

PERIODICITY AND EIGENVALUES OF MATRICES OVER QUASI-MAX-PLUS ALGEBRAS

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Abstract. We extend the notion of a max-plus algebra and study periodicity and eigenvalues of matrices over this new structure thereby generalizing some well-known results on matrices over a max-plus algebra.

1. Introduction

The concept of the standard max-plus algebra over the real numbers has turned out to be a useful tool in applications (discrete event systems, optimal control, game theory) and several other fields of mathematics (matrix theory, combinatorics, asymptotic analysis, geometry). The reader is referred to [4, 9, 12] for details. M. Gavalec [10] introduced the notion of a max-plus algebra in a broader framework (see Section 2 for details) and proved among other things that every irreducible matrix over a max-plus algebra in this new setting is almost linear periodic. We establish a generalization of this result (see Theorem 3.5 below).

Our main concern is an extension of the notion ‘max-plus algebra’ by introducing a quasi-max-plus algebra (see the Definition 2.3). This concept is inspired by a certain dioid over the integers which was introduced in [1] for the description of primitive matrices over polynomial rings. We study periodicity properties and eigenvalues of matrices over a quasi-max-plus algebra (see Theorems 3.7, 3.8 and 3.10). In an appendix we mainly collect some results which are well-known under stronger prerequisites, but which are needed here in a more general setting.

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2. Definition and Examples of Quasi-Max-Plus Algebras

In this paper we always let $(\mathcal{D}, \oplus, \odot)$ be a commutative dioid with neutral elements ε and e , respectively, i.e., \mathcal{D} is a commutative unital semiring with idempotent addition (see [2, Definition 4.1]).¹ As customary we often omit the multiplication sign \odot if there is no fear of confusion. In particular, we write

$$a^n = \underbrace{a \odot \cdots \odot a}_{n \text{ factors}} \quad (a \in \mathcal{D}, n \in \mathbf{N}_{>0})$$

and $a^0 = e$. If not stated otherwise we always use the natural (partial) order on \mathcal{D} , i.e., $a \leq b$ if and only if $a \oplus b = b$ (see [2, Section 4.3.2]). In particular, the natural order is compatible with addition and ε is the minimal element of \mathcal{D} (see [2, Theorem 4.28]).

For the sake of completeness we recall several definitions. The dioid \mathcal{D} is called

- entire if for all $\alpha, \beta \in \mathcal{D}$ the equation $\alpha\beta = \varepsilon$ implies $\alpha = \varepsilon$ or $\beta = \varepsilon$ (see [2, Definition 4.11]),
- cancellative if for $\alpha, \beta, \gamma \in \mathcal{D}$ with $\alpha\beta = \alpha\gamma$ and $\alpha \neq \varepsilon$ we have $\beta = \gamma$,
- archimedean if for all $\alpha, \beta \in \mathcal{D}$ there is some $\gamma \in \mathcal{D}$ such that $\gamma\beta \geq \alpha$ provided $\beta \neq \varepsilon$ (see [2, Definition 4.33]),
- algebraically closed² if for every $\alpha \in \mathcal{D}$ and $n \in \mathbf{N}_{>0}$ the equation $x^n = \alpha$ admits a solution in \mathcal{D} .

Finally we say that \mathcal{D} satisfies the weak stabilization condition (see [8, Definition 1.1.5]) if for all $\alpha, \beta, \lambda, \mu \in \mathcal{D}$ there exist $\gamma, \nu \in \mathcal{D}$ and $N \in \mathbf{N}$ such that for all $n \geq N$ we have

$$\alpha\lambda^n \oplus \beta\mu^n = \gamma\nu^n.$$

A particular class of dioids was introduced under the name ‘extremal algebra’ by J. Nedoma [14] in 1974 and under the name ‘max-plus algebra’ by M. Gavalec [10] in 2000. Here we prefer the more suggestive latter notion. Let $(G, +, \leq)$ be an abelian linearly ordered divisible group with neutral element 0 and $\varepsilon \notin G$ a new element. We call the dioid $(G \cup \{\varepsilon\}, \oplus, \odot)$ the max-plus algebra generated by G where the operations are given by $\oplus = \max$ and $\odot = +$ and ε enjoys the

¹ By abuse of notation we use the same symbols ε and e for the neutral elements of all dioids which occur in the subsequent text.

² In [4] this property is called radicable. In the translation [7] this property is inadvertently named ‘algebraic completeness’.

properties

$$\varepsilon < g \quad (g \in G),$$

and

$$\varepsilon + g = g + \varepsilon = \varepsilon \quad (g \in G \cup \{\varepsilon\}).$$

Correspondingly, the dioid \mathcal{D} is called a max-plus algebra if there exists a commutative linearly ordered divisible group G such that \mathcal{D} is the max-plus algebra generated by G .

EXAMPLE 2.1. Let G be an additive divisible subgroup of the real numbers \mathbf{R} . Then $(G \cup \{-\infty\}, \max, +)$ is a max-plus algebra.³ In particular, the standard max-plus algebra (or simply max-algebra) fits into these settings. For details and applications we refer the reader to [4].

The following elementary properties of max-plus algebras can be verified in a straightforward manner.

PROPOSITION 2.2. *Let \mathcal{D} be a max-plus algebra.*

- (i) *\mathcal{D} is a commutative linearly ordered idempotent semifield⁴ of characteristic zero and with neutral elements ε and e , respectively. Moreover, ε is the minimal element of \mathcal{D} .*
- (ii) *\mathcal{D} is cancellative.*
- (iii) *For every $\alpha \in \mathcal{D}$ and $n \in \mathbf{N}_{>0}$ the equation $x^n = \alpha$ has a unique solution in \mathcal{D} , namely*

$$\alpha/n := \frac{\alpha}{n}.$$

In particular, \mathcal{D} is algebraically closed, and we have $e/n = e$.

- (iv) *Let $\alpha \in \mathcal{D}$ and $n, m \in \mathbf{N}_{>0}$. Then*

$$\frac{\alpha^m}{nm} = \frac{\alpha}{n}.$$

³In connection with the set $\mathbf{R} \cup \{-\infty, \infty\}$ we always use the conventions $(-\infty) \pm (-\infty) = -\infty$ and $-\infty < x, |-\infty| > x$ for all $x \in \mathbf{R}$. Furthermore, every positive real divides $\pm\infty$.

⁴See [2, Definition 3.1].

(v) Let $\alpha, \beta \in \mathcal{D}$ and $n, m \in \mathbf{N}_{>0}$. Then

$$\frac{\alpha}{n} < \frac{\alpha + \beta}{n + m}$$

implies $\alpha/n < \beta/m$.

It is well-known that new dioids can be formed on cartesian products of \mathcal{D} : Either we equip \mathcal{D}^n with componentwise sum and product or we form the free symmetrized dioid on \mathcal{D} (see [8, Sections 1.1 and 2.2]). Based on ideas of [1] we now introduce a different construction thereby extending the notion of a max-plus algebra.

DEFINITION 2.3. The dioid $(\mathcal{D}, \oplus, \odot)$ is called a quasi-max-plus algebra if it satisfies the following properties:

- (i) There exist a max-plus algebra F and a totally ordered set S with at most two elements such that $\mathcal{D} \subseteq F \times S$.
- (ii) The projection on the first component $\pi : \mathcal{D} \rightarrow F$ is a dioid homomorphism which enjoys the following properties:
 - (a) $\pi^{-1}(\{\varepsilon_F\}) = \varepsilon$ and $\pi^{-1}(\{e_F\}) = e$.
 - (b) For all $\alpha, \beta \in \mathcal{D}$ the following implication holds.⁵

$$\pi(\alpha) = \pi(\beta) \Rightarrow (\alpha \oplus \beta)_2 = \max\{\alpha_2, \beta_2\} \quad (\alpha, \beta \in \mathcal{D}).$$

- (iii) The second components of the neutral elements of \mathcal{D} equal the minimal element of S , i.e., we have $\varepsilon_2 = e_2 = \min S$.
- (iv) For all $\alpha \in \mathcal{D}$ we have $(\alpha_1, \min S) \in \mathcal{D}$.
- (v) For all $\alpha, \beta \in \mathcal{D} \setminus \{\varepsilon\}$ we have

$$(\alpha\beta)_2 = \max\{\alpha_2, \beta_2\}.$$

- (vi) \mathcal{D} is algebraically closed.

EXAMPLE 2.4. Plainly, every max-plus algebra can be regarded as a quasi-max-plus algebra: Just take S to be a singleton.

⁵We write x_i for the i -th component of the element x of the cartesian product of a family of sets $(X_i)_{i \in I}$.

In addition to the natural order we impose the lexicographical order on a quasi-max-plus algebra by

$$\alpha <_{\text{lex}} \beta \Leftrightarrow \alpha_1 < \beta_1 \quad \text{or} \quad (\alpha_1 = \beta_1 \text{ and } \alpha_2 < \beta_2).$$

The following basic properties of a quasi-max-plus algebra will be used in the sequel.

PROPOSITION 2.5. *Let \mathcal{D} the quasi-max-plus algebra contained in $F \times S$ where F is a max-plus algebra and S a totally ordered set with at most two elements.*

- (i) *\mathcal{D} is a commutative unital entire semiring of characteristic zero with idempotent addition, and its neutral elements are $(\varepsilon, \min S)$ and $(e, \min S)$, respectively.*
- (ii) *\mathcal{D} is totally ordered by the lexicographical order, and $(\varepsilon, \min S)$ is its minimal element.*
- (iii) *Let $\alpha, \beta \in \mathcal{D}$ with $\alpha \leq \beta$. Then we have $\alpha \leq_{\text{lex}} \beta$.*
- (iv) *Let $\alpha, \beta \in \mathcal{D}$ with $\alpha <_{\text{lex}} \beta$. If $\gamma \in \mathcal{D} \setminus \{\varepsilon\}$ then $\alpha\gamma <_{\text{lex}} \beta\gamma$.*
- (v) *For all $\alpha, \beta \in \mathcal{D}$ and $n \in \mathbb{N}_{>0}$ the equation $\alpha^n = \beta^n$ implies $\alpha = \beta$.*
- (vi) *For every $\alpha \in \mathcal{D}$ and $n \in \mathbb{N}_{>0}$ the equation $x^n = \alpha$ admits a unique solution in \mathcal{D} , namely $(\alpha_1/n, \alpha_2)$, and we also write this solution as*

$$\alpha/n.$$

- (vii) *$\pi(\mathcal{D})$ is an algebraically closed max-plus algebra.*
- (viii) *Let \mathcal{E} be a subdioid of a \mathcal{D} . Then \mathcal{E} is quasi-max-plus algebra if and only if \mathcal{E} is algebraically closed and $(\alpha_1, \min S) \in \mathcal{E}$ for all $\alpha \in \mathcal{E}$.*

PROOF. (i), (vii) This can easily be checked.

(ii) Let $\alpha, \beta \in \mathcal{D}$. By Proposition 2.2 we may assume $\alpha_1 \leq \beta_1$, hence $\alpha \leq_{\text{lex}} \beta$.

(iii) By assumption we have $\alpha \oplus \beta = \beta$, thus $\alpha_1 \oplus \beta_1 = \beta_1$, hence $\alpha_1 \leq \beta_1$. Thus we are done provided $\alpha_1 < \beta_1$ or $S = \emptyset$. Therefore, let us assume $\alpha_1 = \beta_1$ and $S \neq \emptyset$. Then we have $\alpha_2 \leq \beta_2$ by definition.

(iv) Assume $\alpha\gamma \geq_{\text{lex}} \beta\gamma$. Then we have

$$\alpha_1\gamma_1 = (\alpha\gamma)_1 \geq (\beta\gamma)_1 = \beta_1\gamma_1$$

which implies $\alpha_1 \geq \beta_1$. But then we have $\alpha \geq_{\text{lex}} \beta$: Contradiction.

(v) If $\alpha = \varepsilon$ then clearly $\beta = \varepsilon$. Therefore we let $\alpha, \beta \neq \varepsilon$ and assume $\alpha <_{\text{lex}} \beta$. From (iv) we infer

$$\alpha^2 <_{\text{lex}} \alpha\beta <_{\text{lex}} \beta^2$$

and analogously $\alpha^n <_{\text{lex}} \beta^n$: Contradiction. Similarly we show the impossibility of $\alpha >_{\text{lex}} \beta$ and then deduce our assertion.

(vi) The existence of a solution is clear by divisibility. In case $\alpha = \varepsilon$ the only solution is ε . Otherwise, uniqueness follows from the fact that $(\mathcal{D} \setminus \{\varepsilon\})_1$ is contained in a group.

(vii), (viii) Clear by the above. \square

We now construct our principal example, namely a quasi-max-plus algebra over the nonnegative real numbers. On the set

$$\mathcal{R} := ((\mathbf{R}_{\geq 0} \cup \{-\infty\}) \times \{0, 1\}) \setminus \{(-\infty, 1), (0, 1)\}$$

endowed with the usual order relation we introduce two binary operations:

$$(r, a) \oplus (s, b) = (\max\{r, s\}, \delta_+((r, a), (s, b))), \quad \text{and}$$

$$(r, a) \odot (s, b) = (r + s, \delta_\times((r, a), (s, b))),$$

where the functions $\delta_+, \delta_\times : \mathcal{R} \times \mathcal{R} \rightarrow \{0, 1\}$ are defined as follows. First, $\delta_+((r, a), (s, b)) = 1$ if one of the following four conditions is satisfied:

- (i) $\max\{|r - s|, a, b\} = 1$,
- (ii) $r > s + 1$ and $a = 1$,
- (iii) $s > r + 1$ and $b = 1$,
- (iv) $0 < |r - s| < 1$,

otherwise $\delta_+((r, a), (s, b)) = 0$. Second, $\delta_\times((r, a), (s, b)) = \max\{a, b\}$ if $r, s \in \mathbf{R}$, and $\delta_\times((r, a), (s, b)) = 0$, otherwise. Obviously, these definitions extend the one given in [1, Section 6.3].

Now we collect some properties of subsets of \mathcal{R} .

THEOREM 2.6. *Let $T \neq \{0\}$ be an additively closed subset of $\mathbf{R}_{\geq 0}$ and set*

$$\mathcal{F} = ((T \times \{0, 1\}) \setminus \{(0, 1)\}) \cup \{e, \varepsilon\}, \oplus, \odot$$

with $\varepsilon = (-\infty, 0)$ and $e = (0, 0)$.

- (i) \mathcal{F} is a commutative entire archimedean dioid with neutral elements ε and e , respectively.

- (ii) For all $\alpha, \beta \in \mathcal{F}$ with $\beta \neq \varepsilon, e$ there exists $N \in \mathbf{N}$ such that for all $n \geq N$ we have

$$\alpha \oplus \beta^n = \beta^n.$$

- (iii) \mathcal{F} satisfies the weak stabilization condition.
 (iv) If $t/n \in T$ for all $t \in T$ and $n \in \mathbf{N}_{>0}$ then \mathcal{F} is an algebraically closed quasi-max-plus algebra.

PROOF. We leave the rather lengthy, but straightforward verification to the reader. □

REMARK 2.7. In general, \mathcal{F} cannot be embedded into the standard max-plus algebra. Otherwise, for $t \in T \cap (0, 1]$ we would have

$$(t, 1) = (t, 0) \oplus (0, 0) \in \{(t, 0), (0, 0)\}$$

which is impossible.

3. Matrices Over Quasi-Max-Plus Algebras

In this section we let $\mathcal{D} \subseteq F \times S$ be a quasi-max-plus algebra where F is a max-plus algebra and S a linearly ordered set with at most two elements. For $r \in \mathbf{N}_{>0}$ the set of matrices $\mathcal{D}^{r \times r}$ is a dioid and a \mathcal{D} -semimodule where the matrix operations are defined as usual.

Let $A \in \mathcal{D}^{r \times r}$. The digraph⁶ $\mathcal{G}(A)$ is the weighted digraph $([r], E, w)$ with vertex set $[r]$, edge set

$$E = \{(i, j) \in [r]^2 : A_{ij} \neq \varepsilon\}$$

and weight function $w : E \rightarrow \mathcal{D} \setminus \{\varepsilon\}$ with

$$w(i, j) = A_{ij}$$

for all $(i, j) \in E$; here we use the abbreviation $[r] = \{1, \dots, r\}$. We write $q' \subseteq_c q$ if the path q is a cycle extension of the path q' , and we denote by $|q|$ the length of q . We let $P_A^{(n)}(i, j)$ ($P_{A,el}(i, j)$, respectively) be the set of all paths of length n (all elementary paths, respectively) from i to j .

⁶If not stated otherwise we use the terminology of [11].

Let $q = (i_0, i_1, \dots, i_n)$ be a path of positive length n in $\mathcal{G}(A)$. We call $\text{supp}(q) = \{i_0, i_1, \dots, i_n\}$ the support,

$$w(q) = A_{i_0, i_1} \odot \cdots \odot A_{i_{n-1}, i_n}$$

the weight and

$$\bar{w}(q) = \frac{w(q)}{|q|}$$

the mean weight of q (see Proposition 2.5).

Let \mathcal{K} be a subgraph of $\mathcal{G}(A)$. \mathcal{K} is said to be nontrivial if it contains at least one cycle of positive length. The high period of \mathcal{K} is defined by

$$\text{hper}(\mathcal{K}) = \text{gcd}\{|c| : c \text{ cycle of positive length in } \mathcal{K} \text{ and } \bar{w}(c) = \lambda(\mathcal{K})\},$$

if the set on the right hand side is nonvoid, and $\text{hper}(\mathcal{K}) = 0$, otherwise; here⁷

$$\lambda(\mathcal{K}) = \max\{\bar{w}(c) : c \text{ cycle of positive length in } \mathcal{K}\}$$

is the maximal cycle mean weight of \mathcal{K} (cf. [10, p. 169]). Every cycle c in \mathcal{K} with $\bar{w}(c) = \lambda(\mathcal{K})$ is called a critical cycle in \mathcal{K} , and a cycle in $\mathcal{G}(A)$ is critical if it takes the maximal mean weight $\lambda(A) := \lambda(\mathcal{G}(A))$. Further, we use the abbreviation $\text{SCC } \mathcal{G}(A)$ ($\text{SCC}^* \mathcal{G}(A)$, respectively) for the set of strongly connected components (nontrivial strongly connected components, respectively) of $\mathcal{G}(A)$.

Following [10, p. 169] we say that the vertices i and j are highly connected if i and j are contained in a critical cycle; in this case we write $i \equiv_h j$. The subgraphs of $\mathcal{G}(A)$ induced by the classes of \equiv_h are called highly connected components of $\mathcal{G}(A)$. A highly connected component is called trivial if it does not possess a cycle of positive length with cycle mean weight equal to $\lambda(A)$. Analogously as above, we denote by $\text{HCC } \mathcal{G}(A)$ ($\text{HCC}^* \mathcal{G}(A)$, respectively) the set of highly connected components (nontrivial highly connected components, respectively) of $\mathcal{G}(A)$.

LEMMA 3.1. Assume $\lambda(\mathcal{K}) = \lambda$ for all $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$.

- (i) We have $\lambda(A) = \lambda$.
- (ii) Let $\lambda = e$. Then we have $w(c)_1 \leq e$ for all cycles c of $\mathcal{G}(A)$, and for all $\mathcal{K}' \in \text{SCC}^* \mathcal{G}(A)$ there exists a $\mathcal{K} \in \text{HCC}^* \mathcal{G}(A)$ with $\mathcal{K} \subseteq \mathcal{K}'$. Furthermore, if c is a cycle in $\mathcal{G}(A)$ which consists of vertices belonging to critical cycles then c is critical.

⁷Throughout we use the convention $\max \emptyset = \min S$ if (S, \leq) is an ordered set with a minimal element.

PROOF. (i) Let c be a critical cycle in $\mathcal{G}(A)$. It is easy to see that there is a $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$ with $\text{supp}(c) \subseteq \mathcal{K}$. Thus c is a cycle of \mathcal{K} and therefore $\lambda(\mathcal{K}) \geq_{\text{lex}} \bar{w}(c)$. Now the proof can easily be completed.

(ii) Clearly, we have

$$\frac{w(c)_1}{|c|} = \bar{w}(c)_1 \leq e,$$

thus $w(c)_1 \leq e^n = e$. Let $\mathcal{K}' \in \text{SCC}^* \mathcal{G}(A)$ and choose a cycle c in \mathcal{K}' with $\bar{w}(c) = e$. Then all vertices of c are highly connected, thus there is $\mathcal{K} \in \text{HCC}^* \mathcal{G}(A)$ with $\text{supp}(c) \subseteq \mathcal{K}$. Let $k \in \mathcal{K}$. Then k is highly connected to a vertex of c , thus $k \in \mathcal{K}'$, and this shows $\mathcal{K} \subseteq \mathcal{K}'$. The last assertion is clear by [2, Theorem 3.96]. \square

The following two technical lemmas provide an essential step in the proof of Theorem 3.5. For convenience we introduce

$$p_A = \text{lcm}\{\text{hper}(\mathcal{K}) : \mathcal{K} \in \text{HCC}^* \mathcal{G}(A)\}.$$

LEMMA 3.2. *There exists some $M \in \mathbf{N}$ such that for all $n \geq M$ and every $\mathcal{K} \in \text{HCC}^* \mathcal{G}(A)$ there is a critical cycle c in \mathcal{K} with the following properties.*

- (i) $|c| = np_A$
- (ii) *Every elementary critical cycle in \mathcal{K} occurs in c at least once.*

PROOF. This is a straightforward application of a well-known result of elementary number theory (e.g., see [15, Lemma A.3]). \square

LEMMA 3.3. *Let $\lambda(\mathcal{K}) = e$ for all $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$. There exists $N \in \mathbf{N}$ such that for all $n \geq N$, $i, j \in [r]$ and $q_0 \in P_{A, \text{el}}(i, j)$ the following statements hold:*

- (i) *Either there does not exist a cycle extension $q \in P_A(i, j)$ of q_0 with $|q| = n$ or every cycle extension $q \in P_A(i, j)$ of q_0 with $|q| \equiv n \pmod{p_A}$ and*

$$w(q) = \max\{w(q'') : q'' \in P_A^{(|q|)}(i, j), q_0 \subseteq_c q''\}$$

is of the form

$$q = q_0 c_1 \cdots c_t,$$

where c_1, \dots, c_t are critical cycles of $\mathcal{G}(A)$, and we have $w(q) = w(q_0)$ and $|q| \equiv |q_0| \pmod{p_A}$.

- (ii) *If $q \in P_{A, *}(i, j)$ is a cycle extension of q_0 , $m \in \mathbf{N}_{>0}$ and $\hat{q} := q_0 c_1 \cdots c_t \in P_A^{(n+mp_A)}(i, j)$ with critical cycles c_1, \dots, c_t then $\hat{q} \in P_{A, *}^{(n+mp_A)}(i, j)$. Here we*

set

$$P_{A,*}^{(i)}(i, j) = \{q \in P^{(i)}(i, j) : w(q) \geq w(q') \text{ for all } q' \in P_A^{(i)}(i, j)\}$$

for $i \in \mathbb{N}_{>0}$.

PROOF. This can be proved analogously to [13, Lemma 3.2] using Lemma 3.2. \square

In order to mitigate the condition $\lambda(\mathcal{K}) = e$ for all $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$ we need some preparation. The reader is referred to the appendix for the periodicity notions we are using in the sequel.

LEMMA 3.4. *Let \mathcal{D} be a quasi-max-plus algebra, $A \in \mathcal{D}^{r \times r}$ and $B := \pi(A)$.*

- (i) $\mathcal{G}(A) = \mathcal{G}(\pi(A))$, in particular the cycles of $\mathcal{G}(A)$ and $\mathcal{G}(\pi(A))$ coincide.
- (ii) $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$ if and only if $\mathcal{K} \in \text{SCC}^* \mathcal{G}(\pi(A))$.
- (iii) Every critical cycle of $\mathcal{G}(A)$ is a critical cycle of $\mathcal{G}(\pi(A))$, and we have $\lambda(A)_1 = \lambda(\pi(A))$.
- (iv) For each $\mathcal{K} \in \text{HCC}^* \mathcal{G}(A)$ there is a $\mathcal{K}' \in \text{HCC}^* \mathcal{G}(\pi(A))$ with $\mathcal{K} \subseteq \mathcal{K}'$.
- (v) A is irreducible if and only if $\pi(A)$ is irreducible.
- (vi) Assume that A has at least one cycle and $\lambda(A)_2 = \min S$. Then the critical cycles in $\mathcal{G}(A)$ and $\mathcal{G}(\pi(A))$ coincide.
- (vii) If A is almost linear periodic then $\pi(A)$ is almost linear periodic, $\text{ldef}(\pi(A)) \leq \text{ldef}(A)$ and $\text{lper}(\pi(A))$ divides $\text{lper}(A)$, and $\pi(Q_{ij})$ are the entries of a linear factor matrix of $\pi(A)$ where Q is a linear factor matrix of A .
- (viii) If $\lambda(\mathcal{K}) = \lambda(\mathcal{K}')$ for all $\mathcal{K}, \mathcal{K}' \in \text{SCC}^* \mathcal{G}(A)$ then $\lambda(\mathcal{K}) = \lambda(\mathcal{K}')$ for all $\mathcal{K}, \mathcal{K}' \in \text{SCC}^* \mathcal{G}(B)$.

PROOF. (i), (ii) This can easily be checked.

(iii) We only show $\lambda(A)_1 = \lambda(\pi(A))$ using (i). Assume $\lambda(A)_1 > \lambda(\pi(A)) =: \lambda$. Then there exists a cycle c in $\mathcal{G}(A)$ with $\bar{w}(c)_1 > \lambda$, but c is a cycle in $\mathcal{G}(\pi(A))$: Contradiction. Thus we have $\lambda(A)_1 \leq \lambda$. The assumption of strict inequality leads to a cycle c' in $\mathcal{G}(\pi(A))$ with $\bar{w}(c')_1 > \lambda(A)_1$, hence $\bar{w}(c) > \lambda(A)$: Contradiction.

(iv) Let $i, j \in \mathcal{K}$ and c a critical cycle with vertices i, j . Then $\bar{w}(c)_1 = \lambda(\pi(A))$ by (iii), thus there is a $\mathcal{K}' \in \text{HCC}^* \mathcal{G}(\pi(A))$ which contains every vertex of c . We easily check $\mathcal{K} \subseteq \mathcal{K}'$.

(v) Clear by (i).

(vi) In view of (iii) it suffices to show that every critical cycle c in $\mathcal{G}(\pi(A))$ is a critical cycle in $\mathcal{G}(A)$. Suppose to the contrary that there is a cycle c' in $\mathcal{G}(A)$ with $\bar{w}(c) < \bar{w}(c')$. The same relation then holds for the first components because

$$w(c)_2 \geq \lambda(A)_2 = w(c')_2,$$

and we deduce the contradiction

$$\bar{w}(\pi(c)) = \frac{w(\pi(c))}{|c|} = \frac{w(c)}{|c|} < \frac{w(c')_1}{|c'|} = \frac{w(\pi(c'))}{|c'|} = \bar{w}(\pi(c')).$$

(vii) By assumption the sequence $A^* := (A^n)_{n \in \mathbb{N}}$ is almost linear periodic. Let $i, j \in [r]$, $p_{ij} := \text{lper}(A^*)_{ij} \in \mathbb{N}_{>0}$ and $\lambda_{ij} \in \mathcal{D}$ with

$$(A^{n+p_{ij}})_{ij} = (A^n)_{ij} \lambda_{ij}^{p_{ij}} \quad (n > \text{ldef}(A^*)_{ij}).$$

In particular, this holds for the first components, hence $\pi(A)$ is almost linear periodic, and $\text{ldef}(\pi(A)) \leq \text{ldef}(A)$. Furthermore, using Proposition 2.5 we check that $\text{lper}(\pi(A^*))_{ij}$ divides p_{ij} , hence

$$\text{lper}(\pi(A)) = \text{lcm}\{\text{lper}(\pi(A^*))_{ij} : i, j \in [r]\} \mid \text{lcm}\{p_{ij} : i, j \in [r]\} = \text{lper}(A).$$

The proof can now easily be concluded.

(viii) Let $\mathcal{K}, \mathcal{K}' \in \text{SCC}^* \mathcal{G}(B)$. Then $\mathcal{K}, \mathcal{K}' \in \text{SCC}^* \mathcal{G}(A)$ by (ii) and $\lambda(\mathcal{K})_1 = \lambda(\mathcal{K}')_1$ by assumption. This shows that there cannot be a cycle c in \mathcal{K} with $\bar{w}(\pi(c))_1 > \lambda(\mathcal{K}')_1$ which implies our assertion. \square

Now we can state our first main result which slightly extends the structural part of [10, Theorem 3.1] and generalizes a classical theorem on matrices over the standard max-plus algebra [2, Section 3.7].

THEOREM 3.5. *Let \mathcal{D} be a max-plus algebra, $\lambda \in \mathcal{D} \setminus \{\varepsilon\}$ and $A \in \mathcal{D}^{r \times r}$ with $\lambda(\mathcal{K}) = \lambda$ for all $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$. Then A is almost linear periodic, $\text{lper}(A) = p_A$ and $(\text{lfac}(A))_{ij} = \lambda$ for all $i, j \in [r]$. More explicitly, we have*

$$(A^{n+p_A})_{ij} = \lambda^{p_A} (A^n)_{ij} \quad (i, j \in [r])$$

for all sufficiently large n .

PROOF. By Lemma 3.1 we have $\lambda(A) = \lambda$, and for $B := (-\lambda)A$ we have $\lambda(B) = e$ by Lemma 4.7.

Let $p := p_A$ and N be a constant given by Lemma 3.3, $n \geq N$ and $i, j \in [r]$. Assume that there is some $q \in P_{A, \text{el}}(i, j)$. By [10, Lemma 2.1] we have $(B^n)_{ij} = a_q^{(n)}$, and by Lemma 3.3 we have $(B^{n+p_B})_{ij} = a_q^{(n)}$, hence

$$(1) \quad (B^{n+p_B})_{ij} = (B^n)_{ij}.$$

On the other hand, if $P_{A, \text{el}}(i, j) = \emptyset$ then both sides of (1) equal ε by Lemma 3.3.

Thus B is almost linear periodic with $\text{lfac}(B) = \varepsilon$ by Lemma 4.6 and $\text{lper}(B)$ divides p_B by Proposition 2.2. Arguing as in the proof of [10, Lemma 3.3] we see that $\text{lper}(B)$ cannot be smaller than p_B , thus $\text{lper}(B) = p_B$.

An application of Lemma 4.7 concludes the proof. \square

We can certainly recover the first part of [10, Theorem 3.1].

COROLLARY 3.6. *Let \mathcal{D} be a max-plus algebra and $A \in \mathcal{D}^{r \times r}$ be irreducible. Then A is almost linear periodic, and we have $\text{lper}(A) = p_A$ and $\text{lfac}(A) = \lambda(A) \neq \varepsilon$.*

PROOF. As $\text{SCC}^* \mathcal{G}(A)$ is a singleton the assertion drops out of the Theorem. \square

Now we establish the analog of [5, Theorem 2.4].

THEOREM 3.7. *Let \mathcal{D} be a quasi-max-plus algebra with weak stabilization condition. Let \mathcal{E} be a subdioid of \mathcal{D} and $A \in \mathcal{E}^{r \times r}$ be irreducible. Then A and $\pi(A)$ are almost linear periodic and $\lambda(A) \neq \varepsilon$. Furthermore, we have $\text{ldef}(A) \geq \text{ldef}(\pi(A))$, $p_{\pi(A)}$ divides $p := \text{lper}(A)$, $(\lambda(A)_1, \min \mathcal{D}_2)^p \in \mathcal{E}$ and*

$$A^{n+p} = (\lambda(A)_1, \min \mathcal{D}_2)^p A^n \quad (n > \text{ldef}(A)).$$

PROOF. (i) Using Lemma 3.4 (v), (iii) and Corollary 3.6 we find that the matrix $B := \pi(A)$ is irreducible and almost linear periodic, and $\text{lper}(B) = p_B$ and $\text{lfac}(B) = \lambda_1$ with $\lambda := \lambda(A)$.

(ii) Let $i, j \in [r]$. By the above we have

$$(2) \quad (A^{n+tp_B})_{ij} = (B^{n+tp_B})_{ij} = (B^n)_{ij} \lambda_1^{tp_B} = \lambda_1^{tp_B} (A^n)_{ij} \quad (n > \text{ldef}(B), t \in \mathbf{N}_{>0}).$$

Furthermore, from [8, Proposition 1.2.2] we infer the existence of $\lambda_{ij} \in \mathcal{D}$ and $N_{ij}, p_{ij} \in \mathbf{N}_{>0}$ such that

$$(3) \quad (A^{n+t p_{ij}})_{ij} = \lambda_{ij}^t (A^n)_{ij} \quad (n \geq N_{ij}, t \in \mathbf{N}_{>0}).$$

(iii) Let $N = \max\{\text{ldef}(B), N_{ij} : i, j \in [r]\}$ and $p = \text{lcm}\{p_B, p_{ij} : i, j \in [r]\}$. By (2) and (3) we have for all $i, j \in [r]$

$$(A^{n+p})_{ij1} = \lambda_1^p (A^n)_{ij1} = \lambda_{ij1}^{p/p_{ij}} (A^n)_{ij1} \quad (n > N).$$

Now, Proposition 2.2 (ii) yields

$$\lambda_{ij1}^{p/p_{ij}} = \lambda_1^p,$$

hence by (3) and the properties of a quasi-max-plus algebra

$$(A^{n+p})_{ij} = (\lambda_1^p, \lambda_{ij2})(A^n)_{ij} = (\lambda_1, \lambda_{ij2})^p (A^n)_{ij} \quad (n > N).$$

Thus A is almost linear periodic, and from Lemma 3.4 (vii) we know that p_B divides $\text{lper}(A)$ and $\text{ldef}(A) \geq \text{ldef}(B)$.

(iv) In case $\text{Card } \mathcal{D}_2 = 1$ we are done. Otherwise we write $\mathcal{D}_2 = \{0, 1\}$ and show that there is some $M \geq N$ such that

$$(4) \quad (A^{n+p})_{ij} = (\lambda_1, 0)^p (A^n)_{ij} \quad (i, j \in [r], n > M).$$

Note that by the definition of p we have $(\lambda_1, 0)^p \in \mathcal{E}$. We distinguish two cases.

Case 1. For all $i, j \in [r]$ and $n > N$ we have $(A^n)_{ij2} = 0$.

In this case we set $M = N$.

Case 2. There is some $i, j \in [r]$ and $n_{ij} > N$ such that $(A^n)_{ij2} = 1$.

For all i, j with this property we fix some n_{ij} . Now we choose M to be the maximal n_{ij} and check that (4) is satisfied. This completes the proof. \square

Now we study eigenvalues of certain matrices over quasi-max-plus algebras.

THEOREM 3.8. *Let \mathcal{D} be a quasi-max-plus algebra and $A \in \mathcal{D}^{r \times r}$.*

- (i) *If A is nilpotent then ε is the unique eigenvalue of A .*
- (ii) *Let \mathcal{D} satisfy the weak stabilization condition and A be irreducible. Then ε is not an eigenvalue of A , but $(\lambda(A)_1, \min \mathcal{D}_2)$ is an eigenvalue of A .*
- (iii) *Suppose $\mathcal{D}_2 = \{0, 1\}$, $(\lambda(A)_1, 1) \in \mathcal{D}$ and $(v_{11}, 1), \dots, (v_{r1}, 1) \in \mathcal{D}$ for some eigenvector $(v_1, \dots, v_r) \in \mathcal{D}^r$ of A with eigenvalue $(\lambda(A)_1, 0)$. Then $(\lambda(A)_1, 1)$ is an eigenvalue of A with eigenvector $((v_{11}, 1), \dots, (v_{r1}, 1))$.*

PROOF. (i) This is a well-known classical result.

(ii) The first part is clear by Lemma 4.3, and for the second part we closely follow [6, proof of Teorema 1]. By Theorem 3.7 there exist $n, m \in \mathbf{N}$ such that $m > n > 0$ and

$$A^m = \lambda^{m-n} A^n$$

where we set $\lambda := (\lambda(A)_1, \min \mathcal{D}_2) \in \mathcal{D} \setminus \{\varepsilon\}$. Then Theorem 4.5 yields our assertion.

(iii) For each $i \in [r]$ the equation

$$\bigoplus_{j \in [r]} A_{ij} v_j = (\lambda(A)_1, 0) v_i$$

implies

$$\bigoplus_{j \in [r]} A_{ij}(v_{j1}, 1) = (\lambda(A)_1, 1)(v_{i1}, 1). \quad \square$$

We illustrate Theorem 3.8 by two easy examples.

EXAMPLE 3.9. (i) The only eigenvalue of the matrix $(0, 0) \in \mathcal{R}$ is $(0, 0)$, and every element in $\mathcal{R} \setminus \{(-\infty, 0)\}$ is an eigenvector.

(ii) The matrix $(1, 0) \in \mathcal{R}$ has exactly two eigenvalues, namely $(1, 0)$ and $(1, 1)$. In both cases $(x, 1)$ is an eigenvector provided $x > 0$.

Finally, we extend Theorem 3.7 and [1, Proposition 6.19] for the particular dioid \mathcal{R} .

THEOREM 3.10. *Let $T \neq \{0\}$ be an additively closed subset of $\mathbf{R}_{\geq 0}$ and \mathcal{F} be defined as in Theorem 2.6. Further, let $A \in \mathcal{F}^{r \times r}$ be a non-nilpotent matrix and assume $\lambda(\mathcal{K}) = \lambda(\mathcal{K}')$ for all $\mathcal{K}, \mathcal{K}' \in \text{SCC}^* \mathcal{G}(A)$. Then A and $\pi(A)$ are almost linear periodic. Furthermore, we have $\text{ldef}(A) \geq \text{ldef}(\pi(A))$, $(\lambda(A)_1, 0)^p \in \mathcal{F}$ and $p := \text{lper}(A)$ divides $\rho_{\pi(A)}$*

$$A^{n+p} = (\lambda(A)_1, 0)^p A^n \quad (n > \text{ldef}(A)).$$

Thus

$$(\lambda(A)_1, 0)_{i, j \in [r]} \in \mathcal{R}^{r \times r}$$

defines a linear factor matrix of A .

PROOF. As A is non-nilpotent the digraph $\mathcal{G}(A)$ has at least one cycle of positive length. In view of Lemma 3.4 (viii) and Lemma 3.1 (i) we have $\lambda(\mathcal{A}) = \lambda(\mathcal{K}) = \lambda(\mathcal{K}')$ for all $\mathcal{K}, \mathcal{K}' \in \text{SCC}^* \mathcal{G}(B)$ where we set $B := \pi(A)$. We infer from Theorem 3.5 that B is almost linear periodic, $\lambda := \lambda(\mathcal{A})_1$ defines the linear factor of B , and $\text{lper}(B) = p_B$. Thus $(\lambda, 0)^{p_B} \in \mathcal{T}$ and there exists $N \in \mathbb{N}$ such that

$$(5) \quad ((A^{n+p_B})_{ij})_1 = (B^{n+p_B})_{ij} = (B^n)_{ij} \lambda^{p_B} = ((A^n)_{ij})_1 \lambda^{p_B} = ((A^n)_{ij}, (\lambda, 0)_1)^{p_B}.$$

Now, Lemma 3.4 shows that (5) also holds for the second components provided n is large enough. This means that A is almost linear periodic, $(\lambda, 0)_{i,j \in [r]}$ defines a linear factor matrix of A and the relation between the linear defects of A and B is clear by Lemma 3.4. Finally, Lemma 4.1 yields that $\text{lper}(A)$ divides p_B . \square

A central result of [13, Theorem 3.1] can now easily be generalized.

COROLLARY 3.11. *Let $A \in \mathcal{R}^{r \times r}$ and assume that $\mathcal{G}(A)$ has at least one cycle of positive length. The matrix A is eventually periodic if and only if $\lambda(\mathcal{K}) = e$ for all $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$. In this case $\text{per} A$ divides p_A .*

PROOF. We first observe that [13, Theorem 3.1] holds for any matrix over a max-plus algebra whose graph has at least one cycle of positive length.

Let A be eventually periodic. Then $B := \pi(A)$ is eventually periodic, hence for any $\mathcal{K} \in \text{SCC}^* \mathcal{G}(\pi(A))$ we have $\lambda(\mathcal{K})_1 = e$ by [13, Theorem 3.1]. Then Lemma 3.4 (ii) yields $\lambda(\mathcal{K}) = e$ for all $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$.

Conversely, if $\lambda(\mathcal{K}) = e$ for all $\mathcal{K} \in \text{SCC}^* \mathcal{G}(A)$ then Theorem 3.10 yields

$$(A^{n+p})_{ij} = (A^n)_{ij} \quad (n \geq N, i, j \in [r])$$

with some $N \in \mathbb{N}$, and an application of Lemma 4.1 completes the proof. \square

4. Appendix

Let (S, \cdot) be an abelian semigroup and $a^* = (a_n)_{n \in \mathbb{N}}$ be a sequence of elements of S . Following [10, Definition 2.3, 2.4] we say that a^* is almost linear periodic⁸ if there are $N \in \mathbb{N}$, $p \in \mathbb{N}_{>0}$ and $b \in S$ such that

$$(6) \quad a_{n+p} = b^p \cdot a_n \quad (n > N).$$

In this case the smallest $p \in \mathbf{N}_{>0}$ such that there are $N \in \mathbf{N}$ and $b \in S$ with (6) is called the linear period of a^* , and we write $p = \text{lper } a^*$. The minimal $N \in \mathbf{N}$ such that there is some $b \in S$ which satisfies (6) for $p = \text{lper } a^*$ is called the linear defect of a^* , and we write $N = \text{ldef } a^*$. Finally, an element b with (6) for $p = \text{lper } a^*$ and $N = \text{ldef } a^*$ is called a linear factor of a^* . In case b is unique we write $b = \text{lfac } a^*$.

LEMMA 4.1. *Let (S, \cdot) be an abelian cancellative semigroup and assume that for all $x, y \in S$ and $n \in \mathbf{N}_{>0}$ the equation $x^n = y^n$ implies $x = y$. Further, let $a^* = (a_k)_{k \in \mathbf{N}}$ be an almost linear periodic sequence in S , i.e., there exist $b \in S$, $N \in \mathbf{N}$ and $m \in \mathbf{N}_{>0}$ with*

$$a_{n+m} = a_n b^m$$

for every $n > N$.

(i) *We have*

$$a_{n+tm} = a_n b^{tm}$$

for every $t \in \mathbf{N}_{>0}$ and $n > N$.

(ii) $\text{lfac}(a^*) = b$.

(iii) $\text{lper}(a^*)$ divides m .

PROOF. (i) Clear by induction.

(ii) Assume that there are $c \in S$, $M \in \mathbf{N}$ and $k \in \mathbf{N}_{>0}$ with

$$a^{n+k} = a^n c^k$$

for all $n > M$. For large enough n we then have by (i)

$$a_n c^{km} = a_{n+km} = a_n b^{km},$$

which yields $c^{km} = b^{km}$ and then $c = b$.

(iii) Clearly, we have $p := \text{lper}(a^*) \leq m$. Write $m = qp + r$ with $q, r \in \mathbf{N}$, $q > 0$ and $r < p$. If $r = 0$ we are done. Otherwise, applying (i) again we find for large enough n

$$a_n b^{qp+r} = a_n b^m = a_{n+m} = a_{(n+r)+qp} = a_{n+r} b^{qp},$$

⁸ Our definition slightly differs from the one given in [1, Definition 6.1].

hence

$$a_n b^r = a_{n+r},$$

contradicting the definition of p . □

We collect some well-known results for commutative unital semirings. The first one is stated in [3, Theorem 3.24] for matrices over the complex numbers. However, it is easy to see that it remains true in a more general setting (cf. [5, Theorem 2.5], [7, Assertion 7.2]).

THEOREM 4.2 [3, Theorem 3.24]. *Let S be a commutative unital semiring and $M \in S^{r \times r}$. Then there exists a permutation matrix $P \in S^{n \times n}$ such that*

$$(7) \quad PMP^T = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ 0 & M_{22} & \cdots & M_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{nn} \end{pmatrix}.$$

Here $n \geq 1$ and the blocks M_{11}, \dots, M_{nn} that occur in the diagonal in (7) are square matrices which are either 0 or irreducible.⁹ The blocks M_{11}, \dots, M_{nn} are uniquely determined to within simultaneous permutation of their rows and columns, but their ordering in (7) is not necessarily unique. The form on the right hand side of (7) is called the Frobenius normal form of M .

For the study of the ultimate behavior of the sequence of powers of a matrix over S we recall the following definitions. The matrix $A \in S^{r \times r}$ is called

- eventually periodic if the sequence $A^* := (A^n)_{n \in \mathbb{N}}$ is eventually periodic (see for instance [13, Definition 2.4] where the notion ‘almost periodic’ was coined for this property); in this case we write

$$\text{per}(A) = \text{per}(A^*).$$

- almost linear periodic if for all $i, j \in [r]$ the sequence

$$(A^*)_{ij} = ((A_n)_{ij})_{n \in \mathbb{N}} \in S^{\mathbb{N}}$$

⁹Observe the different notion of irreducibility in [3].

is almost linear periodic. In this case we write

$$\text{ldef}(A) = \text{ldef}(A^*), \quad \text{lper}(A) = \text{lper}(A^*).$$

If there is a unique linear factor we write $\text{lfac}(A) = \text{lfac}(A^*)$.

From now on we let \mathcal{D} be a commutative entire dioid. The following result is well-known for the standard max-plus algebra (e.g., see [4]).

LEMMA 4.3. *Let $A \in \mathcal{D}^{r \times r}$ be irreducible and assume $A \neq (\varepsilon)$. Further, let $\lambda \in \mathcal{D}$ be an eigenvalue of A with eigenvector $v \in \mathcal{D}^r$. Then $\lambda \neq \varepsilon$ and $v_i \neq \varepsilon$ for all $i \in [r]$.*

PROOF. By definition we have $v_j \neq \varepsilon$ for some $j \in [r]$. For all $i, k \in [r]$ we have

$$(8) \quad A_{ki}v_i \leq \lambda v_k.$$

Let us assume $\lambda = \varepsilon$. Then by (8) we find

$$A_{ki}v_i = \varepsilon$$

for all $i, k \in [r]$, thus in particular $A_{kj}v_j = \varepsilon$ and then $A_{kj} = \varepsilon$ for all $k \in [r]$ which is impossible.

Let us now assume $v_k = \varepsilon$ for some $k \in [r]$ and pick $n \in \mathbb{N}_{>0}$ such that $(A^n)_{kj} \neq \varepsilon$. Then we are lead to the contradiction

$$\varepsilon < (A^n)_{kj}v_j \leq \lambda^n v_k = \varepsilon. \quad \square$$

LEMMA 4.4. *Let $M \in \mathcal{D}^{r \times r}$, $\lambda \in \mathcal{D} \setminus \{\varepsilon\}$ and $n, m \in \mathbb{N}$ such that $m > n > 0$, $M^n \neq \varepsilon$ and*

$$M^m = \lambda^{m-n} M^n.$$

Then λ is an eigenvalue of M .

PROOF. Our proof is taken from [6, proof of Teorema 1]. For the convenience of the reader we give the details here. Let $z \in \mathcal{D}^r$ such that $y := M^n z \neq \varepsilon$. Then we have

$$(9) \quad M^{m-n}y = M^m z = \lambda^{m-n} M^n z = \lambda^{m-n} y.$$

Therefore, the vector

$$x := \bigoplus_{i=0}^{m-n-1} \lambda^i M^{m-n-1-i} y$$

is not the zero vector because otherwise we have $\lambda^i M^{m-n-1-i} y = \varepsilon$ for all $i = 0, \dots, m-n-1$ which implies

$$\lambda^i M^{m-1-i} z = \varepsilon \quad (i = 0, \dots, m-n-1)$$

yielding the contradiction

$$\lambda^{m-n-1} y = \lambda^{m-n-1} M^{m-1-(m-n-1)} z = \varepsilon.$$

Thus x is an eigenvector of M with eigenvalue λ because by (9) we have

$$\begin{aligned} Mx &= \bigoplus_{i=0}^{m-n-1} \lambda^i M^{m-n-i} y = M^{m-n} y \oplus \bigoplus_{i=1}^{m-n-1} \lambda^i M^{m-n-i} y \\ &= \lambda^{m-n} y \oplus \bigoplus_{i=1}^{m-n-1} \lambda^i M^{m-n-i} y = \bigoplus_{i=1}^{m-n} \lambda^i M^{m-n-i} y = \lambda x. \quad \square \end{aligned}$$

THEOREM 4.5. *Let \mathcal{D} be a commutative entire dioid and $M \in \mathcal{D}^{r \times r}$ be not nilpotent. Further, let $\lambda \in \mathcal{D} \setminus \{\varepsilon\}$ and $n, m \in \mathbf{N}$ such that $m > n > 0$ and*

$$M^m = \lambda^{m-n} M^n.$$

Then λ is an eigenvalue of M .

PROOF. We use induction on r and closely follow the proof of [6, proof of Teorema 1]. If M is irreducible then $M^n \neq \varepsilon$ and we are done by Lemma 4.4. Now, let M be reducible, hence $r > 1$ and by Theorem 4.2 we find a permutation matrix $P \in \mathcal{D}^{r \times r}$ such that we can write

$$PMP^T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with square matrices A, B of smaller size than M . At least one of these matrices is not nilpotent because otherwise M is nilpotent. W.l.o.g. we assume that $A \in \mathcal{D}^{k \times k}$ is not nilpotent. Then we find

$$A^m = \lambda^{m-n} A^n,$$

hence by induction hypothesis there is an eigenvector $a \in \mathcal{D}^k$ of A with eigenvalue λ . Now, the vector $P^T v^T$ with $v := (a, \varepsilon, \dots, \varepsilon) \in \mathcal{D}^r$ is an eigenvector of M with eigenvalue λ . \square

We formally state some results which were implicitly used in [10].

LEMMA 4.6. *Let \mathcal{D} be a max-plus algebra and $A \in \mathcal{D}^{r \times r}$ be almost linear periodic. Then A has a unique linear factor.*

PROOF. Clear by Proposition 2.2 and Lemma 4.1. \square

LEMMA 4.7. *Let \mathcal{D} be a max-plus algebra, $\alpha \in \mathcal{D} \setminus \{\varepsilon\}$, $A \in \mathcal{D}^{r \times r}$ and $B = \alpha A$.*

- (i) *We have $\mathcal{G}(A) = \mathcal{G}(B) =: \mathcal{G}$ and $p_A = p_B$.*
- (ii) *For every path q of positive length in \mathcal{G} we have $\bar{w}_B(q) = \alpha \bar{w}_A(q)$.*
- (iii) *The set of critical cycles w.r.t. \bar{w}_A coincides with the set of critical cycles w.r.t. \bar{w}_B .*
- (iv) *If $\mathcal{K} \in \text{SCC}^* \mathcal{G}$ and $\alpha = -\lambda_A(\mathcal{K})$ then $\lambda_B(\mathcal{K}) = \varepsilon$.*
- (v) *A is almost linear periodic if and only if B is almost linear periodic. In this case we have $\text{lper}(A) = \text{lper}(B)$, $\text{ldef}(A) = \text{ldef}(B)$ and $\text{lfac}(B) = \alpha \text{lfac}(A)$.*

PROOF. (i) Obvious.

(ii) We have

$$\bar{w}_B(q) = \frac{w_B(q)}{|q|} = \frac{w_A(q)\alpha^{|q|}}{|q|} = \bar{w}_A(q)\alpha.$$

(iii) Clear by (i) and (ii).

(iv) This can easily be checked.

(v) Let A be almost linear periodic with $p = \text{lper}(A)$ and $\lambda = \text{lfac}(A)$, hence

$$B^{n+p} = \alpha^{n+p} A^{n+p} = \alpha^n A^n \lambda^p \alpha^p = B^n (\lambda \alpha)^p.$$

for $n > \text{ldef}(A)$. Thus B is almost linear periodic with $\text{lper}(B) \leq p$. However, strict inequality is impossible, hence $\text{lper}(B) = p$. Similarly, we find $\text{ldef}(B) = \text{ldef}(A)$, and finally $\text{lfac}(B) = \lambda \alpha$ in view of Lemma 4.6.

Conversely, let B be almost linear periodic with $q = \text{lper}(B)$ and $\mu = \text{lfac}(B)$, hence

$$\alpha^{n+q} A^{n+q} = B^{n+q} = B^n \mu^q = \alpha^n A^n \mu^q$$

for $n > \text{ldef}(B)$. As \mathcal{D} is cancellative we have

$$A^{n+q} = (-\alpha)^q \alpha^q A^{n+q} = A^n (-\alpha)^q \mu^q = A^n (-\alpha \mu)^q,$$

and the proof can be completed analogously as above. \square

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