Analysis of Nonstationary Navier-Stokes Equations Approximated by Pressure Stabilization Method

Ranmaru Matsui

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Ranmaru Matsui

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Abstract

In this thesis, we consider the nonstationary Navier-Stokes equations approximated by the pressure stabilization method. We can obtain the local in time existence theorem for the approximated Navier-Stokes equations. Moreover we can obtain the error estimate between the solution to the usual Navier-Stokes equations and the Navier-Stokes equations approximated by the pressure stabilization method. We prove these theorem by using maximal regularity theorem. Furthermore, as the application of maximal regularity theorem, we can get the estimates for weak solutions of approximate Navier-Stokes equations.

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Contents

1 Introduction

1.1 Back ground

The mathematical description of fluid flow is given by the following Navier-Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla\pi = f \qquad \nabla\cdot u = 0 & t\in(0,\infty), x\in\Omega, \\ u(0,x) = a & x\in\Omega, \\ u(t,x) = 0 & x\in\partial\Omega, \end{cases} \tag{NS}$$

where the fluid vector fields $u = u(t,x) = (u_1(t,x),\dots,u_n(t,x))$ and the pressure $\pi = \pi(t,x)$ are unknown function, the external force $f = f(t,x)$ is a given vector functions, the initial data a is a given solenoidal function and Ω is some bounded domain. It is well-known that analysis of Navier-Stokes equations (NS) is very important in view of both mathematical analysis and engineering, however the problem concerning existence and regularity of solution to (NS) is unsolved for a long time. One of the difficulty of analysis for (NS) is the pressure term $\nabla\pi$ and incompressible condition $\nabla\cdot u = 0$.

In numerical analysis, some penalty methods (quasi-compressibility methods) are employed as the method to overcome this difficulty. They are methods that eliminate the pressure by using approximated incompressible condition. For example, setting $\alpha > 0$ as a perturbation parameter, we use $\nabla\cdot u = -\pi/\alpha$ in the penalty method, $\nabla\cdot u = \Delta\pi/\alpha$ in the pressure stabilization method and $\nabla\cdot u = -\partial_t\pi/\alpha$ in the pseudocompressible method. In this thesis, we consider the Navier-Stokes equations with incompressible condition approximated by pressure stabilization method. Namely we consider the following equations:

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + (u_\alpha\cdot\nabla)u_\alpha + \nabla\pi_\alpha = f & t\in(0,\infty), x\in\Omega, \\ \nabla\cdot u_\alpha = \Delta\pi_\alpha/\alpha & t\in(0,\infty), x\in\Omega, \\ u_\alpha(0,x) = a_\alpha & x\in\Omega, \\ u_\alpha(t,x) = 0, \qquad \partial_n\pi_\alpha(t,x) = 0 & x\in\partial\Omega. \end{cases} \tag{NSa}$$

(NSa) may be considered as a singular perturbation of (NS). As $\alpha\to\infty$, (NSa) tends to (NS) formally and we cancel the Neumann boundary condition for the pressure.

From the point of view of the maximal regularity theorem, the regularity of solution to the first equation is different from the one of the second equations in (NSa). Therefore, in order to adjust the regularity of the solution to their equations, we consider the following equations instead of approximated incompressible conditions in (NSa):

$$(u_\alpha, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_\alpha, \nabla\varphi)_\Omega \qquad \varphi\in\widehat{W}^1_{q'}(\Omega) \tag{C}$$

for $1 < q < \infty$. We notice that (C) is a weak form of the approximated incompressible condition $\nabla\cdot u_\alpha = \alpha^{-1}\Delta\pi_\alpha$. We call (C) approximated weak incompressible condition in this thesis. Therefore we consider

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + (u_\alpha\cdot\nabla)u_\alpha + \nabla\pi_\alpha = f & t\in(0,\infty), x\in\Omega, \\ u_\alpha(0,x) = a_\alpha & x\in\Omega, \\ u_\alpha(t,x) = 0 & x\in\partial\Omega \end{cases} \tag{NSa$'$}$$

under the approximated weak incompressible condition (C) in L^q-framework ($n/2 < q < \infty$).

1.2 Known result

Pressure stabilization method was first introduced by Brezzi and Pitkäranta [2]. They considered the approximated stationary Stokes equations which are linearlized Navier-Stokes equations with the approximated incompressible condition $\nabla \cdot u_\alpha = \Delta\pi_\alpha/\alpha$. They obtained the following error estimate by using the energy methods:

$$\|u_\alpha - u\|_{H^1(\Omega)} + \|\pi_\alpha - \pi\|_{L^2(\Omega)} \leq C\alpha^{-1/2}\|f\|_{L^2(\Omega)}. \tag{1.1}$$

Nazarov and Specovius-Neugebauer [16] considered the same approximate Stokes problem and derived asymptotically precise estimates for solution to the approximated problem as $\alpha \to \infty$ by using the parameter-dependent Sobolev norms. Their results are not available by the usually applied energy methods. These results introduced above are concerning the stationary Stokes equations and there are few results concerning the nonstationary Stokes equations and Navier-Stokes equations. As far as the authors know, only the result due to Prohl [19] is known as the results concerning the nonstationary problem. In [19], Prohl considered the sharp a priori estimate for the pressure stabilization method under some assumptions and showed the following error estimates:

$$\|u_\alpha - u\|_{L_\infty([0,T],L_2(\Omega))} + \|\tau(\pi_\alpha - \pi)\|_{L_\infty([0,T],W_2^{-1}(\Omega))} \leq C\alpha^{-1},$$
$$\|u_\alpha - u\|_{L_\infty([0,T],W_2^1(\Omega))} + \|\sqrt{\tau}(\pi_\alpha - \pi)\|_{L_\infty([0,T],L_2(\Omega))} \leq C\alpha^{-1/2},$$

where $\tau = \tau(t) = \min(t,1)$. Since their results are proved based on energy method, all of these estimates are in L_2 framework for the space. In this thesis, we shall use the maximal regularity theorem in order to prove the local in time existence theorem and the error estimate in the L_p in time and the L_q in space framework with $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$. Moreover letting P_Ω be the Helmholtz projection in Ω, we consider the following equations :

$$\int_0^T [-(u_\alpha, \partial_t\phi)_\Omega - (u_\alpha, \Delta\phi)_\Omega + (B(u_\alpha, u_\alpha), \phi)_\Omega + \alpha(u_\alpha, \phi)_\Omega]dt = (a_\alpha, \phi(0))_\Omega + \int_0^T (f, \phi)_\Omega dt \tag{WS}$$

for all $\phi \in C_0^\infty([0,T), C_0^\infty(\Omega))$, where $B(u,v)$ defined by

$$B(u,v) = (P_\Omega u \cdot \nabla)v, \tag{NLT}$$

since u_α doesn't satisfy incompressible condition. More precisely, in order to ensure the validity for the pressure stabilization method, we use (NLT) as a modification of the original nonlinearity in (NS) (see [19], [27]). By this setting, we can prove existence theorem of weak solution for (NSa). Lelay introduces weak solution for partial differential equation ([15]) and Hopf constructs weak solution the initial-boundary problem for 3-dimension bounded domain by using Lelay's method ([11]). Hopf's proof is to construct approximate solution and to obtain subsequence in L_2-space which converges weak solution for (NS) by using approximated solution and energy inequality. Therefore, since he used the energy inequality, these estimates are in L_2 framework for the space. This method is developed by Masuda [12]. Masuda proved the existence theorem for domain $\Omega \subset \mathbb{R}^n$ (and the uniqueness for only 2-dimension bounded domain) and the following L_2 framework estimate.

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 dt \leq 2\int_0^t (f,u)_\Omega dt + \|a\|_2^2 \quad (0 \leq t < T).$$

Since the weak solution for (NS) is a strong solution for it in 2-dimension domain, weak solution is required in 3-dimension case. But the uniqueness in this case is not proved until this very day. J. Saal considered existence and regularity of weak solutions for (NS) (with Robin boundary conditions) in $\Omega = \mathbb{R}^n_+$ by using the time L_p and the space L_q estimates for Stokes equations linearized (NS) with Robin boundary conditions and the Duhamel principle. Saal showed the L_2-L_2 estimate :

$$\|u\|^2_{L_\infty((0,T),L_2(\mathbb{R}^n_+))} + \|\nabla u\|_{L_2((0,T),L_2(\mathbb{R}^n_+))} \leq \|a\|^2_{L_2(\mathbb{R}^n_+)} + \int_0^T (f(t), u(t))_{\mathbb{R}^n_+}\,dt$$

and the L_p-L_q estimate :

$$\begin{aligned}&\|\partial_t u\|_{L_p((0,T),L_q(\mathbb{R}^n_+))} + \|\nabla^2 u\|_{L_p((0,T),L_q(\mathbb{R}^n_+))} + \|\nabla \pi\|_{L_p((0,T),L_q(\mathbb{R}^n_+))}\\ &\quad \leq C(\|a\|_{B^{2(1-1/p)}_{q,p}} + \|f\|_{L_p((0,T),L_q(\mathbb{R}^n_+))} + \|u\|^2_{L_\infty((0,T),L_2(\mathbb{R}^n_+))} + \|\nabla u\|^2_{L_2((0,T),L_2(\mathbb{R}^n_+))})\end{aligned}$$

for $n/q + 2/p = n+1$, where $B^{2(1-1/p)}_{q,p}$ is the real interpolation space :

$$B^{2(1-1/p)}_{q,p} = (L_q(\mathbb{R}^n_+), W^2_q(\mathbb{R}^n_+))_{1-1/p.p}.$$

Since his proof is based on maximal regularity for Stokes equations, he obtained L_p-L_q regularity of weak solution for (NS) with Robin boundary condition. In this thesis, using the way of Saal [20], we shall prove the L_p-L_q regularity of weak solution for (NSa) in bounded domain $\Omega \subset \mathbb{R}^n$ and the uniqueness in case of $n = 2$.

1.3 Thesis organization

This thesis consists of the following five sections. In section 2, we present the main results on local in time unique existence of solution to (NSa′) and certain error estimate between the solutions to (NSa′) and (NS) under the weak incompressible condition (Theorem 2.1 and Theorem 2.16). Following the argument due to Shibata and Kubo [24], we can prove the main results by contraction mapping principle with the help of the maximal L_p-L_q regularity theorem. And we state the result of weak solution for (NSa) (Theorem 2.5) by the method of Saal [20] with Hille-Yosida operator. After stating the main results, we present the maximal L_p-L_q regularity theorem for linearlized problem for (NSa′) (Theorem 2.6 and Theorem 2.14) and the theorem concerning the existence of $\mathcal{R}$-bounded solution operator for linearized problem (Theorem 2.11). As was seen in Shibata and Shimizu [26], the maximal L_p-L_q regularity theorem is direct consequence of Theorem 2.11 concerning the generalized resolvent problem for the linearized equations with the help of Weis' operator valued Fourier multiplier theorem (Theorem 2.10), so that the main part of this thesis is to show Theorem 2.11. Moreover another consequence of Theorem 2.11 is the resolvent estimate (Corollary 2.12), which implies the construct of the semigroup $T_\alpha(t)$ for linealized problem for (NSa′). By real interpolation, we obtain some estimates for $T_\alpha(t)$ (Theorem 2.13 and Theorem 2.15). In section 3, as preliminary, we shall introduce some theorems and lemmas which play important role in this thesis. In section 4, we consider the generalized resolvent problem for linearized problem in some bounded domain. For this purpose, we first consider the problem in the whole space case and the half-space case. By using the change of variable with their results, we shall prove the generalized bounded domain cases. In section 5, the following the argument due to Shibata and Kubo [24], we show the local in time existence theorem for (NSa′) and prove the error estimates (Theorem 2.1 and Theorem 2.16). Moreover, as the application of maximal regularity, we prove the existence and regularity theorem of weak solution for (NSa) (Theorem 2.3 and Theorem 2.4), and, in case of $n = 2$, see this solution is unique (Theorem 2.5).

2 Notation and Main Results

Before we describe main theorem, we shall introduce some functional spaces and notations throughout this thesis. As usual $C, M, \ldots$ denote constants that may change from line to line.

Sometimes we would like to express a special dependence on some parameter k. Then we use the notation C_k, M_k, ... or we write it as an argument $C(k), M(k)$, ... For $m \in \{0, 1, \dots, \infty\}$ we denote by $C^m(\Omega)$ the space of all m-times continuously differentiable functions and the space of cut of function

$$C_0^\infty([0,T), C_0^\infty(\Omega)) = \{\phi \in C^\infty([0,T), C_0^\infty(\Omega)) \mid \phi(t,x) = 0 \text{ if } t \text{ belongs to neighborhood of } T\}.$$

For $1 < q < \infty$, let $q' = q/(q-1)$. If $u \in L_q(\Omega)$ and $v \in L_{q'}(\Omega)$, we use the notation $(u,v)_\Omega = \int_\Omega uv dx$ for the dual pairing. For any closed operator A in X, its domain and range are denoted by $D(A)$ and $R(A)$, respectively. Furthermore, we call A a generator, if $\{e^{-tA}\}_{t \geq 0}$ satisfies the semigroup properties. For any two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y and we write $\mathcal{L}(X) = \mathcal{L}(X,X)$ for short. $\mathrm{Hol}(U,X)$ denotes the set of all X-valued holomorphic functions defined on a complex domain U. As the complex domain where a resolvent parameter belongs, we use $\Sigma_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} \mid |\arg\lambda| < \pi - \varepsilon\}$ and $\Sigma_{\varepsilon,\lambda_0} = \{\lambda \in \Sigma_\varepsilon \mid |\lambda| \geq \lambda_0\}$ for $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. For $1 \leq q < \infty$, $L_q(\Omega)$ denotes the Lebesgue space, which consists of all q-integrable functions with its norm $\|\cdot\|_{q,\Omega}$ and $L_\infty(\Omega)$ denotes the space of all functions u that satisfy $\|u\|_{\infty,\Omega} = \mathrm{ess.sup}_{x\in\Omega}|u(x)| < \infty$. $W_q^m(\Omega)$ $(1 \leq q \leq \infty)$ denotes the Sobolev space of order $m \in \mathbb{N}$. Its norm is given by

$$\|u\|_{m,q,\Omega} := \left(\sum_{j=0}^m \|\nabla^j u\|_{q,\Omega}^q\right)^{1/q},$$

where ∇^j is the tensor of all possible j-th order differentials. In particular, for non-negative integer m, we define $H^m(\Omega)$ as $W_2^m(\Omega)$ and $H_0^m(\Omega)$ as closure of infinitely differentiable functions compactly supported in $H^m(\Omega)$. As the time-space Lebesgue space, we use $L_p((0,T), L_q(\Omega)) = \{u \mid \|u\|_{L_p((0,T),L_q(\Omega))} < \infty\}$, where its norm is given by

$$\|u\|_{L_p((0,T),L_q(\Omega))} = \left(\int_0^T \|u(t)\|_{q,\Omega}^p dt\right)^{1/p}$$

If no confusion seems likely, we also write $\|\cdot\|_q = \|\cdot\|_{q,\Omega}$ and $\|\cdot\|_{p,q,T} = \|\cdot\|_{L_p((0,T),L_q(\Omega))}$. and often use the same symbols for denoting the vector and scalar function spaces. For $1 \leq p, q \leq \infty$, $B_{q,p}^{2(1-1/p)}(D)$ denotes the real interpolation space defined by $B_{q,p}^{2(1-1/p)}(D) = (L_q(D), W_q^2(D))_{1-1/p,p}$ (more precisely see Sohr [25]). For a Banach space X, we set

$$\begin{aligned}
L_{p,\gamma_0}(\mathbb{R}, X) &= \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R}, X) \mid \|e^{-\gamma t} f\|_{L_p(\mathbb{R},X)} < \infty, \ (\gamma \geq \gamma_0)\}, \\
L_{p,\gamma_0,(0)}(\mathbb{R}, X) &= \{f(t) \in L_{p,\gamma_0}(\mathbb{R}, X) \mid f(t) = 0 \ (t < 0)\}, \\
W^1_{p,\gamma_0,(0)}(\mathbb{R}, X) &= \{f(t) \in L_{p,\gamma_0,(0)}(\mathbb{R}, X) \mid f'(t) \in L_{p,\gamma_0}(\mathbb{R}, X)\}.
\end{aligned}$$

In order to deal with the pressure term, we use the following functional spaces:

$$\begin{aligned}
L_{q,\mathrm{loc}}(D) &= \{f \mid f|_K \in L_q(K), \ K \text{ is any compact set in } D\}, \\
\widehat{W}_q^1(D) &= \{\theta \in L_{q,\mathrm{loc}}(D) \mid \nabla\theta \in L_q(D)^n\}.
\end{aligned}$$

Since our proof is based on Fourier analysis, we next introduce the Fourier transform and the Laplace transform. We define the Fourier transform, its inverse Fourier transform, the Laplace transform and its inverse Laplace transform by

$$\begin{aligned}
\hat{f}(\xi) &= \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, & \mathcal{F}_\xi^{-1}[f](x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} f(\xi) d\xi, \\
\mathcal{L}_t[f](\lambda) &= \mathcal{F}_t[e^{-\gamma t} f(t)](\tau), & \mathcal{L}_\tau^{-1}[f](t) &= e^{\gamma t} \mathcal{F}_\tau^{-1}[f](t),
\end{aligned}$$

respectively, where $x, \xi \in \mathbb{R}^n$, $\lambda = \gamma + i\tau \in \mathbb{C}$ and $x \cdot \xi$ is usual inner product: $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$. Furthermore, we define the Fourier-Laplace transform by

$$\mathcal{L}_t[\mathcal{F}_x[v(t,x)]](\lambda,\xi) = \mathcal{F}_{t,x}[e^{-\gamma t}v(t,x)](\lambda,\xi) = \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n} e^{-(\lambda t + ix\cdot\xi)}v(t,x)dx\right)dt.$$

By using Fourier transform and Laplace transform, we define $H^s_{p,\gamma_0}(\mathbb{R}, X)$ for a Banach space X. For $\lambda = \gamma + i\tau$, we define the operator Λ^s_γ as

$$(\Lambda^s_\gamma f)(t) = \mathcal{L}^{-1}_\tau[|\lambda|^s \mathcal{L}_t[f](\lambda)](t) = e^{\gamma t}\mathcal{F}^{-1}_\tau[(\tau^2+\gamma^2)^{s/2}\mathcal{F}_t[e^{-\gamma t}f(t)](\tau)](t).$$

For $0 < s < 1$ and $\gamma_0 > 0$, we define the space $H^s_{p,\gamma_0}(\mathbb{R}, X)$ as

$$H^s_{p,\gamma_0}(\mathbb{R}, X) = \{f \in L_{p,\gamma_0}(\mathbb{R}, X) \mid \|e^{-\gamma t}\Lambda^s_\gamma f\|_{L_p(\mathbb{R},X)} < \infty (\forall \gamma \geq \gamma_0)\}.$$

In this thesis, we assume next assumption for our domain Ω.

Assumption 2.1. Let $n/2 < q < \infty$ and $n < r < \infty$. Let Ω be a uniform $W^{2-1/r}_r$ domain introduced in [8] and $L_q(\Omega)$ has the Helmholtz decomposition.

Therefore, the domain Ω has direct sum decomposition. In fact, the space of solenoidal fields in Ω is defined by $L_{q,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, where $C^\infty_{0,\sigma}(\Omega) = \{v \in C^\infty_0(\Omega) \mid \nabla \cdot v = 0\}$. It is well known that $L_{q,\sigma}(\Omega) = \{v \in L_q(\Omega) \mid \nabla \cdot v = 0, v|_{\partial\Omega} = 0\}$ and that this space is complementary in $L_q(\Omega)$ for $1 < q < \infty$. More precisely we obtain the Helmholtz decomposition

$$L_q(\Omega) = L_{q,\sigma}(\Omega) \oplus G_q(\Omega),$$

where $G_q(\Omega) := \{\nabla p \mid p \in \widehat{W}^1_q(\Omega)\}$. Therefore we can define projection operators $P = P_\Omega$ and $Q = Q_\Omega$ (called Helmholtz projection) on $L_q(\Omega)$ to $L_{q,\sigma}(\Omega)$ and $G_q(\Omega)$, respectively, which satisfy

$$u = Pu + \nabla Qu, \qquad \|Pu\|_{q,\Omega} + \|\nabla Qu\|_{q,\Omega} \leq C_{n,q}\|u\|_{q,\Omega}. \tag{HP}$$

We remark that if $q = 2$, $L_2(\Omega)$ has the Helmholtz decomposition for any Ω (see Galdi [10]).

First main result is concerned with the local in time existence theorem for (NSa$'$) with approximated weak incompressible condition (C).

Theorem 2.1. *Let $n \geq 2$, $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$. Let $\alpha > 0$ and $T_0 \in (0,\infty)$. For any $M > 0$, assume that $a_\alpha \in B^{2(1-1/p)}_{q,p}(\Omega)$ and $f \in L_p((0,T_0), L_q(\Omega)^n)$ satisfy*

$$\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|f\|_{L_p((0,T_0),L_q(\Omega)^n)} \leq M. \tag{2.1}$$

Then, there exists $T^* \in (0, T_0)$ depending on only M such that (NSa$'$) *under* (C) *has a unique solution (u_α, π_α) of the following class:*

$$u_\alpha \in W^1_p((0,T^*), L_q(\Omega)^n) \cap L_p((0,T^*), W^2_q(\Omega)^n), \qquad \pi_\alpha \in L_p((0,T^*), \widehat{W}^1_q(\Omega)).$$

Moreover the following estimate holds:

$$\|u_\alpha\|_{L_\infty((0,T^*),L_q(\Omega))} + \|(\partial_t u_\alpha, \nabla^2 u_\alpha, \nabla\pi_\alpha)\|_{L_p((0,T^*),L_q(\Omega))} + \|\nabla u_\alpha\|_{L_r((0,T^*),L_q(\Omega))} \leq C_{n,p,q,T^*}$$

for $1/p - 1/r \leq 1/2$.

Next, we describe our second main result. To do this, we introduce weak solution for (NSa).

Definition 2.2. Let $n \geq 2$ and $T \in [0, \infty)$. We call u_α a weak solution of system (NSa), if u_α belongs to the Lelay-Hopf class i.e. $u_\alpha \in L_\infty((0,T), L_2(\Omega)) \cap L_2((0,T), W_2^1(\Omega)^n)$ and u_α satisfies

$$\int_0^T [-(u_\alpha, \partial_t \phi) - (u_\alpha, \Delta\phi) + \sum_{j=1}^n (\partial_j u_\alpha, u_\alpha^j \phi) + \alpha(u_\alpha, \phi)]dt = (a_\alpha, \phi(0)) + \int_0^T (f, \phi)dt \qquad \text{(WS)}$$

for all $\phi \in C_0^\infty([0,T), C_0^\infty(\Omega))$.

This theorem is based on Hille-Yosida operator and Relich theorem. Namely, we consider the local in time existence theorem, proved by the fixed point theorem, of solution for the integral equation with Hille-Yosida approximation.

Theorem 2.3. *Let $n \geq 2$, $T \in (0, \infty]$. And let*

$$Y = L_\infty((0,T), L_2(\Omega)) \cap L_2((0,T), W_2^1(\Omega)^n).$$

Then, for all $a_\alpha \in L_2(\Omega)$ and $f \in L_2((0,T), L_2(\Omega)^n)$, there exists solution u_α for (WS) *such that the following estimate holds.*

$$\|u_\alpha\|_{\infty,2,T}^2 + \|A_\alpha^{1/2} u_\alpha\|_{2,2,T}^2 \leq \|a_\alpha\|_2^2 + \int_0^t (f(t), u_\alpha(t))dt. \tag{2.2}$$

Next, we shall state L_p-L_q regularity for solution of (WS) depending on the dimension n. This theorem is based on the maximal regularity of A_α and dual problem.

Theorem 2.4. *Let u_α be one of solutions of* (WS)*, which u_α doesn't have to satisfy energy inequality. And let the index p, q satisfy $1 < p, q < \infty$ and $n/q + 2/p = n + 1$. If a_α and f satisfy $a_\alpha \in B_{q,p}^{2(1-1/p)}(\Omega)$ and $f \in L_p((0,T), L_q(\Omega))$, respectively, then there exists a constant C such that the following inequality holds:*

$$\|e^{-\lambda_0 t}(\partial_t u_\alpha, A_\alpha u_\alpha, \nabla\pi_\alpha)\|_{p,q,T} \leq C(\|a_\alpha\|_{B_{q,p}^{2(1-1/p)}} + \|f\|_{p,q,T} + \|u_\alpha\|_{\infty,2,T}^2 + \|A_\alpha^{1/2} u_\alpha\|_{2,2,T}^2). \tag{2.3}$$

This theorem is the existence and uniqueness for solution of (WS) if the dimension n equals to 2.

Theorem 2.5. *Let $T \in (0, \infty]$, $a_\alpha \in L_2(\Omega) \cap W_2^1(\Omega)$ and $f \in L_2((0,T), L_2(\Omega))$. Then, the weak solution obtained by Theorem 2.4 below is unique and satisfies the regularity*

$$\nabla u_\alpha \in L_\infty((0,T), L_2(\Omega)), \qquad \partial_t u_\alpha, \nabla^2 u_\alpha, \nabla\pi_\alpha \in L_2((0,T), L_2(\Omega)).$$

In order to prove Theorem 2.1 and Theorem 2.5, we use maximal L_p-L_q regularity theorem for the following linearized problems corresponding to (NSa$'$):

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + \nabla\pi_\alpha = f & t > 0, x \in \Omega, \\ u_\alpha(t,x) = 0 & x \in \partial\Omega, \\ u_\alpha(0,x) = a_\alpha & x \in \Omega \end{cases} \qquad \text{(Sa}'\text{)}$$

under the approximated weak incompressible condition

$$(u_\alpha, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_\alpha, \nabla\varphi)_\Omega + (g, \nabla\varphi)_\Omega \qquad \varphi \in \widehat{W}_{q'}^1(\Omega). \qquad \text{(Cg)}$$

These main result is based on the following theorem which is concerned with the maximal L_p-L_q regularity for (Sa$'$) under (Cg) with $a_\alpha = 0$.

Theorem 2.6. *Let $1 < p, q < \infty$ and $\alpha > 0$. Then there exists a positive number γ_0 such that the following assertion holds : for any $f, g \in L_{p,\gamma_0,(0)}(\mathbb{R}, L_q(\Omega))$, (Sa$'$) under (Cg) with $a_\alpha = 0$ has a unique solution :*

$$u_\alpha \in L_{p,\gamma_0,(0)}(\mathbb{R}, W_q^2(\Omega)) \cap W^1_{p,\gamma_0,(0)}(\mathbb{R}, L_q(\Omega)), \qquad \pi_\alpha \in L_{p,\gamma_0,(0)}(\mathbb{R}, \widehat{W}_q^1(\Omega)).$$

Moreover, the following estimate holds :

$$\|e^{-\gamma t}(\partial_t u_\alpha, \gamma u_\alpha, \Lambda_\gamma^{\frac{1}{2}} \nabla u_\alpha, \Lambda_{\gamma+\alpha}^{1/2}(\nabla \cdot u_\alpha), \nabla^2 u_\alpha, \nabla \pi_\alpha)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C_{n,p,q} \|e^{-\gamma t}(f, \alpha g)\|_{L_p(\mathbb{R}, L_q(\Omega))}$$

for any $\gamma \geq \gamma_0$.

Remark 2.7. By the property of Helmholtz decomposition, we can solve (Cg) for $u_\alpha, g \in L_q(\Omega)$ and we see $\pi_\alpha = \alpha Q_\Omega(u_\alpha - g)$.

In order to prove Theorem 2.6, we use the operator valued Fourier multiplier theorem due to Weis [29]. This theorem needs $\mathcal{R}$-boundedness of solution operator. To this end, we first introduce the definition of $\mathcal{R}$-boundedness.

Definition 2.8. The family of the operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $f_j \in X$ ($j = 1, \ldots, N$) and for all sequences $\{\gamma_j(u)\}_{j=1}^N$ of independent, symmetric, $\{-1, 1\}$-valued random variables on [0,1], there holds the inequality:

$$\int_0^1 \|\sum_{j=1}^N \gamma_j(u) T_j f_j\|_Y^p du \leq C \int_0^1 \|\sum_{j=1}^N \gamma_j(u) f_j\|_X^p du.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$, which is denoted by $\mathcal{R}(\mathcal{T})$.

Remark 2.9. According to [5], the following properties concerning $\mathcal{R}$-boundedness is known. From Definition 2.8, $\mathcal{R}$-boundedness of the family of operators implies uniform boundedness.

$$\|T\|^p_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X = 1} \|T(x)\|_Y^p \leq \mathcal{R}(\mathcal{T}).$$

Moreover it is well-known that $\mathcal{R}$-bounds behave like norms. Namely, the following properties hold.

(i) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{L}(X, Y)$ be $\mathcal{R}$-bounded. Then $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S})$.

(ii) Let X, Y, Z be Banach spaces and $\mathcal{T} \subset \mathcal{L}(X, Y)$ and $\mathcal{S} \subset \mathcal{L}(Y, Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{S}\mathcal{T}) \leq \mathcal{R}(\mathcal{S})\mathcal{R}(\mathcal{T})$.

The following theorem is the operator valued Fourier multiplier theorem proved by Weis [29] for $X = Y = L_q(\Omega)$.

Theorem 2.10. *Let $1 < p, q < \infty$ and $M(\tau) \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X, Y))$ be satisfy*

$$\mathcal{R}(\{M(\tau) \mid \tau \in \mathbb{R}\backslash\{0\}\}) = c_0 < \infty, \qquad \mathcal{R}(\{|\tau|\partial_\tau M(\tau) \mid \tau \in \mathbb{R}\backslash\{0\}\}) = c_1 < \infty.$$

Then, T_M defined by $[T_M f](t) = \mathcal{F}_\xi^{-1}[M(\tau)\mathcal{F}_x[f](\tau)](t)(f \in \mathcal{S}(\mathbb{R}, X))$ is the bounded operator from $L_p(\mathbb{R}, X)$ to $L_p(\mathbb{R}, Y)$. Moreover, the following estimate holds :

$$\|T_M f\|_{L_p(\mathbb{R}, Y)} \leq C(c_0 + c_1)\|f\|_{L_p(\mathbb{R}, X)} \qquad (f \in L_p(\mathbb{R}, X)),$$

where C is a positive constant depending on p, X.

In order to prove the maximal L_p-L_q regularity theorem with the help of Theorem 2.10, we need the $\mathcal{R}$-boundedness for solution operator to the following generalized resolvent problem

$$\begin{cases} \lambda u_\alpha - \Delta u_\alpha + \nabla \pi_\alpha = f & \text{in } \Omega, \\ u_\alpha = 0 & \text{on } \partial\Omega \end{cases} \tag{RSa$'$}$$

under the approximated weak incompressible condition (Cg), where the resolvent parameter λ varies in $\Sigma_{\varepsilon,\lambda_0}$ $(0 < \varepsilon < \pi/2, \lambda_0 > 0)$.

We can show the existence of the $\mathcal{R}$-boundedness operator to (RSa$'$) under (Cg) as follows:

Theorem 2.11. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Set $X_q(\Omega) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\Omega)\}$, then there exist a $\lambda_0 > 0$ and operator families $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ with*

$$\mathcal{U}(\lambda) \in \mathit{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), W_q^2(\Omega)^n)), \qquad \mathcal{P}(\lambda) \in \mathit{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), \widehat{W}_q^1(\Omega)))$$

such that for any $f, g \in L_q(\Omega)$ and $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, $(u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F)$, where $F = (f, \alpha g)$, is a unique solution to (RSa$'$) *under* (Cg) *and $(\mathcal{U}(\lambda), \mathcal{P}(\lambda))$ satisfies the following estimates :*

$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^{\widetilde{N}})}(\{(\tau\partial_\tau)^\ell (G_{\lambda,\alpha}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq C \ (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^n)}(\{(\tau\partial_\tau)^\ell (\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq C \ (\ell = 0, 1)$$

for $G_{\lambda,\alpha}u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, (\lambda + \alpha)^{1/2}(\nabla \cdot u))$ and $\widetilde{N} = 1 + n + n^2 + n^3$.

By Remark 2.9, we can prove the resolvent estimate for (RSa$'$) under (Cg).

Corollary 2.12. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Let $\lambda_0 > 0$ be a number obtained in Theorem 2.11. For $f, g \in L_q(\Omega)$ and $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, there exists a unique solution (u_α, π_α) to* (RSa$'$) *under* (Cg) *which satisfies the following inequality:*

$$\|(\lambda u_\alpha, \lambda^{1/2}\nabla u_\alpha, \nabla^2 u_\alpha, (\lambda + \alpha)^{1/2}(\nabla \cdot u_\alpha), \nabla\pi_\alpha)\|_{L_q(\Omega)} \leq C\|(f, \alpha g)\|_{L_q(\Omega)}.$$

Let $\mathcal{A}_\alpha$ be the linear operator defined by $\mathcal{A}_\alpha u_\alpha = \Delta u_\alpha - \alpha\nabla Q_\Omega u_\alpha$ and $\mathcal{D}(\mathcal{A}_\alpha) = \{u \in W_q^2(\Omega)^n \mid u|_{\partial\Omega} = 0\}$. By Corollary 2.12 with $g = 0$, we see that $\mathcal{A}_\alpha$ generates the semigroup $\{T_\alpha(t)\}_{t \geq 0}$ on $L_q(\Omega)^n$. Moreover there exists a positive constant $C > 0$ such that for any $a_\alpha \in L_q(\Omega)^n$, $u_\alpha(t) = T_\alpha(t)a_\alpha$ satisfies

$$\|(u_\alpha, t^{1/2}\nabla u_\alpha, t\nabla^2 u_\alpha, t\partial_t u_\alpha)\|_{L_q(\Omega)} \leq Ce^{\lambda_0 t}\|a_\alpha\|_{L_q(\Omega)} \qquad (t \geq 0). \tag{2.4}$$

By the equations (Sa$'$), we have

$$\|\nabla\pi_\alpha\|_{L_q(\Omega)} \leq \|\partial_t u_\alpha\|_{L_q(\Omega)} + \|-\Delta u_\alpha\|_{L_q(\Omega)} \leq Ct^{-1}e^{\lambda_0 t}\|a_\alpha\|_{L_q(\Omega)}. \tag{2.5}$$

On the other hands, since $\pi_\alpha = \alpha Q_\Omega u_\alpha$ is the pressure associated with $u_\alpha = T_\alpha(t)a_\alpha$ and $\nabla\pi_\alpha = \alpha(u_\alpha - P_\Omega u_\alpha)$, (u_α, π_α) enjoys (Sa$'$) under (Cg) and $\nabla\pi_\alpha$ satisfies the following estimate:

$$\|\nabla\pi_\alpha\|_{L_q(\Omega)} = \alpha\|u_\alpha - P_\Omega u_\alpha\|_{L_q(\Omega)} \leq 2\alpha\|u_\alpha\|_{L_q(\Omega)} \leq C\alpha e^{\lambda_0 t}\|a_\alpha\|_{L_q(\Omega)},$$

which implies $\|\nabla\pi_\alpha\|_{L_\infty((0,T),L_q(\Omega))} \leq C\alpha e^{\lambda_0 T}\|a_\alpha\|_{L_q(\Omega)}$. This is the effect of the pressure stabilization method.

By real interpolation, we can see the following maximal L_p-L_q regularity theorem for (Sa$'$) with $f = g = 0$.

Theorem 2.13. *Let $\alpha > 0$ and $1 < p, q < \infty$. Let λ_0 be a number obtained in Theorem 2.11. For $a_\alpha \in B^{2(1-1/p)}_{q,p}(\Omega)$, $u_\alpha = T_\alpha(t)a_\alpha$ satisfies*

$$\begin{aligned}\|e^{-\lambda_0 t}(\partial_t u_\alpha, \nabla^2 u_\alpha)\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)},\\ (\gamma-\lambda_0)^{1/p}\|e^{-\gamma t}u_\alpha\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{L_q(\Omega)},\\ (\gamma-\lambda_0)^{1/(2p)}\|e^{-\gamma t}\nabla u_\alpha\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\end{aligned}$$

for any $\gamma > \lambda_0$. Moreover $\pi_\alpha = \alpha Q_\alpha u_\alpha$ satisfies

$$\begin{aligned}\|e^{-\lambda_0 t}\nabla \pi_\alpha\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)},\\ \|\nabla\pi_\alpha\|_{L_\infty(0,T),L_q(\Omega))} &\le C_{n,p,q}\alpha e^{\lambda_0 T}\|a_\alpha\|_{L_q(\Omega)}\end{aligned}$$

for any $T > 0$.

Next we consider the error estimate between the solution (u,π) to (NS) under the weak incompressible condition $(u,\nabla\varphi)_\Omega = 0$ for $\varphi \in \widehat{W}^1_{q'}(\Omega)$ and solution (u_α,π_α) to (NSa$'$) under (C). To this end, setting $u_E = u - u_\alpha$ and $\pi_E = \pi - \pi_\alpha$, we see that (u_E,π_E) enjoys that

$$\begin{cases}\partial_t u_E - \Delta u_E + \nabla\pi_E + N(u_E,u_\alpha) = 0, & t\in(0,\infty), x\in\Omega,\\ u_E(0,x) = a_E, & x\in\Omega,\\ u_E(t,x) = 0, & x\in\partial\Omega,\end{cases} \tag{NSE}$$

where $N(u_E,u_\alpha) = (u_E\cdot\nabla)u_E + (u_E\cdot\nabla)u_\alpha + (u_\alpha\cdot\nabla)u_E$ and $a_E = a - a_\alpha$ under the approximated weak incompressible condition

$$(u_E,\nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_E,\nabla\varphi)_\Omega + \alpha^{-1}(\nabla\pi,\nabla\varphi)_\Omega \qquad \varphi\in\widehat{W}^1_{q'}(\Omega) \tag{C$_\pi$}$$

for $1 < q < \infty$. In a similar way to Theorem 2.1, we consider (Sa$'$) under (C$_\pi$) for $a_\alpha = a_E$. By Theorem 2.6 with $f = 0$, $g = \alpha^{-1}\nabla\pi$ and Theorem 2.13, we obtain the following theorems :

Theorem 2.14. *Let $1 < p, q < \infty$ and $\alpha > 0$. Let γ_0 be a positive number obtained in Theorem 2.11. If usual Stokes equations under the weak incompressible condition has a unique solution (u,π) in $(L_{p,\gamma_E,(0)}(\mathbb{R}, W^2_q(\Omega)^n)\cap W^1_{p,\gamma_E,(0)}(\mathbb{R}, L_q(\Omega)^n))\times L_{p,\gamma_E,(0)}(\mathbb{R},\widehat{W}^1_q(\Omega))$, and (Sa$'$) under (C$_\pi$) with $a_E = 0$ has a unique solution :*

$$u_E \in L_{p,\gamma_E,(0)}(\mathbb{R}, W^2_q(\Omega)^n)\cap W^1_{p,\gamma_E}((0,\infty), L_q(\Omega)^n), \qquad \pi_E\in L_{p,\gamma_E,(0)}(\mathbb{R},\widehat{W}^1_q(\Omega)).$$

Moreover, the following estimate holds.

$$\|e^{-\gamma t}(\partial_t u_E, \alpha u_E, \Lambda_\gamma^{\frac12}\nabla u_E, \nabla^2 u_E, \Lambda^{1/2}_{\gamma+\alpha}(\nabla\cdot u_E), \nabla\pi_E)\|_{L_p(\mathbb{R},L_q(\Omega))} \le C_{n,p,q}\|e^{-\gamma t}\nabla\pi\|_{L_p(\mathbb{R},L_q(\Omega))}$$

for any $\gamma \ge \gamma_E$.

Theorem 2.15. *Let $1 < p, q < \infty$ and $\alpha > 0$. Let λ_0 be a number obtained in Theorem 2.11. For $a_E \in B^{2(1-1/p)}_{q,p}(\Omega)$, $u_E = T_\alpha(t)a_E$ and $\pi_E = \alpha Q_\Omega u_E - \pi$ satisfy*

$$\begin{aligned}\|e^{-\lambda_0 t}(\partial_t u_E, \nabla^2 u_E, \nabla\pi_E)\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)},\\ (\gamma-\lambda_0)^{1/p}\|e^{-\gamma t}u_E\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_E\|_{L_q(\Omega)},\\ (\gamma-\lambda_0)^{1/(2p)}\|e^{-\gamma t}\nabla u_E\|_{L_p((0,\infty),L_q(\Omega))} &\le C_{n,p,q}\|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\end{aligned}$$

for any $\gamma > \lambda_0$. If $\pi \in L_\infty((0,\infty),\widehat{W}^1_q(\Omega))$, π_E satisfies

$$\|e^{-\lambda_0 t}\nabla\pi_E\|_{L_\infty((0,T),L_q(\Omega))} \le C\alpha\|a_E\|_{L_q(\Omega)} + \|\nabla\pi\|_{L_\infty((0,\infty),L_q(\Omega))}$$

for any $T > 0$.

By above two theorems, we can obtain the following theorem concerned with the error estimates.

Theorem 2.16. *Let $n \geq 2$, $n/2 < q < \infty$, $\max\{1, n/q\} < p < \infty$ and $\alpha > 0$. Let T^* be a positive constant obtained in Theorem 2.1 and (u_α, π_α) be a solution obtained in Theorem 2.1. For any $M > 0$, assume that $a_E \in B^{2(1-1/p)}_{q,p}(\Omega)$ satisfies*

$$\|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq M\alpha^{-1}. \tag{2.6}$$

Then there exists $T^\flat \in (0, T^*)$ such that (NSE) *has a unique solution (u_E, π_E) which satisfies*

$$\|u_E\|_{L_\infty((0,T^\flat),L_q(\Omega))} + \|\nabla u_E\|_{L_r((0,T^\flat),L_q(\Omega))} + \|(\nabla^2 u_E, \partial_t u_E, \nabla \pi_E)\|_{L_p((0,T^\flat),L_q(\Omega))} \leq \frac{C_{n,p,q,T^\flat}}{\alpha} \tag{2.7}$$

for $1/p - 1/r \leq 1/2$.

Remark 2.17. (1) Theorem 2.14 and Theorem 2.15 with $\gamma = \lambda_0 + 1$ means that the error estimate for the Stokes equations. Namely the error estimate is given by

$$\|e^{-\gamma t}(u - u_\alpha)\|_{L_p((0,T),L_q(\Omega))} \leq \frac{C}{\alpha}\|e^{-\gamma t}\nabla\pi\|_{L_p((0,T),L_q(\Omega))} + Ce^{(\lambda_0+1)T}\|a_E\|_{L_q(\Omega)}$$

for any $T > 0$. If $T < \infty$ and $\|e^{-\gamma t}\nabla\pi\|_{L_\infty((0,T),L_q(\Omega))} < \infty$, we see

$$\begin{aligned}\|e^{-\gamma t}(u - u_\alpha)\|_{L_\infty((0,T),L_q(\Omega))} &= \lim_{p\to\infty} \|e^{-\gamma t}(u - u_\alpha)\|_{L_p((0,T),L_q(\Omega))} \\ &\leq \frac{C}{\alpha}\|e^{-\gamma t}\nabla\pi\|_{L_\infty((0,T),L_q(\Omega))} + Ce^{(\lambda_0+1)T}\|a_E\|_{L_q(\Omega)}.\end{aligned}$$

Under assumption (2.6) in Theorem 2.16, we see that there exists a positive constant C depending on T,M and $\|\nabla\pi\|_{L_\infty((0,T),L_q(\Omega))}$ such that

$$\|u - u_\alpha\|_{L_\infty((0,T),L_q(\Omega))} \leq C\alpha^{-1}, \qquad \|\nabla(\pi - \pi_\alpha)\|_{L_\infty((0,T),L_q(\Omega))} \leq C$$

for any $T > 0$.

(2) (2.7) means the following error estimates for the Navier-Stokes equations:

$$\begin{aligned}&\|u - u_\alpha\|_{L_\infty((0,T^\flat),L_q(\Omega))} \leq C\alpha^{-1}, \\ &\|(\nabla^2(u - u_\alpha), \partial_t(u - u_\alpha), \nabla(\pi - \pi_\alpha))\|_{L_p((0,T^\flat),L_q(\Omega))} \leq C\alpha^{-1},\end{aligned}$$

In a similar way to (1), we obtain

$$\|(\nabla^2(u - u_\alpha), \partial_t(u - u_\alpha), \nabla(\pi - \pi_\alpha))\|_{L_\infty((0,T^\flat),L_q(\Omega))} \leq C\alpha^{-1}.$$

In comparison with the result due to Prohl [19], we can extend L_2 framework to L_q framework with respect to the error estimate.

3 Preliminary

In this section, we shall introduce some lemmas and definitions, which plays important role for our proof. Before we describe some propositions and lemmas, we introduce the notation of symbols. Set

$$\begin{aligned}&r = |\xi'|, \qquad \omega_\lambda = \sqrt{\lambda + r^2}, \qquad \omega = \sqrt{\lambda + \alpha + r^2}, \\ &\mathcal{E}(z) = e^{-z(x_n+y_n)}, \qquad \mathcal{M}(a, b, x_n) = \frac{e^{-ax_n} - e^{-bx_n}}{a - b},\end{aligned} \tag{3.1}$$

where $\xi' = (\xi_1, \dots, \xi_{n-1})$. Here $\mathcal{E}(\omega_\lambda)$ is the symbol corresponding to heat equation and $\mathcal{M}(\omega_\lambda, r, x_n)$ is the symbol corresponding to Stokes equations.

We next introduce some lemmas. In order to apply the operator-valued Fourier multiplier theorem proved by Weis [29], we need the $\mathcal{R}$-boundedness of solution operator to (Sa$'$). However since it is difficult to prove $\mathcal{R}$-boundedness directly from its definition, we first introduce the following sufficient condition for showing $\mathcal{R}$-boundedness of solution operator given in Theorem 3.3 in Enomoto and Shibata [7].

Theorem 3.1. *Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Let $m(\lambda, \xi)$ be a function defined on $\Sigma_\varepsilon \times (\mathbb{R}^n \backslash \{0\})$ such that for any multi-index $\beta \in \mathbb{N}_0^n (\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ there exists a constant C_β depending on β and λ such that*

$$|\partial_\xi^\beta m(\lambda, \xi)| \leq C_\beta |\xi|^{-|\beta|}$$

for any $(\lambda, \xi) \in \Sigma_\varepsilon \times (\mathbb{R}^n \backslash \{0\})$. Let K_λ be an operator defined by $[K_\lambda f](x) = \mathcal{F}_\xi^{-1}[m(\lambda, \xi)\mathcal{F}_x[f](\xi)](x)$. Then the set $\{K_\lambda \mid \lambda \in \Sigma_\varepsilon\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^n))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^n))}(\{K_\lambda \mid \lambda \in \Sigma_\varepsilon\}) \leq C \max_{|\beta| \leq n+2} C_\beta$$

with some constant C that depends solely on q and n.

To prove the $\mathcal{R}$-boundedness of the solution operator in $\mathbb{R}^n_+$, we use the following lemma proved by Shibata and Shimizu [26] (see Lemma 5.4 in [26]).

Lemma 3.2. *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Let $m(\lambda, \xi')$ be a function defined on Σ_ε such that for any multi-index $\delta' \in \mathbb{N}_0^{n-1}$ there exists a constant $C_{\delta'}$ depending on δ', ε and N such that*

$$|\partial_{\xi'}^{\delta'} m(\lambda, \xi')| \leq C_{\delta'} r^{-|\delta'|}.$$

Let $K_j(\lambda, m)$ $(j = 1, \dots, 5)$ be the operators defined by

$$\begin{aligned}
[K_1(\lambda, m)g](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m(\lambda, \xi') r \mathcal{E}(\omega_\lambda) \widetilde{g}(\xi', y_n)\right](x') dy_n, \\
[K_2(\lambda, m)g](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m(\lambda, \xi') r^2 \mathcal{M}(\omega_\lambda, r, x_n + y_n) \widetilde{g}(\xi', y_n)\right](x') dy_n, \\
[K_3(\lambda, m)g](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m(\lambda, \xi') |\lambda|^{1/2} r \mathcal{M}(\omega_\lambda, r, x_n + y_n) \widetilde{g}(\xi', y_n)\right](x') dy_n, \\
[K_4(\lambda, m)g](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m(\lambda, \xi') \omega r \mathcal{M}(\omega_\lambda, \omega, x_n + y_n) \widetilde{g}(\xi', y_n)\right](x') dy_n, \\
[K_5(\lambda, m)g](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m(\lambda, \xi') |\lambda|^{1/2} r \mathcal{M}(\omega_\lambda, \omega, x_n + y_n) \widetilde{g}(\xi', y_n)\right](x') dy_n.
\end{aligned}$$

Then, the sets $\{(\tau\partial_\tau)^\ell K_j(\lambda, m) \mid \lambda \in \Sigma_\varepsilon\}$ $(j = 1, \dots, 5, \ell = 0, 1)$ are $\mathcal{R}$-bounded families in $\mathcal{L}(L_q(\mathbb{R}^n_+))$. Moreover, there exists a constant $C_{n,q,\varepsilon}$ such that

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^n_+))}(\{(\tau\partial_\tau)^\ell K_j(\lambda, m) \mid \lambda \in \Sigma_\varepsilon\}) \leq C_{n,q,\varepsilon} \qquad (j = 1, \dots, 5, \ell = 0, 1).$$

This lemma is proved in a similar way to Lemma 5.4 in [26] with the following lemma.

Lemma 3.3. *For $0 < \varepsilon < \pi/2$, let $\lambda \in \Sigma_\varepsilon$.*

(i) *There exist positive constants C_1, C_2 and C_3 depending on ε such that the following inequalities hold:*

$$|\omega_\lambda| \geq C_1(|\lambda|^{1/2} + r), \qquad C_2(\alpha^{1/2} + |\lambda|^{1/2} + r) \leq Re\ \omega \leq C_3(\alpha^{1/2} + |\lambda|^{1/2} + r). \tag{3.2}$$

(ii) *There exist positive constants C such that the following inequalities hold:*

$$\begin{aligned}
&|D_{\xi'}^{\delta'} r^s| \leq C r^{s-|\delta'|},\\
&|D_{\xi'}^{\delta'} \omega_\lambda^s| \leq C(|\lambda|^{1/2} + r)^{s-|\delta'|},\\
&|D_{\xi'}^{\delta'} \omega^s| \leq C(|\lambda|^{1/2} + \alpha^{1/2} + r)^{s-|\delta'|},\\
&|D_{\xi'}^{\delta'} (r+\omega_\lambda)^s| \leq C(|\lambda|^{1/2} + r)^{s} r^{-|\delta'|},\\
&|D_{\xi'}^{\delta'} (r+\omega)^s| \leq C(|\lambda|^{1/2} + \alpha^{1/2} + r)^{s} r^{-|\delta'|},\\
&|D_{\xi'}^{\delta'} (\omega+\omega_\lambda)^s| \leq C(|\lambda|^{1/2} + \alpha^{1/2} + r)^{s} (|\lambda|^{1/2} + r)^{-|\delta'|}
\end{aligned} \tag{3.3}$$

for any $s \in \mathbb{R}$ and multi-index δ'.

(iii) *There exist positive constants C such that the following inequalities hold:*

$$\begin{aligned}
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell e^{-rx_n}\}| &\leq C r^{-|\delta'|} e^{-(1/2)rx_n},\\
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell e^{-\omega_\lambda x_n}\}| &\leq C(|\lambda|^{1/2} + r)^{-|\delta'|} e^{-d(|\lambda|^{1/2}+r)x_n},\\
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell e^{-\omega x_n}\}| &\leq C(\alpha^{1/2} + |\lambda|^{1/2} + r)^{-|\delta'|} e^{-d(\alpha^{1/2}+|\lambda|^{1/2}+r)x_n},\\
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell \mathcal{M}(\omega_\lambda, r, x_n)\}| &\leq C(x_n \ or\ |\lambda|^{-1/2}) e^{-drx_n} r^{-|\delta'|},\\
|D_{\xi'}^{\delta'}\{(\tau\partial_\tau)^\ell \mathcal{M}(\omega_\lambda, \omega, x_n)\}| &\leq C(x_n \ or\ \alpha^{-1/2}) e^{-d(|\lambda|^{1/2}+r)x_n} (|\lambda|^{1/2} + r)^{-|\delta'|}
\end{aligned} \tag{3.4}$$

for $\ell = 0, 1$ and any multi-index δ' and $(\xi', x_n) \in (\mathbb{R}^{n-1}\backslash\{0\}) \times (0, \infty)$, where d is a positive constant independent of ε and δ'.

Proof.

(i) (3.2) are proved by elementary calculation.

(ii) Let $f(t) = t^{s/2}$. By Bell formula, we see

$$D_\xi^\delta r^s = \sum_{\ell=1}^{|\delta|} f^{(\ell)}(r^2) \sum_{\delta_1+\cdots+\delta_\ell=\delta, |\delta_i|\geq 1} \Gamma^\ell_{\delta_1,\ldots,\delta_\ell} (D_\xi^{\delta_1} r^2) \cdots (D_\xi^{\delta_\ell} r^2),$$

where $\Gamma^\ell_{\alpha_1,\ldots,\alpha_\ell}$ is some constant and $f^{(\ell)}(t) = d^\ell f(t)/dt^\ell$. Since $|D_\xi^{\delta_j} r^2| \leq 2r^{2-|\delta_j|}$, we can obtain the first estimate. We can prove the other estimates in a similar way to the first estimate taking the elementary estimate: $|\lambda + |\xi|^2| \geq (\sin\varepsilon)(|\lambda| + |\xi|^2)$ $(0 < \varepsilon < \pi/2,\ \xi \in \mathbb{R}^n)$ into account.

(iii) It is sufficient to prove the last estimate with $\ell = 0$ in (3.4), since we can prove the other estimates similarly.
By $\mathcal{M}(\omega_\lambda, \omega, x_n) = -x_n \int_0^1 e^{-((1-\theta)\omega_\lambda + \theta\omega)x_n} d\theta$ and Bell formula, we have

$$\begin{aligned}
&|D_{\xi'}^{\delta'} e^{-((1-\theta)\omega_\lambda+\theta\omega)x_n}|\\
&\quad \leq C_{\delta'} \sum_{\ell=1}^{|\delta'|} x_n^\ell e^{-(c_1(1-\theta)(|\lambda|^{1/2}+r)+c_2\theta(\alpha^{1/2}+|\lambda|^{1/2}+r))x_n}\\
&\qquad \times ((1-\theta)(|\lambda|^{1/2}+r)^{1-|\delta_1'|} + \theta(\alpha^{1/2}+|\lambda|^{1/2}+r)^{1-|\delta_1'|})\\
&\qquad \times \cdots \times ((1-\theta)(|\lambda|^{1/2}+r)^{1-|\delta_\ell'|} + \theta(\alpha^{1/2}+|\lambda|^{1/2}+r)^{1-|\delta_\ell'|}),
\end{aligned}$$

where we used $|e^{-((1-\theta)\omega_\lambda+\theta\omega)x_n}| = e^{-((1-\theta)\mathrm{Re}\omega_\lambda+\theta\mathrm{Re}\omega)x_n}$. Setting $c = \min(c_1, c_2)$, we see

$$|D_{\xi'}^{\delta'} e^{-((1-\theta)\omega_\lambda+\theta\omega)x_n}| \leq C_{\delta'} e^{-(c/2)((1-\theta)(|\lambda|^{1/2}+r)+\theta(\alpha^{1/2}+|\lambda|^{1/2}+r))x_n}(|\lambda|^{1/2}+r)^{-|\delta'|},$$

which implies

$$\begin{aligned}|D_{\xi'}^{\delta'}\mathcal{M}(\omega_\lambda,\omega,x_n)| &\leq C_{\delta'}\int_0^1 e^{-(c/2)((1-\theta)(|\lambda|^{1/2}+r)+\theta(\alpha^{1/2}+|\lambda|^{1/2}+r))x_n}d\theta x_n(|\lambda|^{1/2}+r)^{-|\delta'|}\\ &= C_{\delta'}\int_0^1 e^{-(c/2)(|\lambda|^{1/2}+r)x_n}e^{-\theta(c/2)\alpha^{1/2}x_n}d\theta x_n(|\lambda|^{1/2}+r)^{-|\delta'|}.\end{aligned}$$

By integrating this right hand side, we have

$$|D_{\xi'}^{\delta'}\mathcal{M}(\omega_\lambda,\omega,x_n)| \leq C_{\delta'}(c/2)^{-1}\alpha^{-1/2}e^{-(c/2)(|\lambda|^{1/2}+r)x_n}(|\lambda|^{1/2}+r)^{-|\delta'|}. \tag{3.5}$$

On the other hands, by $e^{-\theta(c/2)\alpha^{1/2}x_n} \leq 1$, we have

$$|D_{\xi'}^{\delta'}\mathcal{M}(\omega_\lambda,\omega,x_n)| \leq C_{\delta'}x_n e^{-(c/2)(|\lambda|^{1/2}+r)x_n}(|\lambda|^{1/2}+r)^{-|\delta'|}. \tag{3.6}$$

Therefore, we obtain the last estimate with $\ell = 0$ in (3.4). □

By using maximal regularity theorem (Theorem 2.6), we shall prove the existence and uniqueness theorem of strong solution for (NSa) in Section 5. To do this, we prepare some facts shown by this theorem.

Let $(w,\tau) = M_T(f)$ be the solution to

$$\begin{cases} \partial_t w - \Delta w + \nabla\tau = f & x \in \Omega, t \in (0,T),\\ w(0,x) = 0 & x \in \Omega,\\ w(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{3.7}$$

under the approximated weak incompressible condition (C)

For $f \in L_p((0,T), L_q(\Omega))$, let $f_0(t) = f(t)$ $(0 < t < T)$ and $f_0(t) = 0$ $(t \notin (0,T))$. Then, letting (w,τ) be the solution to Stokes equation for $f = f_0$ on $t \in (0,\infty)$, (w,τ) can define on $t \in \mathbb{R}$. Moreover, this solution satisfies $w(t) = \tau(t) = 0$ $(t \leq 0)$ and (3.7) on $t \in (0,T)$. Furthermore, by Theorem 2.6, the following estimate holds: for $0 < S \leq T$,

$$\|\partial_t w\|_{L_p((0,S),L_q(\Omega))} \leq e^{\gamma S}\|e^{-\gamma t}\partial_t w\|_{L_p((0,T),L_q(\Omega))} \leq C_{n,p,q}e^{\gamma S}\|f\|_{L_p((0,T),L_q(\Omega))}. \tag{3.8}$$

Similarly we have

$$\|\nabla^2 w\|_{L_p((0,S),L_q(\Omega))} + \|\nabla\tau\|_{L_p((0,S),L_q(\Omega))} \leq C_{n,p,q}e^{\gamma S}\|f\|_{L_p((0,T),L_q(\Omega))}. \tag{3.9}$$

Moreover taking into account the fact about Bessel potential space:

$$\|e^{-\gamma t}u\|_{L_q(\mathbb{R},X)} \leq C\|e^{-\gamma t}\Lambda_\gamma^\alpha u\|_{L_p(\mathbb{R},X)} \leq C\gamma^{-(\beta-\alpha)}\|e^{-\gamma t}\Lambda_\gamma^\beta u\|_{L_p(\mathbb{R},X)} \tag{3.10}$$

for Banach space X, $1 < p < q < \infty$, $\alpha = 1/p - 1/q$, $\alpha < \beta < \infty$ and $\gamma \geq 0$ and the estimate:

$$\|e^{-\gamma t}u\|_{L_\infty(\mathbb{R},X)} \leq C\|e^{-\gamma t}\Lambda_\gamma^\alpha u\|_{L_p(\mathbb{R},X)}$$

for $0 < \alpha - 1/p < 1$ and $1 < p < \infty$ (see [3]), by Theorem 2.6 we obtain

$$\begin{aligned}&\|\nabla w\|_{L_r((0,S),L_q(\Omega))} + \|w\|_{L_\infty((0,S),L_q(\Omega))}\\ &\quad\leq Ce^{\gamma S}\|e^{-\gamma t}\Lambda_1^\alpha\nabla w\|_{L_q(\mathbb{R},L_q(\Omega))} + Ce^{\gamma S}\|e^{-\gamma t}\Lambda_1^1 w\|_{L_p(\mathbb{R},L_q(\Omega))}\\ &\quad\leq Ce^{\gamma S}\|e^{-\gamma t}\Lambda_1^{1/2}\nabla w\|_{L_p(\mathbb{R},L_q(\Omega))} + Ce^{\gamma S}\|e^{-\gamma t}\Lambda_1^1 w\|_{L_p(\mathbb{R},L_q(\Omega))}\\ &\quad\leq Ce^{\gamma S}\|f\|_{L_p((0,T),L_q(\Omega))},\end{aligned} \tag{3.11}$$

where $1/p - 1/r \le 1/2$.

Letting $\beta = n/(2q)$ and $\ell_k (k = 1, 2, 3)$ are the positive constants satisfying

$$0 < \frac{1}{p} - \frac{1}{\beta p \ell_1} \le \frac{1}{2}, \qquad 0 < \frac{1}{p} - \frac{1}{(1-\beta)p\ell_2} \le \frac{1}{2}, \qquad \beta + \frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3} = 1$$

and setting

$$\gamma = 1/(\ell_3 p), \qquad r_1 = \beta p \ell_1, \qquad r_2 = (1-\beta)p\ell_2, \tag{3.12}$$

by Sobolev embedding theorem and Holder's inequality, we obtain

$$\begin{aligned} \|(v\cdot\nabla)w\|_{L_p((0,S),L_q(\Omega))} \le S^\gamma \|v\|^{1-\beta}_{L_\infty((0,S),L_q(\Omega))} \|\nabla v\|^{\beta}_{L_{r_1}((0,S),L_q(\Omega))} \\ \times \|\nabla w\|^{1-\beta}_{L_{r_2}((0,S),L_q(\Omega))} \|\nabla^2 w\|^{\beta}_{L_p((0,S),L_q(\Omega))} \end{aligned} \tag{3.13}$$

for any $v, w \in W^1_p((0,T), L_q(\Omega)) \cap L_p((0,T), W^2_q(\Omega))$ and $0 < S \le T$.

Moreover, by maximal regularity theorem, we can see the existence theorem (and uniqueness theorem in case of $n = 2$) for weak solution. By using Helmholtz projection, the approximate Stokes operator $A_\alpha = A_{\alpha,\Omega,q}$ in $L_q(\Omega)$ $(1 < q < \infty)$ is defined by

$$A_\alpha u = \Delta u - \alpha\nabla Q u, \quad u \in D(A_\alpha) = D(W^2_q(\Omega)). \tag{3.14}$$

Moreover, we shall introduce Hille-Yosida operator with A_α and its properties, which play an essential role in our proof for weak solution. To do this, we confirm the property of A_α. For the equation

$$\partial_t u_\alpha - A_\alpha u_\alpha + (Pu_\alpha\cdot\nabla)u_\alpha = f, \quad u_\alpha(0) = a_\alpha,$$

by maximal regularity theorem for A_α : for some $\lambda_0 > 0$

$$\|e^{-\lambda_0 t}(\partial_t u_\alpha, \nabla^2 u_\alpha)\|_{L_p((0,\infty),L_q(\Omega))} \le C(\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|e^{-\lambda_0 t} f\|_{L_p((0,\infty),L_q(\Omega))}),$$

letting $f(t,x) = 0$ $(t > T)$, we have

$$\|e^{-\lambda_0 t}(\partial_t u_\alpha, \nabla^2 u_\alpha)\|_{L_p((0,T),L_q(\Omega))} \le C(\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|e^{-\lambda_0 t} f\|_{L_p((0,T),L_q(\Omega))}).$$

Furthermore, we proved the inequality for semigroup $\{T_\alpha(t)\}_{t\ge0}$: for $t > 0$ and $u \in L_q(\Omega)$

$$\|T_\alpha(t)u\|_{q,\Omega} + t^{1/2}\|\nabla T_\alpha(t)u\|_{q,\Omega} + t\|\nabla^2 T_\alpha(t)u\|_{q,\Omega} \le M_{n,q}\|u\|_{q,\Omega},$$

which implies L_p-L_q estimate for approximate Stokes semigroup by Sobolev imbedding theorem

$$\|T_\alpha(t)u\|_{p,\Omega} \le C_{n,p,q} t^{-n/2(1/q-1/p)}\|u\|_{q,\Omega}, \quad \|\nabla T_\alpha(t)u\|_{p,\Omega} \le C_{n,p,q} t^{-1/2-n/2(1/q-1/p)}\|u\|_{q,\Omega}, \tag{3.15}$$

where $t > 0$, $1 < q \le p \le \infty$ and $u \in L_q(\Omega)$. Then, letting $\{T_\alpha(t)\}_{t\ge0}$ be approximate Stokes semigroup in Ω and J_k defined as Hille-Yosida operator :

$$J_k = \left(1 - \frac{A_\alpha}{k}\right)^{-1} \quad (k \in \mathbb{N}),$$

we see, by (3.15),

$$\begin{aligned} \|J_k u_\alpha\|_p \le C\int_0^\infty e^{-t}\left(\frac{t}{k}\right)^{-n/2(1/q-1/p)} dt\|u_\alpha\|_q \le C_1(k)\|u_\alpha\|_q, \\ \|\nabla J_k u_\alpha\|_p \le C\int_0^\infty e^{-t}\left(\frac{t}{k}\right)^{-1/2-n/2(1/q-1/p)} dt\|u_\alpha\|_q \le C_2(k)\|u_\alpha\|_q, \end{aligned} \tag{3.16}$$

where $n/q - n/p < 2$. Note that if $p = q$, the constant $C_1(k)$ is independent of k. And, if $N \in \mathbb{N}$ satisfies $N > 1 + n/4$, we also have

$$\|J_k u_\alpha\|_\infty \le C(k)\|J_k^{N-1} u_\alpha\|_{q_1} \le \cdots \le C(k)\|J_k u_\alpha\|_{q_{N-1}} \le C(k)\|u_\alpha\|_2, \tag{3.17}$$

where $2 \le q_{N-1} \le q_{N-2} \le \cdots \le q_2 \le q_1 \le \infty$ and $n/q_{i+1} - n/q_i < 2$.

Using these instrument, we shall prove weak solution meaning of Definition 2.2 in Section 6. In order to prove the existence theorem of solution satisfying this definition, we prepare a technical lemma. From this lemma, we can estimate the non-linear term.

Lemma 3.4. *Let $\alpha > 0$ and $u \in D(A_\alpha) = H^2(\Omega) \cap H_0^1(\Omega)$.*

(i) *The following relation holds:*

$$\|\nabla u\|_2 \le \|A_\alpha^{1/2} u\|_2. \tag{3.18}$$

(ii) *The following relation holds:*

$$\|\nabla^2 u\|_2 \le C\|A_\alpha u\|_2, \tag{3.19}$$

where C is a positive constant independent of u.

Proof. (i) Since we have

$$(A_\alpha u, u)_\Omega = (\nabla u, \nabla u)_\Omega + \alpha(\nabla Q u, u)_\Omega = \|\nabla u\|_2^2 + \alpha\|\nabla Q u\|_2^2 \ge 0$$

by the properties of the Helmholtz projection : $u = Pu + \nabla Q u$ and $(Pu, \nabla Q u)_\Omega = 0$, we see that A_α is a positive definite self-adjoint operator and that A_α has the square root $A_\alpha^{1/2}$ which satisfies

$$\|A_\alpha^{1/2} u\|_2^2 = \|\nabla u\|_2^2 + \alpha\|\nabla Q u\|_2^2,$$

which implies (3.18).

(ii) In order to prove (3.19), we shall consider the following equations:

$$-\Delta u + \nabla \pi = f, \qquad \text{in } \Omega \tag{3.20}$$

under the weak divergence free condition:

$$(u, \nabla\psi)_\Omega - \alpha^{-1}(\nabla\pi, \nabla\psi)_\Omega = (g, \psi)_\Omega, \tag{3.21}$$

subject to $u|_{\partial\Omega} = 0$ for $f, g \in L_2(\Omega)$. Goal is to show that for $f, g \in L_2(\Omega)$, the solution (u, π) to (3.20) under (3.21) satisfies

$$\|\nabla^2 u\|_2 + \|\nabla\pi\|_2 \le C\left(\|f\|_2 + \alpha\|g\|_2\right). \tag{3.22}$$

If we obtain (3.22), since (3.20) under (3.21) with $g = 0$ is equivalent to $A_\alpha u = f$, we can obtain (3.19).

Taking the fact that there exists $F, G \in L_2(\Omega)$ with $f = \nabla \cdot F, g = \nabla \cdot G$ into account, the weak form of (3.20) under (3.21) is given by

$$\begin{aligned}
(\nabla u, \nabla\varphi)_\Omega + (\nabla\pi, \varphi)_\Omega &= (f, \varphi)_\Omega = (-F, \nabla\varphi)_\Omega, \\
\frac{1}{\alpha}(\nabla\pi, \nabla\psi)_\Omega - (u, \nabla\psi)_\Omega &= (G, \nabla\psi)_\Omega.
\end{aligned}$$

By Helmholtz decomposition, setting $\nabla Q\varphi = \nabla\psi$, we have

$$(\nabla u, \nabla\varphi)_\Omega + \alpha(\nabla Qu, \nabla\varphi)_\Omega = (\nabla Qf, \nabla Q\varphi)_\Omega + \alpha(G, \varphi)_\Omega. \tag{3.23}$$

We consider the only (3.23). For this purpose, set the bilinear form $\mathcal{A}(u, \psi)$ as follows:

$$\mathcal{A}(u, \psi) = (\nabla u, \nabla\psi)_\Omega + \alpha(\nabla Qu, \psi)_\Omega \tag{3.24}$$

for $u, \psi \in H_0^1(\Omega)$. By Schwartz inequality, (HP) and Poincaré, we see

$$|\mathcal{A}(u, \psi)| \le \|\nabla u\|_2 \|\nabla\psi\|_2 + \alpha\|\nabla Qu\|_2 \|\psi\|_2 \le (1 + \alpha)\|u\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}, \tag{3.25}$$

$$\mathcal{A}(u, u) = \|\nabla u\|_2^2 + \alpha\|\nabla Qu\|_2^2 \ge \|\nabla u\|_2^2 \ge C\|u\|_{H^1(\Omega)}^2 \tag{3.26}$$

for $u, \phi \in H_0^1(\Omega)$. By Lax-Milgram theorem, for any $h \in H^{-1}(\Omega)$, there exists a $u \in H_0^1(\Omega)$ uniquely which solves $\mathcal{A}(u, \psi) = (h, \psi)_\Omega$ and $\|u\|_{H^1(\Omega)} \le C\|h\|_{H^{-1}(\Omega)}$. Therefore (3.23) has the solution u in the distribution sense, and u satisfies

$$\|\nabla u\|_2 \le C\left(\|f\|_{H^{-1}} + \alpha\|G\|_{H^{-1}}\right) \le C\left(\|F\|_2 + \alpha\|G\|_2\right).$$

We consider the pressure term π. To this end, let $u \in H_0^1(\Omega)$ be a solution to (3.23) and consider the functional $G : \varphi \mapsto [G, \varphi]$ defined by $[G, \varphi] = (\nabla u, \nabla\varphi)_\Omega + (F, \nabla\varphi)_\Omega$ for $\varphi \in C_0^\infty(\Omega)$. Then we see that $G \in H^{-1}(\Omega)$, which implies that there exists $\pi \in L_2(\Omega)$ with $G = \nabla\pi$ in distribution sense. $\|\pi\|_{L_2}$ is estimated as follows:

$$\|\pi\|_2 \le \|\nabla\pi\|_{H^{-1}} = \|\Delta u + f\|_{H^{-1}} \le C\left(\|\nabla u\|_2 + \|F\|_2\right) \le C\left(\|F\|_2 + \alpha\|G\|_2\right).$$

Therefore we have

$$\|\nabla u\|_2 + \|\pi\|_2 \le C\left(\|F\|_2 + \alpha\|G\|_2\right) \tag{3.27}$$

From now, in a similar way to Kubo and Matsui [13], we shall show that for $f, g \in L_2(\Omega)$, the solution $(u, \pi) \in H^2(\Omega) \times H^1(\Omega)$ to (3.20) under (3.21) satisfying

$$\|\nabla^2 u\|_2 + \|\nabla\pi\|_2 \le C\left(\|f\|_2 + \alpha\|g\|_2\right). \tag{3.28}$$

For this purpose, we need three steps where we treat special cases. In the first step, we consider the case for the whole-space and a half-space. In the second step, we consider the case for a bent half-space. In this case, we reduce to the case for the half-space by a transformation of coordinates. In the third step, we consider the cases for a uniformly $W_r^{2-1/r}$-domain ($n < r < \infty$). In this case, by using localization method we reduce to the case for the whole space, the half-space and bent half-space.

In the first step, we consider the case for $\Omega = \mathbb{R}^n$ and $\mathbb{R}^n_+$. Namely, we consider

$$-\Delta u + \nabla\pi = f$$

under $(u, \nabla\psi)_\Omega - \alpha^{-1}(\nabla\pi, \nabla\psi)_\Omega = -(G, \nabla\psi)_\Omega$, subject to $u|_{\partial\Omega} = 0$ if $\Omega = \mathbb{R}^n_+$. In a similar way to the method due to Kubo and Matsui [13], we see that

$$\|\nabla^2 u\|_{2,\Omega} + \|\nabla\pi\|_{2,\Omega} \le C\left(\|f\|_{2,\Omega} + \alpha\|G\|_{2,\Omega}\right). \tag{3.29}$$

In second step, we consider the case for a bent half-space. For this purpose we shall introduce some notations. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. Denoting $\nabla\Phi = \mathcal{A} + \mathcal{B}(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_{-1} + \mathcal{B}_{-1}(x)$, we assume that $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthogonal matrices with constant coefficients and $\mathcal{B}(x)$ and $\mathcal{B}_{-1}(x)$ are matrices of functions in $W^1_r(\mathbb{R}^n)$ with $n < r < \infty$ such that

$$\|\mathcal{B}\|_{\infty,\mathbb{R}^n} + \|\mathcal{B}_{-1}\|_{\infty,\mathbb{R}^n} \le M_1, \qquad \|\nabla\mathcal{B}\|_{r,\mathbb{R}^n} + \|\nabla\mathcal{B}_{-1}\|_{r,\mathbb{R}^n} \le M_2.$$

We shall choose M_1 small enough later, so that we may assume that $0 < M_1 \le 1 \le M_2$. Let $\mathbb{R}^n_0$ be the boundary of the half-space defined by $\mathbb{R}^n_0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$. Set $\Omega_+ = \Phi(\mathbb{R}^n_+)$ and $\partial\Omega_+ = \Phi(\mathbb{R}^n_0)$. In order to prove the case for the bent half-space, we transfer (3.20) under (3.21) into a problem for the half-space by the change of variable $x = \Phi^{-1}(y)$ with $y \in \Omega_+$ and $x \in \mathbb{R}^n_+$ and by the change of unknowns $\widetilde{u_\alpha}(x) = \mathcal{A}_{-1}(u_\alpha(\Phi(x))), \widetilde{\pi_\alpha}(x) = \mathcal{A}_{-1}(\pi_\alpha(\Phi(x)))$ and $\widetilde{\psi}(x) = \mathcal{A}_{-1}(\psi(\Phi(x)))$. Since $\partial_{y_j} = \sum_{\ell=1}^n(\mathcal{A}_{\ell,j} + \mathcal{B}_{\ell,j}\partial_{x_j})$, we have the following equations:

$$\begin{cases} -\Delta\widetilde{u_\alpha} + \nabla\widetilde{\pi_\alpha} = f_+ + \mathcal{F}(\widetilde{u_\alpha}, \widetilde{\pi_\alpha}) & x \in \mathbb{R}^n_+, \\ \widetilde{u_\alpha} = 0 & x \in \partial\mathbb{R}^n_+ \end{cases}$$

under

$$(\widetilde{u_\alpha}, \nabla\widetilde{\psi})_{\mathbb{R}^n_+} = \alpha^{-1}(\nabla\widetilde{\pi_\alpha}, \nabla\widetilde{\psi})_{\mathbb{R}^n_+} + (G_+, \nabla\widetilde{\psi})_{\mathbb{R}^n_+} + (\mathcal{G}(\widetilde{u_\alpha}, \widetilde{\pi_\alpha}), \widetilde{\psi})_{\mathbb{R}^n_+}$$

for $\widetilde{\psi} \in \widehat{W}^1_{q'}(\mathbb{R}^n_+)$, where $f_+(x) = \mathcal{A}_{-1}(f(\Phi(x)))$ and $G_+(x) = \mathcal{A}_{-1}(G(\Phi(x))) + \mathcal{M}_4 G$. Moreover $\mathcal{F}(\widetilde{u_\alpha}, \widetilde{\pi_\alpha})$ and $\mathcal{G}(\widetilde{u_\alpha}, \widetilde{\pi_\alpha})$ have the following forms:

$$\begin{aligned} &\mathcal{F}(\widetilde{u_\alpha}, \widetilde{\pi_\alpha}) = \mathcal{M}_1\nabla^2\widetilde{u_\alpha} + \mathcal{M}_2\nabla\widetilde{u_\alpha} + \mathcal{M}_3\nabla\widetilde{\pi_\alpha}, \\ &\mathcal{G}(\widetilde{u_\alpha}, \widetilde{\pi_\alpha}) = \alpha^{-1}(\mathcal{M}_4\widetilde{u_\alpha} + \mathcal{M}_5\nabla\widetilde{\pi_\alpha}) \end{aligned}$$

with some matrices of functions $\mathcal{M}_k$ $(k = 1, \dots, 5)$ possessing the estimates

$$\|\mathcal{M}_j\|_{\infty,\mathbb{R}^n_+} \le CM_1, \qquad \|\mathcal{M}_2\|_{r,\mathbb{R}^n_+} + \|\nabla\mathcal{M}_j\|_{r,\mathbb{R}^n_+} \le CM_2$$

for $j = 1, 3, 4, 5$ and $n < r < \infty$. By the results of the case for the half-space, we obtain

$$\begin{aligned} &\|\nabla^2\widetilde{u_\alpha}\|_{2,\mathbb{R}^n_+} + \|\nabla\widetilde{\pi_\alpha}\|_{2,\mathbb{R}^n_+} \\ &\le C\left(\|f_+\|_{2,\mathbb{R}^n_+} + \alpha\|G_+\|_{2,\mathbb{R}^n_+} + \|\mathcal{F}(\widetilde{u_\alpha}, \widetilde{\pi_\alpha})\|_{2,\mathbb{R}^n_+} + \alpha\|\mathcal{G}(\widetilde{u_\alpha}, \widetilde{\pi_\alpha})\|_{2,\mathbb{R}^n_+}\right) \\ &\le C\Big(\|f_+\|_{2,\mathbb{R}^n_+} + \alpha\|G_+\|_{2,\mathbb{R}^n_+} + M_1(\|\nabla^2\widetilde{u_\alpha}\|_{2,\mathbb{R}^n_+} + \|\nabla\widetilde{\pi_\alpha}\|_{2,\mathbb{R}^n_+} + \|\widetilde{u_\alpha}\|_{2,\mathbb{R}^n_+}) \\ &\qquad\qquad + \|\mathcal{M}_2\|_{\infty,\mathbb{R}^n_+}\|\nabla\widetilde{u_\alpha}\|_{2,\mathbb{R}^n_+}\Big). \end{aligned}$$

Taking M_1 sufficient small, we see that

$$\|\nabla^2\widetilde{u_\alpha}\|_{2,\mathbb{R}^n_+} + \|\nabla\widetilde{\pi_\alpha}\|_{2,\mathbb{R}^n_+} \le C\left(\|f_+\|_{2,\mathbb{R}^n_+} + \alpha\|g_+\|_{2,\mathbb{R}^n_+} + \|\widetilde{u_\alpha}\|_{2,\mathbb{R}^n_+} + \|\nabla\widetilde{u_\alpha}\|_{2,\mathbb{R}^n_+}\right),$$

which implies

$$\|\nabla^2 u_\alpha\|_{2,\Omega} + \|\nabla\pi_\alpha\|_{2,\Omega} \le C\left(\|f\|_{2,\Omega} + \alpha\|g\|_{2,\Omega} + \|u_\alpha\|_{2,\Omega} + \|\nabla u_\alpha\|_{2,\Omega}\right)$$

for the case where Ω is the bent half-space.

In the third step, we set $\mathcal{H}^1_j = \Phi^1_j(\mathbb{R}^n_+)$, $\partial\mathcal{H}^1_j = \Phi^1_j(\partial\mathbb{R}^n_+)$, and $\mathcal{H}^2_j = \Phi^2_j(\mathbb{R}^n)$ and set ξ^k_j as the cut-off functions satisfying $0 \le \xi^k_j \le 1$, $\mathrm{supp}\xi^k_j \subset B_{d^k}(x^k_j) = \{x \in \Omega \mid |x - x^k_j| < d^k\}$ for $k = 1, 2$ and $j = 1, 2, \dots$.

Let $f, g \in L_2(\Omega)$, we first consider the following equations:

$$-\Delta u^k_j + \nabla\pi^k_j = \xi^k_j f \qquad x \in \mathcal{H}^k_j$$

under $(u_j^k, \nabla\psi)_{\mathcal{H}_j^k} = \alpha^{-1}(\nabla\pi_j^k, \nabla\psi)_{\mathcal{H}_j^k} + (\xi_j^k G, \nabla\psi)_{\mathcal{H}_j^k}$ for $k = 1, 2$ and we also consider the boundary condition: $u_j^1 = 0$ on $\partial\mathcal{H}_j^1$ for $k = 1$. By the results of the first step and the second step, we can obtain

$$\|\nabla^2 u_j^k\|_{2,\mathcal{H}_j^k} + \|\nabla\pi_j^k\|_{2,\mathcal{H}_j^k} \le C\left(\|\xi_j^k f\|_{2,\mathcal{H}_j^k} + \|\xi_j^k g\|_{2,\mathcal{H}_j^k} + \|u_j^k\|_{2,\mathcal{H}_j^k} + \|\nabla u_j^k\|_{2,\mathcal{H}_j^k}\right)$$

For $f, g \in L_2(\Omega)$, we set

$$u = \sum_{j=1}^{\infty} \xi_j^1 u_j^1 + \sum_{j=1}^{\infty} \xi_j^2 u_j^2, \qquad \pi = \sum_{j=1}^{\infty} \xi_j^1 \pi_j^1 + \sum_{j=1}^{\infty} \xi_j^2 \pi_j^2.$$

Inserting (u, π) into (3.20) and (3.21), we see that

$$-\Delta u + \nabla\pi = f + \widetilde{F}, \qquad (u, \nabla\psi)_\Omega = \frac{1}{\alpha}(\nabla\pi, \nabla\psi)_\Omega + (G, \nabla\psi)_\Omega + (\widetilde{G}, \nabla\psi)_\Omega,$$

where

$$\widetilde{F} = \sum_{k=1}^{2}\sum_{j=1}^{\infty}\left(2(\nabla\xi_j^k) : (\nabla u_j^k) + (\Delta\xi_j^k)u_j^k - (\nabla\xi_j^k)p_j^k\right), \qquad \widetilde{G} = \alpha^{-1}\sum_{k=1}^{2}\sum_{j=1}^{\infty}(\nabla\xi_j^k)\pi_j^k.$$

By the results of the second step, we have

$$\begin{aligned}
&\|\nabla^2 u\|_{2,\Omega} + \|\nabla\pi\|_{2,\Omega} \\
&\le C\sum_{k=1}^{2}\sum_{j=1}^{\infty}\left(\|\nabla^2(\xi_j^k u_j^k)\|_{2,\Omega} + \|\nabla(\xi_j^k \pi_j^k)\|_{2,\Omega}\right) \\
&\le C\sum_{k=1}^{2}\sum_{j=1}^{\infty}\Big((\|\xi_j^k \nabla^2 u_j^k\|_{2,\mathcal{H}_j^k} + \|\xi_j^k \nabla\pi_j^k\|_{2,\mathcal{H}_j^k}) + 2\|\nabla\xi_j^k\|_\infty\|\nabla u_j^k\|_{2,\mathcal{H}_j^k} \\
&\qquad\qquad + \|\nabla^2\xi_j^k\|_\infty(\|u_j^k\|_{2,\mathcal{H}_j^k} + \|\pi_j^k\|_{2,\mathcal{H}_j^k})\Big) \\
&\le C\sum_{k=1}^{2}\sum_{j=1}^{\infty}\left(\|\xi_j^k f\|_{2,\mathcal{H}_j^k} + \alpha\|\xi_j^k G\|_{2,\mathcal{H}_j^k} + \|u_j^k\|_{2,\mathcal{H}_j^k} + \|\nabla u_j^k\|_{2,\mathcal{H}_j^k} + \|\pi_j^k\|_{2,\mathcal{H}_j^k}\right).
\end{aligned}$$

By Poincaré inequality, (3.27) and $\|F\|_{2,\Omega} \le C\|f\|_{2,\Omega}$, we obtain

$$\|\nabla^2 u\|_2 + \|\nabla\pi\|_2 \le C\left(\|f\|_2 + \alpha\|g\|_2\right),$$

which implies (3.19). □

4 Maximal Regularity

Goal of this section is to prove the $\mathcal{R}$-boundedness of the solution operator to the following resolvent problem (RSa′) in Ω:

$$\begin{cases} \lambda u_\alpha - \Delta u_\alpha + \nabla\pi_\alpha = f & \text{in } \Omega, \\ u_\alpha = 0 & \text{on } \partial\Omega, \end{cases} \tag{RSa′}$$

where $\lambda \in \Sigma_{\varepsilon,\lambda_0}(0 < \varepsilon < \pi/2, \lambda_0 > 0)$ under the approximated weak incompressible condition (Cg). Our method is based on cut-off technique. For this purpose, we shall first prove the whole space case. Secondly we shall prove the half-space case by using the result for the whole space case and some lemma introduced in section 3. Next we shall prove the bent half-space case by reducing to the result for the half-space case with the change of variable. Finally we shall prove the bounded domain case by using the result for the whole space and the bent half-space case with cut-off technique.

4.1 Problem in the whole space

In this subsection, we shall prove the following theorem:

Theorem 4.1. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Set $X_q(\mathbb{R}^n) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\mathbb{R}^n)\}$. Then, there exist operator families $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ with*

$$\mathcal{U}(\lambda) \in Hol(\Sigma_\varepsilon, \mathcal{L}(X_q(\mathbb{R}^n), W_q^2(\mathbb{R}^n)^n)), \qquad \mathcal{P}(\lambda) \in Hol(\Sigma_\varepsilon, \mathcal{L}(X_q(\mathbb{R}^n), \widehat{W}_q^1(\mathbb{R}^n)))$$

such that for any $f, g \in L_q(\mathbb{R}^n)^n$ and $\lambda \in \Sigma_\varepsilon$, $(u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F)$, where $F = (f, \alpha g)$, is a unique solution to (RSa$'$) *under* (Cg) *for the case $\Omega = \mathbb{R}^n$ and $(\mathcal{U}(\lambda), \mathcal{P}(\lambda))$ satisfies the following estimates:*

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell(G_{\lambda,\alpha}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \le C \ (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n)^n)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \le C \ (\ell = 0, 1)$$

for $G_{\lambda,\alpha}u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, (\lambda+\alpha)^{1/2}(\nabla \cdot u))$ and $\widetilde{N} = 1 + n + n^2 + n^3$.

Proof. In order to prove the $\mathcal{R}$-boundedness of solution operator by using Theorem 3.1, we shall obtain the solution formula to (RSa$'$) under (Cg) by using Fourier transform. By the property of Helmholtz projection, we know $\nabla\pi_\alpha = \alpha\nabla Q_{\mathbb{R}^n}(u_\alpha - g)$ and $\mathcal{F}[\nabla Q_{\mathbb{R}^n} v] = |\xi|^{-2}\xi(\xi \cdot \widehat{v})$. Applying the Fourier transform to (RSa$'$), we obtain the following solution formula : $u_{\alpha,j}(x) = u_j(x) + u^E_{\alpha,j}(x)$ and $\pi_\alpha(x) = \pi(x) + \pi^E_\alpha(x)$, where (u, π) is the solution to Stokes equations given by

$$u_j(x) = \mathcal{F}_\xi^{-1}\left[\frac{1}{\lambda + |\xi|^2}\widehat{f}_j(\xi)\right](x) - \sum_{k=1}^n \mathcal{F}_\xi^{-1}\left[\frac{\xi_j\xi_k}{(\lambda + |\xi|^2)|\xi|^2}\widehat{f}_k(\xi)\right](x), \tag{4.1}$$

$$\pi(x) = -i\sum_{k=1}^n \mathcal{F}_\xi^{-1}\left[\frac{\xi_k}{|\xi|^2}\hat{f}_k(\xi)\right](x) \tag{4.2}$$

for $j = 1, \dots, n$ and the error term $(u^E_\alpha, \pi^E_\alpha)$ given by

$$u^E_{\alpha,j} = \sum_{k=1}^n \mathcal{F}_\xi^{-1}\left[\frac{\xi_j\xi_k(\hat{f}_k(\xi) - \alpha\hat{g}_k)}{|\xi|^2(\lambda + \alpha + |\xi|^2)}\right](x), \quad \pi^E_\alpha = i\sum_{k=1}^n \mathcal{F}_\xi^{-1}\left[\frac{\xi_k(\lambda + |\xi|^2)(\hat{f}_k(\xi) - \alpha\hat{g}_k(\xi))}{|\xi|^2(\lambda + \alpha + |\xi|^2)}\right](x) \tag{4.3}$$

for $j = 1, \dots, n$. Since in the whole space case, it is well-known that the solution operator to Stokes equations is $\mathcal{R}$-bounded ([26] for detail), we consider the only error term $(u^E_\alpha, \pi^E_\alpha)$. By Leibniz rule, for $\ell = 0, 1$, we obtain

$$\left|(\tau\partial_\tau)^\ell D^\delta_\xi \frac{(\lambda+\alpha)\xi_j\xi_k}{|\xi|^2(\lambda+\alpha+|\xi|^2)}\right| \le C_{\varepsilon,\delta}|\xi|^{-|\delta|}, \quad \left|(\tau\partial_\tau)^\ell D^\delta_\xi \frac{(\lambda+\alpha)^{1/2}\xi_m\xi_j\xi_k}{|\xi|^2(\lambda+\alpha+|\xi|^2)}\right| \le C_{\varepsilon,\delta}|\xi|^{-|\delta|},$$
$$\left|(\tau\partial_\tau)^\ell D^\delta_\xi \frac{\xi_m\xi_n\xi_j\xi_k}{|\xi|^2(\lambda+\alpha+|\xi|^2)}\right| \le C_{\varepsilon,\delta}|\xi|^{-|\delta|}, \quad \left|(\tau\partial_\tau)^\ell D^\delta_\xi \frac{\xi_j\xi_k(\lambda+|\xi|^2)}{|\xi|^2(\lambda+\alpha+|\xi|^2)}\right| \le C_{\varepsilon,\delta}|\xi|^{-|\delta|}, \tag{4.4}$$

which implies from Theorem 3.1

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell(G_{\lambda,\alpha}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \le C \ (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n), L_q(\mathbb{R}^n)^n)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \le C \ (\ell = 0, 1).$$

This completes the proof of Theorem 4.1. □

Remark 4.2. By Theorem 4.1, we see that the existence of the solution (u_α, π_α) to the resolvent problem (RSa′). Moreover by Theorem 2.10 and Remark 2.9, (u_α, π_α) satisfies the following resolvent estimate:

$$\|(\lambda u_\alpha, \lambda^{1/2}\nabla u_\alpha, \nabla^2 u_\alpha, (\lambda+\alpha)^{1/2}(\nabla\cdot u_\alpha), \nabla\pi_\alpha)\|_{L_q(\mathbb{R}^n)} \le C_{n,q,\varepsilon}\|(f, \alpha g)\|_{L_q(\mathbb{R}^n)}.$$

4.2 Problem in the half-space

In this section we shall prove the following theorem:

Theorem 4.3. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Set $X_q(\mathbb{R}^n_+) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\mathbb{R}^n_+)\}$. Then, there exist operator families $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ with*

$$\mathcal{U}(\lambda) \in \mathit{Hol}(\Sigma_\varepsilon, \mathcal{L}(X_q(\mathbb{R}^n_+), W^2_q(\mathbb{R}^n_+)^n), \qquad \mathcal{P}(\lambda) \in \mathit{Hol}(\Sigma_\varepsilon, \mathcal{L}(X_q(\mathbb{R}^n_+), \widehat{W}^1_q(\mathbb{R}^n_+)),$$

such that for any $f, g \in L_q(\mathbb{R}^n_+)^n$ and $\lambda \in \Sigma_\varepsilon$, $(u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F)$, where $F = (f, \alpha g)$, is a unique solution to (RSa′) *under* (Cg) *and $(\mathcal{U}(\lambda), \mathcal{P}(\lambda))$ satisfies the following estimates:*

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\mathbb{R}^n_+)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell(G_{\lambda,\alpha}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \le C \ (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\mathbb{R}^n_+)^n)}(\{(\tau\partial_\tau)^\ell(\nabla\mathcal{P}(\lambda)) \mid \lambda \in \Sigma_\varepsilon\}) \le C \ (\ell = 0, 1)$$

for $G_{\lambda,\alpha}u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, (\lambda+\alpha)^{1/2}(\nabla\cdot u))$ and $\widetilde{N} = 1 + n + n^2 + n^3$.

In order to prove Theorem 4.3 by Lemma 3.2, we shall obtain the solution formula to (RSa′) under (Cg). By density argument, we may let $f, g \in C^\infty_0(\mathbb{R}^n_+)$. In this case, equation (RSa′) under (Cg) is equivalent to the following equations:

$$\begin{cases} \lambda u_\alpha - \Delta u_\alpha + \nabla\pi_\alpha = f, \quad \nabla\cdot u_\alpha - \alpha^{-1}\Delta\pi_\alpha = \nabla\cdot g \quad \text{in } \mathbb{R}^n_+, \\ \qquad u|_{\partial\mathbb{R}^n_+} = 0, \qquad \partial_n\pi_\alpha|_{\partial\mathbb{R}^n_+} = 0. \end{cases} \tag{4.5}$$

We shall obtain the solution formula to (4.5). For this purpose, we extend the external force f and g to the whole space. For $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_n)$, let $F = (f^e_1, \dots, f^e_{n-1}, f^o_n)$ and $G = (g^e_1, \dots, g^e_{n-1}, g^o_n)$, where

$$f^e_j(x) = \begin{cases} f_j(x', x_n) & (x_n > 0) \\ f_j(x', -x_n) & (x_n < 0) \end{cases}, \quad f^o_n(x) = \begin{cases} f_n(x', x_n) & (x_n > 0) \\ -f_n(x', -x_n) & (x_n < 0) \end{cases},$$

where $x' = (x_1, \dots, x_{n-1})$. We consider the resolvent problem with F and G:

$$\lambda U_\alpha - \Delta U_\alpha + \nabla\Theta_\alpha = F, \qquad \nabla\cdot U_\alpha = \alpha^{-1}\Delta\Theta_\alpha + \nabla\cdot G \qquad \text{in } \mathbb{R}^n. \tag{4.6}$$

Here we remark that from the definition of our extension, $(U_\alpha, \Theta_\alpha)$ enjoys the boundary condition

$$U_{\alpha,n}(x', 0) = 0, \qquad \partial_n\Theta_\alpha(x', 0) = 0. \tag{4.7}$$

By the result for the whole space and the definition of our extension, the following estimates hold:

$$\begin{aligned} \|(\lambda U_\alpha, \lambda^{1/2}\nabla U_\alpha, \nabla^2 U_\alpha, (\lambda+\alpha)^{1/2}(\nabla\cdot U_\alpha), \nabla\Theta_\alpha)\|_{L_q(\mathbb{R}^n)} &\le C\|(F, \alpha G)\|_{L_q(\mathbb{R}^n)} \\ &\le C\|(f, \alpha g)\|_{L_q(\mathbb{R}^n_+)}. \end{aligned} \tag{4.8}$$

Setting $u_\alpha = w_\alpha + U_\alpha$ and $\pi_\alpha = \rho_\alpha + \Theta_\alpha$, we see that to solve (4.5) is equivalent to solve

$$\begin{cases} \lambda w_\alpha - \Delta w_\alpha + \nabla\rho_\alpha = 0, \quad \nabla\cdot w_\alpha = \Delta\rho_\alpha/\alpha \qquad \text{in } \mathbb{R}^n_+, \\ \qquad (w_\alpha)_j|_{x_n=0} = h_j|_{x_n=0}, \quad \partial_n\rho_\alpha|_{x_n=0} = 0, \end{cases} \tag{4.9}$$

where $h_j = -(U_\alpha)_j$ for $j = 1, \dots, n-1$ and $h_n = 0$. Applying div and $(\lambda + \alpha - \Delta)\Delta$ to the first equation in (4.9), we obtain

$$(\lambda + \alpha - \Delta)\Delta \rho_\alpha = 0, \qquad (\lambda + \alpha - \Delta)(\lambda - \Delta)\Delta w_\alpha = 0. \tag{4.10}$$

By applying the partial Fourier transform defined by

$$\widetilde{g}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} g(x', x_n)dx'$$

to (4.9) and (4.10) , we have

$$\begin{aligned} \lambda(\widetilde{w_\alpha})_j + r^2(\widetilde{w_\alpha})_j - \partial_n^2(\widetilde{w_\alpha})_j + (i\xi_j)\widetilde{\rho_\alpha} = 0, \\ \lambda(\widetilde{w_\alpha})_n + r^2(\widetilde{w_\alpha})_n - \partial_n^2(\widetilde{w_\alpha})_n + \partial_n\widetilde{\rho_\alpha} = 0, \\ i\xi' \cdot \widetilde{w_\alpha}' + \partial_n(\widetilde{w_\alpha})_n = \alpha^{-1}(-r^2\widetilde{\rho_\alpha} + \partial_n^2\widetilde{\rho_\alpha}), \\ (\widetilde{w_\alpha})_j(\xi', 0) = \widetilde{h}_j(\xi', 0), \qquad (\widetilde{w_\alpha})_n(\xi', 0) = 0, \qquad \partial_n\widetilde{\rho_\alpha}(\xi', 0) = 0 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &(\lambda + \alpha + r^2 - D_n^2)(r^2 - D_n^2)\widetilde{\rho_\alpha} = 0, \\ &(\lambda + \alpha + r^2 - D_n^2)(\lambda + r^2 - D_n^2)(r^2 - D_n^2)\widetilde{w_\alpha} = 0, \end{aligned} \tag{4.12}$$

where $i\xi' \cdot \widetilde{w_\alpha}' = \sum_{j=1}^{n-1}(i\xi_j)(\widetilde{w_\alpha})_j$. Since from (4.12), we see the solution $(\widetilde{w_\alpha}, \widetilde{\rho_\alpha})$ can be expressed by

$$\widetilde{\rho_\alpha} = pe^{-rx_n} + qe^{-\omega x_n}, \qquad (\widetilde{w_\alpha})_j = a_je^{-rx_n} + b_je^{-\omega_\lambda x_n} + c_je^{-\omega x_n} \tag{4.13}$$

for $j = 1, \dots, n$, we shall find the solution to (4.11) having the form (4.13). By substituting (4.13) to (4.11), we see

$$\begin{cases} \lambda a_j + (i\xi_j)p = 0, & -\alpha c_j + (i\xi_j)q = 0, \\ \lambda a_n - rp = 0, & -\alpha c_n - \omega q = 0, \\ i\xi' \cdot a' - ra_n = 0, & i\xi' \cdot b' - \omega_\lambda b_n = 0, \quad i\xi' \cdot c' - \omega c_n = \alpha^{-1}(\alpha + \lambda)q, \\ a_j + b_j + c_j = \widetilde{h}_j, & a_n + b_n + c_n = 0, \qquad\qquad -rp - \omega q = 0 \end{cases}$$

for $j = 1, \dots, n-1$. Setting $\mathcal{A} = \lambda(\omega_\lambda\omega - r^2)$ and $\mathcal{B} = \alpha\omega(\omega_\lambda - r)$, we see

$$\begin{aligned} &p = -\frac{\alpha\lambda\omega i}{r(\mathcal{A}+\mathcal{B})}\xi' \cdot \widetilde{h}', && q = -\frac{r}{\omega}p, \\ &a_j = -\frac{i\xi_j}{\lambda}p, && b_j = \widetilde{h}_j + \frac{i\xi_j}{\lambda}p + \frac{i\xi_j r}{\alpha\omega}p, && c_j = -\frac{i\xi_j r}{\alpha\omega}p, \\ &a_n = \frac{r}{\lambda}p, && b_n = -\frac{r}{\lambda}p - \frac{r}{\alpha}p, && c_n = \frac{r}{\alpha}p. \end{aligned}$$

Therefore, we obtain the solution formula $(\widetilde{w_\alpha})_j = \widetilde{w}_j + \widetilde{w_\alpha}_j^E$ and $\widetilde{\rho_\alpha} = \widetilde{\rho} + \widetilde{\rho_\alpha}^E$, where $(\widetilde{w}, \widetilde{w_\alpha}^E, \widetilde{\rho}, \widetilde{\rho_\alpha}^E)$ is given

$$\begin{aligned} \widetilde{w}_j &= \widetilde{h}_je^{-\omega_\lambda x_n} + \frac{\xi_j}{r}\xi' \cdot \widetilde{h}'\mathcal{M}(\omega_\lambda, r, x_n), \\ \widetilde{w_\alpha}_j^E &= -\frac{\xi_j}{r}\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}\xi' \cdot \widetilde{h}'\mathcal{M}(\omega_\lambda, r, x_n) - \frac{\xi_j}{\omega_\lambda + r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}\xi' \cdot \widetilde{h}'\mathcal{M}(\omega, \omega_\lambda, x_n), \\ \widetilde{w}_n &= i\xi' \cdot \widetilde{h}'\mathcal{M}(\omega_\lambda, r, x_n), \\ \widetilde{w_\alpha}_n^E &= \frac{\mathcal{B}}{\mathcal{A}+\mathcal{B}}i\xi' \cdot \widetilde{h}'\mathcal{M}(\omega_\lambda, r, x_n) - \frac{\alpha\omega\lambda}{(\omega + \omega_\lambda)(\mathcal{A}+\mathcal{B})}i\xi' \cdot \widetilde{h}'\mathcal{M}(\omega, \omega_\lambda, x_n), \\ \widetilde{\rho} &= -\frac{\omega_\lambda + r}{r}i\xi' \cdot \widetilde{h}'e^{-rx_n}, \\ \widetilde{\rho}^E &= \frac{\omega_\lambda + r}{r}\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}i\xi' \cdot \widetilde{h}'e^{-rx_n} + \frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}i\xi' \cdot \widetilde{h}'e^{-\omega x_n}. \end{aligned}$$

Since the symbol $\mathcal{M}(a,b,x_n)$ defined by (3.1) has the following properties:

$$\begin{aligned}
&\partial_n\mathcal{M}(a,b,x_n) = -e^{-ax_n} - b\mathcal{M}(a,b,x_n),\\
&\partial_n^2\mathcal{M}(a,b,x_n) = (a+b)e^{-ax_n} + b^2\mathcal{M}(a,b,x_n)
\end{aligned}$$

and by $g(0) = -\int_0^\infty \partial_n g(y_n)dy_n$, we have

$$\begin{aligned}
\widetilde{h}(\xi',0)e^{-ax_n} &= \int_0^\infty \mathcal{E}(a)(a-D_n)\tilde{h}(\xi',y_n)dy_n,\\
\tilde{h}(\xi',0)\mathcal{M}(a,b,x_n) &= \int_0^\infty \{\mathcal{E}(a)\tilde{h}(y_n) + \mathcal{M}(a,b,x_n+y_n))(b-D_n)\tilde{h}(\xi',y_n)\}dy_n,
\end{aligned}$$

where $\mathcal{E}(z)$ is defined by (3.1). Therefore, setting $\bar{\xi_j} = \xi_j/r$, we obtain

$$\begin{aligned}
&w_j(x)\\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\mathcal{E}(\omega_\lambda)(\omega_\lambda - D_n)\widetilde{h_j}(\xi',y_n)](x')dy_n\\
&+\sum_{k=1}^{n-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_j}\bar{\xi_k}(\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\\
&\qquad + \mathcal{M}(\omega_\lambda,r,x_n+y_n)(r-D_n)r\widetilde{h_k}(\xi',y_n))](x')dy_n,\\
&(w_\alpha)_j^E(x)\\
&= -\sum_{k=1}^{n-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_j}\bar{\xi_k}\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}(\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\\
&\qquad + \mathcal{M}(\omega_\lambda,r,x_n+y_n)(r-D_n)r\widetilde{h_k}(\xi',y_n))](x')dy_n\\
&+\sum_{k=1}^{n-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\frac{r\bar{\xi_j}\bar{\xi_k}}{\omega_\lambda+r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}(\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\\
&\qquad + \mathcal{M}(\omega_\lambda,\omega,x_n+y_n)(\omega-D_n)r\widetilde{h_k}(\xi',y_n))](x')dy_n,\\
&w_n(x)\\
&= \sum_{k=1}^{n-1} i\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_k}(\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\\
&\qquad + \mathcal{M}(\omega_\lambda,r,x_n+y_n)(r-D_n)r\widetilde{h_k}(\xi',y_n))](x')dy_n,\\
&(w_\alpha)_n^E(x)\\
&= \sum_{k=1}^{n-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_k}\frac{i\mathcal{B}}{\mathcal{A}+\mathcal{B}}(\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\\
&\qquad + \mathcal{M}(\omega_\lambda,r,x_n+y_n)(r-D_n)r\widetilde{h_k}(\xi',y_n))](x')dy_n\\
&+\sum_{k=1}^{n-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\bar{\xi_k}\frac{\omega i}{\omega_\lambda+\omega}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}(\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\\
&\qquad + \mathcal{M}(\omega_\lambda,\omega,x_n+y_n)(\omega-D_n)r\widetilde{h_k}(\xi',y_n))](x')dy_n,\\
&\rho(x)\\
&= -\sum_{k=1}^{n-1} i\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\frac{\omega_\lambda+r}{r}\mathcal{E}(r)(r-D_n)r\bar{\xi_k}\widetilde{h_k}(\xi',y_n)](x')dy_n,
\end{aligned}$$

$$
\begin{aligned}
&(\rho_\alpha)^E(x)\\
&= \sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'}[\bar{\xi}_k \frac{\omega_\lambda + r}{r} \frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}} i\mathcal{E}(r)(r - D_n) r\widetilde{h_k}(\xi', y_n)](x') dy_n\\
&+ \sum_{k=1}^{n-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'}[\bar{\xi}_k \frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}} i\mathcal{E}(\omega)(\omega - D_n) r\widetilde{h_k}(\xi', y_n)] dy_n.
\end{aligned} \tag{4.14}
$$

We remark that (w, ρ) is the solution to the usual Stokes equations and (w^E, ρ^E) is the error between the solution to Stokes equations and Stokes equations approximated by pressure stabilization. Since Shibata and Shimizu [26] proved $\mathcal{R}$-boundedness of solution operator to Stokes equations, it is sufficient to consider $(w^E_\alpha, \rho^E_\alpha)$ only. For this purpose, we prepare the following lemma.

Lemma 4.4. *Let $0 < \varepsilon < \pi/2$ and $\alpha > 0$. For any multi-index δ' and $(\lambda, \xi', x_n) \in \Sigma_\varepsilon \times (\mathbb{R}^{n-1}\backslash\{0\}) \times (0, \infty)$, $m(\lambda, \xi') = r(\omega_\lambda + r)^{-1}, \omega(\omega_\lambda + \omega)^{-1}, \mathcal{A}(\mathcal{A}+\mathcal{B})^{-1}, \mathcal{B}(\mathcal{A}+\mathcal{B})^{-1}$ and $\alpha\lambda(\mathcal{A}+\mathcal{B})^{-1}$ enjoy*

$$
|\partial^{\delta'}_{\xi'} m(\lambda, \xi')| \le C r^{-|\delta'|}, \tag{4.15}
$$

where C is a positive constant which is dependent of ε and δ'.

Proof. We first show that $m(\lambda, \xi') = r(\omega_\lambda + r)^{-1}$ and $\omega(\omega_\lambda + \omega)^{-1}$ enjoy (4.15). By Leibniz rule with (3.3), we see

$$
\begin{aligned}
\left| D^{\delta'}_{\xi'} \frac{r}{\omega_\lambda + r} \right| &\le C \sum_{\delta' = \delta'_1 + \delta'_2} r^{1-|\delta'_1|} \frac{r^{-|\delta'_2|}}{|\lambda|^{1/2} + r} \le C r^{-|\delta'|},\\
\left| D^{\delta'}_{\xi'} \frac{\omega}{\omega_\lambda + \omega} \right| &\le C \sum_{\delta' = \delta'_1 + \delta'_2} (|\lambda|^{1/2} + \alpha^{1/2} + r) r^{-|\delta'_1|} \frac{r^{-|\delta'_2|}}{(|\lambda|^{1/2} + \alpha^{1/2} + r)} \le C r^{-|\delta'|}.
\end{aligned}
$$

In order to prove $m(\lambda, \xi') = \mathcal{A}(\mathcal{A}+\mathcal{B})^{-1}, \mathcal{B}(\mathcal{A}+\mathcal{B})^{-1}$ and $\alpha\lambda(\mathcal{A}+\mathcal{B})^{-1}$, we shall consider $D^{\delta'}_{\xi'}(\mathcal{A}+\mathcal{B})$. Since

$$
\mathcal{A}+\mathcal{B} = (\lambda + \alpha)\omega(\omega_\lambda - r) + \lambda r(\omega - r) = \frac{\lambda(\lambda+\alpha)\omega}{\omega_\lambda + r} + \frac{\lambda(\lambda+\alpha) r}{\omega + r},
$$

we have

$$
\begin{aligned}
\left| D^{\delta'}_{\xi'}(\mathcal{A}+\mathcal{B}) \right| &\le C|\lambda|(|\lambda| + \alpha) \left\{ \frac{|\lambda|^{1/2} + \alpha^{1/2} + r}{|\lambda|^{1/2} + r} + \frac{r}{|\lambda|^{1/2} + \alpha^{1/2} + r} \right\} r^{-|\delta'|}\\
&\le C|\lambda|(|\lambda|^{1/2} + \alpha^{1/2})^2 (|\lambda|^{1/2} + \alpha^{1/2} + r)(|\lambda|^{1/2} + r)^{-1} r^{-|\delta'|}.
\end{aligned} \tag{4.16}
$$

Since $|\arg[\omega(\omega + r)/r(\omega_\lambda + r)]| < \pi - \varepsilon$, we know $\omega r^{-1}(\omega + r)(\omega_\lambda + r)^{-1} \in \Sigma_\varepsilon$, which implies that

$$
\begin{aligned}
|\mathcal{A}+\mathcal{B}| &= |\lambda + \alpha||\lambda| \left| \frac{r}{\omega + r} \right| \left| \frac{\omega}{\omega_\lambda + r} \cdot \frac{\omega + r}{r} + 1 \right|\\
&\ge C(|\lambda|^{1/2} + \alpha^{1/2})^2 |\lambda| r (|\lambda|^{1/2} + \alpha^{1/2} + r)^{-1} \left(\left| \frac{\omega}{\omega_\lambda + r} \cdot \frac{\omega + r}{r} \right| + 1 \right)\\
&\ge C(|\lambda|^{1/2} + \alpha^{1/2})^2 |\lambda| (|\lambda|^{1/2} + \alpha^{1/2} + r)(|\lambda|^{1/2} + r)^{-1}.
\end{aligned}
$$

By Bell's formula with (4.16), we obtain

$$
\left| D^{\delta'}_{\xi'}(\mathcal{A}+\mathcal{B})^{-1} \right| \le C|\lambda|^{-1}(|\lambda|^{1/2} + \alpha^{1/2})^{-2}(|\lambda|^{1/2} + \alpha^{1/2} + r)^{-1}(|\lambda|^{1/2} + r) r^{-|\delta'|},
$$

which implies (4.15) for $m(\lambda, \xi') = \mathcal{A}(\mathcal{A}+\mathcal{B})^{-1}, \mathcal{B}(\mathcal{A}+\mathcal{B})^{-1}$ and $\alpha\lambda(\mathcal{A}+\mathcal{B})^{-1}$. □

Proof of Theorem 4.3. We shall prove Theorem 4.3 by Lemma 3.2 with Lemma 4.4. Set $(w_\alpha)^E_{j,k,\ell}(x)(k = 1,\ldots,n-1, \ell = 1,\ldots,6)$ as follows

$$
\begin{aligned}
(w_\alpha)^E_{j,k,1}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\bar{\xi}_j\bar{\xi}_k\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\right](x')dy_n,\\
(w_\alpha)^E_{j,k,2}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\bar{\xi}_j\bar{\xi}_k\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}\mathcal{M}(\omega_\lambda,r,x_n+y_n)r^2\widetilde{h_k}(\xi',y_n)\right](x')dy_n,\\
(w_\alpha)^E_{j,k,3}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\bar{\xi}_j\bar{\xi}_k\frac{\mathcal{A}}{\mathcal{A}+\mathcal{B}}\mathcal{M}(\omega_\lambda,r,x_n+y_n)rD_n\widetilde{h_k}(\xi',y_n)\right](x')dy_n,\\
(w_\alpha)^E_{j,k,4}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{r\bar{\xi}_j\bar{\xi}_k}{\omega_\lambda+r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}\mathcal{E}(\omega_\lambda)r\widetilde{h_k}(\xi',y_n)\right](x')dy_n,\\
(w_\alpha)^E_{j,k,5}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{r\bar{\xi}_j\bar{\xi}_k}{\omega_\lambda+r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}\mathcal{M}(\omega_\lambda,\omega,x_n+y_n)\omega r\widetilde{h_k}(\xi',y_n)\right](x')dy_n,\\
(w_\alpha)^E_{j,k,6}(x) &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{r\bar{\xi}_j\bar{\xi}_k}{\omega_\lambda+r}\frac{\alpha\lambda}{\mathcal{A}+\mathcal{B}}\mathcal{M}(\omega_\lambda,\omega,x_n+y_n)rD_n\widetilde{h_k}(\xi',y_n)\right](x')dy_n.
\end{aligned}
$$

Setting $K_{\alpha,\ell,j}(h_k) = (w_\alpha)^E_{j,k,\ell}(x)$ for $\ell = 1,2,4,5$, by Lemma 3.2, Lemma 4.4 and (4.8), we see that $K_{\alpha,\ell,j}$ is $\mathcal{R}$-bounded. Since $h_k = -(U_\alpha)_k$, $U_\alpha = \mathcal{U}_{\mathbb{R}^n}(\lambda)F$, where $\mathcal{U}_{\mathbb{R}^n}(\lambda)$ is the solution operator in $\mathbb{R}^n$ and $F = (f,\alpha g)$, setting $\mathcal{V}_{j,k,\ell}(\lambda)F = K_{\alpha,j,\ell}((\mathcal{U}_{\mathbb{R}^n}(\lambda)F)_k)$, we see that $G_{\lambda,\alpha}\mathcal{V}_{j,k,\ell}(\lambda)F = K_{\alpha,\ell,j}(G_{\lambda,\alpha}(\mathcal{U}_{\mathbb{R}^n}((\lambda)F)$ is $\mathcal{R}$-bounded by Remark 2.9.

Since Lemma 3.2 and Lemma 4.4 and the relation:

$$\lambda(w_\alpha)^E_{j,k,3}(x) = \int_0^\infty \mathcal{F}^{-1}_{\xi'}[\bar{\xi}_j\bar{\xi}_k\frac{\mathcal{A}}{(\mathcal{A}+\mathcal{B})}\mathcal{M}(\omega_\lambda,r,x_n+y_n)r|\lambda|^{1/2}\frac{\lambda}{|\lambda|}(|\lambda|^{1/2}D_n\widetilde{h_k}(\xi',y_n))](x')dy_n,$$

we see there exists a $\mathcal{R}$-bouned operator $K_{\alpha,3,j}$ such that

$$K_{\alpha,3,j}(|\lambda|^{1/2}D_nh_k) = \lambda(w_\alpha)^E_{j,k,3}(x).$$

Setting $\lambda\mathcal{V}_{j,k,3}(\lambda)F = K_{\alpha,3,j}(|\lambda|^{1/2}D_n(\mathcal{U}_{\mathbb{R}^n}F)_k)$, we see $\lambda\mathcal{V}_{j,k,3}(\lambda)F$ is $\mathcal{R}$-bounded. In a similar way, we can show that $G_{\lambda,\alpha}\mathcal{V}_{j,k,\ell}(\lambda)F(\ell = 3,6)$ is $\mathcal{R}$-bounded. Summing up, setting $(\mathcal{U}(\lambda)F)_j = \sum_{k,\ell}\mathcal{V}_{j,k,\ell}(\lambda)F$ and $\mathcal{U}(\lambda)F = ((\mathcal{U}(\lambda)F)_j)_{j=1,\ldots,n}$, we see $\mathcal{U}(\lambda)F$ is the solution operator in $\mathbb{R}^n_+$ and $G_{\lambda,\alpha}\mathcal{U}(\lambda)F$ is $\mathcal{R}$-bounded.

In the same way, we obtain the results for $(w_\alpha)^E_n(x)$ from the results for $(w_\alpha)^E_j(x)$ and the results for $(\rho_\alpha)^E(x)$ from the equations (RSa$'$) and the results for $(w_\alpha)^E_j(x)$ and $(w_\alpha)^E_n(x)$. □

4.3 Problem in the bent half-space and the bounded domain

Before we describe the theorem for bent half-space, we shall introduce some notations. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. Writing $\nabla\Phi = \mathcal{A} + \mathcal{B}(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_{-1} + \mathcal{B}_{-1}(x)$, we assume that $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthogonal matrices with constant coefficients and $\mathcal{B}(x)$ and $\mathcal{B}_{-1}(x)$ are matrices of functions in $W^1_r(\mathbb{R}^n)$ with $n < r < \infty$ such that

$$\|(\mathcal{B},\mathcal{B}_{-1})\|_{L_\infty(\mathbb{R}^n)} \le M_1, \qquad \|\nabla(\mathcal{B},\mathcal{B}_{-1})\|_{L_r(\mathbb{R}^n)} \le M_2. \tag{4.17}$$

We shall choose M_1 small enough later, so that we may assume that $0 < M_1 \le 1 \le M_2$. Let $\mathbb{R}^n_0$ be the boundary of the half-space defined by $\mathbb{R}^n_0 = \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$. Set $\Omega_+ = \Phi(\mathbb{R}^n_+)$ and $\partial\Omega_+ = \Phi(\mathbb{R}^n_0)$.

Theorem 4.5. *Let $\alpha > 0$, $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Set $X_q(\Omega_+) = \{(F_1, F_2) \mid F_1, F_2 \in L_q(\Omega_+)\}$. Then there exist $M_1 \in (0,1)$, $\lambda_0 \geq 1$ and solution operator families $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ with*

$$\mathcal{U}(\lambda) \in Hol(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega_+), W_q^2(\Omega_+))), \quad \mathcal{P}(\lambda) \in Hol(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega_+), \widehat{W}_q^1(\Omega_+))) \tag{4.18}$$

such that for any $(f, \alpha g) \in X_q(\Omega_+)$ and $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, $(u_\alpha, \pi_\alpha) = (\mathcal{U}(\lambda)F, \mathcal{P}(\lambda)F)$, where $F = (f, \alpha g)$, is a unique solution to problem (RSa$'$) *under* (Cg). *Moreover $(\mathcal{U}(\lambda), \mathcal{P}(\lambda))$ satisfies the following estimates:*

$$\mathcal{R}_{\mathcal{L}(X_q(\Omega_+), L_q(\Omega_+)^{\widetilde{N}})}(\{(\tau\partial_\tau)^\ell G_{\lambda,\alpha}\mathcal{U}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq C \quad (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\Omega_+), L_q(\Omega_+)^n)}(\{(\tau\partial_\tau)^\ell \nabla\mathcal{P}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq C \quad (\ell = 0, 1)$$

for $G_{\lambda,\alpha}u = (\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u, (\lambda+\alpha)^{1/2}(\nabla \cdot u))$ and $\widetilde{N} = 1 + n + n^2 + n^3$.

Proof. In order to prove Theorem 4.5, we transfer (RSa$'$) and (Cg) into a problem in $\mathbb{R}^n_+$ by the change of variable $x = \Phi^{-1}(y)$ with $y \in \Omega_+$ and $x \in \mathbb{R}^n_+$ and by the change of unknowns : $v(x) = \mathcal{A}_{-1}(u_\alpha(\Phi(x)))$, $\rho = \mathcal{A}_{-1}(\pi_\alpha(\Phi(x)))$ and $\psi(x) = \mathcal{A}_{-1}(\varphi(\Phi(x)))$. Since $\partial_{y_j} = \sum_{\ell=1}^n (\mathcal{A}_{\ell,j} + B_{\ell,j})\partial_{x_j}$, employing the same argument to Shibata [21], we have the following equations

$$\begin{cases} \lambda v - \Delta v + \nabla\rho = f_+ + \mathcal{F}(v,\rho) & x \in \mathbb{R}^n_+, \\ v = 0 & x \in \partial\mathbb{R}^n_+ \end{cases} \tag{4.19}$$

under

$$(v, \nabla\psi)_{\mathbb{R}^n_+} = \alpha^{-1}(\nabla\rho, \nabla\psi)_{\mathbb{R}^n_+} + (g_+, \nabla\psi)_{\mathbb{R}^n_+} + (\mathcal{G}(v,\rho), \nabla\psi)_{\mathbb{R}^n_+}, \qquad \psi \in \widehat{W}^1_{q'}(\mathbb{R}^n_+), \tag{4.20}$$

where $f_+(x) = \mathcal{A}_{-1}(f(\Phi(x)))$ and $g_+(x) = \mathcal{A}_{-1}(g(\Phi(x))) + \mathcal{M}_4 g$. Moreover $\mathcal{F}(v,\rho)$ and $\mathcal{G}(v,\rho)$ have the following forms:

$$\mathcal{F}(v,\rho) = \mathcal{M}_1\nabla^2 v + \mathcal{M}_2\nabla v + \mathcal{M}_3\nabla\rho, \qquad \mathcal{G}(v,\rho) = \alpha^{-1}(\mathcal{M}_4 v + \mathcal{M}_5\nabla\rho) \tag{4.21}$$

with some matrices of functions $\mathcal{M}_k$ $(k = 1, \ldots, 5)$ possessing the estimates

$$\|\mathcal{M}_j\|_{L_\infty(\mathbb{R}^n_+)} \leq CM_1, \qquad \|(\mathcal{M}_2, \nabla\mathcal{M}_j)\|_{L_r(\mathbb{R}^n_+)} \leq CM_2 \tag{4.22}$$

for $j = 1, 3, 4, 5$ and $n < r < \infty$. Setting $\mathbb{F}(\lambda)F = \mathcal{F}(\mathcal{U}_{\mathbb{R}^n_+}(\lambda)F, \mathcal{P}_{\mathbb{R}^n_+}(\lambda)F)$ and $\mathbb{G}(\lambda)F = \mathcal{G}(\mathcal{U}_{\mathbb{R}^n_+}(\lambda)F, \mathcal{P}_{\mathbb{R}^n_+}(\lambda)F))$, where $F = (f_+, \alpha g_+)$ and $(\mathcal{U}_{\mathbb{R}^n_+}(\lambda), \mathcal{P}_{\mathbb{R}^n_+}(\lambda))$ is the solution operator in $\mathbb{R}^n_+$, we can obtain, for $\ell = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \mathbb{F}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq \{C(\sigma + M_1) + C_\sigma\lambda_0^{-1/2}\}\kappa_0,$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \alpha\mathbb{G}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq \{C(\sigma + M_1) + C_\sigma\lambda_0^{-1/2}\}\kappa_0,$$

where κ_0 is the $\mathcal{R}$-bound of the half-space case and $\sigma > 0$, by the method due to Shibata [21]. We choose σ and M_1 so small that $C(\sigma + M_1)\kappa_0 \leq 1/8$ and $\lambda_0 \geq 1$ so large that $C_\sigma\lambda_0^{-1/2}\kappa_0 \leq 1/8$. Thus. we have

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\Omega)^{\widetilde{N}})}(\{(\tau\partial_\tau)^\ell \mathbb{F}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq 1/4 \qquad (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+), L_q(\Omega)^{\widetilde{N}})}(\{(\tau\partial_\tau)^\ell \alpha\mathbb{G}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq 1/4 \qquad (\ell = 0, 1).$$

Since $\mathcal{R}$-boundedness implies the usual boundedness (see Remark 2.9), we have

$$\|(\mathbb{F}(\lambda)F, \alpha\mathbb{G}(\lambda)F)\|_{L_q(\mathbb{R}^n_+)} \leq 2^{-1}\|F\|_{L_q(\mathbb{R}^n_+)},$$

where $F = (f, \alpha g)$ for $\lambda \in \Gamma_{\varepsilon,\lambda_0}$. Therefore $\mathcal{R}(\lambda)F = (\mathbb{F}(\lambda)F, \alpha\mathbb{G}(\lambda)F)$ is a contraction map from $X_q(\mathbb{R}^n_+)$ into itself, so that for each $\lambda \in \Gamma_{\varepsilon,\lambda_0}$, $(I+\mathcal{R}(\lambda))^{-1}$ exists and $\|(I+\mathcal{R}(\lambda))^{-1}\|_{\mathcal{L}(X_q(\mathbb{R}^n_+))} \leq 2$. If we define v and ρ by $v = \mathcal{U}_{\mathbb{R}^n_+}(\lambda)(I + \mathcal{R}(\lambda))^{-1}F$ and $\rho = \mathcal{P}_{\mathbb{R}^n_+}(\lambda)(I + \mathcal{R}(\lambda))^{-1}F$, where $F = (f, \alpha g)$, then (v, ρ) is a unique solution to (4.19) under (4.20). Moreover we have

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+),L_q(\Omega))}(\{(\tau\partial_\tau)^\ell(1 + \mathcal{R}(\lambda))^{-1} \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq 2 \qquad (\ell = 0, 1),$$

which implies

$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+),L_q(\Omega))}(\{(\tau\partial_\tau)^\ell G_{\lambda,\alpha}\mathcal{U}_{\mathbb{R}^n_+}(\lambda)(I + \mathcal{R}(\lambda))^{-1} \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq 2\kappa_0, \qquad (\ell = 0, 1),$$
$$\mathcal{R}_{\mathcal{L}(X_q(\mathbb{R}^n_+),L_q(\Omega))}(\{(\tau\partial_\tau)^\ell\nabla\mathcal{P}_{\mathbb{R}^n_+}(\lambda)(I + \mathcal{R}(\lambda))^{-1} \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq 2\kappa_0, \qquad (\ell = 0, 1).$$

By the change of variable $y = \Phi(x)$ transfer (RSa$'$) under (Cg) in the half-sapce case into the bent half-sapce case, we see that $u_\alpha(y) = {}^T\mathcal{A}_{-1}(v(\Phi^{-1}(y)))$ and $\pi_\alpha = {}^T\mathcal{A}_{-1}(\rho(\Phi^{-1}(y)))$ is a unique solution to (RSa$'$) under (Cg) in the bent half-space and we construct an $\mathcal{R}$-bounded solution operator. This completes the proof of Theorem 4.5. □

By using the cut-off technique with Theorem 4.5, we shall prove Theorem 2.11.

Proof of Theorem 2.11. We set $\mathcal{H}^1_j = \Phi^1_j(\mathbb{R}^n_+)$, $\partial\mathcal{H}^1_j = \Phi^1_j(\partial\mathbb{R}^n_+)$ and $\mathcal{H}^2_j = \mathbb{R}^n$ and set ξ^i_j as the cut-off function enjoys $0 \leq \xi^i_j \leq 1$ and $\mathrm{supp}\xi^i_j \subset B_{d^i}(x^i_j) = \{x \in \Omega \mid |x - x^i_j| < d^i\}$. Let $f, g \in L_q(\Omega)$. We consider the two equations

$$\begin{cases} \lambda u^1_j - \Delta u^1_j + \nabla\pi^1_j = \xi^1_j f & x \in \mathcal{H}^1_j, \\ u_j = 0 & x \in \partial\mathcal{H}^1_j \end{cases} \tag{4.23}$$

under

$$(u^1_j, \nabla\varphi)_{\mathcal{H}^1_j} = \alpha^{-1}(\nabla\pi^1_j, \nabla\varphi)_{\mathcal{H}^1_j} + (\xi^1_j g, \nabla\varphi)_{\mathcal{H}^1_j} \qquad \varphi \in \widehat{W}^1_q(\mathcal{H}^1_j) \tag{4.24}$$

and

$$\lambda u^2_j - \Delta u^2_j + \nabla\pi^2_j = \xi^2_j f \qquad x \in \mathcal{H}^2_j \tag{4.25}$$

under

$$(u^2_j, \nabla\varphi)_{\mathcal{H}^2_j} = \alpha^{-1}(\nabla\pi^2_j, \nabla\varphi)_{\mathcal{H}^2_j} + (\xi^2_j g, \nabla\varphi)_{\mathcal{H}^2_j} \qquad \varphi \in \widehat{W}^1_q(\mathcal{H}^2_j). \tag{4.26}$$

By Theorem 4.1 and Theorem 4.5, there exist operator families
$(\mathcal{U}^k_j(\lambda), \mathcal{P}^k_j(\lambda))$ $(k = 1, 2)$ with

$$\mathcal{U}^k_j(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\mathcal{H}^k_j), W^2_q(\mathcal{H}^k_j))),$$
$$\mathcal{P}^k_j(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\mathcal{H}^k_j), \widehat{W}^1_q(\mathcal{H}^k_j)))$$

such that $(u^k_j, \pi^k_j) = (\mathcal{U}^k_j(\lambda)(\xi^k_j f, \alpha\xi^k_j g), \mathcal{P}^k_i(\lambda)(\xi^k_j f, \alpha\xi^k_j g))$ uniquely solves the problem (4.23) under (4.24) and the problem (4.25) under (4.26), respectively. Moreover we see

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(X_q(\mathcal{H}^k_j),L_q(\mathcal{H}^k_j))}(\{(\tau\partial_\tau)^\ell G_{\lambda,\alpha}\mathcal{U}^k_j(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}) \leq \kappa_2, \\ \mathcal{R}_{\mathcal{L}(X_q(\mathcal{H}^k_j),L_q(\mathcal{H}^k_j))}(\{(\tau\partial_\tau)^\ell\nabla\mathcal{P}^k_j(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}) \leq \kappa_2 \end{aligned} \tag{4.27}$$

with some constant κ_2 independent of $j \in \mathbb{N}$. By (4.27), we obtain

$$\|(\lambda u^k_j, \lambda^{1/2}\nabla u^k_j, \nabla^2 u^k_j, (\lambda + \alpha)^{1/2}\nabla\cdot u^k_j, \nabla\pi^k_j)\|_{L_q(\mathcal{H}^k_j)} \leq \kappa_2\|(\xi^k_j f, \alpha\xi^k_j g)\|_{L_q(\mathcal{H}^k_j)}.$$

For $f, g \in L_q(\Omega)$, we set

$$U(\lambda)(f, \alpha g) = \sum_{j=1}^{\infty} \xi_j^1 u_j^1 + \sum_{j=1}^{\infty} \xi_j^2 u_j^2, \qquad P(\lambda)(f, \alpha g) = \sum_{j=1}^{\infty} \xi_j^1 \pi_j^1 + \sum_{j=1}^{\infty} \xi_j^2 \pi_j^2.$$

Inserting $(v, \pi) = (U(\lambda)(f, \alpha g), P(\lambda)(f, \alpha g))$ into (RSa$'$) and (Cg), we have

$$\begin{cases} \lambda v - \Delta v + \nabla \pi = f - \mathcal{V}^1(\lambda)(f, \alpha g), & x \in \Omega, \\ v = 0 & x \in \partial\Omega \end{cases}$$

under

$$(v, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi, \nabla\varphi)_\Omega + (g, \nabla\varphi)_\Omega + (\mathcal{V}^2(\lambda)(f, g), \nabla\varphi)_\Omega$$

with

$$\begin{aligned} \mathcal{V}^1(\lambda)(f, \alpha g) = \sum_{j=1}^{\infty} \big\{ & 2(\nabla\xi_j^1) \cdot (\nabla u_j^1) + (\Delta\xi_j^1) u_j^1 - (\nabla\xi_j^1)\pi_j^1 \\ & + 2(\nabla\xi_j^2) \cdot (\nabla u_j^2) + (\Delta\xi_j^2) u_j^2 - (\nabla\xi_j^2)\pi_j^2 \big\}, \\ \mathcal{V}^2(\lambda)(f, \alpha g) = \alpha^{-1} & \sum_{j=1}^{\infty} \big\{ (\nabla\xi_j^1)\pi_j^1 + (\nabla\xi_j^2)\pi_j^2 \big\}. \end{aligned}$$

Since by Poincare inequality we obtain

$$\|(\nabla\xi_j^k)\pi_j^k\|_{L_q(\Omega)} \le C\|\nabla\pi_j^k\|_{L_q(\Omega)} \le C\alpha\|u\|_{L_q(\Omega)}$$

and $\pi = \alpha Q_\Omega u$, we have $\mathcal{V}^1(\lambda)(f, \alpha g), \mathcal{V}^2(\lambda)(f, \alpha g) \in L_q(\Omega)$ and

$$\|(\mathcal{V}^1(\lambda)(f, \alpha g), \alpha\mathcal{V}^2(\lambda)(f, \alpha g))\|_{L_q(\Omega)} \le C\lambda_0^{-1/2}(1 + \lambda_0^{-1/2} + \alpha\lambda_0^{-1/2})\|(f, \alpha g)\|_{L_q(\Omega)}.$$

Choosing $\lambda_0 \ge 1$ so large that $C\lambda_0^{-1/2}(1 + \lambda_0^{-1/2} + \alpha\lambda_0^{-1/2}) \le 1/2$ and setting $V(\lambda)F = (\mathcal{V}^1(\lambda)F, \mathcal{V}^2(\lambda)F)$, where $F = (f, \alpha g)$, we see that $(I - V(\lambda))^{-1} \in \mathcal{L}(X_q(\Omega))$ exists and $(u, \pi) = (U(\lambda)(I - V(\lambda))^{-1}F, P(\lambda)(I - V(\lambda))^{-1}F)$ is a unique solution to problem (RSa$'$) under (Cg).

Finally we shall show the $\mathcal{R}$-boundedness of solution operator. Let

$$\begin{aligned} \mathbb{U}(\lambda)F &= \sum_{j=1}^{\infty} \xi_j^1 \mathcal{U}_j^1(\lambda)F + \sum_{j=1}^{\infty} \xi_j^2 \mathcal{U}_j^2(\lambda)F, \\ \mathbb{P}(\lambda)F &= \sum_{j=1}^{\infty} \xi_j^1 \mathcal{P}_j^1(\lambda)F + \sum_{j=1}^{\infty} \xi_j^2 \mathcal{P}_j^2(\lambda)F \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}^1(\lambda)F = \sum_{j=1}^{\infty} \big\{ & 2(\nabla\xi_j^1) \cdot (\nabla\mathcal{U}_j^1(\lambda)F) + (\Delta\xi_j^1)\mathcal{U}_j^1(\lambda)F - (\nabla\xi_j^1)\mathcal{P}_j^1(\lambda)F \\ & + 2(\nabla\xi_j^2) \cdot (\nabla\mathcal{U}_j^2(\lambda)F) + (\Delta\xi_j^2)\mathcal{U}_j^2(\lambda)F - (\nabla\xi_j^2)\mathcal{P}_j^2(\lambda)F \big\}, \\ \mathcal{V}^2(\lambda)(f, \alpha g) = \alpha^{-1} & \sum_{j=1}^{\infty} \big\{ (\nabla\xi_j^1)\mathcal{P}_j^1(\lambda)F + (\nabla\xi_j^2)\mathcal{P}_j^2(\lambda)F \big\}, \end{aligned}$$

where $F = (f, \alpha g)$. We see that $\mathbb{U}(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), W_q^2(\Omega)))$ and $\mathbb{P}(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(X_q(\Omega), \widehat{W}_q^1(\Omega)))$ and $(v, \pi) = (\mathbb{U}(\lambda)F, \mathbb{P}(\lambda)F)$, where $F = (f, \alpha g)$ satisfies

$$\begin{cases} \lambda v - \Delta v + \nabla \pi = f - \mathcal{V}^1(\lambda)(f, \alpha g) & x \in \Omega, \\ v = 0 & x \in \partial\Omega \end{cases}$$

under

$$(v, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi, \nabla\varphi)_\Omega + (g, \nabla\varphi)_\Omega + (\mathcal{V}^2(\lambda)(f, g), \nabla\varphi)_\Omega.$$

Since

$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^n)}(\{(\tau\partial_\tau)^\ell \mathcal{V}^1(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le C\lambda_0^{-1/2}(1 + \lambda_0^{-1/2} + \alpha\lambda_0^{-1/2})\kappa_2,$$
$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^n)}(\{(\tau\partial_\tau)^\ell \alpha\mathcal{V}^2(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le C\lambda_0^{-1/2}(1 + \lambda_0^{-1/2} + \alpha\lambda_0^{-1/2})\kappa_2,$$

Choosing $\lambda_0 \ge 1$ so large that $C\lambda_0^{-/2}(1 + \lambda_0^{-1/2} + \alpha\lambda_0^{-1/2})\kappa_2 \le 1/2$, we have

$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell (I - V(\lambda))^{-1} \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le 2.$$

Therefore we obtain

$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell G_{\lambda,\alpha}\mathbb{U}(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le C,$$
$$\mathcal{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell \nabla\mathbb{P}(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le C.$$

We see that $U(\lambda)(I - V(\lambda))^{-1}$ is a required $\mathcal{R}$-bounded solution operator to (RSa′) under (Cg). This completes the proof of Theorem 2.11. □

5 Application of Maximal Regularity

In this section, using the maximal regularity theorem, we shall prove the local in time existence theorem of strong sokution for (NSa) and (NSE) (Theorem 2.1 and Theorem 2.16) by the method due to Shibata-Kubo [24]. Moreover, we shall prove the L_p-L_q regularity in bounded domain $\Omega \subset \mathbb{R}^n$ and the uniqueness of solution for (WS) in case of $n = 2$.

5.1 Proof of Theorem 2.1

Setting $u^* = T_\alpha(t)a_\alpha$ and $\pi^* = \alpha Q_\Omega u_\alpha$, by Theorem 2.13 and (2.5), (u^*, π^*) is the solution to (Sa′) under (Cg) and satisfies

$$\|e^{-\lambda_0 t}(\partial_t u^*, \nabla^2 u^*, \nabla\pi^*)\|_{L_p((0,\infty), L_q(\Omega))} \le C_{n,p,q}\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le CM, \tag{5.1}$$

where $1 < p, q < \infty$ and λ_0 is a positive number obtained in Theorem 2.11. Setting $v_\alpha = u_\alpha - u^*$, and $\rho_\alpha = \pi_\alpha - \pi^*$, we see that what (u_α, π_α) is the solution to (NSa′) under (Cg) is equivalent to what (v_α, ρ_α) is the solution to

$$\left\{ \begin{aligned} \partial_t v_\alpha - \Delta v_\alpha + \nabla\rho_\alpha = f - N_1(v_\alpha) - N_2(u^*) \quad & t \in (0, T), x \in \Omega, \\ v_\alpha(0, x) = 0 \quad & x \in \Omega, \\ v_\alpha(t, x) = 0 \quad & t \in (0, T),\ x \in \partial\Omega \end{aligned} \right. \tag{NSv}$$

under the approximated weak incompressible condition (C), where

$$N_1(v_\alpha, u^*) = (v_\alpha \cdot \nabla)v_\alpha + (u^* \cdot \nabla)v_\alpha + (v_\alpha \cdot \nabla)u^*, \qquad N_2(u^*) = (u^* \cdot \nabla)u^*.$$

In order to prove Theorem 2.1, we consider (NSv) under (C). For this purpose, we set

$$\begin{aligned}\langle (w,\tau)\rangle_T &= \|\partial_t w\|_{L_p((0,T),L_q(\Omega))} + \|\nabla^2 w\|_{L_p((0,T),L_q(\Omega))} + \|\nabla\tau\|_{L_p((0,T),L_q(\Omega))} \\ &\quad + \|w\|_{L_\infty((0,T),L_q(\Omega))} + \|\nabla w\|_{L_{r_1}((0,T),L_q(\Omega))} + \|\nabla w\|_{L_{r_2}((0,T),L_q(\Omega))} \end{aligned} \tag{5.2}$$

with r_1, r_2 is defined by (3.12). By (2.1), (3.8), (3.9) and (3.11), we have

$$\langle M_{T^*}(f))\rangle_{T^*} \leq C_{n,p,q} e^{\lambda_0 T^*} \|f\|_{L_p((0,T^*),L_q(\Omega))} \leq C_{n,p,q} e^{\lambda_0 T^*} M. \tag{5.3}$$

Set $L = C_{n,p,q} e^{\lambda_0 T^*} M$. To prove Theorem 2.1 by contraction mapping principle, we shall define the underlying space $X_{T,L}$ as follows:

$$\begin{aligned} X_{T,L} = \{(w,\tau) \in W^1_p((0,T), L_q(\Omega)^n) \cap L_p((0,T), W^2_q(\Omega)^n)) \\ \times L_p((0,T), \widehat{W}^1_q(\Omega)) \mid w|_{t=0} = 0,\ \langle (w,\tau)\rangle_T \leq 2L\}. \end{aligned} \tag{5.4}$$

Here the constant T is determined later as the sufficiently small constant. We define the map Φ as

$$\Phi(w,\theta) = M_T(f) - M_T(N_1(v_\alpha,u^*)) - M_T(N_2(u^*)),$$

where M_T is the solution operator to (3.7) under (C). We shall prove that Φ is the contraction mapping on $X_{T,L}$. By (3.13) and (5.1) we have

$$\|N_2(u^*)\|_{L_p((0,S),L_q(\Omega))} \leq \|(u^*\cdot\nabla)u^*\|_{L_p((0,S),L_q(\Omega))} \leq C S^\gamma e^{2\lambda_0 S} M^2$$

for $1 < p \leq \infty$ and $n/2 < q < \infty$. By (3.8) the following inequality holds:

$$\langle M_{T^*}(N_2(u^*))\rangle_{T^*} \leq C_{n,p,q} e^{2\lambda_0 T^*} \|N_2(u^*)\|_{L_p((0,T^*),L_q(\Omega))} \leq C_{n,p,q} (T^*)^\gamma e^{2\lambda_0 T^*} M^2 \tag{5.5}$$

for $0 < T^* \leq T_0$. In a similar way, for $(v_\alpha,\rho_\alpha) \in X_{T^*,L}$ we obtain

$$\|N_1(v_\alpha,u^*)\|_{L_p((0,S),L_q(\Omega))} \leq C e^{\lambda_0 T^*} S^\gamma M L,$$

which implies

$$\langle M_{T^*}(N_1(v_\alpha,u^*))\rangle_{T^*} \leq C_{n,p,q} \|N_1(v_\alpha,u^*)\|_{L_p((0,T^*),L_q(\Omega))} \leq C(T^*)^\gamma e^{\lambda_0 T^*} M L. \tag{5.6}$$

Therefore there exists a constant $C = C_{n,p,q,T_0}$ such that

$$\langle \Phi(v_\alpha,\rho_\alpha)\rangle_{T^*} \leq L + C(T^*)^\gamma \left(e^{2\lambda_0 T^*} M^2 + e^{\lambda_0 T^*} M L\right)$$

for $(v_\alpha,\rho_\alpha) \in X_{T^*}$. Taking the time $T^* (\leq T_0)$ sufficiently small such that $C(T^*)^\gamma e^{\lambda_0 T^*} M \leq 1/2$ and $C(T^*)^\gamma e^{2\lambda_0 T^*} M^2 \leq L/2$, we have $\langle \Phi(w,\tau)\rangle_{T^*} \leq 2L$. Therefore, Φ is the mapping on $X_{T^*,L}$. Moreover taking into account the facts:

$$\Phi(w_1,\tau_1) - \Phi(w_2,\tau_2) = M_{T^*}(N_1(w_2,u^*) - N_1(w_1,u^*))$$

and

$$N_1(w_2,u^*) - N_1(w_1,u^*) = ((w_2 - w_1)\cdot\nabla)u^* + (u^*\cdot\nabla)(w_2 - w_1)$$

for $(w_i,\tau_i) \in X_{T^*,L}$ $(i = 1,2)$, by (3.13), (5.1) and (5.4), we can show the following inequality holds:

$$\|N_1(w_2) - N_1(w_1)\|_{L_p((0,T^*),L_q)} \leq C_{n,p,q,T_0} (T^*)^\gamma e^{\lambda_0 T^*} M \langle (w_2,\tau_2) - (w_1,\tau_1)\rangle_{T^*},$$

which implies

$$\langle \Phi(w_1,\tau_1) - \Phi(w_2,\tau_2)\rangle_{T^*} \le C_{n,p,q,T_0}(T^*)^\gamma e^{\lambda_0 T^*} M \langle (w_2,\tau_2) - (w_1,\tau_1)\rangle_{T^*}.$$

Taking T^* sufficiently small such that $C(T^*)^\gamma e^{\lambda_0 T^*} M \le 1/2$ if necessary, we obtain

$$\langle \Phi(w_1,\tau_1) - \Phi(w_2,\tau_2)\rangle_{T^*} \le (1/2)\langle (w_1,\tau_1) - (w_2,\tau_2)\rangle_{T^*}.$$

Therefore, we see that Φ is the contraction mapping on X_{T^*}. By the contraction mapping principle, we see that Φ has fixed point (v_α,ρ_α). Satisfying $\Phi(v_\alpha,\rho_\alpha) = (v_\alpha,\rho_\alpha)$, by (5.5), we see that $(u_\alpha,\pi_\alpha) = (u^* + v_\alpha, \pi^* + \rho_\alpha)$ is the unique solution for (NSa$'$) under (C). Therefore we obtain Theorem 2.1.

5.2 Proof of Theorem 2.16

Let (u^*,π^*) be a solution to (Sa$'$) with $f = g = 0$ and $a_\alpha = a_E$. By Theorem 2.13, the following estimates hold.

$$\|e^{-\lambda_0 t}(\partial_t u^*, \nabla^2 u^*, \nabla \pi^*)\|_{L_p((0,\infty),L_q(\Omega))} \le C_{n,p,q}\|a_E\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le CM\alpha^{-1}, \tag{5.7}$$

where $1 < p,q < \infty$. In order to look for the solution (v_α,ρ_α) of (NSE) as $v_\alpha = u_E - u^*$ and $\rho_\alpha = \pi_E - \pi^*$, we shall obtain the solution to

$$\begin{cases} \partial_t v_\alpha - \Delta v_\alpha + \nabla \rho_\alpha = -N_1(v_\alpha,u^*) - N_2(u^*,u_\alpha) & t \in (0,\infty), x \in \Omega, \\ v_\alpha(0,x) = 0 & x \in \Omega, \\ v_\alpha(t,x) = 0, & x \in \partial\Omega, \end{cases} \tag{NSvE}$$

under the approximated weak incompressible condition (C$_\pi$), where

$$\begin{aligned} N_1(v_\alpha,u^*) &= (v_\alpha\cdot\nabla)v_\alpha + ((u^*+u_\alpha)\cdot\nabla)v_\alpha + (v_\alpha\cdot\nabla)(u^*+u_\alpha), \\ N_2(u^*,u_\alpha) &= (u^*\cdot\nabla)(u^*+u_\alpha) + (u_\alpha\cdot\nabla)u^*. \end{aligned}$$

In a similar way to Theorem 2.1, we shall define underlying space X_{T,L_E} as follows:

$$\begin{aligned} X_{T,L_E} = \{(w,\tau) \in (W^1_p((0,T),L_q(\Omega)^n) \cap L_p((0,T),W^2_q(\Omega)^n)) \\ \times L_p((0,T),\widehat{W}^1_q(\Omega)) \mid w|_{t=0} = 0,\ \alpha\langle (w,\tau)\rangle_T \le L_E\}, \end{aligned} \tag{5.8}$$

where $\langle (w,\tau)\rangle_T$ is defined in (5.2). Setting the map Φ defined by

$$\Phi(w,\theta) = -M_{T^*}(N_1(v_\alpha,u^*)) - M_{T^*}(N_2(u^*,u_\alpha)),$$

where $M_T(f)$ is a solution operator to (3.7) under (C$_\pi$), we shall estimate $N_1(v_\alpha,u^*)$ and $N_2(u^*,u_\alpha)$ in a similar way to Theorem 2.1. Setting $\beta, \ell_k (k = 1,2,3), \gamma, r_i (i = 1,2)$ as the same positive constant in proof of Theorem 2.1, we see

$$\|N_1(v_\alpha,u^*)\|_{L_p((0,S),L_q(\Omega))} \le \frac{CS^\gamma}{\alpha}\left(\frac{1}{\alpha}L_E^2 + \frac{1}{\alpha}e^{\lambda_0 T^*}ML_E + LL_E\right)$$

and

$$\|N_2(u^*,u_\alpha)\|_{L_p((0,S),L_q(\Omega))} \le C\frac{S^\gamma}{\alpha}\left(\frac{1}{\alpha}e^{2\lambda_0 T^*}M^2 + e^{\lambda_0 T^*}ML\right)$$

for $1 < p < \infty$, by (2.6), (3.8) for $0 < T^\flat \le T^*$, the following inequality holds:

$$\alpha\langle M_{T^\flat}(N_1(v_\alpha,u^*) + N_2(u^*,u_\alpha))\rangle_{T^\flat} \le C_{n,p,q,M,L,L_E}(T^\flat)^\gamma.$$

In a similar way to Theorem 2.1, taking $T^\flat$ sufficiently small if necessary, we can prove that Φ is the contraction mapping on $X_{T^\flat,L_E}$. Therefore we obtain Theorem 2.16.

5.3 Proof of Theorem 2.3

In this subsection, we shall prove Theorem 2.3 by using the method due to Saal [20]. Before proving main results, we shall prepare a key lemma. For each $k \in \mathbb{N}$, we consider the following approximate system :

$$\partial_t v + A_\alpha v + (J_k^N Pv \cdot \nabla)v = J_k f := f_k, \quad v(0) = J_k a_\alpha := a_{\alpha,k}. \tag{5.9}$$

By using the fact that A_α is generator of semigroup $\{T_\alpha(t)\}_{t\geq 0}$ on $L_q(\Omega)$ (see Kubo and Matsui [13]), we shall show there exists a fixed point for the following integral equation :

$$\Phi v(t) = T_\alpha(t)a_{\alpha,k} - \int_0^t T_\alpha(t-s)(J_k^N Pv(s)\cdot\nabla)v(s)ds + \int_0^t T_\alpha(t-s)f_k(s)ds. \tag{5.10}$$

Hence, we shall prove the following lemma.

Lemma 5.1. *Let $n \geq 2$, $k \in \mathbb{N}$, $T \in (0,\infty)$ and $X_2 = D(A_\alpha^{1/2}) = H_0^1(\Omega)$. And let $a_\alpha \in L_2(\Omega)$ and $f \in L_2((0,T), L_2(\Omega))$. If N satisfies $N > 1 + n/4$, then (5.10) has a unique solution*

$$v \in C([0,T], X_2) \cap L_2((0,T), D(A_\alpha)) \cap H^1((0,T), L_2(\Omega))$$

enjoying (5.9).

Proof. We fix $k \in \mathbb{N}$. By the definition, we see

$$a_{\alpha,k} \in X_2, \qquad\qquad f_k \in L_2((0,T), L_2(\Omega)) \cap L_2((0,T), X_2).$$

In fact, by (3.16), we obtain

$$\begin{aligned} \|A_\alpha^{1/2} a_{\alpha,k}\|_2^2 &= \|\nabla a_{\alpha,k}\|_2^2 + \alpha\|\nabla Q a_{\alpha,k}\|_2^2 \leq C_k(1+\alpha)\|a_\alpha\|_2^2, \\ \|f_k\|_2 = \|J_k f\|_2 &\leq C\|f\|_2, \qquad \|A_\alpha^{1/2} f_k\|_2 \leq C_k\|f\|_2. \end{aligned} \tag{5.11}$$

In order to get the fixed point for integral equation, choosing $M > 0$ as suitable, we define a function space and its norm as follows:

$$\begin{aligned} B_M &:= \{v \in C([0,T], X_2) \mid v(0) = a_{\alpha,k}, \|v\|_T \leq M\}, \\ \|v\|_T &:= \sup_{t\in[0,T]} (\|v(t)\|_2 + \|A_\alpha^{1/2} v(t)\|_2). \end{aligned}$$

By (3.17) and (3.18), we have

$$\begin{aligned} \|(J_k^N Pv(s)\cdot\nabla)v(s)\|_2 &\leq \|J_k^N Pv(s)\|_\infty \|\nabla v(s)\|_2 \\ &\leq C_k \|Pv(s)\|_2 \|A_\alpha^{1/2} v(s)\|_2 \\ &\leq C_k M^2. \end{aligned} \tag{5.12}$$

Therefore we obtain

$$\|\Phi v\|_T \leq C_k^3 \left(\|a_{\alpha,k}\|_2 + \|f\|_{2,2,T} T^{1/2} + M^2(T + T^{1/2})\right) \tag{5.13}$$

for $v \in B_M$. In fact, by (5.11) we get for $0 < t < T$

$$\begin{aligned} \|\Phi v(t)\|_2 &\leq \|a_{\alpha,k}\|_2 + C\int_0^t \|(J_k^N Pv(s)\cdot\nabla)v(s)\|_2 ds + C\int_0^t \|f_k(s)\|_2 ds \\ &\leq \|a_{\alpha,k}\|_2 + C_k M^2 T + CT^{1/2}\|f\|_{2,2,T}. \end{aligned} \tag{5.14}$$

Similarly, by (5.11) and the L_p-L_q estimate of the gradient of approximate Stokes semigroup proved by Kubo and Matsui[13]:

$$\|\nabla T_\alpha(t)a\|_p \le C_{n,p,q}t^{-1/2-n/2(1/q-1/p)}\|a\|_q$$

for $1 < q \le p < \infty$, $t > 0$ and $a \in L_q(\Omega)$, we can prove the estimate of $\|A_\alpha^{1/2}\Phi v(t)\|_2$ and we obtain (5.13).

Moreover, because of the inequality

$$\begin{aligned}
&\|(J_k^N Pv(s)\cdot\nabla)v(s) - (J_k^N Pw(s)\cdot\nabla)w(s)\|_2 \\
&\le C_k(\|v(s)-w(s)\|_2\|\nabla v(s)\|_2 + \|\nabla(v(s)-w(s))\|_2\|w(s)\|_2) \\
&\le C_k M\|v-w\|_T
\end{aligned}$$

for $v, w \in B_M$, we obtain

$$\|\Phi v - \Phi w\|_T \le C_k^4 M(T+T^{1/2})\|v-w\|_T.$$

Therefore letting M satisfy $C_k^3\|a_{\alpha,k}\|_2 \le M/2$ and T satisfy two inequalities

$$\begin{aligned}
&C_k^3 T^{1/2}\|f\|_{2,2,T} + C_k^3 M^2(T+T^{1/2}) \le M/2, \\
&C_k^4 M(T+T^{1/2}) \le 1/2,
\end{aligned}$$

we see that Φ is a contraction map on B_M. In other words, for sufficiently small T, there exists fixed point u_α of the map Φ on B_M. Since

$$\int_0^T \|(J_k^N Pv(s)\cdot\nabla)v(s)\|_2^2 ds \le C_k\int_0^T \|v(s)\|_2^2\|\nabla v(s)\|_2^2 ds \le C_k M^4 T < \infty,$$

we see $(J_k^N Pv\cdot\nabla)v \in L_2((0,T), L_2(\Omega))$. Therefore by L_p-L_q maximal regularity of A_α, we see $u_\alpha \in L_2((0,T), D(A_\alpha)) \cap H^1((0,T), L_2(\Omega))$, which implies Lemma 5.1 for sufficiently small T.

Next, we shall prove that there exists a global unique solution. For this purpose, we consider the boundedness of $\|v\|_T$. Since by $\mathrm{div} J_k Pv = 0$ we see

$$\begin{aligned}
((J_k Pv\cdot\nabla)v, v)_\Omega &= -(v, (\mathrm{div} J_k Pv)v)_\Omega - (v, (J_k Pv\cdot\nabla)v)_\Omega \\
&= -(v, (J_k Pv\cdot\nabla)v)_\Omega,
\end{aligned}$$

$((J_k Pv\cdot\nabla)v, v) = 0$ holds. Multiplying v to (5.9) and integrating on Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 + \|A_\alpha^{1/2}v(t)\|_2^2 = (f_k(t), v(t))_\Omega. \tag{5.15}$$

Integrating from 0 to t, we see

$$\frac{1}{2}\|v(t)\|_2^2 - \frac{1}{2}\|a_{\alpha,k}\|_2^2 + \int_0^t \|A_\alpha^{1/2}v(s)\|_2^2 ds = \int_0^t (f_k(s), v(s))_\Omega ds. \tag{5.16}$$

Then, we shall estimate the right hand side of (5.16). Using Poincaré inequality and Hölder inequality, then we can estimate

$$\begin{aligned}
\left|\int_0^t (f_k(s), v(s))_\Omega ds\right| &\le \int_0^t \|f_k(s)\|_2\|v(s)\|_2 ds \\
&\le C\|f\|_{2,2,T}\|v\|_{2,2,T} \\
&\le C\|f\|_{2,2,T}\|A_\alpha^{1/2}v\|_{2,2,T} \\
&\le \frac{C}{2\varepsilon}\|f\|_{2,2,T}^2 + \frac{\varepsilon}{2}\|A_\alpha^{1/2}v\|_{2,2,T}^2.
\end{aligned}$$

Therefore, letting $\varepsilon = 1$, we can obtain the estimate

$$\|v(t)\|_2^2 + \|A_\alpha^{1/2} v\|_{2,2,T}^2 \le \|a_{\alpha,k}\|_2^2 + C\|f\|_{2,2,T}^2 \le C(\|a_\alpha\|_2^2 + \|f\|_{2,2,T}^2), \tag{5.17}$$

which implies the boundedness of $\|v(t)\|_2^2$. Finally, we shall prove the boundedness of $\|A_\alpha^{1/2} v(t)\|_2$. Multiplying $A_\alpha v$ to (5.9) and integrating on Ω, we get

$$\begin{aligned}&(\partial_t v(t), A_\alpha v(t))_\Omega + (A_\alpha v(t), A_\alpha v(t))_\Omega \\ &\qquad = (f_k(t), A_\alpha v(t))_\Omega - ((J_k^N P v(t) \cdot \nabla) v(t), A_\alpha v(t))_\Omega.\end{aligned}$$

By integration by parts, we can also obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A_\alpha^{1/2} v(t)\|_2^2 + \|A_\alpha v(t)\|_2^2 \\ &\qquad = (f_k(t), A_\alpha v(t))_\Omega - ((J_k^N P v(t) \cdot \nabla) v(t), A_\alpha v(t))_\Omega.\end{aligned}$$

Since the inequality:

$$\begin{aligned}|((J_k^N P v(t) \cdot \nabla) v(t), A_\alpha v(t))_\Omega| &\le \|(J_k^N P v(t) \cdot \nabla) v(t)\|_2 \|A_\alpha v(t)\|_2 \\ &\le \frac{C_k}{2}\|v(t)\|_2^2 \|A_\alpha^{1/2} v(t)\|_2^2 + \frac{1}{2}\|A_\alpha v(t)\|_2^2,\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A_\alpha^{1/2} v(t)\|_2^2 + \|A_\alpha v(t)\|_2^2 \\ &\le \frac{1}{2}\left(C_k\|v(t)\|_2^2 \|A_\alpha^{1/2} v(t)\|_2^2 + \|f_k(t)\|_2^2\right) + \|A_\alpha v(t)\|_2^2,\end{aligned}$$

which implies

$$\frac{d}{dt}\|A_\alpha^{1/2} v(t)\|_2^2 \le C_k \|v(t)\|_2^2 \|A_\alpha^{1/2} v(t)\|_2^2 + C\|f(t)\|_2^2. \tag{5.18}$$

Inserting (5.17) into (5.18), we have

$$\frac{d}{dt}\|A_\alpha^{1/2} v(t)\|_2^2 \le C\|f(t)\|_2^2 + C_k(\|a_\alpha\|_2^2 + \|f\|_{2,2,T}^2)\|A_\alpha^{1/2} v(t)\|_2^2.$$

Integrating this expression from 0 to t, we get

$$\begin{aligned}&\|A_\alpha^{1/2} v(t)\|_2^2 \\ &\le (\|A_\alpha^{1/2} a_\alpha\|_2^2 + C\|f\|_{2,2,T}^2) + C_k(\|a_{\alpha,k}\|_2^2 + \|f\|_{2,2,T}^2)\int_0^t \|A_\alpha^{1/2} v(s)\|_2^2 ds.\end{aligned}$$

By Gronwall's inequality, we conclude

$$\|A_\alpha^{1/2} v(t)\|_2^2 \le C_k(\|a_\alpha\|_2^2 + \|f\|_{2,2,T}^2)\exp\left[C_k(\|a_\alpha\|_2^2 + \|f\|_{2,2,T})T\right]. \tag{5.19}$$

Therefore, v exists uniquely on arbitrary interval $[0, T]$ and that it admits the claimed regularity properties. □

From here, we shall prove our main results about the existence and regularity theorem for (**??**).

Proof of Theorem 2.3. Let $\{u_k\}$ be a solution of (5.9). By Lemma 5.1 u_k is the bounded in Lelay-Hopf's class: $L_\infty((0,T),L_2(\Omega))\cap L_2((0,T),H^1(\Omega))$. Therefore, there exists a weak limit $u_\alpha \in Y := L_\infty((0,T),L_2(\Omega))\cap L_2((0,T),H^1(\Omega))$. Moreover, since u_k satisfies (5.16), u_α satisfies also (5.19). Consequently, we shall prove u_α is a solution to (WS). For this purpose, let v_k be unique solution of the equation :

$$\partial_t v_k + A_\alpha v_k = f_k \quad t\in(0,T), \quad v_k(0) = a_{\alpha,k}. \tag{5.20}$$

Then, v_k converges strongly in Y. In fact, letting v be a solution of

$$\partial_t v + A_\alpha v = f, \quad v(0) = a_\alpha,$$

we see that $v - v_k$ satisfies

$$\partial_t(v-v_k) + A_\alpha(v-v_k) = f - f_k, \quad v(0)-v_k(0) = a_\alpha - a_{\alpha,k}$$

and

$$\begin{aligned}&\|e^{-\lambda_0 t}(v-v_k)\|_{H^1((0,T),L_2(\Omega))} + \|e^{-\lambda_0 t}(v-v_k)\|_{L_2((0,T),H^2(\Omega))}\\ &\le C\|e^{-\lambda_0 t}(f-f_k)\|_{2,2,T} + \|a_\alpha - a_{\alpha,k}\|_{B^{2(1-1/p)}_{q,p}}\end{aligned}$$

by L_p-L_q maximal regularity of A_α. By $L_2((0,T),H^2(\Omega))\cap H^1((0,T),L_2(\Omega))\subset C^0([0,T],H^1)$ (see [**?**]) and the fact that the continuous map from $H^1((0,T),L_2(\Omega))\cap L_2((0,T),H^2(\Omega))$ to $L_2((0,T),H^1(\Omega))$ is compact (see [28]), we see $\|e^{-\lambda_0 t}(v-v_k)\|_Y \to 0$ as $k\to\infty$. Now, letting $w_k = u_k - v_k$, w_k converges weakly in Y and satisfies the equation

$$\partial_t w_k + A_\alpha w_k = -(J^N_k P u_k\cdot\nabla)u_k \quad t\in(0,T), \quad w_k(0)=0. \tag{5.21}$$

Since the right hand side of first equation of (5.21) is bounded in $L_q((0,T),L_q(\Omega))$ for $q = (n+2)/(n+1)$ by Lemma **??** and L_p-L_q maximal regularity of A_α, we see

$$w_k \in W^1_q((0,T),L_q(\Omega))\cap L_q((0,T),D(A_\alpha))$$

for $T>0$. Here, set $\varphi\in C^\infty_0([0,T),C^\infty_0(\Omega))$ as a test function in the equation (WS) and let $\mathrm{supp}\varphi = K$ as a compact set. In particular, considering

$$w_k \in W^1_q((0,T),L_q(K))\cap L_q((0,T),H^2(K)),$$

by Rellich's theorem, $H^1(K)$ embed $L_q(K)$ in the compact and therefore if $N>4$, we can apply Theorem 2.1 in [28]. Namely, the operator from Y to $L_q((0,T),H^1(K))$ is compact. Therefore w_k converges strongly in $L_q((0,T),H^1(K))$. Therefore since $u_k = w_k + v_k$ converges strongly u_α in $L_q((0,T),H^1(K))$, by using integration by parts, we obtain

$$\begin{aligned}&\left|\int_0^T \left\{(B(u_\alpha,u_\alpha),\phi)_\Omega - ((J^N_k P u_k\cdot\nabla)u_k,\phi)_\Omega\right\}dt\right|\\
&\le \int_0^T \|\nabla(u_k-u_\alpha)\|_{2,K}\|(Pu_k)\phi\|_{2,K}dt + \int_0^T \|u_\alpha\|_{2,K}\|\nabla((Pu_\alpha - J^N_k Pu_k)\phi)\|_{2,K}dt\\
&\le T^{1/q'}\|u_k\|_{\infty,2,T}\|\phi\|_{\infty,\infty,T}\|\nabla(u_k-u_\alpha)\|_{q,2,T}\\
&\quad + T^{1/q'}\|u_\alpha\|_{\infty,2,T}\left(\|\nabla(Pu_\alpha - Pu_k)\|_{q,2,T} + \|\nabla(Pu_k - J^N_k Pu_k)\|_{q,2,T}\right)\|\phi\|_{\infty,\infty,T}\\
&\qquad + T^{1/q'}\|u_\alpha\|_{\infty,2,T}\left(\|Pu_\alpha - Pu_k\|_{q,2,T} + \|Pu_k - J^N_k Pu_k\|_{q,2,T}\right)\|\nabla\phi\|_{\infty,\infty,T}.\end{aligned}$$

By Remark 2.1, the last term of this inequality converges to 0 if $k\to\infty$. Therefore, we see u_α satisfies (WS). □

Proof of Theorem 2.4. If u_α is one of weak solutions to (**??**), by Lemma **??**, we get

$$\|(Pu_\alpha\cdot\nabla)u_\alpha\|_{p,q,T}\le C(\|u_\alpha\|_{\infty,2,T}^2+\|A_\alpha^{1/2}u_\alpha\|_{2,2,T}^2)<\infty$$

for $2/p+n/q=n+1$. Here, letting $F=f-(Pu_\alpha\cdot\nabla)u_\alpha$, we consider the system

$$\partial_t v+A_\alpha v=F,\quad (x\in\Omega,\ t\in(0,T)),\quad v|_{\partial\Omega}=0,\quad v|_{t=0}=a_\alpha. \tag{5.22}$$

By L_p-L_q maximal regularity of A_α, the solution to (5.22) is unique and satisfies

$$\begin{aligned}&\|e^{-\lambda_0 t}\partial_t v\|_{p,q,T}+\|e^{-\lambda_0 t}A_\alpha v\|_{p,q,T}\\&\le C(\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)}+\|f\|_{p,q,T}+\|u_\alpha\|_{\infty,2,T}^2+\|A_\alpha^{1/2}u_\alpha\|_{2,2,T}^2).\end{aligned}$$

Therefore we shall prove $v=u_\alpha$. Since v satisfies (5.22), we have

$$\int_0^T\{-(v(t),\partial_t\phi(t))_\Omega-(v(t),A_\alpha\phi(t))_\Omega\}\,dt=(a_\alpha,\phi(0))_\Omega+\int_0^T(F(t),\phi(t))_\Omega dt \tag{5.23}$$

for any $\phi\in C_0^\infty([0,T),C_0^\infty(\Omega))$. On the other hand, since u_α is the weak solution to (**??**), we have

$$\int_0^T\{-(u_\alpha(t),\partial_t\phi(t))_\Omega-(u_\alpha(t),A_\alpha\phi(t))_\Omega\}\,dt]=(a_\alpha,\phi(0))_\Omega+\int_0^T(F(t),\phi(t))_\Omega dt$$

for any $\phi\in C_0^\infty([0,T),C_0^\infty(\Omega))$. Then, for all $\phi\in C_0^\infty([0,T),C_0^\infty(\Omega))$, we obtain

$$\int_0^T(u_\alpha(t)-v(t),-\partial_t\phi(t)-A_\alpha\phi(t))_\Omega dt=0. \tag{5.24}$$

Let

$$\mathbb{E}_{p',q',T}=\{\phi\in W^1_{p'}((0,T),L_{q'}(\Omega))\cap L_{p'}((0,T),W^2_{q'}(\Omega))\mid \phi|_{t=T}=0\}.$$

Since $C_0^\infty(\Omega)$ is dense in $L_{q'}(\Omega)$, $C_0^\infty([0,T),C_0^\infty(\Omega))$ is dense in $\mathbb{E}_{p',q',T}$. In (5.24), by letting $\phi\in L_{p'}((0,T),L_{q'}(\Omega))$, $\phi_j\in C_0^\infty([0,T),C_0^\infty(\Omega))$ and $\phi_j\to\phi$, (5.24) is hold for any $\phi\in\mathbb{E}_{p',q',T}$. On the other hand, for any $\psi\in C_0^\infty([0,T),C_0^\infty(\Omega))$, dual problem :

$$-\partial_t\phi-A_\alpha\phi=\psi,\quad (x\in\Omega,t\in(0,T)),\quad \phi|_{\partial\Omega}=0,\quad \phi|_{t=T}=0 \tag{5.25}$$

has a unique solution ϕ and we see $\phi\in W^1_r((0,T),L_s(\Omega))\cap L_r((0,T),W^2_s(\Omega)^n)$ for $r,s\in(1,\infty)$. Especially, letting $r=p'$ and $s=q'$, by (5.24), for all $\psi\in C_0^\infty([0,T),C_0^\infty(\Omega))$, we have

$$\int_0^T(u_\alpha(t)-v(t),\psi)_\Omega dt=\int_0^T(u_\alpha(t)-v(t),-\partial_t\phi+A_\alpha\phi)_\Omega dt=0.$$

Therefore, getting

$$\int_0^T(u_\alpha-v,\psi)_\Omega dt=0$$

for all $\psi\in C_0^\infty((0,T),C_0^\infty(\Omega))$ which is dense in $L_{p'}((0,T),L_{q'}(\Omega))$, we obtain

$$\int_0^T(u_\alpha-v,\psi)_\Omega dt=0$$

for all $\psi\in L_{p'}((0,T),L_{q'}(\Omega))$. Then, we see $u_\alpha-v\in L_p((0,T),L_q(\Omega))$and $u_\alpha=v$. With regard to the pressure term $\nabla\pi_\alpha$, by the relation $\nabla\pi_\alpha=\alpha\nabla Qu_\alpha=A_\alpha u_\alpha+\Delta u_\alpha$ (see [13]), we can prove the pressure term $\nabla\pi_\alpha$ satisfies (2.3). □

5.4 Proof of Theorem 2.5

Here we prove the weak solution u_α constructed in Theorem 2.4 is unique if $n = 2$.

Proof. (Proof of Theorem 2.5). Let $T \in (0, \infty]$ and u_k be the approximate sequence constructed in Lemma 5.1. First, we shall prove $\nabla u_\alpha \in L_\infty((0,T), L_2(\Omega))$. For this purpose, we show the constant C_k is independent of k in the inequality (5.19). By the fact that the basic inequality $\|v\|_4 \le C\|\nabla v\|_2^{1/2}\|v\|_2^{1/2}$ holds for $v \in W_2^1(\Omega)$ and Young's inequality, (3.16), (3.18) and (3.19), we can estimate the nonlinear term

$$\begin{aligned}
|((J_k^N Pu_k(t)\cdot\nabla)u_k(t), A_\alpha u_k(t))_\Omega| &\le \|J_k^N Pu_k(t)\|_4\|\nabla u_k(t)\|_4\|A_\alpha u_k(t)\|_2 \\
&\le C\|u_k(t)\|_4\|\nabla u_k(t)\|_4\|A_\alpha u_k(t)\|_2 \\
&\le C\|u_k(t)\|_2^{1/2}\|A_\alpha^{1/2}u_k(t)\|_2\|A_\alpha u_k(t)\|_2^{3/2} \\
&\le \frac{C}{4}\|u_k(t)\|_2^2\|A_\alpha^{1/2}u_k(t)\|_2^4 + \frac{3}{4}\|A_\alpha u_k(t)\|_2^2. \qquad (5.26)
\end{aligned}$$

Hence in the same way as (5.18), we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A_\alpha^{1/2}u_k(t)\|_2^2 + \|A_\alpha u_k(t)\|_2^2 \\
&\le \|f\|_2^2 + \frac{1}{4}\|A_\alpha u_k(t)\|_2^2 + \frac{C}{4}\|u_k\|_2^2\|A_\alpha^{1/2}u_k\|_2^4 + \frac{3}{4}\|A_\alpha u_k\|_2^2 \qquad (5.27)
\end{aligned}$$

for $t \in (0,T)$. Integrating (5.27) and using (5.17) and (3.16), we get

$$\|A_\alpha^{1/2}u_k(t)\|_2^2 \le \|A_\alpha^{1/2}a_\alpha\|_2^2 + 2\|f\|_{2,2,T}^2 + \frac{C}{2}\int_0^t \|u_k\|_2^2\|A_\alpha^{1/2}u_k\|_2^4 dt.$$

Setting $\varphi_k(t) = \|u_k(t)\|_2^2\|A_\alpha^{1/2}u_k(t)\|_2^2$, by (5.17), we have

$$\int_0^t \varphi_k(s)ds \le C\left(\|a_\alpha\|_2^2 + \|f\|_{2,2,T}^2\right)^2.$$

Therefore we see by the Gronwall inequality

$$\|A_\alpha^{1/2}u_k(t)\|_2^2 \le (\|A_\alpha^{1/2}a_\alpha\|_2^2 + 2\|f\|_{2,2,T}^2)\exp\left\{\frac{C}{2}(\|a_\alpha\|_2^2 + \|f\|_{2,2,T}^2)^2\right\},$$

which implies that $A_\alpha^{1/2}u_k \to A_\alpha^{1/2}u$ weakly in $L_\infty((0,T), L_2(\Omega))$.

On the other hands, by the inequality $2ab \le (a^2/\varepsilon) + (\varepsilon b^2)$ for arbitrary $\varepsilon > 0$, we see

$$|(f, A_\alpha u_k)_\Omega| \le 2\|f\|_2^2 + \frac{1}{8}\|A_\alpha u_k\|_2^2.$$

Therefore (5.27) is rewrited

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A_\alpha^{1/2}u_k(t)\|_2^2 + \|A_\alpha u_k\|_2^2 \\
&\le 2\|f\|_2^2 + \frac{1}{8}\|A_\alpha u_k\|_2^2 + \frac{C}{4}\|u_k\|_2^2\|A_\alpha^{1/2}u_k\|_2^2 + \frac{3}{4}\|A_\alpha u_k\|_2^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|A_\alpha^{1/2}u_k\|_{\infty,2,T}^2 + \frac{1}{4}\|A_\alpha u_k\|_{2,2,T}^2 \\
&\le (\|A_\alpha^{1/2}a_\alpha\|_2^2 + 4\|f\|_{2,2,T}^2)\exp\left\{\frac{C}{2}(\|a_\alpha\|_2^2 + \|f\|_{2,2,T}^2)^2\right\}.
\end{aligned}$$

Therefore we see $A_\alpha u_\alpha \in L_2((0,T), L_2(\Omega))$. Furthermore, having $(Pu_\alpha \cdot \nabla)u_\alpha \in L_2((0,T), L_2(\Omega))$ by the estimate

$$\|(Pu_\alpha \cdot \nabla)u_\alpha\|_{2,2,T} \leq (\|u_\alpha\|_{\infty,2,T}\|\nabla u_\alpha\|^2_{\infty,2,T}\|A_\alpha u_\alpha\|_{2,2,T})^{1/2} < \infty,$$

we obtain $\partial_t u_\alpha, \nabla^2 u_\alpha, \nabla \pi_\alpha \in L_2((0,T), L_2(\Omega))$ by the L_p-L_q maximal regularity of A_α.

Next, we shall prove the uniqueness property. Let v be any other weak solution of (**??**). According to Theorem 2.4 and Lemma **??** the difference $w := u_\alpha - v$ satisfies

$$\partial_t w + A_\alpha w + ((Pw \cdot \nabla)u_\alpha) + ((Pv \cdot \nabla)w) = 0 \quad (t \in (0,T)), w(0) = 0. \tag{5.28}$$

in $L_{4/3}((0,T), L_{4/3}(\Omega))$. On the other hand, by $\|w\|_{4,4,T} \leq C\|w\|^{1/2}_{\infty,2,T}\|\nabla w\|^{1/2}_{2,2,T}$ we see w belongs to $L_4((0,T), L_4(\Omega))$ which is the dual space of $L_{4/3}((0,T), L_{4/3}(\Omega))$. Thus considering the dual pairing of w and the first term in (5.28), we obtain

$$\begin{aligned}&(\partial_t w(t), w(t))_\Omega + (A_\alpha w(t), w(t))_\Omega \\ &\qquad + ((Pw(t) \cdot \nabla)u_\alpha(t), w(t))_\Omega + ((Pv(t) \cdot \nabla)w(t), w(t))_\Omega = 0.\end{aligned}$$

It is known that the dual operator P^* of P satisfies $P^* = P$ in L_2. Hence having the property $((Pv \cdot \nabla)w, w) = 0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|A_\alpha^{1/2} w(t)\|_2^2 &= \sum_{j=1}^{2}(\partial_j u_\alpha(t), Pw_j(t)w(t))_\Omega \\ &\leq C\|A_\alpha^{1/2}u_\alpha(t)\|_2\|w(t)\|_4^2 \\ &\leq C\|A_\alpha^{1/2}u_\alpha(t)\|_2\|w(t)\|_2\|A_\alpha^{1/2}w(t)\|_2 \\ &\leq C\|A_\alpha^{1/2}u_\alpha(t)\|_2^2\|w(t)\|_2^2 + \frac{1}{2}\|A_\alpha^{1/2}w(t)\|_2^2\end{aligned}$$

and therefore that

$$\frac{d}{dt}\|w(t)\|_2^2 \leq C\|A_\alpha^{1/2}u_\alpha(t)\|_2^2\|w(t)\|_2^2.$$

By Gronwall inequality and $w(0) = 0$, we get $w \equiv 0$ that is $u = v$. □

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