overline{Y}\cap\overline{T_\sigma}}$. Then it is enough to show the vanishing

$$\mathrm{R}\Gamma(\pi_\sigma^{-1}(\{0\}); \psi_{\pi_\sigma,\lambda}(\mathcal{F}_\sigma)) \simeq 0.$$

According to the general theory of mixed Hodge modules (for the details see e.g. Dimca-Saito [**10**, Section 1.4] etc.), each graded piece of the nearby cycle perverse sheaf $\psi_{\pi_\sigma,\lambda}(\mathcal{F}_\sigma)[n-\dim\sigma-1] \in \mathrm{D}_c^{\mathrm{b}}(\pi_\sigma^{-1}(\{0\}))$ with respect to its weight filtration has a primitive decomposition into some mininal extension perverse sheaves of $\mathcal{L}_\tau[n-\dim\tau] \in \mathrm{D}_c^{\mathrm{b}}(\overline{Y} \cap T_\tau)$, where $\tau$ is a cone in $\Sigma \setminus \Xi$ such that $\sigma \prec \tau$ and $\mathcal{L}_\tau$ is a rank one local system on the $(n-\dim\tau)$-dimensional smooth hypersurface $\overline{Y} \cap T_\tau \subset T_\tau \simeq (\mathbb{C}^*)^{n-\dim\tau+1}$. Moreover the cones $\tau$ appearing in this decomposition should satisfy the following condition. Let $\tau$ be a cone in $\Sigma \setminus \Xi$ such that $\sigma \prec \tau$ and let $\alpha_1, \dots, \alpha_k \in \tau \cap (\mathbb{Z}^{n+1} \setminus \{0\})$ be the primitive vectors on the edges of $\tau$ not contained in the hyperplane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. Set $b_i = q(\alpha_i) > 0$ $(1 \leq i \leq k)$ and $d_\tau = \gcd\{b_i \mid 1 \leq i \leq k\} > 0$. Then, for any $1 \leq i \leq k$ the order of the zero of the function $\pi_\sigma$ along the divisor of $\overline{Y} \cap \overline{T_\sigma}$ associated to the ray containing $\alpha_i$ is equal to $b_i > 0$. Moreover for the cone $\tau$ to appear in the decomposition it should satisfy the condition $\lambda^{d_\tau} = 1$. In such a case, the rank one local system $\mathcal{L}_\tau$ on $\overline{Y} \cap T_\tau$ has the following condition. Let $\rho \in \Sigma \setminus \Xi$ be a ray such that $\tau \cap \rho = \{0\}$ and $\tau(\rho) := \tau + \rho$ is a cone in $\Sigma$. Let $\beta \in \rho \cap (\mathbb{Z}^{n+1} \setminus \{0\})$ be the primitive vector on it. Then for any such $\rho$ the monodromy of the local system $\mathcal{L}_\tau$ around the divisor $\overline{Y} \cap \overline{T_{\tau(\rho)}} \subset \overline{Y} \cap \overline{T_\tau}$ is given by the multiplication by the complex number $\lambda^{-q(\beta)} \in \mathbb{C}$. By cutting the supports of the minimal extension perverse sheaves by the toric stratifications of $\overline{T_\tau}$, for the proof of the theorem it suffices to prove the vanishing

$$\mathrm{R}\Gamma_c(\overline{Y} \cap T_\tau; \mathcal{L}_\tau) \simeq 0$$

for any cone $\tau$ in $\Sigma \setminus \Xi$ such that $\sigma \prec \tau$ and $\lambda^{d_\tau} = 1$. For a cell $F$ in $\mathcal{S}$ let $\widetilde{F} \prec \mathrm{UH}_f$ be the unique compact face of $\mathrm{UH}_f$ such that $F = p(\widetilde{F})$ and $F^\circ \in \Sigma_0$ the cone which corresponds to it in the dual fan $\Sigma_0$. Then for the cone $\tau$ there exists a unique cell $F \in \mathcal{S}$ such that $F \subset \partial P$ and $\mathrm{rel.int}\tau \subset \mathrm{rel.int}F^\circ$. Let $\tau'$ be a cone in $\Sigma \setminus \Xi$ such that $\tau' \subset F^\circ$, $\dim\tau' = \dim F^\circ$ and $\tau \prec \tau'$. Then by the smoothness of the cone $\tau'$ we can easily show the equality $m_F = d_{\tau'}$. By our assumption $\lambda \notin R_f$ we thus obtain $\lambda^{d_{\tau'}} \neq 1$. This implies that there exists a ray $\rho$ of $\tau'$ such that $\tau \cap \rho = \{0\}$ and the primitive vector $\beta \in \rho \cap (\mathbb{Z}^{n+1} \setminus \{0\})$ on it satisfies the condition $\lambda^{-q(\beta)} \neq 1$. Moreover, since the supporting faces of $\tau$ and $\tau'$ in $\mathrm{UH}_f$ coincide and are equal to $\widetilde{F} \prec \mathrm{UH}_f$, we have a product decomposition

$$\overline{Y} \cap T_\tau \simeq Z \times (\mathbb{C}^*)^k$$

for some variety $Z$ and $k > 0$ such that the equation of the divisor $\overline{T_{\tau(\rho)}} \subset \overline{T_\tau}$ corresponds to a coordinate of the torus $(\mathbb{C}^*)^k$. Now the desired vanishing follows from the Künneth formula. This completes the proof. □

By Theorem 4.3.17 we can reprove the concentration in Theorem 4.3.12 as follows.

COROLLARY 4.3.18. *Assume that the family $Y$ is schön. Then for any $\lambda \notin R_f$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ we have the concentration*

$$H_c^j(Y_t; \mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1).$$

*Moreover the equivariant mixed Hodge polynomial* $E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w]$ *for the eigenvalue* $\lambda$ *is concentrated in degree* $n-1$ *in the variable* $w$.

Proof. Assume that $\lambda \notin R_f$. Then the first assertion is already shown in Proposition 4.3.7. However we shall give a new proof to it by using Theorem 4.3.17. By Theorem 4.3.17 for $t \in \mathbb{C}$ such that $0 < |t| \ll 1$ there exist isomorphisms

$$H_c^j(Y_t; \mathbb{C})_\lambda \xrightarrow{\sim} H^j(Y_t; \mathbb{C})_\lambda \qquad (j \in \mathbb{Z}).$$

Since the $(n-1)$-dimensional variety $Y_t$ is affine and smooth, by the (generalized) Poincaré duality the left (resp. right) hand side vanishes for $j < n-1$ (resp. $j > n-1$). We thus obtain the concentration

$$H_c^j(Y_t; \mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1).$$

By applying the proof of Sabbah [**34**, Theorem 13.1] to the above isomorphisms we see that the only non-trivial cohomology group $H_c^{n-1}(Y_t; \mathbb{C})_\lambda$ has a pure weight $n-1$. □

## 4.4. Monodromies and limit mixed Hodge structures of families of hypersurfaces in $\mathbb{C}^n$

Let $f(t, x) = \sum_{v \in \mathbb{Z}_+^n} a_v(t) x^v \in \mathbb{K}[x_1, \dots, x_n]$ ($a_v(x) \in \mathbb{K}$) be a polynomial of $x = (x_1, \dots, x_n)$ over the field $\mathbb{K} = \mathbb{C}(t)$ of rational functions of $t$. Then as in Section 4.3 we can define an (unbounded) polyhedron $\mathrm{UH}_f$ associated to it in $\mathbb{R}_+^n \times \mathbb{R}^1$ and its projection $P = p(\mathrm{UH}_f) \subset \mathbb{R}_+^n$. We call $P$ the Newton polytope of $f \in \mathbb{K}[x_1, \dots, x_n]$. Throughout this section we assume that $\mathrm{dim} P = n$.

Definition 4.4.1 (Stapledon [**44**]). We say that $P$ is convenient if for any coordinate subspace $H$ of $\mathbb{R}^n$ we have $\mathrm{dim} P \cap H = \mathrm{dim} H$.

By this definition, for a convenient polytope $P$ we have $0 \in P$. Let $\Sigma_0$ be the dual fan of $\mathrm{UH}_f$ in $\mathbb{R}^n \times \mathbb{R}_+^1 \subset \mathbb{R}^{n+1}$ and $\nu_f : P \to \mathbb{R}$ the function defining the bottom part of the boundary $\partial \mathrm{UH}_f$ of $\mathrm{UH}_f$. Moreover by the subdivision $\mathcal{S}$ of $P$ into the lattice polytopes $p(\widetilde{F})$ ($\widetilde{F} \prec \mathrm{UH}_f$) we define polynomials $I_f^F(x) \in \mathbb{C}[x_1, \dots, x_n]$ and elements $[V_F \circlearrowleft \hat{\mu}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ for cells $F \in \mathcal{S}$ as in Section 4.3. In this situation, the hypersurface $f^{-1}(0) \subset X = (\mathbb{C}^*)_t \times \mathbb{C}_x^n$ defines a family $Y$ of hypersurfaces of $X_0 = \mathbb{C}_x^n$ over a small punctured disk $B(0; \varepsilon)^*$ ($0 < \varepsilon \ll 1$). By the projection $\pi : (\mathbb{C}^*)_t \times \mathbb{C}_x^n \twoheadrightarrow (\mathbb{C}^*)_t$ for $t \in \mathbb{C}$ such that $0 < |t| < \varepsilon$ we set $Y_t := \pi^{-1}(t) \cap Y \subset \{t\} \times X_0 \simeq X_0 = \mathbb{C}_x^n$. We define also the schönness of the family as in Definition 4.3.1. Then by the proof of the weak Lefschetz theorem (see Danilov-Khovanskii [**4**, Proposition 3.9]) we can prove the following proposition.

Proposition 4.4.2. *Assume that the family* $Y$ *of hypersurfaces in* $X_0 = \mathbb{C}^n$ *is schön and* $P$ *is convenient. Then for* $t \in (\mathbb{C}^*)$ *such that* $0 < |t| \ll 1$ *we have*

$$H_c^j(Y_t; \mathbb{C}) \simeq 0 \qquad (j < n-1)$$

*and the Gysin map*

$$H_c^j(Y_t; \mathbb{C}) \longrightarrow H_c^{j+2}(X_0; \mathbb{C})$$

*associated to the inclusion map* $Y_t \hookrightarrow X_0$ *is an isomorphism (resp. surjective) for* $j > n-1$ *(resp.* $j = n-1$*). Moreover the monodromy* $\Phi_j : H_c^j(Y_t; \mathbb{C}) \xrightarrow{\sim} H_c^j(Y_t; \mathbb{C})$ *is identity for any* $j > n-1$.

We set $P_\infty := \overline{\partial P \cap \mathrm{int}\, \mathbb{R}_+^n} \subset \partial P$.

DEFINITION 4.4.3 (Stapledon [**44**]). We say that the polynomial $f(t,x) = \sum_{v\in\mathbb{Z}_+^n} a_v(t)x^v \in \mathbb{K}[x_1,\dots,x_n]$ satisfies the condition (S) if the function $\nu_f : P \to \mathbb{R}$ is constant on $P_\infty \subset \partial P$.

The following result was proved in Stapledon [**44**] by calculating the related equivariant mixed Hodge polynomials $E_\lambda(Y_t;u,v,w) \in \mathbb{Z}[u,v,w]$ very precisely.

THEOREM 4.4.4 (Stapledon [**44**, Corollary 6.3]). *Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön, $P$ is convenient and $f(t,x) \in \mathbb{K}[x_1,\dots,x_n]$ satisfies the condition* (S). *Then for any $\lambda \neq 1$ the equivariant refined limit mixed Hodge polynomial $E_\lambda(Y_t;u,v,w) \in \mathbb{Z}[u,v,w]$ for the eigenvalue $\lambda$ is concentrated in degree $n-1$ in the variable $w$.*

We can drop the condition (S) and the one on $P$ in Theorem 4.4.4 as follows. We define a finite subset $R_f \subset \mathbb{C}$ by

$$R_f = \bigcup_{F\subset P_\infty} \{\lambda \in \mathbb{C} \mid \lambda^{m_F} = 1\} \subset \mathbb{C}. \tag{27}$$

Then we have the following result.

THEOREM 4.4.5. *Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön. Then for $\lambda \notin R_f$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ the equivariant refined limit mixed Hodge polynomial $E_\lambda(Y_t;u,v,w) \in \mathbb{Z}[u,v,w]$ for the eigenvalue $\lambda$ is concentrated in degree $n-1$ in the variable $w$ and given by*

$$\begin{aligned} E_\lambda(Y_t;u,v,w) &= (-1)^{n-1}w^{n-1}\sum_{p,q} h^{p,q}(H_c^{n-1}(Y_t;\mathbb{C})_\lambda)u^pv^q \\ &= (-1)^{n-1}\frac{w^{n-1}}{uv} l^*_\lambda(P,\nu_f;u,v) \\ &= (-1)^{n-1}\frac{w^{n-1}}{uv}\sum_{F\in\mathcal{S}} v^{\dim F+1} l^*_\lambda(F,\nu_f|_F;uv^{-1})\cdot l_P(\mathcal{S},F;uv). \end{aligned}$$

PROOF. For a possibly empty subset $I \subset \{1,\dots,n\}$, we define a subset $T^I$ of $X_0 = \mathbb{C}^n$ by

$$T^I := \{(x_1,\dots,x_n) \in X_0 \mid x_i = 0\ (i \notin I), x_i \neq 0\ (i\in I)\} \simeq ((\mathbb{C}^*))^{|I|}.$$

Then we have a decomposition $X_0 = \mathbb{C}^n = \bigsqcup_{I\subset\{1,\dots,n\}} T^I$ of $X_0 = \mathbb{C}^n$. We also define a polynomial $f_I \in \mathbb{K}[(x_i)_{i\in I}]$ by substituting 0 into the variable $x_i$ $(i \notin I)$ of $f$, a family of hypersurfaces $Y^I$ of $T^I$ by $Y^I := f_I^{-1}(0) \subset B^* \times T^I$ and a polytope $P^I$ in $\mathbb{R}^I = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_i = 0\ (i \notin I)\} \subset \mathbb{R}^n$ by $P^I := P \cap \mathbb{R}^I$. Then by Lemma 4.3.9 we have

$$E_\lambda(Y_t;u,v,w) = \sum_{I\subset\{1,\dots,n\}} E_\lambda(Y_t^I;u,v,w)$$

for $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$. We shall say that a face $Q \prec P$ of $P$ is relevant if $Q \not\subset P_\infty$. If $Q \prec P$ is relevant, then for any face $\sigma$ of the first quadrant $\mathbb{R}^n_+$ containing $Q$ the face $\sigma \cap P \prec P$ of $P$ is also relevant. Moreover there exist a possibly empty subset $I \subset \{1,\dots,n\}$ such that $Q = P^I$ and $\dim P^I = |I|$. We denote by $S$ the set consisting of possibly empty subsets $I \subset \{1,\dots,n\}$ such that $P^I$ are relevant. Then by Theorem 4.3.11 for $\lambda \notin R_f$ we have

$$uvw^2E_\lambda(Y_t;u,v,w) = \sum_{I\in S}(-1)^{|I|-1}h^*_\lambda(P^I,\nu_f|_{P^I};u,v,w).$$

Moreover for each relevant face $P^I \prec P$ of $P$ by Definition 4.2.7 we have

$$h_\lambda^*(P^I, \nu_f|_{P^I}; u, v, w) = \sum_{Q \prec P^I} w^{\dim Q+1} l_\lambda^*(Q, \nu_f|_Q; u, v) \cdot g([Q, P^I]; uvw^2). \quad (28)$$

If $Q \prec P^I$ is not a relevant face of $P$, then $Q \subset P_\infty$ and for $\lambda \notin R_f$ we have $l_\lambda^*(Q, \nu_f|_Q; u, v) = 0$. Note also that $l_\lambda^*(\emptyset, \nu_f|_\emptyset; u, v) = 0$ for $\lambda \neq 1$. Moreover for any $I, I' \in S$ such that $I' \subset I$ we have $g([P^{I'}, P^I]; uvw^2) = 1$. Hence for each fixed $I' \in S$ we have

$$\sum_{I: I' \subset I} (-1)^{|I|-1} g([P^{I'}, P^I]; uvw^2) = \sum_{I: I' \subset I} (-1)^{|I|-1} = \begin{cases} (-1)^{n-1} & (I' = \{1, \dots, n\}) \\ 0 & (otherwise). \end{cases}$$

We thus obtain

$$\begin{aligned} uvw^2 E_\lambda(Y_t; u, v, w) &= \sum_{I \in S} (-1)^{|I|-1} \sum_{I' \in S: I' \subset I} w^{|I'|+1} l_\lambda^*(P^{I'}, \nu_f|_{P^{I'}}; u, v) \cdot g([P^{I'}, P^I]; uvw^2) \\ &= \sum_{I' \in S} w^{|I'|+1} l_\lambda^*(P^{I'}, \nu_f|_{P^{I'}}; u, v) \cdot \Big\{ \sum_{I: I' \subset I} (-1)^{|I|-1} g([P^{I'}, P^I]; uvw^2) \Big\} \\ &= (-1)^{n-1} w^{n+1} l_\lambda^*(P, \nu_f; u, v). \end{aligned}$$

□

We shall say that a face $\sigma \prec \mathbb{R}^n_+$ of the first quadrant $\mathbb{R}^n_+$ is relevant if the condition $(P \setminus P_\infty) \cap \sigma \neq \emptyset$ is satisfied. It is easy to see that if $\sigma \prec \mathbb{R}^n_+$ is relevant then we have $\dim(P \cap \sigma) = \dim \sigma$. Let $\Sigma_1$ be the fan in $\mathbb{R}^n$ consisting of all the faces of $\mathbb{R}^n_+$ and regard it as the dual fan of the first quadrant $\mathbb{R}^n_+$. Denote by $\Sigma_1^\circ \subset \Sigma_1$ its subset consisting of the dual cones of the relevant faces of $\mathbb{R}^n_+$. Then we can easily see that $\Sigma_1^\circ$ is a subfan of $\Sigma_1$. Denote by $\Omega_0$ the toric variety associated to $\Sigma_1^\circ$. Then $\Omega_0$ is an open subset of $X_0 = \mathbb{C}^n$ and $X_0 \setminus \Omega_0$ is a closed subset in it. Moreover for the action of $T_0 = (\mathbb{C}^*)^n$ on $X_0 = \mathbb{C}^n$ it is a union of some $T_0$-orbits. Set $Y^\circ = Y \cap (\mathbb{C}^* \times \Omega_0) \subset \mathbb{C}^* \times \Omega_0$ and let $\pi^\circ : \mathbb{C}^* \times \Omega_0 \to \mathbb{C}^*$ be the projection.

THEOREM 4.4.6. *Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön. Then for any $\lambda \notin R_f$ the morphism*

$$\psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_{Y^\circ}) \longrightarrow \psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_Y)$$

*induced by the one $\mathbb{C}_{Y^\circ} \to \mathbb{C}_Y$ is an isomorphism. Moreover for such $\lambda$ the morphism*

$$\psi_{t,\lambda}(j_! \mathrm{R}(\pi^\circ)_! \mathbb{C}_{Y^\circ}) \longrightarrow \psi_{t,\lambda}(j_! \mathrm{R}(\pi^\circ)_* \mathbb{C}_{Y^\circ})$$

*induced by the one $\mathrm{R}(\pi^\circ)_! \mathbb{C}_{Y^\circ} \to \mathrm{R}(\pi^\circ)_* \mathbb{C}_{Y^\circ}$ is an isomorphism.*

PROOF. The proof is similar to that of Theorem 4.3.17. By decomposing the normal crossing divisor $X_0 \setminus \Omega_0$ into tori and applying Proposition 4.3.7 and Theorem 4.3.12 to each of them, for $\lambda \notin R_f$ we obtain the vanishing

$$\psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_{Y \setminus Y^\circ}) \simeq 0$$

from which the first assertion follows. Let $\Xi_0$ be the subfan of the dual fan $\Sigma_0$ in $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$ consisting of the cones $\sigma \in \Sigma_0$ contained in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. Then $\Xi_0$ is the dual fan of the $n$-dimensional polytope $P \subset \mathbb{R}^n$. Moreover by the definition of $\Sigma_1^\circ$ we can easily see that $\Sigma_1^\circ$ is a subfan of $\Xi_0$. By this property we can construct a smooth subdivision $\Sigma$ of $\Sigma_0$ such that $\Sigma_1^\circ \subset \Sigma$. Then the toric variety

$X_\Sigma$ associated to $\Sigma$ is a smooth variety containing $\mathbb{C}^* \times \Omega_0$ and the second assertion can be proved as in the proof of Theorem 4.3.17. □

By Theorems 4.4.5 and 4.4.6 we can drop the condition on $P$ in Proposition 4.4.2 as follows. For $\lambda \in \mathbb{C}$ let

$$H_c^j(Y_t;\mathbb{C})_\lambda \subset H_c^j(Y_t;\mathbb{C})$$

be the generalized eigenspace of $\Phi_j$ for the eigenvalue $\lambda$. Similarly for $Y^\circ \subset Y$ we define linear subspaces

$$H_c^j(Y_t^\circ;\mathbb{C})_\lambda \subset H_c^j(Y_t^\circ;\mathbb{C}) \qquad (\lambda \in \mathbb{C}).$$

Then by Theorem 4.4.6 for any $\lambda \notin R_f$ there exist isomorphisms

$$H_c^j(Y_t^\circ;\mathbb{C})_\lambda \simeq H_c^j(Y_t;\mathbb{C})_\lambda \qquad (j \in \mathbb{Z}).$$

Corollary 4.4.7. *Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön. Then for any $\lambda \notin R_f$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ we have the concentration*

$$H_c^j(Y_t;\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1)$$

*and the filtration on the only non-trivial cohomology group $H_c^{n-1}(Y_t;\mathbb{C})_\lambda$ induced by Deligne's weight filtration on $H_c^{n-1}(Y_t;\mathbb{C})$ is concentrated in degree $n-1$.*

Proof. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let

$$H^j(Y_t^\circ;\mathbb{C})_\lambda \subset H^j(Y_t^\circ;\mathbb{C})$$

be the generalized eigenspace of the monodromy $H^j(Y_t^\circ;\mathbb{C}) \xrightarrow{\sim} H^j(Y_t^\circ;\mathbb{C})$. Then by Theorem 4.4.6 for any $\lambda \notin R_f$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ we have isomorphisms

$$H_c^j(Y_t^\circ;\mathbb{C})_\lambda \simeq H^j(Y_t^\circ;\mathbb{C})_\lambda \qquad (j \in \mathbb{Z}).$$

By the proof of Sabbah [**34**, Theorem 13.1] this implies that for any $j \in \mathbb{Z}$ the filtration on $H_c^j(Y_t^\circ;\mathbb{C})_\lambda$ induced by Deligne's weight filtration of $H_c^j(Y_t^\circ;\mathbb{C})$ is concentrated in degree $j$. Then the assertion follows immediately from Theorem 4.4.5. □

Remark 4.4.8. By the proofs of Theorems 4.3.17 and 4.4.6, if the family $Y$ is schön we can also show that for any $\lambda \notin R_f$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ there exist isomorphisms

$$H^j(Y_t;\mathbb{C})_\lambda \simeq H^j(Y_t^\circ;\mathbb{C})_\lambda \qquad (j \in \mathbb{Z}).$$

Since the proof is similar, we omit the details.

By Corollary 4.4.7, we can prove the following formula for the multiplicities of the eigenvalues $\lambda \notin R_f$ in the monodromy $\Phi_{n-1}$ by calculating monodromy zeta functions as in [**23**]. For a cell $F \in \mathcal{S}$ denote by $Q_F \prec P$ the unique face of $P$ such that $\mathrm{rel.int}F \subset \mathrm{rel.int}Q_F$.

Theorem 4.4.9. *Assume that the family $Y$ of hypersurfaces in $X_0 = \mathbb{C}^n$ is schön. Then for $\lambda \notin R_f$ the multiplicity of the factor $t - \lambda$ in the characteristic polynomial of the monodromy*

$$\Phi_{n-1} : H_c^{n-1}(Y_t;\mathbb{C}) \xrightarrow{\sim} H_c^{n-1}(Y_t;\mathbb{C}) \qquad (0 < |t| \ll 1)$$

*is equal to that in*

$$\prod_{F \not\subset P_\infty,\ \dim F = \dim Q_F} (t^{m_F} - 1)^{(-1)^{n-\dim F}\mathrm{Vol}_{\mathbb{Z}}(\widetilde{F})},$$

*where* $\mathrm{Vol}_{\mathbb{Z}}(\widetilde{F}) \in \mathbb{Z}_{>0}$ *is the normalized volume of* $\widetilde{F}$ *with respect to the lattice* $\mathrm{Aff}(\widetilde{F}) \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{\dim F}$ *in* $\mathrm{Aff}(\widetilde{F}) \simeq \mathbb{R}^{\dim F}$.

Moreover by Theorems 4.4.5 and 4.4.6 and Corollary 4.4.7 we can easily obtain results similar to the ones in Corollary 4.3.13, Proposition 4.3.14 and Theorem 4.3.15. In particular as in Theorem 4.3.15, by Corollary 4.4.7 and Theorem 4.4.5 we can describe the numbers $J_{\lambda,m}$ of the Jordan blocks in the middle-dimensional monodromy

$$\Phi_{n-1} : H_c^{n-1}(Y_t;\mathbb{C}) \xrightarrow{\sim} H_c^{n-1}(Y_t;\mathbb{C}) \qquad (0 < |t| \ll 1)$$

for the eigenvalues $\lambda \notin R_f$ with size $m \geq 0$ in terms of $\mathrm{UH}_f$. See Theorem 4.1.3.

## 4.5. Monodromies and limit mixed Hodge structures of families of complete intersection varieties

In this section, we extend our previous results to families complete intersection subvarieties in $((\mathbb{C}^*))^n$ or $\mathbb{C}^n$. Throughout this section, for $1 \leq k \leq n$ let $f_i(t,x)$ $(1 \leq i \leq k)$ be Laurent polynomials $f_i(t,x) = \sum_{v \in \mathbb{Z}^n} a_{i,v}(t)x^v \in \mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$ or polynomials $f_i(t,x) = \sum_{v \in \mathbb{Z}_+^n} a_{i,v}(t)x^v \in \mathbb{K}[x_1, \dots, x_n]$ over the field $\mathbb{K} = \mathbb{C}(t)$. Then the subvariety $f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ in $T = (\mathbb{C}^*)_t \times ((\mathbb{C}^*))_x^n$ or $X = (\mathbb{C}^*)_t \times \mathbb{C}_x^n$ defines a family $Y$ of subvarieties of $T_0 = ((\mathbb{C}^*))_x^n$ or $X_0 = \mathbb{C}^n$ over a small punctured disk $B(0;\varepsilon)^* \subset \mathbb{C}$ $(0 < \varepsilon \ll 1)$. We shall describe its monodromy and limit mixed Hodge structure. As in Section 4.3 we define $\mathrm{UH}_{f_i} \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}_s$ and their second projections $P_i = p(\mathrm{UH}_{f_i}) \subset \mathbb{R}^n$. Set $f = (f_1, \dots, f_k)$ and let

$$\mathrm{UH}_f := \mathrm{UH}_{f_1} + \cdots + \mathrm{UH}_{f_k} \subset \mathbb{R}^{n+1}$$

be the Minkowski sum of $\mathrm{UH}_{f_1}, \dots, \mathrm{UH}_{f_k}$. Set $P = p(\mathrm{UH}_f) = P_1 + \cdots + P_k \subset \mathbb{R}^n$. Throughout this section we assume that $\dim P = n$. By using $\mathrm{UH}_f$, we define a function $\nu_f : P \to \mathbb{R}$, a subdivision $\mathcal{S}$ of $P$ into lattice polytopes and a closed subset $P_\infty \subset P$ as in Sections 4.3 and 4.4. Moreover for each cell $F \in \mathcal{S}$ and $1 \leq i \leq k$ the initial Laurent polynomial $I_{f_i}^F(x) \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ of $f_i$ with respect to $F$ is defined.

Definition 4.5.1 (Stapledon [**44**]). We say that the family $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ of subvarieties of $T_0 = ((\mathbb{C}^*))^n$ or $X_0 = \mathbb{C}^n$ is schön if for any $J \subset \{1, \dots, k\}$ and any cell $F \in \mathcal{S}$ the subvariety $V_F = \bigcap_{j \in J} \{I_j^F = 0\} \subset T_F$ of $T_F \simeq ((\mathbb{C}^*))^{\dim F}$ is a non-degenerate complete intersection (see [**30**]).

It follows easily from the proof of Theorems 4.3.2 and 4.3.17 that if the family $Y$ in $T_0 = ((\mathbb{C}^*))^n$ is schön its generic fiber $Y_t = Y \cap \pi^{-1}(t) \subset T_0$ $(0 < |t| \ll 1)$ is a smooth complete intersection. Moreover by Danilov-Khovanskii [**4**, Theorem 6.4] we obtain the following results. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{Z}$ let

$$H_c^j(Y_t;\mathbb{C})_\lambda \subset H_c^j(Y_t;\mathbb{C})$$

be the generalized eigenspace of the monodromy automorphism

$$\Phi_j : H_c^j(Y_t;\mathbb{C}) \xrightarrow{\sim} H_c^j(Y_t;\mathbb{C})$$

for the eigenvalue $\lambda$.

PROPOSITION 4.5.2. *Let* $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ *be a family of subvarieties in* $T_0 = ((\mathbb{C}^*))^n$ (resp. $X_0 = \mathbb{C}^n$). *Assume that* $Y$ *is schön and* $\dim P_i = n$ *(resp.* $P_i$ *is convenient) for any* $1 \leq i \leq k$. *Then for* $t \in (\mathbb{C}^*)$ *such that* $0 < |t| \ll 1$ *we have*

$$H_c^j(Y_t;\mathbb{C}) \simeq 0 \qquad (j < n-k)$$

*and the Gysin map*

$$H_c^j(Y_t;\mathbb{C}) \longrightarrow H_c^{j+2k}(T_0;\mathbb{C}) \tag{29}$$

$$(\text{resp. } H_c^j(Y_t;\mathbb{C}) \longrightarrow H_c^{j+2k}(X_0;\mathbb{C})) \tag{30}$$

*associated to the inclusion map* $Y_t \hookrightarrow T_0$ (resp. $Y_t \hookrightarrow X_0$) *is an isomorphism for* $j > n-k$ *and surjective for* $j = n-k$. *Moreover the monodromy* $\Phi_j \colon H_c^j(Y_t;\mathbb{C}) \xrightarrow{\sim} H_c^j(Y_t;\mathbb{C})$ *is identity for* $j > n-k$. *In particular, for any* $\lambda \neq 1$ *and* $t \in (\mathbb{C}^*)$ *such that* $0 < |t| \ll 1$ *we have the concentration*

$$H_c^j(Y_t;\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-k).$$

Defining a finite subset $R_f \subset \mathbb{C}$ by using $\partial P, P_\infty \subset P$ as in Sections 4.3 and 4.4, we obtain the following results. In the case where $Y$ is family of subvarieties in $X_0 = \mathbb{C}^n$ we define $\Omega_0 \subset \mathbb{C}^n$, $Y^\circ \subset Y$ and the projection $\pi^\circ : \mathbb{C}^* \times \Omega_0 \to \mathbb{C}^*$ as in Section 4.4.

THEOREM 4.5.3. *Let* $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ *be a family of subvarieties in* $T_0 = ((\mathbb{C}^*))^n$. *Assume that* $Y$ *is schön. Then for any* $\lambda \notin R_f$ *the morphism*

$$\psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_Y) \longrightarrow \psi_{t,\lambda}(j_! \mathrm{R}\pi_* \mathbb{C}_Y)$$

*induced by the one* $\mathrm{R}\pi_!\mathbb{C}_Y \to \mathrm{R}\pi_*\mathbb{C}_Y$ *is an isomorphism.*

COROLLARY 4.5.4. *In the situation of Theorem 4.5.3, for any* $\lambda \notin R_f$ *and* $t \in (\mathbb{C}^*)$ *such that* $0 < |t| \ll 1$ *we have the concentration*

$$H_c^j(Y_t;\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-k)$$

*and the filtration on the only non-trivial cohomology group* $H_c^{n-k}(Y_t;\mathbb{C})_\lambda$ *induced by Deligne's weight filtration on* $H_c^{n-k}(Y_t;\mathbb{C})$ *is concentrated in degree* $n-k$.

THEOREM 4.5.5. *Let* $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ *be a family of subvarieties in* $X_0 = \mathbb{C}^n$. *Assume that* $Y$ *is schön. Then for any* $\lambda \notin R_f$ *the morphism*

$$\psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_{Y^\circ}) \longrightarrow \psi_{t,\lambda}(j_! \mathrm{R}\pi_! \mathbb{C}_Y)$$

*induced by the one* $\mathbb{C}_{Y^\circ} \to \mathbb{C}_Y$ *is an isomorphism. Moreover for such* $\lambda$ *the morphism*

$$\psi_{t,\lambda}(j_! \mathrm{R}(\pi^\circ)_! \mathbb{C}_{Y^\circ}) \longrightarrow \psi_{t,\lambda}(j_! \mathrm{R}(\pi^\circ)_* \mathbb{C}_{Y^\circ})$$

*induced by the one* $\mathrm{R}(\pi^\circ)_!\mathbb{C}_{Y^\circ} \to \mathrm{R}(\pi^\circ)_*\mathbb{C}_{Y^\circ}$ *is an isomorphism.*

We have also a generalization of Remark 4.4.8. By Corollary 4.5.4 and Bernstein-Khovanskii-Kushnirenko's theorem, in the case where $Y$ is family of subvarieties in $T_0 = (\mathbb{C}^*)^n$ we obtain the following formula for the multiplicities of the eigenvalues $\lambda \notin R_f$ in the middle-dimensional monodromy $\Phi_{n-k} \colon H_c^{n-k}(Y_t;\mathbb{C}) \xrightarrow{\sim} H_c^{n-k}(Y_t;\mathbb{C})$ by calculating monodromy zeta functions as in [**23**].

Definition 4.5.6. Let $\Delta_1, \dots, \Delta_n$ be lattice polytopes in $\mathbb{R}^n$. Then we define their normalized ($n$-dimensional) mixed volume $\mathrm{Vol}_{\mathbb{Z}}(\Delta_1, \dots, \Delta_n) \in \mathbb{Z}$ by the formula

$$\mathrm{Vol}_{\mathbb{Z}}(\Delta_1, \dots, \Delta_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{I \subset \{1, \dots, n\} \\ |I|=k}} \mathrm{Vol}_{\mathbb{Z}} \left( \sum_{i \in I} \Delta_i \right) \tag{31}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\ \cdot\ ) = n!\mathrm{Vol}(\ \cdot\ ) \in \mathbb{Z}$ is the normalized ($n$-dimensional) volume with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Let $\Sigma_0$ be the dual fan of $\mathrm{UH}_f$ in $\mathbb{R}^{n+1}$. For a cell $F$ in $\mathcal{S}$ let $\widetilde{F} \prec \mathrm{UH}_f$ be the unique compact face of $\mathrm{UH}_f$ such that $F = p(\widetilde{F})$ and $F^\circ \in \Sigma_0$ the cone which corresponds to it in the dual fan $\Sigma_0$. Then for the supporting faces $\widetilde{F_i} \prec \mathrm{UH}_{f_i}$ of $F^\circ$ in $\mathrm{UH}_{f_i}$ $(1 \le i \le k)$ we have $\widetilde{F_1} + \cdots + \widetilde{F_k} = \widetilde{F}$.

Theorem 4.5.7. *Assume that the family* $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ *of subvarieties in* $T_0 = ((\mathbb{C}^*))^n$ *is schön. Then for* $\lambda \notin R_f$ *the multiplicity of the factor* $t - \lambda$ *in the characteristic polynomial of the middle-dimensional monodromy*

$$\Phi_{n-k} : H_c^{n-k}(Y_t; \mathbb{C}) \xrightarrow{\sim} H_c^{n-k}(Y_t; \mathbb{C}) \qquad (0 < |t| \ll 1)$$

*is equal to that in*

$$\prod_{\mathrm{rel.int} F \subset \mathrm{Int}\, P,\ \dim F = n} (t^{m_F} - 1)^{K_F},$$

*where we set*

$$K_F = \sum_{\substack{m_1, \dots, m_k \ge 1 \\ m_1 + \cdots + m_k = \dim F}} \mathrm{Vol}_{\mathbb{Z}}(\underbrace{\widetilde{F_1}, \dots, \widetilde{F_1}}_{m_1\text{-times}}, \dots, \underbrace{\widetilde{F_k}, \dots, \widetilde{F_k}}_{m_k\text{-times}})$$

*by using the normalized mixed volumes with respect to the lattice* $\mathrm{Aff}(\widetilde{F}) \cap \mathbb{Z}^{n+1} \simeq \mathbb{Z}^{\dim F}$ *in* $\mathrm{Aff}(\widetilde{F}) \simeq \mathbb{R}^{\dim F}$.

For each subset $J \subset \{1, \dots, k\}$, set $\mathbb{R}^J := \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_j = 0\ (j \notin J)\} \simeq \mathbb{R}^{|J|}$ and

$$U_J := \mathrm{Conv}(\bigcup_{j \in J} \{e_j\} \times \mathrm{UH}_{f_j}) \subset \mathbb{R}^J \times \mathbb{R}^{n+1},$$

$$P_J := \mathrm{Conv}(\bigcup_{j \in J} \{e_j\} \times P_j) \subset \mathbb{R}^J \times \mathbb{R}^n,$$

where $e_j = (0, \dots, 0, \overset{j}{\check{1}}, 0, \dots, 0) \in \mathbb{R}^J$ is the standard vector. Obviously, the image of $U_J$ by the projection $p_J \colon \mathbb{R}^J \times \mathbb{R}^n \times \mathbb{R}_s \to \mathbb{R}^J \times \mathbb{R}^n$ is $P_J$. We write $\widetilde{p}$, $\widetilde{U}$ and $\widetilde{P}$ for $p_{\{1,\dots,k\}}$, $U_{\{1,\dots,k\}}$ and $P_{\{1,\dots,k\}}$, respectively. Let $\widetilde{\nu} \colon \widetilde{P} \to \mathbb{R}$ be the function defining the bottom part of the boundary $\partial \widetilde{U}$ of $\widetilde{U}$ and $\widetilde{\mathcal{S}}$ the subdivision of $\widetilde{P}$ by the lattice polytopes $\widetilde{p}(\widetilde{F}) \subset \widetilde{P}$ $(\widetilde{F} \prec \widetilde{U})$. By the assumption that the dimension of $P$ is $n$, we have $\dim \widetilde{U} = n + k$ and $\dim \widetilde{P} = n + k - 1$. We obtain the following generalization of Theorem 4.3.11 to families of complete intersection subvarieties of $T_0 = ((\mathbb{C}^*))^n$.

Recall that for $\lambda \in \mathbb{C}$ we set

$$\varepsilon(\lambda) = \begin{cases} 1 & (\lambda = 1) \\ 0 & (\lambda \neq 1). \end{cases}$$

THEOREM 4.5.8. *Let $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ be a family of subvarieties of $T_0 = ((\mathbb{C}^*))^n$. Assume that $Y$ is schön. Then for any $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ and $\lambda \in \mathbb{C}$ the equivariant refined limit mixed Hodge polynomial $E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w]$ for the eigenvalue $\lambda$ is given by*

$$\begin{aligned}(uvw^2)^k E_\lambda(Y_t; u, v, w) =& \varepsilon(\lambda) \cdot (uvw^2 - 1)^n \\ &+ \sum_{\emptyset \neq J \subset \{1,\dots,k\}} (-1)^{\dim P_J - 1} (uvw^2 - 1)^{n+|J|-1-\dim P_J} \cdot h_\lambda^*(P_J, \widetilde{\nu}|_{P_J}; u, v, w).\end{aligned}$$

PROOF. We prove the assertion only in the case where $\lambda \neq 1$. The proofs for the other cases are similar. We use the Cayley trick of Danilov-Khovanskii [**4**] in its refined form of [**13**]. For sufficiently small $\varepsilon > 0$ we set $B^* = B(0, \varepsilon)^* \subset \mathbb{C}$. Then we have $Y \subset B^* \times T_0$. Moreover we set

$$\Omega := \{(t, (x_1, \dots, x_n), [\alpha_1 : \cdots : \alpha_k]) \in B^* \times T_0 \times \mathbb{P}^{k-1} \mid \sum_{i=1}^{k} \alpha_i f_i(t, x) \neq 0\}.$$

Then there exists a projection $\Omega \to (B^* \times T_0) \setminus Y$ which is a locally trivial fibration with fiber $\mathbb{C}^{k-1}$. Hence it follows from Lemma 4.3.9 that for $t \in B^*$ and any $\lambda \neq 1$ we have

$$\begin{aligned}E_\lambda(\Omega_t; u, v, w) &= (uvw^2)^{k-1} E_\lambda((\{t\} \times T_0) \setminus Y_t; u, v, w) \\ &= (uvw^2)^{k-1}(E_\lambda(\{t\} \times T_0; u, v, w) - E_\lambda(Y_t; u, v, w)) \\ &= -(uvw^2)^{k-1} E_\lambda(Y_t; u, v, w).\end{aligned}$$

For each non-empty subset $J \subset \{1, \dots, k\}$ we define a subset $T_J \simeq ((\mathbb{C}^*))^{|J|-1}$ of $\mathbb{P}^{k-1}$ by

$$T_J := \{[\alpha_1 : \cdots : \alpha_k] \in \mathbb{P}^{k-1} \mid \alpha_j \neq 0 \ (j \in J), \alpha_j = 0 \ (j \notin J)\} \simeq ((\mathbb{C}^*))^{|J|-1}.$$

Moreover we set

$$\begin{aligned}\Omega_J &:= \Omega \cap (B^* \times T_0 \times T_J), \\ Y_J &:= (B^* \times T_0 \times T_J) \setminus \Omega_J \\ &= \{(t, (x_1, \dots, x_n), [\alpha_1 : \cdots : \alpha_k]) \in B^* \times T_0 \times T_J \mid \sum_{j \in J} \alpha_j f_j(t, x) = 0\}.\end{aligned}$$

Then for $t \in B^*$ and $\lambda \neq 1$ we have a decomposition $\Omega_t = \bigsqcup_{J \neq \emptyset} \Omega_{J,t}$ and hence

$$\begin{aligned}E_\lambda(\Omega_t; u, v, w) &= \sum_{J \neq \emptyset} E_\lambda(\Omega_{J,t}; u, v, w) \\ &= \sum_{J \neq \emptyset} (E_\lambda(\{t\} \times T_0 \times T_J; u, v, w) - E_\lambda(Y_{J,t}; u, v, w)) \\ &= -\sum_{J \neq \emptyset} E_\lambda(Y_{J,t}; u, v, w).\end{aligned}$$

We thus obtain the equality

$$(uvw^2)^{k-1}E_\lambda(Y_t;u,v,w)=\sum_{J\neq\emptyset}E_\lambda(Y_{J,t};u,v,w).$$

It is easy to check that $Y_J$ is schön. Note that $U_J$ is $\mathrm{UH}_{\sum_{j\in J}\alpha_j f_j}$. Therefore applying Theorem 4.3.11 to the families $Y_J\subset B^*\times T_0\times T_J$ we obtain

$$(uvw^2)E_\lambda(Y_{J,t};u,v,w)=(-1)^{\mathrm{dim}P_J-1}(uvw^2-1)^{n+|J|-1-\mathrm{dim}P_J}\cdot h^*_\lambda(P_J,\widetilde{\nu}|_{P_J};u,v,w).$$

Now the assertion follows immediately. □

Corollary 4.5.9. *In the situation of Theorem 4.5.8, assume also that* $\mathrm{dim}P_i=n$ *for any* $1\leq i\leq k$. *Then for any* $t\in(\mathbb{C}^*)$ *such that* $0<|t|\ll 1$ *and* $\lambda\in\mathbb{C}$ *we have*

$$(uvw^2)^kE_\lambda(Y_t;u,v,w)=\varepsilon(\lambda)\cdot(uvw^2-1)^n+\sum_{\emptyset\neq J\subset\{1,\dots,k\}}(-1)^{n+|J|}h^*_\lambda(P_J,\widetilde{\nu}|_{P_J};u,v,w).$$

From now on, we consider only families $Y=f_1^{-1}(0)\cap\cdots\cap f_k^{-1}(0)$ of subvarieties of $X_0=\mathbb{C}^n$. The corresponding results for families of subvarieties of $T_0=(\mathbb{C}^*)^n$ can be obtained similarly. For a cell $F\in\widetilde{\mathcal{S}}$ let $\widetilde{\nu_F}\colon \mathrm{Aff}(F)\simeq\mathbb{R}^{\mathrm{dim}F}\longrightarrow\mathbb{R}$ be the (affine) linear extension of $\widetilde{\nu}|_F\colon F\longrightarrow\mathbb{R}$ and define a positive integer $\widetilde{m_F}$ to be the minimal one $m$ for which $m\cdot\widetilde{\nu_F}$ takes only integer values on $\mathrm{Aff}(F)\cap\mathbb{Z}^{n+k}$. Let $\Delta$ be the convex hull of the points $e_1,\dots,e_k$ in $\mathbb{R}^k$. Then $\Delta$ is a $(k-1)$-dimensional lattice simplex and we have $\widetilde{P}\subset\Delta\times\mathbb{R}^n_+$. Now let us set

$$\widetilde{P}_\infty:=\overline{\{\mathrm{Int}(\Delta)\times\mathrm{Int}(\mathbb{R}^n_+)\}\cap\partial\widetilde{P}}\subset\partial\widetilde{P}.$$

Then we define a finite subset $\widetilde{R_f}\subset\mathbb{C}$ by

$$\widetilde{R_f}=\bigcup_{F\subset\widetilde{P}_\infty}\{\lambda\in\mathbb{C}\mid\lambda^{\widetilde{m_F}}=1\}\subset\mathbb{C}.$$

Lemma 4.5.10. *We have* $R_f=\widetilde{R_f}$.

Proof. We shall say that a face $Q\prec\widetilde{P}$ is a side face of $\widetilde{P}$ if its image by the projection $r:\Delta\times\mathbb{R}^n_+\to\Delta$ is equal to $\Delta$. Note also that the inverse image of the barycener of $\Delta$ by the map $r|_{\widetilde{P}}:\widetilde{P}\to\Delta$ is similar to the Minkowski sum $P=P_1+\cdots+P_k$. Hence there exists a natural bijection between the set of the side faces of $\widetilde{P}$ and that of the faces of $P$. Moreover for any cell $F\in\widetilde{\mathcal{S}}$ in $\widetilde{P}_\infty$ there exist another cell $F'\in\widetilde{\mathcal{S}}$ in $\widetilde{P}_\infty$ and a side face $Q\prec\widetilde{P}$ of $\widetilde{P}$ such that $F\prec F'$ and $\mathrm{rel.int}F'\subset\mathrm{rel.int}Q$. Then we have $\widetilde{m_F}|\widetilde{m_{F'}}$. This implies that for the definition of $\widetilde{R_f}$ it suffices to consider only cells $F\in\widetilde{\mathcal{S}}$ in $\widetilde{P}_\infty$ whose relative interiors are contained in those of side faces of $\widetilde{P}$. In fact, there exists also a natural bijection between the set of such cells $F\in\widetilde{\mathcal{S}}$ and that of the cells $G\in\mathcal{S}$ in $P_\infty$. For the cell $F\in\widetilde{\mathcal{S}}$ in $\widetilde{P}_\infty$ let $F_{\mathrm{red}}\in\mathcal{S}$ be the corresponding cell in $P_\infty$. Then it is easy to show that $\widetilde{m_F}=m_{F_{\mathrm{red}}}$. We thus obtain the equality $R_f=\widetilde{R_f}$. □

Now we have the following generalization of Theorem 4.4.5 to families of complete intersection subvarieties of $\mathbb{C}^n$.

THEOREM 4.5.11. *Let $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ be a family of subvarieties of $X_0 = \mathbb{C}^n$. Assume that $Y$ is schön. Then for any $\lambda \notin R_f = \widetilde{R_f}$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ the equivariant refined limit mixed Hodge polynomial $E_\lambda(Y_t; u, v, w) \in \mathbb{Z}[u, v, w]$ for the eigenvalue $\lambda$ is concentrated in degree $n-k$ in the variable $w$ and given by*

$$\begin{aligned} E_\lambda(Y_t; u, v, w) &= (-1)^{n-k} w^{n-k} \sum_{p,q} h^{p,q}(H_c^{n-k}(Y_t; \mathbb{C})_\lambda) u^p v^q \\ &= (-1)^{n-k} \frac{w^{n-k}}{u^k v^k} l_\lambda^*(\widetilde{P}, \widetilde{\nu}; u, v) \\ &= (-1)^{n-k} \frac{w^{n-k}}{u^k v^k} \sum_{F \in \widetilde{\mathcal{S}}} v^{\dim F + 1} l_\lambda^*(F, \widetilde{\nu}|_F; uv^{-1}) \cdot l_{\widetilde{P}}(\widetilde{\mathcal{S}}, F; uv). \end{aligned}$$

*In particular, by setting $u = v = s$ and $w = 1$ we have*

$$\begin{aligned} E_\lambda(Y_t; s, s) &= (-1)^{n-k} \sum_{m \geq 0} \Big( \sum_{p+q=m} h^{p,q}(H_c^{n-k}(Y_t; \mathbb{C})_\lambda) \Big) s^m \\ &= (-1)^{n-k} \frac{1}{s^{2k}} \sum_{F \in \widetilde{\mathcal{S}}} s^{\dim F + 1} l_\lambda^*(F, \widetilde{\nu}|_F; 1) \cdot l_{\widetilde{P}}(\widetilde{\mathcal{S}}, F; s^2). \end{aligned}$$

PROOF. For a possibly empty subset $I \subset \{1, \ldots, n\}$, we define a subset $T^I$ of $X_0 = \mathbb{C}^n$ by

$$T^I := \{(x_1, \ldots, x_n) \in X_0 \mid x_i = 0 \ (i \notin I), x_i \neq 0 \ (i \in I)\} \simeq ((\mathbb{C}^*))^{|I|}.$$

Then we have a decomposition $X_0 = \mathbb{C}^n = \bigsqcup_{I \subset \{1,\ldots,n\}} T^I$ of $X_0 = \mathbb{C}^n$. We also define polynomials $f_{j,I} \in \mathbb{K}[(x_i)_{i \in I}]$ by substituting 0 into the variable $x_i$ $(i \notin I)$ of $f_j$, a family of subvarieties $Y^I$ of $T^I$ by $Y^I := f_{1,I}^{-1}(0) \cap \cdots \cap f_{k,I}^{-1}(0) \subset B^* \times T^I$ and polytopes $P_j^I$ in $\mathbb{R}^I$ by $P_j^I := P_j \cap \mathbb{R}^I$. We set $P^I = P_1^I + \cdots + P_k^I = P \cap \mathbb{R}^I$. Then by Lemma 4.3.9 we have

$$E_\lambda(Y_t; u, v, w) = \sum_{I \subset \{1,\ldots,n\}} E_\lambda(Y_t^I; u, v, w)$$

for $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$. For each non-empty subset $J \subset \{1, \ldots, k\}$ we define a polytope $P_J^I$ in $\mathbb{R}^J \times \mathbb{R}^I$ by $P_J^I := \mathrm{Conv}(\bigcup_{j \in J} \{e_j\} \times P_j^I)$. We shall say that a face $Q \prec \widetilde{P}$ of $\widetilde{P}$ is relevant if $Q \not\subset \widetilde{P}_\infty$. If $Q \prec \widetilde{P}$ is relevant, then for any face $\sigma$ of the polyhedron $\Delta \times \mathbb{R}_+^n$ containing $Q$ the face $\sigma \cap \widetilde{P} \prec \widetilde{P}$ of $\widetilde{P}$ is also relevant. Moreover there exist a possibly empty subset $I \subset \{1, \ldots, n\}$ and a non-empty one $J \subset \{1, \ldots, k\}$ such that $Q = P_J^I$ and $\dim P_J^I = |I| + |J| - 1$. For each $I \subset \{1, \ldots, n\}$ denote by $S^I$ the set consisting of non-empty subsets $J \subset \{1, \ldots, k\}$ such that $P_J^I$ is relevant. Then by Theorem 4.5.8 for $\lambda \notin R_f = \widetilde{R_f}$ we have

$$(uvw^2)^k E_\lambda(Y_t^I; u, v, w) = \sum_{J \in S^I} (-1)^{|I|+|J|} h_\lambda^*(P_J^I, \widetilde{\nu}|_{P_J^I}; u, v, w).$$

Moreover for each relevant face $P_J^I \prec \widetilde{P}$ of $\widetilde{P}$ by Definition 4.2.7 we have

$$h_\lambda^*(P_J^I, \widetilde{\nu}|_{P_J^I}; u, v, w) = \sum_{Q \prec P_J^I} w^{\dim Q + 1} l_\lambda^*(Q, \widetilde{\nu}|_Q; u, v) \cdot g([Q, P_J^I]; uvw^2). \quad (32)$$

If $Q \prec P^I_J$ is not a relevant face of $\widetilde{P}$, then $Q \subset \widetilde{P}_\infty$ for $\lambda \notin R_f = \widetilde{R_f}$ we have $l^*_\lambda(Q, \widetilde{\nu}|_Q; u, v) = 0$. Note also that $l^*_\lambda(\emptyset, \widetilde{\nu}|_\emptyset; u, v) = 0$ for any $\lambda \neq 1$. Thus we may assume $Q$ is not empty set in the right hand side of the equation (32). Hence as in the proof of Theorem 4.4.5, most of terms in the calculation of $(uvw^2)^k E_\lambda(Y_t; u, v, w)$ cancel each other. Eventually, we obtain the desired formula

$$(uvw^2)^k E_\lambda(Y_t; u, v, w) = (-1)^{n+k} w^{n+k} l^*_\lambda(\widetilde{P}; \widetilde{\nu}, u, v).$$

□

As in Corollary 4.4.7, by Theorems 4.5.5 and 4.5.11 we obtain the following result.

Corollary 4.5.12. *In the situation of Theorem 4.5.11, for any $\lambda \notin R_f$ and $t \in (\mathbb{C}^*)$ such that $0 < |t| \ll 1$ we have the concentration*

$$H^j_c(Y_t; \mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-k)$$

*and the filtration on the only non-trivial cohomology group $H^{n-k}_c(Y_t; \mathbb{C})_\lambda$ induced by Deligne's weight filtration on $H^{n-k}_c(Y_t; \mathbb{C})$ is concentrated in degree $n-k$.*

By Theorem 4.5.11 and Corollary 4.5.12, for any $\lambda \notin R_f$ we can describe the Jordan normal form of the middle-dimensional monodromy

$$\Phi_{n-k} : H^{n-k}_c(Y_t; \mathbb{C})_\lambda \xrightarrow{\sim} H^{n-k}_c(Y_t; \mathbb{C})_\lambda$$

as in Theorem 4.3.15. Recall that the dimension of $\widetilde{P}$ is $n+k-1$, and hence for a cell $F \in \widetilde{\mathcal{S}}$ the local $h$-polynomial $l_{\widetilde{P}}(\widetilde{\mathcal{S}}, F; t) \in \mathbb{Z}[t]$ has non-negative coefficients and the symmetry

$$l_{\widetilde{P}}(\widetilde{\mathcal{S}}, F; t) = t^{n+k-1-\dim F} l_{\widetilde{P}}(\widetilde{\mathcal{S}}, F; t^{-1}).$$

Moreover it is unimodal. Hence there exist non-negative integers $l_{F,i}$ $(0 \leq i \leq \lfloor \frac{n+k-1-\dim F}{2} \rfloor)$ such that

$$\begin{aligned} l_{\widetilde{P}}(\widetilde{\mathcal{S}}, F; t) = {} & l_{F,0}(1 + t + t^2 + \cdots + t^{n+k-1-\dim F}) \\ & + l_{F,1}(t + t^2 + \cdots + t^{n+k-1-\dim F-1}) \\ & + l_{F,2}(t^2 + \cdots + t^{n+k-1-\dim F-2}) \\ & + \cdots\cdots . \end{aligned}$$

We set

$$\widetilde{l}_{\widetilde{P}}(\widetilde{\mathcal{S}}, F; t) = \sum_{i=0}^{\lfloor \frac{n+k-1-\dim F}{2} \rfloor} l_{F,i} t^i.$$

Theorem 4.5.13. *Let $Y = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ be a family of subvarieties of $X_0 = \mathbb{C}^n$. Assume that the family $Y$ is schön. For $\lambda \in \mathbb{C}$ and $m \geq 1$ denote by $J_{\lambda,m}$ the number of the Jordan blocks in the monodromy automorphism*

$$\Phi_{n-k} : H^{n-k}_c(Y_t; \mathbb{C}) \xrightarrow{\sim} H^{n-k}_c(Y_t; \mathbb{C}) \qquad (0 < |t| \ll 1)$$

*for the eigenvalue $\lambda$ with size $m$. Then for $\lambda \notin R_f$ we have*

$$\sum_{m=0}^{n-k} J_{\lambda, n-k+1-m} s^{m+2k} = \sum_{F \in \widetilde{\mathcal{S}}} s^{\dim F + 1} l^*_\lambda(F, \widetilde{\nu}|_F; 1) \cdot \widetilde{l}_{\widetilde{P}}(\widetilde{\mathcal{S}}, F; s^2).$$

The multiplicities of the eigenvalues $\lambda \notin R_f$ in the monodromy $\Phi_{n-k}$ are described more simply as follows. For a cell $F \in \mathcal{S}$ let $Q_F \prec P$ be the unique face of $P$ such that $\mathrm{rel.int} F \subset \mathrm{rel.int} Q_F$.

THEOREM 4.5.14. *In the situation of Theorem 4.5.13, for $\lambda \notin R_f$ the multiplicity of the factor $t-\lambda$ in the characteristic polynomial of the middle-dimensional monodromy*

$$\Phi_{n-k} : H_c^{n-k}(Y_t;\mathbb{C}) \xrightarrow{\sim} H_c^{n-k}(Y_t;\mathbb{C}) \qquad (0<|t|\ll 1)$$

*is equal to that in*

$$\prod_{F \not\subset P_\infty,\ \dim F = \dim Q_F} (t^{m_F}-1)^{(-1)^{n-\dim F} K_F},$$

*where we define the integers $K_F$ as in Theorem 4.5.7.*

CHAPTER 5

# Milnor monodromies and mixed Hodge structures for non-isolated hypersurface singularities

## 5.1. Introduction for Chapter 5

The Milnor monodromies of complex hypersurface singularities are important subjects in singularity theory. For isolated hypersurface singular points, we have some algorithms or formulas to compute them. However, for non-isolated singular points, there still remain some difficulties in computing them explicitly. In this chapter, we will show that even for non-isolated singular points, the generalized eigenspaces of the Milnor monodromies for "good" eigenvalues have some nice properties similar to those for isolated singular points. By this result, we give an explicit formula for some parts of the Jordan normal forms of the Milnor monodromies.

Let $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial of $n(\geq 2)$ variables with coefficients in $\mathbb{C}$ such that $f(0) = 0$ and $V := f^{-1}(0) \subset \mathbb{C}^n$ the hypersurface defined by it. We denote by $F_{f,0}$ the Milnor fiber of $f$ at 0 and by

$$\Phi_{j,0} \colon H^j(F_{f,0}; \mathbb{C}) \xrightarrow{\sim} H^j(F_{f,0}; \mathbb{C})$$

the $j$-th Milnor monodromy for $j \in \mathbb{Z}$. If the origin $0 \in V$ is an isolated singular point of $V$, the Milnor fiber $F_{f,0}$ is homotopic to a bouquet of some $(n-1)$-spheres $S^{n-1}$ by a celebrated theorem of Milnor [**27**]. This implies that the reduced cohomology groups $\widetilde{H}^j(F_{f,0}; \mathbb{C})$ vanish except for $j = n-1$. However, if $0 \in V$ is a non-isolated singular point, we can not expect to have such a concentration in general. On the other hand, for a polynomial $f$ non-degenerate at 0, Varchenko [**51**] described explicitly the monodromy zeta function

$$\zeta_{f,0}(t) := \prod_{j \in \mathbb{Z}} \det(\mathrm{Id} - t\Phi_{j,0})^{(-1)^j} \in \mathbb{C}(t)$$

in terms of the Newton polyhedron $\Gamma_+(f)$. If $0 \in V$ is an isolated singular point of $V$, the $(n-1)$-th Milnor monodromy $\Phi_{n-1,0}$ is the only non-trivial one. Therefore, in this case, we obtain a formula for the characteristic polynomial of $\Phi_{n-1,0}$. However, if $0 \in V$ is a non-isolated singular point, Varchenko's formula does not tell us any explicit information about the characteristic polynomial of each Milnor monodromy $\Phi_{j,0}$.

For non-isolated singular points, there is a similar difficulty also for the mixed Hodge structures of the cohomologies of the Milnor fibers. Recall that each cohomology group $H^j(F_{f,0}; \mathbb{Q})$ of $F_{f,0}$ is endowed with a mixed Hodge structure $(H^j(F_{f,0}; \mathbb{Q}), F^\bullet, W_\bullet)$ defined by Steenbrink [**45**] in the case where $0 \in V$ is an isolated singular point and by Navarro [**29**] and M. Saito [**37**] in the case where $0 \in V$ is a non-isolated singular point. For $j \in \mathbb{Z}$ and an eigenvalue $\lambda \in \mathbb{C}$ of the Milnor monodromy $\Phi_{j,0}$, we denote by

$$H^j(F_{f,0}; \mathbb{C})_\lambda \subset H^j(F_{f,0}; \mathbb{C})$$

the generalized eigenspace of $\Phi_{j,0}$ for $\lambda$. For $p, q \in \mathbb{Z}$, we denote by $h_\lambda^{p,q}(H^j(F_{f,0};\mathbb{C}))$ the $(p,q)$-mixed Hodge number for the eigenvalue $\lambda$ of the $j$-th cohomology group of $F_{f,0}$ i.e. the dimension of $\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^j(F_{f,0};\mathbb{C})_\lambda$. For $\lambda \in \mathbb{C}$, we define a polynomial $E_\lambda(F_{f,0};u,v) \in \mathbb{Z}[u,v]$ with coefficients in $\mathbb{Z}$ by

$$E_\lambda(F_{f,0};u,v) := \sum_{p,q\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} (-1)^j h_\lambda^{p,q}(H^j(F_{f,0};\mathbb{C}))u^p v^q.$$

By using a description of the Motivic Milnor fiber of $f$ at 0 (see Theorem 5.3.3), we can describe $E_\lambda(F_{f,0};u,v)$ explicitly in terms of certain polynomials defined by the Newton polyhedron $\Gamma_+(f)$ (see Corollary 5.3.5). Moreover, if $0 \in V$ is an isolated singular point of $V$, as in the previous discussion about Varchenko's formula, we can describe $h_\lambda^{p,q}(H^{n-1}(F_{f,0};\mathbb{C}))$ by our formula. In particular, we obtain an explicit formula for the Jordan normal form of $\Phi_{n-1,0}$ (see Matsui-Takeuchi [**25**]). On the other hand, if $0 \in V$ is a non-isolated singular point, the formula for $E_\lambda(F_{f,0};u,v)$ does not tell us any explicit information about each mixed Hodge number $h_\lambda^{p,q}(H^j(F_{f,0};\mathbb{C}))$.

In this chapter, we follow an idea of Takeuchi-Tibar [**48**] for monodromies at infinity and overcome the above-mentioned difficulties by introducing a finite subset $R_f \subset \mathbb{C}$ of "bad" eigenvalues of the Milnor monodromies (see Definition 5.3.7) as follows.

THEOREM 5.1.1 (see Theorem 5.4.1). *Assume that $f$ is non-degenerate at 0. Then, for any $\lambda \notin R_f$ we have a concentration:*

$$\widetilde{H}^j(F_{f,0};\mathbb{C})_\lambda \simeq 0 \quad (j \neq n-1).$$

Note that a more general but less explicit concentration result was given in [**9**, Corollary 6.1.7]. By this theorem and Varchenko's formula, we can compute the multiplicities of eigenvalues $\lambda \notin R_f$ in $\Phi_{n-1,0}$ as follows.

COROLLARY 5.1.2 (see Corollary 5.4.2). *In the situation of Theorem 5.1.1, for any $\lambda \notin R_f$ the multiplicity of the eigenvalue $\lambda$ in the Milnor monodromy $\Phi_{n-1,0}$ is equal to that of the factor $(1-\lambda t)$ in a rational function*

$$\prod_{\emptyset\neq I\subset\{1,\ldots,n\}} \prod_{i=1}^{k_I} (1-t^{d_{I,i}})^{(-1)^{n-|I|}\mathrm{Vol}_{\mathbb{Z}}(\Gamma_{I,i})}.$$

*For the definitions of $k_I$, $\Gamma_{I,i}$, $d_{I,i}$ and $\mathrm{Vol}_{\mathbb{Z}}(\Gamma_{I,i})$, see Section 5.2.*

Moreover, for such $\lambda$ we obtain the mixed Hodge numbers $h_\lambda^{p,q}(H^{n-1}(F_{f,0};\mathbb{C}))$ by our formula for $E_\lambda(F_{f,0};u,v)$.

If $0 \in V$ is an isolated singular point, the filtration on $H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ induced by the weight filtration coincides with the monodromy filtration of $\Phi_{n-1,0}$. This implies that the Jordan normal form of $\Phi_{n-1,0}$ for an eigenvalue $\lambda$ can be described by the mixed Hodge numbers $h_\lambda^{p,q}(H^{n-1}(F_{f,0};\mathbb{C}))$. On the other hand, to the best of our knowledge, if $0 \in V$ is a non-isolated singular point of $V$, the geometric meaning of the weight filtrations on $H^j(F_{f,0};\mathbb{C})_\lambda$ is not fully understood yet. For this problem, we obtain the following.

THEOREM 5.1.3 (see Theorem 5.4.6). *In the situation of Theorem 5.1.1, for $\lambda \notin R_f$ the filtration on $H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ induced by the weight filtration on $H^{n-1}(F_{f,0};\mathbb{Q})$ coincides with the monodromy filtration of $\Phi_{n-1,0}$ centered at $n-1$.*

We denote by $J_{m,\lambda}$ the number of the Jordan blocks with size $m$ for an eigenvalue $\lambda$ in the Jordan normal form of the Milnor monodromy $\Phi_{n-1,0}$. Combining the above theorem with our description of $h_\lambda^{p,q}(H^{n-1}(F_{f,0};\mathbb{C}))$, for any eigenvalue $\lambda \notin R_f$ we can describe it by using the Newton polyhedron $\Gamma_+(f)$ as follows.

COROLLARY 5.1.4 (see Corollary 5.5.1). *In the situation of Theorems 5.1.1 and 5.1.3, for $\lambda \notin R_f$ we have*

$$\sum_{0\leq k\leq n-1} J_{n-k,\lambda}u^{k+2} = \sum_{F\prec\Gamma_+(f)\colon \text{admissible}} u^{\dim F+2} l_\lambda^*(\Delta_F,\nu;1)\cdot \widetilde{l}_P(\mathcal{S}_\nu,\Delta_F;u^2),$$

*where in the sum $\Sigma$ of the right hand side the face $F$ ranges through the admissible compact ones of $\Gamma_+(f)$ (see Definition 5.3.6). For the definition of the polynomials $l_\lambda^*(\Delta_F,\nu;u)$ and $\widetilde{l}_P(\mathcal{S}_\nu,\Delta_F;u)$, see Definition 4.2.7 and Section 5.5.*

We also apply our results to obtain a formula for the Hodge spectrum of the Milnor fiber $F_{f,0}$ (see Corollary 5.5.3).

## 5.2. Milnor fibration

Let $f \in \mathbb{C}[x_1,\dots,x_n]$ be a polynomial with $f(0)=0$. We consider the Milnor fibration of $f$ at 0 and use the notation in Section 3.4. Recall that we denote by $F_{f,0}$ the Milnor fiber of $f$ at 0 and $\Phi_{j,0}$ the $j$-th Milnor monodromy $\Phi_{j,0}\colon H^j(F_{f,0};\mathbb{C}) \xrightarrow{\sim} H^j(F_{f,0};\mathbb{C})$.

DEFINITION 5.2.1. We define the *monodromy zeta function* $\zeta_{f,0}(t) \in \mathbb{C}(t)$ *of $f$ at* 0 by

$$\zeta_{f,0}(t) := \prod_{j\in\mathbb{Z}} \det(\mathrm{Id} - t\Phi_{j,0})^{(-1)^j} \in \mathbb{C}(t),$$

where Id is the identity map of $H^j(F_{f,0};\mathbb{C})$ to itself.

Since $\Phi_j$ are automorphisms, the polynomials $\det(\mathrm{Id}-t\Phi_{j,0})$ determine the characteristic polynomials of $\Phi_{j,0}$.

Recall we denote by $\Gamma_+(f)$ the Newton polyhedron of $f$. For a subset $I \subset \{1,\dots,n\}$, set

$$\mathbb{R}^I := \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_i = 0\ (i\notin I)\} \simeq \mathbb{R}^{|I|},$$

and let $\Gamma_{I,1},\dots,\Gamma_{I,k_I}$ be the $(|I|-1)$-dimensional compact faces of $\mathbb{R}^I \cap \Gamma_+(f)$. For $1\leq i\leq k_I$, we define an integer $d_{I,i}\in\mathbb{Z}_{>0}$ to be the lattice distance of $\Gamma_{I,i}$ from the origin $0\in\mathbb{R}^I$. Let $\mathrm{Vol}_{\mathbb{Z}}(\Gamma_{I,i})\in\mathbb{Z}_{>0}$ be the $(|I|-1)$-dimensional normalized volume of $\Gamma_{I,i}$. Then we have the following.

THEOREM 5.2.2 (Varchenko [**51**]). *Assume that $f$ is non-degenerate at* 0 *(see Definition 3.4.8). Then we have*

$$\zeta_{f,0}(t) = \prod_{\emptyset\neq I\subset\{1,\dots,n\}} \prod_{i=1}^{k_I} (1-t^{d_{I,i}})^{(-1)^{|I|-1}\mathrm{Vol}_{\mathbb{Z}}(\Gamma_{I,i})}. \tag{33}$$

The monodromy zeta function $\zeta_{f,0}(t)$ being an alternating product of the polynomials $\det(\mathrm{Id}-t\Phi_{j,0})$, we can not compute the eigenvalues of each Milnor monodromy $\Phi_{j,0}$ and their multiplicities by Varchenko's formula in general. Recall that if $0\in V$

is an isolated singular point of $V$ we have $H^j(F_{f,0};\mathbb{C}) = 0$ for $j \neq 0, n-1$ and $\det(\mathrm{Id} - t\Phi_{0,0}) = 1 - t$ (here we assumed $n \geq 2$). In this case we thus obtain

$$\zeta_{f,0}(t) = (1-t) \cdot \det(\mathrm{Id} - t\Phi_{n-1,0})^{(-1)^{n-1}}$$

and can compute the eigenvalues of $\Phi_{n-1,0}$ and their multiplicities by Varchenko's formula. By Proposition 3.4.10, if $f$ is convenient and non-degenerate ate 0, the point $0 \in V$ is a smooth or an isolated singular point. Therefore, in this case, we can describe the eigenvalues of the Milnor monodromy $\Phi_{n-1,0}$ and their multiplicities by the Newton polyhedron $\Gamma_+(f)$. In Section 5.4, we will show that even if $f$ is not convenient, we can compute the multiplicities of some "good" eigenvalues of $\Phi_{n-1,0}$ by Varchenko's formula (see Corollary 5.4.2).

Recall that each cohomology group $H^j(F_{f,0};\mathbb{Q})(\simeq H^j(i_0^*\psi_f(\mathbb{Q}^H_{\mathbb{C}^n})))$ is equipped with a natural mixed Hodge structure (see Section 3.4). If $0 \in V$ is an isolated singular point, the filtration on $H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ induced by the weight filtration is the monodromy filtration of the logarithm $\log \Phi^u_{n-1}$ of $\Phi^u_{n-1}$ centered at $n-1$ (resp. $n$) for $\lambda \neq 1$ (resp. $\lambda = 1$) (see [**45**]). Therefore, we can recover the Jordan normal form of $\Phi_{n-1}$ for the eigenvalue $\lambda$ from the dimensions of the graded pieces $\mathrm{Gr}^W_k H^{n-1}(F_{f,0};\mathbb{C})_\lambda$. On the other hand, if $0 \in V$ is a non-isolated singular point, we can not expect such a relationship between the weight filtration $W_\bullet$ and the Milnor monodromy. In Section 5.4, we will show that even if $f$ is not convenient (so $0 \in V$ may be a non-isolated singular point) for a "good" eigenvalue $\lambda$ the filtration on $H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ induced by $W_\bullet$ coincides with the monodromy filtration (see Theorem 5.4.6).

## 5.3. Motivic Milnor fibers

We use the notations in Section 4.2.1: $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$, $\mathbb{L} := [\mathbb{C} \circlearrowleft \hat{\mu}] \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ and $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$. For a non-constant polynomial $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f(0) = 0$, Denef-Loeser [**8**] defined the *motivic Milnor fiber* $\mathcal{S}_{f,0}$ of $f$ as an object in $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ by using the theory of arc spaces. It is an "incarnation of the Milnor fiber of $f$ at 0" in the ring $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ as we shall see in Theorem 5.3.1 below. Let $X$ be an algebraic variety with a good $\mu_m$-action for some $m \in \mathbb{Z}_{\geq 1}$. The generator of $\mu_m$ defines an automorphism $l \colon X \xrightarrow{\sim} X$ of $X$ such that $l^m$ is the identity map. Each cohomology group $H^j_c(X;\mathbb{Q})$ with compact support is endowed with Deligne's mixed Hodge structure $(H^j_c(X;\mathbb{Q}), F^\bullet, W_\bullet)$ with an automorphism

$$(l^*)^{-1} \colon H^j_c(X;\mathbb{Q}) \xrightarrow{\sim} H^j_c(X;\mathbb{Q}). \tag{34}$$

For $\lambda \in \mathbb{C}$, we denote by

$$H^j_c(X;\mathbb{C})_\lambda \subset H^j_c(X;\mathbb{C})$$

the generalized eigenspace of the automorphism for the eigenvalue $\lambda$. Moreover, for $p, q \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ we define $h^{p,q}_\lambda(H^j_c(X;\mathbb{C})) \in \mathbb{Z}_{\geq 0}$ to be the dimension of $\mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H^j_c(X;\mathbb{C})_\lambda$. Then for $\lambda \in \mathbb{C}$ we can define a ring homomorphism $E_\lambda(\ \cdot\ ; u, v)$ of $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ to the polynomial ring $\mathbb{Z}[u,v]$ which sends $[X \circlearrowleft \hat{\mu}] \in \mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ to the polynomial

$$E_\lambda([X \circlearrowleft \hat{\mu}]; u, v) := \sum_{p,q \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}_\lambda(H^j_c(X;\mathbb{C})) u^p v^q \in \mathbb{Z}[u,v].$$

For an element $\sum_i [X_i \circlearrowleft \hat{\mu}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ and $\lambda \in \mathbb{C}$ we call the polynomial $\sum_i E_\lambda([X_i \circlearrowleft \hat{\mu}]; u, v)$ the *Hodge realization for* $\lambda$ of $\sum_i [X_i \circlearrowleft \hat{\mu}]$. For $\lambda \in \mathbb{C}$ we denote by $E_\lambda(F_{f,0}; u, v)$ the *equivariant Hodge-Deligne polynomial for the eigenvalue* $\lambda$ of the mixed Hodge structures of the cohomology groups of the Milnor fiber $F_{f,0}$ with the automorphisms $\Phi_j$, i.e.

$$E_\lambda(F_{f,0}; u, v) := \sum_{p,q \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} (-1)^j h_\lambda^{p,q}(H^j(F_{f,0}; \mathbb{C})) u^p v^q \in \mathbb{Z}[u, v],$$

where $h_\lambda^{p,q}(H^j(F_{f,0}; \mathbb{C}))$ is the dimension of $\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^j(F_{f,0}; \mathbb{C})_\lambda$. Then we have the following theorem of Denef-Loeser [**8**].

THEOREM 5.3.1 (Denef-Loeser [**8**]). *For any* $\lambda \in \mathbb{C}$ *we have*

$$E_\lambda(\mathcal{S}_{f,0}; u, v) = E_\lambda(F_{f,0}; u, v).$$

Originally, $\mathcal{S}_{f,0}$ is defined abstractly in [**8**] by using the theory of arc spaces. Similarly to section 4.2.1, by using a log resolution of the pair $(\mathbb{C}^n, f^{-1}(0))$, we can describe $\mathcal{S}_{f,0}$ explicitly as follows. Let $Y$ be a smooth algebraic variety over $\mathbb{C}$ and $\pi \colon Y \longrightarrow \mathbb{C}^n$ be a proper morphism such that $\pi^{-1}(V)$ is a normal crossing divisor of $Y$ and $\pi$ induces an isomorphism $Y \setminus \pi^{-1}(V) \xrightarrow{\sim} \mathbb{C}^n \setminus V$. Let

$$\pi^{-1}(V) = E_1 \cup \cdots \cup E_m$$

be the irreducible decomposition of $\pi^{-1}(V)$. We denote by $m_i$ the order of zeros along $E_i$ of $f \circ \pi$. For a subset $I \subset \{1, \ldots, m\}$, we define

$$E_I := \cap_{i \in I} E_i, \quad E_I^\circ := E_I \setminus \cup_{i \notin I} E_i$$

and $m_I := \gcd_{i \in I}(m_i)$. Moreover, we define a covering $\widetilde{E_I^\circ}$ of $E_I^\circ$ in the following way. For a point in $E_I^\circ$, we take a Zariski open neighborhood $U$ of it in $Y$ on which for any $i \in I$ there exists a regular function $h_i$ such that $E_i \cap U = \{h_i = 0\}$. We have $f \circ \pi = f_1 f_2^{m_I}$ on $U$, where we set $f_1 = (f \circ \pi) \prod_{i \in I} h_i^{-m_i}$ and $f_2 = \prod_{i \in I} h_i^{m_i/m_I}$. Note that $f_1$ is a unit on $U$. Then, we have a covering of $E_I^\circ \cap U$ defined by

$$\{(z, y) \in \mathbb{C} \times (E_I^\circ \cap U) \mid z^{m_I} = f_1^{-1}(y)\}. \tag{35}$$

Consider an open covering of $E_I^\circ$ by such affine open sets $E_I^\circ \cap U$. Then by gluing together the varieties (35) in an obvious way, we obtain an $m_I$-fold covering $\widetilde{E_I^\circ}$ of $E_I^\circ$. Moreover by the multiplication of $\exp(2\pi\sqrt{-1}/m_I) \in \mathbb{C}$ to the $z$-coordinate of (35), we can endow $\widetilde{E_I^\circ}$ with a $\mu_{m_I}$-action (and also a $\hat{\mu}$-action induced by it). We denote by $\widetilde{E_{I,0}^\circ}$ the base change of $\widetilde{E_I^\circ} \longrightarrow E_I^\circ$ by $\pi^{-1}(0) \cap E_I^\circ \hookrightarrow E_I^\circ$. Then we obtain the following expression of $\mathcal{S}_{f,0}$.

PROPOSITION 5.3.2 (Denef-Loeser [**8**]). *In the situation as above, we have*

$$\mathcal{S}_{f,0} = \sum_{\emptyset \neq I \subset \{1, \ldots, m\}} (1 - \mathbb{L})^{|I|-1} [\widetilde{E_{I,0}^\circ}] \tag{36}$$

*in* $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

We denote by $\Sigma_0$ the dual fan of $\Gamma_+(f)$ in $\mathbb{R}^n$. Let $\Sigma$ be a smooth subdivision of $\Sigma_0$, and denote by $X_\Sigma$ the toric variety associated with it. We denote by $\Sigma_1$ the fan which consists of all the faces of $\mathbb{R}_{\geq 0}^n$. Note that the toric variety associated with it is $\mathbb{C}^n$. Then, the morphism of fans $\Sigma \longrightarrow \Sigma_1$ induces a morphism of toric varieties

$$\pi \colon X_\Sigma \longrightarrow \mathbb{C}^n.$$

If $f$ is non-degenerate at 0, we can take $\pi\colon X_\Sigma \longrightarrow \mathbb{C}^n$ as a log resolution of $(\mathbb{C}^n, V)$ (In fact, $\pi^{-1}(V)$ is normal crossing only in a neighborhood of $\pi^{-1}(0)$ in $X_\Sigma$. Nevertheless, we can apply Proposition 5.3.2.). Moreover by calculating the right hand side of (36), we can describe the motivic Milnor fiber $S_{f,0}$ explicitly in terms of the Newton boundary $\Gamma_f$ as follows (see Matsui-Takeuchi [**25**, Section 4]). Assume that $f(x) = \sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} a_\alpha x^\alpha$ ($a_\alpha \in \mathbb{C}$) is non-degenerate at 0. For a compact face $F$ of $\Gamma_+(f)$, we define a lattice polytope $\Delta_F \subset \mathbb{R}^n$ by $\Delta_F := \mathrm{Conv}(F \cup \{0\})$. Moreover, we define a polynomial $\widetilde{f_{\Delta_F}}$ to be $f_{\Delta_F} - 1$. Then we define a hypersurface $Z^\circ_F$ in $\mathrm{Spec}\,\mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff}\, F] \simeq (\mathbb{C}^*)^{\dim F}$ by

$$Z^\circ_F = \{x \in \mathrm{Spec}\,\mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff}\, F] \mid f_F(x) = 0\} \subset (\mathbb{C}^*)^{\dim F},$$

and a hypersurface $Z^\circ_{\Delta_F}$ in $\mathrm{Spec}\,\mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff}\, \Delta_F] \simeq (\mathbb{C}^*)^{\dim F+1}$ by

$$Z^\circ_{\Delta_F} = \{x \in \mathrm{Spec}\,\mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff}\, \Delta_F] \mid \widetilde{f_{\Delta_F}}(x) = 0\} \subset (\mathbb{C}^*)^{\dim F+1}.$$

We endow $Z^\circ_F$ with the trivial $\hat{\mu}$-action, and $Z^\circ_{\Delta_F}$ with a good $\hat{\mu}$-action in the following way. Let $\nu_F$ be the linear function on $\mathrm{Aff}\,\Delta_F \simeq \mathbb{R}^{\dim F+1}$ which takes the value 1 on $F$ and $e_F \in (\mathbb{C}^*)^{\dim F+1} \simeq \mathrm{Spec}\,\mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff}\,\Delta_F] \simeq \mathrm{Hom}_{\mathrm{group}}(\mathbb{Z}^n \cap \mathrm{Aff}(\Delta_F), \mathbb{C}^*)$ be the element which corresponds to the group homomorphism

$$\exp\left(2\sqrt{-1}\nu_F(\cdot)\right) \in \mathrm{Hom}_{\mathrm{group}}(\mathbb{Z}^n \cap \mathrm{Aff}(\Delta_F), \mathbb{C}^*).$$

Then $Z^\circ_{\Delta_F}$ is invariant by the multiplication by $e_F$ and hence we can endow $Z^\circ_{\Delta_F}$ with a $\mu_{d_F}$-action. We thus obtain the elements $[Z^\circ_F \circlearrowleft \hat{\mu}]$ and $[Z^\circ_{\Delta_F} \circlearrowleft \hat{\mu}]$ in $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ for any compact face $F \prec \Gamma_+(f)$. For a compact face $F$ of $\Gamma_+(f)$ we define also a subset $I_F \subset \{1, \ldots, n\}$ to be the minimal one such that $F \subset \mathbb{R}^{I_F}$, and set $s_F = |I_F|$. Then we have the following theorem.

THEOREM 5.3.3 (see Matsui-Takeuchi [**25**]). *Assume that $f$ is non-degenerate at 0. Then we have*

$$\mathcal{S}_{f,0} = \sum_{F\prec\Gamma_+(f)\colon \text{compact}} (1-\mathbb{L})^{s_F - \dim F - 1}\Big\{(1-\mathbb{L})\cdot[Z^\circ_F \circlearrowleft \hat{\mu}] + [Z^\circ_{\Delta_F} \circlearrowleft \hat{\mu}]\Big\} \in \mathcal{M}^{\hat{\mu}}_{\mathbb{C}},$$

*where in the sum $\Sigma$ the face $F(\neq \emptyset)$ ranges through the compact ones of $\Gamma_+(f)$.*

This theorem was proved only in the case where $f$ is convenient in [**25**]. However, we can show it even if $f$ is not convenient similarly.

**5.3.1. The Hodge realizations of the motivic Milnor fibers.** Next, we discuss the Hodge realization of $\mathcal{S}_{f,0}$. We recall the proposition stated below for the Hodge realizations of non-degenerate hypersurfaces in the algebraic torus $(\mathbb{C}^*)^n$. Let $g(x) = \sum_{\beta\in\mathbb{Z}^n} c_\beta x^\beta \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ be a non-degenerate Laurent polynomial (see Definition 3.4.8) with $\dim\mathrm{NP}(g) = n$, and $\nu$ a $\mathbb{Q}$-affine function on $P := \mathrm{NP}(g)$ such that $\nu(\beta) \in \mathbb{Z}$ if $c_\beta \neq 0$. The multiplication by the element in $(\mathbb{C}^*)^n$ which correspond to the group homomorphism

$$\exp(2\pi\sqrt{-1}\nu(\ \cdot\ )) \in \mathrm{Hom}_{\mathrm{group}}(\mathbb{Z}^n, \mathbb{C}^*)$$

defines a $\hat{\mu}$-action on the hypersurface

$$Z^\circ := \{x \in (\mathbb{C}^*)^n \mid g(x) = 0\}.$$

We thus obtain an element $[Z^\circ \circlearrowleft \hat{\mu}]$ in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ Set

$$\epsilon(\lambda) = \begin{cases} 1 & (\lambda = 1) \\ \\ 0 & (\lambda \neq 1). \end{cases}$$

Proposition 5.3.4 (Stapledon [**44**], Matsui-Takeuchi [**25**]). *In the situation as above, we have*

$$uvE_\lambda([Z^\circ \circlearrowleft \hat{\mu}]; u, v) = \epsilon(\lambda)(uv-1)^n + (-1)^{n+1} h_\lambda^*(P, \nu; u, v),$$

*for* $\lambda \in \mathbb{C}$.

Let us remark that in [**44**] and [**25**] for an algebraic variety $X$ with a $\mu_m$-action for some $m \in \mathbb{Z}_{\geq 1}$ the authors endowed $H_c^j(X; \mathbb{Q})$ with the inverse automorphism of the one which is defined by (34) in Section 5.3. Therefore, Proposition 5.3.4 is slightly different from the original one in [**44**] and [**25**].

Let $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial such that $f(0) = 0$ and assume that $f$ is non-degenerate at 0. We denote by $P$ the convex hull $\mathrm{Conv}\,(\Gamma_f \cup \{0\})$ of $\Gamma_f \cup \{0\}$ in $\mathbb{R}^n$ and define a piecewise $\mathbb{Q}$-affine function $\nu$ on $P$ which takes the value 0 (resp. 1) at the origin $0 \in \mathbb{R}^n$ (resp. on $\mathrm{Conv}\,(\Gamma_f)$) such that for any compact face $F$ of $\Gamma_+(f)$ the restriction $\nu_F$ of $\nu$ to $\Delta_F$ is linear. Moreover, for a compact face $F$ of $\Gamma_+(f)$ let $0_F$ be the zero function on $F$. Then by Theorem 5.3.3 and Proposition 5.3.4, we can calculate the Hodge realization of the motivic Milnor fiber $S_{f,0}$, and we can describe $E_\lambda(F_{f,0}; u, v)$ as follows.

Corollary 5.3.5. *In the situation as above, for* $\lambda \in \mathbb{C}$ *we have*

$$uvE_\lambda(F_{f,0}; u, v) = \sum_{F \prec \Gamma_+(f):\ \text{compact}} (-1)^{\dim F} \left\{ (1-uv)^{s_F - \dim F} h_\lambda^*(F, 0_F; u, v) + \right.$$
$$\left. (1-uv)^{s_F - \dim F - 1} h_\lambda^*(\Delta_F, \nu_F; u, v) \right\},$$

*where in the sum* $\Sigma$ *the face* $F(\neq \emptyset)$ *ranges through the compact ones of* $\Gamma_+(f)$.

Note that if $\lambda \neq 1$, the polynomial $h_\lambda^*(F, 0_F; u, v)$ is zero. The coefficient of $u^p v^q$ in $E_\lambda(F_{f,0}; u, v)$ being an alternating sum of $h_\lambda^{p,q}(H^j(F_{f,0}; \mathbb{C}))$, for each $j \in \mathbb{Z}$ we can not always compute $h_\lambda^{p,q}(H^j(F_{f,0}; \mathbb{C}))$ by the formula in Corollary 5.3.5. Recall that if $0 \in V$ is an isolated singular point, we have $H^j(F_{f,0}; \mathbb{C}) = 0$ unless $j = 0$ or $n-1$, and $h_\lambda^{p,q}(H^0(F_{f,0}; \mathbb{C})) = 0$ unless $\lambda = 1$ and $(p, q) = (0, 0)$. Therefore, in this case, we can compute each $h_\lambda^{p,q}(H^{n-1}(F_{f,0}; \mathbb{C}))$ by our formula. Even if $f$ is not convenient (in this case, $0 \in V$ may be a non-isolated singular point), we will show later that for "good" eigenvalues we have $H^j(F_{f,0}; \mathbb{C})_\lambda = 0$ $(j \neq n-1)$ and we can compute $h_\lambda^{p,q}(H^{n-1}(F_{f,0}; \mathbb{C}))$ (see Theorem 5.4.1).

Let us explain a symmetry of $E_\lambda(F_{f,0}; u, v)$.

Definition 5.3.6. We say that a compact face $F(\neq \emptyset)$ of $\Gamma_+(f)$ is *extremal* if there exists a non-compact face $G$ of $\Gamma_+(f)$ such that $F \prec G$ and $G$ is not contained in the boundary $\partial \mathbb{R}_{\geq 0}^n$ of $\mathbb{R}_{\geq 0}^n$. If a compact face $F \prec \Gamma_+(f)$ is not extremal, we say that $F$ is *admissible*.

Definition 5.3.7. For a non-constant polynomial $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ such that $f(0) = 0$, we define a finite subset $R_f$ of $\mathbb{C}$ by

$$R_f := \bigcup_{F \prec \Gamma_+(f):\ \text{extremal}} \{\lambda \in \mathbb{C} \mid \lambda^{d_F} = 1\},$$

where $F$ ranges through the extremal compact faces of $\Gamma_+(f)$ and $d_F \in \mathbb{Z}_{\geq 0}$ is the lattice distance of $F$ from the origin $0 \in \mathbb{R}^n$.

EXAMPLE 5.3.8. Consider the case where $n = 2$. Let $f$ be the polynomial $f(x_1, x_2) = x_1^7 + x_1^3 x_2 + x_1^2 x_2^4$. Then the Newton polyhedron $\Gamma_+(f) \subset \mathbb{R}^2$ of $f$ is as in Fig. 1. The union of the bold lines in $\Gamma_+(f)$ in it is the Newton boundary $\Gamma_f$ of $f$. In this case, the only extremal face of $\Gamma_+(f)$ is the 0-dimensional face $\{(2, 4)\}$. Its lattice distance from the origin $(0, 0)$ is 2. Hence, we have

$$R_f = \{1, -1\}.$$

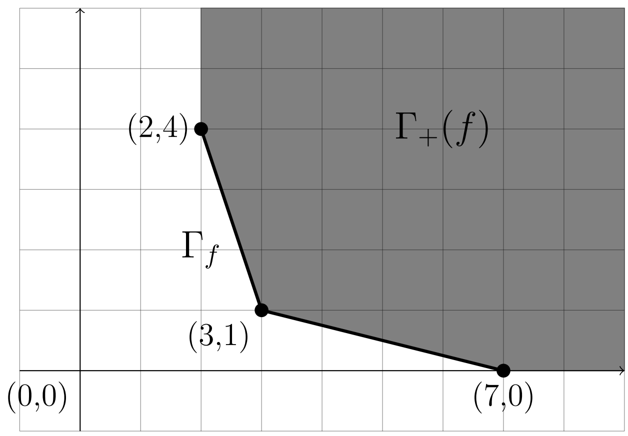


FIGURE 1. The Newton polyhedron of $f = x_1^7 + x_1^3 x_2 + x_1^2 x_2^4$

If $f$ is convenient and $\lambda \neq 1$, then the weight filtration on $H^{n-1}(F_{f,0})$ is the monodromy filtration and we have

$$h_\lambda^{p,q}(H^{n-1}(F_{f,0}; \mathbb{C})) = h_\lambda^{n-1-q, n-1-p}(H^{n-1}(F_{f,0}; \mathbb{C})),$$

for any $p, q \in \mathbb{Z}$. Hence in this case, we have

$$E_\lambda(F_{f,0}; u, v) = (uv)^{n-1} E_\lambda(F_{f,0}; v^{-1}, u^{-1}).$$

However, in the case where $f$ is not convenient, this symmetry does not hold in general. Nevertheless, by Corollary 5.3.5 and some properties in Proposition 4.2.8 of the $h^*$-polynomials we can easily see the following results.

PROPOSITION 5.3.9. *Assume that* $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ *is non-degenerate at* 0 *and is not convenient and* $\dim P = n$. *Then for any* $\lambda \notin R_f$ *we have*

$$uvE_\lambda(F_{f,0}; u, v) = (-1)^{n-1} l_\lambda^*(P, \nu; u, v).$$

*In particular, for* $\lambda \notin R_f$ *we have the symmetry*

$$E_\lambda(F_{f,0}; u, v) = (uv)^{n-1} E_\lambda(F_{f,0}; v^{-1}, u^{-1}).$$

## 5.4. Main theorem

Let $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial such that $f(0) = 0$ and set $V := f^{-1}(0) \subset \mathbb{C}^n$ as before. Throughout this section, we assume that $f$ is non-degenerate at 0 (see Definition 3.4.9). Since our assertion below will become trivial if $\dim P (= \dim \mathrm{Conv}(\Gamma_f \cup \{0\})) < n$, in what follows we assume also that $\dim P = n$. Note that we do not assume $f$ is convenient here, and therefore the origin $0 \in \mathbb{C}^n$ may be a non-isolated singular point of $V$ in general. So we can not expect to have the concentration $\widetilde{H}^j(F_{f,0};\mathbb{C}) \simeq 0$ $(j \neq n-1)$. However, we will prove the following result.

THEOREM 5.4.1. *In the situation as above, for any $\lambda \notin R_f$ (see Definition 5.3.7) we have a concentration*

$$\widetilde{H}^j(F_{f,0};\mathbb{C})_\lambda = 0 \qquad (j \neq n-1).$$

By this theorem and Theorem 5.2.2, we obtain the following corollary.

COROLLARY 5.4.2. *In the situation of Theorem 5.4.1, for any $\lambda \notin R_f$ the multiplicity of the eigenvalue $\lambda$ in the Milnor monodromy $\Phi_{n-1}$ is equal to that of the factor $(1-\lambda t)$ in a rational function*

$$\prod_{\emptyset \neq I \subset \{1,\dots,n\}} \prod_{i=1}^{k_I} (1-t^{d_{I,i}})^{(-1)^{n-|I|}\mathrm{Vol}_{\mathbb{Z}}(\Gamma_{I,i})}.$$

*For the definitions of $k_I$, $\Gamma_{I,i}$, $d_{I,i}$ and $\mathrm{Vol}_{\mathbb{Z}}(\Gamma_{I,i})$, see Section 5.2.*

For the proof of Theorem 5.4.1, we need the following proposition.

PROPOSITION 5.4.3. *In this situation of Theorem 5.4.1, for any $\lambda \notin R_f$, $k \in \mathbb{Z}$ and the inclusion map $j_0\colon \{0\} \hookrightarrow V$ the natural morphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{0\}) = \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbb{C}))$:*

$$j_0^!({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \xrightarrow{\sim} j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))$$

*is an isomorphism.*

PROOF OF PROPOSITION 5.4.3. It suffices to show that for any $\lambda \notin R_f$ the morphism

$$j_0^!\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}) \longrightarrow j_0^{-1}\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}) \tag{37}$$

is an isomorphism. We denote by $\Sigma_0$ the normal fan of the Newton polyhedron $\Gamma_+(f)$. For a subset $I \subset \{1,2,\dots,n\}$ we set $\mathbb{R}^I_{\geq 0} := \mathbb{R}^I \cap \mathbb{R}^n_{\geq 0}$. We construct a smooth subdivision $\Sigma$ of $\Sigma_0$ without subdividing the cones in $\Sigma_0$ of the type $\mathbb{R}^I_{\geq 0}$. Let $X_\Sigma$ be the toric variety associated with $\Sigma$. Recall that $X_\Sigma$ contains $T := (\mathbb{C}^*)^n$ as an open dense subset and it acts naturally on $X_\Sigma$ itself. For a cone $\sigma \in \Sigma$, we denote by $T_\sigma$ the $T$-orbit in $X_\Sigma$ associated with it. Let $\Sigma_1$ be the fan formed by all the faces of $\mathbb{R}^n_{\geq 0}$. Then the toric variety associated with it is $\mathbb{C}^n$. Moreover, the morphism of fans $\Sigma \longrightarrow \Sigma_1$ induces a proper morphism

$$\pi\colon X_\Sigma \longrightarrow \mathbb{C}^n$$

of toric varieties. Set $V' := \pi^{-1}(V)$.

LEMMA 5.4.4. *In the situation as above, we have an isomorphism*

$$\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}) \simeq \psi_{f,\lambda}(\mathrm{R}\pi_*\mathbb{C}_{X_\Sigma})$$

*in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V)$ for any $\lambda \in \mathbb{C}$.*

PROOF. By the construction of $\Sigma$, one can easily show that the morphism $\pi$ induces an isomorphism

$$X_\Sigma \setminus V' \xrightarrow{\sim} \mathbb{C}^n \setminus V.$$

Hence we obtain an isomorphism

$$(\mathrm{R}\pi_*\mathbb{C}_{X_\Sigma})|_{\mathbb{C}^n\setminus V} \simeq \mathbb{C}_{\mathbb{C}^n\setminus V}$$

in $D_c^b(\mathbb{C}^n \setminus V)$. Now the desired assertion immediately follows from the definition of the nearby cycle functor. □

In what follows, we fix $\lambda \notin R_f$. Consider the following distinguished triangle in $D_c^b(\{0\}) \simeq D^b(\mathrm{Mod}(\mathbb{C}))$:

$$(\mathrm{R}\Gamma_{\{0\}}\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}))_0 \longrightarrow \psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n})_0 \longrightarrow (\mathrm{R}\Gamma_{V\setminus\{0\}}\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}))_0 \xrightarrow{+1} .$$

The first arrow in it coincides with the natural morphism

$$j_0^!\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}) \longrightarrow j_0^{-1}\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}). \tag{38}$$

Therefore, it is enough to show that $(\mathrm{R}\Gamma_{V\setminus\{0\}}\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}))_0$ is isomorphic to 0. Consider the Cartesian diagram:

$$\begin{array}{ccc} V\setminus\{0\} & \overset{i}{\hookrightarrow} & V \\ \uparrow{\scriptstyle \pi''} & \square & \uparrow{\scriptstyle \pi'} \\ V'\setminus\pi^{-1}(0) & \overset{i'}{\hookrightarrow} & V', \end{array}$$

where $i, i'$ are the inclusions and $\pi', \pi''$ are the restrictions of $\pi$. Since $\pi'$ and $\pi''$ are proper, we obtain the following isomorphisms, where in the third isomorphism we used Proposition 4.2.11 of Dimca [**9**]:

$$\begin{aligned}
(\mathrm{R}\Gamma_{V\setminus\{0\}}\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}))_0 &= (\mathrm{R}i_*i^{-1}\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}))_0 \\
&\simeq (\mathrm{R}i_*i^{-1}\psi_{f,\lambda}(\mathrm{R}\pi_*\mathbb{C}_{X_\Sigma}))_0 \qquad \text{(by Lemma 5.4.4)} \\
&\simeq (\mathrm{R}i_*i^{-1}\mathrm{R}\pi'_*\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}))_0 \\
&\simeq (\mathrm{R}i_*\mathrm{R}\pi''_*i'^{-1}\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}))_0 \\
&\simeq (\mathrm{R}\pi'_*\mathrm{R}i'_*i'^{-1}\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}))_0 \\
&\simeq \mathrm{R}\Gamma(\pi^{-1}(0); (\mathrm{R}i'_*i'^{-1}\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}))|_{\pi^{-1}(0)}) \qquad (39) \\
&\simeq \mathrm{R}\Gamma(\pi^{-1}(0); (\mathrm{R}\Gamma_{V'\setminus\pi^{-1}(0)}(\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma})))|_{\pi^{-1}(0)}) \qquad (40)
\end{aligned}$$

in $D^b(\mathrm{Mod}(\mathbb{C}))$. The following argument is inspired by the proof of Theorem 4.3.17 in Section 4. Let $\rho_1, \dots, \rho_N$ be the rays in $\Sigma$. By abuse of notation, we will use the same symbol $\rho_i$ for the primitive vector on the ray $\rho_i$. For $1 \le i \le N$, set

$$m_{\rho_i} := \min_{\alpha\in\Gamma_+(f)} \langle \alpha, \rho_i \rangle \in \mathbb{Z}_{\ge 0},$$

where $\langle \alpha, \rho_i \rangle$ is the inner product of $\alpha$ and $\rho_i$ in $\mathbb{R}^n$. We may assume that for some $1 \le l \le N$ we have $m_{\rho_i} \neq 0$ and $\lambda^{m_{\rho_i}} = 1$ if and only if $1 \le i \le l$, and for some $l \le l' \le N$ we have $m_{\rho_i} = 0$ if and only if $l'+1 \le i \le N$. For a cone $\sigma \in \Sigma$ containing some $\rho_i$ with $1 \le i \le l'$, we also set

$$m_\sigma := \gcd_{\substack{1\le i\le l' \\ \rho_i \prec \sigma}} (m_{\rho_i}) \in \mathbb{Z}_{\ge 1}.$$

For $1 \le i \le l'$ we denote by $E_i$ the closure $\overline{T_{\rho_i}}$ of $T_{\rho_i}$ in $X_\Sigma$. Note that the order of zeros of $f\circ\pi$ along the divisor $E_i$ is equal to $m_{\rho_i}$. Let $Z$ be the strict transform of $V$ in $X_\Sigma$. Then $V' = (f\circ\pi)^{-1}(0)$ has the following form

$$V' = E_1 \cup \cdots \cup E_{l'} \cup Z.$$

Since $f$ is non-degenerate at 0, the divisor $V'$ is normal crossing in a neighborhood of $\pi^{-1}(0)$. In particular, $Z$ is smooth there. Thus, for our discussion below we may assume $V'$ is normal crossing. For a subset $I \subset \{1, \ldots, l\}$, we set

$$E_I := \bigcap_{i \in I} E_i, \quad E_I^{\circ} := E_I \setminus \Big( \bigcup_{\substack{1 \leq i \leq l' \\ i \notin I}} E_i \cup Z \Big) \quad \text{and} \quad U_I := E_I \setminus \Big( \bigcup_{l+1 \leq i \leq l'} E_i \cup Z \Big).$$

Moreover, we write $i_I$ and $j_I$ for the inclusion maps $i_I \colon E_I^{\circ} \hookrightarrow E_I$ and $j_I \colon U_I \hookrightarrow E_I$ respectively. We denote by $\iota$ the inclusion map $\iota \colon X_\Sigma \setminus V' \hookrightarrow X_\Sigma$ and define a sheaf $\mathcal{F}_\lambda$ on $X_\Sigma$ by

$$\mathcal{F}_\lambda := \iota_* (f \circ \pi|_{X_\Sigma \setminus V'})^{-1} \mathcal{L}_{\lambda^{-1}},$$

where $\mathcal{L}_{\lambda^{-1}}$ is the $\mathbb{C}$-local system on $\mathbb{C}^*$ of rank 1 whose monodromy is given by the multiplication by $\lambda^{-1}$. Since for $1 \leq i \leq l'$ the monodromy of the local system $(f \circ \pi)^{-1} \mathcal{L}_{\lambda^{-1}}$ around the divisor $E_i$ is given by $\lambda^{-m_{\rho_i}}$ , the restriction of $\mathcal{F}_\lambda$ to $U_I$ is a local system of rank 1. For $I \subset \{1, \ldots, n\}$, we set $\mathcal{F}_{\lambda, I} = \mathcal{F}_\lambda|_{E_I}$ and write $i_{E_I}$ for the inclusion $i_{E_I} \colon E_I \hookrightarrow V'$. Then, by the primitive decomposition of ${}^p\psi_{f \circ \pi, \lambda}(\mathbb{C}_{X_\Sigma}[n])$ each graded piece of ${}^p\psi_{f \circ \pi, \lambda}(\mathbb{C}_{X_\Sigma}[n])$ with respect to the filtration $W_\bullet {}^p\psi_{f \circ \pi, \lambda}(\mathbb{C}_{X_\Sigma}[n])$ is a direct sum of some perverse sheaves

$$i_{E_I *} j_{I!} j_I^{-1} \mathcal{F}_{\lambda, I}[n - |I|] (\simeq i_{E_I *} j_{I*} j_I^{-1} \mathcal{F}_{\lambda, I}[n - |I|])$$

for $I \subset \{1, \ldots, l\}$ (see Section 1.4 of [**10**] and Section 4.2 of [**7**]). Therefore, to show that (40) is isomorphic to 0 in $\mathrm{D^b}(\mathrm{Mod}(\mathbb{C}))$, it suffices to show that for each $I \subset \{1, \ldots, l\}$, we have

$$\mathrm{R}\Gamma(\pi^{-1}(0); (\mathrm{R}\Gamma_{V' \setminus \pi^{-1}(0)}(i_{E_I *} j_{I!} j_I^{-1} \mathcal{F}_{\lambda, I}))|_{\pi^{-1}(0)}) \simeq 0. \tag{41}$$

Fix $I \subset \{1, \ldots, l\}$ such that $E_I \neq \emptyset$. If for some $i \in I$ we have $E_i \subset \pi^{-1}(0)$, the sheaf ${i'}^{-1} i_{E_I *} j_{I!} j_I^{-1} \mathcal{F}_{\lambda, I}$ is zero. For this reason, in what follows we may assume that $E_i \not\subset \pi^{-1}(0)$ for any $i \in I$. Namely, the ray $\rho_i$ is contained in the boundary $\partial \mathbb{R}^n_{\geq 0}$ of $\mathbb{R}^n_{\geq 0}$ for any $i \in I$. By the property of $\mathcal{F}_\lambda$ stated above, we can easily see the isomorphisms on $E_I$:

$$j_{I!} j_I^{-1} \mathcal{F}_{\lambda, I} \simeq j_{I*} j_I^{-1} \mathcal{F}_{\lambda, I} \simeq \mathrm{R} j_{I*} {j_I}^{-1} \mathcal{F}_{\lambda, I} \simeq \mathrm{R}\Gamma_{U_I}(\mathcal{F}_{\lambda, I}). \tag{42}$$

We thus obtain

$$\mathrm{R}\Gamma_{V' \setminus \pi^{-1}(0)}(i_{E_I *} j_{I!} j_I^{-1} \mathcal{F}_{\lambda, I}) \simeq i_{E_I *} \mathrm{R}\Gamma_{U_I \setminus \pi^{-1}(0)}(\mathcal{F}_{\lambda, I}) \simeq i_{E_I *} j'_{I*} {j'_I}^{-1} \mathcal{F}_{\lambda, I},$$

where we write $j'_I$ for the inclusion $j'_I \colon U_I \setminus \pi^{-1}(0) \hookrightarrow E_I$. Note that its restriction to the outside of $U_I$ in a small neighborhood of $\pi^{-1}(0)$ is zero. Therefore, to show (41), it is enough to show that

$$\mathrm{R}\Gamma_c(U_I \cap \pi^{-1}(0); (\mathrm{R} j'_{I*} {j'_I}^{-1} \mathcal{F}_{\lambda, I})|_{U_I \cap \pi^{-1}(0)}) \simeq 0. \tag{43}$$

Since $E_I \neq \emptyset$, the cone $\tau$ generated by the rays $\{\rho_i\}_{i \in I}$ is in $\Sigma$. The subvariety $U_I \cap \pi^{-1}(0)$ of $X_\Sigma$ is the union of $T_\sigma \setminus Z$ for the cones $\sigma \in \Sigma$ satisfying the following condition $(\star)$:

$$(\star) \begin{cases} \text{(i)} & \tau \text{ is a face of } \sigma, \\ \text{(ii)} & \mathrm{rel.int}\, \sigma \subset \mathrm{Int}\, \mathbb{R}^n_{\geq 0} \quad \text{and} \\ \text{(iii)} & \text{any ray } \rho_i \in \Sigma \text{ contained in } \sigma \text{ satisfies } \lambda^{m_{\rho_i}} = 1, \end{cases}$$

where $\mathrm{rel.int}\,\sigma$ is the relative interior of $\sigma$ and $\mathrm{Int}\,\mathbb{R}^n_{\geq 0}$ is the interior of $\mathbb{R}^n_{\geq 0}$. Note that such $\sigma$ may contain some rays $\rho_i$ such that $i > l'$. Therefore, to show (43), we shall show that for any $\sigma \in \Sigma$ with the condition $(\star)$ we have

$$\mathrm{R}\Gamma_c(T_\sigma \setminus Z; (\mathrm{R}j'_{I*}j'^{-1}_I \mathcal{F}_{\lambda,I})|_{T_\sigma \setminus Z}) \simeq 0. \tag{44}$$

In what follows, we fix a cone $\sigma \in \Sigma$ with the condition $(\star)$. Let $\widetilde{\sigma} \in \Sigma_0$ be the unique cone in $\Sigma_0$ such that $\mathrm{rel.int}\,\sigma \subset \mathrm{Int}\,\widetilde{\sigma}$ and $F(\widetilde{\sigma}) \prec \Gamma_+(f)$ the face of $\Gamma_+(f)$ which corresponds to it. By the condition $\mathrm{rel.int}\,\sigma \subset \mathrm{Int}\,\mathbb{R}^n_{\geq 0}$, the face $F(\widetilde{\sigma})$ is compact. We denote by $d_{F(\widetilde{\sigma})}$ the lattice distance of $F(\widetilde{\sigma})$ from the origin $0 \in \mathbb{R}^n$. Then we can easily show the following assertion.

LEMMA 5.4.5. *Assume that* $\dim\sigma = \dim\widetilde{\sigma}$. *Then we have* $m_\sigma = d_{F(\widetilde{\sigma})}$.

Suppose that $\dim\sigma = \dim\widetilde{\sigma}$. Since for any $i \in I$ we have $\rho_i \subset \partial\mathbb{R}^n_{\geq 0}$ and $m_{\rho_i} > 0$, there exists a non-compact face $G$ of $\Gamma_+(f)$ containing $F(\widetilde{\sigma})$ and $G \not\subset \partial\mathbb{R}^n_{\geq 0}$. Then, by Lemma 5.4.5 and the assumption that $\lambda \notin R_f$, we have $\lambda^{m_\sigma} = \lambda^{d_{F(\widetilde{\sigma})}} \neq 1$. Therefore, there exists a ray $\rho_i$ in $\sigma$ such that $\lambda^{m_{\rho_i}} \neq 1$. This contradicts our condition $(\star)$. Hence we have

$$\dim\sigma < \dim\widetilde{\sigma}.$$

Take a cone $\sigma' \in \Sigma$ such that $\sigma \prec \sigma' \subset \widetilde{\sigma}$ and $\dim\sigma' = \dim\widetilde{\sigma}$. Then by the argument as above, we have $\lambda^{m_{\sigma'}} \neq 1$. Thus, by the condition $(\star)$ it follows that there exists a ray $\rho_i \prec \sigma'$ such that $\rho_i \not\prec \sigma$ and

$$\lambda^{m_{\rho_i}} \neq 1. \tag{45}$$

Moreover, take a $n$-dimensional cone $\sigma'' \in \Sigma$ such that $\sigma' \prec \sigma''$. Let $\mathrm{Edge}(\tau)$, $\mathrm{Edge}(\sigma)$, $\mathrm{Edge}(\sigma')$ and $\mathrm{Edge}(\sigma'')$ be the sets of edges (i.e rays $\rho_i$) of the smooth cones $\tau$, $\sigma$, $\sigma'$ and $\sigma''$ respectively. We assume that for some $1 \leq i_1 \leq i_2 < i_3 \leq i_4 \leq n$ we have

$$\begin{aligned} \mathrm{Edge}(\tau) &= \{\xi_1, \dots, \xi_{i_1}\}, \\ \mathrm{Edge}(\sigma) &= \{\xi_1, \dots, \xi_{i_2}\}, \\ \mathrm{Edge}(\sigma') &= \{\xi_1, \dots, \xi_{i_3}\}, \\ \mathrm{Edge}(\sigma'') &= \{\xi_1, \dots, \xi_n\} \end{aligned}$$

($\xi_i \in \Sigma$). Set $s_1 := i_1$, $s_2 := i_2 - i_1$, $s_3 := i_3 - i_2$, $s_4 := n - i_3$. Note that by the condition $\sigma \precneqq \sigma'$ we have $s_3 > 0$. Since $\sigma''$ is a smooth cone, the affine open subset $\mathbb{C}^n(\sigma'') \simeq \mathbb{C}^n$ of $X_\Sigma$ associated with $\sigma''$ has a natural decomposition:

$$\mathbb{C}^n(\sigma'') = \mathbb{C}^{s_1} \times \mathbb{C}^{s_2} \times \mathbb{C}^{s_3} \times \mathbb{C}^{s_4}.$$

Let

$$(x_1, \dots, x_{s_1}, y_1, \dots, y_{s_2}, z_1, \dots, z_{s_3}, w_1, \dots, w_{s_4})$$

be the corresponding coordinates of $\mathbb{C}^n(\sigma'')$. In $\mathbb{C}(\sigma'') \simeq \mathbb{C}^n$, we have

$$\begin{aligned} E_I &= \{0\} \times \mathbb{C}^{s_2} \times \mathbb{C}^{s_3} \times \mathbb{C}^{s_4} \qquad \text{and} \\ T_\sigma &= \{0\} \times \{0\} \times (\mathbb{C}^*)^{s_3} \times (\mathbb{C}^*)^{s_4}. \end{aligned}$$

For any $i_2 + 1 \leq i \leq i_3$ the function $\langle \xi_i, \cdot \rangle$ is constant on $F(\widetilde{\sigma})$. Since $f$ is non-degenerate at 0, $T_\sigma \cap Z$ is smooth and its defining polynomial in $T_\sigma$ can be described by

$$f \circ \pi = z_1^{m_{\xi_{i_2+1}}} \dots z_{s_3}^{m_{\xi_{i_3}}} g(w_1, \dots, w_{s_4}),$$

where $g(w_1, \dots, w_{s_4}) \in \mathbb{C}[w_1, \dots, w_{s_4}]$. We denote by $W$ the zero set of $g(w_1, \dots, w_{s_4})$ in $(\mathbb{C}^*)^{s_4}$. Then we have

$$T_\sigma \cap Z = \{0\} \times \{0\} \times (\mathbb{C}^*)^{s_3} \times W,$$

and

$$T_\sigma \setminus Z = \{0\} \times \{0\} \times (\mathbb{C}^*)^{s_3} \times ((\mathbb{C}^*)^{s_4} \setminus W).$$

Let $p_3$ be the projection $p_3 \colon T_\sigma \setminus Z = (\mathbb{C}^*)^{s_3} \times ((\mathbb{C}^*)^{s_4} \setminus W) \longrightarrow (\mathbb{C}^*)^{s_3}$, and $p_4$ be the projection $p_4 \colon T_\sigma \setminus Z = (\mathbb{C}^*)^{s_3} \times ((\mathbb{C}^*)^{s_4} \setminus W) \longrightarrow (\mathbb{C}^*)^{s_4} \setminus W$. We define $\mathcal{L}_3$ by the $\mathbb{C}$-local system on $(\mathbb{C}^*)^{s_3}$ of rank 1 whose monodromy around the divisor $\{(z_1, \dots, z_{s_3}) \in \mathbb{C}^{s_3} \mid z_t = 0\}$ is given by the multiplication by $\lambda^{-m_{\xi_{(i_2+t)}}}$ for each $1 \leq t \leq s_3$. Note that by (45) there exists $i_2 + 1 \leq i \leq i_3$ such that

$$\lambda^{-m_{\xi_i}} \neq 1. \tag{46}$$

By the definition of $\mathcal{F}_{\lambda,I}$, one can show that there exist a $\mathbb{C}$-local system $\mathcal{L}_4$ on $(\mathbb{C}^*)^{s_4} \setminus W$ and a complex $C^\bullet$ of $\mathbb{C}$-vector spaces such that

$$(\mathrm{R}j'_{I*} j_I'^{-1} \mathcal{F}_{I,\lambda})|_{T_\sigma \setminus Z} \simeq C^\bullet \otimes_{\mathbb{C}} p_3^{-1}\mathcal{L}_3 \otimes_{\mathbb{C}} p_4^{-1}\mathcal{L}_4.$$

Recall that for any non-trivial $\mathbb{C}$-local system $\mathcal{L}$ on $\mathbb{C}^*$ of rank 1, we have $H^j(\mathbb{C}^*; \mathcal{L}) \simeq 0$ for all $j \in \mathbb{Z}$. Hence, by the Künneth formula and (46) we deduce the vanishing (44). This completes the proof that the morphism (37) is isomorphism and the proof of Proposition 5.4.3. □

Proof of Theorem 5.4.1. Recall that the complex ${}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]) = \psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n-1]) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V)$ is a perverse sheaf. By the fact that the functor $j_0^{-1} \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{0\})$ (resp. $j_0^! \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{0\})$) is right (resp. left) $t$-exact, it follows from Proposition 5.4.3 that $j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))$ is a perverse sheaf on $\{0\}$. Hence the cohomology group

$$H^j(j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))) \simeq H^{j+n-1}(F_{f,0}; \mathbb{C})_\lambda$$

vanishes for $j \neq 0$. We thus obtain the desired concentration. □

Recall that the cohomology groups $H^j(F_{f,0}; \mathbb{Q})$ of the Milnor fiber are endowed with mixed Hodge structures. Since we do not assume here that $f$ is convenient, we can not expect that their weight filtrations coincide with the monodromy filtrations in general. However, we will obtain the following result.

Theorem 5.4.6. *In the situation of Theorem 5.4.1, for any $\lambda \notin R_f$ the filtration on $H^{n-1}(F_{f,0}; \mathbb{C})_\lambda$ induced by the weight filtration on $H^{n-1}(F_{f,0}; \mathbb{Q})$ coincides with the monodromy filtration of the logarithm of the unipotent part of $\Phi_{n-1,\lambda}$ centered at $n-1$.*

Remark 5.4.7. Theorems 5.4.1 and 5.4.6 explain the reason why the coefficients of the polynomial $E_\lambda(F_{f,0}; u, v)$ for $\lambda \notin R_f$ satisfy the symmetry in Proposition 5.3.9.

For the proof of Theorems 5.4.6, we need a generalization of Proposition 5.4.3 below. In what follows, we shall freely use the notations in the proof of Proposition 5.4.3. Let $W_\bullet({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))$ be the filtration on the perverse sheaf ${}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])$ defined by the weight filtration of the mixed Hodge module $\psi_f^H(\mathbb{Q}^H_{\mathbb{C}^n}[n])$. More generally, by using the exact functors $W_k \colon \mathrm{MHM}(X) \to \mathrm{MHM}(X)$ for an object $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}\mathrm{MHM}(X)$ we can define new ones $W_k\mathcal{M}^\bullet$ in $\mathrm{D}^{\mathrm{b}}\mathrm{MHM}(X)$.

PROPOSITION 5.4.8. *In this situation of Theorems 5.4.6, for any* $\lambda \notin R_f$, $k \in \mathbb{Z}$ *and the inclusion map* $j_0\colon \{0\} \hookrightarrow V$ *the natural morphism in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{0\}) = \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbb{C}))$:

$$j_0^! W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \longrightarrow j_0^{-1} W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))$$

*is an isomorphism.*

PROOF. It is enough to show that

$$\mathrm{R}\Gamma_{V\setminus\{0\}}(W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])))_0 \simeq 0$$

for any $k \in \mathbb{Z}$. Moreover, since by Lemma 5.4.4 we have $\mathrm{R}\pi'_*({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n])) \simeq {}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])$, it suffices to show that

$$\mathrm{R}\Gamma_{V\setminus\{0\}}(W_k \mathrm{R}\pi'_* \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n])))_0 \simeq 0$$

for any $i, k \in \mathbb{Z}$. Thus we have only to show that

$$\mathrm{R}\Gamma_{V\setminus\{0\}}(W_k({}^p\mathcal{H}^j \mathrm{R}\pi'_* \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n]))))_0 \simeq 0 \tag{47}$$

for any $i, j, k \in \mathbb{Z}$, where for $\mathcal{F}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathrm{V})$ we denote by ${}^p\mathcal{H}^j(\mathcal{F}^\bullet)$ the $j$-th perverse cohomology of $\mathcal{F}^\bullet$. Note that ${}^p\mathcal{H}^j \mathrm{R}\pi'_* \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n]))$ has a pure weight $i+j$. Therefore, we have

$$W_k({}^p\mathcal{H}^j \mathrm{R}\pi'_* \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n]))) \simeq \begin{cases} 0 & (k < i+j) \\ {}^p\mathcal{H}^j \mathrm{R}\pi'_* \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n])) & (k \geq i+j). \end{cases}$$

Eventually, to show the vanishing (47), it is enough to show that for any $i, j \in \mathbb{Z}$ we have

$$\mathrm{R}\Gamma_{V\setminus\{0\}}({}^p\mathcal{H}^j \mathrm{R}\pi'_* \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n])))_0 \simeq 0. \tag{48}$$

Note that a perverse sheaf $\mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n]) \oplus \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\bar{\lambda}}(\mathbb{C}_{X_\Sigma}[n])$ is the complexification of the underlying perverse sheaf of a pure Hodge module. Then, by the decomposition theorem for the proper map $\pi'$ and this perverse sheaf, $\mathrm{R}\pi'_* \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n])$ is decomposed into a direct sum of perverse sheaves with some shifts. Therefore, to show (48), it remains for us to show that

$$\mathrm{R}\Gamma_{V\setminus\{0\}}(\mathrm{R}\pi'_* \mathrm{Gr}^W_i({}^p\psi_{f\circ\pi,\lambda}(\mathbb{C}_{X_\Sigma}[n])))_0 \simeq 0$$

for any $i \in \mathbb{Z}$. This was already proved in the proof of Proposition 5.4.3. □

For the proof of Theorem 5.4.6, we also need the following lemma.

LEMMA 5.4.9. *For any* $\lambda \notin R_f$ *and* $k \in \mathbb{Z}$ *we have an isomorphism in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{0\}) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbb{C}))$:

$$W_k j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \simeq j_0^{-1} W_k\left({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])\right).$$

PROOF. For $k \in \mathbb{Z}$ we have an exact sequence in $\mathrm{Perv}(V)$:

$$0 \to W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \to {}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]) \to {}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])/W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \to 0. \tag{49}$$

By Proposition 5.4.3 and this sequence, the natural morphism

$$j_0^!\left({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])/W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))\right) \longrightarrow j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])/W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])))$$

is an isomorphism. Thus, by the proof of Theorem 5.4.1 the complexes $j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))$, $j_0^{-1}W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))$ and $j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])/W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])))$

are perverse sheaves on $\{0\}$, i.e. their (perverse) cohomologies are concentrated in the degree 0. Therefore, applying the functor $j_0^{-1}$ to the sequence (49), we obtain an exact sequence

$$0 \to j_0^{-1}W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \to j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \\ \to j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])/W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))) \to 0 \quad (50)$$

in $\mathrm{Perv}(\{0\}) \simeq \mathrm{Mod}(\mathbb{C})$. Since the functor $j_0^!$ preserves the property that a complex of mixed Hodge modules has weights $> k$, $j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])/W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))) \simeq j_0^!({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])/W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])))$ has weights $> k$. Therefore, taking $W_k$ of the sequence (50), we obtain

$$W_k j_0^{-1}W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \simeq W_k j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])). \quad (51)$$

On the other hand, since the functor $j_0^{-1}$ preserves the property that a complex of mixed Hodge modules has weights $\leq k$, $j_0^{-1}W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]))$ has weights $\leq k$. Hence we have

$$W_k j_0^{-1}W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \simeq j_0^{-1}W_k({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])). \quad (52)$$

Combining the isomorphisms (51) and (52), we get the desired isomorphism. □

Proof of Theorem 5.4.6. Assume that $\lambda \notin R_f$. We denote by $N$ the logarithm of the unipotent part of the monodromy automorphism of ${}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])$, and by $N_0$ its restriction to 0, i.e. the logarithm operator of the unipotent part of $\Phi_{n-1}$ of $j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \simeq H^{n-1}(F_{f,0};\mathbb{C})_\lambda$. Recall that for any $k \in \mathbb{Z}_{\geq 1}$ we have

$$N^k \colon \mathrm{Gr}^W_{n-1+k}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \xrightarrow{\sim} \mathrm{Gr}^W_{n-1-k}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])). \quad (53)$$

Applying the functor $j_0^{-1}$ to the both sides of (53), by Proposition 5.4.8 and Lemma 5.4.9 we obtain

$$N_0^k \colon \mathrm{Gr}^W_{n-1+k} j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \xrightarrow{\sim} \mathrm{Gr}^W_{n-1-k} j_0^{-1}({}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n])) \quad (54)$$

that is

$$N_0^k \colon \mathrm{Gr}^W_{n-1+k} H^{n-1}(F_{f,0};\mathbb{C})_\lambda \xrightarrow{\sim} \mathrm{Gr}^W_{n-1-k} H^{n-1}(F_{f,0};\mathbb{C})_\lambda$$

for any $k \in \mathbb{Z}_{\geq 1}$. This implies that the weight filtration on $H^{n-1}(F_{f,0};\mathbb{C})_\lambda$ coincides with the monodromy filtration centered at $n-1$. □

Remark 5.4.10. We can also prove Theorem 5.4.6 in the following way. First, we show the following general fact: for a mixed Hodge module $\mathcal{M}$ on $\mathbb{C}^n$ if the natural morphism $H^j j_0^! \mathcal{M} \to H^j j_0^* \mathcal{M}$ is an isomorphism for any $k \in \mathbb{Z}$, then we have the following:

(i) For $k \neq 0$, we have $H^k j_0^* \mathcal{M} = 0$.
(ii) The natural morphism $(j_0)_* H^0 j_0^* \mathcal{M} \to \mathcal{M}$ is a monomorphism in $\mathrm{MHM}(\mathbb{C}^n)$ and $(j_0)_* H^0 j_0^* \mathcal{M}$ is a direct summand of $\mathcal{M}$.

Let $\mathcal{F}$ be the underlying perverse sheaf of $\mathcal{M}$. We remark that the natural morphism $H^k j_0^! \mathcal{M} \to H^k j_0^* \mathcal{M}$ is an isomorphism for any $k \in \mathbb{Z}$ if and only if the natural morphism $j_0^! \mathcal{F} \to j_0^{-1}\mathcal{F}$ is an isomorphism since the functor $\mathrm{MHM}(\mathbb{C}^n) \longrightarrow \mathrm{Perv}(\mathbb{C}^n)$ is faithful. Next, for $\lambda \notin R_f$ we consider the sub $\mathbb{R}$-mixed Hodge module $\mathcal{M}$ of $\psi_f^H(\mathbb{R}^H_{\mathbb{C}^n}[n])$ such that the complexification of the underlying perverse sheaf is $\mathcal{F} = {}^p\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n}[n]) \oplus {}^p\psi_{f,\bar{\lambda}}(\mathbb{C}_{\mathbb{C}^n}[n])$. Note that $\lambda \notin R_f$ implies $\bar{\lambda} \notin R_f$ by definition. Then, we can apply the above fact to $\mathcal{M}$ by Proposition 5.4.3, and we thus obtain ${}^p\mathcal{H}^k(j_0^{-1}\mathcal{F}) = 0$ $(k \neq 0)$ and the filtration $(j_0)_*(W_\bullet {}^p\mathcal{H}^0 j_0^{-1}\mathcal{F}) (= W_\bullet (j_0)_* {}^p\mathcal{H}^0 j_0^{-1}\mathcal{F})$ is a direct summand of $W_\bullet\mathcal{F}$. In this way, we can deduce the isomorphism (54) from the isomorphism (53), and we get Theorem 5.4.6.

REMARK 5.4.11. Let $f \in \mathbb{C}\{x_1, \dots, x_n\}$ be a convergent power series with $f(0) = 0$. In this case, we can also define the notions in Section 5.2: the Milnor fiberation, the Milnor fiber $F_{f,0}$, the Milnor monodromies $\Phi_j \colon H^j(F_{f,0};\mathbb{C}) \xrightarrow{\sim} H^j(F_{f,0};\mathbb{C})$, the Newton polyhedron $\Gamma_+(f)$, the non-degeneracy at 0 and the finite set $R_f \subset \mathbb{C}$, similarly to the case where $f$ is a polynomial. We can also consider a (analytic) mixed Hodge module $\psi_f(\mathbb{Q}^H_{\mathbb{C}^n}[n])$ and a mixed Hodge structure $H^k j_0^*(\psi_f(\mathbb{Q}^H_{\mathbb{C}^n}[n]))$, whose underlying vector space is $H^{k+n-1}(F_{f,0};\mathbb{Q})$ for $k \in \mathbb{Z}$. Assume that $f$ is non-degenerate at 0. Then, even in this setting we can prove Proposition 5.4.3, Theorem 5.4.1, Proposition 5.4.8, Lemma 5.4.9 and Theorem 5.4.6 in the same way (We remark that we do not have the 6-operations between the derived categories of (analytic) mixed Hodge modules on analytic spaces in general. Therefore we have to be careful to use $\mathrm{D^bMHM}(\mathbb{C}^n)$ in the proof of Lemma 5.4.9. Nevertheless, in this setting, we have the functors $j_0^*$ and $j_0^!$ as in the case of the derived categories of algebraic mixed Hodge modules (see section 2.30 of [**37**]). Thus, the same proofs work even in the case where $f$ is a covergent power series.).
The proof in Remark 5.4.10 works even in the analytic setting. Therefore, also in this way, we obtain Theorem 5.4.1 and 5.4.6 for a convergent power series $f$.

## 5.5. Applications

In this section, we apply Theorems 5.4.1 and 5.4.6 to compute the Jordan normal forms of the Milnor monodromies and the Hodge spectra. Let $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial such that $f(0) = 0$. Assume that it is non-degenerate at 0. Let $P$ be the convex hull of $\Gamma_f \cup \{0\}$. Since our formula below will become trivial in the case when the dimension of $P$ less than $n$, in what follows we assume that the dimension of $P$ is equal to $n$. Moreover, since the case where $f$ is convenient was already treated by Matsui-Takeuchi [**25**] and M. Saito [**35**], we assume that $f$ is not convenient in this section. Then $R_f$ is not empty and contains $1 \in \mathbb{C}$. For $\lambda \notin R_f$, by Proposition 5.3.9, we have

$$u^2 E_\lambda(F_{f,0}; u, u) = (-1)^{n-1} l^*_\lambda(P, \nu; u, u),$$

where $\nu$ is the piecewise linear function on $P$ defined in Section 5.3.1. Recall that $\mathcal{S}_\nu$ is the polyhedral subdivision of $P$ defined by $\nu$. By the definition of the $h^*$-polynomial, for $\lambda \notin R_f$ we have

$$l^*_\lambda(P, \nu; u, u) = \sum_{F \prec \Gamma_+(f)\colon \text{admissible}} u^{\dim \Delta_F + 1} l^*_\lambda(\Delta_F, \nu; 1) \cdot l_P(\mathcal{S}_\nu, \Delta_F; u^2),$$

where in the sum $\Sigma$ the face $F$ ranges through the compact admissible ones of $\Gamma_+(f)$. The polynomial $l_P(\mathcal{S}_\nu, \Delta_F; t)$ is symmetric and unimodal centered at $(n - \dim F - 1)/2$, i.e. if $a_i \in \mathbb{Z}$ is the coefficient of $t^i$ in $l_P(\mathcal{S}_\nu, \Delta_F; t)$ we have $a_i = a_{n-\dim F-1-i}$ and $a_i \leq a_j$ for $0 \leq i \leq j \leq (n - \dim F - 1)/2$. Therefore, it can be expressed in the form

$$l_P(\mathcal{S}_\nu, \Delta_F; t) = \sum_{i=0}^{\lfloor (n-1-\dim F)/2 \rfloor} \widetilde{l}_{F,i}(t^i + t^{i+1} + \cdots + t^{n-1-\dim F - i}),$$

for some non-negative integers $\widetilde{l}_{F,i} \in \mathbb{Z}_{\geq 0}$. We set

$$\widetilde{l}_P(\mathcal{S}_\nu, \Delta_F, t) := \sum_{i=0}^{\lfloor (n-1-\dim F)/2 \rfloor} \widetilde{l}_{F,i} t^i.$$

For $m \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{C}$ we denote by $J_{m,\lambda}$ the number of the Jordan blocks in $\Phi_{n-1}$ with size $m$ for the eigenvalue $\lambda$. Then by Theorems 5.4.1 and 5.4.6 we obtain the following formula for them.

Corollary 5.5.1. *In the situation as above, for any $\lambda \notin R_f$ we have*

$$\sum_{0 \leq k \leq n-1} J_{n-k,\lambda} u^{k+2} = \sum_{F \prec \Gamma_+(f):\ admissible} u^{\dim \Delta_F + 1} l^*_\lambda(\Delta_F, \nu; 1) \cdot \widetilde{l}_P(\mathcal{S}_\nu, \Delta_F; u^2),$$

*where in the sum $\Sigma$ of the right hand side the face $F$ ranges through the admissible ones of $\Gamma_+(f)$.*

Finally, we introduce our formula for the Hodge spectrum of $f$ at the origin 0.

Definition 5.5.2. (i) We define a Puiseux polynomial $\mathrm{sp}_{f,0}(t)$ with coefficients in $\mathbb{Z}$ by

$$\mathrm{sp}_{f,0}(t) = (-1)^{n-1} \sum_{\alpha \in \mathbb{Q} \cap [0,n]} \left\{ \sum_{j \in \mathbb{Z}} (-1)^j \dim \mathrm{Gr}_F^{\lfloor n-\alpha \rfloor} \widetilde{H^j}(F_{f,0}; \mathbb{C})_{\exp(-2\pi\sqrt{-1}\alpha)} \right\} t^\alpha,$$

where $\mathrm{Gr}_F^{\lfloor n-\alpha \rfloor} \widetilde{H^j}(F_{f,0}; \mathbb{C})_{\exp(-2\pi\sqrt{-1}\alpha)}$ is the graded piece with respect to the Hodge filtration of the mixed Hodge structure of $\widetilde{H}^j(F_{f,0}; \mathbb{Q})$. We call it the *Hodge spectrum of $f$ at* 0.

(ii) For $\beta \in (0,1) \cap \mathbb{Q}$ we set $\lambda = \exp(2\pi\sqrt{-1}\beta)$ and we define a Puiseux polynomial $\mathrm{sp}^\lambda_{f,0}(t)$ by

$$\mathrm{sp}^\lambda_{f,0}(t) = (-1)^{n-1} \sum_{i=0}^{n-1} \left\{ \sum_{j \in \mathbb{Z}} (-1)^j \dim \mathrm{Gr}_F^{\lfloor n-\beta-i \rfloor} \widetilde{H^j}(F_{f,0}; \mathbb{C})_{\lambda^{-1}} \right\} t^{\beta+i}.$$

Since $f$ is non-degenerate at 0, by setting $v = 1$ in Corollary 5.3.5 we can express $\mathrm{sp}_{f,0}(t)$ and $\mathrm{sp}^\lambda_{f,0}(t)$ in terms of $\Gamma_f$. Moreover, if $\lambda = \exp(2\pi\sqrt{-1}\beta)$ is not in $R_f$, $\mathrm{sp}^\lambda_{f,0}(t)$ can be rewritten much more simply as follows. For a compact face $F$ of $\Gamma_+(f)$, we define a cone $\mathrm{Cone}(F) \subset \mathbb{R}^n$ by $\mathrm{Cone}(F) := \mathbb{R}_{\geq 0} F$ and the linear function $h_F$ on $\mathrm{Cone}(F)$ which takes the value 0 at the origin $0 \in \mathbb{R}^n$ and the value 1 on $F$. Moreover, for $\beta \in (0,1) \cap \mathbb{Q}$ we define a Puiseux polynomial $P_{F,\beta}(t)$ by

$$P_{F,\beta}(t) := \sum_{i=0}^{+\infty} \#\{v \in \mathrm{Cone}(F) \cap \mathbb{Z}^n_{\geq 0} \mid h_F(v) = \beta + i\} t^{\beta+i}.$$

Then we obtain the following formula, which generalizes the one for $\mathrm{sp}_{f,0}(t)$ in the case where $0 \in V$ is an isolated singular point obtained by M. Saito [**35**]. For the corresponding result for the monodromies at infinity, see Theorem 5.16 of Matsui-Takeuchi [**24**].

Corollary 5.5.3. *In the situation as above, assume moreover that $\lambda$ is not in $R_f$. Then we have*

$$\mathrm{sp}^\lambda_{f,0}(t) = \sum_{F \prec \Gamma_+(f):\ admissible} (-1)^{n-1-\dim F} (1-t)^{s_F} P_{F,\beta}(t),$$

*where in the sum* $\Sigma$ *the face* $F$ *ranges through the admissible ones of* $\Gamma_+(f)$ *and* $s_F \in \mathbb{Z}_{\geq 1}$ *is the integer defined in Section 5.3.*

PROOF. By Theorem 5.4.1, for $\beta \in (0,1)\cap\mathbb{Q}$ such that $\lambda = \exp(2\pi\sqrt{-1}\beta) \notin R_f$ we have the concentration

$$\widetilde{H^j}(F_{f,0};\mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1).$$

Moreover, for $0 \leq i \leq n-1$ we have

$$\begin{aligned}
\mathrm{dimGr}_F^{\lfloor n-i-\beta\rfloor}\widetilde{H}^{n-1}(F_{f,0};\mathbb{C})_{\lambda^{-1}} &= \sum_{k\in\mathbb{Z}} \mathrm{dimGr}_F^{\lfloor n-i-\beta\rfloor}\mathrm{Gr}_k^W\widetilde{H}^{n-1}(F_{f,0};\mathbb{C})_{\lambda^{-1}} \\
&= \sum_{k\in\mathbb{Z}} \mathrm{dimGr}_F^{n-1-\lfloor n-i-\beta\rfloor}\mathrm{Gr}^W_{2(n-1)-k-\lfloor n-\alpha\rfloor}\widetilde{H}^{n-1}(F_{f,0};\mathbb{C})_\lambda \\
&= \mathrm{dimGr}_F^i\widetilde{H}^{n-1}(F_{f,0};\mathbb{C})_\lambda,
\end{aligned}$$

where in the second (resp. third) equality we used Theorem 5.3.9 and Proposition 4.2.8 (ii) (resp. $n-1-\lfloor n-i-\beta\rfloor = i$). Then we have

$$\begin{aligned}
\mathrm{sp}^\lambda_{f,0}(t) &= \sum_{i=0}^{n-1} \mathrm{dimGr}_F^{\lfloor n-i-\beta\rfloor}H^{n-1}(F_{f,0};\mathbb{C})_{\lambda^{-1}}t^{\beta+i} \\
&= \sum_{i=0}^{n-1} \mathrm{dimGr}_F^i H^{n-1}(F_{f,0};\mathbb{C})_\lambda t^{\beta+i} \\
&= (-1)^{n-1}E_\lambda(F_{f,0};t,1)t^\beta \\
&= \frac{1}{t} l^*_\lambda(P,\nu;t)t^\beta \qquad \text{(by Propositions 5.3.9 and 4.2.8 (i))},
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
& t^{\beta-1}l^*_\lambda(P,\nu;t) \\
&= t^{\beta-1} \sum_{F\prec\Gamma_+(f):\,\text{admissible}} \left\{(-1)^{n-\dim\sigma(\Delta_F)}(t-1)^{\dim\sigma(\Delta_F)-\dim\Delta_F}h^*_\lambda(\Delta_F,\nu;t)\right\} \\
&= t^{\beta-1} \sum_{F\prec\Gamma_+(f):\,\text{admissible}} \Bigg\{(-1)^{n-\dim\sigma(\Delta_F)}(t-1)^{\dim\sigma(\Delta_F)-\dim\Delta_F}(1-t)^{\dim\Delta_F+1} \\
&\qquad\qquad \cdot \sum_{m\geq 0} f_\lambda(P,\nu;m)t^m\Bigg\} \\
&= t^{\beta-1} \sum_{F\prec\Gamma_+(f):\,\text{admissible}} \left\{(-1)^{n-1-\dim F}(1-t)^{\dim\sigma(\Delta_F)+1}\sum_{m\geq 0} f_\lambda(P,\nu;m)t^m\right\} \\
&= t^{\beta-1} \sum_{F\prec\Gamma_+(f):\,\text{admissible}} \Bigg\{(-1)^{n-1-\dim F}(1-t)^{\dim\sigma(\Delta_F)} \\
&\qquad\qquad \cdot \sum_{m\geq 1} \lfloor (f_\lambda(P,\nu;m) - f_\lambda(P,\nu;m-1)\rfloor)t^m\Bigg\} \\
&= \sum_{F\prec\Gamma_+(f):\,\text{admissible}} (-1)^{n-1-\dim F}(1-t)^{\dim\sigma(\Delta_F)}P_{F,\beta}(t),
\end{aligned}$$

where in the sums $\Sigma$ the faces $F$ range through the admissible ones of $\Gamma_+(f)$. Since for an admissible face $F \prec \Gamma_+(f)$ we have $\dim \sigma(\Delta_F) = s_F$, this completes the proof. $\square$

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