# ON FUNCTOR POINTS OF AFFINE SUPERGROUPS

AKIRA MASUOKA AND TAIKI SHIBATA

*Dedicated to Professor Susan Montgomery in honor of her distinguished career*

Abstract. To construct an affine supergroup from a Harish-Chandra pair, Gavarini [2] invented a natural method, which first constructs a group functor and then proves that it is representable. We give a simpler and more conceptual presentation of his construction in a generalized situation, using Hopf superalgebras over a superalgebra. As an application of the construction, given a closed super-subgroup of an algebraic supergroup, we describe the normalizer and the centralizer, using Harish-Chandra pairs. We also prove a tensor product decomposition theorem for Hopf superalgebras, and describe explicitly by cocycle deformation, the difference which results from the two choices of dualities found in literature.

Key Words: affine supergroup, algebraic supergroup, Hopf superalgebra, Harish-Chandra pair

Mathematics Subject Classification (2000): 14L15, 14M30, 16T05

## 1. Introduction

1.1. **Basic definitions.** In this paper we work over a non-zero commutative ring $\Bbbk$. The unadorned $\otimes$ is the tensor product over $\Bbbk$. A $\Bbbk$-module is said to be $\Bbbk$-*finite*, if it is finitely generated.

The word "super" is a synonym of "graded by the group $\mathbb{Z}_2 = \{0, 1\}$". Therefore, a $\Bbbk$-*supermodule* is a $\Bbbk$-module $V$ graded by $\mathbb{Z}_2$ so that $V = V_0 \oplus V_1$. When we say that $v$ is an element of $V$, we assume that it is homogeneous, and denote its degree by $|v|$. If $i = |v|$, the element or the component $V_i$ is said to be *even* or *odd*, according to $i = 0$ or $i = 1$. We say that $V$ is *purely even* if $V = V_0$, and is *purely odd* if $V = V_1$. The dual $\Bbbk$-module $V^*$ of $V$ is again a $\Bbbk$-supermodule so that $(V^*)_i = V_i^*$, $i = 0, 1$.

The $\Bbbk$-supermodules form a symmetric tensor category $\mathsf{SMod}_\Bbbk$ with respect to the tensor product $\otimes$, the unit object $\Bbbk$ and the super-symmetry

$$c_{V,W} : V \otimes W \to W \otimes V, \quad c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v. \tag{1.1}$$

Super-objects are objects, such as algebra object or Hopf-algebra object, defined in $\mathsf{SMod}_\Bbbk$. They are called with "super" attached, so as (*Hopf*) *superalgebras*. Ordinary objects, such as (Hopf) algebras, are regarded as purely even super-objects. We let

$$\mathsf{SAlg}_\Bbbk, \quad \mathsf{Alg}_\Bbbk$$

denote the category of super-commutative superalgebras and its full subcategory consisting of all commutative algebras, respectively. A *group functor* is a group-valued functor defined on $\mathsf{SAlg}_\Bbbk$ or $\mathsf{Alg}_\Bbbk$.

Working over an arbitrary commutative ring, we should be careful to define super-commutativity. By saying that a superalgebra $A$ is *super-commutative*, we require that $a^2 = 0$ for all odd elements $a \in A_1$, in addition to the usual requirement that the product on $A$ should be invariant, composed with the super-symmetry.

1.2. **Algebraic supergroups and Harish-Chandra pairs.** The notion of affine or algebraic groups defined in [6, Part I, 2.1] is directly generalized to the super context, as follows. An *affine supergroup* is a representable group functor $\mathbf{G}$ defined on $\mathsf{SAlg}_{\Bbbk}$. The super-commutative superalgebra $\mathcal{O}(\mathbf{G})$ which represents $\mathbf{G}$ necessarily has a Hopf superalgebra structure which arises uniquely from the group structure on $\mathbf{G}$. An affine supergroup $\mathbf{G}$ is called an *algebraic supergroup* if $\mathcal{O}(\mathbf{G})$ is finitely generated.

We let $\mathsf{ASG}_{\Bbbk}$ denote the category of algebraic supergroups. Given $\mathbf{G} \in \mathsf{ASG}_{\Bbbk}$, we have

$$G = \mathbf{G}_{ev} \text{ and } \mathfrak{g} = \mathrm{Lie}(\mathbf{G}),$$

where $G$ is the algebraic group defined as a restricted group functor by $G = \mathbf{G}|_{\mathsf{Alg}_{\Bbbk}}$, and $\mathfrak{g}$ is the Lie superalgebra of $\mathbf{G}$. Very roughly speaking, a *Harish-Chandra pair* is such a pair $(G, \mathfrak{g})$ with some additional structures involved; the notion is due to Kostant [7], and an irredundant definition will be reproduced from [12, 14] in Section 6.1 when $\Bbbk$ is a field of characteristic $\neq 2$. In this last situation the last cited articles reformulated the result [11, Theorem 29], which is formulated in purely Hopf-algebraic terms, so that $\mathbf{G} \to (\mathbf{G}_{ev}, \mathrm{Lie}(\mathbf{G}))$ gives a category equivalence

$$\mathsf{ASG}_{\Bbbk} \xrightarrow{\approx} \mathsf{HCP}_{\Bbbk}, \tag{1.2}$$

where $\mathsf{HCP}_{\Bbbk}$ denotes the category of Harish-Chandra pairs; the result was applied in several papers including [11, 12, 13, 3, 14]. An analogous category equivalence for super Lie groups had been proved 30 years before by Koszul [8] in the $C^\infty$ situation, and slightly before by Vishnyakova [16] in the complex analytic situation; see also [1, Section 7.4]. Very recently, the same result was proved by Hoshi and the first-named author [5] for super Lie groups over a complete field of characteristic $\neq 2$. For the above cited result [11, Theorem 29] in the algebraic situation, crucial is the result [10, Theorem 4.5] that for every $\mathbf{G} \in \mathsf{ASG}_{\Bbbk}$, the Hopf superalgebra $\mathcal{O}(\mathbf{G})$ decomposes so that

$$\mathcal{O}(\mathbf{G}) \simeq \mathcal{O}(\mathbf{G}_{ev}) \otimes \wedge(\mathfrak{g}_1^*) \tag{1.3}$$

as a left $\mathcal{O}(\mathbf{G}_{ev})$-comodule superalgebra with counit, where $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. In the previous work [13] the authors proved the category equivalence above when $\Bbbk$ is a commutative ring which is 2-torsion free, restricting the objects of $\mathsf{ASG}_{\Bbbk}$ to those $\mathbf{G}$ such that $\mathcal{O}(\mathbf{G})$ decomposes as above. Gavarini [2] proved the same result more generally when $\Bbbk$ is an arbitrary commutative ring, requiring Lie superalgebras to have an additional structure, 2-*operations*.

The quasi-inverse of the functor (1.2) constructed in [11, 13] is close to the one constructed by Koszul and others for super Lie groups. Gavarini's quasi-inverse, which is our concern, is new and natural; it first constructs a group functor and then proves that it is representable. He defines the groups of functor points by generators and relations.

We remark that in the appendix of [13], Gavarini's category equivalence was re-proved by using the method of the cited article, to supplement the original proof which skipped some necessary arguments.

1.3. **Three purposes.** This paper is written for three purposes.

The first one is to give a simpler and more conceptual presentation of Gavarini's construction cited above. Given a Harish-Chandra pair $(G, \mathfrak{g})$ and $A \in \mathsf{SAlg}_{\Bbbk}$, we construct the group $\mathbf{\Gamma}(A)$ which shall be the functor points of the desired affine supergroup $\mathbf{\Gamma}$, in two steps. Recall that the universal envelope $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$ is naturally a Hopf superalgebra, whence the base extension $A \otimes \mathbf{U}(\mathfrak{g})$ to $A$ is the Hopf superalgebra over $A$. Consider the following elements in this last Hopf superalgebra over $A$:

$$e(a, v) = 1 \otimes 1 + a \otimes v, \quad f(\epsilon, x) = 1 \otimes 1 + \epsilon \otimes x,$$

where $a \in A_1$, $v \in \mathfrak{g}_1$, $x \in \mathfrak{g}_0$, and $\epsilon \in A_0$ with $\epsilon^2 = 0$. Each of these elements, being the sum of the identity element and an even primitive whose tensor-square is zero, is an even grouplike; see Lemma 4.1. The first step of our construction is to construct the group $\mathbf{\Sigma}(A)$ generated by these even grouplikes. The subgroup $F(A_0)$ generated by all $f(\epsilon, x)$, being included in $A_0 \otimes U(\mathfrak{g}_0)$, has a natural group map into the group $G(A_0)$ of $A_0$-points of $G$. The second step is to construct the desired $\mathbf{\Gamma}(A)$ from $\mathbf{\Sigma}(A)$ by what we call the *base extension* along the group map $F(A_0) \to G(A_0)$; this notion of base extensions of groups is defined and discussed in Section 2. Our construction will be seen to be natural, when one thinks of the natural pairing of $A \otimes \mathbf{U}(\mathfrak{g})$ with $\mathcal{O}(\mathbf{\Gamma})$; see Lemma 3.1. The construction is done in Section 4, in a more generalized situation, aiming at an application to super Lie groups; see Remark 4.14.

We have described the contents of Sections 2 and 4. Section 3 is devoted to discussing some basic results on super-objects that include the comparison of dualities explained in the next subsection.

Section 5 starts with the subsection in which we re-prove Gavarini's category equivalence cited above, using our method of construction. This aims to supplement again Gavarini's original proof; see Remark 5.6. Recall from the second paragraph of Section 1.2 that the algebraic supergroups $\mathbf{G}$ over an arbitrary commutative ring $\Bbbk$, for which the category equivalence will be re-proven, are assumed so that $\mathcal{O}(\mathbf{G})$ decomposes as in (1.3). The second purpose of ours, which is achieved in Section 5.2, is to prove, along the line of our renewed proof, that the assumption above is necessarily satisfied if $\mathcal{O}(\mathbf{G}_{ev})$ is $\Bbbk$-flat; see Theorem 5.7. This theorem benefits two results cited in the second paragraph of Section 1.2. In fact it improves our previous result in [13], in which $\Bbbk$ is assumed to be 2-torsion free, while the proof gives an alternative proof of [10, Theorem 4.5], in which $\Bbbk$ is assumed to be a field; see Remark 5.8.

In the final Section 6, which consists of 2 subsections, we suppose that $\Bbbk$ is a field of characteristic $\neq 2$. In Section 6.1 we reproduce the category equivalence (1.2) from [12, 14], giving the irredundant definition of Harish-Chandra pairs referred to above. In Section 6.2, given an algebraic supergroup $\mathbf{G}$ and its closed super-subgroup $\mathbf{H}$, we describe the normalizer $\mathcal{N}_{\mathbf{G}}(\mathbf{H})$ and the centralizer $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$ in terms of Harish-Chandra pairs; see Theorem 6.6. Gavarini's construction is quite useful to discuss group-theoretical properties of affine supergroups, and it applies to prove the last cited theorem. This application is indeed the third purpose of ours.

1.4. **Comparing dualities.** We will make it clear that there are two choices, when we discuss the duality of Hopf superalgebras; this may not have been clearly recognized so far. If the simpler duality is chosen, as was done by [11, 12, 13, 14], one defines a pairing $\langle\ ,\ \rangle : \wedge(W^*) \times \wedge(W) \to \Bbbk$ between the exterior algebras on a $\Bbbk$-finite free module $W$ and on its dual $W^*$, as usually so that

$$\langle v_1 \wedge \cdots \wedge v_n,\ w_1 \wedge \cdots \wedge w_m \rangle = \delta_{n,m} \det\big(v_i(w_j)\big), \quad m, n \geq 0, \tag{1.4}$$

where $v_i \in W^*$, $w_i \in W$. We have to choose the other one, replacing (1.4) with (3.4) below, since the simpler duality does not work well for Hopf superalgebras over $A \in \mathsf{SAlg}_{\Bbbk}$, in general. In Section 3.2 this circumstance is explained, and the difference caused by choices is described in terms of *cocycle deformations*. Fortunately, the category equivalence (1.2) obtained with our choice of duality coincides with the one obtained before in [12, 13, 14], up to an involutive category isomorphism $\mathsf{HCP}_{\Bbbk} \to \mathsf{HCP}_{\Bbbk}$, as will be seen in Remarks 5.5 and 6.2.

## 2. Base extension of groups

Suppose that the quintuple

$$(\Sigma,\ F,\ G,\ i,\ \alpha)$$

consists of groups $\Sigma$, $F$ and $G$, a group map $i : F \to G$, and anti-group map $\alpha : G \to \mathrm{Aut}(\Sigma)$ such that

(A1) $F$ is a subgroup of $\Sigma$,
(A2) $\varphi^{i(f)} = f^{-1}\varphi f$ for all $f \in F$, $\varphi \in \Sigma$,
(A3) $f^g \in F$ and $i(f^g) = g^{-1} i(f) g$,

where $f \in F$, $g \in G$, $\varphi \in \Sigma$, and $\varphi^g$ stands for $\alpha(g)(\varphi)$. Suppose that $F$ and $G$ act on $\Sigma$ and $G$, respectively, from the right by inner automorphisms. Then (A2) reads that $i$ preserves the actions on $\Sigma$, while (A3) reads that $F$ is $G$-stable, and $i$ is $G$-equivariant.

Let $G \ltimes \Sigma$ be the semi-direct product given by $\alpha$, and set

$$\Xi = \{(i(f), f^{-1}) \in G \ltimes \Sigma \mid f \in F\}.$$

Then one sees from (A2)–(A3) that $\Xi$ is a normal subgroup of $G \ltimes \Sigma$; in particular, $\Sigma$ centralizes $\Xi$. We let

$$\Gamma = \Gamma(\Sigma, F, G, i, \alpha)$$

denote the quotient group $G \ltimes \Sigma/\Xi$.

**Lemma 2.1.** *We have the following.*

(1) *The composite $G \to G \ltimes \Sigma \to \Gamma$ of the inclusion with the quotient map is an injection, through which we will regard $G$ as a subgroup of $\Gamma$.*

(2) *The composite* $\Sigma \to G \ltimes \Sigma \to \Gamma$ *of the inclusion with the quotient map induces a bijection* $F\backslash\Sigma \to G\backslash\Gamma$ *between the sets of right cosets.*

*Proof.* Choose arbitrarily a set $X \subset F$ of representatives of $F\backslash\Sigma$. Then the product map $p : F \times X \to \Sigma$, $p(f,x) = fx$ is a bijection, through which we will identify $\Sigma$ with $F \times X$. Then we have $G \ltimes \Sigma = (G \ltimes F) \times X$ as left $G \ltimes F$-sets. Note $\Xi \subset G \ltimes F$ and that the canonical map $G \to G \ltimes F/\Xi = \Xi\backslash G \ltimes F$ is an isomorphism. The direct product with $\mathrm{id}_X$ gives a left $G$-equivariant bijection, $q : G \times X \to (\Xi\backslash G \ltimes F) \times X = \Gamma$. The injectivity of $q$ yields Part 1, since one may choose $X$ so as containing the identity element. The equivariant bijections $p$ and $q$ induce bijections, $\overline{p} : X \to F\backslash\Sigma$, $\overline{q} : X \to G\backslash\Gamma$. We see that $\overline{q} \circ \overline{p}^{-1} : F\backslash\Sigma \to G\backslash\Gamma$ is the bijection claimed by Part 2. □

Taking into account the property shown in Part 2 above we say:

**Definition 2.2.** $\Gamma$ is the *base extension* of $\Sigma$ along $i : F \to (G, \alpha)$. Here we suppose that $i$ is a morphism of groups acting on $\Sigma$, bearing in mind the action of $F$ by inner automorphisms.

## 3. Basic results on super-objects

In what follows we work over a non-zero commutative ring $\Bbbk$. This $\Bbbk$ is supposed to be arbitrary unless otherwise specified.

3.1. **Pairings.** Recall from Section 1.1 that $\mathsf{SAlg}_\Bbbk$ denotes the category of those superalgebras $A$ which are *super-commutative* in the sense that $A_0$ is central in $A$, and $a^2 = 0$ for all $a \in A_1$; see [2, Section 2.1.1], for example. Let $A \in \mathsf{SAlg}_\Bbbk$. An $A$-*supermodule* is a left $A$-module object in $\mathsf{SMod}_\Bbbk$; this is identified with the right $A$-module object which is defined on the same $\Bbbk$-supermodule, say $M$, by $ma := (-1)^{|a||m|}am$, $a \in A$, $m \in M$. Given $A$-supermodules $M$, $N$, let $M \otimes_A N$ denote the quotient $\Bbbk$-supermodule of $M \otimes N$ defined by the relations

$$ma \otimes n = m \otimes an, \quad a \in A,\ m \in M,\ n \in N.$$

This is naturally an $A$-supermodule. The $A$-supermodules form a symmetric tensor category $A$-$\mathsf{SMod}$, where the tensor product is the $\otimes_A$ just defined, and the unit object is $A$. The symmetry is the one induced from the super-symmetry $c_{M,N}$ (see (1.1)), and it will be denoted by the same symbol. A Hopf superalgebra over $A$ is a Hopf-algebra object in $A$-$\mathsf{SMod}$. The structure maps of a Hopf superalgebra $\mathcal{H}$ over $A$ will be denoted by

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H},\ \Delta(h) = h_{(1)} \otimes h_{(2)}, \quad \varepsilon : \mathcal{H} \to A, \quad \mathcal{S} : \mathcal{H} \to \mathcal{H}.$$

A *pairing* between objects $M$ and $N$ in $A$-$\mathsf{SMod}$ is a morphism $M \otimes_A N \to A$ in $A$-$\mathsf{SMod}$, which will be often presented as $\langle\ ,\ \rangle : M \times N \to A$, $\langle m,\ n\rangle$ = the value of $m \otimes n$. The *tensor product* with another pairing $\langle\ ,\ \rangle : M' \otimes_A N' \to A$ is the pairing between $M \otimes_A M'$ and $N \otimes_A N'$ which is defined to be the composite

$$(M \otimes_A M') \otimes_A (N \otimes_A N') \to (M \otimes_A N) \otimes_A (M' \otimes_A N') \to A \otimes_A A = A$$

of $\mathrm{id}_M \otimes_A c_{M',N} \otimes_A \mathrm{id}_{N'}$ with $\langle\ ,\ \rangle \otimes_A \langle\ ,\ \rangle$. Explicitly, it is defined by

$$\langle m \otimes m',\ n \otimes n'\rangle = (-1)^{|m'||n|}\langle m,\ n\rangle\, \langle m',\ n'\rangle, \tag{3.1}$$

where $m \in M$, $m' \in M'$, $n \in N$, $n' \in N'$. We remark that if $A = \Bbbk$, then the sign $(-1)^{|m'||n|}$ above can be replaced by either $(-1)^{|m||n'|}$, $(-1)^{|m||m'|}$ or $(-1)^{|n||n'|}$.

Let $\mathcal{L}$, $\mathcal{H}$ be Hopf superalgebras over $A$. A pairing $\langle\ ,\ \rangle : \mathcal{L} \times \mathcal{H} \to A$ is called a *Hopf pairing*, if we have

$$\langle x,\ hk\rangle = \langle \Delta(x),\ h \otimes k\rangle, \quad \langle xy,\ h\rangle = \langle x \otimes y,\ \Delta(h)\rangle, \tag{3.2}$$

$$\langle x,\ 1\rangle = \varepsilon(x), \quad \langle 1,\ h\rangle = \varepsilon(h), \tag{3.3}$$

where $x, y \in \mathcal{L}$, $h, k \in \mathcal{H}$. On the right-hand sides of (3.2) appears the tensor product of two copies of the pairing. The conditions imply

$$\langle \mathcal{S}(x),\ h\rangle = \langle x,\ \mathcal{S}(h)\rangle, \quad x \in \mathcal{L},\ h \in \mathcal{H}.$$

Just as in the non-super situation, the set

$$\mathsf{Gpl}(\mathcal{L}) := \{g \in \mathcal{L}_0 \mid \Delta(g) = g \otimes_A g,\ \varepsilon(g) = 1\}$$

of all even grouplikes in $\mathcal{L}$ is a group under the product of $\mathcal{L}$, and the set

$$\mathsf{SAlg}_A(\mathcal{H}, A)$$

of all superalgebra maps $\mathcal{H} \to A$ over $A$ is a group under the convolution product [15, Page 6]. Our construction of affine supergroups is inspired by the following simple fact, which is easy to prove.

**Lemma 3.1.** *A Hopf pairing $\langle\ ,\ \rangle : \mathcal{L} \times \mathcal{H} \to A$ induces the group map*

$$\mathsf{Gpl}(\mathcal{L}) \to \mathsf{SAlg}_A(\mathcal{H}, A), \quad g \mapsto \langle g,\ -\rangle.$$

Here is a typical example of Hopf pairings over $\Bbbk$.

**Example 3.2** (cf. (1.4), [11, Eq. (5)])**.** Let $W$ be a $\Bbbk$-module which is $\Bbbk$-finite free. We regard the exterior algebra $\wedge(W)$ on $W$ as a (super-commutative) Hopf superalgebra in which every element in $W$ is a (square-zero) odd primitive. We have another such Hopf superalgebra $\wedge(W^*)$. A Hopf pairing $\langle\ ,\ \rangle : \wedge(W^*) \times \wedge(W) \to \Bbbk$ is defined by

$$\langle v_1 \wedge \cdots \wedge v_n,\ w_1 \wedge \cdots \wedge w_m\rangle = \delta_{n,m}\,(-1)^{\binom{n}{2}} \det\big(v_i(w_j)\big), \quad m, n \geq 0, \tag{3.4}$$

where $v_i \in W^*$, $w_i \in W$. By convention we have $\binom{0}{2} = \binom{1}{2} = 0$.

3.2. **Comparing dualities.** In the situation above we suppose $A = \Bbbk$, and consider super-objects and pairings over $\Bbbk$.

Suppose that $\mathcal{H}$ is a super-coalgebra. Then we make the dual $\Bbbk$-supermodule $\mathcal{H}^*$ uniquely into a superalgebra so that the canonical pairing $\mathcal{H}^* \times \mathcal{H} \to \Bbbk$ satisfies the second equations of (3.2), (3.3). This is the same as saying that the pairing $\mathcal{H} \times \mathcal{H}^* \to \Bbbk$, with the sides switched, satisfies the first equations of (3.2), (3.3). The identity of $\mathcal{H}^*$ is the counit of $\mathcal{H}$, and the product is given by

$$pq(h) = (-1)^{|p||q|}\, p(h_{(1)})\, q(h_{(2)}), \quad p, q \in \mathcal{H}^*,\ h \in \mathcal{H}.$$

We denote this superalgebra by $\mathcal{H}^{\bar{*}}$.

Similarly, if $\mathcal{H}$ is Hopf superalgebra which is $\Bbbk$-finite projective, we make $\mathcal{H}^*$ uniquely into a Hopf superalgebra denoted by $\mathcal{H}^{\bar{*}}$, so that $\mathcal{H}^* \times \mathcal{H} \to \Bbbk$ or $\mathcal{H} \times \mathcal{H}^* \to \Bbbk$ is a Hopf pairing. We call $\mathcal{H}^{\bar{*}}$ the *dual Hopf superalgebra* of

$\mathcal{H}$. Since the Hopf pairing given in Example 3.2 is non-degenerate, it follows that the Hopf superalgebras $\wedge(W)$ and $\wedge(W^*)$ are dual to each other.

Suppose that

$$\langle\ ,\ \rangle : V \times W \to \Bbbk, \quad \langle\ ,\ \rangle : V' \times W' \to \Bbbk$$

are pairings over $\Bbbk$. We remark that in the articles [11, 12, 13], the tensor product of pairings is supposed to be the ordinary one

$$\langle v \otimes w,\ v' \otimes w' \rangle_{\mathrm{ord}} = \langle v,\ w\rangle\, \langle v',\ w'\rangle,$$

just as in the non-super situation. This is justified since it holds that $\langle\ ,\ \rangle_{\mathrm{ord}} \circ (c_{V,W} \otimes \mathrm{id}_{W' \otimes V'}) = \langle\ ,\ \rangle_{\mathrm{ord}} \circ (\mathrm{id}_{V \otimes W} \otimes c_{W',V'})$; see the proof of [11, Corollary 3] and the following remark. Over $A \in \mathsf{SAlg}_\Bbbk$ in general, this is not true any more. Therefore, we chose the definition as in (3.1), so that we have $\langle\ ,\ \rangle \circ (c_{M,N} \otimes_A \mathrm{id}_{N' \otimes_A M'}) = \langle\ ,\ \rangle \circ (\mathrm{id}_{M \otimes_A N} \otimes_A c_{N',M'})$, indeed. Due to these different choices, the Hopf pairing given by (3.4) is different from the ordinary one given by (1.4) or [11, Eq. (5)]. Note also that the dual (Hopf) superalgebras given above are different from those given in the cited articles. We are going to clarify this difference.

Let $\Bbbk^\times$ denote the multiplicative group of all units in $\Bbbk$, and regard it as a trivial module over the group $\mathbb{Z}_2 = \{0, 1\}$. Then the map $\sigma : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \Bbbk^\times$ defined by

$$\sigma(i, j) = (-1)^{ij}, \quad i, j \in \mathbb{Z}_2$$

is a 2-cocycle. Therefore, the identity functor

$$\mathsf{SMod}_\Bbbk \to \mathsf{SMod}_\Bbbk, \quad V \mapsto V = {}_\sigma V$$

together with the tensor structure

$$\begin{aligned} {}_\sigma V \otimes {}_\sigma W \to {}_\sigma(V \otimes W),\ v \otimes w \mapsto \sigma(|v|, |w|)\, v \otimes w, \\ \mathrm{id} : \Bbbk \to \Bbbk = {}_\sigma\Bbbk \end{aligned} \tag{3.5}$$

form a tensor equivalence. One sees that this preserves the super-symmetry, and it is an involution since $\sigma(i, j)^2 = 1$, $i, j \in \mathbb{Z}_2$. It follows that if $\mathcal{H}$ is a super-object, e.g. a Hopf superalgebra, over $\Bbbk$, then ${}_\sigma\mathcal{H}$ is such an object, and ${}_\sigma({}_\sigma\mathcal{H}) = \mathcal{H}$. This ${}_\sigma\mathcal{H}$ is called the (*cocycle*) *deformation* of $\mathcal{H}$ by $\sigma$; see [9, Section 1.1], for example.

If $\Bbbk$ contains a square root $\sqrt{-1}$ of $-1$, then $\sigma$ is the coboundary of

$$\nu : \mathbb{Z}_2 \to \Bbbk^\times, \quad \nu(0) = 1,\ \nu(1) = \sqrt{-1}.$$

It follows that

$${}_\sigma V \mapsto V, \quad v \mapsto \nu(|v|)\, v$$

gives a natural isomorphism from the tensor equivalence ${}_\sigma(\ )$ given by $\sigma$ to the identity tensor functor, whence the deformation ${}_\sigma\mathcal{H}$ by $\sigma$ is naturally isomorphic to the original $\mathcal{H}$, in this case.

Given two pairings over $\Bbbk$ as above, we have

$$\begin{aligned} \langle v \otimes w,\ v' \otimes w' \rangle &= \langle \sigma(|v|, |w|)\, v \otimes w,\ v' \otimes w' \rangle_{\mathrm{ord}} \\ &= \langle v \otimes w,\ \sigma(|v'|, |w'|)\, v' \otimes w' \rangle_{\mathrm{ord}}; \end{aligned}$$

see (3.5). This shows that the dual (Hopf) superalgebra $\mathcal{H}^{\bar{*}}$ of a super-coalgebra (or $\Bbbk$-finite projective Hopf superalgebra) $\mathcal{H}$ coincides with the deformation ${}_\sigma(\mathcal{H}^*)$ of the one $\mathcal{H}^*$ treated in [11, 12, 13].

3.3. **Base extensions.** Let $A \in \mathsf{SAlg}_{\Bbbk}$. Given $V \in \mathsf{SMod}_{\Bbbk}$, we let

$$V_A = A \otimes V \in A\text{-}\mathsf{SMod}$$

denote that base extension to $A$. Given a pairing $\langle\ ,\ \rangle : V \times W \to \Bbbk$ over $\Bbbk$, we let

$$\langle\ ,\ \rangle_A : V_A \times W_A \to A$$

denote the base extension to $A$. This is a pairing over $A$. The base extension to $A$ of a Hopf superalgebra over $\Bbbk$ is a Hopf superalgebra over $A$. The base extension to $A$ of a Hopf pairing over $\Bbbk$ is a Hopf pairing over $A$.

3.4. **Lie superalgebras.** A *Lie superalgebra* (over $\Bbbk$) is an object $\mathfrak{g}$ given a morphism $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, both in $\mathsf{SMod}_{\Bbbk}$, such that

(B1) $[u, u] = 0,\ u \in \mathfrak{g}_0$,
(B2) $[[v, v], v] = 0,\ v \in \mathfrak{g}_1$,
(B3) $[\ ,\ ] \circ (\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}}) = 0$,
(B4) $[[\ ,\ ],\ ] \circ (\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}}) = 0$.

In the last two equations, $c_{V,W} : V \otimes W \to W \otimes V$ denotes the supersymmetry (1.1). We call $[\ ,\ ]$ the *super-bracket* of the Lie superalgebra.

As is well known, (B1) ensures the equality (B3) restricted to $\mathfrak{g}_0 \otimes \mathfrak{g}_0$. Recall that a *Lie algebra* is a $\Bbbk$-module given a bracket which satisfies (B1) and the Jacobi identity, that is, (B4) in the purely even situation; it is, therefore, the same as a purely even Lie superalgebra. It follows that if $\mathfrak{g}$ is a Lie superalgebra, then $\mathfrak{g}_0$ is a Lie algebra.

Let $\mathfrak{g}$ be a Lie superalgebra.

A 2-*operation* [2, Definition 2.2.1] on $\mathfrak{g}$ is a map $(\ )^{\langle 2\rangle} : \mathfrak{g}_1 \to \mathfrak{g}_0$ such that

(B5) $(\lambda v)^{\langle 2\rangle} = \lambda^2 v^{\langle 2\rangle}$,
(B6) $(v + w)^{\langle 2\rangle} = v^{\langle 2\rangle} + [v, w] + w^{\langle 2\rangle}$,
(B7) $[v^{\langle 2\rangle}, z] = [v, [v, z]]$,

where $\lambda \in \Bbbk$, $v, w \in \mathfrak{g}_1$, $z \in \mathfrak{g}$.

Given $R \in \mathsf{Alg}_{\Bbbk}$, the base extension $\mathfrak{g}_R = R \otimes \mathfrak{g}$ is naturally a Lie superalgebra over $R$.

**Lemma 3.3** ([13, Proposition A.3])**.** *Assume that $\mathfrak{g}_1$ is $\Bbbk$-free. Given a 2-operation $(\ )^{\langle 2\rangle} : \mathfrak{g}_1 \to \mathfrak{g}_0$, there uniquely exists a map*

$$(\ )_R^{\langle 2\rangle} : (\mathfrak{g}_1)_R \to (\mathfrak{g}_0)_R$$

*such that*

$$\Big(\sum_{i=1}^{n} c_i \otimes v_i\Big)_R^{\langle 2\rangle} = \sum_{i=1}^{n} c_i^2 \otimes v_i^{\langle 2\rangle} + \sum_{i<j} c_i c_j \otimes [v_i, v_j]$$

*for every element $\sum_{i=1}^{n} c_i \otimes v_i \in R \otimes \mathfrak{g}_1$. This $(\ )_R^{\langle 2\rangle}$ is a 2-operation on the Lie superalgebra $\mathfrak{g}_R$ over $R$.*

Let $\mathfrak{g}$ be a Lie superalgebra equipped with a 2-operation. The tensor algebra $\mathbf{T}(\mathfrak{g})$ on $\mathfrak{g}$ uniquely turns into a Hopf superalgebra in which every even or odd element of $\mathfrak{g}$ is an even or odd primitive, respectively. We let $\mathbf{U}(\mathfrak{g})$ denote the quotient Hopf superalgebra of $\mathbf{T}(\mathfrak{g})$ divided by the superideal generated by the homogeneous primitives

$$zw - (-1)^{|z||w|} wz - [z, w], \quad v^2 - v^{\langle 2\rangle}, \tag{3.6}$$

where $z, w \in \mathfrak{g}$, and $v \in \mathfrak{g}_1$. With the construction applied to $\mathfrak{g}_0$, obtained is the universal enveloping algebra $U(\mathfrak{g}_0)$, as is well known. The inclusion $\mathfrak{g}_0 \to \mathfrak{g}$ induces a Hopf superalgebra map $U(\mathfrak{g}_0) \to \mathbf{U}(\mathfrak{g})$, through which $\mathbf{U}(\mathfrak{g})$ turns into a left (and right) $U(\mathfrak{g}_0)$-module.

Given an element $v$ of $\mathfrak{g}$, we will denote its natural image in $\mathbf{U}(\mathfrak{g})$ by the same symbol $v$.

**Proposition 3.4** ([13, Corollary A.6]). *Let* $\mathfrak{g}$ *be as above. Assume*

(C) $\mathfrak{g}_0$ *is* $\Bbbk$*-finite projective, and* $\mathfrak{g}_1$ *is* $\Bbbk$*-finite free.*

*Choose arbitrarily a* $\Bbbk$*-free basis* $v_1, \dots, v_n$ *of* $\mathfrak{g}_1$. *Then the left* $U(\mathfrak{g}_0)$*-module* $\mathbf{U}(\mathfrak{g})$ *is free with the free basis*

$$v_{i_1} v_{i_2} \dots v_{i_r}, \tag{3.7}$$

*where* $1 \le i_1 < i_2 < \cdots < i_r \le n$, $r \ge 0$.

It is known that if $\mathfrak{g}_0$ is $\Bbbk$-flat, then the canonical map $\mathfrak{g}_0 \to U(\mathfrak{g}_0)$ is injective. Combined with the proposition above, it follows that under (C), the canonical map $\mathfrak{g} \to \mathbf{U}(\mathfrak{g})$ is injective, and $\mathfrak{g}_1 \to \mathbf{U}(\mathfrak{g}) \leftarrow U(\mathfrak{g}_0)$ are $\Bbbk$-linearly split injections.

**Remark 3.5** (see [13, Lemma A.2]). (1) Assume that $\Bbbk$ is 2-torsion free in the sense that $2 : \Bbbk \to \Bbbk$ is injective. Let $\mathfrak{g}$ be a Lie superalgebra which satisfies

(C′) $\mathfrak{g}_0$ is $\Bbbk$-flat, and $\mathfrak{g}_1$ is $\Bbbk$-free.

There exists a 2-operation on $\mathfrak{g}$ if and only if for every $v \in \mathfrak{g}_1$, the element $[v, v]$ is 2-divisible in $\mathfrak{g}_0$, that is, it is the double of some element of $\mathfrak{g}_0$; this last element is uniquely determined, and is denoted by $\frac{1}{2}[v, v]$. If this is the case, then

$$v^{\langle 2 \rangle} := \frac{1}{2}[v, v], \quad v \in \mathfrak{g}_1 \tag{3.8}$$

defines a unique 2-operation on $\mathfrak{g}$, and the same result [13, Proposition 3.4] as Proposition 3.4 above is proved.

(2) Assume that $\Bbbk$ is a field of characteristic $\neq 2$. Then the results above can apply: every Lie superalgebra has the unique 2-operation defined by (3.8), and one may not refer to such operations any more.

Let $\mathbf{G}$ be an affine supergroup, and set $\mathbf{O} = \mathcal{O}(\mathbf{G})$, the super-commutative Hopf superalgebra which represents $\mathbf{G}$. We let

$$\mathbf{O}^+ = \mathrm{Ker}(\varepsilon : \mathbf{O} \to \Bbbk)$$

denote the augmentation super-ideal of $\mathbf{O}$. The *Lie superalgebra* $\mathrm{Lie}(\mathbf{G})$ of $\mathbf{G}$ is defined by

$$\mathrm{Lie}(\mathbf{G}) = (\mathbf{O}^+/(\mathbf{O}^+)^2)^*$$

as an object in $\mathsf{SMod}_\Bbbk$. We suppose

$$\mathrm{Lie}(\mathbf{G}) \subset \mathbf{O}^{\bar{*}}.$$

Indeed, $\mathrm{Lie}(\mathbf{G})$ is identified with the $\Bbbk$-super-submodule of $\mathbf{O}^{\bar{*}}$ which consists of the elements $z$ of $\mathbf{O}^{\bar{*}}$ such that

$$z(hk) = z(h)\varepsilon(k) + \varepsilon(h)z(k), \quad h, k \in \mathbf{O}.$$

**Lemma 3.6** ([13, Proposition A.7]). Lie($\mathbf{G}$) *is a Lie superalgebra under the super-bracket*

$$[z, w] := zw - (-1)^{|z||w|}wz, \quad z, w \in \mathrm{Lie}(\mathbf{G}),$$

*and a* 2*-operation on* Lie($\mathbf{G}$) *is given by the square map*

$$(\ )^2 : \mathrm{Lie}(\mathbf{G})_1 \to \mathrm{Lie}(\mathbf{G})_0, \quad v \mapsto v^2.$$

*Here the products $zw$, $wz$ and the square $v^2$ are computed in $\mathbf{O}^{\bar{*}}$.*

We define Lie($\mathbf{G}$) to be the Lie superalgebra equipped with the 2-operation, as above. One sees that Lie then gives a functor from the category of affine supergroups to the category of Lie superalgebras equipped with 2-operation. A morphism of the latter category is a morphism in $\mathsf{SMod}_{\Bbbk}$ which preserves the super-bracket and the 2-operation.

**Remark 3.7.** Let $\mathfrak{g}$ be a Lie superalgebra. Note from Section 3.2 that the deformation ${}_\sigma\mathfrak{g}$ by $\sigma$ is the object $\mathfrak{g}$ in $\mathsf{SMod}_{\Bbbk}$ which is given the super-bracket

$${}_\sigma[z, w] := (-1)^{|z||w|}[z, w], \quad z, w \in \mathfrak{g}$$

deformed from the original super-bracket $[z, w]$. If $\mathfrak{g}$ is equipped with a 2-operation, we suppose that ${}_\sigma\mathfrak{g}$ is equipped with the deformed 2-operation

$$v^{\,\sigma\langle 2\rangle} := -v^{\langle 2\rangle}, \quad v \in \mathfrak{g}_1.$$

This indeed defines a 2-operation on ${}_\sigma\mathfrak{g}$, as is easily seen.

Let $\mathbf{G}$ be an affine supergroup. As is seen from the last paragraph of Section 3.2, the definition of Lie($\mathbf{G}$) above is different from the one in [13, Appendix]. In fact, the two Lie($\mathbf{G}$) are the deformations of each other by $\sigma$.

3.5. *$G$*-**supermodules.** Let $G$ be an affine group, and set $O = \mathcal{O}(G)$.

Let $V \in \mathsf{SMod}_{\Bbbk}$. A *right $G$-supermodule structure* on $V$ is an anti-morphism from $G$ to the group functor $\mathsf{GL}(V)$ which assigns to each $R \in \mathsf{Alg}_{\Bbbk}$, the group $\mathrm{GL}_R(V_R)$ of all $R$-super-linear automorphisms of $V_R$. Such a structure, say $\alpha$, arises uniquely from a left $O$-super-comodule structure

$$\rho_\alpha : V \to O \otimes V, \quad \rho_\alpha(v) = v_{(-1)} \otimes v_{(0)} \tag{3.9}$$

so that

$$\alpha_R(g)(1 \otimes v) = g(v_{(-1)}) \otimes v_{(0)}, \quad v \in V,\ g \in G(R),$$

where $R \in \mathsf{Alg}_{\Bbbk}$. Note that $\rho_\alpha(v) = \alpha_O(\mathrm{id}_O)(1 \otimes v)$. We will write $v^g$ for $\alpha_R(g)(1 \otimes v)$, and $u^g$ for $\alpha_R(g)(u)$, $u \in V_R$.

Similarly, for left $G$-supermodule structures we will write as ${}^g w$.

Let $W \in \mathsf{SMod}_{\Bbbk}$, and suppose that it is $\Bbbk$-finite projective. A left $G$-supermodule structure on $W$ is *transposed* to $W^*$ so that

$$v^g(w) := v({}^g w), \quad v \in W^*,\ w \in W,\ g \in G(R). \tag{3.10}$$

This defines a right $G$-supermodule structure on $W^*$. Here we understand that the $v$ in the equation represents the element $1 \otimes v \in (W^*)_R$ or its image through the canonical isomorphism $(W^*)_R \simeq \mathrm{Hom}_R(W_R, R)$.

The right (or left) $G$-supermodules form a symmetric tensor category with respect to the tensor product $\otimes$, the unit object $\Bbbk$ and the super-symmetry.

## 4. Construction of affine supergroups based on Gavarini's idea

In this section the base commutative ring $\Bbbk$ remains arbitrary.

4.1. **The group $\mathbf{\Sigma}(A)$.** Let $\mathfrak{g}$ be a Lie superalgebra which satisfies (C); see Proposition 3.4. Suppose that it is equipped with a 2-operation.

Let $A \in \mathsf{SAlg}_{\Bbbk}$. We have the group $\mathsf{Gpl}(\mathbf{U}(\mathfrak{g})_A)$ of all even grouplikes in the Hopf superalgebra $\mathbf{U}(\mathfrak{g})_A = A \otimes \mathbf{U}(\mathfrak{g})$ over $A$. As is seen from the paragraph following Proposition 3.4, the canonical maps

$$A_0 \otimes \mathfrak{g}_0 \to A \otimes U(\mathfrak{g}_0) \to A \otimes \mathbf{U}(\mathfrak{g}) \leftarrow A \otimes \mathfrak{g}_1$$

are all injections, which we will regard as inclusions. We define even elements $e(a, v)$, $f(\epsilon, x)$ of $A \otimes \mathbf{U}(\mathfrak{g})$ by

$$e(a, v) = 1 \otimes 1 + a \otimes v, \quad f(\epsilon, x) = 1 \otimes 1 + \epsilon \otimes x, \tag{4.1}$$

where $a \in A_1$, $v \in \mathfrak{g}_1$, $x \in \mathfrak{g}_0$, and $\epsilon \in A_0$ with $\epsilon^2 = 0$. Note that $e(\lambda a, v) = e(a, \lambda v)$, $f(\lambda\epsilon, x) = f(\epsilon, \lambda x)$ for $\lambda \in \Bbbk$.

**Lemma 4.1.** *The elements $e(a, v)$, $f(\epsilon, x)$ are contained in $\mathsf{Gpl}(\mathbf{U}(\mathfrak{g})_A)$, and we have*

$$e(a, v)^{-1} = e(-a, v), \quad f(\epsilon, x)^{-1} = f(-\epsilon, x), \quad e(0, v) = 1 = f(0, x).$$

*Proof.* This follows since $a \otimes v$ and $\epsilon \otimes x$ are such even primitives $z$ that are tensor-square zero, $z \otimes_A z = 0$. □

**Lemma 4.2.** *Let $a, b \in A_1$, $u, v \in \mathfrak{g}_1$, $x, y \in \mathfrak{g}_0$, and $\epsilon, \eta \in A_0$ with $\epsilon^2 = \eta^2 = 0$. Then the following relations hold in $\mathsf{Gpl}(\mathbf{U}(\mathfrak{g})_A)$.*

(i) $e(a, u)\, e(b, v) = f(-ab, [u, v])\, e(b, v)\, e(a, u)$
(ii) $e(a, v)\, e(b, v) = f(-ab, v^{\langle 2\rangle})\, e(a + b, v)$
(iii) $e(a, v)\, f(\epsilon, x) = f(\epsilon, x)\, e(a, v)\, e(\epsilon a, [v, x])$
(iv) $f(\epsilon, x)\, f(\eta, y) = f(\eta, y)\, f(\epsilon, x)\, f(\epsilon\eta, [x, y])$

*Proof.* These follow by direct computation. □

In particular, $e(a, u)$ and $e(b, v)$ (resp., $e(a, v)$ and $f(\epsilon, x)$; resp., $f(\epsilon, x)$ and $f(\eta, y)$) commute with each other if $ab = 0$ or $[u, v] = 0$ (resp., if $\epsilon a = 0$ or $[v, x] = 0$; resp., if $\epsilon\eta = 0$ or $[x, y] = 0$).

Let $\mathbf{\Sigma}(A)$ denote the subgroups of $\mathsf{Gpl}(\mathbf{U}(\mathfrak{g})_A)$ generated by all the elements $e(a, v)$, $f(\epsilon, x)$ defined by (4.1). Let $F(A_0)$ denote the subgroup of $\mathbf{\Sigma}(A)$ generated by all $f(\epsilon, x)$.

**Proposition 4.3.** *We have the following.*

(1) $F(A_0) = \mathbf{\Sigma}(A) \cap U(\mathfrak{g}_0)_{A_0}$.
(2) *Choose arbitrarily a $\Bbbk$-free basis $v_1, \ldots, v_n$ of $\mathfrak{g}_1$. Then every element of $\mathbf{\Sigma}(A)$ is uniquely expressed in the form*

$$f\, e(a_1, v_1)\, e(a_2, v_2) \ldots e(a_n, v_n), \tag{4.2}$$

*where $f \in F(A_0)$, and $a_i \in A_1$, $1 \le i \le n$.*

*Proof.* First, we prove the uniqueness of expression in Part 2. Note that $F(A_0) \subset A \otimes U(\mathfrak{g}_0)$. By Proposition 3.4 $\mathbf{U}(\mathfrak{g})_A$ has the elements given by (3.7) as left $A \otimes U(\mathfrak{g}_0)$-free basis. Suppose that one element has two expressions, $f\, e(a_1, v_1) \ldots e(a_n, v_n)$, $f'\, e(a'_1, v_1) \ldots e(a'_n, v_n)$. Then the comparison of coefficients of the free basis elements $1 \otimes 1$, $1 \otimes v_i$ shows that $f = f'$, $fa_i = f'a_i'$, and so $a_i = a_i'$, which proves the uniqueness.

The argument also shows that the element $f\, e(a_1, v_1) \dots e(a_n, v_n)$ is in $A \otimes U(\mathfrak{g}_0)$ if and only if $a_i = 0$, $1 \leq i \leq n$. Therefore, once the possibility of expression in Part 2 is shown, Part 1 follows. □

The proof of the last mentioned possibility which was given in an earlier version of this paper was wrong. Alexandr Zubkov kindly pointed out this, showing a correct proof which is reproduced essentially as follows.

**Lemma 4.4** (A. Zubkov)**.** *In the situation of Proposition 4.3, choose arbitrarily a super-ideal $\mathfrak{a}$ of $A$. For each integer $k \geq 0$, let $\mathbf{\Sigma}_k$ (resp., $F_k$) denote the subgroup of $\mathbf{\Sigma}(A)$ (resp., of $F(A_0)$) which is generated by the elements $e(a, v)$ and $f(\epsilon, x)$ (resp., the elements $f(\epsilon, x)$), where $a \in \mathfrak{a}^k \cap A_1$, $v \in \mathfrak{g}_1$, $x \in \mathfrak{g}_0$, and $\epsilon \in \mathfrak{a}^k \cap A_0$ with $\epsilon^2 = 0$.*

(1) *We have*
$$[\mathbf{\Sigma}_k, \mathbf{\Sigma}_\ell] \subset \mathbf{\Sigma}_{k+\ell}, \quad [F_k, F_\ell] \subset F_{k+\ell} \ \textit{for all}\ k, \ell \geq 0.$$

(2) *Each $\mathbf{\Sigma}_k \subset \mathbf{\Sigma}(A)$ (resp., $F_k \subset F(A_0)$) is a normal subgroup such that $\mathbf{\Sigma}_k \supset F_k$, $k \geq 0$, and*
$$\mathbf{\Sigma}(A) = \mathbf{\Sigma}_0 \supset \mathbf{\Sigma}_1 \supset \mathbf{\Sigma}_2 \supset \dots; \quad F(A_0) = F_0 \supset F_1 \supset F_2 \supset \dots$$

(3) *Let $v \in \mathfrak{g}_1$, $a \in \mathfrak{a}^k \cap A_1$ and $b \in \mathfrak{a}^\ell \cap A_1$, where $k$, $\ell \geq 0$ are integers. Then we have*
$$e(a, v)\, e(b, v) \equiv e(a + b, v) \bmod F_{k+\ell}.$$

(4) *Choose arbitrarily a $\Bbbk$-free basis $v_1, \dots, v_n$ of $\mathfrak{g}_1$. Fix an integer $k > 1$. Then every element of $\mathbf{\Sigma}_1$ is congruent modulo $\mathbf{\Sigma}_k$ to a product*
$$f\, e(a_1, v_1)\, e(a_2, v_2) \dots e(a_n, v_n),$$
*where $f \in F_1$, and $a_i \in \mathfrak{a} \cap A_1$, $1 \leq i \leq n$.*

*Proof.* (1), (3) These follow from Lemmas 4.1–4.2.

(2) This follows from (1).

(4) We prove by induction on $k$. If $v = \sum_{i=1}^n \lambda_i v_i$ with $\lambda_i \in \Bbbk$, then
$$e(a, v) = e(\lambda_1 a, v_1)\, e(\lambda_2 a, v_2) \dots e(\lambda_n a, v_n).$$
Therefore, $\mathbf{\Sigma}_1$ is generated by $F_1$ and all the elements $e(a, v_i)$, $1 \leq i \leq n$, where $a_i \in \mathfrak{a} \cap A_1$. Since $\mathbf{\Sigma}_1/\mathbf{\Sigma}_2$ is abelian by (1) the desired result for $k = 2$ follows by (3).

An analogous result on the group $\mathbf{\Sigma}_k/\mathbf{\Sigma}_{k+1}$ which is seen to be abelian, combined with the induction hypothesis, shows that every element of $\mathbf{\Sigma}_1$ is congruent modulo $\mathbf{\Sigma}_{k+1}$ to a product
$$f\, e(a_1, v_1) \dots e(a_n, v_n)\, f'\, e(a_1', v_1) \dots e(a_n', v_n),$$
where $f \in F_1$, $f' \in F_k$, $a_i \in \mathfrak{a} \cap A_1$ and $a_i' \in \mathfrak{a}^k \cap A_1$, $1 \leq i \leq n$. This is congruent to
$$ff'\, e(a_1, v_1)\, e(a_1', v_1) \dots e(a_n, v_n)\, e(a_n', v_n)$$
since $\mathbf{\Sigma}_k/\mathbf{\Sigma}_{k+1}$ is central in $\mathbf{\Sigma}_1/\mathbf{\Sigma}_{k+1}$ by (1). The desired result for $k + 1$ follows by (3). □

*Proof of Proposition 4.3 (Continued).* To express as a desired product, an element, say $h$, which is the product of any order of elements

$$e(\tau_i, v_i),\ 1 \le i \le n;\ f(\epsilon_j, x_j),\ 1 \le j \le m,$$

where $v_i \in \mathfrak{g}_1$, $x_j \in \mathfrak{g}_0$, we may suppose that $\tau_i$ and $\epsilon_j$ are variables, or more precisely, we may suppose

$$A = \Bbbk[\epsilon_1, \dots, \epsilon_m]/(\epsilon_1^2, \dots, \epsilon_m^2) \otimes \wedge(\tau_1, \dots, \tau_n)$$

in which $\tau_i$, $1 \le i \le n$, are odd variables; for arbitrary $\tau_i'$ and $\epsilon_j'$ in $B$, say, one has only to specialize the obtained expression $h = f\, e(a_1, v_1) \dots e(a_n, v_n)$ via $\mathbf{\Sigma}(A) \to \mathbf{\Sigma}(B)$ induced by $A \to B$, $\tau_i \mapsto \tau_i'$, $\epsilon_j \mapsto \epsilon_j'$. Lemma 4.4, applied to this $A$ and the super-deal $\mathfrak{a} = (\epsilon_1, \dots, \epsilon_m, \tau_1, \dots, \tau_n)$, gives the desired expression, since $\mathfrak{a}$ is nilpotent, and so $\mathbf{\Sigma}_k$ is trivial for $k \gg 0$. $\square$

4.2. **The group $\mathbf{\Gamma}(A)$.** Retain the situation as above.

Let $G$ be an affine group. The right adjoint action $G \times G \to G$, $(h, g) \mapsto g^{-1}hg$ is dualized to the left $G$-module structure on $\mathcal{O}(G)$ defined by

$$ {}^g c = g^{-1}(c_{(1)})\, c_{(2)}\, g(c_{(3)}), \quad g \in G(R),\ c \in \mathcal{O}(G), \tag{4.3}$$

where $R \in \mathsf{Alg}_\Bbbk$. This makes $\mathcal{O}(G)$ into a Hopf-algebra object in the symmetric tensor category $G$-$\mathsf{Mod}$ of left $G$-modules.

Recall that $\mathfrak{g}$ is a Lie superalgebra equipped with a 2-operation, and it satisfies (C). Let $\mathsf{Aut}_{Lie}(\mathfrak{g})$ denote the subgroup functor of $\mathsf{GL}(\mathfrak{g})$ (see Section 3.5) that assigns to each $R \in \mathsf{Alg}_\Bbbk$, the group $\mathrm{Aut}_{R\text{-}Lie}(\mathfrak{g}_R)$ of all $R$-Lie-superalgebra automorphisms preserving $(\ )_R^{\langle 2\rangle}$; see Lemma 3.3.

We are going to work in a more general situation than will be needed to discuss a category equivalence in Section 5.1; see Remark 4.14 for the reason.

Suppose that we are given a pairing and an anti-morphism,

$$\langle\ ,\ \rangle : \mathfrak{g}_0 \times \mathcal{O}(G) \to \Bbbk, \quad \alpha : G \to \mathsf{Aut}_{Lie}(\mathfrak{g}). \tag{4.4}$$

As in (3.9), let us write as $\rho_\alpha(z) = z_{(-1)} \otimes z_{(0)}$, $z \in \mathfrak{g}$. We assume that

(D1) $[z, x] = \langle x,\ z_{(-1)}\rangle\, z_{(0)}$,
(D2) $\langle x,\ cd\rangle = \langle x,\ c\rangle\, \varepsilon(d) + \varepsilon(c)\, \langle x,\ d\rangle$, and
(D3) $\langle x^g,\ c\rangle_R = \langle x,\ {}^g c\rangle_R$,

where $x \in \mathfrak{g}_0$, $z \in \mathfrak{g}$, $c, d \in \mathcal{O}(G)$ and $g \in G(R)$, $R \in \mathsf{Alg}_\Bbbk$.

By (D2) we have the map

$$\mathfrak{g}_0 \to \mathrm{Lie}(G)\, (\subset \mathcal{O}(G)^*), \quad x \mapsto \langle x,\ -\rangle. \tag{4.5}$$

This is a Lie algebra map, since we see from (D1) for even $z$ and (D3) that

$$\begin{aligned}\langle [x, y],\ c\rangle &= \langle x, c_{(2)}\rangle\, \langle y,\ \mathcal{S}(c_{(1)})c_{(3)}\rangle \\ &= \langle x \otimes y,\ \Delta(c)\rangle - \langle y \otimes x,\ \Delta(c)\rangle,\end{aligned}$$

where $x, y \in \mathfrak{g}_0$, $c \in \mathcal{O}(G)$. Therefore, it uniquely extends to an algebra map $U(\mathfrak{g}_0) \to \mathcal{O}(G)^*$, with which is associated the Hopf pairing

$$\langle\ ,\ \rangle : U(\mathfrak{g}_0) \times \mathcal{O}(G) \to \Bbbk \tag{4.6}$$

that uniquely extends the given pairing.

Recall $A \in \mathsf{SAlg}_{\Bbbk}$. By Lemma 3.1 the base extension to $A_0$ of the last Hopf pairing gives rise to the group map

$$\mathsf{Gpl}(U(\mathfrak{g}_0)_{A_0}) \to \mathsf{Alg}_{\Bbbk}(\mathcal{O}(G), A_0) = G(A_0), \quad g \mapsto \langle g, - \rangle_{A_0},$$

whose restriction to $F(A_0)$ we denote by

$$i_{A_0} = i : F(A_0) \to G(A_0).$$

**Lemma 4.5.** *Let $R \in \mathsf{Alg}_{\Bbbk}$ and $g \in G(R)$. Then $\alpha_R(g) \in \mathrm{Aut}_{R\text{-}Lie}(\mathfrak{g}_R)$ uniquely extends to an automorphism of the Hopf superalgebra $\mathbf{U}(\mathfrak{g})_R$ over $R$.*

*Proof.* One sees that $\alpha_R(g)$ uniquely extends an automorphism of the $R$-Hopf superalgebra $\mathbf{T}(\mathfrak{g})_R$. It is easy to see that the automorphism stabilizes the super-ideal of $\mathbf{T}(\mathfrak{g})_R$ generated by the elements $zw - (-1)^{|z||w|}wz - [z, w]$ in (3.6). To see that it stabilizes the super-ideal generated by all elements in (3.6), let $v \in \mathfrak{g}_1$, and suppose $v^g = \sum_i c_i \otimes v_i \in R \otimes \mathfrak{g}_1$. Then the desired result will follow if one compares the following two:

$$(v^{\langle 2 \rangle})^g = (v^g)_R^{\langle 2 \rangle} = \sum_i c_i^2 \otimes v_i^{\langle 2 \rangle} + \sum_{i<j} c_i c_j \otimes [v_i, v_j],$$

$$(v^2)^g = (v^g)^2 = \sum_i c_i^2 \otimes v_i^2 + \sum_{i<j} c_i c_j \otimes (v_i v_j + v_j v_i).$$

□

The assignment of the above extended automorphism to $g \in G(R)$ gives rise to an anti-morphism from $G$ to the automorphism group functor of $\mathbf{U}(\mathfrak{g})$, which we denote again by

$$\alpha : G \to \mathsf{Aut}_{Hopf}(\mathbf{U}(\mathfrak{g})).$$

Given $g \in G(A_0)$, the base extension $(\alpha_{A_0}(g))_A$ of $\alpha_{A_0}(g) \in \mathrm{Aut}_{A_0\text{-}Hopf}\,(\mathbf{U}(\mathfrak{g})_{A_0})$ along $A_0 \to A$ is an automorphism of the Hopf superalgebra $\mathbf{U}(\mathfrak{g})_A$ over $A$. As before, we will write $u^g$ for $(\alpha_{A_0}(g))_A(u)$, where $u \in \mathbf{U}(\mathfrak{g})_A$, $g \in G(A_0)$. Since the action stabilizes $\mathbf{\Sigma}(A)$, as is seen from the next lemma, it follows that $g \mapsto (\alpha_{A_0}(g))_A|_{\Sigma(A)}$ defines a anti-group map from $G(A_0)$ to the automorphism group of the group $\mathbf{\Sigma}(A)$, which we denote by

$$\alpha_A : G(A_0) \to \mathrm{Aut}(\mathbf{\Sigma}(A)).$$

**Lemma 4.6.** *Let $g \in G(A_0)$. Let $e(a, v)$ and $f(\epsilon, x)$ be as before. Suppose*

$$\rho_\alpha(v) = \sum_{i=1}^n c_i \otimes v_i \in \mathcal{O}(G) \otimes \mathfrak{g}_1, \quad \rho_\alpha(x) = \sum_{j=1}^m d_j \otimes x_j \in \mathcal{O}(G) \otimes \mathfrak{g}_0.$$

*Then we have*

(1) $e(a, v)^g = e(ag(c_1), v_1)\, e(ag(c_2), v_2) \dots e(ag(c_n), v_n)$,
(2) $f(\epsilon, x)^g = f(\epsilon g(d_1), x_1)\, f(\epsilon g(d_2), x_2) \dots f(\epsilon g(d_m), x_m)$.

This is easy to see. We remark that $F(A_0)$ is $G(A_0)$-stable by Part 2.

**Proposition 4.7.** *The quintuple*

$$(\mathbf{\Sigma}(A),\ F(A_0),\ G(A_0),\ i_{A_0},\ \alpha_A)$$

*satisfies Conditions (A1)–(A3) given in Section 2.*

*Proof.* Since the last remark shows that the first half of (A3) is satisfied, it remains to verify (A2) and the second half of (A3).

Choose $g \in G(A_0)$, and let $f = f(\epsilon, x)$. Note

$$i(f)(c) = \varepsilon(c)1 + \epsilon\langle x,\ c\rangle, \quad c \in \mathcal{O}(G).$$

Then by using (D3) we see

$$\begin{aligned} i(f^g)(c) &= \varepsilon(c)1 + \epsilon\,\langle x^g,\ c\rangle_{A_0} = \varepsilon(c)1 + \epsilon\,\langle x,\ {}^g c\rangle_{A_0} \\ &= \varepsilon(c)1 + \epsilon\, g^{-1}(c_{(1)})\,\langle x,\ c_{(2)}\rangle\, g(c_{(3)}) \\ &= (g^{-1} i(f) g)(c), \end{aligned}$$

which verifies the second half of (A3). By using (D1) we see

$$\begin{aligned} e(a,v)^{i(f)} &= 1 \otimes 1 + a\, i(f)(v_{(-1)}) \otimes v_{(0)} \\ &= 1 \otimes 1 + a \otimes v + \epsilon a \otimes \langle x,\ v_{(-1)}\rangle v_{(0)} \\ &= 1 \otimes 1 + a \otimes v + \epsilon a \otimes [v, x] \\ &= e(a,v)\, e(\epsilon a, [v,x]), \end{aligned}$$

and similarly,

$$f(\eta, y)^{i(f)} = f(\eta, y)\, f(\epsilon\eta, [y, x]).$$

These, combined with (iii)–(iv) of Lemma 4.2, verify (A2). □

**Definition 4.8.** $\mathbf{\Gamma}(A)$ denotes the base extension of $\mathbf{\Sigma}(A)$ along $i_{A_0}$ : $F(A_0) \to (G(A_0), \alpha_A)$; see Definition 2.2.

In $\mathbf{\Gamma}(A)$, the natural images of $e(a, v)$ and of elements $g \in G(A_0)$ will be denoted by the same symbols.

**Proposition 4.9.** *Choose arbitrarily a $\Bbbk$-free basis $v_1, \dots, v_n$ of $\mathfrak{g}_1$. Then every element of $\mathbf{\Gamma}(A)$ is uniquely expressed in the form*

$$g\, e(a_1, v_1)\, e(a_2, v_2) \dots e(a_n, v_n),$$

*where $g \in G(A_0)$, $a_i \in A_1$, $1 \le i \le n$.*

*Proof.* This follows from Proposition 4.3 (2) and the proof of Lemma 2.1 (2). □

Gavarini's original construction starts with constructing by generators and relation the group which shall be the functor points of the desired affine supergroup; see the group $G_{\mathcal{P}}(A)$ defined by [2, Definition 4.3.2]. Let us prove that the group, which is essentially the same as $\mathbf{\Gamma}'(A)$ below, is isomorphic to our $\mathbf{\Gamma}(A)$, though the result will not be used in the subsequent argument.

**Lemma 4.10.** *Choose arbitrarily a $\Bbbk$-free basis $v_1, \dots, v_n$ of $\mathfrak{g}_1$. Let $E(A_1)$ denote the free group on the set of the symbols*

$$e_j(a),\ 1 \le j \le n,\ a \in A_1,$$

*and let $\mathbf{\Gamma}'(A)$ denote the quotient group of the free product $G(A_0) * E(A_1)$ divided by the relations*

(i) $e_j(a)\, e_k(b) = i(f(-ab, [v_j, v_k]))\, e_k(b)\, e_j(a),\ \ k < j,$

(ii) $e_j(a)\, e_j(b) = i(f(-ab, v_j^{\langle 2\rangle}))\, e_j(a+b),$

(iii) *$e_j(a)\,g = g\,e_1(ag(c_{j1}))\dots e_n(ag(c_{jn}))$, where $g \in G(A_0)$, and we suppose $\rho_\alpha(v_j) = \sum_{k=1}^n c_{jk} \otimes v_k$.*

*Then*

$$e_j(a) \mapsto e(a, v_j), \quad 1 \le j \le n,\ a \in A_1$$

*gives an isomorphism $\mathbf{\Gamma}'(A) \xrightarrow{\simeq} \mathbf{\Gamma}(A)$ which is identical on $G(A_0)$.*

*Proof.* It is easy to see that the assignment above gives an epimorphism. By Proposition 4.9, $g\,e(a_1, v_1)\dots e(a_n, v_n) \mapsto g\,e_1(a_1)\dots e_n(a_n)$ well defines a section. This section is surjective, since one sees just as proving Proposition 4.3 that every element of $\mathbf{\Gamma}'(A)$ is expressed in the form $g\,e_1(a_1)\dots e_n(a_n)$, where $g \in G(A_0)$, $a_j \in A_1$, $1 \le j \le n$. The surjectivity of the section proves that the epimorphism is an isomorphism. □

4.3. **The affine supergroup $\mathbf{\Gamma}$.** Retain the situation as above. One sees easily that

$$A \mapsto \mathbf{\Gamma}(A)$$

defines a group functor $\mathbf{\Gamma}$ on $\mathsf{SAlg}_\Bbbk$. Moreover, we see:

**Proposition 4.11.** *This $\mathbf{\Gamma}$ is an affine supergroup, represented by the supercommutative superalgebra*

$$\mathbf{O} := \mathcal{O}(G) \otimes \wedge(\mathfrak{g}_1^*). \tag{4.7}$$

*Proof.* Choose a $\Bbbk$-free basis $v_1, \dots, v_n$ of $\mathfrak{g}_1$, as above. Let $w_1, \dots, w_n$ denote the dual basis of $\mathfrak{g}_1^*$. Proposition 4.9 gives the bijection

$$G(A) \times A_1^n \xrightarrow{\simeq} \mathbf{\Gamma}(A), \quad (g, a_1, \dots, a_n) \mapsto g\,e(a_1, v_1)\dots e(a_n, v_n), \tag{4.8}$$

which is seen to be natural in $A$. To an element $(g, a_1, \dots, a_n) \in G(A) \times A_1^n$, assign the superalgebra map $\phi : \mathbf{O} \to A$ defined by

$$\phi(c) = g(c),\ c \in \mathcal{O}(G), \quad \phi(w_i) = a_i,\ 1 \le i \le n.$$

This indeed defines such a map since every element in $A_1$ is required to be square-zero. The assignment above is in fact a bijection

$$G(A) \times A_1^n \xrightarrow{\simeq} \mathsf{SAlg}_\Bbbk(\mathbf{O}, A) \tag{4.9}$$

which is natural in $A$. This proves the proposition. □

**Remark 4.12.** Note that $G$, regarded as $A \mapsto G(A_0)$, is a subgroup functor of $\mathbf{\Gamma}$. Let $\mathbf{G}_\mathrm{a}^{-n}$ denote the functor which assigns to $A \in \mathsf{SAlg}_\Bbbk$ the additive group $A_1^n$. Note that this $\mathbf{G}_\mathrm{a}^{-n}$ is represented by $\wedge(\mathfrak{g}_1^*)$. One sees that the bijection (4.8) gives rise to a left $G$-equivariant isomorphism $G \times \mathbf{G}_\mathrm{a}^{-n} \xrightarrow{\simeq} \mathbf{\Gamma}$ of functors which preserves the identity element.

The superalgebra $\mathbf{O}$ has a unique Hopf-superalgebra structure that makes the composite $\mathbf{\Gamma}(A) \xrightarrow{\simeq} \mathsf{SAlg}_\Bbbk(\mathbf{O}, A)$ of the bijections (4.8) and (4.9) into an isomorphism of group functors. In particular, the counit is the tensor product

$$\varepsilon \otimes \varepsilon : \mathcal{O}(G) \otimes \wedge(\mathfrak{g}_1^*) \to \Bbbk$$

of the counits of the Hopf superalgebras $\mathcal{O}(G)$ and $\wedge(\mathfrak{g}_1^*)$, as is seen from Remark 4.12. It follows that

$$\mathbf{O}^+/(\mathbf{O}^+)^2 = \mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2 \oplus \mathfrak{g}_1^*,$$

which is dualized to the identification

$$\mathrm{Lie}(\mathbf{\Gamma}) = \mathrm{Lie}(G) \oplus \mathfrak{g}_1$$

of $\Bbbk$-supermodules.

Let $i' : \mathfrak{g}_0 \to \mathrm{Lie}(G)$ denote the Lie algebra map given by (4.5). Let $\mathrm{Der}(\mathfrak{g})$ denote the Lie algebra of $\Bbbk$-super-linear derivations on $\mathfrak{g}$. The morphism $\alpha$ given in (4.4) induces the anti-Lie algebra map

$$\alpha' : \mathrm{Lie}(G) \to \mathrm{Der}(\mathfrak{g}), \quad \alpha'(x)(z) = x(z_{(-1)})\, z_{(0)},$$

where $x \in \mathrm{Lie}(G),\ z \in \mathfrak{g}$. We remark that by (D1), the composite $\alpha' \circ i' : \mathfrak{g}_0 \to \mathrm{Der}(\mathfrak{g})$ coincides with the right adjoint representation.

**Proposition 4.13.** *We have the following.*

(1) *The super-bracket on* $\mathrm{Lie}(\mathbf{\Gamma}) = \mathrm{Lie}(G) \oplus \mathfrak{g}_1$ *is given by*

$$[(x,u),(y,v)] = \big([x,y] + i'([u,v]),\ \alpha'(y)(u) - \alpha'(x)(v)\big),$$

*where* $x, y \in \mathrm{Lie}(G)$, $u, v \in \mathfrak{g}_1$.

(2) $i' \oplus \mathrm{id} : \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \to \mathrm{Lie}(G) \oplus \mathfrak{g}_1 = \mathrm{Lie}(\mathbf{\Gamma})$ *is a Lie superalgebra map which preserves the* 2*-operation.*

*Proof.* (1) We see from Remark 4.12 that $\mathcal{O}(G)$ is a quotient Hopf superalgebra of $\mathbf{O}$ through $\mathrm{id} \otimes \varepsilon : \mathbf{O} = \mathcal{O}(G) \otimes \wedge(\mathfrak{g}_1^*) \to \mathcal{O}(G)$, and $G$ is thus a closed super-subgroup of $\mathbf{\Gamma}$; see the paragraph following the proof. It follows that $\mathrm{Lie}(G)$ is a Lie super-subalgebra of $\mathrm{Lie}(\mathbf{\Gamma})$ through the inclusion $\mathrm{Lie}(G) \to \mathrm{Lie}(G) \oplus \mathfrak{g}_1$.

It remains to compute $[v_1, v_2]$ in $\mathrm{Lie}(\mathbf{\Gamma})$, where $v_1, v_2 \in \mathfrak{g}_1$, or $v_1 \in \mathfrak{g}_1$, $v_2 \in \mathrm{Lie}(G)$. If elements $\tau \in A$ and $v \in \mathrm{Lie}(\mathbf{\Gamma})$ satisfy $\tau^2 = 0$ and $|\tau| = |v|$, then

$$g(\tau, v) : \mathbf{O} \to A, \quad g(\tau, v)(h) = \varepsilon(h)1 + \tau\, v(h),\ h \in \mathbf{O}$$

is an element in $\mathbf{\Gamma}(A)$ with inverse $g(-\tau, v)$. This coincides with $e(\tau, v)$ if $|\tau| = |v| = 1$. Note that $g(\tau, v) = i(f(\tau, x))$, if $|\tau| = 0$ and $v = i'(x)$ with $x \in \mathfrak{g}_0$. Given elements $g_1 = g(\tau_1, v_1)$, $g_2 = g(\tau_2, v_2)$ as above, then the commutator $(g_1, g_2) = g_1 g_2 g_1^{-1} g_2^{-1}$ coincides with

$$g((-1)^{|\tau_1||\tau_2|}\tau_1\tau_2,\ [v_1, v_2]),$$

from which we will see the desired values of $[v_1, v_2]$.

First, suppose that $A = \wedge(\tau_1, \tau_2)$, where $\tau_i$, $i = 1, 2$, are odd variables. Let $u, v \in \mathfrak{g}_1$. Since we have $(e(\tau_1, u), e(\tau_2, v)) = g(-\tau_1\tau_2,\ i'([u, v]))$ by (i) of Lemma 4.2, it follows that

$$[(0,u),(0,v)] = (i'([u,v]), 0).$$

Next, suppose that $A = \Bbbk[\tau_1]/(\tau_1^2) \otimes \wedge(\tau_2)$, where $\tau_1$ (resp., $\tau_2$) is an even (resp., odd) variable. Let $y, v \in \mathrm{Lie}(\mathbf{\Gamma})$ with $y$ even and $v$ odd. Note $g(\pm\tau_1, y) \in G(A_0)$. Since we see from (1) of Lemma 4.6 that

$$(g(-\tau_1, y), e(\tau_2, v)) = e(\tau_2, v)^{g(\tau_1, y)}\, e(-\tau_2, v) = e(\tau_1\tau_2, \alpha'(y)(v)),$$

it follows that

$$[(0,v),(y,0)] = (\alpha'(y)(v), 0).$$

(2) By Part 1 and the remark given above the proposition it remains to prove that the map preserves the 2-operation. Suppose again that $A = \wedge(\tau_1, \tau_2)$. Then we see from (ii) of Lemma 4.2 that

$$g(-\tau_1\tau_2,\ i'(v^{\langle 2\rangle})) = e(\tau_1, v)\, e(\tau_2, v)\, e(-(\tau_1 + \tau_2), v).$$

This last equals $g(-\tau_1\tau_2,\ v^2)$, which proves the desired result. □

Recall that a *closed super-subgroup* of an affine supergroup $\mathbf{G}$ is a subgroup functor of $\mathbf{G}$ which is represented by a quotient Hopf superalgebra of $\mathcal{O}(\mathbf{G})$.

**Remark 4.14.** We have worked in a general situation as above, aiming at an application to super Lie groups over a complete field of characteristic $\neq 2$; see [5]. Suppose that $\mathfrak{G}$ is such a super Lie group. Then a Lie group, $\mathfrak{G}_{red}$, is naturally associated with it. Let $\mathcal{R}(\mathfrak{G}_{red})$ be the commutative Hopf algebra of all analytic representative functions on $\mathfrak{G}_{red}$; this is not necessarily finitely generated. The corresponding affine group and the Lie superalgebra $\mathrm{Lie}(\mathfrak{G})$ of $\mathfrak{G}$ have a natural pairing and an anti-morphism as in (4.4), which satisfy (D1)–(D3). In [5] the resulting affine supergroup $\mathbf{\Gamma}$ is used and is proved to be *universal algebraic hull* of $\mathfrak{G}$ (see [4, p.1141]) in the sense that $\mathbf{\Gamma}$-supermodules are naturally identified with analytic $\mathfrak{G}$-supermodules.

## 5. The category equivalence over a commutative ring

We continue to suppose that $\Bbbk$ is an arbitrary non-zero commutative ring.

5.1. **Re-proving Gavarini's equivalence.** Let $\mathbf{G}$ be an affine supergroup, and set $\mathbf{O} = \mathcal{O}(\mathbf{G})$. Recall from [13, Section 2.5], for example, that the *associated affine group* $\mathbf{G}_{ev}$ is the restricted group functor $\mathbf{G}|_{\mathsf{Alg}_\Bbbk}$ defined on $\mathsf{Alg}_\Bbbk$. This is represented by the largest purely even quotient Hopf superalgebra

$$\overline{\mathbf{O}} := \mathbf{O}/\mathbf{O}\mathbf{O}_1\ (= \mathbf{O}_0/\mathbf{O}_1^2) \tag{5.1}$$

of $\mathbf{O}$, so that $\overline{\mathbf{O}} = \mathcal{O}(\mathbf{G}_{ev})$. This $\mathbf{G}_{ev}$ is also regarded as the closed super-subgroup of $\mathbf{G}$ which assigns to $A \in \mathsf{SAlg}_\Bbbk$ the group $\mathbf{G}(A_0)$. Let

$$W^{\mathbf{O}} := \mathbf{O}_1/\mathbf{O}_0^+\mathbf{O}_1,\ \text{ where } \mathbf{O}_0^+ = \mathbf{O}_0 \cap \mathbf{O}^+, \tag{5.2}$$

as in [10]. Since $\mathbf{O}_0^+/((\mathbf{O}_0^+)^2 + \mathbf{O}_1^2) \simeq \overline{\mathbf{O}}^+/(\overline{\mathbf{O}}^+)^2$, we have

$$\mathbf{O}^+/(\mathbf{O}^+)^2 \simeq \overline{\mathbf{O}}^+/(\overline{\mathbf{O}}^+)^2 \oplus W^{\mathbf{O}},$$

which is dualized to

$$\mathrm{Lie}(\mathbf{G}) \simeq \mathrm{Lie}(\mathbf{G}_{ev}) \oplus (W^{\mathbf{O}})^*;$$

see [13, Lemma 4.3]. It follows that

$$\mathrm{Lie}(\mathbf{G})_0 \simeq \mathrm{Lie}(\mathbf{G}_{ev}), \quad \mathrm{Lie}(\mathbf{G})_1 = (W^{\mathbf{O}})^*.$$

The former is the canonical Lie-algebra isomorphism induced from the embedding $\overline{\mathbf{O}}^* \subset \mathbf{O}^{\bar{*}}$, through which we will identify as

$$\mathrm{Lie}(\mathbf{G})_0 = \mathrm{Lie}(\mathbf{G}_{ev}).$$

Just as for (4.3), the right adjoint action $\mathbf{G} \times \mathbf{G}_{ev} \to \mathbf{G}$, $(f, g) \mapsto g^{-1}fg$ is dualized to the left $\mathbf{G}_{ev}$-supermodule structure on $\mathbf{O}$ defined by

$${}^g h = g^{-1}(h_{(1)})\, h_{(2)}\, g(h_{(3)}), \quad g \in \mathbf{G}_{ev}(R),\ h \in \mathbf{O}, \tag{5.3}$$

where $R \in \mathsf{Alg}_{\Bbbk}$. This makes $\mathbf{O}$ into a Hopf-algebra object in the symmetric tensor category $\mathbf{G}_{ev}$-$\mathsf{SMod}$ of left $\mathbf{G}_{ev}$-supermodules; see Section 3.5.

Let us recall the definitions [2, Definitions 3.2.6, 4.1.2] of two categories, following mostly the formulation of [13, Appendix].

First, let $(\text{gss-fsgroups})_{\Bbbk}$ denote the category of the affine supergroups $\mathbf{G}$ such that when we set $\mathbf{O} = \mathcal{O}(\mathbf{G})$,

(E1) $W^{\mathbf{O}}$ is $\Bbbk$-finite free,
(E2) $\overline{\mathbf{O}}^+/(\overline{\mathbf{O}}^+)^2$ is $\Bbbk$-finite projective, and
(E3) there exists a counit-preserving isomorphism $\mathbf{O} \simeq \overline{\mathbf{O}} \otimes \wedge(W^{\mathbf{O}})$ of left $\overline{\mathbf{O}}$-comodule superalgebras.

A morphism in $(\text{gss-fsgroups})_{\Bbbk}$ is a natural transformation of group functors. The conditions above re-number those (E1)–(E3) given in [13, Appendix].

**Remark 5.1.** It will be proved by Theorem 5.7 in the next subsection that an affine supergroup $\mathbf{G}$ necessarily satisfies (E3), if it satisfies (E1) and (E2), and if $\overline{\mathbf{O}}\,(= \mathcal{O}(\mathbf{G}_{ev}))$ is $\Bbbk$-flat.

Next, to define the category $(\text{sHCP})_{\Bbbk}$, let $(G, \mathfrak{g})$ be a pair of an affine group $G$ and a Lie superalgebra $\mathfrak{g}$ equipped with a 2-operation. Suppose that $\mathfrak{g}_1$ is $\Bbbk$-finite free, and is given a right $G$-module structure. Suppose in addition,

(F1) $\mathfrak{g}_0 = \text{Lie}(G)$,
(F2) $\mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2$ is $\Bbbk$-finite projective, so that $\mathfrak{g}_0 = \text{Lie}(G)$ is necessarily $\Bbbk$-finite projective, and has the right $G$-module structure (see (3.10), and (5.5) below) determined by

$$x^g(c) = x({}^g c), \quad x \in \mathfrak{g}_0,\ c \in \mathcal{O}(G), \tag{5.4}$$

where ${}^g c = g^{-1}(c_{(1)})\, c_{(2)}\, g(c_{(3)})$, as in (4.3),
(F3) the left $\mathcal{O}(G)$-comodule structure $\mathfrak{g}_1 \to \mathcal{O}(G) \otimes \mathfrak{g}_1$, $v \mapsto v_{(-1)} \otimes v_{(0)}$ on $\mathfrak{g}_1$ corresponding to the given right $G$-module structure satisfies

$$[v, x] = x(v_{(-1)})\, v_{(0)}, \quad v \in \mathfrak{g}_1,\ x \in \mathfrak{g}_0,$$

(F4) the restricted super-bracket $[\ ,\ ]|_{\mathfrak{g}_1 \otimes \mathfrak{g}_1} : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$ is $G$-equivariant, and
(F5) the right $G$-module structure preserves the 2-operation, or explicitly,

$$(v_R^{\langle 2 \rangle})^g = (v^g)_R^{\langle 2 \rangle}, \quad v \in (\mathfrak{g}_1)_R,\ g \in G(R),$$

where $R \in \mathsf{Alg}_{\Bbbk}$.

As for the last (F5) we have replaced the original [13, Appendix, (F5)] with the equivalent one given soon after.

Finally, let $(\text{sHCP})_{\Bbbk}$ denote the category of all those pairs $(G, \mathfrak{g})$ which satisfy (F1)–(F5) above. A morphism $(G, \mathfrak{g}) \to (G', \mathfrak{g}')$ in $(\text{sHCP})_{\Bbbk}$ is a pair $(\gamma, \delta)$ of a morphism $\gamma : G \to G'$ of affine groups and a Lie superalgebra map $\delta = \delta_0 \oplus \delta_1 : \mathfrak{g} \to \mathfrak{g}'$, such that

(F6) the Lie algebra map $\text{Lie}(\gamma)$ induced from $\gamma$ coincides with $\delta_0$,
(F7) $(\delta_1)_R(v^g) = \delta_1(v)^{\gamma_R(g)}$, $v \in \mathfrak{g}_1$, $g \in G(R)$, where $R \in \mathsf{Alg}_{\Bbbk}$, and
(F8) $\delta_0(v^{\langle 2 \rangle}) = \delta_1(v)^{\langle 2 \rangle}$, $v \in \mathfrak{g}_1$.

Let us reproduce from [2] functors between the two categories just defined,

$$\Phi : (\mathrm{gss\text{-}fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}, \quad \Psi : (\mathrm{sHCP})_{\Bbbk} \to (\mathrm{gss\text{-}fsgroups})_{\Bbbk},$$

which are denoted by $\Phi_g$, $\Psi_g$ in [2].

First, let $\mathbf{G}$ be an object in $(\mathrm{gss\text{-}fsgroups})_{\Bbbk}$. Set $\mathbf{O} = \mathcal{O}(\mathbf{G})$. Consider the pair

$$(G, \mathfrak{g}) := (\mathbf{G}_{ev}, \mathrm{Lie}(\mathbf{G})),$$

giving to $\mathfrak{g}_1$ the right $G$-module structure determined by

$$v^g(h) = v({}^g h), \quad v \in \mathfrak{g}_1,\ h \in \mathbf{O},\ g \in G(R), \tag{5.5}$$

where $R \in \mathsf{Alg}_{\Bbbk}$, and ${}^g h$ is given by (5.3). To see that this indeed defines a right $G$-module structure, note that the left $G$-module structure on $\mathbf{O}$ given by (5.3) induces such a structure on $\mathbf{O}^+/(\mathbf{O}^+)^2$, and the induced structure is transposed to $\mathfrak{g}$, since $\mathbf{O}^+/(\mathbf{O}^+)^2$ is $\Bbbk$-finite projective by (E1)–(E2); see (3.10). What is given by (5.5) is precisely the restriction to $\mathfrak{g}_1$ of the transposed structure, while the restriction to $\mathfrak{g}_0$ coincides with the one given by (5.4). It is now easy to see that the pair satisfies (F1)–(F4). Recall that $\mathfrak{g}$ is equipped with the 2-operation which arises from the square map on $\mathbf{O}^{\bar{*}}$. Then one verifies (F5), using the fact that the $G$-module structure on $\mathbf{O}$ preserves the coproduct; cf. [13, Lemma A.9]. Therefore, $(G, \mathfrak{g}) \in (\mathrm{sHCP})_{\Bbbk}$. We let

$$\Phi(\mathbf{G}) = (\mathbf{G}_{ev}, \mathrm{Lie}(\mathbf{G})).$$

One sees easily that this indeed defines a functor.

**Remark 5.2.** Following [2, Defintion 2.3.3], let $(\mathrm{fsgroups})_{\Bbbk}$ denote the category of those affine supergroup which satisfy (E1) and (E2). This includes $(\mathrm{gss\text{-}fsgroups})_{\Bbbk}$ as a full subcategory. Note that Condition (E3) was not used above, to define the functor $\Phi$. In fact we have thus defined a functor $\Phi : (\mathrm{fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}$, as is formulated by [2, Proposition 4.1.3]. This last functor will be used to prove Theorem 5.7 in the next subsection.

Next, to construct $\Psi$, we prove:

**Lemma 5.3.** *Let $\mathbf{\Gamma}$ be the affine supergroup constructed in Section 4, and set $\mathbf{O} = \mathcal{O}(\mathbf{\Gamma})$. Then we have*

$$\mathbf{\Gamma}_{ev} = G, \quad W^{\mathbf{O}} = \mathfrak{g}_1^*,$$

*where $G$ and $\mathfrak{g}$ are those given in Section 4 from which $\mathbf{\Gamma}$ is constructed. Moreover, $\mathbf{\Gamma}$ satisfies (E1) and (E3) above.*

*Proof.* From Remark 4.12 and the following argument we see that (4.7) gives an identification $\mathcal{O}(\mathbf{\Gamma}) = \mathcal{O}(G) \otimes \wedge(\mathfrak{g}_1^*)$ of left $\mathcal{O}(G)$-comodule superalgebras with counit. This implies the desired results. □

Finally, let $(G, \mathfrak{g}) \in (\mathrm{sHCP})_{\Bbbk}$. Choose these $G$ and $\mathfrak{g}$ as those in Section 4. By (F1)–(F2), $\mathfrak{g}$ satisfies (C); see Proposition 3.4. The given right $G$-module structure on $\mathfrak{g}_1$, summed up with such a structure on $\mathfrak{g}_0$ determined by (5.4), gives rise to an anti-morphism, say $\alpha$, from $G$ to $\mathsf{Aut}_{Lie}(\mathfrak{g})$; see [13, Remark 4.5 (2)]. This $\alpha$, together with the canonical pairing

$$\langle\ ,\ \rangle : \mathfrak{g}_0 \times \mathcal{O}(G) \to \Bbbk, \quad \langle x,\ c\rangle = x(c),$$

satisfy (D1)–(D3), as is easily seen. We remark that Lie algebra map $i' : \mathfrak{g}_0 \to \mathrm{Lie}(G)$ given by (4.5) is now the identity. The construction of Section 4 gives an affine supergroup $\mathbf{\Gamma}$, which satisfies (E1)–(E3) by Lemma 5.3. Indeed, by (F2) it satisfies (E2) as well, since $\mathbf{\Gamma}_{ev} = G$. Define $\Psi(G, \mathfrak{g})$ to be this $\mathbf{\Gamma}$ in $(\text{gss-fsgroups})_{\Bbbk}$. As is easily seen, $\Psi$ defines a functor.

**Theorem 5.4** ([2, Theorem 4.3.14])**.** *We have a category equivalence*

$$(\text{gss-fsgroups})_{\Bbbk} \approx (\text{sHCP})_{\Bbbk}.$$

*In fact the functors $\Phi$ and $\Psi$ constructed above are quasi-inverse to each other.*

*Proof.* Let $\mathbf{G} \in (\text{gss-fsgroups})_{\Bbbk}$, and set

$$(G, \mathfrak{g}) = \Phi(\mathbf{G}), \quad \mathbf{\Gamma} = \Psi \circ \Phi(\mathbf{G}).$$

Just as for (4.6) we see that there uniquely exists a Hopf paring

$$\langle\ ,\ \rangle : \mathbf{U}(\mathfrak{g}) \times \mathcal{O}(\mathbf{G}) \to \Bbbk$$

such that $\langle z,\ h\rangle = z(h)$, $z \in \mathfrak{g}$, $h \in \mathcal{O}(\mathbf{G})$. Suppose $A \in \mathsf{SAlg}_{\Bbbk}$. Recall that $\mathbf{\Gamma}(A)$ is a quotient of the group $G(A_0) \ltimes \mathbf{\Sigma}(A)$ of semi-direct product. Since $\mathbf{\Sigma}(A) \subset \mathsf{Gpl}(\mathbf{U}(\mathfrak{g})_A)$, the last pairing induces, after base extension to $A$, a group map

$$\mathbf{\Sigma}(A) \to \mathsf{SAlg}_{\Bbbk}(\mathcal{O}(\mathbf{G}), A) = \mathbf{G}(A). \tag{5.6}$$

Lemma 4.6 gives the following equations in $\mathbf{\Sigma}(A)$:

$$e(a, v)^g = 1 \otimes 1 + a\, v^g,\ f(\epsilon, x)^g = 1 \otimes 1 + \epsilon\, x^g, \quad g \in G(A_0). \tag{5.7}$$

By definitions of $\Phi$ and $\Psi$, the $G$-actions on $\mathfrak{g}$ which appear on the right-hand sides are determined by

$$\langle z^g,\ h\rangle_{A_0} = \langle z,\ {}^g h\rangle_{A_0}, \quad z \in \mathfrak{g},\ h \in \mathcal{O}(\mathbf{G}),\ g \in G(A_0),$$

where ${}^g h = g^{-1}(h_{(1)})\, h_{(2)}\, g(h_{(3)})$, as in (5.3). It follows that the group map (5.6) is right $G(A_0)$-equivariant, where we suppose that $G(A_0) = \mathbf{G}(A_0)$ acts on $\mathbf{G}(A)$ by inner automorphisms. Therefore, the group map together with the embedding $G(A_0) \to \mathbf{G}(A)$ uniquely extend to $G(A_0) \ltimes \mathbf{\Sigma}(A) \to \mathbf{G}(A)$. It factors through $\mathbf{\Gamma}(A) \to \mathbf{G}(A)$, since $\mathbf{\Gamma}(A)$ is the quotient group of $G(A_0) \ltimes \mathbf{\Sigma}(A)$ divided by the relations

$$(i(f(\epsilon, x)), 1) = (1, f(\epsilon, x)), \quad x \in \mathfrak{g}_0,\ \epsilon \in A_0,\ \epsilon^2 = 0,$$

and $i : F(A_0) \to G(A_0)$ is now the restriction of the canonical map $\mathsf{Gpl}(U(\mathfrak{g}_0)_{A_0}) \to G(A_0)$. The group map $\mathbf{\Gamma}(A) \to \mathbf{G}(A)$, being natural in $A$, gives rise to a morphism $\mathbf{\Gamma} \to \mathbf{G}$. This morphism is natural in $\mathbf{G}$, as is easily seen. In fact, it is a natural isomorphism by [13, Lemma 4.26]; see also Remark 5.12 below. Indeed, the assumptions required by the cited lemma are satisfied, since $\mathbf{\Gamma}$ and $\mathbf{G}$ satisfy (E3), the morphism $\mathbf{\Gamma} \to \mathbf{G}$ restricts to the identity $\mathbf{\Gamma}_{ev} \to \mathbf{G}_{ev}$, and the map $\mathfrak{g}_1 = \mathrm{Lie}(\mathbf{\Gamma})_1 \to \mathrm{Lie}(\mathbf{G})_1$ induced from the pairing above is the identity. We conclude $\Psi \circ \Phi \simeq \mathrm{id}$.

Let $(G, \mathfrak{g}) \in (\text{sHCP})_{\Bbbk}$, and set $\mathbf{\Gamma} = \Psi(G, \mathfrak{g})$. Recall that for this $\mathbf{\Gamma}$, the Lie algebra map $i' : \mathfrak{g}_0 \to \mathrm{Lie}(G)$ given by (4.5) is the identity. By Lemma 5.3 and Proposition 4.13 we have the natural identifications

$$G = \mathbf{\Gamma}_{ev}, \quad \mathfrak{g} = \mathrm{Lie}(\mathbf{\Gamma})$$

of affine groups and of Lie superalgebras equipped with 2-operation. Let $R \in \mathsf{Alg}_{\Bbbk}$. To conclude $\Phi \circ \Psi = \mathrm{id}$, we wish to prove that given $v \in \mathfrak{g}_1$ and $g \in G(R)$, the result $v^g \in (\mathfrak{g}_1)_R$ by the $G$-action associated with the original $(G, \mathfrak{g})$ coincides the one given by (5.5) for $\mathbf{\Gamma}$. Suppose $A = R \otimes \wedge(\tau)$, where $\tau$ is an odd variable. Note $A_0 = R$. Just as in (5.7) we have $e(\tau, v)^g = 1 \otimes 1 + \tau\, v^g$ in $\mathbf{\Gamma}(A)$. This, evaluated at $h \in \mathcal{O}(\mathbf{\Gamma})$, gives $\tau\, v({}^g h) = \tau\, v^g(h)$, which shows the desired result. $\square$

**Remark 5.5.** Let $(G, \mathfrak{g}) \in (\mathrm{sHCP})_{\Bbbk}$, and recall that this $\mathfrak{g}$ is equipped with a 2-operation, say $(\ )^{\langle 2 \rangle}$. Replace $(\mathfrak{g}, (\ )^{\langle 2 \rangle})$ with the cocycle deformation $({}_{\sigma}\mathfrak{g}, (\ )^{\sigma\langle 2 \rangle})$ by $\sigma$ (see Remark 3.7), keeping the right $G$-module structure on the odd component unchanged. Then we see $(G, {}_{\sigma}\mathfrak{g}) \in (\mathrm{sHCP})_{\Bbbk}$, and that $(G, \mathfrak{g}) \mapsto (G, {}_{\sigma}\mathfrak{g})$ gives an involutive category isomorphism, which we denote by

$$(\mathrm{id}, {}_{\sigma}(\ )) : (\mathrm{sHCP})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}.$$

As was remarked in Introduction, Gavarini's category equivalence was re-proved also in [13, Appendix], using an older construction of affine supergroups. Due to different choice of tensor products of pairings, the category equivalence $\mathbf{P}' : (\mathrm{gss\text{-}fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}$ obtained there is slightly different from the $\Phi$ above. In fact, we see

$$\mathbf{P}' = (\mathrm{id}, {}_{\sigma}(\ )) \circ \Phi. \tag{5.8}$$

**Remark 5.6.** The argument of Gavarini [2] seems incomplete at some points, as is pointed out below. See also [13, Remark A.11].

(1) To construct the functor $\Phi_g : (\mathrm{gss\text{-}fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}$, and prove $\Phi_g \circ \Psi_g = \mathrm{id}$ in [2, Proposition 4.1.3, Theorem 4.3.14], the article takes no account of 2-operations or $G$-supermodule structures on Lie superalgebras.

(2) The functoriality of $\Psi_g : (\mathrm{sHCP})_{\Bbbk} \to (\mathrm{gss\text{-}fsgroups})_{\Bbbk}$ (see [2, Proposition 4.3.9 (2)]) is proved, indeed, if one replaces the original definition of $\Psi_g$ by the group $G_{\mathcal{P}}(A)\, (= \Psi_g(\mathcal{P}))$ given in [2, Definition 4.3.2] (and referred to before Lemma 4.10), with the definition by the alternative $G^{\bullet}_{\mathcal{P}}(A)$ given in [2, Remark 4.3.3 (c)]. Nevertheless, in view of the equations preceding our Lemma 4.1, the relation $(1 + (c\eta)Y) = (1 + \eta(cY))$, $c \in \Bbbk$, is missing to define the group $G^{\bullet}_{\mathcal{P}}(A)$ in the last cited remark.

5.2. **Tensor product decomposition.** Let $\mathbf{G}$ be an affine supergroup, and set $\mathbf{O} = \mathcal{O}(\mathbf{G})$. Recall from (5.1) and (5.2) the definitions of $\overline{\mathbf{O}}$ and $W^{\mathbf{O}}$. As was announced in Remark 5.1 we prove the following theorem. Note that the conclusion below is the same as (E3).

**Theorem 5.7.** *Assume that $\overline{\mathbf{O}}\,(= \mathcal{O}(\mathbf{G}_{ev}))$ is $\Bbbk$-flat. Then there exists a counit-preserving isomorphism $\mathbf{O} \simeq \overline{\mathbf{O}} \otimes \wedge(W^{\mathbf{O}})$ of left $\overline{\mathbf{O}}$-comodule superalgebras, if*

(E1) *$W^{\mathbf{O}}$ is $\Bbbk$-finite free, and*

(E2) *$\overline{\mathbf{O}}^{+}/(\overline{\mathbf{O}}^{+})^2$ is $\Bbbk$-finite projective.*

**Remark 5.8.** (1) Let $(\mathrm{gss\text{-}fsgroups})'_{\Bbbk}$ denote the category of the affine supergroups $\mathbf{G}$ which satisfy (E1), (E2) and

(E0) $\mathcal{O}(\mathbf{G}_{ev})$ is $\Bbbk$-flat.

This category is a full subcategory of $(\text{gss-fsgroups})_{\Bbbk}$ by Theorem 5.7. Let $(\text{sHCP})'_{\Bbbk}$ denote the full subcategory of $(\text{sHCP})_{\Bbbk}$ which consists of the objects $(G, V)$ such that

(F0) $\mathcal{O}(G)$ is $\Bbbk$-flat.

One sees that the category equivalence given by Theorem 5.4 restricts to

$$(\text{gss-fsgroups})'_{\Bbbk} \approx (\text{sHCP})'_{\Bbbk}.$$

(2) Suppose that $\Bbbk$ is 2-torsion free, or namely, $2 : \Bbbk \to \Bbbk$ is an injection. In this special situation, essentially the same category equivalence as given by Theorem 5.4 was proved by [13, Theorem 4.22]; one need not there refer to 2-operations. To be more precise, considered there is the category $\mathsf{ASG}$ of the algebraic supergroups $\mathbf{G}$ which satisfy (E0) as well as (E1)–(E3); see [13, Section 4.3]. However, (E3) can be removed from the last conditions, since it is ensured by Theorem 5.7. To define $\mathsf{ASG}$ in [13], one can thus weaken the condition that $\mathbf{O} = \mathcal{O}(\mathbf{G})$ is *split* [13, Definition 2.1] to the one that $W^{\mathbf{O}}$ is $\Bbbk$-free. See [13, Note added in proof].

(3) Suppose that $\Bbbk$ is a field of characteristic $\neq 2$. Then the conclusion of Theorem 5.7 holds for any finitely generated super-commutative Hopf superalgebra $\mathbf{O}$, since the assumptions are then necessarily satisfied. The result was in fact proved by [10, Theorem 4.5] for any $\mathbf{O}$ that is not necessarily finitely generated. The proof uses Hopf crossed products, and is crucial when $\mathbf{O}$ is finitely generated. The proof below gives an alternative proof of the cited theorem in this crucial case.

The rest of this subsection is devoted to proving the theorem. The proof is divided into 3 steps.

5.2.1. *Step 1.* Recall from Remark 5.2 that the functor $\Phi$ is defined on the category $(\text{fsgroups})_{\Bbbk}$ including $(\text{gss-fsgroups})_{\Bbbk}$, which consists of the affine supergroup satisfying (E1) and (E2).

Let $\mathbf{G} \in (\text{fsgroups})_{\Bbbk}$, and set $\mathbf{\Gamma} = \Psi \circ \Phi(\mathbf{G})$, as in the proof of Theorem 5.4. The argument in the cited proof which shows that we have a natural morphism $\mathbf{\Gamma} \to \mathbf{G}$ of affine supergroups is valid. Let

$$\phi : \mathbf{\Gamma} \to \mathbf{G} \tag{5.9}$$

denote the morphism. We will prove that this $\phi$ is an isomorphism, assuming that $\overline{\mathbf{O}}$ is $\Bbbk$-flat. This proves the theorem, since $\mathbf{\Gamma}$ satisfies (E3).

5.2.2. *Step 2.* We need some general Hopf-algebraic argument. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the semigroup of non-negative integers. An $\mathbb{N}$-graded $\Bbbk$-module $V = \bigoplus_{n=0}^{\infty} V(n)$ is regarded as a $\Bbbk$-supermodule so that $V_0 = \bigoplus_{n \text{ even}} V(n)$, $V_1 = \bigoplus_{n \text{ odd}} V(n)$. The $\mathbb{N}$-graded $\Bbbk$-modules form a symmetric tensor category $\mathsf{GrMod}_{\Bbbk}$ with respect to the super-symmetry.

Let $\mathsf{ConnAlg}_{\Bbbk}$ denote the category of the commutative algebra objects $\mathbf{B}$ in $\mathsf{GrMod}_{\Bbbk}$ such that $\mathbf{B}(0) = \Bbbk$; the $\mathsf{Conn}$ expresses "connected", meaning $\mathbf{B}(0) = \Bbbk$.

Fix a commutative Hopf algebra $O$. Note that $O$ is a commutative Hopf-algebra object in $\mathsf{GrMod}_{\Bbbk}$ which is trivially graded, $O(0) = O$. A *graded left $O$-comodule* is a left $O$-comodule object in $\mathsf{GrMod}_{\Bbbk}$. The graded left $O$-comodules form a symmetric tensor category $O$-$\mathsf{GrComod}$. Let $O$-$\mathsf{NGrComodAlg}$

denote the category of the commutative algebra objects $\mathbf{A}$ in $O$-$\mathsf{GrComod}$ such that $\mathbf{A}(0) = O$; the $\mathsf{NGr}$ expresses "neutrally graded", meaning $\mathbf{A}(0) = O$. Note that every such object is an (ordinary) left $O$-Hopf module [15, Page 15] with respect to the left multiplication by $O$.

Here, commutative algebra objects may not satisfy the condition that every odd elements should be square-zero.

Given $\mathbf{B} \in \mathsf{ConnAlg}_{\Bbbk}$, the tensor product

$$O \otimes \mathbf{B}$$

of graded algebras, given the left $O$-comodule structure $\Delta \otimes \mathrm{id}_{\mathbf{B}}$, is an object in $O$-$\mathsf{NGrComodAlg}$. Moreover, this constructs a functor

$$O\otimes : \mathsf{ConnAlg}_{\Bbbk} \to O\text{-}\mathsf{NGrComodAlg}.$$

**Proposition 5.9.** *This functor is a category equivalence.*

*Proof.* Given $\mathbf{A} \in O$-$\mathsf{NGrComodAlg}$,

$$\mathbf{A}/O^+\mathbf{A}$$

is naturally an object in $\mathsf{ConnAlg}_{\Bbbk}$. One sees that this constructs a functor. We wish to show that this is a quasi-inverse of the functor $O\otimes$. We have to prove that the two composites of the functors are naturally isomorphic to the identity functors. For one composite this is easy. For the remaining, let $\mathbf{A} \in O$-$\mathsf{NGrComodAlg}$. Set $\mathbf{B} = \mathbf{A}/O^+\mathbf{A}$, and let $\pi : \mathbf{A} \to \mathbf{B}$ denote the natural projection. We see that the left $O$-comodule structure $\mathbf{A} \to O \otimes \mathbf{A}$, $a \mapsto a_{(-1)} \otimes a_{(0)}$ on $\mathbf{A}$ induces the morphism

$$\mathbf{A} \to O \otimes \mathbf{B}, \quad a \mapsto a_{(-1)} \otimes \pi(a_{(0)}) \tag{5.10}$$

in $O$-$\mathsf{NGrComodAlg}$ which is natural in $\mathbf{A}$. It remains to prove that this is an isomorphism. As was remarked before, $\mathbf{A}$ is a left $O$-Hopf module, and the morphism above is in fact a morphism of Hopf modules. The fundamental theorem for Hopf modules [15, 1.9.4, Page 15] holds over an arbitrary base ring $\Bbbk$, and can now apply to see that (5.10) is an isomorphism; explicitly, the inverse is induced from $O \otimes \mathbf{A} \to \mathbf{A}$, $c \otimes a \mapsto cS(a_{(-1)})a_{(0)}$. $\square$

Let $\mathbf{O}$ be a super-commutative Hopf superalgebra. Set $O = \overline{\mathbf{O}}$, and assume that this $O$ is $\Bbbk$-flat. Let $O$-$\mathsf{SComod}$ denote the symmetric tensor category of left $O$-super-comodules. The flatness assumption ensures that this category is abelian; see [6, Part I, 2.9]. Indeed, the $\Bbbk$-linear kernel $Z$ of a morphism $V \to U$ turns to be a sub-object of $V$, since we have $O \otimes Z \subset O \otimes V$, and the composite $Z \hookrightarrow V \to O \otimes V$ of the inclusion with the structure on $V$ factors through $O \otimes Z$.

Let $I = \mathbf{O}\mathbf{O}_1$, so that we have $\mathbf{O}/I = O$. Note that $\mathbf{O}$ is naturally a commutative algebra object in $O$-$\mathsf{SComod}$, and the super-ideals $I^n$, $n > 0$, are sub-objects of $\mathbf{O}$ in $O$-$\mathsf{SComod}$. It follows that

$$\mathrm{gr}\,\mathbf{O} = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$$

is an object in $O$-$\mathsf{NGrComodAlg}$. To see this, note $\mathrm{gr}\,\mathbf{O}(0) = O$. Moreover, $I^n/I^{n+1} = \mathbf{O}_1^n/\mathbf{O}_1^{n+2}$, and so $\mathrm{gr}\,\mathbf{O}(n)$ is purely odd (resp., even) if $n$ is odd (resp., even).

Let $\mathbf{B} = \operatorname{gr} \mathbf{O}/O^+(\operatorname{gr} \mathbf{O})$ denote the object in $\mathsf{ConnAlg}_{\Bbbk}$ which corresponds to $\operatorname{gr} \mathbf{O}$ through the category equivalence given in (the proof of) Proposition 5.9. It is easy to see the following (see [10, Proposition 4.3 (1)]):

**Lemma 5.10.** *The composite of natural maps*

$$W^{\mathbf{O}} = \mathbf{O}_1/\mathbf{O}_0^+\mathbf{O}_1 \to \mathbf{O}_1/\mathbf{O}_1^3 = \operatorname{gr} \mathbf{O}(1) \to \mathbf{B}(1)$$

*is an isomorphism.*

5.2.3. *Step 3.* Let $\mathbf{O}$ be a super-commutative Hopf superalgebra. Note that the constructions of the associated $\overline{\mathbf{O}}$ and $W^{\mathbf{O}}$ are functorial.

Assume that $\mathbf{O}$ satisfies (E1) and (E3). Assume that $\overline{\mathbf{O}}$ is $\Bbbk$-flat. Let $\mathbf{O}'$ be a super-commutative Hopf superalgebra, and let $\psi : \mathbf{O}' \to \mathbf{O}$ is a Hopf superalgebra map. It naturally induces

$$\overline{\psi} : \overline{\mathbf{O}'} \to \overline{\mathbf{O}}, \quad W^{\psi} : W^{\mathbf{O}'} \to W^{\mathbf{O}}.$$

**Proposition 5.11.** *If these two maps are bijections, then $\psi$ is an isomorphism.*

*Proof.* We may suppose $\overline{\mathbf{O}'} = \overline{\mathbf{O}} = O$ and $\overline{\psi} = \mathrm{id}_O$, where $O$ is a commutative $\Bbbk$-flat Hopf algebra. We see that $\psi$ induces a morphism $\operatorname{gr}(\psi) : \operatorname{gr} \mathbf{O}' \to \operatorname{gr} \mathbf{O}$ in $O$-$\mathsf{NGrComodAlg}$. Let $\xi : \mathbf{B}' \to \mathbf{B}$ be the corresponding morphism between the corresponding objects in $\mathsf{ConnAlg}_{\Bbbk}$.

We wish to show that $\xi$ is an isomorphism. By Lemma 5.10, $\xi(1) : \mathbf{B}'(1) \to \mathbf{B}(1)$ is identified with $W^{\psi}$. Since $\mathbf{O}$ satisfies (E3), we see that $\operatorname{gr} \mathbf{O} = O \otimes \wedge(W^{\mathbf{O}})$, and so $\mathbf{B} = \wedge(W^{\mathbf{O}})$. It follows that $\xi$ has a unique section in $\mathsf{ConnAlg}_{\Bbbk}$, since $\xi(1)$ is an isomorphism, and $\mathbf{B}'$ is super-commutative, with the odd elements being square-zero. Note that $\mathbf{B}'$ is generated by $\mathbf{B}'(1)$, since $\operatorname{gr} \mathbf{O}'$ is generated by $O = \operatorname{gr} \mathbf{O}'(0)$ and $\operatorname{gr} \mathbf{O}'(1)$. This implies that the section is an isomorphism, proving the desired result.

It follows that $\operatorname{gr}(\psi)$ is an isomorphism, and $\operatorname{gr} \mathbf{O}'(n) = \operatorname{gr} \mathbf{O}(n) = 0$ for $n \gg 0$. Therefore, $\psi$ is an isomorphism. □

**Remark 5.12.** In the situation of Proposition 5.11, suppose in addition that $\mathbf{O}'$ satisfies (E3), and remove the assumption that $\overline{\mathbf{O}}$ is $\Bbbk$-flat. Then the same result as the proposition follows easily from [13, Lemma 4.26]. The result was essentially used to prove [13, Theorem A.11] in the last paragraph of the proof.

Let us return to the natural morphism $\phi : \mathbf{\Gamma} \to \mathbf{G}$ in (5.9), assuming that $\overline{\mathcal{O}(\mathbf{G})}$ is $\Bbbk$-flat. Consider $\mathcal{O}(\phi) : \mathcal{O}(\mathbf{G}) \to \mathcal{O}(\mathbf{\Gamma})$. In view of the proof of Theorem 5.4 (see the last part of the first paragraph), the induced $\overline{\mathcal{O}(\mathbf{G})} \to \overline{\mathcal{O}(\mathbf{\Gamma})}$ and $W^{\mathcal{O}(\mathbf{G})} \to W^{\mathcal{O}(\mathbf{\Gamma})}$ are both the identity maps. It follows that $\overline{\mathcal{O}(\mathbf{\Gamma})}$ is $\Bbbk$-flat. Since $\mathbf{\Gamma}$ satisfies (E1) and (E3), Proposition 5.11, applied to $\mathcal{O}(\phi)$, proves that $\phi$ is an isomorphism, as desired.

## 6. The category equivalence over a field

In this section we suppose that $\Bbbk$ is a field of characteristic $\neq 2$.

6.1. **Reformulation.** We let $\mathsf{ASG}_{\Bbbk}$ denote the category of algebraic supergroups over $\Bbbk$. This coincides with the full subcategory of $(\text{gss-fsgroups})_{\Bbbk}$ consisting of the objects which are algebraic supergroups. By [10, Theorem 4.5] (or Theorem 5.7 above) every object in $\mathsf{ASG}_{\Bbbk}$ satisfies (E3), in particular; see Remark 5.8 (3).

Note from Remark 3.5 (2) that we may not refer to 2-operations on Lie superalgebras. The definition of $(\text{sHCP})_{\Bbbk}$ then contains redundancy in (F1). In other words one can remove $\mathfrak{g}_0$ from the definition since it is determined by $G$. Following [12, 14], we modify the definition of Harish-Chandra pairs as follows; one will see in the next subsection that this modified definition is suitable at least to describe sub-objects.

A *Harish-Chandra pair* is a pair $(G, V)$ of an algebraic group $G$ and a finite-dimensional right $G$-module $V$ which is equipped with a $G$-equivariant linear map $[\ ,\ ] : V \otimes V \to \mathrm{Lie}(G)$ such that

(G1) $[v, v'] = [v', v], \quad v, v' \in V,$

(G2) $v \triangleleft [v, v] = 0, \quad v \in V.$

When we say that $[\ ,\ ]$ is $G$-equivariant, $\mathrm{Lie}(G)$ is regarded as a right $G$-module as was done in (5.4). In (G2), $\triangleleft$ represents the right $\mathrm{Lie}(G)$-Lie module structure on $V$ defined by

$$v \triangleleft x = x(v_{(-1)})\, v_{(0)}, \quad v \in V,\ x \in \mathrm{Lie}(G), \tag{6.1}$$

where $V \to \mathcal{O}(G) \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$ denotes the left $\mathcal{O}(G)$-comodule structure corresponding to the right $G$-module structure on $V$. A *morphism* $(\phi, \psi) : (G_1, V_1) \to (G_2, V_2)$ of Harish-Chandra pairs consists of a morphism $\phi : G_1 \to G_2$ of algebraic groups and a linear map $\psi : V_1 \to V_2$ such that

(G3) $\psi$ is $G_1$-equivariant, with $V_2$ regarded as a $G_1$-module through $\phi$,

(G4) $[\psi(v), \psi(v')] = \mathrm{Lie}(\phi)([v, v']), \quad v, v' \in V.$

We let $\mathsf{HCP}_{\Bbbk}$ denote the category of Harish-Chandra pairs over $\Bbbk$.

This category $\mathsf{HCP}_{\Bbbk}$ is isomorphic to the full subcategory of $(\text{sHCP})_{\Bbbk}$ consisting of the objects $(G, \mathfrak{g})$ in which $G$ is an algebraic group. To describe an explicit category isomorphism, let $(G, V) \in \mathsf{HCP}_{\Bbbk}$. Define $\mathfrak{g} := \mathrm{Lie}(G) \oplus V \in \mathsf{SMod}_{\Bbbk}$ with $\mathfrak{g}_0 = \mathrm{Lie}(G)$, $\mathfrak{g}_1 = V$. Give to $\mathfrak{g}$ the bracket on $\mathrm{Lie}(G)$ as well as the structure $[\ ,\ ]$ of $(G, V)$, and define $[v, x] := v \triangleleft x$ for $v$, $x$ as in (6.1). Then $\mathfrak{g}$ turns into a Lie superalgebra. Retain the right $G$-module structure on $\mathfrak{g}_1 = V$. One sees that $(G, V) \mapsto (G, \mathfrak{g})$ gives the desired category isomorphism. The inverse is given by $(G, \mathfrak{g}) \mapsto (G, \mathfrak{g}_1)$, where the $\mathfrak{g}_1$ of the latter is given the restricted super-bracket and the original $G$-module structure.

Now, let $\mathbf{G} \in \mathsf{ASG}_{\Bbbk}$. Then $\mathbf{G}_{ev}$ is an algebraic group, and the Lie superalgebra $\mathrm{Lie}(\mathbf{G})$ is finite-dimensional. Regard the odd component $\mathrm{Lie}(\mathbf{G})_1$ of the Lie superalgebra as the right $\mathbf{G}_{ev}$-module defined by (5.5). Restrict the super-bracket on $\mathrm{Lie}(\mathbf{G})$ to the odd component, and give it to the pair $(\mathbf{G}_{ev}, \mathrm{Lie}(\mathbf{G})_1)$. Then the pair turns into a Harish-Chandra pair, and it corresponds to $\Phi(\mathbf{G})$ in $(\text{sHCP})_{\Bbbk}$ through the category isomorphism above. By Theorem 5.4 we have:

**Theorem 6.1.** $\mathbf{G} \mapsto (\mathbf{G}_{ev}, \mathrm{Lie}(\mathbf{G})_1)$ *gives a category equivalence*

$$\mathsf{ASG}_{\Bbbk} \xrightarrow{\approx} \mathsf{HCP}_{\Bbbk}.$$

Essentially the same result was already given in [12, 14]; see Remark 6.2 for a subtle difference caused by choice of dualities. As an advantage here, we have obtained an explicit quasi-inverse of the functor above, which is essentially the same as $\Psi$ in Section 5.1. Therefore, every algebraic supergroup can be realized as $\mathbf{\Gamma}$ constructed in Section 4. This realization is useful when we discuss group-theoretical properties of algebraic supergroups, as will be shown in the next subsection.

**Remark 6.2.** A category equivalence between $\mathsf{ASG}_{\Bbbk}$ and $\mathsf{HCP}_{\Bbbk}$ is given by [12, Theorem 6.5] and [14, Theorem 3.2], which both reformulate the result [11, Theorem 29] formulated in purely Hopf-algebraic terms. Given $(G, V) \in \mathsf{HCP}_{\Bbbk}$, denote now it by $(G, V, [\ ,\ ])$, indicating the structure. Replacing $[\ ,\ ]$ with $-[\ ,\ ]$, we still have $(G, V, -[\ ,\ ]) \in \mathsf{HCP}_{\Bbbk}$. Moreover, $(G, V, [\ ,\ ]) \mapsto (G, V, -[\ ,\ ])$ gives an involutive category isomorphism $\mathsf{HCP}_{\Bbbk} \to \mathsf{HCP}_{\Bbbk}$. The equivalence given by Theorem 6.1, composed with the last isomorphism, coincides with the equivalence cited above, as is seen from (5.8).

6.2. **An application.** Throughout in this subsection we let $\mathbf{G} \in \mathsf{ASG}_{\Bbbk}$, and let $(G, V)$ be the associated Harish-Chandra pair. We suppose that $\mathbf{G}$ is realized as the $\mathbf{\Gamma}$ which is constructed as in Section 4 from $G$, $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$, the canonical pairing $\mathfrak{g}_0 \times \mathcal{O}(G) \to \Bbbk$ and the right $G$-supermodule structure on $\mathfrak{g}$ defined by (5.4) and (5.5).

**Definition 6.3.** Let $(H, W)$ be a pair of closed subgroup $H \subset G$ and a sub-vector space $W \subset V$. We say that $(H, V)$ is a *sub-pair* of the Harish-Chandra pair $(G, V)$, if

(H1) $W$ is $H$-stable in $V$, and
(H2) $[W, W] \subset \mathrm{Lie}(H)$,

where $[\ ,\ ]$ is the structure of $(G, V)$.

If $\mathbf{H}$ is a closed super-subgroup of $\mathbf{G}$, then the associated Harish-Chandra pair $(H, W)$, with the right $H$-module structure on $W$ as well as the structure $[\ ,\ ]$ forgotten, is a sub-pair of $(G, V)$. In this case we say that the sub-pair $(H, W)$ *corresponds to* $\mathbf{H}$. The assignment $\mathbf{H} \mapsto (H, W)$ as above gives a bijection from the set of all closed super-subgroups of $\mathbf{G}$ to the set of all sub-pairs of $(G, V)$.

**Lemma 6.4.** *Let $(H, W)$ be the sub-pair of $(G, V)$ corresponding to a closed super-subgroup $\mathbf{H} \subset \mathbf{G}$. Given $v \in V$, the following are equivalent:*

(i) $v \in W$;
(ii) $e(a, v) \in \mathbf{H}(A)$ *for arbitrary* $A \in \mathsf{SAlg}_{\Bbbk}$ *and* $a \in A_1$;
(iii) $e(a, v) \in \mathbf{H}(A)$ *for some* $A \in \mathsf{SAlg}_{\Bbbk}$ *and* $a \in A_1$ *with* $a \neq 0$.

*Proof.* We only prove (iii) $\Rightarrow$ (i), since the rest is obvious.

Suppose that $e(a, v) \in \mathbf{H}(A)$ with $a \in A_1$, but $v \notin W$. Given an arbitrary basis $w_1, \dots, w_r$ of $W$, one can extend it, adding $v$ and others, to a basis $w_1, \dots, w_r, v, \dots$ of $V$. By Proposition 4.9, $e(a, v)$, being an element in $\mathbf{H}(A)$, is expressed uniquely in the form

$$e(a, v) = h\, e(a_1, w_1) \dots e(a_r, w_r), \tag{6.2}$$

where $h \in H(A_0)$ and $a_i \in A_1$, $1 \le i \le r$. The cited proposition gives analogous expressions of elements of $\mathbf{G}(A)$ which use the extended basis. Regarding (6.2) as two such expressions of one element, we have $a = 0$. $\square$

Just as in the non-super situation we define as follows, and obtain the next lemma; see [6, Part I, 2.6].

Let $\mathbf{H} \subset \mathbf{G}$ be a closed super-subgroup. The *normalizer* $\mathcal{N}_{\mathbf{G}}(\mathbf{H})$ (resp., the *centralizer* $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$) of $\mathbf{H}$ in $\mathbf{G}$ is the subgroup functor of $\mathbf{G}$ whose $A$-points consists of the elements $g \in \mathbf{G}(A)$ such that for every $A \to A'$ in $\mathsf{SAlg}_{\Bbbk}$, the natural image $g_{A'}$ of $g$ in $\mathbf{G}(A')$ normalizes (resp., centralizes) $\mathbf{H}(A')$.

**Lemma 6.5.** *$\mathcal{N}_{\mathbf{G}}(\mathbf{H})$ and $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$ are closed super-subgroups of $\mathbf{G}$. Moreover, $\mathcal{N}_{\mathbf{G}}(\mathbf{H})$ (resp., $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$) is the largest closed super-subgroup of $\mathbf{G}$ whose $A$-points normalize (resp., centralize) $\mathbf{H}(A)$ for every $A \in \mathsf{SAlg}_{\Bbbk}$.*

Let $\mathbf{H} \subset \mathbf{G}$ be a closed super-subgroup, and let $(H, W)$ be the corresponding sub-pair of $(G, V)$.

Recall that the *stabilizer* $\mathrm{Stab}_G(W)$ (resp., the *centralizer* $\mathrm{Cent}_G(W)$) of $W$ in $G$ is the largest closed subgroup of $G$ that makes $W$ into a module (resp., a trivial module) over it.

Let $\rho_H : V \to \mathcal{O}(H) \otimes V$ denote the left $\mathcal{O}(H)$-comodule structure on $V$ corresponding to the restricted right $H$-module structure on $V$. Define

$$\mathrm{Inv}_H(V/W) := \{v \in V \mid \rho_H(v) - 1 \otimes v \in \mathcal{O}(H) \otimes W\}.$$

This is the largest $H$-submodule of $V$ including $W$ whose quotient $H$-module by $W$ is trivial. The definition makes sense, replacing $W$ with any $H$-submodule, say $U$, of $V$. We will use $\mathrm{Inv}_H(V) = \mathrm{Inv}_H(V/0)$ when $U = 0$.

When $L = \mathrm{Lie}(H)$ or 0, we define

$$(L : W) := \{v \in V \mid [v, W] \subset L\},$$

where [ , ] is the structure of $(G, V)$.

**Theorem 6.6.** *Let $\mathbf{H} \subset \mathbf{G}$ and $(H, W) \subset (G, V)$ be as above.*

(1) *The sub-pair of $(G, V)$ corresponding to $\mathcal{N}_{\mathbf{G}}(\mathbf{H})$ is*

$$(\mathcal{N}_G(H) \cap \mathrm{Stab}_G(W),\ \mathrm{Inv}_H(V/W) \cap (\mathrm{Lie}(H) : W)).$$

(2) *The sub-pair of $(G, V)$ corresponding to $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$ is*

$$(\mathcal{Z}_G(H) \cap \mathrm{Cent}_G(W),\ \mathrm{Inv}_H(V) \cap (0 : W)).$$

*Proof.* In each part let us denote by $(K, Z)$ the desired sub-pair.

(1) First, we prove

$$K \subset \mathcal{N}_G(H) \cap \mathrm{Stab}_G(W), \quad Z \subset \mathrm{Inv}_H(V/W) \cap (\mathrm{Lie}(H) : W). \tag{6.3}$$

Note that $K$ normalizes $\mathbf{H}$ in $\mathbf{G}$. Then it follows that $K$ normalizes $H = \mathbf{H}_{ev}$ in $G = \mathbf{G}_{ev}$, whence $K \subset \mathcal{N}_G(H)$. It also follows that the right $G$-supermodule structure on $\mathrm{Lie}(\mathbf{G})$, restricted to a right $K$-supermodule structure, stabilizes $\mathrm{Lie}(\mathbf{H})$, whence $K \subset \mathrm{Stab}_G(W)$.

Since $[\mathrm{Lie}(\mathbf{H}), \mathrm{Lie}(\mathcal{N}_{\mathbf{G}}(\mathbf{H}))] \subset \mathrm{Lie}(\mathbf{H})$, we have $[W, Z] \subset \mathrm{Lie}(H)$, whence $Z \subset (\mathrm{Lie}(H) : W)$.

To prove $Z \subset \mathrm{Inv}_H(V/W)$, choose $z \in Z$. We may suppose $z \notin W$. Let $A = \mathcal{O}(H) \otimes \wedge(\tau)$ with $\tau$ an odd variable. We have an $A$-point $e(\tau, z)$ of

$\mathcal{N}_{\mathbf{G}}(\mathbf{H})$ by Lemma 6.4. Given a basis $w_1, \ldots, w_r$ of $W$, we extend it, adding $z$ and others, to a basis $w_1, \ldots, w_r, z, u_1, \ldots, u_s$ of $V$. Present $\rho_H(z)$ as

$$\rho_H(z) = \sum_{i=1}^{r} a_i \otimes w_i + b \otimes z + \sum_{i=1}^{s} c_i \otimes u_i \in \mathcal{O}(H) \otimes V.$$

Let $h \in H(A_0)$ be $\mathrm{id}_{\mathcal{O}(H)}$. Using Lemmas 4.2 and 4.6, one computes

$$\begin{aligned} &e(\tau, z)\, h\, e(\tau, z)^{-1} = \\ &\quad h\, e(a_1\tau, w_1) \ldots e(a_r\tau, w_r)\, e((b-1)\tau, w_r)\, e(c_1\tau, u_1) \ldots e(c_s\tau, u_s). \end{aligned} \tag{6.4}$$

Since this is contained in $\mathbf{H}(A)$, it follows by the same argument as proving Lemma 6.4 that $b = 1$ and $c_i = 0$, $1 \le i \le s$, whence $Z \subset \mathrm{Inv}_H(V/W)$. We have thus proved (6.3).

Next, to prove the converse inclusions, choose $\phi : A \to A'$ from $\mathsf{SAlg}_{\Bbbk}$.

Let $g$ be an $A$-point of $\mathcal{N}_G(H) \cap \mathrm{Stab}_G(W)$. Then $g_{A'}$ normalizes $H(A')$. Given $a \in A'_1$ and $w \in W$, we have

$$e(a, w)^{g_{A'}} = 1 \otimes 1 + a\, w^g \in \mathbf{H}(A'),$$

and the same result with $g$ replaced by $g^{-1}$ holds. This proves $g \in K(A)$.

Let $v \in \mathrm{Inv}_H(V/W) \cap (\mathrm{Lie}(H) : W)$ and $0 \ne a \in A_1$. To see that $v \in Z$, we wish to prove, using Lemma 6.4, that $e(a, v)$ is an $A$-point of $\mathcal{N}_{\mathbf{G}}(\mathbf{H})$. Note that the $A'$-point $e(a, v)_{A'}$ of its image is $e(\phi(a), v)$. Given $h \in H(A')$, the same argument as proving (6.4) shows $e(a, v)_{A'}\, h\, e(a, v)_{A'}^{-1} \in \mathbf{H}(A')$, since $v^h - v \in W_{A'_0}$. Given $w \in W$ and $b \in A'_1$, we see by Lemma 4.2 (i) that

$$e(a, v)_{A'}\, e(b, w)\, e(a, v)_{A'}^{-1} = i(f(-\phi(a)b, [v, w]))\, e(b, w) \in \mathbf{H}(A'),$$

since $[v, w] \in \mathrm{Lie}(H)$. The last two conclusions prove the desired result.

(2) We only prove

$$K \subset \mathcal{Z}_G(H) \cap \mathrm{Cent}_G(W), \quad Z \subset \mathrm{Inv}_H(V) \cap (0 : W).$$

The converse inclusions follow by modifying slightly the second half of the proof of Part 1.

Since $K$ centralizes $\mathbf{H}$ in $\mathbf{G}$. it follows that $K$ centralizes $H$ in $G$, whence $K \subset \mathcal{Z}_G(H)$. It also follows that the restricted right $K$-supermodule structure on $\mathrm{Lie}(\mathbf{G})$ centralizes $\mathrm{Lie}(\mathbf{H})$, whence $K \subset \mathrm{Cent}_G(W)$.

Since $[\mathrm{Lie}(\mathbf{H}), \mathrm{Lie}(\mathcal{Z}_{\mathbf{G}}(\mathbf{H}))] = 0$, we have $[W, Z] = 0$, whence $Z \subset (0 : W)$. The argument which proved $Z \subset \mathrm{Inv}_H(V/W)$ above, modified with $W$ replaced by 0, shows $Z \subset \mathrm{Inv}_H(V)$. □

Suppose that $\mathbf{G} = \mathbf{H}$, and so $G = H$, $V = W$. Then Part 2 above reads:

**Corollary 6.7.** *Let $\mathbf{G}$ and $(G, V)$ be as above. The sub-pair of $(G, V)$ corresponding to the center $\mathcal{Z}(\mathbf{G}) = \mathcal{Z}_{\mathbf{G}}(\mathbf{G})$ of $\mathbf{G}$ is*

$$(\mathcal{Z}(G) \cap \mathrm{Cent}_G(V),\ \mathrm{Inv}_G(V) \cap (0 : V)).$$

The algebraic-group component of this sub-pair was obtained by [14, Proposition 7.1] in an alternative way.

*Note added in revision.* In present paper the term "algebraic (super)group" is used in a restricted sense to indicate *affine* algebraic (super)groups (see Section 1.2), and the category equivalence theorem, Theorem 6.1, concerns affine algebraic supergroups. Very recently Alexandr Zubkov and the first-named author generalized this theorem to not necessarily affine, algebraic supergroups over a field of characteristic $\neq 2$, and showed that the results in Section 6.2 above hold for those supergroups, more generally. Details will be contained in a forthcoming joint paper.

## Acknowledgments

The first-named author was supported by JSPS Grant-in-Aid for Scientific Research (C) 26400035. The second-named author was supported by Grant-in-Aid for JSPS Fellows 26 · 2022. The authors thank Alexandr Zubkov for his valuable comments on an earlier version of the paper, which include a corrected proof of Proposition 4.3 (2).

## References

[1] C. Carmeli, L. Caston, R. Fioresi, *Mathematical foundations of supersymmetry*, EMS Series of Lectures in Mathematics, European Math. Soc., Zürich, 2011.
[2] F. Gavarini, *Global splittings and super Harish-Chandra pairs for affine supergroups*, Trans. Amer. Math. Soc. **368** (2016), 3973–4026.
[3] A. N. Grishkov, A. N. Zubkov, *Solvable, reductive and quasireductive supergroups*, J. Algebra **452** (2016), 448–473.
[4] G. Hochschild, G. D. Mostow, *Complex analytic groups and Hopf algebras*, Amer. J. Math. **91** (1969), 1141–1151.
[5] M. Hoshi, A. Masuoka, *Super Lie groups and Harish-Chandra pairs over a complete field*; preprint arXiv:1706.02839v2.
[6] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs, vol. 107, Amer. Math. Soc., Providence, RI, 2003.
[7] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Lecture Notes in Mathematics vol. 570, Springer-Verlag, Berlin/Heidelberg/New York, 1977, pp.177–306.
[8] J.-L. Koszul, *Graded manifolds and graded Lie algebras*, Proceedings of the International Meeting on Geometry and Physics (Florence, 1982), pp.71–84, Pitagora, Bologna, 1982.
[9] A. Masuoka, *Abelian and non-abelian second cohomologies of quantized enveloping algebras*, J. Algebra **320** (2008), 1–47.
[10] A. Masuoka, *The fundamental correspondences in super affine groups and super formal groups*, J. Pure Appl. Algebra **202** (2005), 284–312.
[11] A. Masuoka, *Harish-Chandra pairs for algebraic affine supergroup schemes over an arbitrary field*, Transform. Groups **17** (2012), no. 4, 1085–1121.
[12] A. Masuoka, *Hopf algebraic techniques applied to super algebraic groups*, Proceedings of Algebra Symposium (Hiroshima, 2013), pp. 48–66, Math. Soc. Japan, 2013; available at arXiv: 1311.1261v2.
[13] A. Masuoka, T. Shibata, *Algebraic supergroups and Harish-Chandra pairs over a commutative ring*, Trans. Amer. Math. Soc. **369** (2017), 3443–3481.
[14] A. Masuoka, A. N. Zubkov, *Solvability and nilpotency for algebraic supergroups*, J. Pure Appl. Algebra **221** (2017), 339–365.
[15] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Conf. Series in Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.
[16] E . G. Vishnyakova, *On complex Lie supergroups and split homogeneous supermanifolds*, Transform. Groups **16** (2011), no. 1, 265–285.

Akira Masuoka, Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
*E-mail address*: akira@math.tsukuba.ac.jp

Taiki Shibata, Department of Applied Mathematics, Okayama University of Science, 1-1 Ridaicho, Kita-ku, Okayama 700-0005, Japan
*E-mail address*: shibata@xmath.ous.ac.jp