Some Generalizations of Radon Transforms

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1 Classical Radon Transform

1.1 Preliminary

For $1 \leq p < \infty$, $L^p(\mathbf{R}^n)$ denote

$$L^p(\mathbf{R}^n) = \Big\{ f : ||f||_{L^2(\mathbf{R}^n)} = \Big(\int_{\mathbf{R}^n} |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty \Big\}.$$

Also, the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ is the function space of C^∞-functions whose derivatives are rapidly decreasing, i.e.,

$$\mathcal{S}(\mathbf{R}^n) = \Big\{ f \in C^\infty(\mathbf{R}^n) : \lim_{|x|\to\infty} x^\alpha \partial^\beta f(x) = 0 \ \ \forall \alpha, \beta \in \mathbf{Z}^n_+ \Big\}.$$

For $f \in L^2(\mathbf{R}^n)$, the Fourier transform and the inverse Fourier transform are given by

$$\mathcal{F}[f](\xi)\Big(= \hat{f}(\xi) \Big) = \int_{\mathbf{R}} f(x) e^{-ix\cdot\xi} dx,$$
$$\mathcal{F}^{-1}[f](\xi)\Big(= \mathcal{F}^*[f](\xi) \Big) = \frac{1}{(2\pi)^n} f(x) e^{ix\cdot\xi} d\xi.$$

The Hilbert transform of a function $u(t)$ is defined as

$$\mathcal{H}[u](t) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau} d\tau = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|t-\tau|>\varepsilon} \frac{u(\tau)}{t-\tau} d\tau.$$

1.2 Properties of classical Radon transforms

We first introduce the definition of the classical Radon transform.

Definition 1. *Let $n \geq 2$. The Radon transform of a function $f \in \mathcal{S}(\mathbf{R}^n)$ is defined by the formula*

$$\mathcal{R}(f)(t,\gamma) = \int_{\mathbf{R}^n} f(x) \delta(x \cdot \gamma - t) dx, \tag{1}$$

where $\gamma \in S^{n-1}$, $t \in \mathbf{R}$ and δ is the Dirac delta function on $\mathbf{R}^1$.

Here, S^{n-1} is the unit sphere. Obviously, $\mathcal{R}(f)$ is an even function on $S^{n-1} \times \mathbf{R}$, that is $\mathcal{R}(f)(-t,-\gamma) = \mathcal{R}(f)(t,\gamma)$. If $n = 2$, then the classical Radon transform is called the (2-dimensional) X-ray transform. Note that the (n-dimensional) X-ray transform is given by

$$\mathcal{X}(f)(x,\gamma) = \int_{-\infty}^{\infty} f(x + p\gamma) dp,$$

where $\gamma \in S^{n-1}$ and $x \in \mathbf{R}^n$. The above formula means the integral of $f \in \mathcal{S}(\mathbf{R}^n)$ over the straight line through x with direction γ.
Then, the following result connecting with the Fourier transform is known:

Fourier Slice Theorem *For all $s \in \mathbf{R}$ and $\gamma \in S^{n-1}$, the Radon transform $\mathcal{R}(f)$ satisfies*

$$\mathcal{F}_t[\mathcal{R}(f)](s,\gamma) = \mathcal{F}_x[f](s\gamma). \tag{2}$$

Proof The left hand side of (2) can be computed as follows,

$$\begin{aligned}\mathcal{F}_t[\mathcal{R}(f)](s,\gamma) &= (2\pi)^{-1/2}\int_{\mathbf{R}^n} e^{-its}\mathcal{R}(f)(t,\gamma)dt \\ &= (2\pi)^{-1/2}\int_{\mathbf{R}} e^{-its}\int_{\mathbf{R}^n}\delta(x\cdot\gamma - t)dxdt \\ &= \int_{\mathbf{R}^n} f(x)e^{-ix\cdot(s\gamma)}dx = \mathcal{F}_x[f](s\gamma).\ \Box\end{aligned}$$

The Fourier slice theorem enables us to deduce the inverse of (1). Also for generalized Radon transforms such as the d-plane transform, this sort of equality would be a crucial to derive range properties (see [13], [15]). Next, we consider the reconstruction formula of f from $\mathcal{R}(f)$. We also define the dual transform for $\varphi(t,\gamma) \in \mathcal{S}(\mathbf{R}\times S^{n-1})$

$$\mathcal{R}^*(\varphi)(x) = \int_{S^{n-1}} \varphi(x\cdot\gamma,\gamma)d\sigma(\gamma),$$

where $d\sigma(\gamma)$ is the normalized Haar measure on S^{n-1}. Indeed, it formally holds that

$$\begin{aligned}\Big(\mathcal{R}(f),\varphi\Big)_{L^2(\mathbf{R}\times S^{n-1})} &= \int_{-\infty}^{\infty}\int_{S^{n-1}}\int_{\mathbf{R}^n} f(x)\delta(t-\gamma\cdot x)dx\overline{\varphi(t,\gamma)}dtd\sigma(\gamma) \\ &= \int_{\mathbf{R}^n} f(x)\overline{\int_{S^{n-1}}\varphi(\gamma\cdot x,\gamma)d\sigma(\gamma)dx} \\ &= \Big(f,\mathcal{R}^*(\varphi)\Big)_{L^2(\mathbf{R}\times S^{n-1})}.\end{aligned}$$

Define the Laplace-Bertrami operator $(-\Delta)^{\frac{n-1}{2}}$ by

$$(-\Delta)^{\frac{n-1}{2}}f(x) = \mathcal{F}^*_\xi[|\xi|^{n-1}\mathcal{F}_x f](x).$$

Then we have the following theorem:

Theorem 1. *Let $n \geq 2$. The classical Radon transform for $f \in \mathcal{S}(\mathbf{R}^n)$*

$$\mathcal{R}(f)(t,\gamma) = \int_{\mathbf{R}^n} f(x)\delta(x\cdot\gamma - t)dx,$$

and its dual transform for $\varphi \in \mathcal{S}(\mathbf{R}\times S^{n-1})$

$$\mathcal{R}^*(\varphi)(x) = \int_{S^{n-1}} \varphi(x\cdot\gamma,\gamma)d\sigma(\gamma),$$

satisfy the reconstruction formula:

$$f(x) = \frac{1}{2(2\pi)^{n-1}}(-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*\mathcal{R}(f)(x). \tag{3}$$

Proof From the definitions, we get

$$\begin{aligned}\mathcal{R}^*\mathcal{R}f(x) &= \int_{S^{n-1}}\mathcal{R}(f)(x\cdot\gamma,\gamma)d\sigma(\gamma) \\ &= \int_{S^{n-1}}\frac{1}{2\pi}\int_{\mathbf{R}} e^{isx\cdot\gamma}\hat{f}(s\gamma)dsd\sigma(\gamma) \\ &= \frac{1}{2\pi}\int_{S^{n-1}}\Big\{\int_0^\infty e^{isx\cdot\gamma}\hat{f}(s\gamma)ds + \int_{-\infty}^0 e^{isx\cdot\gamma}\hat{f}(s\gamma)ds\Big\}d\sigma(\gamma) \\ &= \frac{1}{\pi}\int_{S^{n-1}}\int_0^\infty e^{ix\cdot s\gamma}\hat{f}(s\gamma)dsd\sigma(\gamma).\end{aligned}$$

The polar coordinates $\xi = s\gamma$ gives

$$\mathcal{R}^*\mathcal{R}f(x) = \frac{1}{\pi}\int_{\mathbf{R}^n} e^{ix\cdot\xi}|\xi|^{1-n}\hat{f}(\xi)d\xi.$$

Operating $(-\Delta)^{\frac{n-1}{2}}$, we obtain

$$(-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*\mathcal{R}f(x) = \frac{(2\pi)^n}{\pi}\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)d\xi$$

Thus, we have

$$f(x) = \frac{1}{2(2\pi)^{n-1}}(-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*\mathcal{R}(f)(x). \square$$

2 Fractional Radon Transform

2.1 Fractional Fourier Transform

The fractional Fourier transform is a generalization of classical Fourier transform. It was introduced by Namias in the context of quantum mechanics as a tool for solving several types of Schrödinger equation. Recently, Mendlovic and Ozaktas et al [28] showed that the fractional Fourier transform can be successfully applied in optics. The fractional Fourier domains interpolate continuously between time and frequency domains. Let $C_\alpha = (\frac{e^{i\alpha}}{2\pi i \sin\alpha})^{1/2}$, where $-\pi/2 < \arg C_\alpha < \pi/2$. Define that

$$K_\alpha(x,\xi) = \begin{cases} \delta(x-\xi) & (\alpha \in 2\pi\mathbf{Z}), \\ \delta(x+\xi) & (\alpha+\pi \in 2\pi\mathbf{Z}), \\ (C_\alpha)^n \exp\Big\{\dfrac{i(|x|^2+|\xi|^2)}{2\tan\alpha} - \dfrac{ix\cdot\xi}{\sin\alpha}\Big\} & (\text{otherwise}), \end{cases}$$

where δ is the Dirac delta function on $\mathbf{R}^n$. For $f \in L^2(\mathbf{R}^n)$ the αth fractional Fourier transform is defined with the integral kernel $K_\alpha(x,\xi)$ as follows:

$$\mathcal{F}^{(\alpha)}[f](\xi) = \int_{\mathbf{R}^n_x} K_\alpha(x,\xi)f(x)dx.$$

$K_\alpha(x,\xi)$ satisfies the following properties:

- Symmetry: $K_\alpha(x,\xi) = K_\alpha(\xi,x)$.
- Inverse: $K_{-\alpha}(x,\xi) = K^*_\alpha(x,\xi)$.
- Additivity: $K_{\alpha+\beta}(x,\xi) = \int_{\mathbf{R}^n_y} K_\alpha(x,y)K_\beta(y,\xi)dy$.
- Periodicity: $K_0(x,\xi) = K_{2\pi}(x,\xi) = \delta(x-\xi)$.
- Generalizability: $K_{\pi/2}(x,\xi) = \frac{1}{(2\pi)^{n/2}}\exp(-ix\cdot\xi)$.

Generalizability means that the fractional Fourier transform is a generalization of the standard Fourier transform. By the Inverse property, we can obtain the inversion formula of the fractional Fourier transform and write

$$\mathcal{F}^{(-\alpha)}[\varphi](x)(= \mathcal{F}^{(\alpha)*}[\varphi](x)) = \int_{\mathbf{R}^n_\xi} K_{-\alpha}(\xi,x)\varphi(\xi)d\xi.$$

Moreover, we introduce the αth order fractional Fourier series. Let $T > 0$ and $W = 2\pi/T$. For $f \in L^2(\mathbf{R}_t)$ satisfying supp$f \subset [-T/2, T/2]$, the αth order fractional Fourier series is given by the following (see [2]):

$$f(t) = WC_{-\alpha}|\sin\alpha| \sum_{k\in\mathbf{Z}} \mathcal{F}^{(\alpha)}(kW\sin\alpha)\mathbf{e}_k^{(-\alpha)}(t), \tag{5}$$

where $\{\mathbf{e}_k^{(-\alpha)}\}_{k\in\mathbf{Z}}$ is the orthonormal basis in $L^2(-T/2, T/2)$ defined as

$$\mathbf{e}_k^{(-\alpha)}(t) := \frac{K_{-\alpha}(t, kW\sin\alpha)}{\sqrt{T}} = \frac{1}{\sqrt{T}} \exp\Big\{ -\frac{i(t^2 + (kW\sin\alpha)^2)}{2\tan\alpha} + ikWt \Big\}. \tag{6}$$

At the end of this section, we show the direct and simple relation of the fractional Fourier transform to the Wigner transform. The Wigner transform of a function f is given by

$$\mathcal{W}[f](t,\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f\Big(t + \frac{t'}{2}\Big) \overline{f\Big(t - \frac{t'}{2}\Big)} e^{-it'\omega} dt' \tag{7}$$

which also can be defined in frequency domains:

$$\mathcal{W}[f](t,\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \mathcal{F}[f]\Big(\omega + \frac{\omega'}{2}\Big) \overline{\mathcal{F}[f]\Big(\omega - \frac{\omega'}{2}\Big)} e^{it\omega'} d\omega',$$

where $\overline{z}$ is the complex conjugate of the complex number z. We easily see that

$$\int_{\mathbf{R}} \mathcal{W}[f](t,\omega) d\omega = |f(x)|^2 \quad \text{and} \quad \int_{\mathbf{R}} \mathcal{W}[f](t,\omega) dt = |\mathcal{F}[f](\omega)|^2.$$

Then, the following result connecting with the fractional Fourier transform and the Radon transform is known:

Theorem 2. *Let $0 < \alpha < \pi/2$. For $f \in L^2(\mathbf{R})$ the Wigner transform $\mathcal{W}$ satisfies*

$$\mathcal{R}(\mathcal{W}[f](t,\omega))(s,\alpha) = |\mathcal{F}^{(\alpha)}[f](s)|^2. \tag{8}$$

The left hand side of (8) is so-called "Radon-Wigner transform" (see [28]).

2.2 Fractional Radon Transform and its Properties

In this section, we generalize the classical Radon transform by using the fractional Fourier transform. In 1960 Semyanistyi [32] proved that the Dirac delta function δ on $\mathbf{R}$ is the weak limit of $h_\alpha|\tau|^{\alpha-1}$ ($h_\alpha := \frac{\Gamma((1-\alpha)/2)}{2^\alpha \pi^{1/2}\Gamma(\alpha/2)}$) as $\alpha \to 0$ and the following αth order fractional Radon transform can be considered:

$$\mathcal{R}_\alpha(f)(t,\gamma) = h_\alpha \int_{\mathbf{R}_x^n} f(x)|x\cdot\gamma - t|^{\alpha-1} dx. \tag{9}$$

(9) satisfies $\mathcal{F}_t[\mathcal{R}_\alpha(f)](s,\gamma) = \mathcal{F}_x[|D|^{-\alpha}f](s\gamma)$ instead of the Fourier slice theorem (2) since $\mathcal{F}^*[|p|^{\alpha-1}](s) = h_\alpha^{-1}|s|^{-\alpha} (= h_\alpha^{-1}|\gamma s|^{-\alpha})$, where $|D|^{-\alpha}$ is defined by $|D|^{-\alpha}f(x) = \mathcal{F}_\xi^*[|\xi|^{-\alpha}\mathcal{F}_x f](x)$. On the other hand, we are concerned with another αth order fractional Radon transform satisfying the following relation (linked to the Fourier slice theorem (2) by the αth order fractional Fourier transform):

$$\mathcal{F}_t^{(\alpha)}[\mathcal{R}^{(\alpha)}(f)](s,\gamma) = \mathcal{F}_x^{(\alpha)}[f](s\gamma). \tag{10}$$

Based on this identity, we define the αth order fractional Radon transform

$$\mathcal{R}^{(\alpha)}(f)(t,\gamma) = (C_\alpha)^{1-n}\mathcal{F}_s^{(\alpha)*}\Big[\mathcal{F}_x^{(\alpha)}[f](s\gamma)\Big](t,\gamma). \tag{11}$$

Remark 3. *In particular when $\alpha = \pi/2$, by the Fourier slice theorem the definition (11) coincides with the classical Radon transform.*

The classical Radon transform has been generalized with various manifolds, e.g., Funk transform, Horocycle transform, etc. (see [14], [18]). We consider (11) not only for $|\gamma| = 1$ but also for all $\gamma \in \mathbf{R}^n\backslash\{0\}$. For $|\gamma| = 1$, Zelevsky and Mendlovic [36] first gave a definition of the fractional Radon transform without oscillatory Fourier integrals. Starting from (10), we shall derive such a representation for all $\gamma \in \mathbf{R}^n\backslash\{0\}$. We prove the following:

Theorem 4. [11] *Let $n \geq 2$, $-\pi/2 < \alpha < \pi/2$ and $K_\gamma^{(\alpha)} = \frac{1}{(i\pi(1-|\gamma|^2)\sin 2\alpha)^{\frac{1}{2}}}$. For $f \in \mathcal{S}(\mathbf{R}^n)$ the αth order fractional Radon transform $\mathcal{R}^{(\alpha)}(f)$ satisfying the relation*

$$\mathcal{F}_t^{(\alpha)}[\mathcal{R}^{(\alpha)}(f)](s,\gamma) = (C_\alpha)^{1-n}\mathcal{F}_x^{(\alpha)}[f](s\gamma) \quad (s \in \mathbf{R},\ \gamma \in \mathbf{R}^n\backslash\{0\}) \tag{12}$$

and $\mathcal{R}^{(\pm\pi/2)}(f) = \mathcal{R}(f)$, is given by

$$\mathcal{R}^{(\alpha)}(f)(t,\gamma) = K_\gamma^{(\alpha)} \int_{\mathbf{R}_x^n} f(x) \exp\Big\{\frac{i(|x|^2 - t^2)}{2\tan\alpha} + \frac{i(|\gamma|x\cdot\gamma - t)^2}{(1-|\gamma|^2)\sin 2\alpha}\Big\}dx \tag{13}$$

in the case of $|\gamma| \neq 1, 0$, and

$$\mathcal{R}^{(\alpha)}(f)(t,\gamma) = \mathcal{R}\Big(f(x)\exp\Big\{\frac{i(|x|^2-t^2)}{2\tan\alpha}\Big\}\Big)(t,\gamma) \tag{14}$$

in the case of $|\gamma| = 1$. For the special case of for $|\gamma| = 1$, we have

$$f(x) = 2^{-n}\pi^{1-n}\exp\Big\{\frac{-i|x|^2}{2\tan\alpha}\Big\}(-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*\Big(\exp\Big\{\frac{it^2}{2\tan\alpha}\Big\}\mathcal{R}^{(\alpha)}(f)(t,\gamma)\Big)(x).$$

Remark 5. *The representation (14) in the case of $|\gamma| = 1$ is the same as the definition given by [36]. Moon[26] investigated isometry and stability estimates in some functional spaces for (14).*

Remark 6. *For all $\gamma \in \mathbf{R}^n\backslash\{0\}$, the classical Radon transform is generalized as the so-called "Radon transform on the affine plane"*

$$\tilde{\mathcal{R}}(f)(t,\gamma) := |\gamma|^{-1}\mathcal{R}(f)\Big(\frac{t}{|\gamma|}, \frac{\gamma}{|\gamma|}\Big), \tag{15}$$

where $\mathcal{R}(f)(t/|\gamma|, \gamma/|\gamma|)$ in the right hand side is defined by (3) (see [14], p. 5 and [32], p. 131). The changes of $t/|\gamma|$ and $\gamma/|\gamma|$ would be reasonable by regarding the line $x\cdot\gamma = t$ as $x \cdot \frac{\gamma}{|\gamma|} = \frac{t}{|\gamma|}$. However, we have to pay attention to the multiplication by $|\gamma|^{-1}$ which is required for the Fourier slice theorem (12) with $\alpha = \pi/2$. We remark that (14) generalized as (15), i.e.,

$$\tilde{\mathcal{R}}^{(\alpha)}(f)(t,\gamma) = |\gamma|^{-1}\mathcal{R}\Big(f(x)\exp\Big\{\frac{i(|x|^2-t^2)}{2\tan\alpha}\Big\}\Big)\Big(\frac{t}{|\gamma|}, \frac{\gamma}{|\gamma|}\Big) \tag{16}$$

does not satisfy the relation (12) with $-\pi/2 < \alpha < \pi/2$ in the case of $|\gamma| \neq 1, 0$. So, (16) never coincides with (13) which preserves the property (12).

Proof of Theorem 4 Taking the orthonormal basis $\{\gamma/|\gamma|, \mathbf{e}_1^\gamma, \cdots, \mathbf{e}_{n-1}^\gamma\}$ in $\mathbf{R}^n$ we try to change the representation (11) as follows:

$$
\begin{aligned}
&\mathcal{F}_x^{(\alpha)}[f](s\gamma) = (C_\alpha)^n \int_{\mathbf{R}^n} f(x) \exp\Big\{\frac{i(|x|^2+|s\gamma|^2)}{2\tan\alpha} - \frac{i(s\gamma)\cdot x}{\sin\alpha}\Big\}dx \\
= \;& (C_\alpha)^n \int_{\mathbf{R}^n} f\Big((t|\gamma|)\frac{\gamma}{|\gamma|} + y_1\mathbf{e}_1^\gamma + \cdots + y_{n-1}\mathbf{e}_{n-1}^\gamma\Big) \exp\Big\{\frac{i\{(t|\gamma|)^2+|y|^2\} + is^2|\gamma|^2}{2\tan\alpha} \\
&\qquad - \frac{i(s\gamma)\cdot\{(t|\gamma|)\frac{\gamma}{|\gamma|} + y_1\mathbf{e}_1^\gamma + \cdots + y_{n-1}\mathbf{e}_{n-1}^\gamma\}}{\sin\alpha}\Big\}dyd(t|\gamma|) \\
= \;& (C_\alpha)^n \int_{\mathbf{R}^n} f\Big(t\frac{\gamma}{|\gamma|} + y_1\mathbf{e}_1^\gamma + \cdots + y_{n-1}\mathbf{e}_{n-1}^\gamma\Big) \\
&\times \exp\Big\{\frac{i\{t^2+|y|^2\} + is^2|\gamma|^2}{2\tan\alpha} - \frac{i(s\gamma)\cdot\{t\frac{\gamma}{|\gamma|} + y_1\mathbf{e}_1^\gamma + \cdots + y_{n-1}\mathbf{e}_{n-1}^\gamma\}}{\sin\alpha}\Big\}dydt \\
= \;& (C_\alpha)^n \int_{\mathbf{R}_t} \exp\Big\{\frac{i(t^2+s^2|\gamma|^2)}{2\tan\alpha} - \frac{ist|\gamma|}{\sin\alpha}\Big\} \int_{\mathbf{R}_y^{n-1}} f\Big(t\frac{\gamma}{|\gamma|} + y_1\mathbf{e}_1^\gamma + \cdots + y_{n-1}\mathbf{e}_{n-1}^\gamma\Big) \exp\Big\{\frac{i|y|^2}{2\tan\alpha}\Big\}dydt \\
= \;& (C_\alpha)^{n-1}\mathcal{F}_t^\alpha\Big[\int_{\mathbf{R}_y^{n-1}} f\Big(t\frac{\gamma}{|\gamma|} + y_1\mathbf{e}_1^\gamma + \cdots + y_{n-1}\mathbf{e}_{n-1}^\gamma\Big) \exp\Big\{\frac{i|y|^2}{2\tan\alpha}\Big\}dy\Big](s|\gamma|,\gamma).
\end{aligned}
$$

For the special case of $|\gamma| = 1$, by (10) $\mathcal{R}^{(\alpha)}(f)$ can be rewritten as

$$
\mathcal{R}^{(\alpha)}(f)(t,\gamma) = \int_{\mathbf{R}_y^{n-1}} f(t\gamma + y_1\mathbf{e}_1^\gamma + \cdots + y_{n-1}\mathbf{e}_{n-1}^\gamma) \exp\Big\{\frac{i|y|^2)}{2\tan\alpha}\Big\}dy. \tag{17}
$$

By (10), we get

$$
\mathcal{F}_t^{(\alpha)}[\mathcal{R}^{(\alpha)}(f)](s,\gamma) = \mathcal{F}_x^{(\alpha)}[f](s\gamma) = \mathcal{F}_t^{(\alpha)}[\mathcal{R}^{(\alpha)}(f)(t,\gamma/|\gamma|)](s|\gamma|,\gamma).
$$

Now, let us define the composition operator $S_{|\gamma|}^{(\alpha)}$ by $S_{|\gamma|}^{(\alpha)}[g(t)] := \mathcal{F}_s^{(\alpha)*}[\mathcal{F}_t^{(\alpha)}[g(t)](s|\gamma|)]$. Then, $\mathcal{R}^{(\alpha)}(f)$ is also rewritten as

$$
\mathcal{R}^{(\alpha)}(f)(t,\gamma) = S_{|\gamma|}^{(\alpha)}\Big[\mathcal{R}^{(\alpha)}(f)\Big(t,\frac{\gamma}{|\gamma|}\Big)\Big](t,\gamma). \tag{18}
$$

Next, we consider the solution to the Schrödinger equation $i\partial_p\psi(p,q) = -\partial_q^2\psi(p,q)$ with $\psi(0,q) = \psi_0(q)$. This solution is given by

$$
\psi(p,q) = \frac{1}{(4\pi ip)^{1/2}} \int \psi_0(\eta)\exp\{\frac{i(q-\eta)^2}{4p}\}d\eta,
$$

where we interpret $\pm i^{1/2}$ as $e^{\pm i\pi/4}$ (see [7]). Noting that $i\partial_p\hat{\psi}(p,s) = s^2\hat{\psi}(p,s)$, we also get

$$
\psi(p,q) = \frac{1}{(2\pi)^{1/2}} \int \hat{\psi}_0(s)\exp\{-is^2p + isq\}ds,
$$

and see that

$$
\int_{\mathbf{R}_s} \hat{\psi}_0(s)\exp\{-is^2p + isq\}ds = \frac{1}{(2ip)^{1/2}} \int_{\mathbf{R}_\eta} \psi_0(\eta)\exp\Big\{\frac{i(q-\eta)^2}{4p}\Big\}d\eta. \tag{19}
$$

Hence, putting $p = \frac{1-|\gamma|^2}{2\tan\alpha}$ and $q = \frac{t}{\sin\alpha}$, we have for $\alpha \neq \pm\pi/2$

$$
\begin{aligned}
& S^{(\alpha)}_{|\gamma|}[g(t)] \\
= \ & \frac{1}{2\pi\sin\alpha}\int_{\mathbf{R}_s}\int_{\mathbf{R}_\tau} g(\tau)\exp\Big\{\frac{i(\tau^2+|s\gamma|^2-s^2-t^2)}{2\tan\alpha}-\frac{i(\tau s|\gamma|-st)}{\sin\alpha}\Big\}d\tau ds \\
= \ & \frac{1}{2\pi\sin\alpha}\int_{\mathbf{R}_s}\int_{\mathbf{R}_\tau} g(\tau)\exp\Big\{\frac{i(\tau^2-t^2)}{2\tan\alpha}-\frac{is|\gamma|\tau}{\sin\alpha}\Big\}d\tau\exp\Big\{\frac{-is^2(1-|\gamma|^2)}{2\tan\alpha}+\frac{ist}{\sin\alpha}\Big\}ds \\
= \ & \frac{|\sin\alpha|}{(2\pi)^{1/2}|\gamma|\sin\alpha}\int_{\mathbf{R}_s}\Big\{\frac{1}{(2\pi)^{1/2}}\int_{\mathbf{R}_\tau} e^{-is\eta}g\Big(\frac{\eta\sin\alpha}{|\gamma|}\Big)\exp\Big\{\frac{i((\frac{\eta\sin\alpha}{|\gamma|})^2-t^2)}{2\tan\alpha}\Big\}d\eta\Big\}\exp\{-is^2p+isq\}ds \\
= \ & \frac{|\sin\alpha|}{(4\pi ip)^{1/2}|\gamma|\sin\alpha}\int_{\mathbf{R}_\eta} g\Big(\frac{\eta\sin\alpha}{|\gamma|}\Big)\exp\Big\{\frac{i((\frac{\eta\sin\alpha}{|\gamma|})^2-t^2)}{2\tan\alpha}\Big\}\exp\Big\{\frac{i(q-\eta)^2}{4p}\Big\}d\eta \\
= \ & \frac{1}{(4\pi ip)^{1/2}\sin\alpha}\int_{\mathbf{R}_\tau} g(\tau)\exp\Big\{\frac{i(\tau^2-t^2)}{2\tan\alpha}\Big\}\exp\Big\{\frac{i(q-\frac{\tau|\gamma|}{\sin\alpha})^2}{4p}\Big\}d\tau \\
= \ & K^{(\alpha)}_\gamma\int_{\mathbf{R}_\tau} g(\tau)\exp\Big\{\frac{i(\tau^2-t^2)}{2\tan\alpha}+\frac{i(|\gamma|\tau-t)^2}{(1-|\gamma|^2)\sin 2\alpha}\Big\}d\tau,
\end{aligned}
$$

and for the special case of $\alpha = \pm\pi/2$

$$
\begin{aligned}
S^{(\pm\pi/2)}_{|\gamma|}[g(t)](t,\gamma) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}_s} e^{\pm ist}\mathcal{F}^{(\pm\pi/2)}_t\Big[g(t)\Big](s|\gamma|)ds \\
= \ \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}_{s'}} & e^{\pm is'(t/|\gamma|)}\mathcal{F}^{(\pm\pi/2)}_t\Big[g(t)\Big](s')\frac{ds'}{|\gamma|} = \frac{1}{|\gamma|}g\Big(\frac{t}{|\gamma|}\Big).
\end{aligned} \tag{20}
$$

In the case of $|\gamma| \neq 1$, we find that

$$
\begin{aligned}
\mathcal{R}^{(\alpha)}(f)(t,\gamma) &= S^{(\alpha)}_{|\gamma|}\Big[\mathcal{R}^{(\alpha)}(f)\Big(t,\frac{\gamma}{|\gamma|}\Big)\Big](t,\gamma) \\
= \ K^{(\alpha)}_\gamma &\int_{\mathbf{R}_\tau}\int_{x\in\mathcal{P}_{\tau,\gamma}} f(x)\exp\Big\{\frac{i(|x|^2-t^2)}{2\tan\alpha}+\frac{i(|\gamma|\tau-t)^2}{(1-|\gamma|^2)\sin 2\alpha}\Big\}dxd\tau \\
= \ K^{(\alpha)}_\gamma &\int_{\mathbf{R}^n_x} f(x)\exp\Big\{\frac{i(|x|^2-t^2)}{2\tan\alpha}+\frac{i(|\gamma|x\cdot\gamma-t)^2}{(1-|\gamma|^2)\sin 2\alpha}\Big\}dx.
\end{aligned} \tag{21}
$$

In the case of $|\gamma| = 1$, noting that $x = t\gamma + y_1\mathbf{e}^\gamma_1 + \cdots + y_{n-1}\mathbf{e}^\gamma_{n-1}$ and $|y|^2 = |x-t\gamma|^2 = |x|^2 - 2tx\cdot\gamma + t^2|\gamma|^2 = |x|^2 - t^2$, we are allowed to write (17) as

$$
\mathcal{R}^{(\alpha)}(f)(t,\gamma) = \mathcal{R}\Big(f(x)\exp\Big\{\frac{i(|x|^2-t^2)}{2\tan\alpha}\Big\}\Big)(t,\gamma). \tag{22}
$$

By (3), we obtain the inversion formula of (22),

$$
f(x) = 2^{-n}\pi^{1-n}\exp\Big\{\frac{-i|x|^2}{2\tan\alpha}\Big\}(-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*\Big(\exp\Big\{\frac{it^2}{2\tan\alpha}\Big\}\mathcal{R}^{(\alpha)}(f)(t,\gamma)\Big)(x). \ \Box
$$

Remark 7. *(15) and (20) give* $\mathcal{R}(f)(t,\gamma) = S^{(\pm\pi/2)}_{|\gamma|}[\mathcal{R}^{(\pm\pi/2)}(f)(t,\gamma/|\gamma|)](t,\gamma)$*, which coincides with (18) for* $\alpha = \pm\pi/2$*.*

The last reconstruction formula is not the one we desire in the sense that it has been described with the dual transform of not $\mathcal{R}^{(\alpha)}(f)$ but $\mathcal{R}(f)$. The inversion formula for $|\gamma| = 1$ had been proved via the αth order fractional dual transform of $\mathcal{R}^{(\alpha)}(f)$ (see [8], [26]). To get an inversion formula for all $|\gamma| \neq 1, 0$, we also use the αth order fractional dual transform of $\mathcal{R}^{(\alpha)}(f)$ and a suitable operator instead of the Laplacian in the next section.
Finally, at the end of this section, let us introduce the result of applying the fractional Radon transform and its reconstruction formula to the pictures. Figure (Original image) is the original sample data, Figures give the images of fractional Radon transform of the sample data, and Figure (Reconstruct image) is the image reconstructed from the αth order fractional Radon transform data with $\alpha = \frac{\pi}{4}$. In each image, the horizontal line represents the angle of γ from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$ and the vertical line represents the parameter t.

Figure 1: Sample Data

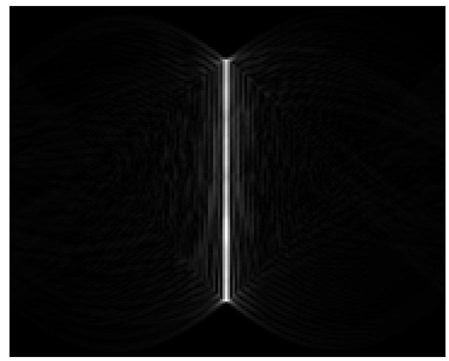

Figure 2: $\alpha = \frac{1}{12}\pi$

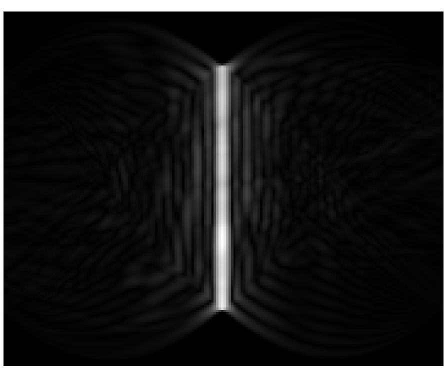

Figure 3: $\alpha = \frac{1}{6}\pi$

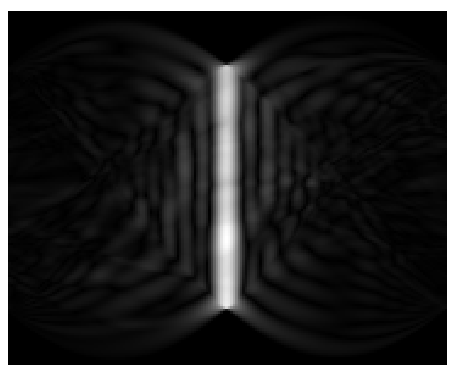

Figure 4: $\alpha = \frac{1}{4}\pi$

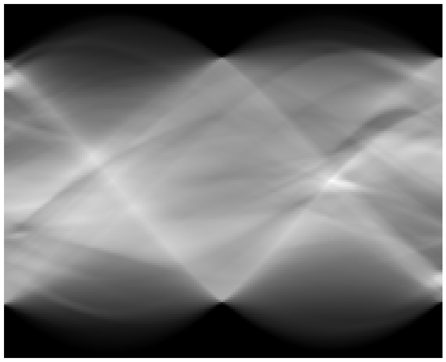

Figure 5: $\alpha = \frac{1}{2}\pi$

Figure 6: Reconstruction of Sample Data

2.3 Dual Transform and Laplace-Bertrami Operator

In this section we discuss the αth order fractional dual Radon transforms for $|\gamma| = 1$ and for $|\gamma| = r > 0$. $\mathcal{F}_x^{(\alpha)*}$ for the relation (10) would not give a reconstruction formulae, since only one direction variable $s\gamma$ is substituted into the right hand side. In fact, the reconstruction formula of some generalized Radon transforms are derived via the dual transforms and the Laplace operator (see, e.g., [18], [21]). In the case of the Funk transform regarded as a generalization of the Radon transform, its dual transform and the Riemann-Liouville fractional derivative operator play an important role in reconstructing the original function (see [32], [33]). Therefore, our strategy for the αth order fractional Radon transform is to find the dual transform and a suitable differential operator.
In the case of $|\gamma| = 1$, by Theorem 13 we note that the $\mathcal{R}^{(\alpha)}(f)$ is represented by the classical Radon transform with δ on $\mathbf{R}$, i.e.,

$$\mathcal{R}^{(\alpha)}(f)(t,\gamma) = \int_{\mathbf{R}^n} \delta(t - \gamma \cdot x) f(x) \exp\{\frac{i(|x|^2 - t^2)}{2\tan\alpha}\} dx.$$

Therefore, we find the dual transform

$$\mathcal{R}^{(\alpha)*}(\varphi)(x) = \int_{S^{n-1}} \exp\Big\{\frac{i((\gamma \cdot x)^2 - |x|^2)}{2\tan\alpha}\Big\} \varphi(\gamma \cdot x, \gamma) d\sigma(\gamma).$$

Indeed, it formally holds that

$$\begin{aligned} \Big(\mathcal{R}^{(\alpha)}(f), \varphi\Big)_{L^2(\mathbf{R}\times S^{n-1})} &= \int_{-\infty}^{\infty} \int_{S^{n-1}} \int_{\mathbf{R}^n} \exp\Big\{ - \frac{i(t^2 - |x|^2)}{2\tan\alpha} \Big\} \delta(t - \gamma \cdot x) f(x) dx \overline{\varphi(t,\gamma)} dt d\sigma(\gamma) \\ &= \int_{\mathbf{R}^n} f(x) \overline{\int_{S^{n-1}} \exp\Big\{\frac{i((\gamma \cdot x)^2 - |x|^2)}{2\tan\alpha}\Big\} \varphi(\gamma \cdot x, \gamma) d\sigma(\gamma)} dx = \Big(f, \mathcal{R}^{(\alpha)*}(\varphi)\Big)_{L^2(\mathbf{R}^n)}. \end{aligned}$$

Define the Laplace-Beltrami operator in the sense of the αth order fractional Fourier transform $\mathcal{F}^{(\alpha)}$ by $-\Delta^{(\alpha)} := \mathcal{F}^{(\alpha)*}|\zeta|^2\mathcal{F}^{(\alpha)}$. More generally, as for pseudo-differential operators in the sense of $\mathcal{F}^{(\alpha)}$, see [29].

Proposition 1. *Let $m \in \mathbf{N}$. Define that*

$$p_m(x,\xi) = (\cos^{2m}\alpha) e^{-i\xi\cdot(x + 2^{-1}\xi\tan\alpha)} (-\Delta_\xi)^m e^{i\xi\cdot(x+2^{-1}\xi\tan\alpha)}. \quad (23)$$

Then, we have $(-\Delta^{(\alpha)})^m = p_m(x, D_x)$*, in particular when* $m = 1$*,*

$$-\Delta^{(\alpha)} = -(\sin^2\alpha)\Delta - i(\sin\alpha\cos\alpha)(2x\cdot\nabla_x + n) + (\cos^2\alpha)|x|^2. \tag{24}$$

Remark 8. *The one-dimensional Laplace-Beltrami operator* $-\Delta^{(\alpha)}$ *has the nonnegative eigenvalues* $\{(kW\sin\alpha)^2\}_{k\in\mathbf{Z}}$ *and the eigenfunctions (6) for the orthonormal basis in* $L^2(-T/2, T/2)$*. [29] considers the 1st order differential operator* $(\partial_x - ix\cot\alpha)$ *to formulate the* α*th order fractional Fourier analysis, and studies a heat type of equation with* $(\partial_x - ix\cot\alpha)\circ(\partial_x - ix\cot\alpha)$ *which coincides with* $\Delta^{(\alpha)}$ *up to a constant additive factor.*

Proof Let us decompose the fractional Fourier transforms as

$$(-\Delta^{(\alpha)})^m = \mathcal{F}^{(\pi/2)*}_{\xi\to x}\Big\{\mathcal{F}^{(\alpha-\pi/2)*}_{\zeta\to\xi}|\zeta|^{2m}\mathcal{F}^{(\alpha-\pi/2)}_{\xi'\to\zeta}\Big\}\mathcal{F}^{(\pi/2)}_{x\to\xi'}.$$

Then, considering the oscillatory integrals, by the change of variables $y = -\zeta/\cos\alpha$ and $\eta = \xi' - \xi$ we obtain

$$\begin{aligned}
&\mathcal{F}^{(\alpha-\pi/2)*}_{\zeta\to\xi}|\zeta|^{2m}\mathcal{F}^{(\alpha-\pi/2)}_{\xi'\to\zeta}\hat{f}(\xi') \\
=\;& \frac{1}{(2\pi)^n\cos^n\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\hat{f}(\xi')|\zeta|^{2m}\exp\Big\{\frac{i(|\xi|^2-|\xi'|^2)}{2\cot\alpha}+\frac{i\zeta\cdot(\xi'-\xi)}{\cos\alpha}\Big\}d\xi' d\zeta \\
=\;& \frac{\cos^{2m}\alpha}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\Big\{e^{-iy\cdot\eta}|y|^{2m}\Big\}\hat{f}(\eta+\xi)\exp\Big\{\frac{i(|\xi|^2-|\eta+\xi|^2)}{2\cot\alpha}\Big\}d\eta dy \\
=\;& \frac{\cos^{2m}\alpha}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}e^{-iy\cdot\eta}(-\Delta_\eta)^m\Big\{\hat{f}(\eta+\xi)\exp\Big\{\frac{i(|\xi|^2-|\eta+\xi|^2)}{2\cot\alpha}\Big\}\Big\}d\eta dy \\
=\;& \cos^{2m}\alpha(-\Delta_\eta)^m\Big\{\hat{f}(\eta+\xi)\exp\Big\{\frac{i(|\xi|^2-|\eta+\xi|^2)}{2\cot\alpha}\Big\}\Big\}\Big|_{\eta=0} \\
=\;& \cos^{2m}\alpha\exp\Big\{\frac{i|\xi|^2}{2\cot\alpha}\Big\}(-\Delta_\xi)^m\Big\{\hat{f}(\xi)\exp\Big\{\frac{-i|\xi|^2}{2\cot\alpha}\Big\}\Big\},
\end{aligned}$$

here we used the properties of oscillatory integrals : $\int\int e^{-iy\cdot\eta}y^\alpha g(y,\eta)d\eta dy = \int\int e^{-iy\cdot\eta}D^\alpha_\eta g(y,\eta)d\eta dy$ and $\frac{1}{(2\pi)^n}\int\int e^{-iy\cdot\eta}h(\eta)d\eta dy = h(0)$ (see, e.g., [23]). Hence, the symbol of $(-\Delta^{(\alpha)})^m$ in the sense of pseudo-differential operator theory is $(\cos^{2m}\alpha)e^{-i\xi\cdot(x+2^{-1}\xi\tan\alpha)}(-\Delta_\xi)^m e^{i\xi\cdot(x+2^{-1}\xi\tan\alpha)}$, indeed

$$\begin{aligned}
&(-\Delta^{(\alpha)})^m f = \frac{\cos^{2m}\alpha}{(2\pi)^{n/2}}\int_{\mathbf{R}^n}e^{ix\cdot\xi}\exp\Big\{\frac{i|\xi|^2}{2\cot\alpha}\Big\}(-\Delta_\xi)^m\Big\{\hat{f}(\xi)\exp\Big\{\frac{-i|\xi|^2}{2\cot\alpha}\Big\}d\xi \\
=\;& \frac{\cos^{2m}\alpha}{(2\pi)^{n/2}}\int_{\mathbf{R}^n}(-\Delta_\xi)^m\Big\{e^{ix\cdot\xi}\exp\Big\{\frac{i|\xi|^2}{2\cot\alpha}\Big\}\Big\}\hat{f}(\xi)\exp\Big\{\frac{-i|\xi|^2}{2\cot\alpha}\Big\}d\xi \\
=\;& \frac{1}{(2\pi)^{n/2}}\int_{\mathbf{R}^n}e^{ix\cdot\xi}\Big\{(\cos^{2m}\alpha)e^{-i\xi\cdot(x+2^{-1}\xi\tan\alpha)}(-\Delta_\xi)^m e^{i\xi\cdot(x+2^{-1}\xi\tan\alpha)}\Big\}\hat{f}(\xi)d\xi.
\end{aligned}$$

In particular the case when $m = 1$ implies (24), since the symbol is given by

$$\begin{aligned}
(\cos^2\alpha)e^{-i\xi\cdot(x+2^{-1}\xi\tan\alpha)}(-\Delta_\xi)e^{i\xi\cdot(x+2^{-1}\xi\tan\alpha)} &= \cos^2\alpha\{|x|^2+|\xi\tan\alpha|^2+2x\cdot\xi\tan\alpha - ni\tan\alpha\} \\
&= (\sin^2\alpha)|\xi|^2+(\sin\alpha\cos\alpha)(2\xi\cdot x - ni)+(\cos^2\alpha)|x|^2.\Box
\end{aligned}$$

2.4 Reconstruction Formula

In this section, we consider the reconstruction formula of the αth order fractional Radon transform. First, we prove the following theorem in the case of $|\gamma| = 1$:

Theorem 9. [11] *Let $n \geq 2$ and $-\pi/2 < \alpha < \pi/2$. The αth order fractional Radon transform for $f \in \mathcal{S}(\mathbf{R}^n)$*

$$\mathcal{R}^{(\alpha)}(f)(t,\gamma) = \mathcal{R}\Big(f(x)\exp\Big\{\frac{i(|x|^2 - t^2)}{2\tan\alpha}\Big\}\Big)(t,\gamma) \quad \Big((t,\gamma) \in \mathbf{R} \times S^{n-1}\Big)$$

and its dual transform for $\varphi \in L^2(\mathbf{R} \times S^{n-1})$

$$\mathcal{R}^{(\alpha)*}(\varphi)(x) = \int_{S^{n-1}} \exp\Big\{\frac{i((\gamma\cdot x)^2 - |x|^2)}{2\tan\alpha}\Big\}\varphi(\gamma\cdot x,\gamma)d\sigma(\gamma) \quad \Big(x \in \mathbf{R}^n\Big)$$

satisfy the reconstruction formula:

$$f(x) = \frac{1}{2(2\pi\sin\alpha)^{n-1}}(-\Delta^{(\alpha)})^{\frac{n-1}{2}}\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x). \tag{25}$$

Proof Noting that $\int_{S^{n-1}} G(\gamma)d\sigma(\gamma) = \int_{S^{n-1}} G(-\gamma)d\sigma(\gamma)$, we have

$$\begin{aligned}
&\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x) = \int_{S^{n-1}} \exp\Big\{\frac{i((\gamma\cdot x)^2 - |x|^2)}{2\tan\alpha}\Big\}\mathcal{R}^{(\alpha)}(f)(\gamma\cdot x,\gamma)d\sigma(\gamma)\\
=\ & \int_{S^{n-1}} \exp\Big\{\frac{i((\gamma\cdot x)^2 - |x|^2)}{2\tan\alpha}\Big\}\\
&\times C_\alpha^{1-n}C_{-\alpha}\int_{\mathbf{R}_\lambda} \exp\Big\{-\frac{i((\gamma\cdot x)^2 + \lambda^2)}{2\tan\alpha} + \frac{i(\gamma\cdot x)\lambda}{\sin\alpha}\Big\}\mathcal{F}^{(\alpha)}[f](\lambda\gamma)d\lambda d\sigma(\gamma)\\
=\ & C_\alpha^{1-n}C_{-\alpha}\int_{S^{n-1}}\int_0^\infty \exp\Big\{-\frac{i(|x|^2 + \lambda^2)}{2\tan\alpha} + \frac{i(\lambda\gamma)\cdot x}{\sin\alpha}\Big\}\mathcal{F}^{(\alpha)}[f](\lambda\gamma)d\lambda d\sigma(\gamma)\\
&+C_\alpha^{1-n}C_{-\alpha}\int_{S^{n-1}}\int_{-\infty}^0 \exp\Big\{-\frac{i(|x|^2 + \lambda^2)}{2\tan\alpha} + \frac{i(\lambda(-\gamma))\cdot x}{\sin\alpha}\Big\}\mathcal{F}^{(\alpha)}[f](\lambda(-\gamma))d\lambda d\sigma(\gamma)\\
=\ & 2C_\alpha^{1-n}C_{-\alpha}\int_{S^{n-1}}\int_0^\infty \exp\Big\{-\frac{i(|x|^2 + \lambda^2)}{2\tan\alpha} + \frac{i(\lambda\gamma)\cdot x}{\sin\alpha}\Big\}\mathcal{F}^{(\alpha)}[f](\lambda\gamma)d\lambda d\sigma(\gamma)\\
=\ & 2C_\alpha^{1-n}C_{-\alpha}\int_{\mathbf{R}^n_\zeta} \exp\Big\{-\frac{i(|x|^2 + |\zeta|^2)}{2\tan\alpha} + \frac{i\zeta\cdot x}{\sin\alpha}\Big\}\mathcal{F}^{(\alpha)}[f](\zeta)|\zeta|^{1-n}d\zeta\\
=\ & 2(2\pi\sin\alpha)^{n-1}\mathcal{F}^{(\alpha)*}\Big[\mathcal{F}^{(\alpha)}[f](\zeta)|\zeta|^{1-n}\Big](x).
\end{aligned}$$

By the inversion formula of the αth fractional Fourier transform, we obtain

$$f = \frac{1}{2(2\pi\sin\alpha)^{n-1}}\mathcal{F}^{(\alpha)*}|\zeta|^{n-1}\mathcal{F}^{(\alpha)}\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x). \ \square$$

Next, we shall consider (13) in the case of $|\gamma| = r(\neq 1)$ to reconstruct $f(x)$. By the operator theory, we find the dual fractional Radon transform

$$\mathcal{R}^{(\alpha)*}(\varphi)(x) = \overline{K_r^{(\alpha)}}\int_{S_r^{n-1}}\int_{\mathbf{R}_t}\varphi(t,\gamma)\exp\{\frac{i(t^2 - |x|^2)}{2\tan\alpha} - \frac{i(rx\cdot\gamma - t)^2}{(1-r^2)\sin 2\alpha}\}dtd\sigma(\gamma)$$

Theorem 10. [11] Let $n \geq 2$, $-\pi/2 < \alpha < \pi/2$, $0 < r \neq 1$ and $K_r^{(\alpha)} = \frac{1}{(i\pi(1-r^2)\sin 2\alpha)^{\frac{1}{2}}}$. The αth order fractional Radon transform for $f \in \mathcal{S}(\mathbf{R}^n)$

$$\mathcal{R}^{(\alpha)}(f)(t,\gamma) = K_r^{(\alpha)} \int_{\mathbf{R}_x^n} f(x) \exp\Big\{\frac{i(|x|^2 - t^2)}{2\tan\alpha} + \frac{i(rx\cdot\gamma - t)^2}{(1-r^2)\sin 2\alpha}\Big\}dx \quad \Big((t,\gamma) \in \mathbf{R} \times S_r^{n-1}\Big),$$

and its dual transform for $\varphi \in L^2(\mathbf{R} \times S_r^{n-1})$

$$\mathcal{R}^{(\alpha)*}(\varphi)(x) = \overline{K_r^{(\alpha)}} \int_{S_r^{n-1}} \int_{\mathbf{R}_t} \varphi(t,\gamma) \exp\Big\{\frac{i(t^2 - |x|^2)}{2\tan\alpha} - \frac{i(rx\cdot\gamma - t)^2}{(1-r^2)\sin 2\alpha}\Big\}dtd\sigma(\gamma) \quad \Big(x \in \mathbf{R}^n\Big),$$

satisfy the reconstruction formula:

$$f(x) = \frac{1}{2r^{n-2}(2\pi\sin\alpha)^{n-1}}(-\Delta^{(\alpha)})^{\frac{n-1}{2}}\Big\{\exp\Big\{\frac{i(r^2-1|x|^2)}{2\tan\alpha}\Big\}\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(\frac{x}{r})\Big\}. \tag{26}$$

Proof From the definitions in case of $|\gamma| = r(\neq 1)$, we have

$$\begin{aligned}
&\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x) \\
= \ & C_\alpha^{1-n} \int_{S_r^{n-1}} \int_{\mathbf{R}_t} \overline{K_r^{(\alpha)}} \mathcal{F}_s^{(\alpha)*}\Big[\mathcal{F}_x^{(\alpha)}[f](s\gamma)\Big](t,\gamma) \exp\Big\{\frac{i(t^2 - |x|^2)}{2\tan\alpha} - \frac{i(rx\cdot\gamma - t)^2}{(1-r^2)\sin 2\alpha}\Big\}dtd\sigma(\gamma) \\
= \ & -\frac{|\sin\alpha|2^{1/2}C_\alpha^{1-n}C_{-\alpha}}{(i(r^2-1)\sin 2\alpha)^{\frac{1}{2}}} \int_{S_r^{n-1}} \exp\Big\{-\frac{i(rx\cdot\gamma)^2}{(1-r^2)\sin 2\alpha}\Big\} \\
&\times \int_{\mathbf{R}_t} \Big\{\frac{1}{(2\pi)^{1/2}} \int_{\mathbf{R}_\eta} e^{-it\eta} \exp\Big\{-\frac{i(|x|^2 + (\eta\sin\alpha)^2)}{2\tan\alpha}\Big\}\mathcal{F}_x^{(\alpha)}[f](-\eta\sin\alpha\gamma)d\eta\Big\} \\
&\times \exp\Big\{-\frac{it^2}{(1-r^2)\sin 2\alpha} + \frac{2itrx\cdot\gamma}{(1-r^2)\sin 2\alpha}\Big\}dtd\sigma(\gamma).
\end{aligned}$$

Putting (19) with $p = \frac{1}{(1-r^2)\sin 2\alpha}$, $q = \frac{2rx\cdot\gamma}{(1-r^2)\sin 2\alpha}$ and t instead of s and similarly to Theorem 9, we obtain

$$\begin{aligned}
\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x) &= \frac{-|\sin\alpha|C_\alpha^{1-n}C_{-\alpha}}{(p(1-r^2)\sin 2\alpha)^{\frac{1}{2}}} \int_{S_r^{n-1}} \exp\Big\{-\frac{i(rx\cdot\gamma)^2}{(1-r^2)\sin 2\alpha}\Big\} \\
&\times \int_{\mathbf{R}_\eta} \exp\Big\{-\frac{i(|x|^2 + (\eta\sin\alpha)^2)}{2\tan\alpha}\Big\}\mathcal{F}_x^{(\alpha)}[f](-\eta\sin\alpha\gamma)\exp\Big\{\frac{i(q-\eta)^2}{4p}\Big\}d\eta d\sigma(\gamma) \\
&= C_\alpha^{1-n}C_{-\alpha} \int_{S_r^{n-1}} \int_{\mathbf{R}_s} \exp\Big\{-\frac{i(|x|^2 + s^2r^2)}{2\tan\alpha} + \frac{isrx\cdot\gamma}{\sin\alpha}\Big\}\mathcal{F}_x^{(\alpha)}[f](s\gamma)dsd\sigma(\gamma) \\
&= 2C_\alpha^{1-n}C_{-\alpha}r^{n-2} \int_{S^{n-1}} \int_0^\infty \exp\Big\{-\frac{i(|x|^2 + \lambda^2)}{2\tan\alpha} + \frac{ir(\lambda\gamma)\cdot x)}{\sin\alpha}\Big\}\mathcal{F}^{(\alpha)}[f](\lambda\gamma)d\lambda d\sigma(\gamma) \\
&= 2C_\alpha^{1-n}C_{-\alpha}r^{n-2} \int_{\mathbf{R}_\zeta^n} \exp\Big\{-\frac{i(|x|^2 + |\zeta|^2)}{2\tan\alpha} + \frac{ir\zeta\cdot x}{\sin\alpha}\Big\}\mathcal{F}^{(\alpha)}[f](\zeta)|\zeta|^{1-n}d\zeta. \qquad (27)
\end{aligned}$$

Hence, it follows that

$$\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x) = 2(2\pi\sin\alpha)^{n-1}r^{n-2}\mathcal{F}^{(\alpha)*}\Big[\mathcal{F}^{(\alpha)}[f](\zeta)\exp\Big\{\frac{i(r-1)\zeta\cdot x}{\sin\alpha}\Big\}|\zeta|^{1-n}\Big](x).$$

Thus, we have

$$f = \frac{1}{2r^{n-2}(2\pi\sin\alpha)^{n-1}}[\mathcal{F}^{(\alpha)*}\{\exp\{\frac{i(1-r)\zeta\cdot x}{\sin\alpha}\}|\zeta|^{n-1}\}\mathcal{F}^{(\alpha)}]\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x).$$

Moreover, by using the dilation operator $M_r g(x) = g(x/r)$, we proceed to change (27) into

$$\begin{aligned}
&\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x)\\
=\ & 2C_\alpha^{1-n}C_{-\alpha}r^{n-2}M_r^{-1}\int_{\mathbf{R}^n_\zeta}\exp\Big\{-\frac{i(r^2|x|^2+|\zeta|^2)}{2\tan\alpha}+\frac{i\zeta\cdot x}{\sin\alpha}\Big\}\mathcal{F}^{(\alpha)}[f](\zeta)|\zeta|^{1-n}d\zeta\\
=\ & 2(C_\alpha C_{-\alpha})^{1-n}r^{n-2}M_r^{-1}\exp\Big\{\frac{i(1-r^2)|x|^2}{2\tan\alpha}\Big\}\mathcal{F}^{(\alpha)*}\Big[\mathcal{F}^{(\alpha)}[f](\zeta)|\zeta|^{1-n}\Big](x),
\end{aligned}$$

and get

$$f = \frac{1}{2r^{n-2}(2\pi\sin\alpha)^{n-1}}\Delta^{(\alpha)\frac{n-1}{2}}\exp\{\frac{i(r^2-1)|x|^2}{2\tan\alpha}\}M_r\mathcal{R}^{(\alpha)*}\mathcal{R}^{(\alpha)}(f)(x).\ \square$$

2.5 Wave Equation

In this section we try to apply the αth order fractional Radon transform to a wave equation. It is well known that the solutions of the wave equation can be consisted by the calssical Radon transform. In fact, as for the classical Radon transform, let us put for $j = 0, 1$

$$\mathcal{W}_n(u_j)(t,\gamma) := \begin{cases} 2^{-1}(2\pi i)^{1-n}\partial_t^{n-1-j}\mathcal{R}(u_j)(t,\gamma) & n \text{ odd},\\ 2^{-1}(2\pi i)^{1-n}\mathcal{H}\Big(\partial_t^{n-1-j}\mathcal{R}(u_j)(t,\gamma)\Big) & n \text{ even},\end{cases}$$

where $\mathcal{H}$ is the Hilbert transform. We see that

$$u(t,x) = \int_{S^{n-1}}\Big\{\mathcal{W}_n(u_0)(x\cdot\gamma+t,\gamma)+\mathcal{W}_n(u_1)(x\cdot\gamma+t,\gamma)\Big\}d\sigma(\gamma)$$

solves the following Cauchy problem:

$$\begin{cases}\partial_t^2 u - \Delta_x u = 0 & \Big((t,x)\in(0,\infty)\times\mathbf{R}^n\Big),\\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x) & \Big(x\in\mathbf{R}^n\Big).\end{cases}$$

For the αth order fractional Radon transform, we also get the following:

Theorem 11. *Define that for $j = 0, 1$*

$$\mathcal{W}_n^{(\alpha)}(u_j)(t,\gamma) := \begin{cases} 2^{-1}(2\pi i)^{1-n}q_{n,j}^{(\alpha)}(t,\partial_t)\mathcal{R}^{(\alpha)}(u_j)(t,\gamma) & n \text{ odd},\\ 2^{-1}(2\pi i)^{1-n}\mathcal{H}\Big(q_{n,j}^{(\alpha)}(t,\partial_t)\mathcal{R}^{(\alpha)}(u_j)(t,\gamma)\Big) & n \text{ even},\end{cases}$$

where $q_{n,j}^{(\alpha)}(x,t,\partial_t) = \exp\{\frac{i(t^2-|x|^2)}{2\tan\alpha}\}(\partial_t^{n-1-j} - [\partial_t^{n-1-j}, \exp\{\frac{-it^2}{2\tan\alpha}\}]\exp\{\frac{it^2}{2\tan\alpha}\})$. Then,

$$u^{(\alpha)}(t,x) = \int_{S^{n-1}}\{\mathcal{W}_n^{(\alpha)}(u_0)(x\cdot\gamma+t,\gamma)+\mathcal{W}_n^{(\alpha)}(u_1)(x\cdot\gamma+t,\gamma)\}d\sigma(\gamma)$$

solves the Cauchy problem

$$\begin{cases}\partial_t^2 u^{(\alpha)} - \frac{1}{\sin^2\alpha}\Delta_x^{(\alpha)}u^{(\alpha)} = 0 & \Big((t,x)\in(0,\infty)\times\mathbf{R}^n\Big),\\ u^{(\alpha)}(0,x) = u_0(x),\ \partial_t u^{(\alpha)}(0,x) = u_1(x) & \Big(x\in\mathbf{R}^n\Big).\end{cases}$$

$\mathcal{R}^{(\alpha)}(f)(t,\gamma)$ is suitable to deal with the multi-dimensional case of the generalized wave equation based on $\Delta_x^{(\alpha)}$ in analogy to the classical Radon transform.
Proof Let $c_n = 2^{-1}(2\pi i)^{1-n}$ and put $v_j(x) := \exp\left\{\frac{i|x|^2}{2\tan\alpha}\right\}u_j(x)$. By (14) we find that when n is odd

$$\begin{aligned}\mathcal{W}_n^{(\alpha)}(u_j)(t,\gamma) &= c_n q_{n,j}^{(\alpha)}(x,t,\partial_t)\mathcal{R}\Big(u_j(x)\exp\left\{\frac{i(|x|^2-t^2)}{2\tan\alpha}\right\}\Big)(t,\gamma)\\ &= c_n q_{n,j}^{(\alpha)}(x,t,\partial_t)\Big(\exp\left\{\frac{-it^2}{2\tan\alpha}\right\}\mathcal{R}(v_j)(t,\gamma)\Big) = \exp\left\{-\frac{i|x|^2}{2\tan\alpha}\right\}\mathcal{W}_n(v_j)(t,\gamma),\end{aligned}$$

and when n is even $\mathcal{W}_n^{(\alpha)}(u_j)(t,\gamma) = \exp\{-\frac{i|x|^2}{2\tan\alpha}\}\mathcal{W}_n(v_j)(t,\gamma)$, here we remark that $[\partial_t^{n-1-j}, \exp\left\{\frac{-it^2}{2\tan\alpha}\right\}]$ in $q_{n,j}^{(\alpha)}$ denotes the commutator of the differential operator ∂_t^{n-1-j} and the multiplication operator $\exp\{\frac{-it^2}{2\tan\alpha}\}$. Thus, we see that

$$v^{(\alpha)}(t,x) = \exp\left\{\frac{i|x|^2}{2\tan\alpha}\right\}\int_{S^{n-1}}\Big\{\mathcal{W}_n^{(\alpha)}(u_0)(x\cdot\gamma+t,\gamma)+\mathcal{W}_n^{(\alpha)}(u_1)(x\cdot\gamma+t,\gamma)\Big\}d\sigma(\gamma)$$

solves

$$\begin{cases}\partial_t^2 v^{(\alpha)} - \Delta_x^{(\alpha)} v^{(\alpha)} = 0 & \Big((t,x)\in(0,\infty)\times\mathbf{R}^n\Big),\\ v^{(\alpha)}(0,x) = v_0(x),\ \partial_t v^{(\alpha)}(0,x) = v_1(x) & \Big(x\in\mathbf{R}^n\Big).\end{cases}$$

Hence, putting $u^{(\alpha)}(t,x) = \exp\{-\frac{i|x|^2}{2\tan\alpha}\}v^{(\alpha)}(t,x)$, by (24) we get

$$\begin{aligned}\Delta_x v^{(\alpha)}(t,x) &= \Delta_x(\exp\{\frac{i|x|^2}{2\tan\alpha}\}u^{(\alpha)}(t,x)) = \exp\{\frac{i|x|^2}{2\tan\alpha}\}(\Delta_x + \frac{2ix}{\tan\alpha}\cdot\nabla_x + \frac{in}{\tan\alpha} - \frac{|x|^2}{\tan^2\alpha})u^{(\alpha)}(t,x)\\ &= \exp\{\frac{i|x|^2}{2\tan\alpha}\}\frac{1}{\sin^2\alpha}\Delta_x^{(\alpha)}u^{(\alpha)}(t,x).\end{aligned}$$

This proves Theorem 2.7. □

Remark 12. *Theorem 11 characterizes the αth order fractional Radon transform $\mathcal{R}^{(\alpha)}(f)(t,\gamma)$. Also for a generalized heat equation based on $(\partial_x - ix\cot\alpha)\circ(\partial_x - ix\cot\alpha)\sim\Delta_x^{(\alpha)}$, the fundamental solution is given by [29].*

3 Known Results of Various Transforms

In this section, for application to images we concern two-dimenssional transforms (including dilation, translation and rotation). We shall use letters in bold font to denote two-dimensional vectors. Let

$$a\in\mathbf{R}\backslash\{0\},\ \mathbf{a}=(a_1,a_2)\in\mathbf{R}^2\backslash\{0\},\ b\in\mathbf{R},\ \mathbf{b}=(b_1,b_2)\in\mathbf{R}^2,\ \omega\in(-\pi,\pi],$$

and put

$$\boldsymbol{\gamma}_\omega = (\cos\omega,\sin\omega),\quad \boldsymbol{\gamma}_\omega^\perp = \boldsymbol{\gamma}_{\omega-\pi/2} = (\sin\omega,-\cos\omega),\quad R_\omega = \begin{pmatrix}\cos\omega & -\sin\omega\\ \sin\omega & \cos\omega\end{pmatrix}.$$

• **Wavelet transform** The continuous wavelet transform is given by

$$\mathcal{W}_\Psi f(\mathbf{b},a,\omega) := \frac{1}{|a|}\int_{\mathbf{R}^2_{\mathbf{x}}}\overline{\Psi\Big(R_\omega\Big(\frac{\mathbf{x}-\mathbf{b}}{a}\Big)\Big)}f(\mathbf{x})d\mathbf{x}$$

which is useful for representing point singularities (but not so efficient for detecting line singularities). The continuous wavelet transform was pioneered by Morlet and Grossmann [17]. Recently like the Fourier transform, it is one of the important tools in time-frequency analysis.

• **Ridgelet transform** Candés [3] and Donoho [6] proposed the ridgelet transform

$$\mathcal{R}_\psi f(b,a,\omega) := \frac{1}{|a|^{1/2}} \int_{\mathbf{R}^2_{\mathbf{x}}} \overline{\psi\Big(\frac{\gamma_\omega \cdot \mathbf{x} - b}{a}\Big)} f(\mathbf{x}) d\mathbf{x},$$

which can be obtained by the composition of the one-dimensional wavelet transform and the X-ray transform (Radon transform), and extracts directional features for straight-line singularities.

• **Windowed X-ray transform (windowed Radon transform)** Kaiser [21] proposed the windowed X-ray transform for the relativistic quantum theory. Afterwards it realized a generalization of the Analytic-Signal transform (see also [22], [35]).

$$\mathcal{WX}_\psi f(\mathbf{x},\mathbf{v}) = \int_{\mathbf{R}} \overline{\psi(p)} f(\mathbf{x}+p\mathbf{v}) dp.$$

By the Parseval's formula we find that the windowed X-ray transform (windowed Radon transform) represents with the kernel:

$$\mathcal{WX}_\psi f(\mathbf{x},\mathbf{v}) = \int_{\mathbf{R}^2_{\mathbf{b}}} \overline{K_\psi(\mathbf{b}-\mathbf{x},\mathbf{v})} f(\mathbf{b}) d\mathbf{b},$$

where the integral kernel as follows:

$$K_\psi(\mathbf{b}-\mathbf{x},\mathbf{v}) := \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i(\mathbf{b}-\mathbf{x})\cdot\boldsymbol{\xi}} \hat{\psi}(\mathbf{v}\cdot\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

Moreover we give the inversion formulas of these transforms. The following condition plays a central role to the inversion formulae:

Admissibility condition Let be $\psi \in L^2(\mathbf{R}^1)$. We say ψ satisfies the admissibility condition if

$$C_\psi = \int_{\mathbf{R}} \frac{|\hat{\psi}(s)|^2}{|s|} ds$$

converges.

• **Inversion formula of the wavelet transform** If ψ is admissible, for $f \in L^2(\mathbf{R}^2)$, the inversion formula of the wavelet transform is given by

$$C_\psi f(\mathbf{x}) = \int_{\mathbf{R}^1_a} \int_{\mathbf{R}^2} \psi^{(a,\mathbf{b})}(t) \mathcal{W}_\psi f(\mathbf{b},a) \frac{da d\mathbf{b}}{a^2}. \tag{28}$$

In 2014, Lebedeva and Postnikov [24] gave another inverse formula of the one-dimensional wavelet transform when $C_\psi := -\overline{\psi(0)} \neq 0$ and $\psi \in L^2(\mathbf{R})$:

$$C_\psi f(q) \;\; = \;\; \mathcal{H}_b\Big[\int_{\mathbf{R}_a} \partial_b \mathcal{W}_\psi f(b,a) |a|^{-1/2} da\Big](q). \tag{29}$$

• **Inversion formula of the ridgelet transform** For $f \in L^2(\mathbf{R}^2)$, the inversion formula of the ridgelet transform is given by

$$D_\psi f(\mathbf{x}) = \int_{S^1} \int_{\mathbf{R}} \int_{\mathbf{R}} \mathcal{R}_\psi f(b,a,\omega) \psi\Big(\frac{\gamma_\omega \cdot \mathbf{x} - b}{a}\Big) \frac{da d\omega}{|a|^3},$$

if a constant D_ψ exists:

$$D_\psi = 2(2\pi)\int_{\mathbf{R}} \frac{|\hat{\psi}(s)|^2}{|s|^2} ds.$$

• **Inversion formula of the windowed X-ray transform** If ψ is admissible, the inversion formula of the windowed X-ray transform is given by

$$C_\psi f(\mathbf{x}) = \int_{\mathbf{R}^2_{\mathbf{v}}}\int_{\mathbf{R}^2_{\mathbf{y}}} \mathcal{WX}_\psi f(\mathbf{y},\mathbf{v}) K_\psi(\mathbf{x}-\mathbf{y},\mathbf{v}) \frac{d\mathbf{y}d\mathbf{v}}{|\mathbf{v}|^2}. \tag{30}$$

4 Double Ridgelet Transform

In this section we propose a generalization of the ridgelet transform.
We consider the composition of the one-dimensional wavelet transform and the windowed X-ray transform.

$$\mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b},\mathbf{a},\omega) = \frac{1}{|a_1|^{1/2}} \int_{\mathbf{R}_q} \overline{\psi\Big(\frac{q-b}{a_1}\Big)} |a_2|^{1/2} \mathcal{WX}_{\tilde{\psi}} f(q\boldsymbol{\gamma}_\theta, a_2, \omega) dq. \tag{31}$$

In [12], (31) is called "**double windowed ridgelet transform**". The following theorem means that (31) has the form with the kernel of double windows.

Theorem 13. [12] *Let $\psi, \tilde{\psi} \in L^2(\mathbf{R}^1)$ and $\boldsymbol{\gamma}_\theta = (\cos\theta, \sin\theta)$, $\boldsymbol{\gamma}_\omega^\perp = (\sin\omega, -\cos\omega)$. Then, for $f \in L^2(\mathbf{R}^2_{\mathbf{x}})$ the double windowed ridgelet transform defined by (31) is represented as*

$$\begin{aligned}
\mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b},\mathbf{a},\omega) &= \frac{|a_1 a_2|^{1/2}}{(2\pi)^2} \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}} \overline{\hat{\psi}(a_1\boldsymbol{\gamma}_\theta\cdot\boldsymbol{\xi}) \hat{\tilde{\psi}}(a_2\boldsymbol{\gamma}_\omega^\perp\cdot\boldsymbol{\xi})} \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi} \\
&= \frac{1}{|a_1 a_2|^{1/2}\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta} \int_{\mathbf{R}^2_{\mathbf{x}}} \overline{\psi\Big(\frac{\boldsymbol{\gamma}_\omega\cdot(\mathbf{x}-\mathbf{b})}{a_1\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta}\Big) \tilde{\psi}\Big(\frac{\boldsymbol{\gamma}_\theta^\perp\cdot\mathbf{x}}{a_2\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta}\Big)} f(\mathbf{x}) d\mathbf{x},
\end{aligned}$$

where $\mathbf{b} := b\boldsymbol{\gamma}_\theta$ with $b \in \mathbf{R}$ and $\theta \in (-\pi/2, \pi/2]$, $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2\backslash\{0\}$ and $\omega \in (-\pi, \pi]$ such that $\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta \neq 0$.

Remark 14. *In particular when $\mathbf{b} = \mathbf{b}_\omega := b\boldsymbol{\gamma}_\omega$ with $\theta = \omega$, we have*

$$\mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}_\omega,\mathbf{a},\omega) = \frac{1}{|a_1 a_2|^{1/2}} \int_{\mathbf{R}^2_{\mathbf{x}}} \overline{\psi\Big(\frac{\boldsymbol{\gamma}_\omega\cdot\mathbf{x}-b}{a_1}\Big) \tilde{\psi}\Big(\frac{\boldsymbol{\gamma}_\omega^\perp\cdot\mathbf{x}}{a_2}\Big)} f(\mathbf{x}) d\mathbf{x},$$

which corresponds to the ridgelet transform, if $\tilde{\psi} \equiv 1$.

Proof We change the windowed X-ray transform (windowed Radon transform) with the additional coefficient $|a|^{1/2}$ as

$$\begin{aligned}
|a|^{1/2}\mathcal{WX}_\psi f(\mathbf{b}, a, \omega) &= |a|^{1/2} \int_{\mathbf{R}_p} \overline{\psi(p)} f\Big(\mathbf{b} + ap(\sin\omega, -\cos\omega)\Big) dp \\
&= \frac{|a|^{1/2}}{(2\pi)^2} \int_{\mathbf{R}_p} \overline{\psi(p)} \int_{\mathbf{R}^2_\xi} e^{i\mathbf{b}\cdot\xi} e^{iap\boldsymbol{\gamma}_\omega^\perp\cdot\boldsymbol{\xi}} \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi} dp \\
&= \frac{|a|^{1/2}}{(2\pi)^2} \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}} \overline{\hat{\psi}(a\boldsymbol{\gamma}_\omega^\perp\cdot\boldsymbol{\xi})} \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}.
\end{aligned} \tag{32}$$

Substituting this windowed X-ray transform with $\tilde{\psi}$ into (31), we get

$$\begin{aligned}
\mathcal{WR}_{\psi,\tilde{\psi}}f(\mathbf{b},\mathbf{a},\omega) &= \frac{|a_2|^{1/2}}{(2\pi)^2\sqrt{|a_1|}}\int_{\mathbf{R}_q}\overline{\psi\Big(\frac{q-b}{a_1}\Big)}\Big\{\int_{\mathbf{R}^2_\xi} e^{iq\gamma_\theta\cdot\boldsymbol{\xi}}\overline{\hat{\tilde{\psi}}(a_2\gamma_\omega^\perp\cdot\boldsymbol{\xi})}\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi}\Big\}dq \\
&= \frac{|a_2|^{1/2}}{(2\pi)^2\sqrt{|a_1|}}\int_{\mathbf{R}^2_{\boldsymbol{\xi}}}\Big\{\overline{\int_{\mathbf{R}_q} e^{-iq\gamma_\theta\cdot\boldsymbol{\xi}}\psi\Big(\frac{q-b}{a_1}\Big)dq}\Big\}\overline{\hat{\tilde{\psi}}(a_2\gamma_\omega^\perp\cdot\boldsymbol{\xi})}\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi} \\
&= \frac{|a_1a_2|^{1/2}}{(2\pi)^2}\int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}}\overline{\hat{\psi}(a_1\gamma_\theta\cdot\boldsymbol{\xi})\hat{\tilde{\psi}}(a_2\gamma_\omega^\perp\cdot\boldsymbol{\xi})}\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi}.
\end{aligned}$$

Putting $\hat{\Psi}_{\psi,\tilde{\psi}}(\boldsymbol{\xi}) := \hat{\psi}(\xi_1)\otimes\hat{\tilde{\psi}}(\xi_2)$, $G := \begin{pmatrix}\gamma_\theta \\ \gamma_\omega^\perp\end{pmatrix}$ and $A := \begin{pmatrix} a_1 & 0 \\ 0 & a_2\end{pmatrix}$, we see that

$$\hat{\Psi}_{\psi,\psi'}(AG\boldsymbol{\xi}) = \hat{\psi}(a_1\gamma_\theta\cdot\boldsymbol{\xi})\hat{\tilde{\psi}}(a_2\gamma_\omega^\perp\cdot\boldsymbol{\xi}).$$

Therefore, it follows that

$$\begin{aligned}
\mathcal{WR}_{\psi,\psi'}f(\mathbf{b},\mathbf{a},\omega) &= \frac{|a_1a_2|^{1/2}}{(2\pi)^2}\int_{\mathbf{R}^2_\xi} e^{i\mathbf{b}\cdot\boldsymbol{\xi}}\overline{\hat{\Psi}_{\psi,\tilde{\psi}}(AG\boldsymbol{\xi})}\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi} \\
&= \frac{|a_1a_2|^{1/2}}{(2\pi)^2}\int_{\mathbf{R}^2_\xi}\overline{\int_{\mathbf{R}^2_\mathbf{y}} e^{-i(G^*A^*\mathbf{y}+\mathbf{b})\cdot\boldsymbol{\xi}}\Psi_{\psi,\tilde{\psi}}(\mathbf{y})d\mathbf{y}}\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^2|a_1a_1|^{1/2}\gamma_\omega\cdot\gamma_\theta}\int_{\mathbf{R}^2_\xi}\overline{\int_{\mathbf{R}^2_\mathbf{x}} e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\Psi_{\psi,\tilde{\psi}}\Big(A^{*-1}G^{*-1}(\mathbf{x}-\mathbf{b})\Big)d\mathbf{x}}\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi},
\end{aligned}$$

since

$$G^{*-1} = \begin{pmatrix}\cos\theta & \sin\omega \\ \sin\theta & -\cos\omega\end{pmatrix}^{-1} = \frac{1}{\cos(\omega-\theta)}\begin{pmatrix}\cos\omega & \sin\omega \\ \sin\theta & -\cos\theta\end{pmatrix} = \frac{1}{\gamma_\omega\cdot\gamma_\theta}\begin{pmatrix}\gamma_\omega \\ \gamma_\theta^\perp\end{pmatrix}.$$

Noting that $\Psi_{\psi,\tilde{\psi}}(\mathbf{x}) = \psi(x_1)\otimes\tilde{\psi}(x_2)$, we have

$$\begin{aligned}
&\mathcal{WR}_{\psi,\psi'}f(\mathbf{b},\mathbf{a},\omega) \\
&= \frac{1}{|a_1a_2|^{1/2}\gamma_\omega\cdot\gamma_\theta}\int_{\mathbf{R}^2_\mathbf{x}}\overline{\Psi_{\psi,\tilde{\psi}}\Big(A^{*-1}G^{*-1}(\mathbf{x}-\mathbf{b})\Big)}f(\mathbf{x})d\mathbf{x} \\
&= \frac{1}{|a_1a_2|^{1/2}\gamma_\omega\cdot\gamma_\theta}\int_{\mathbf{R}^2_\mathbf{x}}\overline{\Psi_{\psi,\tilde{\psi}}\Big(\frac{1}{\gamma_\omega\cdot\gamma_\theta}A^{*-1}\Big\{\begin{pmatrix}\gamma_\omega\cdot\mathbf{x} \\ \gamma_\theta^\perp\cdot\mathbf{x}\end{pmatrix} - \begin{pmatrix}\gamma_\omega\cdot\mathbf{b} \\ 0\end{pmatrix}\Big\}\Big)}f(\mathbf{x})d\mathbf{x} \\
&= \frac{1}{|a_1a_2|^{1/2}\gamma_\omega\cdot\gamma_\theta}\int_{\mathbf{R}^2_\mathbf{x}}\overline{\psi\Big(\frac{\gamma_\omega\cdot(\mathbf{x}-\mathbf{b})}{a_1\gamma_\omega\cdot\gamma_\theta}\Big)\tilde{\psi}\Big(\frac{\gamma_\theta^\perp\cdot\mathbf{x}}{a_2\gamma_\omega\cdot\gamma_\theta}\Big)}f(\mathbf{x})d\mathbf{x}.\ \Box
\end{aligned}$$

4.1 Features in the Frequency Space $\mathbf{R}^2_\xi$

From a point of view of the partition in the frequency space $\mathbf{R}^2_\xi$, we study the features of the double windowed ridgelet transform and other transforms.

• **Double windowed ridgelet transform**

$$\mathcal{WR}_{\psi,\tilde{\psi}}f(\mathbf{b},\mathbf{a},\omega) = \frac{|a_1a_2|^{1/2}}{(2\pi)^2}\int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}}\overline{\hat{\psi}(a_1\gamma_\theta\cdot\boldsymbol{\xi})\hat{\tilde{\psi}}(a_2\gamma_\omega^\perp\cdot\boldsymbol{\xi})}\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi}$$

has the window $\hat{\psi}(a_1\boldsymbol{\gamma}_\theta \cdot \boldsymbol{\xi})$ rotating according to $\mathbf{b} := b\boldsymbol{\gamma}_\theta$ and another window $\hat{\tilde{\psi}}(a_2\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi})$ moving independently.

(A) Both ψ_A and $\tilde{\psi}_A$ are mother wavelets.
(B) ψ_B is a father wavelet (scaling function) and $\tilde{\psi}_B$ is a mother wavelet.
(C) ψ_C is a mother wavelet and $\tilde{\psi}_C$ is a father wavelet (scaling function).
(D) Both ψ_D and $\tilde{\psi}_D$ are father wavelets (scaling functions).

By the Parseval's theorem and the Fourier slice theorem we also get the representations of other transforms:

- **Wavelet transform**

$$\mathcal{W}_\Psi f(\mathbf{b}, a, \omega) := \frac{|a|}{(2\pi)^2} \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}} \overline{\hat{\Psi}(aR_\omega\boldsymbol{\xi})} \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

- **X-ray transform (Radon transform)**

$$\mathcal{X} f(b, \theta) = \frac{1}{2\pi} \int_{\mathbf{R}_s} e^{ibs} \hat{f}(s\boldsymbol{\gamma}_\theta) ds.$$

- **Windowed X-ray transform (windowed Radon transform)**

$$\mathcal{W}\mathcal{X}_\psi f(\mathbf{b}, a, \omega) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}} \overline{\hat{\psi}(a\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi})} \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

- **Ridgelet transform**

$$\mathcal{R}_\psi f(b, a, \omega) = \frac{1}{|a|^{1/2}} \int_{\mathbf{R}_q} \overline{\psi\Big(\frac{q-b}{a}\Big)} \mathcal{X} f(q, \omega) dq = \frac{|a|^{1/2}}{2\pi} \int_{\mathbf{R}_s} e^{ibs} \overline{\hat{\psi}(as)} \hat{f}(s\boldsymbol{\gamma}_\omega) ds.$$

4.2 Reconstruction Formula

In this section we consider a reconstruction formulae of the double windowed ridgelet transform. Moreover, we discuss the transforms $\mathcal{W}\mathcal{R}_{\psi,\tilde{\psi}} f$ which have the windows of the four types.

4.2.1 Case (A)

Since we have inserted the additional coefficient $|a|^{1/2}$ to $\mathcal{W}\mathcal{X}_\psi f$, considering the additional coefficient $\frac{1}{2\pi|a|^{1/2}}$, we get the inversion formula of the windowed X-ray transform:

$$C_{\tilde{\psi}} f(\mathbf{x}) = \frac{1}{2\pi} \int_{S_\omega} \int_{-\pi/2}^{\pi/2} \int_{\mathbf{R}_a} \int_{\mathbf{R}_q} K_{\tilde{\psi}}(\mathbf{x} - q\boldsymbol{\gamma}_\theta, a, \omega) \mathcal{W}\mathcal{X}_\psi f(q\boldsymbol{\gamma}_\theta, a, \omega) \frac{q dq da d\omega d\theta}{|a|^{3/2}}, \tag{33}$$

here we used the polar coordinates $\mathbf{v} = |a|\boldsymbol{\gamma}_\omega^\perp \in \mathbf{R}^2$ with $a \in \mathbf{R}$ and $\omega \in (-\pi, \pi]$ and $q\boldsymbol{\gamma}_\theta$ with $q \in \mathbf{R}_q$ and $\theta \in (-\pi/2, \pi/2]$ in (30).

Remark 15. *The integration in* $\mathbf{v}$ *has been replaced by the integrations in* a *and* ω. *But in fact, the integration in* ω *yields only the constant* 2π. *So, we can reduce (30) to*

$$C_{\tilde{\psi}} f(\mathbf{x}) = \int_{-\pi/2}^{\pi/2} \int_{\mathbf{R}_a} \int_{\mathbf{R}_q} K_{\tilde{\psi}}(\mathbf{x} - q\boldsymbol{\gamma}_\theta, a, \omega) \mathcal{WX}_\psi f(q\boldsymbol{\gamma}_\theta, a, \omega) \frac{q dq da d\theta}{|a|^{3/2}}. \tag{34}$$

Then, we prove the following:

Proposition 2. [12] *Let* $\psi, \tilde{\psi} \in L^2(\mathbf{R})$ *satisfying*

$$C_\psi := \int_{\mathbf{R}} \frac{|\hat{\psi}(s)|^2}{|s|} ds < \infty \quad \text{and} \quad C_{\tilde{\psi}} := \int_{\mathbf{R}} \frac{|\hat{\tilde{\psi}}(s)|^2}{|s|} ds < \infty,$$

and $\boldsymbol{\gamma}_\theta = (\cos\theta, \sin\theta)$, $\boldsymbol{\gamma}_\omega^\perp = (\sin\omega, -\cos\omega)$. *The inverse formula of the double windowed ridgelet transform* $\mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega)$ *for* $\mathbf{b} := b\boldsymbol{\gamma}_\theta$ *with* $b \in \mathbf{R}$ *and* $\theta \in (-\pi/2, \pi/2]$, $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2 \backslash \{0\}$ *and* $\omega \in (-\pi, \pi]$ *such that* $\boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta \neq 0$, *is given by*

$$C_\psi C_{\tilde{\psi}} f(\mathbf{x}) = \int_{\mathbf{R}^2_{\mathbf{a}}} \int_{\mathbf{R}^2_{\mathbf{b}}} \tilde{\psi}\Big(\frac{\boldsymbol{\gamma}_\theta^\perp \cdot \mathbf{x}}{a_2 \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta}\Big) \psi\Big(\frac{\boldsymbol{\gamma}_\omega \cdot (\mathbf{x} - \mathbf{b})}{a_1 \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta}\Big) \mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega) \frac{\boldsymbol{\gamma}_\omega \cdot \mathbf{x} \, d\mathbf{b} d\mathbf{a}}{|a_1 a_2|^{5/2} (\boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta)^2}.$$

Remark 16. *Indeed, by (28), (34), we can easily verify that (see, [12])*

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} \int_{\mathbf{R}_{a_2}} \int_{\mathbf{R}_q} K_{\tilde{\psi}}(\mathbf{x} - q\boldsymbol{\gamma}_\theta, a_2, \omega) \int_{\mathbf{R}_{a_1}} \int_{\mathbf{R}_b} \psi\Big(\frac{q - b}{a_1}\Big) \mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega) \frac{q db da_1 dq da_2 d\theta}{|a_1|^{5/2} |a_2|^{3/2}} \\ = \ & C_\psi C_{\tilde{\psi}} f(\mathbf{x}). \end{aligned} \tag{35}$$

Proof of Proposition 2 Let us simplify the inverse formula (35) as

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} \int_{\mathbf{R}_{a_2}} \int_{\mathbf{R}_q} K_{\tilde{\psi}}(\mathbf{x} - q\boldsymbol{\gamma}_\theta, a_2, \omega) \int_{\mathbf{R}_{a_1}} \int_{\mathbf{R}_b} \psi\Big(\frac{q - b}{a_1}\Big) \mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega) \frac{q db da_1 dq da_2 d\theta}{|a_1|^{5/2} |a_2|^{3/2}} \\ = & \int_{\mathbf{R}^2_{\mathbf{a}}} \int_{\mathbf{R}^2_{\mathbf{b}}} \int_{\mathbf{R}^2_{\boldsymbol{\eta}}} e^{i(\mathbf{x} - b\boldsymbol{\gamma}_\theta) \cdot \boldsymbol{\eta}} \hat{\tilde{\psi}}(a_2 \boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\eta}) \Big\{ b\hat{\psi}(a_1 \boldsymbol{\gamma}_\theta \cdot \boldsymbol{\eta}) + a_1 (\hat{p\psi})(a_1 \boldsymbol{\gamma}_\theta \cdot \boldsymbol{\eta}) \Big\} d\boldsymbol{\eta} \\ & \times \mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega) \frac{d\mathbf{b} d\mathbf{a}}{|a_1 a_2|^{3/2}} \\ = & \int_{\mathbf{R}^2_{\mathbf{a}}} \int_{\mathbf{R}^2_{\mathbf{b}}} \frac{(2\pi)^2}{|a_1 a_2| \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta} \tilde{\psi}\Big(\frac{\boldsymbol{\gamma}_\theta^\perp \cdot \mathbf{x}}{a_2 \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta}\Big) \Big\{ b\psi\Big(\frac{\boldsymbol{\gamma}_\omega \cdot (\mathbf{x} - \mathbf{b})}{a_1 \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta}\Big) + a_1 \frac{\boldsymbol{\gamma}_\omega \cdot (\mathbf{x} - \mathbf{b})}{a_1 \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta} \psi\Big(\frac{\boldsymbol{\gamma}_\omega \cdot (\mathbf{x} - \mathbf{b})}{a_1 \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta}\Big) \Big\} \\ & \times \mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega) \frac{d\mathbf{b} d\mathbf{a}}{|a_1 a_2|^{3/2}} \\ = & \int_{\mathbf{R}^2_{\mathbf{a}}} \int_{\mathbf{R}^2_{\mathbf{b}}} \tilde{\psi}\Big(\frac{\boldsymbol{\gamma}_\theta^\perp \cdot \mathbf{x}}{a_2 \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta}\Big) \psi\Big(\frac{\boldsymbol{\gamma}_\omega \cdot (\mathbf{x} - \mathbf{b})}{a_1 \boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta}\Big) \mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega) \frac{\boldsymbol{\gamma}_\omega \cdot \mathbf{x} \, d\mathbf{b} d\mathbf{a}}{|a_1 a_2|^{5/2} (\boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta)^2}, \end{aligned}$$

here we used

$$\begin{aligned} \int_{\mathbf{R}_q} \psi\Big(\frac{q - b}{a_1}\Big) e^{-iq\boldsymbol{\gamma}_\theta \cdot \boldsymbol{\eta}} q dq &= |a_1| e^{-ib\boldsymbol{\gamma}_\theta \cdot \boldsymbol{\eta}} \int_{\mathbf{R}_p} \psi(p) e^{-ipa_1 \boldsymbol{\gamma}_\theta \cdot \boldsymbol{\eta}} (b + a_1 p) dp \\ &= |a_1| e^{-ib\boldsymbol{\gamma}_\theta \cdot \boldsymbol{\eta}} \Big\{ b\hat{\psi}(a_1 \boldsymbol{\gamma}_\theta \cdot \boldsymbol{\eta}) + a_1 (\hat{p\psi})(a_1 \boldsymbol{\gamma}_\theta \cdot \boldsymbol{\eta}) \Big\} \end{aligned}$$

and

$$
\begin{aligned}
& a_1 \int_{\mathbf{R}^2_{\boldsymbol{\eta}}} e^{i(\mathbf{x}-b\boldsymbol{\gamma}_\theta)\cdot\boldsymbol{\eta}} \hat{\tilde{\psi}}(a_2\boldsymbol{\gamma}^\perp_\omega\cdot\boldsymbol{\eta})(\widehat{p\psi})(a_1\boldsymbol{\gamma}_\theta\cdot\boldsymbol{\eta})d\boldsymbol{\eta} \\
= & \frac{(2\pi)^2 a_1}{|a_1a_2|\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta} \frac{\boldsymbol{\gamma}_\omega\cdot(\mathbf{x}-\mathbf{b})}{a_1\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta} \psi\Big(\frac{\boldsymbol{\gamma}_\omega\cdot(\mathbf{x}-\mathbf{b})}{a_1\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta}\Big)\tilde{\psi}\Big(\frac{\boldsymbol{\gamma}^\perp_\theta\cdot\mathbf{x}}{a_2\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta}\Big) \\
= & \frac{(2\pi)^2}{|a_1a_2|\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta}\Big\{\frac{\boldsymbol{\gamma}_\omega\cdot\mathbf{x}}{\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta} - b\Big\}\psi\Big(\frac{\boldsymbol{\gamma}_\omega\cdot(\mathbf{x}-\mathbf{b})}{a_1\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta}\Big)\tilde{\psi}\Big(\frac{\boldsymbol{\gamma}^\perp_\theta\cdot\mathbf{x}}{a_2\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta}\Big).
\end{aligned}
$$

This proves Proposition 2. □

4.2.2 Case (B)

In this case, the mother wavelet $\widetilde{\psi}_B$ satisfies the admissible condition and the father wavelet ψ_B is not equal to zero at the origin. We can rewrite (29) by:

$$
(29) = \int_{\mathbf{R}_a} |D_q| \mathcal{W}_\psi f(q,a)|a|^{-1/2} da, \tag{36}
$$

where

$$
|D_b|g(b) = \frac{1}{2\pi}\int_{\mathbf{R}_\lambda} e^{i\lambda b}|\lambda|\hat{g}(\lambda)d\lambda.
$$

Then, we can also prove the following:

Proposition 3. [12] *Let $\psi, \psi', \tilde{\psi} \in L^2(\mathbf{R})$ satisfying*

$$
C_\psi := -\overline{\psi(0)} \neq 0 \quad \text{and} \quad C_{\tilde{\psi}} := \int_{\mathbf{R}} \frac{|\hat{\tilde{\psi}}(s)|^2}{|s|} ds < \infty,
$$

and $\boldsymbol{\gamma}_\theta = (\cos\theta, \sin\theta)$, $\boldsymbol{\gamma}^\perp_\omega = (\sin\omega, -\cos\omega)$. The inverse formula of the double windowed ridgelet transform $\mathcal{WR}_{\psi,\tilde{\psi}}f(\mathbf{b},\mathbf{a},\omega)$ for $\mathbf{b} := b\boldsymbol{\gamma}_\theta$ with $b \in \mathbf{R}$ and $\theta \in (-\pi/2, \pi/2]$, $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2\backslash\{0\}$ and $\omega \in (-\pi, \pi]$ such that $\boldsymbol{\gamma}_\omega\cdot\boldsymbol{\gamma}_\theta \neq 0$, is given by

$$
C_\psi C_{\tilde{\psi}} f(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbf{R}^2_{\mathbf{b}}}\int_{\mathbf{R}^2_{\mathbf{a}}} K_{\tilde{\psi}}(\mathbf{x}-\mathbf{b}, a_2, \omega)|D_b|\mathcal{WR}_{\psi,\tilde{\psi}}f(\mathbf{b},\mathbf{a},\omega)\frac{d\mathbf{b}d\mathbf{a}}{|a_1|^{1/2}|a_2|^{3/2}},
$$

where

$$
K_{\tilde{\psi}}(\mathbf{x}-\mathbf{b}, a_2, \omega) = \int_{\mathbf{R}^2_{\boldsymbol{\eta}}} e^{i(\mathbf{x}-\mathbf{b})\cdot\boldsymbol{\eta}}\hat{\tilde{\psi}}(a_2\boldsymbol{\gamma}^\perp_\omega\cdot\boldsymbol{\eta})d\boldsymbol{\eta}
$$

Proof of Proposition 3 By (34) and (36) we simplify the inverse formula as

$$
\begin{aligned}
& \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\int_{\mathbf{R}_{a_2}}\int_{\mathbf{R}_q} K_{\tilde{\psi}}(\mathbf{x}-q\boldsymbol{\gamma}_\theta, a_2, \omega)\int_{\mathbf{R}_{a_1}} |D_q|\mathcal{WR}_{\psi,\tilde{\psi}}f(q\boldsymbol{\gamma}_\theta,\mathbf{a},\omega)\frac{da_1 q dq da_2 d\theta}{|a_1|^{1/2}|a_2|^{3/2}} \\
= & \frac{1}{2\pi}\int_{\mathbf{R}^2_{\mathbf{a}}}\int_{\mathbf{R}^2_{\mathbf{b}}} K_{\tilde{\psi}}(\mathbf{x}-\mathbf{b}, a_2, \omega)|D_b|\mathcal{WR}_{\psi,\tilde{\psi}}f(\mathbf{b},\mathbf{a},\omega)\frac{d\mathbf{b}d\mathbf{a}}{|a_1|^{1/2}|a_2|^{3/2}}.
\end{aligned}
$$

In the last equality, we changed the polar coordinate $q\boldsymbol{\gamma}_\theta$ into the Cartesian coordinate $\mathbf{b}$. This proves Proposition 3. □

4.2.3 Case (C)

In this case, the mother wavelet ψ_A satisfies the admissible condition and the father wavelet $\tilde{\psi}_A$ is not equal to zero at the origin. Hence we need to consider the inversion formula of the windowed X-ray transform which has the father wavelet.

Lemma 1. *Let $\tilde{\psi} \in L^2(\mathbf{R})$ satisfying*

$$C_{\tilde{\psi}} := 2\pi\overline{\tilde{\psi}(0)} \neq 0,$$

and $\boldsymbol{\gamma}_\omega^\perp = (\sin\omega, -\cos\omega)$. The inversion formula of the windowed X-ray transform $\mathcal{WX}_{\tilde{\psi}}f(\mathbf{b}, a, \omega)$ for $\mathbf{b} \in \mathbf{R}^2$, $a \in \mathbf{R}\backslash\{0\}$ and $\omega \in (-\pi, \pi]$, is given by

$$C_{\tilde{\psi}} f(\mathbf{x}) = \int_{\mathbf{R}_a} |\boldsymbol{\gamma}_\omega^\perp \cdot D_\mathbf{b}| \mathcal{WX}_{\tilde{\psi}} f(\mathbf{b}, a, \omega) da d\omega \Big|_{\mathbf{b}=\mathbf{x}},$$

where

$$|\boldsymbol{\gamma}_\omega^\perp \cdot D_\mathbf{b}| f(\mathbf{b}) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}} |\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi}| \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

Proof of Lemma 1 By (32) we obtain

$$\begin{aligned}
\int_{\mathbf{R}_a} |\boldsymbol{\gamma}_\omega^\perp \cdot D_\mathbf{b}| \mathcal{WX}_{\tilde{\psi}} f(\mathbf{b}, a, \omega) da &= \frac{1}{(2\pi)^2} \int_{\mathbf{R}_a} \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} |\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi}| e^{i\mathbf{b}\cdot\boldsymbol{\xi}} \overline{\hat{\tilde{\psi}}(a\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi})} \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi} da \\
&= \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}} \Big\{ \int_{\mathbf{R}_a} \overline{\hat{\tilde{\psi}}(a\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi})} |\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi}| da \Big\} \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi} \\
&= \frac{\overline{\tilde{\psi}(0)}}{2\pi} \int_{\mathbf{R}^2_{\boldsymbol{\xi}}} e^{i\mathbf{b}\cdot\boldsymbol{\xi}} \hat{f}(\boldsymbol{\xi}) d\boldsymbol{\xi},
\end{aligned}$$

here we used

$$\int_{\mathbf{R}_a} \overline{\hat{\tilde{\psi}}(a\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi})} |\boldsymbol{\gamma}_\omega^\perp \cdot \boldsymbol{\xi}| da = \int_{\mathbf{R}_a} \overline{\hat{\tilde{\psi}}(a)} da = 2\pi\overline{\tilde{\psi}(0)}.$$

Substituting $\mathbf{b} = \mathbf{x}$, we get

$$2\pi\overline{\tilde{\psi}(0)} f(\mathbf{x}) = \int_{\mathbf{R}_a} (\boldsymbol{\gamma}_\omega^\perp \cdot D_\mathbf{b})^2 \mathcal{WX}_{\tilde{\psi}} f(\mathbf{b}, a, \omega) da \Big|_{\mathbf{b}=\mathbf{x}}. \quad \square$$

Now, by Lemma 1 we prove the following:

Proposition 4. *Let $\psi, \tilde{\psi} \in L^2(\mathbf{R})$ satisfying*

$$C_\psi := \int_\mathbf{R} \frac{|\hat{\psi}(s)|^2}{|s|} ds < \infty \quad \text{and} \quad C_{\tilde{\psi}} := 2\pi\overline{\tilde{\psi}(0)} \neq 0,$$

and $\boldsymbol{\gamma}_\theta = (\cos\theta, \sin\theta)$, $\boldsymbol{\gamma}_\omega^\perp = (\sin\omega, -\cos\omega)$. The inverse formula of the double windowed ridgelet transform $\mathcal{WR}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega)$ for $\mathbf{b} := b\boldsymbol{\gamma}_\theta$ with $b \in \mathbf{R}$ and $\theta \in (-\pi/2, \pi/2]$, $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2\backslash\{0\}$ and $\omega \in (-\pi, \pi]$ such that $\boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta \neq 0$, is given by

$$C_\psi C_{\tilde{\psi}} f(\mathbf{x}) = \int_{\mathbf{R}^2_\mathbf{a}} \int_{\mathbf{R}_b} |\boldsymbol{\gamma}_\omega^\perp \cdot D_\mathbf{x}| \psi\Big(\frac{q-b}{a_1}\Big) \mathcal{WR}_{\psi,\tilde{\psi}} f(b\boldsymbol{\gamma}_\theta, \mathbf{a}, \omega) \frac{db d\mathbf{a}}{|a_1|^{5/2}}$$

for $\mathbf{x} = q\boldsymbol{\gamma}_\theta \in \mathbf{R}^2$ with $q \in \mathbf{R}$ and $\theta \in (-\pi/2, \pi/2]$.

Proof of Proposition 4 By (28) and Lemma 1 we see that

$$\int_{\mathbf{R}_{a_2}} |\boldsymbol{\gamma}_\omega^\perp \cdot D_{\mathbf{x}}| \int_{\mathbf{R}_{a_1}} \int_{\mathbf{R}_b} \psi\Big(\frac{q-b}{a_1}\Big) \mathcal{W}\mathcal{R}_{\psi,\tilde{\psi}} f(b\boldsymbol{\gamma}_\theta, \mathbf{a}, \omega) \frac{dbda_1da_2}{|a_1|^{5/2}}$$
$$= \int_{\mathbf{R}^2_{\mathbf{a}}} \int_{\mathbf{R}_b} |\boldsymbol{\gamma}_\omega^\perp \cdot D_{\mathbf{x}}| \psi\Big(\frac{q-b}{a_1}\Big) \mathcal{W}\mathcal{R}_{\psi,\tilde{\psi}} f(b\boldsymbol{\gamma}_\theta, \mathbf{a}, \omega) \frac{dbd\mathbf{a}}{|a_1|^{5/2}}. \ \square$$

4.2.4 Case (D)

In the last case, the father wavelets ψ_D and $\tilde{\psi}_D$ are not equal to zero at the origin. Then, we prove the following:

Proposition 5. *Let $\psi, \tilde{\psi} \in L^2(\mathbf{R})$ satisfying*

$$C_\psi := -\overline{\psi(0)} \neq 0 \quad \text{and} \quad C_{\tilde{\psi}} := 2\pi \overline{\tilde{\psi}(0)} \neq 0,$$

and $\boldsymbol{\gamma}_\theta = (\cos\theta, \sin\theta)$, $\boldsymbol{\gamma}_\omega^\perp = (\sin\omega, -\cos\omega)$. The inverse formula of the double windowed ridgelet transform $\mathcal{W}\mathcal{R}_{\psi,\tilde{\psi}} f(\mathbf{b}, \mathbf{a}, \omega)$ for $\mathbf{b} := b\boldsymbol{\gamma}_\theta$ with $b \in \mathbf{R}$ and $\theta \in (-\pi/2, \pi/2]$, $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2 \backslash \{0\}$ and $\omega \in (-\pi, \pi]$ such that $\boldsymbol{\gamma}_\omega \cdot \boldsymbol{\gamma}_\theta \neq 0$, is given by

$$C_\psi C_{\tilde{\psi}} f(\mathbf{x}) = \int_{\mathbf{R}^2_{\mathbf{a}}} |\boldsymbol{\gamma}_\omega^\perp \cdot D_{\mathbf{x}}| |D_q| \mathcal{W}\mathcal{R}_{\psi,\tilde{\psi}} f(\mathbf{x}, \mathbf{a}, \omega) |a_1|^{-1/2} d\mathbf{a}$$

for $\mathbf{x} = q\boldsymbol{\gamma}_\theta \in \mathbf{R}^2$ with $q \in \mathbf{R}$ and $\theta \in (-\pi/2, \pi/2]$.

Proof of Proposition 5 By (36) and Lemma 1 we see that

$$\int_{\mathbf{R}_{a_2}} |\boldsymbol{\gamma}_\omega^\perp \cdot D_{\mathbf{x}}| \int_{\mathbf{R}_{a_1}} |D_q| \mathcal{W}\mathcal{R}_{\psi,\tilde{\psi}} f(q\boldsymbol{\gamma}_\theta, \mathbf{a}, \omega) |a_1|^{1/2} da_1 da_2$$
$$= \int_{\mathbf{R}^2_{\mathbf{a}}} |\boldsymbol{\gamma}_\omega^\perp \cdot D_{\mathbf{x}}| |D_q| \mathcal{W}\mathcal{R}_{\psi,\tilde{\psi}} f(\mathbf{x}, \mathbf{a}, \omega) |a_1|^{-1/2} d\mathbf{a}. \ \square$$

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