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TIME DECAY ESTIMATES OF SOLUTIONS TO THE MIXED PROBLEM FOR HEAT EQUATIONS IN A HALF SPACE

By

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1 Introduction

The Cauchy problem for the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & t > 0, x \in \mathbf{R}^n, \\ u|_{t=0} = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

has a solution

$$u(t,x) = \frac{1}{\sqrt{4\pi t}^n} \int_{\mathbf{R}^n} e^{-|x-y|^2/4t} u_0(y)\, dy \tag{1.2}$$

which has the following three estimates for $t > 0$

$$\|u(t)\|_{L^\infty} \le \frac{c_n}{t^{n/2}} \|u_0\|_{L^1}, \tag{1.3}$$

$$\|u(t)\|_{L^p} \le c_{n,p} \|u_0\|_{L^p}, \quad 1 \le p \le \infty, \tag{1.4}$$

and

$$\|u(t)\|_{L^p} \le \frac{c_{n,p,q}}{t^{(n/2)(q^{-1}-p^{-1})}} \|u_0\|_{L^q}, \quad 1 \le q < p \le \infty, \tag{1.5}$$

where

$$\|u\|_{L^p} = \left(\int_{\mathbf{R}^n} |u(x)|^p\, dx \right)^{1/p}.$$

(1.3) and (1.4) follow immediately from (1.2). We can derive (1.5) from (1.3) and (1.4) by use of interpolation (see Proposition 2.1 below).

Dirichlet and Neumann problem in a half space $\mathbf{R}^n_+ = \{x = (x', x_n), x' \in \mathbf{R}^{n-1}, x_n > 0\}$ has a solution respectively as

$$u_D(t,x) = \frac{1}{\sqrt{4\pi t}^n} \int_{\mathbf{R}^n_+} [e^{-(|x'-y'|^2-(x_n+y_n)^2)/4t} - e^{-(|x'-y'|^2-(x_n-y_n)^2)/4t}] u_0(y)\, dy \tag{1.6}$$

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and

$$u_N(t,x) = \frac{1}{\sqrt{4\pi t}^n} \int_{\mathbf{R}^n_+} [e^{-(|x'-y'|^2-(x_n+y_n)^2)/4t} + e^{-(|x'-y'|^2-(x_n-y_n)^2)/4t}] u_0(y)\,dy. \quad (1.7)$$

So we can see that $u_D(t,x)$ and $u_N(t,x)$ also satisfy (1.3), (1.4) and (1.5).

In this article we shall investigate the boundary conditions which are generalizations of Dirichlet and Neumann conditions and with which the solution of heat equation in a half space $\mathbf{R}^n_+$ satisfies (1.3), (1.4) and (1.5).

Let us consider

$$\begin{cases} \partial_t u - \Delta u = 0, \quad t > 0,\ x \in \mathbf{R}^n_+, \\ Bu := \left(c\partial_{x_n} + \sum_{j=1}^{n-1} b_j \partial_{x_j} + d \right) u|_{x_n=0} = 0, \quad t > 0,\ x' \in \mathbf{R}^{n-1}, \\ u|_{t=0} = u_0(x), \quad x \in \mathbf{R}^n_+, \end{cases} \quad (1.8)$$

where c, b_j and d are complex constants.

Our aim is to give the integral expressions of the solutions to the mixed problem (1.8) and to derive by use of these expression the time decay estimates (1.3)–(1.5) of the solutions of (1.8) under the strong Lopatinski condition (below).

The above problem (1.8) is well posed in $L^2(\mathbf{R}^n_+)$ if the boundary operator B satisfies

$$B(-\sqrt{\lambda + |\xi'|^2}, \xi') = -c\sqrt{\lambda + |\xi'|^2} + i\sum_{j=1}^{n-1} b_j\xi_j + d \neq 0, \quad (1.9)$$

for $\lambda \in \mathbf{C}$ such that $\Re\lambda > c_1$ ($\exists c_1 \in \mathbf{R}$) and for $\xi' \in \mathbf{R}^{n-1}$, which is called a Lopatinski condition, where we choose the branch of $\sqrt{\lambda + |\xi'|^2}$ such that $\Re\sqrt{\lambda + |\xi'|^2} > 0$ for $\Re\lambda + |\xi'|^2 > 0$. Then the solution of (1.8) satisfies

$$\|u(t)\|_{L^2(R^n_+)} \le C(t)\|u_0\|_{L^2(R^n_+)}, \quad t > 0,\ \exists C(t) > 0. \quad (1.10)$$

However in order to get the estimate (1.3), (1.4) and (1.5) we need a stronger condition than (1.9), that is, there is $c_0 > 0$ such that

$$B(-\sqrt{\lambda + |\xi'|^2}, \xi') \neq 0, \quad \Re\lambda \ge -c_0|\xi'|^2, \quad \xi' \in \mathbf{R}^{n-1}\backslash 0. \quad (1.11)$$

We call our condition (1.11) the strong Lopatinski condition. For simplicity we denote $b = (b_1, \ldots, b_{n-1})$, $b\xi' = \sum_{j=1}^{n-1} b_j\xi_j$ and $\zeta_0 = \frac{ib\xi'+d}{c}$. When $\Im(\frac{b}{c}) \neq 0$ and

$c \neq 0$, there is $\xi' \in \mathbf{R}^{n-1}$ such that $\Re\zeta_0 = \Re\{\frac{ib\xi'+d}{c}\} > 0$. Then we can find $\lambda_0 \in \mathbf{C}$ satisfying for such ξ'

$$B(-\sqrt{\lambda_0 + |\xi'|^2}, \xi') = -c\sqrt{\lambda_0 + |\xi'|^2} + i\sum_{j=1}^{n-1} b_j\xi_j + d = 0. \tag{1.12}$$

For $\lambda_0 \in \mathbf{C}$ satisfying (1.12) then we have

$$B(-\sqrt{\lambda + |\xi'|^2}, \xi') = \frac{-c(\lambda - \lambda_0)}{\sqrt{\lambda_0 + |\xi'|^2} + \sqrt{\lambda + |\xi'|^2}}, \quad \lambda \in \mathbf{C}, \tag{1.13}$$

from which λ_0 is determined uniquely. On the other hand we note that if $\Re\zeta_0 = \Re\{\frac{ib\xi'+d}{c}\} < 0$ holds, then there is no solution of (1.12).

Now we can see that the strong Lopatinski condition (1.11) implies the following theorem.

THEOREM 1.1. *Assume that the strong Lopatinski condition* (1.11) *is valid.*

(0) *When* $c = 0$, *the condition* (1.11) *implies that* $d \neq 0$ *is valid and that* $b = 0$ *or if* $b \neq 0$ *the two real vectors* $\Im(\frac{b}{d})$ *and* $\Re(\frac{b}{d})$ *are parallel. In this case the boundary condition is equivalent to Dirichlet condition.*

(1) *Let* $c \neq 0$. *Then the condition* (1.11) *implies that* $|\xi'|^2 - (\Im(\frac{b\xi'}{c}))^2 + (\Re(\frac{b\xi'}{c}))^2 \geq c_0|\xi'|^2$, $\Re\lambda_0 \leq -c_0|\xi'|^2$ *for all* $\xi' \in \mathbf{R}^{n-1}$ *and* $(\Im(\frac{d}{c}))^2 - (\Re(\frac{d}{c}))^2 \geq 0$ *hold.*

Then the following three cases occur.

(i) *When* $(\Im(\frac{d}{c}))^2 - (\Re(\frac{d}{c}))^2 > 0$ *is valid, there is* $c_1 > 0$ *such that* $\Re\lambda_0 \leq -c_1(1 + |\xi'|^2)$ *for* $\xi' \in \mathbf{R}^{n-1}$.

(ii) *When* $(\Im(\frac{d}{c}))^2 = (\Re(\frac{d}{c}))^2 \neq 0$ *is valid*, $\Re(\frac{d}{c})\Im(\frac{b\xi'}{c}) + \Im(\frac{d}{c})\Re(\frac{b\xi'}{c}) = 0$, $\lambda_0 = -|\xi|^2 + 2i(\Re(\frac{d}{c}) - \Im(\frac{b\xi'}{c}))((\Im(\frac{d}{c}) + \Re(\frac{b\xi'}{c}))$ *and* $\zeta_0 = (\frac{\Im(d/c)}{\Re(d/c)} + i)(\Im(\frac{d}{c}) + \Re(\frac{b\xi'}{c}))$ *hold. Moreover if* $\Re d < 0$, *then* $\lambda_0 \leq -c_1(1 + |\xi'|^2)$ *holds for* $\Re\zeta_0 \geq 0$.

(iii) *If* $(\Im(\frac{d}{c}))^2 = (\Re(\frac{d}{c}))^2 = 0$, *that is*, $d = 0$, *then* $\zeta_0 = i\frac{b\xi'}{c}$ *holds.*

(2) *Let* $c \neq 0$. *It holds that* $\zeta_0 = ib\xi' + d$ *satisfies*

$$\Re(\zeta_0^2) - |\xi'|^2 \leq -c_0|\xi'|^2, \quad \xi' \in \mathbf{R}^{n-1}.$$

The proof of this theorem will be given in the section 2.

Since we consider the equation (1.8) in the half space $\mathbf{R}^{n-1}$, it is natural to use the Fourier transform with respect to the tangential variable x'. We follows the notation used in Ukai [6]. Let $f(x', x_n)$ be a function defined on $\mathbf{R}^n_+$. Then its Fourier transform in x' is defined by

$$\hat{f}(\xi', x_n) = \int_{R^{n-1}} e^{-ix'\xi'} f(x', x_n)\, dx'$$

and Fourier inverse transform by

$$f(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{R^{n-1}} e^{i\xi' x'} \hat{f}(x', x_n)\, dx'.$$

In the sequel we shall drop the hat in $\hat{f}$, if there is no confusion. Thus we shall use the same symbol f for f and its Fourier transform $\hat{f}$.

We can give the expression of the solution of the mixed problem (1.8) as follows.

THEOREM 1.2. *Assume that the strong Lopatinski condition* (1.11) *is vaild. Let* $\zeta_0 = \frac{ib\xi' + d}{c}$ *and* $\tilde{c} > \max\{\Re\zeta_0, 0\}$. *Then we have the Fourier image* $u(t, \xi', x_n)$ *with respect to* x' *of the solution of* (1.8) *if* $c = 0$

$$u(t, \xi', x_n) = u_D(t, \xi', x_n) \tag{1.14}$$

where $u_D(t, \xi', x_n)$ *is the Fourier transform of* $u_D(t, x)$ *given by* (1.6) *and if* $c \neq 0$

$$u(t, \xi', x_n) = u_M(t, \xi', x_n) + u_D(t, \xi', x_n), \tag{1.15}$$

where

$$u_M(t, \xi', x_n) = \int_0^\infty \hat{K}(t, \xi', x_n + y_n) u_0(\xi', y_n)\, dy_n \tag{1.16}$$

and $\hat{K}(t, \xi', x_n) = 2\frac{\partial}{\partial x_n} \hat{K}_1(t, \xi', x_n)$ *and* $\hat{K}_1(t, \xi', x_n)$ *is given by,*

$$K_1(t, \xi', x_n) = \frac{1}{2\pi i} \int_{\Re\zeta = \tilde{c}} \frac{e^{(\zeta^2 - |\xi'|^2)t - x_n\zeta}}{\zeta - \zeta_0}\, d\zeta. \tag{1.17}$$

Moreover we can also express

$$u(t, \xi', x_n) = -u_M^0(t, \xi', x_n) + u_N(t, \xi', x_n) \tag{1.18}$$

where

$$u_M^0(t, \xi', x_n) = \int_0^\infty \frac{1}{2\pi i} \int_{\Re\zeta = \tilde{c}} 2\zeta_0 \frac{e^{(\zeta^2 - |\xi'|^2)t - (x_n + y_n)\zeta}}{\zeta - \zeta_0}\, d\zeta u_0(\xi', y_n)\, dy_n \tag{1.19}$$

and $u_N(t, \xi', x_n)$ *is the Fourier transform of* $u_N(t, x)$ *given by* (1.7).

The proof of this theorem will be given in the section 3.

It is trivial that $u_D(t,x)$ and $u_N(t,x)$ satisfy (1.3) and (1.4) clearly. So in Theorems below it suffices to prove that u_M or u_M^0 satisfies (1.3) and (1.4) in the case of $c \neq 0$.

Now we can mention the following main theorems.

THEOREM 1.3. *Let k be a non negative integer. Assume that the strong Lopatinski condition* (1.11) *is valid. When $\left(\Im\left(\frac{d}{c}\right)\right)^2 = \left(\Re\left(\frac{d}{c}\right)\right)^2 \neq 0$ occurs, moreover we assume $\Re\left(\frac{d}{c}\right) < 0$. Then the solution of* (1.8) *satisfies*

$$\|\nabla_x^k u(t)\|_{L^\infty(\mathbf{R}_+^n)} \leq C_k t^{-(n+k)/2} \|u_0\|_{L^1(\mathbf{R}_+^n)}, \quad t > 0. \tag{1.20}$$

Moreover we assume the space dimension $n \geq 3$. Then

$$\|\nabla_x^k u(t)\|_{L^p(\mathbf{R}_+^n)} \leq C_k t^{-k/2} \|u_0\|_{L^p(\mathbf{R}_+^n)}, \quad t > 0,\ 1 \leq p \leq \infty, \tag{1.21}$$

and

$$\|\nabla^k u(t)\|_{L^p(\mathbf{R}_+^n)} \leq C t^{-k/2-(n/2)(q^{-1}-p^{-1})} \|u_0\|_{L^q(\mathbf{R}_+^n)}, \quad t > 0,\ 1 \leq q \leq p \leq \infty, \tag{1.22}$$

are satisfied, where $\nabla^k u = \{\partial_x^\alpha u; |\alpha| = k\}$.

The proof of (1.20) and (1.21) of this theorem will be given in the section 4. We can derive (1.22) from (1.20) and (1.21) evidently by use of the interpolation theorem.

When $\left(\Im\left(\frac{d}{c}\right)\right)^2 = \left(\Re\left(\frac{d}{c}\right)\right)^2 \neq 0$ occurs, In order that (1.20) with $k = 0$ in Theorem 1.3 holds the condition $\Re\left(\frac{d}{c}\right) < 0$ is necessary in the sense of the following theorem.

THEOREM 1.4. *Assume that the strong Lopatinski condition* (1.11) *is valid and that $\left(\Im\left(\frac{d}{c}\right)\right)^2 = \left(\Re\left(\frac{d}{c}\right)\right)^2 \neq 0$ and $\Re\left(\frac{d}{c}\right) > 0$ hold. Then there is a initial datum $u_0 \in L^1(\mathbf{R}_+^n)$ such that the solution of* (1.8) *does not satisfy* (1.20) *with $k = 0$.*

We shall prove this theorem in the section 5.

When $1 < p < \infty$, we can prove (1.21) in Theorem 1.3 without the aditional condition $\Re\left(\frac{d}{c}\right) < 0$ in the case of $\left(\Im\left(\frac{d}{c}\right)\right)^2 = \left(\Re\left(\frac{d}{c}\right)\right)^2 \neq 0$. In fact, we can prove the following theorem, applying the L^p boundedness of singular integral operators of Calderón-Zugmund [1] to the expression (1.19) (also see (6.7)) of solution of the equation (1.8). However it is noted that Calderón-Zugmund theorem is not applicable to the case of $p = 1, \infty$.

THEOREM 1.5. *Let $1 < p < \infty$. Assume that the strong Lopatinski conditions* (1.11) *is valid. Then the solution of* (1.8) *satisfies*

$$\|\nabla_x^k u(t)\|_{L^p} \le \frac{C}{\sqrt{t}^k} \|u_0\|_{L^p}, \quad t > 0, \ k = 0, 1, \ldots. \tag{1.23}$$

We shall prove Theorem 1.5 in the section 6.

It should be remarked that there are any works about the mixed problem in $\mathbf{R}_+^n$ and in the exterior domain of heat equation with which boundary conditions are Dirichlet, Neumann and Bobin. We refer, for example, S. Jimbo and S. Sakaguchi [3], K. Ishige [2] and their references.

Forthcoming paper we shall derive the time decay estimates of solutions to the mixed problem for Stokes equation in a half space by use of the expression of solutions to the mixed problem for the heat equation and the Ukai's formula [6] of solutions for the Stokes equation in a half space.

2 Proof of Theorem 1.1 and Preliminaries

We begin to prove Theorem 1.1. When $c = 0$, $B(-\alpha(\lambda, \xi'), \xi') = ib\xi' + d$. Therefore the condition (1.11) implies that $d \neq 0$, $d(1 + i\frac{b\xi'}{d}) = d(1 - \Im(\frac{b\xi'}{d}) + i(\Re\frac{b\xi'}{d})) \neq 0$ for any $\xi'(\neq 0) \in \mathbf{R}^{n-1}$ and consequently we get our conclution that $d \neq 0$ and $\Re\frac{b}{d}$ is parallel to $\Im\frac{b}{d}$. Hence we get Dirichlet condition $u(\xi', 0) = 0$ from the boundary condition $Bu|_{x_n=0} = (1 - \Im\frac{b\xi'}{d} + i\Re\frac{b\xi'}{d})u(\xi', 0) = 0$.

Next we investigate the case (1). We may assume $c = 1$ without loss of generality. Let $\Re\zeta_0 = -\Im b\xi' + \Re d \ge 0$. Then $B(-\sqrt{\lambda_0 + |\xi'|^2}, \xi') = -\sqrt{\lambda_0 + |\xi'|^2} + ib\xi' + d = 0$ is equivalent to $-\Re\sqrt{\lambda_0 + |\xi'|^2} - \Im b\xi' + \Re d = 0$ and $-\Im\sqrt{\lambda_0 + |\xi'|^2} + \Re b\xi' + \Im d = 0$, that is,

$$\Re\sqrt{\lambda_0 + |\xi'|^2} = \sqrt{\frac{\mu_0 + |\xi'|^2 + \sqrt{(\mu_0 + |\xi'|^2)^2 + \sigma_0^2}}{2}} = \Re d - \Im b\xi' \tag{2.1}$$

and

$$\Im\sqrt{\lambda_0 + |\xi'|^2} = \pm\sqrt{\frac{-\mu_0 - |\xi'|^2 + \sqrt{(\mu_0 + |\xi'|^2)^2 + \sigma_0^2}}{2}} = \Re b\xi' + \Im d, \tag{2.2}$$

where $\lambda_0 = \mu_0 + i\sigma_0$ and $\pm$ means the sign of $\sigma_0 \neq 0$. When $\sigma_0 = 0$, we have $\Re\sqrt{\lambda_0 + |\xi'|^2}\Im\sqrt{\lambda_0 + |\xi'|^2} = 0$. We note that there exists (μ_0, σ_0) satisfying (2.1) and (2.2), only if $\Re\zeta_0 = \Re d - \Im b\xi' \ge 0$. Then we get from (2.1) and (2.2)

$$\mu_0 = -|\xi'|^2 + (\Re \mathrm{d} - \Im \mathrm{b}\xi')^2 - (\Re \mathrm{b}\xi' + \Im \mathrm{d})^2,$$
$$\sigma_0 = 2(\Re \mathrm{d} - \Im \mathrm{b}\xi')(\Re \mathrm{b}\xi' + \Im \mathrm{d}). \tag{2.3}$$

The assumption (1.11) implies

$$\begin{aligned}\mu_0 &= -|\xi'|^2 + (\Re \mathrm{d} - \Im \mathrm{b}\xi')^2 - (\Re \mathrm{b}\xi' + \Im \mathrm{d})^2 \\ &= -\{|\xi'|^2(1 - (\Im \mathrm{b}\omega')^2 + (\Re \mathrm{b}\omega')^2) + 2|\xi'|(\Re \mathrm{d}\Im \mathrm{b}\omega' + \Im \mathrm{d}\mathrm{Reb}\omega') \\ &\quad + (\Im \mathrm{d})^2 - (\Re \mathrm{d})^2\} \leq -c_0|\xi'|^2, \end{aligned}\tag{2.4}$$

for $\xi \in \mathbf{R}^{n-1}\backslash\{0\}$ with $\Re\zeta_0 = \Re \mathrm{d} - \Im \mathrm{b}\xi' \geq 0$, where $\omega' = \frac{\xi'}{|\xi'|}$. However μ_0 is a funcion of $(\Re\zeta_0)^2$. Therefore (2.4) is valid for $\Re\zeta_0 < 0$, that is, for all $\xi' \in \mathbf{R}^{n-1}$. So we can see from (2.4) that $(\Im \mathrm{d})^2 - (\Re \mathrm{d})^2 \geq 0$ and $|\xi'|^2 - (\Im \mathrm{b}\xi')^2 + (\Re \mathrm{b}\xi')^2 \geq c_0|\xi'|^2$ for all $\xi' \in \mathbf{R}^{n-1}$.

(1)-(i). Let $(\Im \mathrm{d})^2 - (\Re \mathrm{d})^2 > 0$. Since $1 - (\Im \mathrm{b}\omega')^2 + (\Re \mathrm{b}\omega')^2 \geq c_0 > 0$ holds for $|\omega'| = 1$. We can write

$$\begin{aligned}\mu_0 &= -(1 - (\Im \mathrm{b}\omega')^2 + (\Re \mathrm{b}\omega')^2)\left\{|\xi'| + \frac{\Re \mathrm{d}\Im \mathrm{b}\omega' + \Im \mathrm{d}\Re \mathrm{b}\omega'}{1 - (\Im \mathrm{b}\omega')^2 + (\Re \mathrm{b}\omega')^2}\right\}^2 \\ &\quad + \frac{(\Re \mathrm{d}\Im \mathrm{b}\omega' + \Im \mathrm{d}\Re \mathrm{b}\omega')^2}{1 - (\Im \mathrm{b}\omega')^2 + (\Re \mathrm{b}\omega')^2} - ((\Im \mathrm{d})^2 - (\Re \mathrm{d})^2) \leq -c_0|\xi'|^2 \end{aligned}\tag{2.5}$$

for all ξ'. We can prove

$$c_1 = (\Im d)^2 - (\Re d)^2 - \sup_{|\omega'|=1} \frac{(\Re d\Im b\omega' + \Im d\Re b\omega')^2}{1 - (\Im b\omega')^2 + (\Re b\omega')^2} > 0. \tag{2.6}$$

In fact, assume for some ω'

$$d_0 = \frac{(\Re d\Im b\omega' + \Im d\Re b\omega')^2}{1 - (\Im b\omega')^2 + (\Re b\omega')^2} = (\Im d)^2 - (\Re d)^2 > 0.$$

Taking $|\xi| = -\frac{\Re \mathrm{d}\Im \mathrm{b}\omega' + \Im \mathrm{d}\Re \mathrm{b}\omega'}{1-(\Im \mathrm{b}\omega')^2 + (\Re \mathrm{b}\omega')^2}$ in the equality in (2.5) we get

$$\begin{aligned}0 &= \frac{(\Re \mathrm{d}\Im \mathrm{b}\omega' + \Im \mathrm{d}\Re \mathrm{b}\omega')^2}{1 - (\Im \mathrm{b}\omega')^2 + (\Re \mathrm{b}\omega')^2} - ((\Im \mathrm{d})^2 - (\Re \mathrm{d})^2) \\ &\leq -c_0\left|\frac{\Re \mathrm{d}\Im \mathrm{b}\omega' + \Im \mathrm{d}\Re \mathrm{b}\omega'}{1 - (\Im \mathrm{b}\omega')^2 + (\Re \mathrm{b}\omega')^2}\right|^2\end{aligned}$$

which implies

$$\left|\frac{\Re d\Im b\omega' + \Im d\Re b\omega'}{1-(\Im b\omega')^2+(\Re b\omega')^2}\right|^2 = 0.$$

This contradicts to $d_0 \neq 0$. It follows from (2.5) and (2.6) that we have $\mu_0 \leq -\frac{1}{2}(c_0|\xi'|^2 + c_1)$.

(1)-(ii). Let $(\Im d)^2 = (\Re d)^2 \neq 0$. Then it follows from (2.4) that $\Re d\Im b\omega' + \Im d\Re b\omega' = 0$ must hold for $\omega' = \frac{\xi'}{|\xi'|}$. Hence we obtain $\Re d\Im b + \Im d\Re b = 0$ and consequently $\lambda_0 = -|\xi'|^2 + 2i(\Re d - \Im b\xi')(\Re b\xi' + \Im d)$ from (2.4) and $\zeta_0 = (\frac{\Im d}{\Re d} + i)(\Im d) + \Re(b\xi'))$ hold. If $\Re d < 0$ and $\Re\zeta_0 = -\Im b\xi' + \Re d > 0$, then $|\xi'| \geq c_2 > 0$ $(c_2 > 0)$ must be valid. Hence we have $\Re\lambda_0 \leq -\frac{c_0}{2}(|\xi'|^2 + c_2^2)$.

(1)-(iii) is trivial.

(2). If $\Re\zeta_0 \geq 0$, we have λ_0 satisfying $\sqrt{\lambda_0 + |\xi'|^2} = \zeta_0$ which implies $\lambda_0 + |\xi'|^2 = (\zeta_0)^2$. Hence we get from the strong Lopatinski condition $\Re(\zeta_0)^2 - |\xi'|^2 = \Re\lambda_0 \leq -c_0|\xi'|^2$, for $\Re\zeta_0 \geq 0$. On the other hand $\Re(\zeta_0)^2 - |\xi'|^2$ is a function of $(\Re\zeta_0)^2$ and so $\Re(\zeta_0)^2 - |\xi'|^2 \leq -c_0|\xi'|^2$ holds also for $\Re\zeta_0 \leq 0$. Thus we have proved (2) and consequently we have complete the proof of Theorem 1.1. Q.E.D.

In oder to prove Theorem 1.3–1.5 we need the lemmas below.

LEMMA 2.1. *Let $\zeta \in \mathbf{C}$ and $\rho \in \mathbf{R}$. Then*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{it\rho}}{t - i\zeta}\,dt = \begin{cases} e^{-\zeta\rho}H(\rho), & \Re\zeta > 0, \\ -e^{-\zeta\rho}H(-\rho), & \Re\zeta < 0. \end{cases} \tag{2.7}$$

Here H is the Heaviside function such that $H(\rho) = 1$ for $\rho > 0$ and $= 0$ for $\rho < 0$.

PROOF. Let $\Re\zeta > 0$. Then Fourier transform of $e^{-\zeta\rho}H(\rho)$ is given by

$$\int_{-\infty}^{\infty} e^{-i\rho t}[e^{-\zeta\rho}H(\rho)]\,d\rho = \int_0^{\infty} e^{(-it-\zeta)\rho}\,d\rho$$
$$= \frac{1}{i(t - i\zeta)}.$$

Hence we get (2.7) taking the inverse Fourier transform of $\frac{1}{i(t-i\zeta)}$. We can show similarly (2.7) for $\Re\zeta < 0$. Q.E.D.

LEMMA 2.2. *Let* $\zeta_0 \in \mathbf{C}$, $c \in \mathbf{R}$, $t > 0$ *and* $y \geq 0$. *Then*

$$\frac{1}{2\pi i}\int_{Re\zeta=c}\frac{e^{\zeta^2 t-\zeta y}}{\zeta-\zeta_0}\,d\zeta=\begin{cases}\dfrac{e^{c^2t-cy}}{2\pi\sqrt{4\pi t}}\displaystyle\int_{-\infty}^{0}e^{-(\zeta_0-c)z-(y-2ct-z)^2/4t}\,dz, & c-Re\zeta_0>0\\ \left(=\dfrac{1}{2\pi\sqrt{4\pi t}}\displaystyle\int_{-\infty}^{0}e^{-\zeta_0 z-(y-z)^2/4t}\,dz\right), & \\ -\dfrac{e^{c^2t-cy}}{2\pi\sqrt{4\pi t}}\displaystyle\int_{0}^{\infty}e^{-(\zeta_0-c)z-(y-2ct-z)^2/4t}\,dz, & c-Re\zeta_0<0\\ \left(=-\dfrac{1}{2\pi\sqrt{4\pi t}}\displaystyle\int_{0}^{\infty}e^{-\zeta_0 z-(y-z)^2/4t}\,dz\right). & \end{cases}\tag{2.8}$$

PROOF. Let $c - \Re\zeta_0 > 0$. Then we can see

$$\begin{aligned}&\frac{1}{2\pi i}\int_{Re\zeta=c}\frac{e^{\zeta^2 t-y\zeta}}{\zeta-\zeta_0}\,d\zeta\\ &=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{(i\theta+c)^2t-y(i\theta+c)}}{i\theta+c-\zeta_0}\,i\,d\theta\\ &=\frac{1}{2\pi i}e^{c^2t-cy}\int_{-\infty}^{\infty}\left[e^{-t\theta^2}\left(\frac{1}{\theta-i(c-\zeta_0)}\right)\right]e^{-i(y-2ct)\theta}\,d\theta\\ &=\frac{1}{2\pi i}\frac{e^{c^2t-cy}}{2\pi}\int_{-\infty}^{\infty}(e^{-t\theta^2})^\wedge(y-2ct-z)\left(\frac{1}{\theta-i(c-\zeta_0)}\right)^\wedge(z)\,dz.\end{aligned}\tag{2.9}$$

where $f^\wedge$ means Fourier transform of f and we used $(fg)^\wedge = \frac{1}{2\pi}\hat{f} * \hat{g}$. On the other hand it follows from (2.7) that for $\Re(c - \zeta_0) > 0$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\theta-i(c-\zeta_0)}e^{-iz\theta}\,d\theta=e^{(c-\zeta_0)z}H(-z).$$

Besides, noting

$$\int_{-\infty}^{\infty}e^{-t\theta^2}e^{-i(y-2ct-z)\theta}\,d\theta=\frac{1}{\sqrt{4\pi t}}e^{-(y-2ct-z)^2/4t},$$

we get the part of $c - \Re\zeta_0 > 0$ in (2.8) of Lemma 2.2 from (2.9). We can show Lemma 2.2 similarly in the case of $c - \Re\zeta_0 < 0$. The relation

$$\frac{(y-2ct-z)^2}{4t}=\frac{(y-z)^2}{4t}-c(y-z)+c^2t\tag{2.10}$$

yields the equality in the brackets of (2.8). Q.E.D.

Let j be a nongegative integer. It follows from Lemma 2.2 that we can see easily

$$\frac{1}{2\pi i}\int_{Re\zeta=c}\frac{(-\zeta)^j e^{\zeta^2 t-\zeta y}}{\zeta-\zeta_0}\,d\zeta$$

$$=\begin{cases}\dfrac{1}{2\pi\sqrt{4\pi t}}\displaystyle\int_{-\infty}^{0} e^{-\zeta_0 z}\partial_z^j e^{-(y-z)^2/4t}\,dz, & c-Re\zeta_0>0\\[2ex] \dfrac{-1}{2\pi\sqrt{4\pi t}}\displaystyle\int_{0}^{\infty} e^{-\zeta_0 z}\partial_z^j e^{-(y-z)^2/4t}\,dz, & c-Re\zeta_0<0.\end{cases} \tag{2.11}$$

We remark that for $c_1<\Re\zeta_0<c_2$ Cauchy formula gives

$$\frac{1}{2\pi i}\int_{Re\zeta=c_2}\frac{e^{\zeta^2 t-\zeta y}}{\zeta-\zeta_0}\,d\zeta=e^{\zeta_0^2 t-\zeta_0 y}+\frac{1}{2\pi i}\int_{Re\zeta=c_1}\frac{e^{\zeta^2 t-\zeta y}}{\zeta-\zeta_0}\,d\zeta. \tag{2.12}$$

To get our Theorem 1.5, we need the boundedness in $L^p(\mathbf{R}^l)$ of singular integral operators of which proof is in Calderón and Zygmund [1].

LEMMA 2.3. *Let $1<p<\infty$. Assume that $\hat{R}(\xi)\in C^\infty(\mathbf{R}^l_\xi\backslash\{0\})$ satisfies*

$$|\partial_\xi^\gamma\hat{R}(\xi)|\le C_\gamma|\xi|^{-|\gamma|},\quad \xi\neq 0, \tag{2.13}$$

for any multi-index $\gamma\in\mathbf{N}^l$ (denotes the set of multi-index $\gamma=(\gamma_1,\dots,\gamma_l)$ with $\gamma_j\ge 0$) with $|\gamma|\le\left[\frac{n}{2}\right]+1$. Put

$$Ru(x)=v.p\int_{R^l}R(x-y)u(y)\,dy,$$

where $R(x)$ means the Fourier inverse transform of $\hat{R}$. Then there is $C>0$ such that

$$\|Ru\|_{L^p}\le C\max_{|\gamma|\le[n/2]+1}C_\gamma\|u\|_{L^p},$$

for any $u\in L^p$.

LEMMA 2.4. *Let $a(\xi)=\sum_{j,k=1}^n a_{jk}\xi_j\xi_k$ be a polynomial satisfying $\Re a(\xi)\ge c_0|\xi|^2$, $(c_0>0)$. Then for any $\varepsilon>0$ and for $\alpha\in\mathbf{N}^n$ there is $C_{\alpha\varepsilon}>0$ such that*

$$|\partial_\xi^\alpha e^{-a(\xi)t}|\le C_{\alpha\varepsilon}\sqrt{t}^{|\alpha|}e^{-(1-\varepsilon)\Re a(\xi)t},\quad t>0,\ \xi\in\mathbf{R}^n. \tag{2.14}$$

PROOF. Put $Q^\alpha(\xi,t) = e^{a(\xi)t}\partial_\xi^\alpha e^{-a(\xi)t}$. We shall prove by induction of α.

$$|\partial_\xi^\gamma Q^\alpha(\xi,t)|e^{-\varepsilon\Re a(\xi)t} \leq C_{\gamma\alpha\varepsilon}\sqrt{t}^{|\alpha|+|\gamma|}, \quad \gamma \in \mathbf{N}^n \tag{2.15}$$

for $t > 0$, $\xi \in \mathbf{N}^n$. When $|\alpha| = 1$, we have $\partial_\xi^\alpha = \partial_{\xi_j}$ for some j and so we get $Q^\alpha = \sum_k (a_{jk} + a_{kj})\xi_k$. Hence we get $\partial_\xi^\gamma Q^\alpha(\alpha,t) = 0$ for $|\gamma| \geq 2$ and consequently (2.15) with $|\alpha| = 1$ is trivial for $|\gamma| \geq 2$. While

$$|\partial_\xi^\gamma Q^\alpha(\alpha,t)|e^{-\varepsilon\Re a(\xi)t} \leq C_{\gamma\alpha}|\xi|^{1-|\gamma|}te^{-\varepsilon c_0|\xi|^2 t} \leq C_{\gamma\alpha\varepsilon}\sqrt{t}^{1+|\gamma|}$$

for $|\alpha| = 1$, $|\gamma| \leq 1$ which implies (2.14) with $|\alpha| = 1$. We assume that (2.15) is valid for $|\alpha| \geq 1$. Noting that $Q^{e+\alpha}(\xi,t) = Q^e(\xi,t)Q^\alpha(\xi,t) + \partial_\xi^e Q^\alpha(\xi,t)$ for $|e| = 1$, we obtain

$$\begin{aligned}|\partial_\xi^\gamma Q^{\alpha+e}(\xi,t)|e^{-\varepsilon\Re a(\xi)t} &\leq C_\gamma \sum_{\gamma'\leq\gamma} |\partial_\xi^{\gamma-\gamma'} Q^e|e^{-\varepsilon\Re a(\xi)t/2}|\partial_\xi^{\gamma'} Q^\alpha|e^{-\varepsilon\Re a(\xi)t/2} \\ &\quad + |\partial_\xi^{\gamma+e} Q^\alpha|e^{-\varepsilon\Re a(\xi)t} \\ &\leq \left\{\sum_{\gamma'} C_\gamma C_{e(\gamma-\gamma')(\varepsilon/2)} + C_{(\gamma+e)\alpha\varepsilon}\right\}\sqrt{t}^{|\gamma|+|\alpha|+1},\end{aligned}$$

which implies (2.15) with $\alpha + e$. Q.E.D.

We remark that if $a(\xi)$ satisfies (2.14), then

$$|\partial_\xi^\alpha(b(\xi)e^{-a(\xi)t})| \leq C_{\alpha m\varepsilon}\sqrt{t}^{|\alpha|-m}e^{-(1-\varepsilon)\Re a(\xi)t}, \quad t > 0, \tag{2.16}$$

for any homogenuous polynomial $b(\xi)$ of order m and that if $a(\xi)$ satisfies $\Re a(\xi) \geq c_0(1 + |\xi|^2)$, $(c_0 > 0)$, we can prove analogously to (2.14)

$$|\partial_\xi^\alpha(b(\xi)e^{-a(\xi)t})| \leq C_{\alpha m\varepsilon}\sqrt{t}^{|\alpha|-m}e^{-(1-\varepsilon)\Re a(\xi)t} \tag{2.17}$$

for any polynomial $b(\xi)$ of order m.

Applying Hölder inequality we can prove easily the following lemma.

LEMMA 2.5. *Let $1 \leq p \leq \infty$ and Ω be an open domain in $\mathbf{R}^l$ and $H(x,y)$ be a mesurable function defined in $\Omega \times \Omega$. Assume that $H(x,y)$ satisfies*

$$\int_\Omega |H(x,y)|\,dx \leq C, \quad \int_\Omega |H(x,y)|\,dy \leq C.$$

Define $Hu(x) = \int_\Omega H(x, y)u(y)\,dy$. *Then we have*

$$\|Hu\|_{L^p(\Omega)} \le C\|u\|_{L^p(\Omega)},$$

for any $u \in L^p(\Omega)$.

To prove Theorem 1.5 we need the following lemma of which proof is given by Yuzawa [7].

LEMMA 2.6. *Let* Ω *be a domain in* **C**. *Assume tat* $f(z)$ *is holomorphic function* Ω, γ *is a closed curve in* Ω *and* $\{\lambda_1, \lambda_2, \dots, \lambda_d\}$ *is a subset of points in the interior of* γ. *Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\lambda)}{\prod_{j=1}^d(\lambda - \lambda_j)}\,d\lambda$$

$$= \int_0^1 \cdots \int_0^1 \theta_2\theta_3^2\theta_4^3 \cdots \theta_{d-1}^{d-2} f^{(d-1)}(q(\lambda_1, \lambda_2, \dots, \lambda_d; \theta))\,d\theta_1 d\theta_2 \cdots d\theta_{d-1}, \tag{2.18}$$

where

$$\begin{aligned} q(\lambda_1, \lambda_2, \dots, \lambda_d; \theta) = {} & \theta_1\theta_2 \cdots \theta_{d-1}\lambda_1 + (1 - \theta_1)\theta_2 \cdots \theta_{d-1}\lambda_2 \\ & + (1 - \theta_2)\theta_3 \cdots \theta_{d-1}\lambda_3 + \cdots + (1 - \theta_{d-2})\theta_{d-1}\lambda_{d-1} \\ & + (1 - \theta_{d-1})\lambda_d. \end{aligned}$$

We remark that applying the above lemma of the case of $d = 2$, we can see easily that if f satisfies $\sup_{c_2 \le \mu \le c_1}|f(\mu + iA)| = o(|A|^2)$, $|A| \to \infty$, we get for $c_1 > \Re\lambda_1 \ge \Re\lambda_2 > c_2$

$$\begin{aligned} & \frac{1}{2\pi i}\int_{\Re\lambda = c_1} \frac{f(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)}\,d\lambda \\ & \quad = \int_0^1 f'(\theta\lambda_2 + (1 - \theta)\lambda_1)\,d\theta + \frac{1}{2\pi i}\int_{\Re\lambda = c_2} \frac{f(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)}\,d\lambda \end{aligned}$$

which will be applied to the proof of Theorem 1.5 in the section 6.

PROPOSITION 2.1 (Riesz-Thorin interpolation theorem). *Let* T *be a linear mapping from* L^{q_i} *to* L^{p_i} *satisfying* $\|Tf\|_{p_i} \le M_i\|f\|_{q_i}$, $i = 0, 1$. *Then for each* $f \in L^{p_0} \cap L^{p_1}$, *and for each* $t \in (0, 1)$, $Tf \in L^{p_t}$ *and* $\|Tf\|_{p_t} \le M_0^{1-t}M_1^t\|f\|_{q_t}$ *hold, where* $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$ *and* $\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$.

The proof of this proposition can be seen in M. Reed and B. Simon [4] (Theorem IX.17).

Using the above proposition we can derive (1.5) from (1.3) and (1.4). In fact, we denote by T a linear operator defined as $Tu_0 = u(t)$ where $u(t)$ is the solution of the mixed problem (1.8). Then it follows from (1.3) and (1.4) that T satisfies

$$\|Tu_0\|_\infty \le Ct^{-n/2}\|u_0\|_1$$

and

$$\|Tu_0\|_r \le C\|u_0\|_r.$$

Take $p_0 = \infty$, $q_0 = 1$ and $p_1 = r$, $q_1 = r$. Then it follows from Proposition 2.1 that

$$\|Tu_0\|_{p_t} \le Ct^{-(n/2)(1-t)}\|u_0\|_{q_t},$$

where $\frac{1}{p_t} = \frac{t}{r}$, $\frac{1}{q_t} = 1 - t + \frac{t}{r}$. Here we choose $1 - t = \frac{1}{q} - \frac{1}{p}$, $p_t = p$ and $q_t = q$, that is, we choose t, r such that $t = 1 - \frac{1}{q} + \frac{1}{p}$ and $r = \left(1 - \frac{1}{q} + \frac{1}{p}\right)p$. Then the last estimate means (1.5).

3 Solution Formulas for Mixed Problem to Heat Equation in Half Space

First we shall show that the fundamental solutions $E(t)$ of the mixed problem (1.8) are given as follows;

$$E(t)u_0(\xi', x_n) = \int_0^\infty E(t, \xi', x_n, y_n)u_0(\xi', y_n)\, dy_n, \tag{3.1}$$

where

$$E(t, \xi', x_n, y_n) = \frac{1}{2\pi i}\int_{Re\lambda = c} e^{\lambda t}E(\lambda, \xi', x_n, y_n)\, d\lambda. \tag{3.2}$$

Here we take a complex variable $\lambda = \mu + i\sigma$ and the symbol $E(\lambda, \xi', x_n, y_n)$ is given by

$$\begin{aligned} E(\lambda, \xi', x_n, y_n) = &-\frac{e^{-(x_n+y_n)\sqrt{\lambda+|\xi'|^2}}}{2\sqrt{\lambda+|\xi'|^2}} \\ &+\frac{e^{-(x_n-y_n)\sqrt{\lambda+|\xi'|^2}}}{2\sqrt{\lambda+|\xi'|^2}}H(x_n - y_n) \\ &+\frac{e^{(x_n-y_n)\sqrt{\lambda+|\xi'|^2}}}{2\sqrt{\lambda+|\xi'|^2}}H(y_n - x_n), \end{aligned} \tag{3.3}$$

if $c=0$, and

$$E(\lambda,\xi',x_n,y_n)=\frac{e^{-(x_n+y_n)\sqrt{\lambda+|\xi'|^2}}}{B(-\sqrt{\lambda+|\xi'|^2},\xi')}-\frac{e^{-(x_n+y_n)\sqrt{\lambda+|\xi'|^2}}}{2\sqrt{\lambda+|\xi'|^2}}$$

$$+\frac{e^{-(x_n-y_n)\sqrt{\lambda+|\xi'|^2}}}{2\sqrt{\lambda+|\xi'|^2}}H(x_n-y_n)$$

$$+\frac{e^{(x_n-y_n)\sqrt{\lambda+|\xi'|^2}}}{2\sqrt{\lambda+|\xi'|^2}}H(y_n-x_n), \tag{3.4}$$

if $c\neq 0$, where we denote by H Heaviside function such that $H(z)=1$ for $z>0$ and $=0$ for $z<0$.

We shall derive the formula (3.1)–(3.4). Let $u(t,x)$ be a solution of (1.8) and denote by $u(t,\xi',x_n)$ Fourier transform of u with respect to x'. Denote by U Laplace transform of u, that is,

$$U(\lambda,\xi',x_n)=\int_0^\infty e^{-\lambda t}u(t,\xi',x_n)\,dt,$$

which satisfies from (1.8)

$$\left(\lambda+|\xi'|^2-\frac{\partial^2}{\partial x^2}\right)U(\lambda,\xi',x_n)=u_0(\xi',x_n), \tag{3.5}$$

and the boundary condition,

$$BU=\left(c\frac{\partial}{\partial x_n}+ib\xi'+d\right)U(\lambda,\xi',0)=0. \tag{3.6}$$

Denote $\mu=\Re\lambda$, $\sigma=\Im\lambda$ and $\alpha=\sqrt{\lambda+|\xi'|^2}$. We can choose a bounded solution of (3.5) as follows,

$$U(\lambda,\xi',x_n)=e^{-\alpha(\lambda,\xi')x_n}U(\lambda,\xi',0)-\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_0^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi,y_n)\,dy_n$$

$$+\frac{e^{\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_{x^n}^\infty e^{-\alpha(\lambda,\xi')y_n}\hat{u}_0(\xi',y_n)\,dy_n$$

$$+\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_0^{x_n} e^{\alpha(\lambda,\xi')y_n}u_0(\xi',y_n)\,dy_n. \tag{3.7}$$

When $c = 0$, $BU = 0$ implies $U(\lambda, \xi', 0) = 0$. It follows from (3.7) that we have

$$\begin{aligned} U(\lambda, \xi', x_n) = &-\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_0^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi', y_n)\,dy_n \\ &+\frac{e^{\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_{x_n}^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi, y_n)\,dy_n \\ &+\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_0^{x_n} e^{\alpha(\lambda,\xi')y_n}u_0(\xi', y_n)\,dy_n. \end{aligned} \tag{3.8}$$

When $c = 1$ (we may assume $c = 1$ without loss of generarity), by operating B to (3.7) we see that the boundary condition $BU = 0$ yields

$$U(\lambda, \xi', 0) = \frac{1}{-\alpha(\lambda,\xi') + ib\cdot\xi' + d}\int_0^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi', y_n)\,dy_n. \tag{3.9}$$

Therefore inserting $U(\lambda, \xi', 0)$ given by (3.9) into (3.7), we obtain

$$\begin{aligned} U(\lambda, \xi', x_n) = &\left\{\frac{e^{-\alpha(\lambda,\xi')x_n}}{-\alpha(\lambda,\xi') + ib\cdot\xi' + d}\right. \\ &\left.-\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\right\}\int_0^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi', y_n)\,dy_n \\ &+\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_0^{x_n} e^{\alpha(\lambda,\xi')y_n}u_0(\xi', y_n)\,dy_n \\ &+\frac{e^{\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_{x_n}^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi', y_n)\,dy_n. \end{aligned} \tag{3.10}$$

Since

$$u(t, x) = \frac{1}{2\pi i}\int_{\Re\lambda = c_1 > 0} U(\lambda, \xi', x_n)\,d\lambda$$

we get (3.1)–(3.4) from (3.8) and (3.10).

LEMMA 3.1. *Assume* $B(-\sqrt{\lambda + |\xi'|^2}, \xi') = -\sqrt{\lambda + |\xi'|^2} + ib\xi' + d$ *satisfies the strong Lopatinski condition* (1.11). *Let* $t > 0$, $y > 0$, $c_1 > 0$, $c_2 > 0$ *and* $\Re\zeta_0 < \tilde{c}$. *Then we have*

$$\frac{1}{2\pi i}\int_{\Re\lambda=c_1}\frac{e^{\lambda t-y\sqrt{\lambda+|\xi'|^2}}}{B(-\sqrt{\lambda+|\xi'|^2},\xi')}\,d\lambda$$

$$=\frac{-1}{2\pi i}\int_{\Re\zeta=\tilde c}2\zeta\frac{e^{(-|\xi'|^2+\zeta^2)t-\zeta y}}{\zeta-\zeta_0}\,d\zeta \tag{3.11}$$

$$=\frac{-1}{2\pi i}\int_{\Re\zeta=\tilde c}2\zeta_0\frac{e^{(-|\xi'|^2+\zeta^2)t-\zeta y}}{\zeta-\zeta_0}\,d\zeta-\frac{1}{\sqrt{\pi t}}e^{-|\xi'|^2t-y^2/4t} \tag{3.12}$$

and

$$\frac{1}{2\pi i}\int_{\Re\lambda=c}\frac{e^{\lambda t-y\sqrt{\lambda+|\xi'|^2}}}{\sqrt{\lambda+|\xi'|^2}}\,d\lambda=\frac{1}{\sqrt{\pi t}}e^{-|\xi'|^2t-y^2/4t}. \tag{3.13}$$

PROOF. First we shall prove (3.11). Put $\Gamma=\{\lambda\in\mathbf{C};\Re\sqrt{\lambda+|\xi'|^2}=\tilde c\}$. It suffices to prove

$$\frac{1}{2\pi i}\int_{\Re\lambda=c_1}\frac{e^{\lambda t-y\sqrt{\lambda+|\xi'|^2}}}{B(-\sqrt{\lambda+|\xi'|^2},\xi')}\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{\lambda t-y\sqrt{\lambda+|\xi'|^2}}}{B(-\sqrt{\lambda+|\xi'|^2},\xi')}\,d\lambda. \tag{3.14}$$

In fact, by the change of variable $\zeta=\sqrt{\lambda+|\xi'|^2}$ we can get (3.11) from (3.14). Let $A\gg 1$. Denote $\Gamma_1(A)=\{\lambda\in\mathbf{C};\Re\lambda=c_1;|\Im\lambda|\le A\}$, $\Gamma_2(A)=\{\lambda\in\mathbf{C};-|\xi'|^2+\tilde c^2-\frac{A^2}{4\tilde c^2}\le\Re\lambda\le c_1,\Im\lambda=A\}$, $\Gamma_3(A)=\{\lambda\in\mathbf{C};-|\xi'|^2+\tilde c^2-\frac{A^2}{4\tilde c^2}\le\Re\lambda\le c_1,\Im\lambda=-A\}$ and $\Gamma_4(A)=\{\lambda\in\Gamma;-A\le\Im\lambda\le A\}$. Then since there is no zero of $B(-\sqrt{\lambda+|\xi'|^2},\xi)$ in the interior domain enclosed by boundaries $\bigcup_{j=1}^4\Gamma_j(A)$, Cauchy formula gives

$$\frac{1}{2\pi i}\sum_{k=1}^4\int_{\Gamma_k(A)}\frac{e^{\lambda t-y\sqrt{\lambda+|\xi'|^2}}}{B(-\sqrt{\lambda+|\xi'|^2},\xi')}\,d\lambda=0. \tag{3.15}$$

Therefore in order to prove (3.14) it suffices to show that when $A\to\infty$,

$$\int_{\Gamma_2(A)}\frac{e^{\lambda t-y\sqrt{\lambda+|\xi'|^2}}}{B(-\sqrt{\lambda+|\xi'|^2},\xi')}\,d\lambda\to 0 \tag{3.16}$$

and

$$\int_{\Gamma_3(A)}\frac{e^{\lambda t-y\sqrt{\lambda+|\xi'|^2}}}{B(-\sqrt{\lambda+|\xi'|^2},\xi')}\,d\lambda\to 0. \tag{3.17}$$

Let us show (3.16). Put $\mu(A) = -|\xi'|^2 + \tilde{c}^2 - \frac{A^2}{4\tilde{c}^2}$. Since $\Im\lambda = A > 0$ and $|B(-\sqrt{\lambda + |\xi'|^2}, \xi')| \geq \sqrt{A} - C(\xi)$ for $\lambda \in \Gamma_2(A)$ we have

$$\left| \int_{\Gamma_2(A)} \frac{e^{\lambda t - y\sqrt{\lambda + |\xi'|^2}}}{B(-\sqrt{\lambda + |\xi'|^2}, \xi')} d\lambda \right| \leq \int_{-\infty}^{c_1} \frac{e^{\mu t}}{\sqrt{A} - C(\xi)} d\mu$$

$$\leq \frac{1}{\sqrt{A} - C(\xi)} \frac{e^{c_1 t}}{t} \to 0, \quad (A \to \infty)$$

Thus we can obtain (3.16). Analogously we can show (3.17). Put $\zeta_0 = ib\xi' + d$. Changing the variale $\zeta = \sqrt{\lambda + |\xi'|^2}$, we have

$$\frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t - y\sqrt{\lambda + |\xi'|^2}}}{B(-\sqrt{\lambda + |\xi'|^2})} d\lambda = \frac{1}{2\pi i} \int_{\Re\zeta = \tilde{c}} \frac{e^{(\zeta^2 - |\xi'|^2)t - y\zeta}}{\zeta_0 - \zeta} 2\zeta \, d\zeta$$

$$= \frac{-2}{2\pi i} \int_{\Re\zeta = \tilde{c}} e^{(\zeta^2 - |\xi'|^2)t - y\zeta} \, d\zeta$$

$$+ \frac{2\zeta_0}{2\pi i} \int_{\Re\zeta = \tilde{c}} \frac{e^{(\zeta^2 - |\xi'|^2) - y\zeta}}{\zeta_0 - \zeta} d\zeta. \tag{3.18}$$

On the other hand, changing the variable $\zeta = \tilde{c} + i\theta$ again, we can get

$$\frac{1}{2\pi i} \int_{\Re\zeta = \tilde{c}} e^{\zeta^2 t - y\zeta} \, d\zeta = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{(\tilde{c} + i\theta)^2 t - y(\tilde{c} + i\theta)} i \, d\theta$$

$$= \frac{e^{\tilde{c}^2 t - y\tilde{c}}}{2\pi} \int_{-\infty}^{\infty} e^{-\theta^2 t + i(2\tilde{c}t - y)\theta} \, d\theta$$

$$= \frac{e^{\tilde{c}^2 t - y\tilde{c}}}{2\sqrt{\pi}} \frac{1}{\sqrt{t}} e^{-(2\tilde{c}t - y)^2/4t} = \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}. \tag{3.19}$$

Here we used the equality

$$\int_{-\infty}^{\infty} e^{-\theta^2 t + i(2\tilde{c}t - y)\theta} \, d\theta = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-(2\tilde{c}t - y)^2/4t}. \tag{3.20}$$

Thus we can get (3.12) from (3.18). Next we shall prove (3.13). Analogously to (3.11) we have

$$\frac{1}{2\pi i} \int_{Re\lambda = c} \frac{e^{\lambda t - y\sqrt{\lambda + |\xi'|^2}}}{\sqrt{\lambda + |\xi'|^2}} d\lambda = \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t - y\sqrt{\lambda + |\xi'|^2}}}{\sqrt{\lambda + |\xi'|^2}} d\lambda.$$

Taking the change of variable $\zeta=\sqrt{\lambda+|\xi'|^2}$, we can see from (3.19)

$$\frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t-y\sqrt{\lambda+|\xi'|^2}}}{\sqrt{\lambda+|\xi'|^2}}\,d\lambda=\frac{1}{2\pi i}\int_{Re\zeta=\bar{c}}\frac{e^{(\zeta^2-|\xi'|^2)t-y\zeta}}{\zeta}2\zeta\,d\zeta=\frac{1}{\sqrt{\pi t}}e^{-t|\xi'|^2-y^2/4t},$$

which means (3.13). Thus we have proved Lemma 3.1.

PROOF OF THEOREM 1.2. When $c=0$, it follows from (3.1) and (3.3) that we get by use of (3.13)

$$\begin{aligned}
u(t,\xi',x_n)&=\frac{1}{2\pi i}\int_{\Re\lambda=c}e^{\lambda t}\left\{\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_0^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi',y_n)\,dy_n\right.\\
&\qquad-\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_0^{x_n}e^{\alpha(\lambda,\xi')y_n}u_0(\xi',y_n)\,dy_n\\
&\qquad\left.-\frac{e^{\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_{x_n}^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi',y_n)\,dy_n\right\}d\lambda\\
&=\int_0^\infty\left\{\frac{1}{\sqrt{4\pi t}}\left(e^{-|\xi'|^2t-(x_n+y_n)^2/4t}-e^{-|\xi'|^2t-(x_n-y_n)^2/4t}\right)\right\}u_0(t,\xi',y_n)\,dy_n\\
&=u_D(t,\xi',x_n).
\end{aligned}\tag{3.21}$$

This means (1.14). When $c=1$, it follows from (3.1), (3.4) and (3.11) in Lemma 3.1 that we can see

$$\begin{aligned}
&u(t,\xi',x_n)\\
&=\frac{1}{2\pi i}\int_{\Re\lambda=c}e^{\lambda t}\left\{\left(\frac{e^{-\alpha(\lambda,\xi')x_n}}{-\alpha(\lambda,\xi')+ib\cdot\xi'+d}-\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\right)\int_0^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi',y_n)\,dy_n\right.\\
&\qquad+\frac{e^{-\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_0^{x_n}e^{\alpha(\lambda,\xi')y_n}u_0(\xi',y_n)\,dy_n\\
&\qquad\left.+\frac{e^{\alpha(\lambda,\xi')x_n}}{2\alpha(\lambda,\xi')}\int_{x_n}^\infty e^{-\alpha(\lambda,\xi')y_n}u_0(\xi',y_n)\,dy_n\right\}d\lambda\\
&=\int_0^\infty\left\{\frac{-1}{2\pi i}\int_{\Re\lambda=c}2\zeta\frac{e^{(\zeta^2-|\xi'|^2)t-(x_n+y_n)\zeta}}{\zeta-\zeta_0}\,d\zeta\right.\\
&\qquad\left.+\frac{1}{\sqrt{4\pi t}}\left(e^{-|\xi'|^2t-(x_n-y_n)^2/4t}-e^{-|\xi'|^2t-(x_n+y_n)^2/4t}\right)\right\}u_0(t,\xi',y_n)\,dy_n,
\end{aligned}\tag{3.22}$$

which means (1.15) and moreover we get (1.18) from (3.12) and (3.22). Q.E.D.

4 Proof of Theorem 1.3

We shall prove Theorem 1.3. We may assume $c = 1$ without loss of generality.

PROOF OF (1.20) OF THEOREM 1.3. It follows from (1.16) in Theorem 1.2 that we can express u_M as

$$\partial_{x'}^{\gamma}\partial_{x_n}^{j}u_M(t,x) = \int_{R_+^n} \partial_{x'}^{\gamma}\partial_{x_n}^{j}K(t,x'-y',x_n+y_n)u_0(y)\,dy, \tag{4.1}$$

where (1.17) gives

$$\partial_{x'}^{\gamma}\partial_{x_n}^{j}K(t,x',x_n) = \frac{1}{(2\pi)^{n-1}}\int_{R^{n-1}} e^{ix'\xi'}(i\xi')^{\gamma}\partial_{x_n}^{j}\hat{K}(t,\xi',x_n)\,d\xi' \tag{4.2}$$

and

$$\partial_{x_n}^{j}\hat{K}(t,\xi',x_n) = \frac{1}{2\pi i}\left(\frac{\partial}{\partial x_n}\right)^{j+1}\int_{\Re\zeta=\tilde{c}} \frac{2e^{(\zeta^2-|\xi'|^2)t-x_n\zeta}}{\zeta-\zeta_0}\,d\zeta. \tag{4.3}$$

In order to prove that $\partial_{x'}^{\gamma}\partial_{x_n}^{j}u_M(t,x)$ given in (4.1) satisfies (1.20), it suffices to show that

$$|\partial_{x'}^{\gamma}\partial_{x_n}^{j}K(t,x',x_n)| \le \int_{R^{n-1}} |(i\xi')^{\gamma}\partial_{x_n}^{j}\hat{K}(t,\xi',x_n)|\,d\xi' \le Ct^{-(n+|\gamma|+j)/2}, \tag{4.4}$$

for $t > 0$, $x \in \mathbf{R}_+^n$.

First we consider the case of $\Re\zeta_0 < 0$. In this case we can apply the first equality of (2.8) with $c = 0$ in Lemma 2.2 to (4.3) and so we get

$$\partial_{x_n}^{j}\hat{K}(t,\xi',x_n) = \frac{e^{-|\xi'|^2 t}}{\pi\sqrt{4\pi t}}\int_{-\infty}^{0} e^{-\zeta_0 z}\left(\frac{\partial}{\partial x_n}\right)^{j+1} e^{-(x_n-z)^2/4t}\,dz, \tag{4.5}$$

Taking account that Lemma 2.4 implies

$$\int_{-\infty}^{\infty}\left|\left(\frac{\partial}{\partial z}\right)^{j+1} e^{-(x_n-z)^2/4t}\right| dz \le Ct^{-j/2}, \quad t > 0, \tag{4.6}$$

we can estimate easily

$$\begin{aligned}\int_{\Re\zeta_0<0} |(i\xi')^{\gamma}\partial_{x_n}^{j}\hat{K}(t,\xi',x_n)|\,d\xi' &\le Ct^{-(j+1)/2}\int_{R^{n-1}} |\xi'|^{|\gamma|}e^{-|\xi'|^2 t}\,d\xi' \\ &\le Ct^{-(n+|\gamma|+j)/2}.\end{aligned} \tag{4.7}$$

Next we consider the case of $\Re\zeta_0 \geq 0$. In this case we can use (2.12) with $c_2 = \tilde{c}$, $c_1 = 0$ and the second equality of (2.8) and we get

$$\partial_{x_n}^j \hat{K}(t,\xi',x_n) = 2(-\zeta_0)^{j+1} e^{\lambda_0 t - \zeta_0 x_n} + \frac{1}{2\pi i}\left(\frac{\partial}{\partial x_n}\right)^{j+1} \int_{\Re\zeta=0} 2\frac{e^{(-|\xi'|^2+\zeta^2)t-\zeta x_n}}{\zeta-\zeta_0}\, d\zeta$$

$$= 2(-\zeta_0)^{j+1} e^{\lambda_0 t - \zeta_0 x_n} - \frac{e^{-|\xi'|^2 t}}{\pi\sqrt{4\pi t}} \int_0^\infty e^{-\zeta_0 z}\left(\frac{\partial}{\partial x_n}\right)^{j+1} e^{-(x_n - z)^2/4t}\, dz. \quad (4.8)$$

In the case (i) of (1) in Theorem 1.1, taking account that $\Re\lambda_0 \leq -(c_0 + \varepsilon_0|\xi'|^2)$ is valid, by use of (4.6) we have from (4.8)

$$|\partial_{x_n}^j \hat{K}(t,\xi',x_n)| \leq C\{(1+|\xi|)^{j+1} e^{-(c_0+\varepsilon_0|\xi'|^2)t} + t^{-(j+1)/2} e^{-|\xi'|^2 t}\}$$

$$\leq Ct^{-(j+1)/2} e^{-\varepsilon_0|\xi'|^2 t}, \quad (4.9)$$

for $\Re\zeta_0 > 0$, which implies

$$\int_{\Re\zeta_0>0} |(i\xi')^\gamma \partial_{x_n}^j \hat{K}(t,\xi',x_n)|\, d\xi' \leq C\int_{R^{n-1}} |\xi'|^{|\gamma|} t^{-(j+1)/2} e^{-\varepsilon_0|\xi'|^2 t}\, d\xi'$$

$$\leq Ct^{-(n+|\gamma|+j)/2}. \quad (4.10)$$

Next we consider the case (ii). Namely $(\Im d)^2 = (\Re d)^2 \neq 0$ and the assumption $\Re d < 0$ are valid. Then there is $\varepsilon_0 > 0$ such that $\Re\zeta_0 < 0$ for $|\xi'| \leq \varepsilon_0$, which case is considered already in (4.7). If $\Re\zeta_0 > 0$, then $|\xi'| \geq \varepsilon_0$ must hold and consequently $\Re\lambda_0 \leq -c_1(1+|\xi'|^2)$ holds. Hence by use of (4.8) we can show (4.4) in this case similarly to (4.9). Next we consider the csae (iii), that is, $d = 0$, $\zeta_0 = ib\xi'$. So $|\zeta_0| \leq C|\xi'|$ holds. When $\Re\zeta_0 > 0$, $\Re\lambda_0 \leq -\varepsilon_0|\xi'|^2$ holds. Hence we have by use of (4.8)

$$|\partial_{x_n}^j \hat{K}(t,\xi',x_n)| \leq C(|\xi'|^{j+1} e^{-c_0|\xi'|^2 t} + t^{-(j+1)/2} e^{-|\xi'|^2 t})$$

which implies

$$\int |(i\xi')^\gamma \partial_{x_n}^j \hat{K}(t,\xi',x_n)|\, d\xi' \leq C\int_{R^{n-1}} (|\xi'|^{j+1} e^{-c_0|\xi'|^2 t} + t^{-(j+1)/2} e^{-|\xi'|^2 t})|\xi'|^{|\gamma|}\, d\xi'$$

$$\leq Ct^{-(n+|\gamma|+j)/2}.$$

Thus we have completed the proof of (1.20) of Theorem 1.3.

PROOF OF (1.21) OF THEOREM 1.3. We shall begin to state the following proposition which is inspired by the idea of Theorem 3.2 in Shibata and Shimizu [7].

PROPOSITION 4.1. *$\hat{K}_1(t, \xi', y)$ given by* (1.17) *satisfies*

$$|(\partial_{\xi'} + iby)^\alpha\{(\xi')^\gamma \partial_y^k \hat{K}_1(t, \xi', y)\}|$$
$$\leq C_{\alpha\gamma k}\sqrt{t}^{|\alpha|-k-|\gamma|}(e^{-(\varepsilon_0|y^2)/t} + e^{-\varepsilon_0|\xi'|y})e^{-\varepsilon_0|\xi'|^2 t},$$
$$t > 0,\ y > 0,\ \xi' \in \mathbf{R}^{n-1}\backslash 0, \tag{4.11}$$

for $\alpha, \gamma \in \mathbf{N}^{n-1}$, $k = 0, 1, 2, \ldots$ and some $\varepsilon_0 > 0$.

Let $K_1(t, x', x_n)$ be the Fourier inverse transform in ξ' of $\hat{K}_1(t, \xi', x_n)$. It follows from Proposition 4.1 that we can prove the following proposition.

PROPOSITION 4.2. *Assume $n \geq 3$ and let k be a non negative integer. Then $K(t, x', x_n) = 2\frac{\partial}{\partial x_n}K_1(t, x', x_n)$ satisfies*

$$\int_{R_+^n} |\nabla_x^k K(t, x', x_n)|\, dx' dx_n \leq \frac{C_k}{\sqrt{t}^k}, \quad t > 0, \tag{4.12}$$

where we denote $\nabla_x^k = \{\partial_{x'}^\alpha \partial_{x_n}^j; |\alpha| + j = k\}$.

This proposition and Lemma 2.5 imply (1.23) of Theorem 1.5. Because the solution of (1.8) is given by

$$\nabla_x^k u(t, x) = \int_{R_+^n} \nabla_x^k K(t, x' - y', x_n + y_n) u_0(x)\, dy' dy_n + \nabla^k u_D(t, x).$$

PROOF OF PROPOSITION 4.2. We shall prove that (4.11) implies (4.12). In fact, noting that

$$(-ix' + ibx_n)^\alpha \partial_{x'}^\gamma \partial_{x_n}^k K(t, x', x_n)$$
$$= \frac{1}{(2\pi)^{n-1}} \int e^{ix'\xi'}(\partial_{\xi'} + ibx_n)^\alpha\{(i\xi')^\gamma \partial_{x_n}^{k+1}\hat{K}_1(t, \xi', x_n)\}\, d\xi'$$

holds, we get by use of (4.11)

$$|\partial_{x'}^\gamma \partial_{x_n}^k K(t, x', x_n)| \leq \frac{C_{|\alpha|\gamma k}}{\sqrt{t}^{|\gamma|+k+1}}\left(\frac{\sqrt{t}}{|x' - bx_n|}\right)^{|\alpha|}$$
$$\times \int_{R^{n-1}} (e^{-\varepsilon_0 x_n^2/t} + e^{-\varepsilon_0|\xi'|x_n})e^{-\varepsilon_0|\xi'|^2 t}\, d\xi' \tag{4.13}$$

for any $\alpha \in \mathbf{N}^{n-1}$. Therefore using (4.13) with $|\alpha| = 0$ and with $|\alpha| = n$ and taking account that the assumption $n \geq 3$ and Fubini's Theorem imply

$$\int_0^\infty \int_{R^{n-1}} (e^{-\varepsilon_0 x_n^2/t} + e^{-\varepsilon_0 |\xi'| x_n}) e^{-\varepsilon_0 |\xi'|^2 t} \, d\xi' dx_n \leq C\sqrt{t}^{2-n} \int_0^\infty (r^{n-2} + r^{n-3}) \, dr$$
$$\leq C\sqrt{t}^{2-n},$$

we get

$$\int_{R^{n-1}} \int_0^\infty |\partial_{x'}^\gamma \partial_{x_n}^k K(t, x', x_n)| \, dx' dx_n$$
$$= \int_{|x' - \Re b x_n| \leq \sqrt{t}} \int_0^\infty |\partial_{x'}^\gamma \partial_{x_n}^k K(t, x', x_n)| \, dx' dx_n$$
$$+ \int_{|x' - \Re b x_n| \geq \sqrt{t}} \int_0^\infty |\partial_{x'}^\gamma \partial_{x_n}^k K(t, x', x_n)| \, dx' dx_n$$
$$\leq \frac{C_{|\alpha|\gamma k}}{\sqrt{t}^{|\gamma|+k+n-1}} \left\{ \int_{|x'| \leq \sqrt{t}} dx' + \int_{|x'| \geq \sqrt{t}} \left(\frac{\sqrt{t}}{|x'|} \right)^n dx' \right\} \leq \frac{C_{n\gamma k}}{\sqrt{t}^{|\gamma|+k}},$$

which means (4.12). Q.E.D.

PROOF OF PROPOSITION 4.1. It follows from the strong Lopatinski condition and the assumption $\Re d < 0$ in the case $(\Im d)^2 = (\Re d)^2 \neq 0$ that it suffices to consider the three cases below.

(a) $\Re \zeta_0 \leq \varepsilon |\xi'|$.

(b) $\Re \zeta_0 \geq \varepsilon |\xi'|$ and $\Re \lambda_0 \leq -c_1(1 + |\xi'|^2)$, $(c_1 > 0)$.

(c) $\Re \zeta_0 \geq \varepsilon |\xi'|$ and $|\zeta_0| \leq C|\xi'|$.

Here $\varepsilon > 0$ is determined later.

In fact, Since $\Re \zeta_0 \leq \varepsilon |\xi'|$ means (a), it suffices to consider only the case of $\Re \zeta_0 \geq \varepsilon |\xi'|$. In the case (i) of (1) in Theorem 1.1 $\Re \lambda_0 \leq -c_0(1 + |\xi'|^2)$ is valid. Hence this case means (b). In the case of (ii) of (1) in Theorem 1.1 we assume $\Re d < 0$. If $\Re \zeta_0 \geq \varepsilon |\xi'|$, then $|\xi'| \geq \varepsilon_0 > 0$ must hold. Hence we have $c_1 > 0$ such that $\Re \lambda_0 \leq -c_1(1 + |\xi'|^2)$, because $\Re \lambda_0 \leq -c_0 |\xi|^2$ holds from the strong Lopatinski condition. Hence this case is contained in the case (b). In the case (iii) of (1) of Theorem 1.1 we have $\zeta_0 = ib\xi'$ and so $|\zeta_0| \leq |b| \, |\xi'|$. This means the case (c), if $\Re \zeta_0 \geq \varepsilon |\xi'|$. Q.E.D.

We shall begin to prove (4.11). First we consider the case (a), that is, $\Re\zeta_0 \le \varepsilon|\xi'|$. Then it follows from the first term with $c = 0$ in (2.8) that we can express

$$\hat{K}_1(t, \xi', y) = \frac{e^{-|\xi'|^2 t}}{2\pi\sqrt{4\pi t}} \int_{-\infty}^{0} e^{-\zeta_0 z} e^{-(y-z)^2/4t}\, dz. \tag{4.14}$$

Applying (2.16) to $\partial_{\xi'}^{\alpha}((\xi')^{\gamma} e^{-|\xi'|^2 t})$ and to $|y-z|^{|\alpha|}\partial_y^k e^{-(y-z)^2/4t}$ we obtain

$$|\partial_{\xi'}^{\alpha}((\xi')^{\gamma} e^{-|\xi'|^2 t})| \le C_{\alpha\gamma\varepsilon} \sqrt{t}^{|\alpha-|\gamma||} e^{-(1-\varepsilon)|\xi'|^2} \tag{4.15}$$

and

$$|y-z|^{|\alpha|}|\partial_y^k e^{-(y-z)^2/4t}| \le C_{\alpha\varepsilon k} \sqrt{t}^{|\alpha|-k} e^{-(1-\varepsilon)((y-z)^2/4t)} \tag{4.16}$$

and using the relation $\frac{(y-2ct-z)^2}{4t} = \frac{(y-z)^2}{4t} - c(y-z) + c^2 t$, we can estimate by use of (4.15) and (4.16)

$$\begin{aligned}
&|(\partial_{\xi'} + iby)^{\alpha}(\xi')^{\gamma}\partial_y^k \hat{K}_1(t, \xi', y)| \\
&= \left|\left\{\frac{1}{2\pi\sqrt{4\pi t}} \int_{-\infty}^{0} e^{-\zeta_0 z}(ib(y-z) + \partial_{\xi'})^{\alpha}((\xi')^{\gamma} e^{-|\xi'|^2 t})\partial_y^k e^{-(y-z)^2/4t}\, dz\right\}\right| \\
&\le \frac{C_\alpha}{\sqrt{t}}\left\{\sum_{\alpha' \le \alpha} |\partial_{\xi'}^{\alpha-\alpha'}((\xi')^{\gamma} e^{-|\xi'|^2 t})| \int_{-\infty}^{0} |y-z|^{|\alpha'|} e^{-\Re\zeta_0 z}|\partial_y^k e^{-(y-z)^2/4t}|\, dz\right. \\
&\le C_{\alpha\gamma k} \frac{e^{-(1-\varepsilon)|\xi'|^2)t}}{\sqrt{t}^{(-|\alpha|+|\gamma|+k+1)}} \int_{-\infty}^{0} e^{-\Re\zeta_0 z - (1-\varepsilon/2)(y-z)^2/4t}\, dz \\
&\le C_{\alpha\gamma k} \frac{e^{(1-\varepsilon)((c^2-|\xi'|^2)t - cy)}}{\sqrt{t}^{(-|\alpha|+|\gamma|+k+1)}} \int_{-\infty}^{0} e^{((1-\varepsilon)c - \Re\zeta_0)z - (1-\varepsilon)(y-2cy-z)^2/4t} e^{-(\varepsilon/2)(y^2+z^2)/4t}\, dz \\
&\le \frac{C_{\alpha\gamma k}}{\sqrt{t}^{(-|\alpha|+|\gamma|+k)}} e^{-\varepsilon_2|\xi'|^2 t - \varepsilon_3 y},
\end{aligned} \tag{4.17}$$

which implies (4.11) in the case (a), where we choose $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$ and $c = \varepsilon_1|\xi'|$ such that $(1-\varepsilon)c - \Re\zeta_0 \ge (1-\varepsilon)c - \varepsilon|\xi'|0$, $(1-\varepsilon)\varepsilon_1 = \varepsilon_3$ and $(1-\varepsilon)(c^2 - |\xi'|^2) = -\varepsilon_2|\xi'|^2$ $(\varepsilon_2 > 0)$.

Next we consider the case of (b). It follows from (2.12) and the second term of (2.8) with $c = 0$ that we have

$$\hat{K}_1(t, \xi', y) = e^{\lambda_0 t - \zeta_0 y} - \frac{e^{-|\xi'|^2 t}}{2\pi\sqrt{4\pi t}} \int_{0}^{\infty} e^{-\zeta_0 z} e^{-(y-z)^2/4t}\, dz. \tag{4.18}$$

Hence noting $(\partial_{\xi'}+iby)^{\alpha}e^{-\zeta_0 y}=e^{-\zeta_0 y}\partial_{\xi'}^{\alpha}$ and $(\partial_{\xi'}+iby)^{\alpha}e^{-\zeta_0 z}=e^{-\zeta_0 z}(\partial_{\xi'}+ib(y-z))^{\alpha}$ we can see

$$\begin{aligned}
&(\partial_{\xi'}+iby)^{\alpha}((\xi')^{\gamma}\partial_y^k\hat{K}_1(t,\xi',y))\\
&\quad=(\partial_{\xi'}+iby)^{\alpha}\left\{(\xi')^{\gamma}\zeta_0^k e^{\lambda_0 t-\zeta_0 y}-\frac{(\xi')^{\gamma}e^{-|\xi'|^2 t}}{2\pi\sqrt{4\pi t}}\int_0^{\infty}e^{-\zeta_0 z}\partial_y^k e^{-(y-z)^2/4t}\,dz\right\}.\\
&\quad=e^{-\zeta_0 y}\partial_{\xi'}^{\alpha}((\xi')^{\gamma}\zeta_0^k e^{\lambda_0 t})\\
&\qquad+\sum_{\alpha'\le\alpha}C_{\alpha\alpha'}\frac{\partial_{\xi'}^{\alpha'}((\xi')^{\gamma}e^{-|\xi'|^2 t})}{2\pi\sqrt{4\pi t}}\int_0^{\infty}e^{-\zeta_0 z}(ib(y-z))^{\alpha-\alpha'}\partial_y^k e^{-(y-z)^2/4t}\,dz. \qquad (4.19)
\end{aligned}$$

We can prove by use of (4.15) and (4.16) that the second term above satisfies (4.11). In fact, taking account again that the relation $\frac{(y-2ty-z)^2}{4t}=\frac{(y-z)^2}{4t}-c(y-z)+c^2t$ we get by use of (4.15) and (4.16) analogously to (4.17)

$$\begin{aligned}
&\sum_{\alpha'\le\alpha}\left|\frac{\partial_{\xi'}^{\alpha'}((\xi')^{\gamma}e^{-|\xi'|^2 t})}{2\pi\sqrt{4\pi t}}\int_0^{\infty}e^{-\Re\zeta_0 z}(ib(y-z))^{\alpha-\alpha'}\partial_y^k e^{-(y-z)^2/4t}\,dz\right|\\
&\quad\le C_{\alpha\gamma\varepsilon}\sqrt{t}^{|\alpha|-|\gamma|-k}e^{(1-\varepsilon)((c^2-|\xi'|^2)t-cy)}\frac{1}{\sqrt{t}}\int_0^{\infty}e^{((1-\varepsilon)c-\Re\zeta_0)z}e^{-(1-\varepsilon)((y-2ty-z)^2/4t)}\,dz\\
&\quad\le C_{\alpha\gamma\varepsilon}\sqrt{t}^{|\alpha|-|\gamma|-k}e^{-\varepsilon_2|\xi'|^2t-\varepsilon_3|\xi'|y}, \qquad (4.20)
\end{aligned}$$

where we choose $\varepsilon>0$, $\varepsilon_1>0$, $\varepsilon_2>0$, $\varepsilon_3>0$ and $c=\varepsilon_1|\xi'|$ such that $(1-\varepsilon)c-\Re\zeta_0\le(1-\varepsilon)c-\varepsilon|\xi'|\le 0$, $(1-\varepsilon)\varepsilon_1=\varepsilon_3$ and $(1-\varepsilon)(c^2-|\xi'|^2)=-\varepsilon_2|\xi'|^2$.

Noting $\Re\lambda_0\le -c_0(1+\varepsilon_0|\xi'|^2)$ and using (2.17) we can prove that $|\partial_{\xi'}^{\alpha}((\xi')^{\gamma}\zeta_0^k e^{\lambda_0 t})|\le C_{\alpha}\sqrt{t}^{|\alpha|-|\gamma|-k}e^{-c_0(1+|\xi'|^2)t/2}$ is valid. Hence we see the first term of the right side in (4.19) satisfies

$$|e^{-\zeta_0 y}\partial_{\xi'}^{\alpha}\{(\xi')^{\gamma}\zeta_0^k e^{\lambda_0 t}\}|\le C_{\alpha k}\sqrt{t}^{|\alpha|-|\gamma|-k}e^{-(c_0/2)(1+|\xi'|^2)t}e^{-\varepsilon|\xi'|y} \qquad (4.21)$$

which and (4.20) imply (4.11) for the case (b).

Finally we consider the case (c), that is, $\varepsilon|\xi'|\le\Re\zeta_0$, $|\zeta_0|\le C|\xi'|$ and $\Re\lambda_0\le -c_0|\xi'|^2$ are valid. In this case we use again (4.19). By use of (2.16) we can show

$$|\partial_{\xi'}^{\alpha}\{(\xi')^{\gamma}\zeta_0^k e^{\lambda_0 t}\}|\le C_{\alpha\gamma k}\sqrt{t}^{|\alpha|-|\gamma|-k}e^{-(c_0/2)|\xi'|^2}.$$

Therefore we can see similarly to the case (b) that the first and second term of (4.19) satisfies (4.21) and (4.20) respectively. This shows (4.11) in the case (c). Thus we have proved (4.11) for all cases. Q.E.D.

5 Proof of Theorem 1.4

We shall prove Theorem 1.4 in this section. We shall show that $u_M^0(t,x)$ given by (1.19) in Theorem 1.2 does not satisfy (1.3) if $(\Im \frac{d}{c})^2 = (\Re \frac{d}{c})^2 \neq 0$ and $\Re \frac{d}{c} > 0$. We may assume $c = 1$. Then there is $\varepsilon > 0$ such that $\Re \zeta_0 > \frac{\Re d}{2}$ holds for $|\xi'| \leq \varepsilon$. We note that there is λ_0 the solution of equation $B(-\sqrt{\lambda + |\xi'|^2}, \xi') = 0$. Hence we have from (4.8)

$$\begin{aligned}\hat{K}(t,\xi',x_n) &= 2\zeta_0 e^{\lambda_0 t - \zeta_0 x_n} - \frac{1}{\pi\sqrt{4\pi t}} e^{-|\xi'|^2 t} \int_0^\infty e^{-\zeta_0 z} \frac{\partial}{\partial z} e^{-(x_n - z)^2/4t}\, dz \\ &= \hat{K}_1(t,\xi',x_n) + \hat{K}_2(t,\xi',x_n),\end{aligned} \tag{5.1}$$

where $\hat{K}_1(t,\xi',x_n) = 2\zeta_0 e^{\lambda_0 t - \zeta_0 x_n}$ and

$$\hat{K}_2(t,\xi',x_n) = -\frac{1}{\pi\sqrt{4\pi t}} e^{-|\xi'|^2 t} \int_0^\infty e^{-\zeta_0 z} \frac{\partial}{\partial z} e^{-(x_n - z)^2/4t}\, dz.$$

It is easy to check $|\hat{K}_2(t,\xi',x_n)| \leq \frac{C}{\sqrt{t}} e^{-|\xi'|^2 t}$ from (4.6). So $\hat{K}_2(t,\xi',x_n)$ satisfies

$$\int_{|\xi'| \leq \varepsilon} |\hat{K}_2(t,\xi',x_n)|\, d\xi' \leq Ct^{-n/2}. \tag{5.2}$$

On the other hand, we decompose $\hat{K}_1$ as

$$\begin{aligned}\frac{1}{2}\hat{K}_1(t,\xi',x_n) &= \zeta_0 e^{\lambda_0 t - \zeta_0 x_n} \\ &= ib\xi' e^{\lambda_0 t - \zeta_0 x_n} + d(e^{\lambda_0 t - \zeta_0 x_n} - e^{\lambda_0 t - d x_n}) + d e^{\lambda_0 t - d x_n} \\ &= \hat{K}_1^1(t,\xi',x_n) + K_1^2(t,\xi',x_n) + K_1^\infty(t,\xi',x_n).\end{aligned}$$

Define $K_i(t,x',x_n)$ by

$$K_i(t,x',x_n) = \int_{|\xi'| \leq \varepsilon} e^{ix'\xi'} \hat{K}_i(t,\xi',x_n)\, d\xi', \quad j = 1,2,$$

$$K_1^j(t,x',x_n) = \int_{|\xi'| \leq \varepsilon} e^{ix'\xi'} \hat{K}_1^j(t,\xi',x_n)\, d\xi', \quad j = 1,2,\infty$$

and put

$$u_j(t,x) = \frac{1}{(2\pi)^{n-1}} \int_{R_+^n} K_j(t,x'-y',x_n+y_n) u_0(y)\, dy, \quad j = 1,2,$$

$$u_1^j(t,x) = \frac{1}{(2\pi)^{n-1}} \int_{R_+^n} K_1^j(t,x'-y',x_n+y_n) u_0(y)\, dy, \quad j = 1,2,\infty,$$

and so we see $u = u_1^1 + u_1^2 + u_1^\infty + u_2$, where u_2 satisfies (1.3) from (5.2). We shall prove that u_1^1 and u_1^2 satisfy (1.3), and however we shall show that u_1^∞ does not satisfy (1.3) if we choose u_0 suitably. We begin to prove that

$$|K_1^j(t, x', x_n)| \le Ct^{-n/2}, \quad j = 1, 2. \tag{5.3}$$

In fact, $\hat{K}_1^1$ satisfies

$$|\hat{K}_1^1(t, \xi', x_n)| = |ib\xi' e^{\lambda_0 t - \zeta_0 x_n}| \le C|\xi| e^{-|\xi'|^2 t},$$

which implies (5.3) for $j = 1$ and taking account that $\Re\lambda_0 = -|\xi'|^2$ and $\zeta_0 = ib\xi' + d$, we can see $\hat{K}_1^2$ satisfies

$$\begin{aligned} |\hat{K}_1^2(t, \xi', x_n)| &= |d|\,|e^{\lambda_0 t - \zeta_0 x_n} - e^{\lambda_0 t - dx_n}| \\ &\le |d|\,|e^{\Re\lambda_0 t}(e^{-(ib\xi' + d)x_n} - e^{-dx_n})| \\ &\le C|\xi'| x_n e^{-|\xi'|^2 t + (|\Im b\xi'| - \Re d)x_n} \le C|\xi'| x_n e^{-|\xi'|^2 t - (\Re d/2)x_n} \le C|\xi'| e^{-|\xi'|^2 t} \end{aligned}$$

holds for $|\xi'| \le \varepsilon$, because $\Re d > 0$, and consequently yields (5.3) for $j = 2$. Finally we shall prove that the last term $u_1^\infty(t, x)$ does not satisfiy (1.3). In fact, noting that $\lambda_0 = -|\xi'|^2 + 2i(-\Re b\xi' \Im b\xi' + 2\Re d \Re b\xi' + \Re d \Im d)$, we have

$$\begin{aligned} u_1^\infty(t, x) &= \frac{d}{(2\pi)^{n-1}} \int_{R_+^n} K_1^\infty(t, x' - y', x_n + y_n) u_0(y)\, dy \\ &= \frac{d}{(2\pi)^{n-1}} \int_{R^{n-1}} \int_{R_+^n} e^{i(x' - y')\xi'} e^{\lambda_0 t - d(x_n + y_n)} u_0(y)\, d\xi' dy \\ &= \frac{d}{(2\pi)^{n-1}} e^{2i\Re d \Im dt - dx_n} \int_{R^{n-1}} \int_{R_+^n} e^{i(x' - y' + 4\Re d \Re bt)\xi'} \\ &\quad \times e^{-|\xi'|^2 t - i\Re b\xi' \Im b\xi' t - dy_n} u_0(y)\, d\xi' dy. \end{aligned} \tag{5.4}$$

Hence we get

$$\begin{aligned} u_1^\infty(t, -4\Re d \Re bt, 0) &= \frac{d}{(2\pi t^{1/2})^{n-1}} e^{2i\Re d \Im dt} \\ &\quad \times \int_{R^{n-1}} \int_{R_+^n} e^{-i(y'\xi'/t) - |\xi'|^2 - 2i\Re b\xi' \Im b\xi' - dy_n} u_0(y)\, d\xi' dy \\ &= \frac{d}{(2\pi t^{1/2})^{n-1}} e^{2i\Re d \Im dt}(A + o(1)), \quad t \to \infty \end{aligned}$$

which satisfies

$$|u_1^\infty(t, -4\Re d\Re bt, 0)| \geq \frac{|d|}{2(2\pi\sqrt{t})^{n-1}}|A|, \quad t \geq \exists t_0 > 0$$

and so does not satisfiy (1.3), if we choose $u_0 \in L^1(R_+^n)$ such that

$$A = \int_{R^{n-1}}\int_{R_+^n} e^{-|\xi'|^2 - 2i\Re b\xi'\Im b\xi' - dy_n} u_0(y)\, d\xi' dy \neq 0$$

and the support of $\hat{u}_0(\xi', x_n)$ contained in the set $\{|\xi'| \leq \varepsilon\}$. Thus we have proved Theorem 1.4.

Finally we remark that (1.21) with $k = 1$, $p = 1, \infty$ in general does not hold if $\Re d > 0$ in the case of $(\Im d)^2 = (\Re d)^2 \neq 0$. We shall give an example. Let $B = \partial_{x_n} + 1 + i$. Then we have $\zeta_0 = d = 1 + i$ and $\lambda_0 = -|\xi'|^2 + 2i$. Hence we have from (5.4)

$$\begin{aligned}
\partial_{x_n}^k u_1^\infty(t, x) &= \int_{R^{n-1}}\int_0^\infty 2\zeta_0^{1+k}\hat{K}(t, \xi', x_n + y_n) u_0(\xi', y_n)\, dy_n d\xi' \\
&= \int_{R^{n-1}}\int_0^\infty 2(1+i)^{1+k} e^{ix'\xi' - (1+i)(x_n + y_n) - |\xi'|^2 t + 2it} u_0(\xi', y_n)\, d\xi' dy_n \\
&= \frac{2(1+i)^{1+k} e^{2it - (1+i)x_n}}{(4\pi t)^{(n-1)/2}} \int_{R^{n-1}}\int_0^\infty e^{-|x' - y'|^2/4t - (1+i)y_n} u_0(y', y_n)\, dy' dy_n.
\end{aligned}$$

If we take u_0 such that $e^{ix_n} u_0 = 1$, we see

$$|\partial_{x_n}^k u_1^\infty(t, 0)| = 2|1 + i|^{k+1} \int_0^\infty e^{-y_n}\, dy_n \neq 0.$$

On the other hand (1.21) with $p = \infty$ implies

$$|\partial_{x_n}^k u_1^\infty(t, 0)| \leq \|\nabla_x u_1^\infty(t)\|_{L^\infty} \leq \frac{c_n}{t^{k/2}}\|u_0\|_{L^\infty} = \frac{c_n}{t^{k/2}}, \quad t > 0,\ k \geq 1.$$

The above two inequalities contradict.

Besides if we take $u_0 \in L^1$ such that $e^{-(1+i)y_n} u_0(y', y_n) = e^{-|y|^2}$, we see

$$\begin{aligned}
\int_{R_+^n} |\partial_{x_n}^k u_1^\infty(t, x)|\, dx &= 2|1 + i|^{1+k} \int_0^\infty e^{-x_n}\, dx_n \\
&\quad \times \frac{1}{(4\pi t)^{(n-1)/2}} \int_{R^{n-1}} e^{-|x'|^2/4t}\, dx' \int_{R_+^n} e^{-|y|^2}\, dy' dy_n \\
&= 2|1 + i|^{1+k} \int_0^\infty e^{-x_n}\, dx_n \int_{R_+^n} e^{-|y|^2}\, dy' dy_n \neq 0, \quad t > 0
\end{aligned}$$

which contradicts to (1.21) with $p=1$, $k \geq 1$. However in this example

$$\|u_1^{\infty}(t)\|_{L^p} \leq C\|u_0\|_{L^p}, \quad t>0,\ 1 \leq p \leq \infty$$

holds. Because

$$\int_{R_+^n} |K(t,x',x_n)|\,dx = \frac{|d|}{\sqrt{\pi}^{n-1}} \int_{R^{n-1}} e^{-|x'|^2}\,dx' \int_0^{\infty} e^{-x_n}\,dx_n$$

is finite.

6 Proof of Theorem 1.5

In this section we shall prove Theorem 1.5. We may assume $c=1$ without loss of generality. We shall show that u_M^0 given by (1.19) satisfies (1.23). The Fourier transform $u_M^0(t,\xi',x_n)$ of u_M^0 is given by

$$u_M^0(t,\xi',x_n) = \int_0^{\infty} \frac{1}{2\pi i} \int_{\Re\zeta=\tilde{c}} 2\zeta_0 \frac{e^{(\zeta^2-|\xi'|^2)t-(x_n+y_n)\zeta}}{\zeta-\zeta_0}\,d\zeta u_0(\xi',y_n)\,dy_n, \tag{6.1}$$

where $\tilde{c} > \max\{0, \Re\zeta_0\}$. Applying (2.7) to $e^{-(x_n+y_n)\zeta}$ we get for $x_n+y_n>0$, $\Re\zeta_0>0$

$$e^{-(x_n+y_n)\zeta} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi_n(x_n+y_n)}}{\zeta+i\xi_n}\,d\xi_n \tag{6.2}$$

and inseting the above relation into (6.1) we obtain

$$u_M^0(t,\xi',x_n) = \int_0^{\infty}\int_{\Re\zeta=\tilde{c}} \frac{2\zeta_0}{(2\pi)^2 i}\int_{-\infty}^{\infty} \frac{e^{(\zeta^2-|\xi'|^2)t+i(x_n+y_n)\xi_n}}{(\zeta-\zeta_0)(\zeta+i\xi_n)}\,d\xi_n d\zeta u_0(\xi',y_n)\,dy_n, \tag{6.3}$$

which implies

$$\begin{aligned}
\partial_{x'}^{\gamma}\partial_{x_n}^{j} u_M^0(t,x) &= \frac{1}{(2\pi)^{n-1}}\int_{R^{n-1}} e^{ix\xi}(i\xi')^{\gamma}\partial_{x_n}^{j} u_M^0(t,\xi',x_n)\,d\xi' \\
&= \frac{1}{(2\pi)^n}\int_{R^n} e^{ix\xi}(i\xi')^{\gamma}\int_0^{\infty}\frac{1}{2\pi i}\int_{\Re\zeta=\tilde{c}} 2\zeta_0(-\zeta)^j \frac{e^{(\zeta^2-|\xi'|^2)t+iy_n\xi_n}}{(\zeta-\zeta_0)(\zeta+i\xi_n)}\,d\zeta \\
&\quad \times u_0(\xi',y_n)\,dy_n d\xi \\
&= \frac{1}{(2\pi)^n}\int_{R^n} e^{ix\xi}\hat{K}_+(t,\xi)F(eu_0)(\xi)\,d\xi,
\end{aligned} \tag{6.4}$$

where $eu_0(y) = 0,\ y_n > 0,\ = u_0(y', -y_n),\ y_n < 0$, $F(eu_0)(\xi)$ means Fourier image of $eu_0(x)$ and

$$\hat{K}_+(t,\xi) = \frac{1}{2\pi i}\int_{\Re\zeta = c > \max\{\Re\zeta_0, 0\}} 2\zeta_0 \frac{(i\xi')^\gamma(-\zeta)^j e^{(\zeta^2 - |\xi'|^2)t}}{(\zeta-\zeta_0)(\zeta+i\xi_n)}\,d\zeta. \tag{6.5}$$

Let $\chi(t)$ be in $C_0^\infty(R)$ such that $\chi(t) = 1,\ |t| < 1,\ = 0,\ |t| > 2$ and put for $\varepsilon > 0$

$$\chi_0(\xi) = 1 - \chi\left(\frac{|\zeta_0 + i\xi_n|}{\varepsilon|\xi|}\right), \quad \chi_1(\xi) = \chi\left(\frac{|\zeta_0 + i\xi_n|}{\varepsilon|\xi|}\right).$$

Then takng account that $|\partial_\xi^\alpha|\zeta_0 + i\xi_n|| \le C_\alpha|\xi|^{1-|\alpha|}$ holds if $\varepsilon \le \frac{|\zeta_0+i\xi_n|}{|\xi|} \le 2\varepsilon$, we can see easily that χ_j satisfies

$$|\partial_\xi^\alpha \chi_j(\xi)| \le C_\alpha|\xi|^{-|\alpha|}, \quad j = 0, 1,\ \xi \in \mathbf{R}^n\backslash 0 \tag{6.6}$$

PROPOSITION 6.1. *$u_M^0(t,x)$ given by (6.4) satisfies*

$$\begin{aligned}\partial_x^{\gamma,j} u_M^0(t,x) &= \frac{1}{(2\pi)^n}\int_{R^n} e^{ix\xi}\chi_0(\xi)(\hat{K}_+(t,\xi) - \hat{K}_-(t,\xi))F(eu_0)(\xi)\,d\xi \\ &\quad - \frac{1}{(2\pi)^n}\int_{R^n} e^{ix\xi}2\zeta_0\chi_1(\xi)(i\xi')^\gamma \int_0^1 (j(-\zeta_\theta)^{j-1} + 2(-\zeta_\theta)^{j+1}t)e^{\zeta_\theta^2 t}\,d\theta \\ &\quad \times e^{-|\xi'|^2 t}F(eu_0)(\xi)\,d\xi,\end{aligned} \tag{6.7}$$

where $\zeta_\theta = \theta\zeta_0 - (1-\theta)i\xi_n$ and

$$\hat{K}_\pm(t,\xi) = \frac{1}{2\pi i}\int_{\Re\zeta = c_\pm} 2\zeta_0 \frac{(i\xi')^\gamma(-\zeta)^j e^{(\zeta^2 - |\xi'|^2)t}}{(\zeta-\zeta_0)(\zeta+i\xi_n)}\,d\zeta, \tag{6.8}$$

here we choose $c_\pm$ such that $c_- < \min\{0, \Re\zeta_0\}$ and $c_+ > \max\{0, \Re\zeta_0\}$.

PROOF. Since we can write $\hat{K}_+(t,\xi) = (\chi_0 + \chi_1)\hat{K}_+(t,\xi)$, in order to get (6.7) it suffices to prove that $\chi_1\hat{K}_+(t,\xi)$ satisfies

$$\begin{aligned}&\int_{R^n} e^{ix\xi}\chi_1\hat{K}_+(t,\xi)F(eu_0)(\xi)\,d\xi \\ &\quad = -\int_{R^n} e^{ix\xi}\chi_0\hat{K}_-(t,\xi)F(eu_0)(\xi)\,d\xi - \int_{R^n} e^{ix\xi}2\zeta_0\chi_1(\xi) \\ &\qquad \times \int_0^1 (j(-\zeta_\theta)^{j-1} + 2(-\zeta_\theta)^{j+1}t)e^{\zeta_\theta^2 t}\,d\theta e^{-|\xi'|^2 t}F(eu_0)(\xi)\,d\xi.\end{aligned} \tag{6.9}$$

In fact, it follows from (2.19) with $c_1 = c_+$, $c_2 = c_-$ that we obtain

$$\int_{R^n} e^{ix\xi}\chi_1 \hat{K}_+(t,\xi)F(eu_0)(\xi)\,d\xi$$

$$= \int_{R^n} e^{ix\xi}\chi_1 \hat{K}_-(t,\xi)F(eu_0)(\xi)\,d\xi - \int_{R^n} e^{ix\xi}2\zeta_0\chi_1(\xi)$$

$$\times \int_0^1 (j(-\zeta_\theta)^{j-1} + 2(-\zeta_\theta)^{j+1}t)te^{\zeta_\theta^2 t}\,d\theta e^{-|\xi'|^2 t}F(eu_0)(\xi)\,d\xi. \tag{6.10}$$

On the other hand, since it follows from (2.7) that for $\Re\zeta < 0$, $x_n > 0$ and $y_n > 0$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(ix+y_n)\xi_n}}{\zeta + i\xi_n}\,d\xi_n = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i(x-y_n)\xi_n}}{\xi_n - i\zeta}\,d\xi_n$$

$$= -e^{-\zeta(x_n+y_n)}H(-(x_n+y_n)) = 0,$$

we can see by use of (6.8)

$$\int_{R^n} e^{ix\xi}\chi_1 \hat{K}_-(t,\xi)F(eu_0)(\xi)\,d\xi$$

$$= \int_{R^n} e^{ix'\xi' + ix_n\xi_n}\chi_1(\xi)\frac{1}{2\pi i}\int_{\Re\zeta = c_-} 2\zeta_0 \frac{(i\xi')^\gamma(-\zeta)^j e^{(\zeta^2 - |\xi'|^2)t}}{(\zeta - \zeta_0)(\zeta + i\xi_n)}\,d\zeta F(eu_0)(\xi)\,d\xi' d\xi_n$$

$$= \int_{R^{n-1}}\int_0^\infty e^{ix'\xi'}\frac{1}{2\pi i}\int_{\Re\zeta = c_-} 2\zeta_0 \frac{(i\xi')^\gamma(-\zeta)^j e^{(\zeta^2 - |\xi'|^2)t}}{(\zeta - \zeta_0)}\,d\zeta$$

$$\times \int_{-\infty}^{\infty} \chi_1(\xi)\frac{e^{i(x_n+y_n)\xi_n}}{(\zeta + i\xi_n)}\,d\xi_n u_0(\xi', y_n)\,d\xi'$$

$$= \int_{R^n} e^{ix'\xi' + ix_n\xi_n}(\chi_1 - 1)\frac{1}{2\pi i}\int_{\Re\zeta = c_-} 2\zeta_0 \frac{(i\xi')^\gamma(-\zeta)^j e^{(\zeta^2 - |\xi'|^2)t}}{(\zeta - \zeta_0)(\zeta + i\xi_n)}\,d\zeta F(eu_0)(\xi)\,d\xi,$$

which implies (6.9) together with (6.10). Q.E.D.

Decompose

$$\frac{1}{(\zeta - \zeta_0)(\zeta + i\xi_n)} = \frac{1}{\zeta_0 + i\xi_n}\left\{\frac{1}{\zeta - \zeta_0} - \frac{1}{\zeta + i\xi_n}\right\}.$$

We can write

$$\hat{\chi}_0 K_{\pm}(t,\xi) = \frac{2\zeta_0}{2\pi i(\zeta_0 + i\xi_n)}\chi_0(\xi)\int_{\substack{(-\zeta)^j \\ \Re\zeta = c_{\pm}}} e^{(\zeta^2 - |\xi'|^2)t}\left\{\frac{1}{\zeta - \zeta_0} - \frac{1}{\zeta + i\xi_n}\right\} d\zeta. \quad (6.11)$$

Put

$$\hat{G}(t,\xi) = \frac{2\zeta_0}{\pi(\zeta_0 + i\xi_n)}\chi_0(\xi), \quad (6.12)$$

$$\hat{K}^1_{0\pm}(t,\xi') = \frac{1}{2\pi i}\int_{\Re\zeta = c_{\pm}} (i\xi')^{\gamma}(-\zeta)^j e^{(\zeta^2 - |\xi'|^2)t}\frac{1}{(\zeta - \zeta_0)}\, d\zeta \quad (6.13)$$

and

$$\begin{aligned}\hat{K}^2_{0\pm}(t,\xi) &= \frac{1}{2\pi i}\int_{\Re\zeta = c_{\pm}} (i\xi')^{\gamma}(-\zeta)^j e^{(\zeta^2 - |\xi'|^2)t}\frac{1}{\zeta + i\xi_n}\, d\zeta \\ &= e^{-|\xi'|^2 t}(i\xi')^{\gamma}\hat{g}_{\pm}(t,\xi_n), \end{aligned} \quad (6.14)$$

where $g_{\pm}(t,\xi_n) = \frac{1}{2\pi i}\int_{\Re\zeta = c_{\pm}} e^{\zeta^2 t}\frac{(-\zeta)^j}{\zeta + i\xi_n}\, d\zeta$. Moreover we put

$$\hat{R}(t,\xi) = -2\zeta_0 (i\xi')^{\gamma}\chi_1(\xi)\int_0^1 (j(-\zeta_\theta)^{j-1} + 2(-\zeta_\theta)^{j+1}t)e^{\zeta_\theta^2 t}\, d\theta e^{-|\xi'|^2 t}. \quad (6.15)$$

Then the relation

$$\hat{K}_+(t,\xi) = \hat{G}(t,\xi)\{\hat{K}^1_{0+}(t,\xi') - \hat{K}^1_{0-}(t,\xi') + \hat{K}^2_{0-}(t,\xi) - \hat{K}^2_{0+}(t,\xi)\} + \hat{R}(t,\xi) \quad (6.16)$$

holds. Therefore in order to prove that u_M satisfies (1.23) it suffices to show the following lemma.

Lemma 6.1. *$\hat{G}(t,\xi)$, $\hat{K}^1_{0\pm}(t,\xi')$, $\hat{g}_{\pm}(t,\xi_n)$ and $\hat{R}(t,\xi)$ satisfy the following estimates.*

(1) $|\partial_\xi^\alpha \hat{G}(t,\xi)| \le C_\alpha |\xi|^{-|\alpha|}$,
for $\xi(\ne 0) \in \mathbf{R}^n$ *and for all* $\alpha \in \mathbf{N}^n$.

(2) $|\partial_{\xi'}^\alpha \hat{K}^1_{0\pm}(t,\xi')| \le C_\alpha t^{-(|\gamma|+j)/2}|\xi'|^{-|\alpha|}$,
for $\xi'(\ne 0) \in \mathbf{R}^{n-1}$ *and for all* $\alpha \in \mathbf{N}^{n-1}$.

(3) $\left|\left(\frac{d}{d\xi_n}\right)^k \hat{g}_{\pm}(t,\xi_n)\right| \le C_k t^{-j/2}|\xi_n|^{-k}$, $k = 0, 1, \dots$
for $\xi_n(\ne 0) \in \mathbf{R}$.

(4) $|\partial_\xi^\alpha \hat{R}(t,\xi)| \le C_\alpha t^{-(|\gamma|+j)/2}|\xi|^{-|\alpha|}$,
for $\xi(\ne 0) \in \mathbf{R}^n$ *and for all* $\alpha \in \mathbf{N}^n$.

PROOF. (1). Noting that ξ in the support of χ_0 implies $|\zeta_0 + i\xi_n| \geq \varepsilon|\xi|$ and that $\zeta_0 = ib\xi' + d$ is valid, we can see easily

$$\left|\partial_\xi^\alpha\left\{\frac{\zeta_0}{\zeta_0 + i\xi_n}\right\}\right| \leq C_\alpha|\xi|^{-|\alpha|}, \quad \forall\alpha \tag{6.17}$$

holds and so we get (1) by use of (6.6).

(2). It follows from (2.12) with $y = 0$ that

$$\hat{K}_{0+}^1(t,\xi') = (i\xi')^\gamma(-\zeta_0)^j e^{(\zeta_0^2 - |\xi'|^2)t} + \hat{K}_{0-}^1(t,\xi') \tag{6.18}$$

If $\Re\zeta_0 \geq 0$, we get from (2.11) with $y = 0$

$$\hat{K}_{0-}^1(t,\xi') = \frac{-(i\xi')^\gamma}{2\pi\sqrt{4\pi t}} e^{-|\xi'|^2 t}\int_0^\infty e^{-\zeta_0 z}\partial_z^j e^{-z^2/4t}\,dz. \tag{6.19}$$

It follows from (2) of Theorem 1.1 that $\Re\zeta_0^2 - |\xi'|^2 = \lambda_0 \leq -c_0|\xi'|^2$ for $\xi' \in \mathbf{R}^{n-1}$. Hence we can see by use of Lemma 2.4

$$\begin{aligned}|\partial_{\xi'}^\alpha((i\xi')^\gamma e^{(\zeta_0^2 - |\xi'|^2)t})| &\leq C_\alpha\sum_{\alpha' \leq \alpha}|\xi'|^{|\gamma - \alpha'|}\sqrt{t}^{|\alpha - \alpha'|}e^{-(c_0/2)|\xi'|^2 t} \leq C_\alpha|\xi'|^{|\gamma| - |\alpha|}e^{-(c_0/4)|\xi'|^2 t}\\ &\leq C_\alpha|\xi|^{-|\alpha|}t^{-|\gamma|/2}e^{-(c_0/8)|\xi'|^2 t}, \quad \forall\alpha.\end{aligned}$$

Moreover we see

$$\begin{aligned}|\partial_{\xi'}^\alpha e^{-\zeta_0 z}|e^{-z^2/4t} &= |b^\alpha z^{|\alpha|}e^{-\zeta_0 z}|e^{-z^2/4t}\\ &\leq C_\alpha|z|^{|\alpha|}e^{-z^2/4t} \leq C_\alpha\sqrt{t}^{|\alpha|}e^{-z^2/8t}\end{aligned}$$

and so we get from (6.19)

$$\begin{aligned}|\partial_\xi^\alpha\hat{K}_{0-}^1(t,\xi')| &= \left|\partial_{\xi'}^\alpha\left\{(i\xi')^\gamma e^{-|\xi'|^2 t}\frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\zeta_0 z}\partial_z^j e^{-z^2/4t}\,dz\right\}\right|\\ &\leq \sum_{\alpha' \leq \alpha}C_{\alpha,\alpha'}|\partial_{\xi'}^{\alpha'}\{(i\xi')^\gamma e^{-|\xi'|^2 t}\}|\frac{1}{\sqrt{t}}\int_0^\infty|\partial_{\xi'}^{\alpha - \alpha'}e^{-\zeta_0 z}\partial_z^j e^{-z^2/4t}|\,dz\\ &\leq C_\alpha\sum_{\alpha' \leq \alpha}|\xi'|^{-|\alpha'|}e^{-(1/2)|\xi'|^2 t}\sqrt{t}^{|\alpha - \alpha'| - |\gamma| - j} \leq \frac{C_\alpha}{\sqrt{t}^{|\gamma| + j}}|\xi'|^{-|\alpha|},\end{aligned}$$

for $\Re\zeta_0 \geq 0$. Hence $\hat{K}^1_{0+}(t,\xi')$ also satisfies (2) for $\Re\zeta_0 \geq 0$ from (6.18). When $\Re\zeta_0 < 0$, we can take $c_+ = 0$ in (6.13) and so we get by use of the first term of (2.8) with $y = 0$

$$\hat{K}^1_{0+}(t,\xi') = \frac{(i\xi')^\gamma}{\sqrt{4\pi t}} e^{-|\xi'|^2 t} \int_{-\infty}^0 e^{-\zeta_0 z} \partial_z^j e^{-z^2/4t}\, dz.$$

Therefore we can prove analogously to K^1_{0-} in the case of $\Re\zeta_0 \geq 0$ that $\hat{K}^1_{0+}(t,\xi')$ satiafies (2) for $\Re\zeta_0 < 0$ and consequently we can see that $\hat{K}^1_{0-}(t,\xi')$ satiafies (2) for $\Re\zeta_0 < 0$ from the relation (6.18). Thus we have proved (2).

(3). Let us consider g_+. By use of the first term of (2.11) with $y = 0$ and with $\zeta_0 = i\xi_n$ we can write

$$\begin{aligned} g_+(t,\xi_n) &= \frac{1}{2\pi i} \int_{\Re\zeta = c_-} e^{\zeta^2 t} \frac{(-\zeta)^j}{\zeta + i\xi_n}\, d\zeta \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^0 e^{-i\xi_n z} \partial_z^j e^{-z^2/4t}\, dz. \end{aligned}$$

and consequently we see

$$\begin{aligned} \left(\frac{d}{d\xi_n}\right)^k \hat{g}_+(t,\xi_n) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^0 (-iz)^k e^{-i\xi_n z} \partial_z^j e^{-z^2/4t}\, dz \\ &= \frac{1}{\sqrt{4\pi t}} (-i\xi_n)^{-k} \int_{-\infty}^0 \left\{ \left(\frac{d}{dz}\right)^k e^{-i\xi_n z} \right\} \{(-iz)^k \partial_z^j e^{-z^2/4t}\}\, dz \\ &= \frac{1}{\sqrt{4\pi t}} (-i\xi_n)^{-k} \int_{-\infty}^0 e^{-i\xi_n z} \left(-\frac{d}{dz}\right)^k \{(-iz)^k \partial_z^j \partial_z^j e^{-z^2/4t}\}\, dz. \end{aligned}$$

Moreover we can see

$$\left| \left(\frac{d}{dz}\right)^k \{(-iz)^k \partial_z^j e^{-z^2/4t}\} \right| \leq C_k \sqrt{t}^{-j} e^{-z^2/8t},$$

which implies (3) for g_+. Analogously we can see that g_- satisfies (2).

(4). Since $|\zeta_0 + i\xi_n| \leq 2\varepsilon|\xi|$ for ξ in the support of χ_1, we see $|\zeta_0| \leq (1 + 2\varepsilon)|\xi|$, $|\zeta_\theta| \leq C|\xi|$ and moreover

$$\begin{aligned} &\Re(i\xi_n\theta - (1-\theta)\zeta_0)^2 - |\xi'|^2 \\ &= \Re(i\xi_n - (1-\theta)(\zeta_0 + i\xi_n))^2 - |\xi'|^2 \\ &= -|\xi|^2 - \Re\{2i\xi_n(1-\theta)(\zeta_0 + i\xi_n) - (1-\theta)^2(\zeta_0 + i\xi_n)^2\} \\ &\leq -|\xi|^2 + 8\varepsilon^2|\xi|^2 \leq -\frac{1}{2}|\xi|^2, \end{aligned}$$

if we choose $\varepsilon > 0$ such that $1 - 8\varepsilon^2 \geq \frac{1}{2}$. Hence we get by use of Lemma 2.4

$$|\partial_\xi^\alpha (i\xi')^\gamma e^{(i\xi_n\theta+(1-\theta)\zeta_0)^2 t-|\xi'|^2 t}| \leq C_\alpha \sum_{\alpha'\leq\alpha} |\xi|^{|\gamma|-|\alpha'|}\sqrt{t}^{|\alpha-\alpha'|}e^{-(1/3)|\xi|^2 t}$$

$$\leq \frac{C_\alpha}{\sqrt{t}^{|\gamma|}}|\xi|^{-|\alpha|}e^{-(1/4)|\xi|^2 t}$$

and consequently we can estimate

$$\left|\partial_\xi^\alpha\left\{\zeta_0\int_0^1((j((-\zeta_\theta)^{j-1}+2(-\zeta_\theta)^{j+1}t)e^{(i\xi_n\theta+(1-\theta)\zeta_0)^2 t-|\xi'|^2 t}\,d\theta\right\}\right|$$

$$\leq C_\alpha(|\xi|^j+|\xi|^{j+2}t)|\xi|^{-|\alpha|}e^{-(1/4)|\xi|^2 t} \leq \frac{C_\alpha}{\sqrt{t}^j}|\xi|^{-|\alpha|}, \tag{6.20}$$

for ξ in the support of χ_1 and for $t > 0$. Therefore we get by use of (6.6)

$$|\partial_\xi^\alpha \hat{R}(t,\xi)| \leq \sum C_{\alpha\alpha'}|\partial_\xi^{\alpha-\alpha'}(i\xi')^\gamma\chi_1|$$

$$\times\left|\partial_\xi^{\alpha'}\left\{\zeta_0\int_0^1((j((-\zeta_\theta)^{j-1}+2(-\zeta_\theta)^{j+1}t)e^{(i\xi_n\theta+(1-\theta)\zeta_0)^2 t-|\xi'|^2 t}\,d\theta\right\}\right|$$

$$\leq C_\alpha|\xi|^{|\gamma|+j-|\alpha|}e^{-(1/5)|\xi|^2 t} \leq \frac{C_\alpha}{\sqrt{t}^{|\gamma|+j}}|\xi|^{-|\alpha|}, \quad \alpha\in N^n,$$

which means (4). Thus we have completed the proof of Lemma 6.1. Q.E.D.

Let $K_\pm(t,x)$, $G(t,x)$, $K_{0\pm}^l(t,x')$, $(l=1,2)$, $g_\pm(t,x_n)$ and $R(t,x)$ be Fourier inverse transform of $\hat{K}_\pm(t,\xi)$, $\hat{G}(t,\xi)$, $\hat{K}_{0\pm}^1(t,\xi')$, $\hat{g}_\pm(t,\xi_n)$ and $\hat{R}(t,\xi)$ respectively and denote by $G(t)$, $K_{0\pm}^l(t)$ $(l=1,2)$, $g_\pm(t)$ and $R(t)$ the singular integral operators with the kernel $G(t,x)$, $K_{0\pm}^l(t,x')$ $(l=1,2)$, $\hat{g}_\pm(t,x_n)$ and $R(t,x)$ respectively, where (6.14) means $K_{0\pm}^2(t) = D_{x'}^\gamma e^{\Delta' t}g_\pm(t,x_n)$. Then it follows from Lemma 6.1 and Lemma 2.3 that all singular integral operators $G(t)$, $K_{0\pm}^l(t)$ $(l=1,2)$ and $R(t)$ are bounded in $L^p(R^n)$ and consequently it follows from (6.16) and Lemma 6.1 that $\partial_x^{\gamma,j}u_M = K_+(t)(eu_0) = \{G(t)(K_{0+}^1(t)-K_{0-}^1(t)+K_{0-}^2(t)-K_{0+}^2(t))+R(t)\}(eu_0)$ satisfies (1.23). Thus we have proved Theorem 1.5.

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