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INTERACTIVE INFINITE MARKOV PARTICLE SYSTEMS WITH JUMPS

By

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Abstract. In [2] we investigated independent infinite Markov particle systems as measure-valued Markov processes with jumps, and we gave sample path properties and martingale characterizations. In particular, we investigated the exponent of Hölder-right continuity in case that the motion process is *absorbing* α*-stable motion on* $(0, \infty)$ *with* $0 < \alpha < 2$, that is, time-changed absorbing Brownian motions on $(0, \infty)$ by the increasing $\alpha/2$-stable Lévy processes.

In the present paper we shall extend the results to the case of simple interactive infinite Markov particle systems. We also consider the absorbing stable motion on a half space $H = \mathbf{R}^{d-1} \times (0, \infty)$ as a motion process.

1. Settings and Previous Results

In this section we give the general settings and the known results which are given in [2] in order to describe the main results in §3 and §4.

Let S be a domain of $\mathbf{R}^d$. Let $(w(t), P_x)_{t \geq 0, x \in S}$ be a S-valued Markov process having life time $\zeta(w) \in (0, \infty]$ such that $w \in \mathbf{D}([0, \zeta(w)) \to S)$, i.e., $w : [0, \zeta(w)) \to S$ is right continuous and has left-hand limits. For convenience, we fix an extra point $\Delta \notin S$ and set $w(t) = \Delta$ if $t \geq \zeta(w)$. Moreover we shall extend functions f on S to on $\{\Delta\}$ by $f(\Delta) = 0$, if necessary.

We use the following notations: Let $S \subset \mathbf{R}^d$ be a domain.

- If $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$, then $\partial^k_{i_1 \cdots i_k} = \partial^k/(\partial x_{i_1} \cdots \partial x_{i_k})$, $\partial^k_i = \partial^k/(\partial x^k_i)$ and $\partial_i = \partial^1_i$ for each $k = 0, 1, \ldots,$ $i = 1, \ldots, d$. Moreover $\partial_t = \partial/\partial t$ for time $t \geq 0$.

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- $f \in C_c \equiv C_c(S) \overset{\text{def}}{\Longleftrightarrow} f$ is a continuous function on S with compact support in S, and $C_c^\infty \equiv C_c^\infty(S) := C_c(S) \cap C^\infty(S)$.
- For each integer $k \geq 0$, $C_b^k := C_b^k(\mathbf{R}^d)|_S$, that is, $f \in C_b^k \overset{\text{def}}{\Longleftrightarrow} f$ is a restriction to S of k-times continuously differentiable function on $\mathbf{R}^d$ with bounded derivatives of order between 0 and k. Moreover $f \in C_0 \overset{\text{def}}{\Longleftrightarrow} f$ is continuous on S and $f(x) \to 0$ whenever $x \to \partial S$ or $|x| \to \infty$. Furthermore $C_b := C_b^0$, $C_b^\infty := \bigcap_k C_b^k$, $C_0^k := C_0 \cap C_b^k$ and $C_0^\infty := \bigcap_k C_0^k$.
- For a function space D on S, $f \in D^+ \overset{\text{def}}{\Longleftrightarrow} f \in D$; $f \geq 0$.
- $\langle \mu, f \rangle := \int_S f(x)\mu(dx)$ for a function f on S and a measure μ on S.

The following two assumptions are the same as in [2].

ASSUMPTION 1. *Let $(P_t)_{t \geq 0}$ be the transition semigroup of $(w(t), P_x)$, i.e., $P_t f(x) = E_x[f(w(t)) : t < \zeta]$.*

(i) *(P_t) is a strongly continuous nonnegative contraction semigroup on $(C_0, \|\cdot\|_\infty)$ with generator $(A, \mathscr{D}(A))$, where $\|f\|_\infty = \sup_{x \in S}|f(x)|$.*
(ii) *$C_c^\infty \subset \mathscr{D}(A)$ and there is a strictly positive function $g_0 \in C_0^\infty$ such that $g_0 \in \mathscr{D}(A)$ and that $g_0^{-1}Af \in C_b$ with $g_0^{-1} = 1/g_0$ for every $f \in C_c^\infty$ and $f = g_0$.*
(iii) *$\sup_{t \leq T}\|g_0^{-1}P_t g_0\|_\infty < \infty$ for every $T > 0$.*

Under this assumption we introduce a function space $D_{g_0} \subset \mathscr{D}(A)$ as follows:

$$f \in D_{g_0} \overset{\text{def}}{\Longleftrightarrow} f \in \mathscr{D}(A) \quad \text{such that } \|g_0^{-1}f\|_\infty < \infty \text{ and } \|g_0^{-1}Af\|_\infty < \infty.$$

Clearly $g_0 \in D_{g_0}$ and $C_c^\infty \subset D_{g_0}$. Moreover since C_c^∞ is dense in C_0 and $P_t C_c^\infty \subset D_{g_0}$ for every $t \geq 0$, D_{g_0} is a core for A. However, D_{g_0} may be too large, so we further need the following assumption:

ASSUMPTION 2. *There exist a bounded function $g_1 \in C^\infty$; $g_1 \geq g_0 (> 0)$ and a core $D \subset D_{g_0}$ (we denote $D = D_g$ with $g = (g_0, g_1)$) satisfying the following:*

(i) *If $f \in D_g$, then $\lim_{t \downarrow 0} \frac{1}{t}(P_t f^2(x) - f(x)^2)$ exists for each $x \in S$ (we also denote the limit as $Af^2(x)$), $\partial_t P_t f^2(x) = AP_t f^2(x) = P_t Af^2(x)$ ($\to Af^2(x)$ as $t \downarrow 0$ for each $x \in S$), $Af^2 \in C_b$ and $\|g_1^{-1}Af^2\|_\infty < \infty$.*
(ii) *For each $T > 0$, $\sup_{t \in [0,T]}\|g_1^{-1}P_t g_1\|_\infty < \infty$.*
(iii) *For each $0 < s < T$, $\sup_{t \in [s,T]}\|g_0^{-1}P_t g_1\|_\infty < \infty$.*
(iv) *There exist constants $0 \leq \gamma < 1$, $\delta > 0$ such that $\sup_{0 \leq t \leq \delta} t^\gamma\|g_0^{-1}P_t g_1\|_\infty < \infty$.*
(v) *$g_0 \in D_g$.*

All through the present paper we suppose that Assumption 1 and 2 are fulfilled. We shall sometime use the notation $\|\cdot\|_{g_0} = \|\cdot/g_0\|_\infty$.

Let $\mathscr{M}_{g_0} \equiv \mathscr{M}_{g_0}(S)$ be a space of counting measures on S defined as $\mu \in \mathscr{M}_{g_0} \overset{\text{def}}{\Longleftrightarrow} \mu = \sum_n \delta_{x_n}$ such that $\langle \mu, g_0 \rangle < \infty$ and

$$\mu_n \to \mu \quad \text{in } \mathscr{M}_{g_0} \overset{\text{def}}{\Longleftrightarrow} \sup \langle \mu_n, g_0 \rangle < \infty,$$

$$\langle \mu_n, f \rangle \to \langle \mu, f \rangle \quad \text{for all } f \in C_c \text{ and } f = g_0,$$

where $C_c \equiv C_c(S)$ denotes the space of all continuous functions with compact supports on S. Then $\mathscr{M}_{g_0}$ is metrizable and separable.

We mainly consider the case that the generator has the form

$$A = A^c + A^d, \tag{1.1}$$

with

$$A^c f(x) = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x) \partial_{ij}^2 f(x) + \sum_{i=1}^{d} b^i(x) \partial_i f(x).$$

$$A^d f(x) = \int_{S \setminus \{x\}} [f(y) - f(x) - \nabla f(x) \cdot (y - x) I(|y - x| < 1)] \nu(x, dy)$$

$$- k(x) f(x) + \sum_{i=1}^{d} c^i(x) \partial_i f(x)$$

for $f \in D_g$. where $a^{ij}, b^i \in C_b(S)$, (a^{ij}) is positive definite, $k(x) \geq 0$ denotes the killing rate by jumps, $(c^i(x))$ depends on jumps, and $\nu(x, dy)$ is the Lévy kernel on $S \times (S \setminus \{x\})$ satisfying that

$$\sup_{x \in S} \int_{S \setminus \{x\}} (1 \wedge |y - x|^2) \nu(x, dy) < \infty.$$

Independent IMPS; Results in [2]

Let $(X_t, \mathbf{P}_\mu)$ be an (indistinguishable) independent Markov particle system (IMP) associated with the motion process $(w(t), P_x)$, i.e., for many independent motions $(w_n(t), P_{x_n}) \overset{(d)}{=} (w(t), P_{x_n})$,

$$X_t = \sum_n \delta_{w_n(t)}|_S \quad \text{if } \mu = \sum_n \delta_{x_n} \text{ on } S, \quad \text{and} \quad \mathbf{P}_\mu = \prod_n P_{x_n}.$$

The generator $\mathscr{L}_0$ of this particle system is given by the following: for $f \in C_c^\infty$,

$$\mathscr{L}_0 e^{-\langle \cdot, f \rangle}(\mu) = -\langle \mu, e^f A(1-e^{-f})\rangle e^{-\langle \mu, f\rangle}$$
$$= -\langle \mu, Af - \Gamma f\rangle e^{-\langle \mu, f\rangle},$$

where $\Gamma f := Af - e^f A(1-e^{-f})$. In fact, let $\{\mathscr{F}_t\}$ be the filtration generated by $\{X_t\}$ and set

$$V_t f(x) = -\log P_x[\exp -f(w(t))] = -\log\{1 - P_t(1-e^{-f})(x)\}.$$

We have that if $0 \le s < t$, then

$$\mathrm{E}_\mu[e^{-\langle X_t, f\rangle} | \mathscr{F}_s] = \exp[-\langle X_s, V_{t-s} f\rangle].$$

It is easy to see that $(V_t)_{t \ge 0}$ is a nonnegative contraction semigroup on C_0 and that by (ii) of Assumption 1 if $f \in C_c^\infty$, then $1 - e^{-f} \in C_c^\infty \subset D_g$, hence we have

$$\partial_t V_t f = \frac{P_t A(1-e^{-f})}{1 - P_t(1-e^{-f})} = \frac{AP_t(1-e^{-f})}{1-P_t(1-e^{-f})} = e^{V_t f} A(1 - e^{-V_t f})$$
$$\to e^f A(1-e^{-f}) = Af - \Gamma f \quad (t \downarrow 0).$$

Note that since $V_t f \le P_t f$ (by Jensen's inequality), Γ is nonnegative;

$$\Gamma f = Af - \partial_t V_t f|_{t=0+} = \lim_{t \downarrow 0} \frac{1}{t}[(P_t f - f) - (V_t f - f)] \ge 0$$

and that for each $f \in C_c^\infty$, $v_t = V_t f$ is the unique solution to the following equation:

$$\partial_t v_t = e^{v_t} A(1 - e^{-v_t}), \quad v_0 = f$$

(because $u_t := 1 - e^{-v_t}$ satisfies $\partial_t u_t = Au_t$, $u_0 = 1 - e^{-f}$ and the unique solution is given as $u_t = P_t(1-e^{-f})$). Moreover if $Av_t(x)$ is well-defined for $t > 0$, $x \in S$, then

$$\partial_t v_t = Av_t - \Gamma v_t, \quad v_0 = f.$$

By using the Markov property and by induction we have

PROPOSITION 1 (Prop. 1 in [1]). *For every* $0 \le t_1 \le \cdots \le t_n$ *and* $f_i \in D_g^+$, $i = 1, 2, \ldots, n$,

$$\mathbf{E}_\mu[\langle X_{t_1}, f_1\rangle \cdots \langle X_{t_n}, f_n\rangle]$$
$$\leq \prod_{i=1}^{n}\langle \mu, P_{t_i}f_i\rangle + C_1^{(n)}\sum_{i=1}^{n}\prod_{j\neq i}\langle \mu, P_{t_j}f_j\rangle$$
$$+ C_2^{(n)}\sum_{i_1\neq i_2}\prod_{j\neq i_1,i_2}\langle \mu, P_{t_j}f_j\rangle + \cdots + C_{n-1}^{(n)}\sum_{j=1}^{n}\langle \mu, P_{t_j}f_j\rangle + C_n^{(n)},$$

where $C_k^{(n)}$, $k = 1,\ldots,n$ *are positive constants, depending on* $(n, \{\|f_i\|_\infty\}_{i\leq n})$.

We introduce a non-negative operator Q as $Qf = Af^2 - 2fAf$ for $f \in D_g$, which is well-defined by (i) of Assumption 2 and plays an important role to investigate the exponents of Hölder (right) continuity. The non-negativity follows from $(P_tf^2 - f^2) - 2f(P_tf - f) \geq (P_tf)^2 - 2fP_tf + f^2 = (P_tf - f)^2 \geq 0$.

THEOREM 1 (Th. 2.3 and Cor. 2.1 in [2]). *Let* $(w(t), P_x)$ *be a discontinuous Markov process in* $\mathbf{D}([0,\zeta(w)) \to S)$ *with transition semigroup* (P_t) *satisfying Assumption* 1 *and* 2. *Let* $\mu \in \mathscr{M}_{g_0}$. *The following holds with* $\mathbf{P}_\mu$*-probability one.*

- (i) $\{\langle X_t, g_0\rangle\}$ *is* $((1-\gamma)/2 - \varepsilon)$*-Hölder right continuous at* $t = 0$ *for sufficiently small* $\varepsilon > 0$, *where the constant* $0 \leq \gamma < 1$ *is in* (iv) *of Assumption* 2.
- (ii) *If* $\langle \mu, g_1\rangle < \infty$, *in particular, if* $g_1(x) = g_0(x)$ *then* $\{\langle X_t, g_0\rangle\}$ *is* $(1/2 - \varepsilon)$*-Hölder right continuous at* $t = 0$ *for sufficiently small* $\varepsilon > 0$.
- (iii) *For each fixed* $t_0 > 0$, $\{\langle X_t, g_0\rangle\}$ *is* $(1/2 - \varepsilon)$*-Hölder right continuous at* $t = t_0$ *for sufficiently small* $\varepsilon > 0$.

2. Sampling Replacement Markov Particle Systems

Let $\mu = \sum_n \delta_{x_n} \in \mathscr{M}_{g_0}$. Let $(Y_t, \mathbf{P}_\mu^Y)$ be a sampling replacement Markov particle system associated with the motion process $(w(t), P_x)$, sampling replacement rate $\lambda > 0$ and sampling replacement probability $q(d(m,n)) = \sum_{k,l} p_{k,l}\delta_{(k,l)}(d(m,n))$ on $\mathbf{N}^2$, where $p_{k,l} \geq 0$, $p_{k,k} = 0$ and $\sum p_{k,l} = 1$. Each particle first moves independently each other. After a λ-exponential random time, two particles are selected randomly, for example, m-th and n-th particles are selected with probability $p_{m,n}$, and at that time the m-th particle jumps to the place of the n-th particle. Then the m-th particle moves independently. And these operations are continued. We denote each particle by $w_n^*(t)$ such that $w_n^*(0) = x_n$, and hence $Y_t = \sum_n \delta_{w_n^*(t)}$. Note that if (P_t) is non-conservative, then it is possible that the dead particles come life again.

Recall $(X_t, \mathbf{P}_\mu)$ is the independent Markov particle system with the motion process $(w(t), P_x)$. For $f \in D_g$, set $L_t^Y(\mu) = \mathbf{E}_\mu^Y[\exp -\langle Y_t, f\rangle]$ and $L_t(\mu) = \mathbf{E}_\mu[\exp -\langle X_t, f\rangle]$. Then

$$L_t(\mu) = e^{-\langle \mu, V_t f\rangle} \quad \text{with} \quad V_t f(x) = -\log E_x[e^{-f(w(t))}] = -\log(1 - P_t(1 - e^{-f})).$$

It is easy to see that $L_t^Y(\mu)$ satisfies the following equation:

$$L_t^Y(\mu) = e^{-\lambda t} L_t(\mu) + \lambda \int_0^t ds e^{-\lambda s} \int_{\mathbf{N}^2} q(d(m,n)) \mathbf{P}_s(\Theta_{m,n} L_{t-s})(\mu),$$

where $(\mathbf{P}_t)$ is the transition semigroup of $(X_t, \mathbf{P}_\mu)$ and $\Theta_{m,n}$ is an operator such that it makes the m-th particle jump to the place of n-th particle of $\mu = \sum \delta_k \in \mathscr{M}_{g_0}$ on a class of all functions $F(\mu)$ and it is defined by $\Theta_{m,n} F(\mu) = F(\mu^{m,n})$ with $\mu^{m,n} = \mu - \delta_{x_m} + \delta_{x_n}$. Note that $\mathbf{P}_s \Theta_{m,n} = \Theta_{m,n} \mathbf{P}_s$ holds. The solution is given as

$$(2.1) \quad L_t^Y(\mu) = \mathbf{T}_t e^{-\langle \cdot, V_t f\rangle}(\mu) \quad \text{with } \mathbf{T}_t = \sum_k e^{-\lambda t} \frac{(\lambda t)^k}{k!} \left(\int_{\mathbf{N}^2} q(d(m,n)) \Theta_{m,n} \right)^k,$$

where $\mathbf{T}_t$ is an operator on a class of functions $F(\mu)$ with polynomial growth of $\langle \mu, f_1\rangle, \langle \mu, f_2\rangle, \dots, \langle \mu, f_n\rangle$ $(f_i \in D_g)$ and

$$\left(\int_{\mathbf{N}^2} q(d(m,n)) \Theta_{m,n} \right)^k F(\mu) = \left(\int_{\mathbf{N}^2} q(d(m,n)) \Theta_{m,n} \right)^{k-1} \sum_{m,n \in \mathbf{N}} p_{m,n} F(\mu^{m,n}).$$

The generator $\mathscr{L}^Y$ of this particle system is given by the following: for $f \in C_c^\infty$,

$$\begin{aligned} \mathscr{L}^Y e^{-\langle \cdot, f\rangle}(\mu) &= \mathscr{L}_0 e^{-\langle \cdot, f\rangle}(\mu) + \lambda \int (e^{-\langle \mu^{m,n}, f\rangle} - e^{-\langle \mu, f\rangle}) q(d(m,n)) \\ &= -\Big\{ \langle \mu, Af - \Gamma f\rangle + \lambda \int \langle \delta_{x_n} - \delta_{x_m}, f\rangle q(d(m,n)) \\ &\qquad - \lambda \int [e^{-\langle \delta_{x_n} - \delta_{x_m}, f\rangle} - 1 + \langle \delta_{x_n} - \delta_{x_m}, f\rangle] q(d(m,n)) \Big\} e^{-\langle \mu, f\rangle} \end{aligned}$$

(more general formula of $\mathscr{L}^Y F(\mu)$ is given in §5). We have the following result. Recall that we denote the particles of Y_t by $w_n^*(t)$, i.e., $Y_t = \sum \delta_{w_n^*(t)}$. Note that $w_n^*(t)$ moves like as $w_n(t)$ during the jump times.

THEOREM 2 (Semi-martingale Representation of Y_t). *Under Assumption 1 and 2 for (P_t), if $\mu \in \mathcal{M}_{g_0}$, then $(Y_t, \mathbf{P}_\mu^Y)$ is an $\mathcal{M}_{g_0}$-valued Markov process with sample paths in $\mathbf{D}([0,\infty) \to \mathcal{M}_{g_0})$ satisfying the following:*

(i) *$\{\langle Y_t, g_0 \rangle\}$ has the same exponent of Hölder right continuity as in Theorem 1.*

(ii) *If the motion process $(w(t), P_x)$ has generator A of the form as in (1.1), then for $f \in D_g$,*

$$\langle Y_t, f\rangle = \langle Y_0, f\rangle + \int_0^t \left\{ \langle Y_s, Af\rangle + \lambda \int \langle \delta_{w_n^*(s)} - \delta_{w_m^*(s)}, f\rangle q(d(m,n)) \right\} ds$$
$$+ M_t^c(f) + M_t^d(f),$$

where

$$M_t^c(f) \quad \text{is a continuous } L^2\text{-martingale}$$

with quadratic variation $\langle\!\langle M^c(f)\rangle\!\rangle_t = \int_0^t \langle Y_s, Q^c f\rangle\, ds = 2\int_0^t \langle Y_s, \Gamma^c f\rangle\, ds$ and

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_{g_0}^\pm} \langle \mu, f\rangle \tilde{N}(ds, d\mu) \quad \text{is a purely discontinuous martingale}$$

where $\tilde{N} = N - \hat{N}$ is the martingale measure with

$$N(ds, d\mu) = \sum_{u:\Delta Y_u \neq 0} \delta_{(u, \Delta Y_u)}(ds, d\mu): \quad \text{the jump measure of } \{Y_t\},$$

$$\hat{N}(ds, d\mu) = ds \left\{ \int Y_s(dx) \left(\int \nu(x, dy) \delta_{(\delta_y - \delta_x)} + k(x) \delta_{-\delta_x} \right) \right.$$
$$\left. + \lambda \int q(d(m,n)) \delta_{(\delta_{w_n^*(s)} - \delta_{w_m^*(s)})} \right\} (d\mu): \quad \text{the compensator of } N.$$

PROOF. The proof is the same as the independent case (Proof of Theorem 2.4 in [2]). However, we need some computations. First the Markov property can be shown by mathematical induction. For $t_1 < t_2$, $f_1, f_2 \in C_c^\infty$, let $L_{t_1,t_2}^{f_1,f_2}(\mu) = \mathbf{E}_\mu^Y[\exp(-\langle Y_{t_1}, f_1\rangle - \langle Y_{t_2}, f_2\rangle)]$. Recall that $L_t^Y(\mu) = L_t^f(\mu) = \mathbf{E}_\mu^Y[\exp -\langle Y_t, f\rangle]$ satisfies (2.1) and the solution is given as $L_t^f(\mu) = \mathbf{T}_t\mathbf{P}_t[\exp -\langle \cdot, f\rangle](\mu)$. Hence it is easy to see that $L_{t_1,t_2}^{f_1,f_2}(\mu)$ satisfies the following equation:

$$L_{t_1,t_2}^{f_1,f_2}(\mu) = e^{-\lambda t_1}\mathbf{P}_{t_1}(e^{-\langle \cdot, f_1\rangle} L_{t_2-t_1}^{f_2})(\mu)$$
$$+ \lambda \int_0^{t_1} ds e^{-\lambda s} \int q(d(m,n)) \mathbf{P}_s(\Theta_{m,n} L_{t_1-s,t_2-s}^{f_1,f_2})(\mu).$$

The solution is given as

$$L^{f_1,f_2}_{t_1,t_2}(\mu) = \mathbf{T}_{t_1}\mathbf{P}_{t_1}(e^{-\langle \cdot, f_1\rangle} L^{f_2}_{t_2-t_1})(\mu)$$
$$= \mathbf{E}^Y_\mu[e^{-\langle Y_{t_1}, f_1\rangle}\mathbf{E}^Y_{Y_{t_1}}[e^{-\langle Y_{t_2-t_1}, f_2\rangle}]].$$

Therefore by induction, for every $n \in \mathbf{N}$, if $t_1 < t_2 < \cdots < t_n$, $f_1, \dots, f_n \in C_c^\infty$, then it holds that

$$\mathbf{E}^Y_\mu[\exp(-\langle Y_{t_1}, f_1\rangle - \cdots - \langle Y_{t_n}, f_n\rangle)]$$
$$= \mathbf{E}^Y_\mu[e^{-\langle Y_{t_1}, f_1\rangle}\mathbf{E}^Y_{Y_{t_1}}[e^{-\langle Y_{t_2-t_1}, f_2\rangle}\cdots \mathbf{E}^Y_{Y_{t_{n-1}}}[e^{-\langle Y_{t_n-t_{n-1}}, f_n\rangle}]\cdots]].$$

Next we shall show that $(Y_t, \mathbf{P}^Y_\mu)$ satisfies a moment inequality of the same type as in Proposition 1.

PROPOSITION 2. *Let $T > 0$ and $n \in \mathbf{N}$. For every $0 \le t_1 \le \cdots \le t_n \le T$ and $f_i \in D_g^+$, $i = 1, 2, \dots, n$,*

$$\mathbf{E}^Y_\mu[\langle Y_{t_1}, f_1\rangle \cdots \langle Y_{t_n}, f_n\rangle]$$
$$\le \prod_{i=1}^n \langle \mu, P_{t_i} f_i\rangle + C^{(n)}_{1,T}\sum_{i=1}^n \prod_{j\ne i}\langle \mu, P_{t_j} f_j\rangle$$
$$+ C^{(n)}_{2,T}\sum_{i_1\ne i_2}\prod_{j\ne i_1,i_2}\langle \mu, P_{t_j} f_j\rangle + \cdots + C^{(n)}_{n-1,T}\sum_{j=1}^n \langle \mu, P_{t_j} f_j\rangle + C^{(n)}_{n,T},$$

where $C^{(n)}_{k,T}$, $k = 1, \dots, n$ are positive constants depending on $(n, T, \{\|f_i\|_\infty\}_{i\le n})$.

PROOF. For simplicity, we use notations $f_m = f(x_m)$ and $\|\cdot\| = \|\cdot\|_\infty$. Since $\Theta_{m,\ell}\langle \mu, f\rangle = \langle \mu, f\rangle + f_\ell - f_m \le \langle \mu, f\rangle + \|f\|$, we have for every $k \in \mathbf{N}$,

$$\left(\int_{\mathbf{N}^2} q(d(m,\ell))\Theta_{m,\ell}\right)^k (\langle \mu, f_1\rangle \cdots \langle \mu, f_n\rangle) \le (\langle \mu, f_1\rangle + k\|f_1\|)\cdots(\langle \mu, f_n\rangle + k\|f_n\|).$$

Moreover if we denote by $M(j;\lambda t)$ the j-th moment of λt-Poisson distribution, then

$$\mathbf{T}_t(\langle \cdot, f_1\rangle \cdots \langle \cdot, f_n\rangle)(\mu)$$
$$= \sum_{k\ge 0} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left(\int_{\mathbf{N}^2} q(d(m,\ell))\Theta_{m,\ell}\right)^k (\langle \cdot, f_1\rangle \cdots \langle \cdot, f_n\rangle)(\mu)$$
$$\le \sum_{j=0}^n M(n-j;\lambda t)\sum_{\{i_1,\dots,i_j\}\subset\{1,\dots,n\}} \langle \mu, f_{i_1}\rangle \cdots \langle \mu, f_{i_j}\rangle \|f_{i_{j+1}}\|\cdots\|f_{i_n}\|.$$

Therefore by applying Proposition 1 and the above inequality to

$$\mathbf{E}_{\mu}^{Y}[\langle Y_{t_1}, f_1\rangle \cdots \langle Y_{t_n}, f_n\rangle] = \mathbf{T}_t(\mathbf{E}.[\langle X_{t_1}, f_1\rangle \cdots \langle X_{t_n}, f_n\rangle])(\mu),$$

we can get the desired inequality. ■

Thus the proof can be proceeded by the same way as in the independent case (see §4 in [2]). In that way we can get the following result. For $\mu = \sum_n \delta_{x_n} \in \mathscr{M}_{g_0}$, $f \in C_c^\infty$, let

$$\Psi(\mu; f) = \langle \mu, Af - \Gamma f\rangle - \lambda \int [e^{-\langle \delta_{x_m} - \delta_{x_n}, f\rangle} - 1] q(d(m,n)).$$

THEOREM 3. *For* $f \in C_c^\infty$,

$$e^{-\langle Y_t, f\rangle} - e^{-\langle Y_0, f\rangle} - \int_0^t \mathscr{L}^Y e^{-\langle \cdot, f\rangle}(Y_s)\, ds$$

is a $\mathbf{P}_\mu^Y$*-martingale. Moreover*

$$H_t(f) = \exp\left[-\langle Y_t, f\rangle + \int_0^t \Psi(Y_s; f)\, ds\right]$$

is also a $\mathbf{P}_\mu^Y$*-martingale.*

PROOF. By the same way as in the proof of Theorem 4.1 in [2] we have if $s < t$, then

$$\begin{aligned}
\partial_t \mathbf{E}_\mu^Y[e^{-\langle Y_t, f\rangle} \mid \mathscr{F}_s] &= \partial_t \mathbf{T}_{t-s}(e^{-\langle \cdot, V_{t-s} f\rangle})(Y_s) \\
&= \partial_{u=0+} \mathbf{T}_{t-s+u}(e^{-\langle \cdot, V_{t-s+u} f\rangle})(Y_s) \\
&= \partial_{u=0+} \mathbf{E}_\mu^Y[\mathbf{T}_u e^{-\langle \cdot, V_u f\rangle}(Y_t) \mid \mathscr{F}_s] \\
&= \mathbf{E}_\mu^Y[\partial_{u=0+} \mathbf{T}_u e^{-\langle \cdot, V_u f\rangle}(Y_t) \mid \mathscr{F}_s] \\
&= \mathbf{E}_\mu^Y[\mathscr{L}^Y e^{-\langle \cdot, f\rangle}(Y_t) \mid \mathscr{F}_s].
\end{aligned}$$

■

By using the above results it is not difficult to prove the semi-martingale representation of Y_t as of X_t in [2]. In fact, for $f \in C_c^\infty$, $\langle Y_t, f\rangle$ is a special semi-martingale, thus,

$$\langle Y_t, f\rangle = \langle Y_0, f\rangle + C_t(f) + M_t^c(f) + \tilde{N}_t(f) + N_t(f),$$

where $C_t(f)$ is a continuous process of locally bounded variation, $M_t^c(f)$ is a continuous L^2-martingale with quadratic variation $\langle\langle M^c(f)\rangle\rangle_t$, and

$$\tilde{N}_t(f) = \int_0^t \int_{\mathscr{M}^\pm} \langle \mu, f\rangle I(\|\mu\| < 1)\tilde{N}(ds, d\mu),$$

$$N_t(f) = \int_0^t \int_{\mathscr{M}^\pm} \langle \mu, f\rangle I(\|\mu\| \geq 1)N(ds, d\mu)$$

with the jump measure N of $\{Y_t\}$, its compensator $\hat{N}$ and $\tilde{N} = N - \hat{N}$.

If we set

$$B_t(f) = C_t(f) + \int_0^t \int_{\{\|\mu\|\geq 1\}} \langle \mu, f\rangle \hat{N}(ds, d\mu)$$
$$+ \lambda \int_0^t ds \int \langle \delta_{w_n(s)} - \delta_{w_m(s)}, f\rangle q(d(m, n)),$$

then by applying Ito's formula for $Z_t(f)$ we can get

$$-dB_t(f) + \frac{1}{2} d\langle\langle M^c(f)\rangle\rangle_t + \int [e^{-\langle \mu, f\rangle} - 1 + \langle \mu, f\rangle]\hat{N}(dt, d\mu)$$
$$= -\Psi(Y_t; f)\, dt$$
$$= \left\{ -\langle Y_t, Af\rangle + \langle Y_t, \Gamma f\rangle + \lambda \int (e^{-\langle \delta_{w_n^*(t)} - \delta_{w_m^*(t)}, f\rangle} - 1) q(d(m, n)) \right\} dt$$
$$= \left\{ -\left[\langle Y_t, Af\rangle + \lambda \int \langle \delta_{w_n^*(t)} - \delta_{w_m^*(t)}, f\rangle q(d(m, n)) \right] + \langle Y_t, \Gamma^c f\rangle \right.$$
$$\left. + \langle Y_t, \Gamma^d f\rangle + \lambda \int [e^{-\langle \delta_{w_n^*(t)} - \delta_{w_m^*(t)}, f\rangle} - 1 + \langle \delta_{w_n^*(t)} - \delta_{w_m^*(t)}, f\rangle] q(d(m, n)) \right\} dt$$

Thus we have

$$B_t(f) = \int_0^t \langle Y_s, Af\rangle\, ds + \lambda \int \langle \delta_{w_n^*(t)} - \delta_{w_m^*(t)}, f\rangle q(d(m, n)),$$

$$\langle\langle M^c(f)\rangle\rangle_t = 2\int_0^t \langle Y_s, \Gamma^c f\rangle\, ds = \int_0^t \langle Y_s, Q^c f\rangle\, ds$$

and

$$\hat{N}(ds, d\mu) = ds\left\{\int Y_s(dx)\left(\int \nu(x, dy)\delta_{(\delta_y - \delta_x)} + k(x)\delta_{-\delta_x}\right)\right.$$
$$\left. + \lambda\int q(d(m,n))\delta_{(\delta_{w_n^*(s)} - \delta_{w_m^*(s)})}\right\}(d\mu).$$

Therefore the proof is completed. ■

3. Martingale Problems for $\mathscr{L}^Y$

The following assumption is needed to prove the well-posedness of martingale problems.

ASSUMPTION 3. *For each* $f \in (C_c^\infty)^+$, $AV_tf = -A\log(1 - P_t(1 - e^{-f}))$ *is well-defined and* AV_tf *is continuous in* t *under the norm* $\|\cdot/g_1\|_\infty$, *i.e.*,

$$\|(AV_tf - AV_{t_0}f)/g_1\|_\infty \to 0 \quad (t \to t_0).$$

In the following we suppose that the generator A of the motion process has the form of (1.1).

For $\eta \in \mathscr{M}_{g_0}$, let $F(\eta) = \Phi(\langle\eta, f_1\rangle, \ldots, \langle\eta, f_n\rangle) \in \mathscr{D}_0 \overset{\text{def}}{\Longleftrightarrow} \Phi(x) \in C^\infty(\mathbf{R}^n)$ is a polynomial growth function with polynomial growth derivatives of all orders and $f_i \in D_g$, $i = 1, \ldots, n$. For this $F(\eta)$, the generator $\mathscr{L}_0$ of X_t will be extended to the following form:

$$\mathscr{L}_0F(\eta) = \sum_{i=1}^n \partial_i\Phi(\langle\eta, f_1\rangle, \ldots, \langle\eta, f_n\rangle)\langle\eta, Af_i\rangle$$
$$+ \frac{1}{2}\sum_{i,j=1}^n \partial_{ij}^2\Phi(\langle\eta, f_1\rangle, \ldots, \langle\eta, f_n\rangle)\langle\eta, Q^c(f_i, f_j)\rangle$$
$$+ \int_S\left\{\int_{S\setminus\{x\}} \nu(x, dy)\left[\Phi(\langle\eta, f_1\rangle + f_1(y) - f_1(x), \ldots, \langle\eta, f_n\rangle\right.\right.$$
$$+ f_n(y) - f_n(x)) - \Phi(\langle\eta, f_1\rangle, \ldots, \langle\eta, f_n\rangle)$$
$$\left. - \sum_{i=1}^n \partial_i\Phi(\langle\eta, f_1\rangle, \ldots, \langle\eta, f_n\rangle)(f_i(y) - f_i(x))\right]$$

$$+ k(x)\left[\Phi(\langle\eta, f_1\rangle - f_1(x), \dots, \langle\eta, f_n\rangle - f_n(x)) \right.$$

$$- \Phi(\langle\eta, f_1\rangle, \dots, \langle\eta, f_n\rangle)$$

$$\left. + \sum_{i=1}^{n} \partial_i\Phi(\langle\eta, f_1\rangle, \dots, \langle\eta, f_n\rangle) f_i(x)\right]\Bigg\}\eta(dx),$$

where

$$Q^c(f,g)(x) = \sum_{i,j} a^{ij}(x)\partial_i f(x)\partial_j g(x).$$

For $F(\eta) \in \mathscr{D}_0$, the generator $\mathscr{L}^Y$ of Y_t will be extended to

$$\mathscr{L}^Y F(\eta) = \mathscr{L}_0 F(\eta) + \lambda \int (\Theta_{m,n} F(\eta) - F(\eta)) q(d(m,n)).$$

THEOREM 4 (Martingale Problem for $(\mathscr{L}^Y, \mathscr{D}_0, \mu)$). *Under Assumption* 1, 2 *and* 3, *suppose that the generator* A *is given as in* (1.1). *Let* $\mu \in \mathscr{M}_{g_0}$.

(i) $\mathbf{P}_\mu^Y(Y_0 = \mu) = 1$ *holds and for each* $F(\mu) = \Phi(\langle\mu, f_1\rangle, \dots, \langle\mu, f_n\rangle) \in \mathscr{D}_0$,

$$M_t^F = F(Y_t) - F(Y_0) - \int_0^t \mathscr{L}^Y F(Y_s)\, ds \quad \text{is } \mathbf{P}_\mu^Y\text{-martingale.}$$

(ii) *If there is a probability measure* $\mathbf{Q}_\mu$ *on* $\mathbf{D} = \mathbf{D}([0,\infty) \to \mathscr{M}_{g_0})$ *such that the canonical process* $\tilde{Y}_t(\omega) = \omega(t)$ $(\omega \in \mathbf{D})$ *satisfies the same conditions as* $(Y_t, \mathbf{P}_\mu^Y)$ *in* (i) *and*

$$\int_0^t \langle \tilde{Y}_s, g_1\rangle\, ds < \infty \quad \mathbf{Q}_\mu\text{-a.s. for all } t \geq 0,$$

then $\mathbf{Q}_\mu = \mathbf{P}_\mu^Y \circ Y^{-1}$ *on* $\mathbf{D}$, *that is, martingale problem for* $(\mathscr{L}^Y, \mathscr{D}_0, \mu)$ *on* $\mathbf{D}$ *is well-posed.*

PROOF. The proof is essentially the same as the independent case (see §5 in [2]). However, the computations are more complicated, so we give the outline of the proof. (i) is easily obtained. We show (ii). We always fix $f \in C_c^\infty$, $T > 0$, and set $\Psi_t^T(\eta) = \mathbf{T}_{T-t}(\exp -\langle\cdot, V_{T-t}f\rangle)(\eta)$ $(0 \leq t \leq T, \eta \in \mathscr{M}_{g_0})$. It is no difficult to show that $\{\Psi_t^T(\tilde{Y}_t)\}_{t \leq T}$ is a $\mathbf{Q}_\mu$-martingale. In fact, by using Ito's formula

$$d(\exp -\langle \tilde{Y}_t, f\rangle) = -\langle \tilde{Y}_t, Af - \Gamma f\rangle e^{-\langle \tilde{Y}_t, f\rangle} dt$$
$$+ \lambda \int q(d(m,n))(\Theta_{m,n} - I)e^{-\langle \cdot, f\rangle}(\tilde{Y}_t)\, dt + d(\mathbf{Q}_\mu\text{-martingale}).$$

Since $\mathbf{T}_t$ is a bounded operator, we have (set $v_t = v_t^T = V_{T-t}f$ again)

$$d(\Psi_t^T(\tilde{Y}_t)) = \mathbf{T}_{T-t}\Big(-\lambda \int q(d(m,n))(\Theta_{m,n} - I)e^{-\langle \cdot, v_t\rangle} - \langle \cdot, \partial_t v_t\rangle e^{-\langle \cdot, v_t\rangle}$$
$$- \langle \cdot, Av_t - \Gamma v_t\rangle e^{-\langle \cdot, v_t\rangle} + \lambda \int q(d(m,n))(\Theta_{m,n} - I)e^{-\langle \cdot, v_t\rangle}\Big)(\tilde{Y}_t)\, dt$$
$$+ d(\mathbf{Q}_\mu\text{-martingale})$$
$$= \mathbf{T}_{T-t}(-\langle \cdot, \partial_t v_t + Av_t - \Gamma v_t\rangle e^{-\langle \cdot, v_t\rangle})(\tilde{Y}_t) + d(\mathbf{Q}_\mu\text{-martingale})$$
$$= d(\mathbf{Q}_\mu\text{-martingale})$$

Hence for $0 \le s < t \le T$, we have

$$\mathbf{Q}_\mu[\Psi_t^T(\tilde{Y}_t) \mid \mathscr{F}_s] = \Psi_s^T(\tilde{Y}_s)$$

and set $T = t$, then

$$\mathbf{Q}_\mu[e^{-\langle \tilde{Y}_t, f\rangle} \mid \mathscr{F}_s] = \mathbf{T}_{t-s}e^{-\langle \cdot, V_{t-s}f\rangle}(\tilde{Y}_s).$$

Therefore $\mathbf{P}_\mu = \mathbf{Q}_\mu$ on $\mathbf{D}$. ∎

4. Multi-Dimensional Absorbing Stable Motions on a Half Space

In §3 of [2] as a motion process we considered absorbing Brownian motion and absorbing stable motion on $(0, \infty)$ and discussed the Hölder (right) continuities of $\{X_t\}$. It is possible to consider absorbing motions on $H = \mathbf{R}^{d-1} \times (0, \infty)$ and we can get the same results as in Theorem 3.1 and in Corollary 3.1 of [2]. For the absorbing Brownian motion, it is not so difficult and essentially done in [1]. So in this section we only discuss the absorbing stable motion on H.

For a function f on H, let $\bar{f}$ be an extension of f to on $\mathbf{R}^d$ defined as

$$\bar{f}(x) = \begin{cases} f(x) & (x_d > 0), \\ f(\tilde{x}, 0+) = 0 & (x_d = 0), \\ -f(\tilde{x}, -x_d) & (x_d < 0), \end{cases}$$

where $x = (\tilde{x}, x_d) \in H$. Note that if $x \in H$, then $\bar{f}(x) = f(x)$. The generator $A^- \equiv A^{-,\alpha}$ of absorbing α-stable motion $(w^-(t), P_x^-) \equiv (w^{-,\alpha}(t), P_x^{-,\alpha})$ on H is given as $A^{-,\alpha}f(x) = A^\alpha \bar{f}(x)$; ($A^-$ is the same as L^- in §4 of [1], however, in which we have some miss-prints)

$$(4.1)\quad A^{-,\alpha}f(x) = c\int_{\mathbf{R}^d\setminus\{0\}} [\bar{f}(x+y) - \bar{f}(x) - \nabla\bar{f}(x)\cdot yI(|y|<1)]\frac{dy}{|y|^{d+\alpha}}$$

$$= c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_{-x_d}^{x_d} [f(x+y) - f(x) - \nabla f(x)\cdot yI(|y|<1)]\frac{dy_d}{|y|^{d+\alpha}}$$

$$+ c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_{x_d}^{\infty} [f(x+y) - f(\tilde{x}+\tilde{y}, y_d - x_d) - 2f(x)]\frac{dy_d}{|y|^{d+\alpha}}$$

with some positive constant c, where in the last term the integral corresponding to $\nabla f(x)\cdot y$ is equal to zero by the symmetric property (of course, it is integrable). We can also write that if $0 < \alpha < 1$, then

$$A^{-,\alpha}f(x) = c\int_{\mathbf{R}^d\setminus\{x\}} [\bar{f}(y) - \bar{f}(x)]\frac{dy}{|y-x|^{d+\alpha}}$$

$$= c\int_{\mathbf{R}^{d-1}} d\tilde{y}\left\{\int_0^\infty [f(y) - f(x)]K(x,y)\,dy_d - 2f(x)\int_0^\infty \frac{dy_d}{|(\tilde{y}-\tilde{x}, y_d + x_d)|^{d+\alpha}}\right\},$$

$$= c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_0^\infty [f(y) - f(x)]K(x,y)\,dy_d - f(x)k(x),$$

and that if $1 \le \alpha < 2$, then

$$A^{-,\alpha}f(x) = c\int_{\mathbf{R}^d\setminus\{x\}} [\bar{f}(y) - \bar{f}(x) - \nabla\bar{f}(x)\cdot(y-x)I(|y-x|<1)]\frac{dy}{|y-x|^{d+\alpha}}$$

$$= c\int_{\mathbf{R}^{d-1}} d\tilde{y}\left\{\int_0^\infty [f(y) - f(x) - \nabla f(x)\cdot(y-x)I(|y-x|<1)]K(x,y)\,dy_d\right.$$

$$+ \int_0^\infty [-2f(x) - \nabla f(x)\cdot(y-x)I(|y-x|<1)$$

$$- \nabla f(x)\cdot(\tilde{y}-\tilde{x}, -y_d - x_d)I(|(\tilde{y}-\tilde{x}, y_d + x_d)| < 1)]$$

$$\left.\times\frac{dy_d}{|(\tilde{y}-\tilde{x}, y_d + x_d)|^{d+\alpha}}\right\}$$

$$= c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_0^\infty [f(y) - f(x) - \nabla f(x)\cdot(y-x)I(|y-x|<1)]K(x,y)\,dy_d$$

$$- f(x)k(x) + \nabla f(x)\cdot c(x),$$

where

$$K(x,y) = \frac{I(y\neq x)}{|y-x|^{d+\alpha}} - \frac{1}{|(\tilde{y}-\tilde{x}, y_d + x_d)|^{d+\alpha}},$$

$$k(x) = k(x_d) = 2c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_{x_d}^\infty \frac{dy_d}{|y|^{d+\alpha}}$$

and

$$c(x) = c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_0^\infty [-(\tilde{y}-\tilde{x}, -y_d - x_d)I(|(\tilde{y}-\tilde{x}, y_d + x_d)| < 1)$$

$$- (y-x)I(|y-x|<1)]\frac{dy_d}{|(\tilde{y}-\tilde{x}, y_d+x_d)|^{d+\alpha}}$$

$$= c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_0^\infty [(\tilde{y}, y_d + x_d)I(|(\tilde{y}, y_d + x_d)| < 1)$$

$$- (\tilde{y}, y_d - x_d)I(|(\tilde{y}, y_d - x_d)| < 1)]\frac{dy_d}{|(\tilde{y}, y_d + x_d)|^{d+\alpha}}.$$

Let $h_0(v)$ be a C^∞-function on $(0,\infty)$ such that $0 < h_0 \le 1$ on $(0,\infty)$, $h_0(v) = v$ for $v \in (0, 1/2]$ and $h_0(v) = 1$ for $v \ge 1$. Let $d < p < d+\alpha$. Set $g_p(x) = (1+|x|^2)^{-p/2}$ and $g_{p,0}(x) := g_p(x)h_0(x_d)$ for $x \in H$. Let $f \in C_p \overset{\text{def}}{\iff} f \in C(\mathbf{R}^d)|_H$; $\|f/g_p\|_\infty < \infty$. $f \in C_{p,0} \overset{\text{def}}{\iff} f \in C(\mathbf{R}^d)|_H$; $\|f/g_{p,0}\| < \infty$. Moreover set

$$f \in C^3_{p,0} \overset{\text{def}}{\iff} f \in C^3_b(\mathbf{R}^d)|_H;$$

$$\text{for } i,j \neq d,\ f, \partial_d^2 f, \partial_i f, \partial_{ij}^2 f \in C_{p,0} \text{ and } \partial_d f, \partial_{id}^2 f \in C_p.$$

Then we can take $D_g = C^3_{p,0}$.

Moreover for each $0 < \alpha < 2$, $Q^- f \equiv Q^{-,\alpha} f = Af^2 - fAf$ is given by the following formula:

$$(4.2)\quad Q^{-,\alpha}f(x) = c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_{-x_d}^{x_d} [f(x+y) - f(x)]^2\frac{dy_d}{|y|^{d+\alpha}}$$

$$+ c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_{x_d}^\infty [\{f(x+y) - f(\tilde{y}+\tilde{x}, y_d - x_d)\}$$

$$\{f(x+y)+f(\tilde{y}+\tilde{x},y_d-x_d)-2f(x)\}+2f(x)^2]\frac{dy_d}{|y|^{d+\alpha}}$$

$$=c\int_{\mathbf{R}^{d-1}}d\tilde{y}\int_0^\infty[f(y)-f(x)]K(x,y)\,dy_d+f(x)^2k(x).$$

THEOREM 5. *Let $d \geq 1$, $d < p < d+\alpha$, $\mu \in \mathcal{M}_{g_{p,0}}$ and let the motion process be absorbing α-stable motion on H with $0 < \alpha < 2$. Let $\varepsilon > 0$ denote an arbitrary small number.*

(i) *For $(X_t, \mathbf{P}_\mu)$ the following holds.*
  (a) *Under $\mathbf{P}_\mu$, $\{\langle X_t, g_{p,0}\rangle\}$ is $(1/(2(\alpha \vee 1)) - \varepsilon)$-Hölder right continuous at $t=0$. Moreover in case of $1 < \alpha < 2$, if $\langle \mu, g_1\rangle < \infty$ with $g_1(x) = g_p(x)h_0(x_d)^{2-\alpha}$, then $\{\langle X_t, g_{p,0}\rangle\}$ is $(1/2-\varepsilon)$-Hölder right continuous at $t=0$.*
  (b) *If $t_0 > 0$, then under $\mathbf{P}_\mu$, $\{\langle X_t, g_{p,0}\rangle\}$ is $(1/2-\varepsilon)$-Hölder right continuous at $t = t_0$ for every $0 < \alpha < 2$.*

(ii) *For $(Y_t, \mathbf{P}_\mu^Y)$ the same results hold as above.*

PROOF. Let $d \geq 2$. The proof is proceeded in the same way as the case of $d=1$. It suffices to check that the conditions in Assumption 1 and 2 are fulfilled with $g_0 = g_{p,0}$ and with suitable $g_1 \in C^\infty$, $0 \leq \gamma < 1$ as follows. Let $h_1 \in C^\infty$; $0 < h_1 \leq 1$, $h_1(v) = v\log(1/v)$ for $v \in (0, 1/e]$ and $h_1(v) = 1$ for $v \geq 1$.

(i) If $0 < \alpha < 1$, then $g_1(x) = g_{p,0}(x)$, $\gamma = 0$.
(ii) If $\alpha = 1$, then $g_1(x) = g_p(x)h_1(x_d)$, $\gamma = \delta$ for any small $0 < \delta < 1$.
(iii) If $1 < \alpha < 2$, then $g_1(x) = g_p(x)h_0(x_d)^{2-\alpha}$, $\gamma = 1 - 1/\alpha$.

Note that as $x_d \downarrow 0$,

$$g_1(x) \sim x_d \ (0 < \alpha < 1), \quad \sim x_d \log(1/x_d) \ (\alpha = 1), \quad \sim x_d^{2-\alpha} \ (1 < \alpha < 2).$$

For simplicity of the notations we omit the superscript "α" as $P_t^{-,\alpha} = P_t^-$, $A^{-,\alpha} = A^-$. We shall show the following. Since they imply $\|g_{p,0}P_t^- g_1\|_\infty \leq Ct^{-\gamma}$, we can get the $((1-\gamma)/2 - \epsilon)$-Hölder right continuity.

(C1) $C_{p,0}^3 \subset \mathscr{D}(A^-)$, $P_t^- C_c^\infty \subset C_{p,0}^3$ for every $t \geq 0$, $\sup_{t\geq 0, 0<x_d\leq 1}|x_d^{-1}P_t^- g_{p,0}(x)| < \infty$ and $A^- C_{p,0}^3 \subset C_{p,0}$ (these imply Assumption 1 and that $C_{p,0}^3$ is a core).

(C2) For every $f \in C^3_{p,0}$, $\partial_t P^-_t f^2(x) = A^- P^-_t f^2(x) = P^-_t A^- f^2(x)$ $(x \in H)$, $A^- f^2 \in C_b$ and $\|g_1^{-1} Q^- f\|_\infty < \infty$ (these imply (i) of Assumption 2).

(C3) For each $0 < \beta \le 1$, $\sup_{t \ge 0} P^-_t(y_d^\beta)(x) \le 2(1+\beta)x_d^\beta$ for all $x \in H$ (this implies (ii) of Assumption 2).

(C4) For each $0 < \beta \le 1$, $\sup_{0 < x_d \le 1} x_d^{-1} P^-_t(y_d^\beta)(x) \le C_\beta t^{-(1-\beta)/\alpha}$ with a constant $C_\beta > 0$ depending only on β (this implies (iii), (iv) of Assumption 2).

Note that we take $\gamma = (1-\beta)/\alpha$ in Assumption 2. More exactly, if $0 < \alpha < 1$, then $\beta = 1$, i.e., $\gamma = 0$. If $\alpha = 1$, then $\beta = 1 - \delta$ for any small $0 < \delta < 1$, i.e., $\gamma = \delta$. If $1 < \alpha < 2$, then $\beta = 2 - \alpha$, i.e., $\gamma = 1 - 1/\alpha$. (C3) and (C4) can be shown in a way similar to the case of $d = 1$; (B3) and (B4) in [2] by using the following. For the density $p^\alpha(x)$ of the rotation invariant α-stable motion on $\mathbf{R}^d$ starting from 0, $p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha}x)$ and $p_1^\alpha(x) \le C(1 \wedge |x|^{-d-\alpha})$. The transition density $p_t^-(x, y) \equiv p_t^{-,\alpha}(x, y)$ of absorbing α-stable motion on H is given as

$$p_t^-(x, y) = p_t^\alpha(y - x) - p_t^\alpha(\tilde{y} - \tilde{x}, y_d + x_d) = -\int_{-x_d}^{x_d} \partial_d p_t^\alpha(\tilde{y} - \tilde{x}, y_d + v)\, dv.$$

We also use the following result.

$$\int_H z_d^{\beta-1} p_1^\alpha(\tilde{z}, z_d + u)\, dz \quad \text{is bounded in } u \in \mathbf{R}.$$

From these results we can get (C3), (C4).

In each (C1), (C2), the claims except the last one can be shown by the same way as in $d = 1$. In order to show the last claims of (C1), (C2), it is enough to prove that for each $f \in C^3_{p,0}$, there is a constant $C > 0$ such that

$$|A^- f(x)| \le C x_d \quad \text{for } 0 < x_d \le 1/2 \quad \text{and} \quad Q^- f(x) \le C g_1(x) \quad \text{for all } x \in H.$$

Let $0 < x_d \le 1/2$. For A^- we use the formula (4.1). In the following we decompose as $A^- f = (J_{1,1} + J_{1,2}) + (J_2 + J_3)$ and we shall show each term has order of x_d^2, x_d, x_d, x_d, respectively. The main calculus is of $J_{1,2}$ $(1 < \alpha < 2)$ and J_3. In the first term of (4.1) we divide the integral area to $\{|y| \ge 1\}$, $\{|y| < 1\}$ and denote the corresponding terms by $J_{1,1}(x)$, $J_{1,2}(x)$ respectively. In the following we use the same symbols C', C'' as any positive finite constants which are independent of x. First note that if $|y| \ge 1$ and $|y_d| \le x_d \le 1/2$, then $|\tilde{y}|^2 \ge 1 - x_d^2 \ge 3/4 =: b$. By $|f(x)| \le C x_d$,

$$
\begin{aligned}
|J_{1,1}(x)| &= \left| c\int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{-x_d}^{x_d} [f(x+y)-f(x)]I(|y|\geq 1)\frac{dy_d}{|y|^{d+\alpha}}\right| \\
&\leq 2c\int_{|\tilde{y}|\geq b} d\tilde{y}\int_0^{x_d} C(2x_d+y_d)\frac{dy_d}{|y|^{d+\alpha}} \\
&\leq 2cC\int_{|\tilde{y}|\geq b}\frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha}}\int_0^{x_d}(2x_d+y_d)\,dy_d \\
&\leq 2cC\int_b^{\infty}\frac{dr}{r^{2+\alpha}}\int_0^{x_d}(2x_d+y_d)\,dy_d \\
&= \frac{5cC}{1+\alpha}b^{-1-\alpha}x_d^2 = C'x_d^2.
\end{aligned}
$$

Next note that if $|y|<1$, then $|\tilde{y}|<1$ and that for some $\theta\in(0,1)$,

$$
|f(x+y)-f(x)-\nabla f(x)\cdot y| = \frac{1}{2}|f^{(2)}(x+\theta y)\cdot y^2| \leq \frac{1}{2}\|f^{(2)}\|_{\infty}|y|^2.
$$

If $0<\alpha<1$, then $|y|^{d-2+\alpha}\geq|\tilde{y}|^{d-2+\alpha}$ by $d-2+\alpha>0$, and

$$
\begin{aligned}
|J_{1,2}(x)| &= \left| c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_{-x_d}^{x_d}[f(x+y)-f(x)-\nabla f(x)\cdot y]I(|y|<1)\frac{dy_d}{|y|^{d+\alpha}}\right| \\
&\leq c\int_{|\tilde{y}|<1} d\tilde{y}\int_{-x_d}^{x_d}|f(x+y)-f(x)-\nabla f(x)\cdot y|\frac{dy_d}{|y|^{d+\alpha}} \\
&\leq c\int_{|\tilde{y}|<1} d\tilde{y}\int_0^{x_d}\|f^{(2)}\|_{\infty}|y|^2\frac{dy_d}{|y|^{d+\alpha}} \\
&\leq c\int_{|\tilde{y}|<1}\frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha-2}}\|f^{(2)}\|_{\infty}x_d \\
&\leq c\int_0^1 r^{-\alpha}\,dr\|f^{(2)}\|_{\infty}x_d \\
&= \frac{c\|f^{(2)}\|_{\infty}}{1-\alpha}x_d. = C'x_d.
\end{aligned}
$$

On the other hand if $1\leq\alpha<2$, then by using

$$
f(x+y)-f(x)-\nabla f(x)\cdot y = \frac{1}{2}f^{(2)}(x)\cdot y^2+\frac{1}{6}f^{(3)}(x+\theta y)\cdot y^3
$$

with some $\theta \in (0,1)$, and the corresponding integral to $\sum_{i=1}^{d-1} \partial_{id}^2 f(x) y_i y_d$ is equal to zero by symmetric property in y_d, we have

$$|J_{1,2}(x)| \le c \int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} \left\{ \sum_{i,j=1}^{d-1} |\partial_{ij}^2 f(x)|\,|y_i|\,|y_j| + |\partial_d^2 f(x)| y_d^2 \right.$$
$$\left. + \frac{1}{3} \|f^{(3)}\|\,|y|^3 \right\} \frac{dy_d}{|y|^{d+\alpha}}.$$

Let $0 < \epsilon < 2-\alpha$ and set $\alpha_\epsilon := \alpha + \epsilon \in (1,2)$, then $|y|^{d+\alpha-2} \ge |\tilde{y}|^{d-1-\epsilon} |y_d|^{-1+\alpha_\epsilon}$. By $|\partial_{ij}^2 f(x)|, |\partial_d^2 f(x)| \le C x_d$ for $i, j \ne d$, corresponding terms to $f^{(2)}$ are less than or equal to

$$Cx_d \int_{|\tilde{y}|<1} \frac{d\tilde{y}}{|\tilde{y}|^{d-1-\epsilon}} \int_0^{x_d} \frac{dy_d}{y_d^{-1+\alpha_\epsilon}} = Cx_d \int_0^1 r^{\epsilon-1}\, dr \frac{x_d^{2-\alpha_\epsilon}}{2-\alpha_\epsilon} = \frac{C}{(2-\alpha_\epsilon)\epsilon} x_d^{3-\alpha_\epsilon}.$$

For the last term, by $d \ge 2$, $\alpha \ge 1$, i.e., $d + \alpha - 3 \ge 0$, we have $|y|^{d+\alpha-3} \ge |\tilde{y}|^{d+\alpha-3}$. Hence the last term is less than or equal to

$$\int_{|\tilde{y}|<1} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha-3}} \|f^{(3)}\|_\infty x_d = \int_0^1 r^{1-\alpha}\, dr \|f^{(3)}\|_\infty x_d = \frac{\|f^{(3)}\|_\infty}{2-\alpha} x_d.$$

These estimates imply $|J_{1,2}(x)| \le C' x_d$. In the second term of (4.1) we also divide the integral area to $\{|y| \ge 1\}$, $\{|y| < 1\}$ and denote the corresponding terms by $J_2(x)$, $J_3(x)$ respectively. For $J_2(x)$, by

$$|f(x+y) - f(\tilde{y} + \tilde{x}, y_d - x_d)| \le 2x_d \|\partial_d f\|_\infty$$

and $|f(x)| \le C x_d$, we have

$$|J_2(x)| = \left| c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^\infty [f(x+y) - f(\tilde{x}+\tilde{y}, y_d - x_d) - 2f(x)] I(|y| \ge 1) \frac{dy_d}{|y|^{d+\alpha}} \right|$$
$$\le c \int_H 2(\|\partial_d f\|_\infty + C) x_d I(|y| \ge 1) \frac{dy}{|y|^{d+\alpha}}$$
$$\le C' x_d \int_{|y| \ge 1} \frac{dy}{|y|^{d+\alpha}} = C'' x_d.$$

For $J_3(x) = c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^\infty [f(x+y) - f(\tilde{y}+\tilde{x}, y_d - x_d) - 2f(x)] I(|y| < 1) \frac{dy_d}{|y|^{d+\alpha}}$,

$$f(x+y) - f(\tilde{x}+\tilde{y}, y_d - x_d) - 2f(x)$$
$$= [f(x+y) - f(\tilde{x}+\tilde{y}, y_d - x_d) - 2f(\tilde{x}+\tilde{y}, x_d)] - 2[f(\tilde{x}+\tilde{y}, x_d) - f(x)].$$

For the first term, by the same way as in case of $d = 1$ (in $J_3(x)$) corresponding to the variable y_d, we have

$$|f(x+y) - f(\tilde{x}+\tilde{y}, y_d - x_d) - 2f(x)|$$
$$\leq 2\|\partial_d^3 f\|_\infty x_d(y_d^2 + x_d^2) + C(y_d + x_d)x_d y_d + 2Cx_d^3.$$

For the second term, note that

$$f(\tilde{x}+\tilde{y}, x_d) - f(x) = \sum_{i=1}^{d-1} \partial_i f(\tilde{x}, x_d) y_i + \frac{1}{2}\sum_{i,j=1}^{d-1} \partial_{ij}^2 f(\tilde{x}+\theta\tilde{y}, x_d) y_i y_j.$$

and $|\partial_{ij}^2 f(\tilde{x}+\theta\tilde{y}, x_d)| \leq Cx_d$ for $i, j \leq d$. Moreover note that by the symmetric property in y_i we have

$$\int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^{\infty} \sum_{i=1}^{d-1} \partial_i f(\tilde{x}, x_d) y_i I(|y| < 1) \frac{dy_d}{|y|^{d+\alpha}}$$
$$= \sum_{i=1}^{d-1} \partial_i f(\tilde{x}, x_d) \int_{x_d}^{1} dy_d \int_{|\tilde{y}| < \sqrt{1-y_d^2}} \frac{y_i \, d\tilde{y}}{|y|^{d+\alpha}} = 0.$$

Let $0 < \epsilon < 2 - \alpha$ and set $\alpha_\epsilon := \alpha + \epsilon \in (0, 2)$, then $|y|^{d+\alpha} \geq |\tilde{y}|^{d-1-\epsilon}|y_d|^{1+\alpha_\epsilon}$ by $d \geq 2$. Thus we can get the following: by $x_d \leq y_d$,

$$|J_3(x)| \leq c\int_{|\tilde{y}|<1} d\tilde{y} \int_{x_d}^{1} [2\|\partial_d^3 f\|_\infty x_d(y_d^2 + x_d^2) + C(y_d + x_d)x_d y_d + 2Cx_d^3] \frac{dy_d}{|y|^{d+\alpha}}$$
$$+ c\int_{|\tilde{y}|<1} d\tilde{y} \int_{x_d}^{1} \frac{1}{2}\sum_{i,j=1}^{d-1} |\partial_{ij}^2 f(\tilde{x}+\theta\tilde{y}, x_d) y_i y_j| \frac{dy_d}{y_d^{d+\alpha}}$$
$$\leq C' x_d \int_{|\tilde{y}|<1} \frac{d\tilde{y}}{|\tilde{y}|^{d-1-\epsilon}} \int_{x_d}^{1} (y_d^2 + x_d y_d + x_d^2) \frac{dy_d}{y_d^{1+\alpha_\epsilon}}$$
$$+ c\int_{|\tilde{y}|<1} \frac{1}{2} Cx_d |\tilde{y}|^2 \, d\tilde{y} \int_0^1 \frac{dy_d}{|y|^{d+\alpha}}$$
$$\leq 3C' x_d \frac{1}{\epsilon}\int_0^1 y_d^{1-\alpha_\epsilon} \, dy_d + \frac{cC}{2} x_d \int_{|\tilde{y}|<1} d\tilde{y} |\tilde{y}|^2 \int_0^1 \frac{dy_d}{|y|^{d+\alpha}}.$$

For the second term if $0 < \alpha < 1$, then by $|y|^{d+\alpha} \geq |\tilde{y}|^d y_d^\alpha$,

$$\int_{|\tilde{y}|<1} d\tilde{y}|\tilde{y}|^2 \int_0^1 \frac{dy_d}{|y|^{d+\alpha}} \leq \int_{|\tilde{y}|<1} \frac{|\tilde{y}|^2}{|\tilde{y}|^d} \, d\tilde{y} \int_0^1 \frac{dy_d}{y_d^\alpha} = \frac{1}{1-\alpha},$$

or if $1 \le \alpha < 2$, then by $|y|^{d+\alpha} \ge |\tilde{y}|^{d+1-\epsilon} y_d^{\alpha_\epsilon - 1}$,

$$\int_{|\tilde{y}|<1} d\tilde{y}|\tilde{y}|^2 \int_0^1 \frac{dy_d}{|y|^{d+\alpha}} \le \int_{|\tilde{y}|<1} \frac{|\tilde{y}|^2}{|\tilde{y}|^{d+1-\epsilon}} d\tilde{y} \int_0^1 \frac{dy_d}{y_d^{\alpha_\epsilon - 1}} = \frac{1}{\epsilon(2-\alpha_\epsilon)}.$$

Therefore

$$|J_3(x)| \le \begin{cases} \left(\frac{3C'}{\epsilon(2-\alpha_\epsilon)} + \frac{cC}{2(1-\alpha)}\right) x_d & (0 < \alpha < 1) \\ \frac{6C' + cC}{2\epsilon(2-\alpha_\epsilon)} x_d & (1 \le \alpha < 2) \end{cases}$$

$$= C'' x_d.$$

Therefore we have $|A^- f(x)| \le C'' x_d$.

Next in order to show $Q^- f(x) \le C g_1(x)$, it suffices to prove that there is a constant $C > 0$ such that for $0 < x_d \le 1$, if $0 < \alpha < 1$, then $Q^- f(x) \le C x_d$, if $\alpha = 1$, then $Q^- f(x) \le C x_d \log(1/x_d)$ if $1 < \alpha < 2$, then $Q^- f(x) \le C x_d^{2-\alpha}$. We use the first formula of (4.2). In the following we decompose as $Q^- f = (R_1 + R_2) + (S_1 + S_2)$ and we shall show each R_1, R_2, S_1 has order of x_d^3, $x_d^{2-\alpha}$, x_d respectively, and the main parts is S_2. In the first term of the right hand side of (4.2), we divide the integral area of $\mathbf{R}^{d-1}$ to $\{|\tilde{y}| \ge 1\}$, $\{|\tilde{y}| < 1\}$ and denote the corresponding terms by $R_1(x)$, $R_2(x)$ respectively. By $f(x) \le C x_d$, we have

$$\begin{aligned} R_1(x) &= c \int_{|\tilde{y}| \ge 1} d\tilde{y} \int_{-x_d}^{x_d} [f(x+y) - f(x)]^2 \frac{dy_d}{|y|^{d+\alpha}} \\ &\le 2c \int_{|\tilde{y}| \ge 1} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha}} \int_0^{x_d} C^2 (2x_d + y_d)^2 \, dy_d \\ &= C x_d^3 \end{aligned}$$

For R_2, by $|\partial_i f(x)| \le C x_d$ if $i \ne d$,

$$\begin{aligned} R_2(x) &= 2c \int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} [f(x+y) - f(x)]^2 \frac{dy_d}{|y|^{d+\alpha}} \\ &\le 2c \int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} \left[C(x_d + y_d) \sum_{i=1}^{d-1} y_i + \|\partial_d f\|_\infty y_d \right]^2 \frac{dy_d}{|y|^{d+\alpha}} \\ &\le C \int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} [(x_d^2 + y_d^2)|\tilde{y}|^2 + y_d^2] \frac{dy_d}{|y|^{d+\alpha}} \end{aligned}$$

In the above we first consider the last term (which is the main term), i.e.,

$$\int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} y_d^2 \frac{dy_d}{|y|^{d+\alpha}} = \int_0^{x_d} dy_d y_d^2 \left(\int_{|\tilde{y}|<y_d} + \int_{y_d \le |\tilde{y}|<1} \right) \frac{d\tilde{y}}{|y|^{d+\alpha}} =: R_{2,1}(x) + R_{2,2}(x).$$

For $R_{2,1}$, let $\alpha_\epsilon = \alpha + \epsilon < 2$ be the same as before. By $|y|^{d+\alpha} \ge |\tilde{y}|^{d-1-\epsilon}|y_d|^{1+\alpha_\epsilon}$,

$$\int_{|\tilde{y}|<y_d} \frac{d\tilde{y}}{|\tilde{y}|^{d-1-\epsilon}} = \int_0^{y_d} r^{\epsilon-1}\, dr = \frac{y_d^\epsilon}{\epsilon}.$$

Hence

$$R_{2,1}(x) \le \int_0^{x_d} \frac{y_d^2}{y_d^{1+\alpha+\epsilon}} \frac{y_d^\epsilon}{\epsilon}\, dy_d = \frac{1}{\epsilon}\int_0^{x_d} y_d^{1-\alpha}\, dy_d = \frac{1}{(2-\alpha)\epsilon} x_d^{2-\alpha}.$$

For $R_{2,2}$, by $|y|^{d+\alpha} \ge |\tilde{y}|^{d+\alpha}$ and

$$\int_{y_d \le |\tilde{y}|<1} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha}} = \int_{y_d}^1 \frac{dr}{r^{2+\alpha}} = \frac{1}{1+\alpha}(y_d^{-1-\alpha} - 1) \le \frac{1}{1+\alpha} y_d^{-1-\alpha}.$$

Hence

$$R_{2,2}(x) \le \int_0^{x_d} y_d^2 \frac{1}{1+\alpha} y_d^{-1-\alpha}\, dy_d = \frac{1}{1+\alpha}\int_0^{x_d} y_d^{1-\alpha}\, dy_d = \frac{1}{(1+\alpha)(2-\alpha)} x_d^{2-\alpha}.$$

Furthermore we can show more easily that the other terms of R_2 are $o(x_d^2)$, In fact, by $|y|^{d+\alpha} \ge |\tilde{y}|^{d-1+\alpha_\epsilon}|y_d|^{1-\epsilon}$,

$$\int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} (x_d^2 + y_d^2)|\tilde{y}|^2 \frac{dy_d}{|y|^{d+\alpha}} \le \int_{|\tilde{y}|<1} \frac{|\tilde{y}|^2}{|\tilde{y}|^{d-1+\alpha_\epsilon}}\, d\tilde{y} \int_0^{x_d} (x_d^2 + y_d^2) \frac{dy_d}{y_d^{1-\epsilon}} = Cx_d^{2+\epsilon}.$$

Therefore we have $R_2(x) \le Cx_d^{2-\alpha}$ for all $0 < \alpha < 2$.

In the second term of the right hand side of (4.2), we divide the integral area to $\{|y| \ge 1\}$, $\{|y| < 1\}$ and denote the corresponding terms by $S_1(x)$, $S_2(x)$ respectively. For S_1, by

$$|f(x+y) - f(\tilde{y} + \tilde{x}, y_d - x_d)| \le 2x_d \|\partial_d f\|_\infty, \tag{4.3}$$

we have

$$S_1(x) \le (2x_d\|\partial_d f\|_\infty \cdot 3\|f\|_\infty + 2Cx_d^2) \int_{|y|\ge 1} \frac{dy_d}{|y|^{d+\alpha}} \le Cx_d.$$

For S_2, by (4.3) and by $|f(x)| \le Cx_d$,

$$|\{f(x+y)-f(\tilde{y}+\tilde{x},y_d-x_d)\}\{f(x+y)+f(\tilde{y}+\tilde{x},y_d-x_d)-2f(x)\}+2f(x)^2|$$
$$\le 2\|\partial_d f\|_\infty x_d\cdot C(x_d+y_d)+2Cx_d^2$$
$$\le Cx_d(x_d+y_d).$$

Hence, noting that $\{|y|<1\}\subset\{|\tilde{y}|<1\}\times\{|y_d|<1\}$,

$$|S_2(x)|\le Cx_d\int_{|\tilde{y}|<1}d\tilde{y}\int_{x_d}^{1}(x_d+y_d)\frac{dy_d}{|y|^{d+\alpha}}\le 2Cx_d\int_{|\tilde{y}|<1}d\tilde{y}\int_{x_d}^{1}y_d\frac{dy_d}{|y|^{d+\alpha}}.$$

By the same way as in R_2, we can show the desired estimate as follows. Let

$$\int_{x_d}^{1}dy_dy_d\left(\int_{|\tilde{y}|<y_d}+\int_{y_d\le|\tilde{y}|<1}\right)\frac{d\tilde{y}}{|y|^{d+\alpha}}=:(S_{2,1}(x)+S_{2,2}(x)).$$

Then $|S_2(x)|\le Cx_d(S_{2,1}(x)+S_{2,2}(x))$. Let $0<\epsilon<2-\alpha$. By $|y|^{d+\alpha}\ge|\tilde{y}|^{d-1-\epsilon}|y_d|^{1+\alpha_\epsilon}$,

$$S_{2,1}(x)\le\int_{x_d}^{1}\frac{dy_d}{y_d^{1+\alpha_\epsilon}}y_d\int_{|\tilde{y}|<y_d}\frac{d\tilde{y}}{|\tilde{y}|^{d-1+\epsilon}}=\int_{x_d}^{1}\frac{dy_d}{y_d^{\alpha+\epsilon}}\frac{y_d^{\epsilon}}{\epsilon}=\frac{1}{\epsilon}\int_{x_d}^{1}\frac{dy_d}{y_d^{\alpha}}.$$

That is, if $0<\alpha<1$, then $S_{2,1}(x)\le C$, if $\alpha=1$, then $S_{2,1}(x)\le C\log(1/x_d)$, if $0<\alpha<1$, then $S_{2,1}(x)\le Cx_d^{1-\alpha}$. Moreover for $S_{2,2}$, as in $R_{2,2}$, by $\int_{y_d\le|\tilde{y}|<1}d\tilde{y}/|\tilde{y}|^{d+\alpha}\le y_d^{-1-\alpha}/(1+\alpha)$,

$$S_{2,2}(x)\le\int_{x_d}^{1}dy_dy_d\int_{y_d\le|\tilde{y}|<1}\frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha}}\le\int_{x_d}^{1}y_d\frac{1}{1+\alpha}y_d^{-1-\alpha}\,dy_d=\frac{1}{1+\alpha}\int_{x_d}^{1}y_d^{-\alpha}\,dy_d.$$

Thus $S_{2,2}$ satisfies the same estimates as $S_{2,1}$. By $|S_2(x)|\le Cx_d(S_{2,1}(x)+S_{2,2}(x))$, we have if $0<\alpha<1$, then $S_2(x)\le Cx_d$, if $\alpha=1$, then $S_2(x)\le Cx_d\log(1/x_d)$ if $1<\alpha<2$, then $S_2(x)\le Cx_d^{2-\alpha}$. These imply our desired result. ■

By $P_t^-C_c^\infty\subset C_{p,0}^3$, the following result for martingale problem is obtained by the same way as in $d=1$.

THEOREM 6. *Let $\mu\in\mathcal{M}_{g_{p,0}}$. The martingale problems for $(\mathcal{L}_0,\mathcal{D}_0,\mu)$, $(\mathcal{L}^Y,\mathcal{D}_0,\mu)$ associated with absorbing stable motion on H are well-posed.*

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