# Brane Geometry from Matrix Models

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# Contents

# 1. Introduction

As its name suggests, the dynamical objects in string theory are “strings”, namely one-dimensional objects. The difference between string theory and conventional field theories of point particles is just the dimension of dynamical objects. The only one difference, however, leads to a great difference: String theory can describe quantum gravity while field theories cannot due to the non-renormalizablity of gravity. String theory is the only consistent quantum theory of gravity so far. There are five consistent types of string theory: type I, type IIA, type IIB and heterotic $SO(32)$ and $E_8 \times E_8$. In addition to these types, the existence of an eleven-dimensional theory, which is called M-theory, has been conjectured [1]. It is known that the five types of string theory and M-theory are related by duality and compactifications of space-time. It is hence widely believed that these string theories and M-theory just correspond to various different aspects of a single fundamental theory.

Apart from the existence of such mysterious fundamental theory, we still do not have a complete formulation of string theory in the first place. The all types of string theory are formulated in the framework of perturbation theory so far. The perturbation series of string theory are asymptotic series such that the radius of convergence is zero as in usual quantum field theories. Such asymptotic series never give the complete formulation of the theory and the current formulation is trustable only when the coupling constant is very small. With such perturbative framework, we can not understand non-perturbative phenomenons and physics at strong coupling. In particular, we can never solve long-standing but interesting problems with strong gravitational interactions such as the emergence of our universe, compactification of space-time and so on. We therefore need to find a non-perturbative formulation of string theory. Perhaps, understanding the non-perturbative formulation also unravels the above-mentioned fundamental theory which unifies all types of string theory and M-theory.

The fundamental dynamical objects in non-perturbative formulation of string theory are not necessarily strings. Even in the perturbative framework, besides strings, there exist yet another several-dimensional extended objects, which is referred to as *branes*. The typical example of branes are D-branes, which is originally introduced as hypersurfaces on which the Dirichlet boundary conditions are imposed for endpoints of open strings [2]. They carry charges of various gauge potentials in string theory. The important point is that D-branes always exist in string theory: through some dualities in string theory, pictures without D-branes are actually equivalent to dual pictures with D-branes. This means that the string theory is not just a theory of strings but also contains D-branes as fundamental objects. In the case of M-theory, strings are no longer the relevant degrees of freedom because the theory of strings is only consistent in the critical dimension $d = 10$. Two-branes, namely two-dimensional objects which is also called *membranes*, rather than strings are considered as the fundamental objects in the eleven-dimensional theory.

Concerning M-theory, a profound conjecture has been known. The statement is that M-theory in the light-cone frame is exactly equivalent to the large-$N$ limit of the non-relativistic low energy effective theory of $N$ D0-branes [3]. The low energy theory of $N$ D0-branes is just a supersymmetric matrix quantum mechanics, dynamical variables of which are nine $N \times N$ matrices and the fermionic superpartners. It is no wonder that D0-branes are the basic degrees of freedom in M-theory because D0-branes give the minimum scale of length in M-theory [4][5]. The amazing points in this conjecture are that the all of M-theory is described in terms of only D0-branes and the action is just matrix quantum mechanics.. Although the basic degrees of freedom are only D0-branes, the matrix model can describe higher dimensional objects such as membranes. Indeed, the matrix model can also be interpreted as a regularization of the membrane theory [21]. The important point is that M-theory is clearly non-perturbatively defined by the large-$N$ limit of the matrix model if this conjecture is correct. In addition, matrices naturally describe the multi-particle states. This means that the second-quantization can also be realized by the matrix model.

Inspired by the conjecture, various formulations in terms of matrices have been proposed also in the context of string theory. As a typical example, it has been conjectured that the type IIB string theory can be non-perturbatively defined by the large-$N$ reduced model of the ten-dimensional $\mathcal{N} = 1$ super Yang-Mils theory [7]. However, we have not understood the mechanism of reproducing all the fundamental objects in string theory including various branes in terms of matrices completely. There are only a few concrete examples in which the relation between matrices and branes have been fully understood. Thus, the formulation of string and M- theory by the matrix model remains a matter of “conjecture” so far.

In the matrix-model formulations of string and M- theories, the matrix variables play the most essential role. The difficulties in understanding the matrix models are also due to matrices, however. The matrix variables form “noncommutative” algebra and so matrix models are written in terms of the noncommutative geometry [8][9]. The noncommutative geometry grows independently of the context of string theory and are expected as a candidate describing the quantum structure of space-time. It is interesting to look back the history of the transition of dynamical objects; we start from particles, namely zero-dimensional objects, through strings and extended objects, and then reach to D0-branes, namely zero-dimensional objects again. This is definitely not a repeat of history, however. Particles are described in terms of conventional commutative geometry. On the other hand, D0-branes are described in term of noncommutative geometry. We may reach to completely new understanding of the fundamental structure of space-time.

In order to understand matrix models, we need to understand the noncommutative geometry which is written in terms of the matrix variables. If matrix models actually describe string and M-theories, we must be able to extract all the information about various branes from the matrices. There are a lot of studies, that propose a systematic method of describing the "shape" of branes in terms of matrices, have been proposed so far. However, we understand very little about the relation between the matrices and the geometric objects on branes, for example Riemann metrics, gauge connections, etc. In this paper, we focus on a recently proposed method that describes the shape of branes using the notion of coherent states. In the method, matrices are converted into the state vectors in the Hilbert space on which the matrices act. Then, by introducing *Berry's connection* and *information metric*, which are geometric objects defined on the Hilbert space of a quantum mechanics, we construct a Riemann metric and gauge connection on the branes associated by the matrices in the method [10].

The organization of this thesis is as follows: In the section 2, we briefly see various branes in string and M- theory and the dynamics of D-branes which play important role in the matrix-model formulation. In the section 3, we review the conjecture that M-theory in a frame is completely described by the matrix models, and in particular we see how the statement is justified. We also review the relation between matrix variables and M2- and M5- barnes based on some concrete examples. In the section 4, we introduce a method to relate given matrices with corresponding branes. In the section 5, based on the method which is introduced in the section 4, we show that Berry's connection and information metric give gauge connection and Riemann metric on the corresponding branes in terms of given matrices. Also, we show that these objects give a Kähler structure on a single membrane in the context of M-theory.

# 2. Branes in string and M- theories

As I mentioned in the previous section, various kinds of branes exist in string theory. D-branes are famous examples of branes, but not all Branes are D-branes. For example, strings are 1-branes, but they are not D1-branes. In this subsection, we study what kinds of branes exist in string theory.

## 2.1. Gauge potentials and branes

The most simple way to see the existence of branes may be focusing on the gauge potentials in the theory. Let $C$ be the world-line of a particle and $x^\mu$, $\mu = 0, 1, 2, 3$ are the components of a embedding from $C$ to a four dimensional space-time. In the electromagnetism theory, the point particle couples to a one-form gauge potential $A$ in the form of

$$q \int_C A, \tag{2.1.1}$$

where $q$ is a electric charge carried by the particle. This is an electric coupling of a zero-brane to the one-form gauge potential. We can straightforwardly generalize this kind of couplings to the case for higher rank form potentials. Let us consider a theory with a $p$-form gauge potential $C^{(p)}$ in $d$ space-time dimensions. We define the field strength of $C^{(p)}$ by $F^{(p+1)} = dC^{(p)}$. As the exterior derivative $d$ is nilpotent: $d^2 = 0$, $F^{(p)}$ is clearly invariant under the gauge transformation

$$C^{(p)} \to C^{(p)} + d\lambda^{(p-1)}, \tag{2.1.2}$$

where $\lambda^{(p-1)}$ is a $(p-1)$-form. Note that in the case of $p = 1$, the gauge transformation (2.1.2) reduces to the U(1) gauge transformation for the one-form gauge potential. We therefore can interpret $C^{(p)}$ as a generalization of the electromagnetic one-form gauge potential $A$. As with (2.1.1), the $p$-form gauge potential naturally couples to a $(p-1)$-brane in the form of

$$q_p \int_{\mathcal{V}_p} C^{(p)}, \tag{2.1.3}$$

where $q_p$ is a conserved charge associated with $C^{(p)}$. Here $\mathcal{V}_p$ is the $p$-dimensional hypersurface in the space-time, which is swept out by the $p$-brane, and is referred to as *world-volume* of the

$(p-1)$-brane. We call this coupling the electric coupling of the $(p-1)$-brane to the $p$-form gauge potential.

There is another type of branes in the theory with $C^{(p)}$. Let $*F^{(p+1)} = (*F)^{d-p-1}$ be the Hodge dual of $F^{(p+1)}$. By the analogy with that of electromagnetic 1-form gauge potential, the kinetic term of $C^{(p)}$ in the action is given in the form of

$$-\frac{1}{2}\int F \wedge *F, \tag{2.1.4}$$

where the integral is taken over the space-time. The free equation of motion for $C^{(p)}$ is given by

$$d * F^{(p+1)} = 0. \tag{2.1.5}$$

In addition, the nilpotency of $d$ means that $F^{(p+1)}$ satisfy the Bianchi identity

$$dF^{(p+1)} = 0. \tag{2.1.6}$$

We find that replacing $F^{(p+1)}$ with $*F^{(p+1)}$ and vice versa simply switch the equation of motion (2.1.5) and the Bianchi identity (2.1.6). The picture in terms of $F^{(p+1)}$ is equivalent to the picture in terms of $*F^{(p+1)}$. This equivalence corresponds to the electric-magnetic duality of Maxwell's equation in the case of $p = 1$. Let us consider the dual picture in terms of $*F^{(p+1)}$. In this picture, (2.1.5) and (2.1.6) are interpreted as the Bianchi identity and free equation of motion for $*F^{(p+1)}$, respectively. We can solve (2.1.5) in terms of a $(d-p-2)$-form gauge potential by $*F^{(p+1)} = dC^{(d-p-2)}$. The new gauge potential $C^{(d-p-2)}$ is dual to the original potential $C^{(p)}$. The same discussion as (2.1.3) leads to the electric coupling of $C^{(d-p-2)}$ to a $(d-p-3)$-brane as

$$q_{d-p-2}\int_{\mathcal{V}_{d-p-2}} C^{(d-p-2)}. \tag{2.1.7}$$

We call this coupling the magnetic coupling of the $(p-1)$-brane to the $p$-form potential. We therefore conclude that for the theory with $p$-form gauge potential in $d$ space-time dimensions, $(p-1)$-branes and $(d-p-3)$-branes, which electrically and magnetically couple to the $p$-form gauge potential, respectively, naturally exists.

## 2.2. Branes in closed superstring theory

We see what kinds of branes exist in string theory based on the above discussion. Although we can see the existence of branes even in bosonic string theory, we are interested in superstring theory which has the critical dimension $d = 10$.

All the types of superstring theory are formulated by Neveu-Schwarz-Ramond approach which introduces world-sheet fermions. There are two possible boundary conditions for the world-sheet fermions; periodic and antiperiodic conditions, which are called Ramond (R) and Neveu-Schwarz (NS), respectively. For closed strings, there are two modes of world-sheet fermions corresponding to right-moving and left-moving waves, and we can choose the boundary conditions for each modes separately. As a result, there are four sectors for closed strings: NS-NS, R-R, NS-R, R-NS. The states in the NS-NS and R-R sectors correspond to the space-time bosonic fields and the states in the NS-R and R-NS sectors correspond to the space-time fermionic fields.

We focus on the type IIA and typ IIB theory which are closed superstring theories. In both cases, the massless bosonic fields in the NS-NS sector are the scalar $\Phi$, two-form gauge potential $B^{(2)}$ and symmetric two-form $G$, which are the dilaton, Kalb-Ramond field and Riemann metric in the space-time, respectively. We therefore find that one-branes and five-branes, which electrically and magnetically couple to the NS-NS gauge potential $B^{(2)}$, respectively, naturally exist in the type IIA and typ IIB theory. The one-branes are nothing but the closed strings. The world-sheet action of closed string action indeed include the coupling term to $B^{(2)}$ in the form of (2.1.3). The five-branes are referred to as NS5-branes and are still mysterious objects. The NS5-branes are closely related to five-branes in M-theory as we will see in the next section.

In the R-R sector, the type IIA and typ IIB theory have different fields. The massless bosonic fields in the R-R sector for each theory are following:

$$\begin{aligned} &\mathrm{IIA} : C^{(1)},\ C^{(3)}, \\ &\mathrm{IIB} : C^{(0)},\ C^{(2)},\ C^{(4)}, \end{aligned} \tag{2.2.1}$$

where $C^{(p)}$ is a $p$-form gauge potential. We therefore find that the following branes, which couple to the R-R gauge potentials, naturally exist in each theory:

$$\begin{aligned} &\mathrm{IIA} : \text{0-, 2-, 4- and 6-branes}, \\ &\mathrm{IIB} : (-1)\text{-, 1-, 3-, 5- and 7-branes}, \end{aligned} \tag{2.2.2}$$

In the type IIA theory, the 0- and 2-branes electrically couple to $C^{(1)}$ and $C^{(3)}$, respectively, and the 4- and 6-branes magnetically couple to them, respectively. In the type IIB theory, The $(-1)$-, 1- and 3-branes electrically couple to $C^{(0)}$, $C^{(2)}$ and $C^{(4)}$, respectively, and the 5- and 7-branes magnetically couple to them, respectively. The branes in (2.2.2) are called Dirichlet branes, or D-branes [11]. In particular, the $(-1)$-brane in the type IIB theory is referred to as D-instanton.

D-branes have a special characteristic as distinct from strings and NS5-branes: the end points of all open strings must lie on them [2]. In other words, Dirichlet and Neumann boundary conditions are imposed on the embedding coordinates in the direction normal and tangent to branes, respectively. The important point is that we can switch a Neumann boundary condition to a Dirichlet boundary condition, and vice versa, by performing *T-duality* transformations, which is a duality in string theory. Since $T$-duality transformations do not change the mass spectrum for strings, the dual picture is physically identical to the original picture. This also means that pictures without D-branes are identical to dual pictures with D-branes. One may wonder why D-branes exist in the type II theories which do not contain open strings. $T$-duality transformations in the type II theories indeed never lead to pictures with D-branes. However, we can find the type II theories with D-branes, according to the following process [2]: We start from the type I theory, which is the open plus closed superstring theory, with D-branes. Since the end points of all open strings lie on the D-branes, open strings can not separate far away from them, and therefore the local physics is described by closed superstring theory. This means that the type I theory with D-branes looks like the closed superstring theory containing D-branes.

Since the D-branes also impose the boundary condition on left-moving and right-moving waves, the scattering amplitudes of closed strings from the D-branes are invariant only half the supersymmetries of the type II theories with 16 supercharges.

## 2.3. The low energy effective action for D-branes

First of all, we consider the action for a $p$-brane in ten-dimensional flat space-time. Let $x^\mu$, $\mu = 0, 1, \dots, 9$ are the embedding coordinates from the world-volume $\mathcal{V}_{p+1}$ to the target space-time and $\sigma^a$, $a = 0, 1, \dots, p$ are local coordinates on $\mathcal{V}_{p+1}$. We define a Riemann metric on $\mathcal{V}_{p+1}$ by the pullback of the flat metric $\eta_\mu = \mathrm{diag}(- + + \cdots +)$ on the target space-time induced by $x^\mu$:

$$h_{ab} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b}. \tag{2.3.1}$$

The simplest local quantity written in terms of $x^\mu$, which is invariant under the Poincaré transformation in the target space and the diffeomorphism on $\mathcal{V}_{p+1}$, is the infinitesimal volume element on $\mathcal{V}_{p+1}$: $d^{p+1}\sigma \sqrt{-\det h_{ab}}$. The simplest action for the $p$-brane is therefore given by

$$S_p = -T_p \int_{\mathcal{V}_{p+1}} d^{p+1}\sigma \sqrt{-\det h_{ab}}, \tag{2.3.2}$$

where $T_p$ is the tension of the $p$-brane with mass dimension equal to $p + 1$. This is a geometric generalization of the Nambu-Goto action, which is the simplest action for single string. Note that only $10 - (p + 1) = 9 - p$ components of $x^\mu$ are dynamical degrees of freedom because we can gauge-fix the $(p + 1)$ components of them by the diffeomorphism symmetry. A useful gauge choice is the static gauge, in which the first $p + 1$ components of $x^\mu$ are identified with $\sigma^a$. In the static gauge, the induced metric $h_{ab}$ is locally written in terms of the other $9 - p$ components $x^I$, $I = p + 1, p + 2, \dots, 9$ as

$$h_{ab} = \eta_{ab} + \delta_{IJ} \frac{\partial x^I}{\partial \sigma^a} \frac{\partial x^J}{\partial \sigma^b}. \tag{2.3.3}$$

Note that the action (2.3.2) in the static gauge reduces to

$$-T_p \int_{\mathcal{V}_{p+1}} d^{p+1}\sigma \left(\frac{1}{2}\partial_a x^I \partial^a x_I + \cdots\right). \tag{2.3.4}$$

Here we have dropped a constant term in the action, and $\cdots$ stands for the higher order terms of $\partial_a x^I$. This means that the action (2.3.2) is interpreted as a nonlinear generalization of the action for the $9-p$ massless scalar fields living on $\mathcal{V}_{p+1}$.

Next, we consider the action for a D$p$-brane in ten-dimensional flat space-time. We use the same symbol $x^\mu$ and $\mathcal{V}_{p+1}$ as the case of a $p$-brane to denote the embedding coordinates and the world-volume of the D$p$-brane, respectively. In the case of the D$p$-brane, there are additional massless fields on $\mathcal{V}_{p+1}$: the U(1) gauge potentials $A_a$ that correspond to the tangential components of the massless modes of open strings, whose end points lie on the D$p$-brane. The transverse components of the massless modes correspond to the $9-p$ dynamical components of $x^\mu$. The relation between (2.3.2) and (2.3.4) suggests that the low energy dynamics of D$p$-brane is captured by the nonlinear generalization of the action for the U(1) gauge potentials coupled to the massless scalar fields on $\mathcal{V}_{p+1}$. Such action, which is known as Dirac-Born-Infeld action, is given by

$$S_{\mathrm{D}_p} = -T_{\mathrm{D}p} \int_{\mathcal{V}_{p+1}} d^{p+1}\sigma \sqrt{-\det\left(h_{ab} + 2\pi\alpha' F_{ab}\right)}, \tag{2.3.5}$$

where $T_{\mathrm{D}_p}$ is the tension of the D$p$-brane and $F_{ab} = (dA)_{ab}$ is the field strength of $A_a$. In the static gauge, this action indeed reduces to

$$-(2\pi\alpha')^2 T_{\mathrm{D}_p} \int_{\mathcal{V}_{p+1}} d^{p+1}\sigma \left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a \phi^I \partial^a \phi_I + \cdots\right), \tag{2.3.6}$$

where we have rescaled $x^I$ as $\phi_I := x^I/2\pi\alpha'$. Here we have again dropped a constant term in the action, and $\cdots$ stands for the higher order terms of $\partial_a x^I$ or $2\pi\alpha'$. Note that the first two terms of this action are identified with the U(1) Yang-Mils action in ten space-time dimensions, which is the low energy effective action of open strings, by the dimensional reduction: assuming that all fields are independent of the transverse coordinates to $\mathcal{V}_{p+1}$. This identification result from the equivalence between dimensionally reducing and imposing the Dirichlet boundary conditions on open strings. The Yang-Mils coupling constant $g_{\mathrm{YM}}$ are related to the D$p$-brane tension $T_{\mathrm{D}_p}$ as

$$g_{\mathrm{YM}}^2 = \frac{1}{(2\pi\alpha')^2 T_{\mathrm{D}_p}}. \tag{2.3.7}$$

Since the Yang-Mils action describe the low energy modes of open strings, $g_{\mathrm{YM}}$ is proportional to the dimensionless open string coupling constant, which is equal to the square of the closed string coupling constant $g_s$. The relation (2.3.7) therefore means that $T_{\mathrm{D}_p}$ is proportional to

$1/g_s$. The following precise formula is known [12]:

$$T_{\mathrm{D}_p} = \frac{1}{(2\pi)^p g_s \alpha'^{(p+1)/2}}. \tag{2.3.8}$$

In the weak coupling limit for $g_s$, the D-brane are become very heavy and are regarded as a fixed hypersurface in the space-time. On the other hand, in the strong coupling limit, they become very light and completely change the aspect of the spectrum of the theory. This implies that D-branes are non-perturbative degrees of freedom in string theory.

The dynamics of the D-brane is affected when it moves in a more general background which is created by closed string modes. In order to find the low energy effective action, we have only to focus on the coupling of the D-brane to massless fields in the closed string sectors: the dilaton $\Phi$, Kalb-Ramond field $B_{\mu\nu}$ and Riemann metric $G_{\mu\nu}$ in the NS-NS sector, and gauge potentials $C^{(p)}$ in the R-R sector. By the coupling to the NS-NS fields, the Dirac-Born-Infeld action (2.3.5) is modified as

$$\sqrt{-\det h_{ab}} \longrightarrow e^{-\Phi_0}\sqrt{-\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} \tag{2.3.9}$$

where $G_{ab}$ and $B_{ab}$ are the pullback of $G_{\mu\nu}$ and $B_{\mu\nu}$ induced by $x^\mu$, respectively:

$$G_{ab} = G_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}, \quad B_{ab} = B_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}. \tag{2.3.10}$$

Here we have defined the sifted dilaton as $\Phi_0 := \Phi - \log g_s$ such that its expectation value is vanishing, which means that the tension of the D$p$-brane is given by (2.3.8) even in this case. Note that the expectation value of dilaton determines the string coupling constant: $g_s = e^{\langle\Phi\rangle}$. The coupling to the R-R fields is given by the Chern-Simons like term, which also couples to $B_\mu$ and $F_{ab}$. The lowest term in the low energy limit and for small derivatives $\partial_a x^I$ is given by the same form as (2.1.3).

We next consider the case of $N$ D$p$-branes. In this case, each end point of open strings can attach to one of the $N$ D-branes separately. Since there are $N^2$ ways for each open string to attach the D-branes, each mode of open strings has $N^2$ degrees of freedom. In particular, when all of the $N$ D-branes are coincide, there are $N^2$ massless modes, which correspond to $N^2$ space-time gauge potentials. It is natural to package the $N^2$ fields in $N \times N$ Hermitian matrices, whose components are

$$(A_a)_{ij}, \quad i, j = 1, 2, \ldots, N. \tag{2.3.11}$$

The diagonal components correspond to the $N$ gauge potentials associated to the open strings whose both ends attach to the same D-branes, respectively. The embedding coordinates $x^I$ also become $N \times N$ Hermitian matrices since they are the transverse components of massless modes of open strings. It is known that in the low energy limit, the effective action of $N$ D$p$-branes is given by the dimensional reduction to $p+1$ dimension of $\mathcal{N} = 1$ supersymmetric Yang-Mils theory with gauge group U($N$) in static gauge [17].

## 2.4. M-theory and M-branes

Since the D-branes holds half of the supersymmetry the mass of single D$p$-brane exactly equals to the D$p$-brane charge (by the BPS bound)

$$\tau_p = \frac{1}{g(2\pi)^p \alpha'^{(p+1)/2}}. \tag{2.4.1}$$

This relation is guaranteed by the supersymmetry algebra and so not modified by perturbative and nonperturbative effects. (2.4.1) means that at weak coupling D$p$-branes are heavy but at strong coupling become light. Let us focus on D0-branes in the type IIA theory because they give the lowest mass at strong coupling. Since there is no force between BPS objects D0-branes can make bound states without bound energy. So, for $n$ D0-branes there are infinite bound states with mass

$$n\tau_0 = \frac{n}{g\alpha'^{1/2}}, \quad n \in \mathbf{Z}. \tag{2.4.2}$$

In the strong coupling limit these states become infinite number of massless states and change the aspect of the spectrum of the theory completely. This spectrum is a very similar to the spectrum of the infinite tower of Kaluza-Klein states in compactified theories on $S^1$. (2.4.2) matches actually the mass of Kaluza-Klein states in compactification on $S^1$ of radius

$$R = g\alpha'^{1/2}. \tag{2.4.3}$$

This matching suggests that the type IIA theory at strong coupling is described by an eleven-dimensional theory. The existence of such eleven-dimensional theory is compatible with the critical dimension $d = 10$ of superstring theory because the critical dimension holds only for perturbative string theory and allow the existence of dimensions invisible at weak coupling. The compact dimension of radius (2.4.3) is exactly invisible at weak coupling. On the other hand $d = 11$ is the maximum in which supersymmetry including graviton can exist. So, eleven-dimension is a unique candidate except ten-dimension in which superstring theory can be described.

By the comaptification, the eleven-dimensional gravitational coupling is related to the ten-dimensional coupling as $\kappa_{11}^2 = 2\pi R\kappa_{10}^2$. We define the eleven-dimensional Planck mass by the convention $2\kappa_{11}^2 = (2\pi)^8 M_{11}^{-9}$. In this convention, we find

$$M_{11} = g^{-1/3}\alpha'^{-1/2}. \tag{2.4.4}$$

This gives the fundamental length scale of the eleven-dimensional theory. The coupling and Regge slope in the type IIA theory are written in terms of the eleven-dimensional Planck mass (2.4.4) and the radius of the compactification (2.4.3) as

$$g = (M_{11}R)^{3/2}, \quad \alpha' = M_{11}^{-3}R^{-1}. \tag{2.4.5}$$

The low energy effective theory of the M-theory must be the eleven-dimensional supergravity. The bosonic fields in the eleven-dimensional supergravity are the eleven-dimensional metric and three-form $A^{(3)}$. These fields actually correspond to the bosonic fields in the type IIA theory in one-to-one by

$$\begin{aligned} ds^2 &= e^{-\frac{2}{3}\phi(x)} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4}{3}\phi(x)} (dx^{10} + C^{(1)}_\mu(x) dx^\mu)^2, \\ A^{(3)} &= \frac{1}{6} d^\mu \wedge dx^\nu \wedge dx^\rho C^{(3)}_{\mu\nu\rho}(x) + \frac{1}{2} d^\mu \wedge dx^\nu \wedge dx^{10} B^{(2)}_{\mu\nu}(x), \end{aligned} \tag{2.4.6}$$

where $\mu, \nu, \rho = 0, 2, \ldots, 9$. In the type IIA theory $\phi$, $B^{(2)}$, $C^{(1)}$ and $C^{(3)}$ correspond to the dilaton, $B$-field, R-R one-form and R-R three-form respectively. There are also extended objects which couple to 3-form $A^{(3)}$ in the eleven-dimensional theory. The corresponding electric or magnetic charge carriers are two- or five-dimensional objects which are called M2- or M5-branes respectively. We can check that the classical solution of the equation of motion of the eleven-dimensional supergravity actually have two- and five-dimensional objects. The M2- and M5-branes are related to the fundamental strings and (even-dimensional) D-branes in the type IIA theory by the compactification: for example, fundamental strings is M2-branes wrapped on the conpactified dimension.

# 3. Matrix-model formulation

## 3.1. A conjecture: “M” is for matrix

The original conjecture in [3] is based on the compactified M-theory on a spatial circle of radius $R$ with momentum $P = N/R$ in the periodic dimension. The statement is that in the limit $N, R, P \to \infty$ the (original noncompact) M-theory exactly is equivalent to the nonrelativistic low energy effective theory of $N$ D0-branes described by

$$S = \int_0^\infty dt\, \mathrm{tr}\left[\frac{1}{2R}\dot{X}^i\dot{X}_i - \frac{R}{4}[X^i, X^j]^2 - \theta^T\dot{\theta} - R\theta^T\Gamma_i[\theta, X^i]\right]. \tag{3.1.1}$$

Although original conjecture is based on the context of the large-$N$ limit it was later stated in [18] that the matrix model (3.1.1) describes the conpactified M-theory on a light-like circle of radius $R$ with momentum $P^+ = N/R$. We see from now how these statements are justified by following the excellent discussion in [19] which combine these statements.

Let us consider a compactification on a light-like circle of radius $R$ by identifying

$$\begin{pmatrix} x \\ t \end{pmatrix} \cong \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} \frac{R}{\sqrt{2}} \\ -\frac{R}{\sqrt{2}} \end{pmatrix}, \tag{3.1.2}$$

where $x$ is a spatial coordinate. This identification means $x^- \cong x^- - R$, where we have defined $x^\pm = t \pm x$. so the light-cone momentum $P^+$ is quantized. The comapctification (3.1.2) can be interpreted as the limit $R_s \to 0$ of a compactification on “almost light-like” circle of radius $R_s$ by identifying

$$\begin{pmatrix} x \\ t \end{pmatrix} \cong \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{R^2}{2} + R_s^2} \\ -\frac{R}{\sqrt{2}} \end{pmatrix}. \tag{3.1.3}$$

This comactification is the identification on almost light-like but space-like circle, so taking the Lorentz boost with

$$\beta = \frac{R}{\sqrt{R^2 + 2R_s^2}}, \tag{3.1.4}$$

we get a comapctification on a spatial circle of radius $R_s$ on the $x$-axis with the identification

$$\begin{pmatrix} x \\ t \end{pmatrix} \cong \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} R_s \\ 0 \end{pmatrix}. \tag{3.1.5}$$

In this way, the Lorentz invariance guarantees the equivalence between the compactification on the light-like circle (3.1.2) and the comapctification on the spatial circle (3.1.5) in the limit $R_s \to 0$.

The light-cone momentum $P^-$ is rescaled by the Lorentz boost (3.1.4) as well as the radius of the light-like circle (3.1.2), so $P^-$ is proportional to $R$. In the case of $R_s \ll R$, the scale factor is given by

$$P^- \to \frac{1-\beta}{\sqrt{1-\beta^2}} P^- \approx \frac{1}{\sqrt{2}} \frac{R_s}{R} P^-, \tag{3.1.6}$$

so the boosted $P^-$ which gives the energy scale in the system with (3.1.5) is independent of $R$ and of order $R_s$.

Le us consider the M-theory compactified on the space-like circle (3.1.5) with momentum $P = N/R$ in compact dimension, which equivalent to the M-theory compactified on the light-like circle (3.1.2) in the limit $R_s \to 0$. For small $R_s$, this theory has energy scale of order $R_s M_{11}^2$ ($M_{11}^2$ is inserted by dimensional analysis) based on the discussion (3.1.6). So, for fixed $M_{11}$ the limit $R_s \to 0$ means vanishing energy scale and leads very complicated theory. In order to hold the energy scale to be of order one, we introduce new eleven-dimensional Planck mass defined by

$$R_s \widetilde{M}_{11}^2 = R M_{11}^2. \tag{3.1.7}$$

This means $\widetilde{M}_{11}$ goes to infinity as $R_s \to 0$ holding $R_s \widetilde{M}_{11}^2$ fixed. Therefore we find that the M-theory with Planck mass $M_{11}$ compactified on the light-like circle of radius $R$ and momentum $P^+ = N/R$ is equivalent to the M-theory with Planck mass $\widetilde{M}_{11}$ compactified on the space-like circle of radius $R_s$ and momentum $P = N/R_s$ in the limit

$$\begin{aligned} &R_s \to 0, \\ &\widetilde{M}_{11} \to \infty, \\ &R_s \widetilde{M}_{11}^2 = R M_{11}^2 = \text{fixed}. \end{aligned} \tag{3.1.8}$$

The later theory is related to the type IIA theory in which the string coupling and Regge slope are given by

$$\begin{aligned} g &= (R_s \widetilde{M}_{11})^{3/2} = R_s^{3/4} (R M_{11}^2), \\ \alpha' &= R_s^{-1} \widetilde{M}_{11}^{-3} = R_s^{1/2} (R M_{11}^2)^{-3/2}. \end{aligned} \tag{3.1.9}$$

In the limit $R_s \to 0$ with finite $R$ and $M_{11}$, the string coupling becomes zero and so all strings decouple. Only dynamical objects in this theory are $N$ D0-branes because the coupling for D$p$-branes is

$$g_{\mathrm{D}p}^2 = \frac{1}{(2\pi\alpha')^2 \tau_p} = (2\pi)^{p-2} g \alpha'^{(p-3)/2} \tag{3.1.10}$$

and of order one in the limit $R_s \to 0$ only for $p = 0$. The vanishing of the Regge slope means that the $N$ D0-branes are described by those low energy effective action, namely, the Born-Infeld action. Furthermore the action is approximated to minimal supersymetric Yang-mils form because higher powers of derivatives have higher power of $R_s$ and suppressed by the limit $R_s \to 0$.

## 3.2. Branes from matrix models

### M2-branes and matrix regularization

Let us consider a relativistic bosonic membrane in $d$-dimensional flat space-time $\mathbf{R}^{d-1,1}$ (see [20] for a review). The membrane moving in $(d-1)$-dimensional space sweeps out a trajectory described by a three-dimensional hypersurface in $\mathbf{R}^{d-1,1}$. The world-volume is a three-dimensional submanifold $\mathcal{V}$ of $\mathbf{R}^{d-1,1}$ and so described by the embedding $x : \mathcal{V} \to \mathbf{R}^{d-1,1}$. We assume that $\mathcal{V}$ is the form of $\mathbf{R} \times \Sigma$ where $\Sigma$ is some orientable closed surface of fixed topology. Let $\tau$, $\sigma^a$, $a = 1, 2$ are local coordinates on $\mathbf{R}$ and $\Sigma$, respectively. In such coordinate system, the motion of the membrane is described by a set of $d$ bosonic fields $x^\mu(\tau, \sigma^1, \sigma^2)$, $\mu = 0, 1, \cdots, d-1$, which are the components of $x$. In the light-cone frame, only the transverse components with $i = 1, 2, \cdots, d-2$ are dynamical. After some gauge-fixing of diffeomorphism symmetry, the Hamiltonian of this theory is given by

$$H = \frac{\nu T}{4} \int d^2\sigma \left( \dot{x}^i \dot{x}_i + \frac{2}{\nu^2} \{x^i, x^j\} \{x^i, x^j\} \right), \tag{3.2.1}$$

where $T = (2\pi)^3 l_p$ is a membrane tension and $\nu$ is an arbitrary normalization constant, and also $\{\cdot, \cdot\}$ is a Poisson bracket on $\Sigma$. The Poisson bracket is gauge-fixed at each $\tau$ as

$$\{f, g\} = \epsilon^{ab} \partial_a f \partial_b g, \tag{3.2.2}$$

for $f, g \in C^\infty(\Sigma)$. By the gauge-fixing, the Hamiltonian (3.2.1) is invariant under only time-independent area-preserving diffeomorphism: coordinate transformations $\sigma^a \to \sigma'^a(\sigma^1, \sigma^2)$ such that $J = \det \partial_a \sigma'^b = 1$.

In the case for $d = 11$ and $\nu = N$, we notice that (3.2.1) corresponds to the bosonic part of the matrix model (3.1.1) by [22]

$$x^i \to X^i, \quad \{\cdot, \cdot\} \to -iN[\cdot, \cdot], \quad \frac{1}{2\pi} \int d^2\sigma \to \frac{1}{N} \mathrm{Tr}. \tag{3.2.3}$$

To clarify the meaning of this operation, we focus on the algebraic structure of $C^\infty(\Sigma)$. For the functions in $C^\infty(\Sigma)$, two types of multiplication are now defined. One is by the pointwise product,

$(f,g) \mapsto f \cdot g$, and the other is by the Poisson bracket, $(f,g) \mapsto \{f,g\}$. The former multiplication endows $C^\infty(\Sigma)$ with associative commutative structure, on the other hand the latter multiplication endows that with Lie algebraic structure. Furthermore, these multiplications satisfy the "Leibniz rule",

$$\{f, g \cdot h\} = \{f,g\} \cdot h + g \cdot \{f,h\}, \tag{3.2.4}$$

for $\forall f,g,h \in C^\infty(\Sigma)$. The algebra $C^\infty(\Sigma)$ together with such two multiplications is called *Poisson algebra*.

On the other hand, also for $N \times N$ Hermitian matrices, two types of multiplication are defined. One is by the matrix product, and the other is by the commutator. The algebra of $N \times N$ Hermitian matrices together with these multiplication is nothing but the Lie algebra $\mathfrak{su}(N)$ of $\mathrm{SU}(N)$. Although these multiplications also satisfy the Leibniz rule, $\mathfrak{su}(N)$ is not Poisson algebra because the matrix product is associative but "noncommutative". The operation (3.2.3) therefore approximates the Poisson algebra of $C^\infty(\Sigma)$ by the noncommutative algebra of $N \times N$ Hermitian matrices. This approximation is called *matrix regularization*.

#### Matrix regularization

Before we introduce concrete examples of the matrix regularization, we review the more formal definition of the matrix regularization for orientable closed surfaces $\Sigma$ embedded in flat space $\mathbf{R}^{10}$. Let $\{T_N\}$ be a sequence of linear mappings from $C^\infty(\Sigma)$ to $N \times N$ Hermitian matrices and $\hbar(N)$ be a real-valued strictly positive decreasing function such that $\lim_{N\to\infty} N\hbar(N) < \infty$. If $\{T_N\}$ satisfies the following conditions for $\forall f,h \in C^\infty(\Sigma)$ we call the pair $(T_N, \hbar_N)$ a matrix regularization of $\Sigma$: [21][23]

$$\lim_{N\to\infty} \|T_N(f)\| < \infty, \tag{3.2.5}$$

$$\lim_{N\to\infty} \|T_N(fh) - T_N(f)T_N(h)\| = 0, \tag{3.2.6}$$

$$\lim_{N\to\infty} \|[T_N(f), T_N(h)] - i\hbar(N)T_N(\{f,h\})\| = 0, \tag{3.2.7}$$

$$\lim_{N\to\infty} 2\pi\hbar_N \mathrm{Tr} T_N(f) = \int_\Sigma f\omega, \tag{3.2.8}$$

where $\|\cdot\|$ denote the norm for $T_N(f) \in M_N(\mathbf{C})$.

Symplectic structures are essential in the procedure of the matrix regularization. Actually, the definition of the matrix regularization is straightforwardly generalized to any symplectic manifold which is not necessarily two-dimensional manifold. Note that as shown in Appendix (B.1) it holds that

$$\lim_{N\to\infty} 2\pi\hbar_N(N) \mathrm{Tr} \hat{f}_N \hat{h}_N = \int_\Sigma f h\, \omega. \tag{3.2.9}$$

where $\{\hat{f}_N\}$ is a sequence of $N \times N$ matrices satisfying $\lim_{N\to\infty} \|\hat{f}_N - T_N(f)\| = 0$. This relation is useful in rewriting the action of bosonic membrane in terms of matrices. Let $T_N(x^i(t)) = \hbar(N)X^i(t)$. In the local coordinates such that $\sqrt{g} = 1$, the relation (3.2.9) means that

$$\begin{aligned} \lim_{N\to\infty} 2\pi\hbar_N(N)\mathrm{Tr}\dot{X}^i\dot{X}_i &= \int d^2\sigma \dot{x}^i\dot{x}_i, \\ \lim_{N\to\infty} 2\pi\hbar_N(N)\mathrm{Tr}[X^i, X^j]\,[X_i, X_j] &= \int d^2\sigma \{x^i, x^j\}\,\{x_i, x_j\}\,. \end{aligned} \tag{3.2.10}$$

These equations explicitly show the relation between the matrix model and the theory of membranes.

## 3.3. Examples of matrix regularization

Here, we consider concrete examples of the matrix regularization.

**Example:** $S^2$

Let us consider two-sphere $S^2$ embedded in three-dimensional flat space $\mathbf{R}^3 \subset \mathbf{R}^{10}$ with the coordinates $x^i$ where $i = 1, 2, 3$. The standard embedding is defined by the constraint

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = r^2, \tag{3.3.1}$$

where $r$ is the radius of the sphere. The embedding $x^i$ are expressed in terms of the polar coordinates as

$$x^1 = r\sin\theta\cos\varphi, \quad x^2 = r\sin\theta\cos\varphi, \quad x^3 = r\cos\theta, \tag{3.3.2}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. Let $f(\theta, \phi)$ be an arbitrary function on $S^2$. We can uniquely expand the function in terms of the spherical harmonics $Y_{\ell m}(\theta, \varphi)$ as

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \chi_{\ell m} Y_{\ell m}(\theta, \varphi), \tag{3.3.3}$$

where $\chi_{\ell m}$ are complex coefficients. $Y_{\ell m}(\theta, \phi)$ can also be expanded in terms of polynomial in coordinates $x^i$,

$$Y_{\ell m}(\theta, \phi) = r^{-\ell}\sum_{k=0}^{\ell}\sum_{a_i=1,2,3} t^{\ell m}_{a_1 a_2\cdots a_k} x^{a_1}x^{a_2}\cdots x^{a_k}, \tag{3.3.4}$$

where the coefficients $t^{m}_{a_1 a_2\cdots a_k}$ are totally symmetric. Since the spherical harmonics are irreducible tensors and satisfy $Y^\dagger_{\ell m} = (-1)^m Y_{\ell -m}$, the coefficients also satisfy

$$\begin{aligned} t^{\ell m *}_{a_1 a_2\cdots a_k} &= (-1)^m t^{\ell -m}_{a_1 a_2\cdots a_k}, \\ \sum_{a_i=1,2,3} t^{\ell m}_{a_1 a_2\cdots a_k} R^{a_1}{}_{b_1} R^{a_2}{}_{b_2}\cdots R^{a_k}{}_{b_k} &= \sum_{m'=-\ell}^{\ell} t^{\ell m'}_{b_1 b_2\cdots b_k} D^{\ell}_{m'm}(R), \end{aligned} \tag{3.3.5}$$

where $R_{ab}$ and $D^{\ell}_{m'm}(R)$ are matrix elements of $R \in \mathrm{SO}(3)$ and the spin $\ell$ representation of $R$, respectively.

We define a linear mapping $T_N$ by [24]

$$T_N(x^i) := X^i := \frac{2r}{\sqrt{N^2-1}} L^i \tag{3.3.6}$$

where $L^i$ are the $N$-dimensional irreducible representations of $SU(2)$ generators. The motivation to choose these matrices is that they satisfy an algebraic relation

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = r^2 \mathbf{1}_N \tag{3.3.7}$$

similar to (3.3.1), because $L^i L_i$ is the Casimir operator on $S^2$ with eigenvalue $(N^2-1)/4$. Note that $X^i$ satisfy

$$U(R) X_i U^{\dagger}(R) = R_{ij} X_j, \tag{3.3.8}$$

where $R \in \mathrm{SO}(3)$ is a rotation matrix and $U(R)$ is the spin $J$ representation of $R$ with $N = 2J+1$. We also define *matrix spherical harmonics* in the same form as (3.3.4),

$$T_{\ell m} := c_{\ell N} r^{-\ell} \sum_{k=0}^{\ell} \sum_{a_i=1,2,3} t^{\ell m}_{a_1 a_2 \cdots a_k} X^{a_1} X^{a_2} \cdots X^{a_k}, \tag{3.3.9}$$

where $c_{\ell N}$ is a normalization constant which depends on $\ell$ and $N$. Combining the property of $t^{m}_{a_1 a_2 \cdots a_k}$ with (3.3.7), we have

$$\begin{aligned} T^{\dagger}_{\ell m} &= (-1)^m T_{\ell -m}, \\ U(R) T_{\ell m} U^{\dagger}(R) &= \sum_{m'=-\ell}^{\ell} T_{\ell m'} D^{\ell}_{m'm}(R) \end{aligned} \tag{3.3.10}$$

Th second relation implies that $T_{\ell m}$ are irreducible tensors of rank $\ell$ as well as the spherical harmonics. Let

$$|Js\rangle, \quad s = -J, -J+1, \ldots, J \tag{3.3.11}$$

are the standard basis for the spin $J$ representation. By the Wigner-Eckart theorem, the matrix elements of $T_{\ell m}$ are given by the form of

$$\langle Js | T_{\ell m} | Js' \rangle = (-1)^{-J+s'} \sqrt{N} C^{\ell m}_{JsJ-s'}, \tag{3.3.12}$$

where $C^{\ell m}_{JsJ-s'}$ are the Clebsch-Gordan coefficients, and we choose the normalization constant $c_{\ell N}$ such that

$$\frac{1}{N} \mathrm{Tr}(T^{\dagger}_{\ell m} T_{\ell' m'}) = \delta_{\ell \ell'} \delta_{m m'}. \tag{3.3.13}$$

Through the expansion (3.3.3), we define the linear mapping from arbitrary function in $C^\infty(S^2)$ to $N \times N$ Hermitian matrices by

$$T_N(Y_{\ell m}(\theta,\varphi)) = \begin{cases} T_{\ell m} & \ell \leq N-1 \\ 0 & \ell > N-1 \end{cases} . \tag{3.3.14}$$

This mapping satisfy the condition of matrix regularization of $S^2$ (see Appendix **??**).

The Poisson bracket for $x^i$ is of the form $\{x^i, x^j\} = r\epsilon^{ijk}x^k$. On the other hand, the commutator for $X^i$ is given by the SU(2) algebra as

$$[X^i, X^j] = \frac{2i}{\sqrt{N^2-1}} r\epsilon^{ijk}X^k. \tag{3.3.15}$$

Therefore, we find $[T_N(x^i), T_N(x^j)] = i\hbar T_N(\{x^i, x^j\})$ where $\hbar = 2/\sqrt{N^2-1}$, and the third condition (3.2.5) holds for embedding $x^i$.

#### **Example:** $T^2$

Let us consider a torus $T^2$ embedded in four-dimensional flat space $\mathbf{R}^4 \subset \mathbf{R}^{10}$ with coordinates $x^i$ where $i = 1, 2, 3, 4$. The torus defined by the two constraints

$$(x^1)^2 + (x^2)^2 = u^2, \quad (x^3)^2 + (x^4)^2 = v^2. \tag{3.3.16}$$

The embedding $x^i$ are expressed in terms of two real parameters as

$$x^1 = u\sin\alpha, \quad x^2 = u\cos\alpha, \quad x^3 = v\sin\beta, \quad x^4 = v\cos\beta, \tag{3.3.17}$$

where $0 \leq \alpha, \beta \leq 2\pi$. Let $f(\alpha,\beta)$ be an arbitrary function on the torus. We can uniquely expand the function in terms of the Fourier modes $Y_{nm}(\alpha,\beta) := e^{in\alpha + im\beta}$ as

$$f(\alpha,\beta) = \sum_{n,m=-\infty}^{\infty} \chi_{nm} Y_{nm}(\alpha,\beta), \tag{3.3.18}$$

where $\chi_{nm}$ are complex coefficients.

We introduce two $N \times N$ unitary matrices $U$ and $V$ whose elements with respect to a orthonormal basis $|a\rangle$ are given by

$$\begin{aligned} \langle a|U|b\rangle &= \delta_{ab}\, e^{2\pi i a/N}, \\ \langle a|V|b\rangle &= \delta_{a+1b}, \end{aligned} \tag{3.3.19}$$

where $a, b = 0, 1, \ldots, N-1$, and $\delta_{Nb} := \delta_{0b}$ for all $b$. These matrices satisfy the relation

$$UV = \omega^{-1} VU \tag{3.3.20}$$

where $\omega := e^{\frac{2\pi i}{N}}$. Using these matrices, we define a linear mapping $T_N$ by [25][26]

$$
\begin{aligned}
T_N(x^1) &:= X^1 := \frac{u}{\sqrt{2}}(U^\dagger + U),\\
T_N(x^2) &:= X^2 := \frac{iu}{\sqrt{2}}(U^\dagger - U),\\
T_N(x^3) &:= X^3 := \frac{v}{\sqrt{2}}(V^\dagger + V),\\
T_N(x^4) &:= X^4 := \frac{iv}{\sqrt{2}}(V^\dagger - V).
\end{aligned}
\tag{3.3.21}
$$

These matrices clearly satisfy relations

$$
(X^1)^2 + (X^2)^2 = u^2\mathbf{1}_N, \quad (X^3)^2 + (X^4)^2 = v^2\mathbf{1}_N, \tag{3.3.22}
$$

similar to (3.3.16). We define the linear mapping from arbitrary function in $C^\infty(T^2)$ to $N \times N$ Hermitian matrices by

$$
T_N(Y_{nm}(\alpha,\beta)) = \begin{cases} \omega^{nm/2}U^nV^m & |n|,|m| \leq N-1 \\ 0 & |n| > N-1 \text{ or } |m| > N-1 \end{cases}. \tag{3.3.23}
$$

This mapping satisfy the condition of the matrix regularization of $T^2$.

For example, since the matrix elements of the factor $U^nV^m$ are given by

$$
\langle a|U^nV^m|b\rangle = \sum_{c=0}^{N-1} \langle a|U^n|c\rangle\ \langle c|V^m|b\rangle = \delta_{ab+m}\omega^{an}, \tag{3.3.24}
$$

we can check the fourth condition as

$$
\frac{1}{N}\mathrm{Tr}(\omega^{nm/2}U^nV^m) = \frac{1}{N}\delta_{m0}\sum_{a=0}^{N-1}\omega^{an} = \delta_{n0}\delta_{m0} = \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta\, Y_{nm}(\alpha,\beta), \tag{3.3.25}
$$

where the second equal sign follows from $\omega^N = 1$.

## 3.4. M5-brane

Unlike M2-branes, little is known about the relation between matrices and M5-branes. The main reason is that the world-volume action of a five-brane is not constructed so far. Since we do not have the theory of five-brane, some systematic procedure of relating five-branes to matrix configuration, such as matrix regularization, has not been known.

There are two way of appearing of M5-branes in the type IIA theory: it is wrapped or unwrapped around the compact direction, which is referred to as "longitudinal" or "transverse", respectively. In the type IIA theory, longitudinal M5-branes appear as D4-branes and transverse

M5-branes appear as NS5-branes. Fortunately, regarding longitudinal M5-branes, some matrix configurations have been concretely proposed. As a example of such configuration, we introduce the matrix configuration corresponding to a four-spherical longitudinal M5-brane [29]. Let $G^{(n)i}$, $(i = 1, 2, 3, 4, 5)$ are Hermitian matrices defined by the $n$-fold symmetric tensor product of the five dimensional Euclidean gamma matrices $\Gamma^i$:

$$G^{(n)i} = (\Gamma^i \otimes \mathbf{1}_4 \otimes \cdots \otimes \mathbf{1}_4 + \cdots + \mathbf{1}_4 \otimes \cdots \otimes \mathbf{1}_4 \otimes \Gamma^i)_{sym}. \tag{3.4.1}$$

where $\mathbf{1}$ is the $4 \times 4$ identity matrix, and the symbol "$sym$" means that $G^{(n)i}$ are restricted to the completely symmetrized tensor product space with dimension

$$N := \begin{pmatrix} n+3 \\ 3 \end{pmatrix} = \frac{(n+1)(n+2)(n+3)}{6}. \tag{3.4.2}$$

Using $G^{(n)i}$, we define $N \times N$ Hermitian matrices as

$$X_i = \frac{r}{n} G^{(n)i}. \tag{3.4.3}$$

Note that the product $\sum_i (G^{(n)i})^2$ is the Casimir operator with eigenvalue $n(n+4)$. We therefore find

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 = r^2 \mathbf{1}_n + O(1/n), \tag{3.4.4}$$

similar to the constraint of $S^4$ of radius $r$ embedded in $\mathbf{R}^5$. This configuration agree with the longitudinal M5-branes charge in matrix models [27][28].

On the other hand, transverse M5-branes are mysterious so far. Little has been reported on the construction of matrix configuration corresponding to transverse M5-branes. In particular, it is known that there are no charge for transverse M5-branes in the supersymmetry algebra of matrix models [28]. The description of transverse M5-branes in matrix models therefore is known as a longstanding puzzle. For example, see [29] for the discussion concerning to transverse M5-branes.

# 4. From matrices to geometry

## 4.1. Coherent states and classical space

In the formulation proposed in [30], the given matrices are interpreted as coordinate operators. That is, $d$ matrices $X^\mu$ are interpreted as quantized version of the embedding coordinates from a surface $\mathcal{M}$ to target space $\mathbf{R}^D$. The surface $\mathcal{M}$ is just what we want to find. Let $\mathcal{H}$ be the $N$-dimensional Hilbert space on which the given matrices act. For normalized states $|\psi\rangle \in \mathcal{H}$, we define the position of $|\psi\rangle$ in $\mathbf{R}^D$ by

$$x^\mu(\,|\psi\rangle) = \langle\psi|X^\mu|\psi\rangle\,. \tag{4.1.1}$$

In such interpretation, we can see the problem of finding $\mathcal{M}$ from the given matrices as the inverse problem of quantization.

#### Canonical coherent states

Let us consider the two-dimensional classical phase space, namely, $\mathbf{R}^2$. By the Dirac quantization, the classical phase space is replaced by the quantum system consisting of coordinate operators $\hat{x}^i$ with the Heisenberg algebra $[\hat{x}^1, \hat{x}^2] = i\hbar$ and a Hilbert space on which $x^i$ act. In the quantum system, the position of a point on the phase space is given by the expectation values of $\hat{x}^i$ with respect to some state which corresponds to the point. Since $\hat{x}^1$ and $\hat{x}^2$ do not commute each other, however, we can not simultaneously get the two coordinates of the point without uncertainty. The phase space therefore looks fuzzy in the quantum system. In the classical limit $\hbar \to 0$, the algebra becomes $[\hat{x}^1, \hat{x}^2] = 0$, and there exist the simultaneous eigenstates of $\hat{x}^i$ which one-to-one correspond to the points on the classical phase space. We can reproduce the classical phase space by gathering the expectation values of $\hat{x}^i$ with respect to such states. This means that the specific states, which become the simultaneous eigenstates in the classical limit, have information about the "shape" of the classical phase space. We recall that the canonical coherent states saturate the uncertainly inequality, and in particular the standard deviation of $\hat{x}^i$ with respect to them are both vanishing in the classical limit. They therefore become just the simultaneous eigenstates of $\hat{x}^i$ in the classical limit. Note that the canonical coherent states are defined as the ground states of a Hamiltonian of the form

$$H(y) = \frac{1}{2}(\hat{x}^1 - y^1)^2 + \frac{1}{2}(\hat{x}^2 - y^2)^2, \tag{4.1.2}$$

where $y^i$ are two-dimensional real parameters. In terms of the creation-annihilation operators,

$$\hat{a} := (2\hbar)^{-\frac{1}{2}}(\hat{x}^1 + i\hat{x}^2), \tag{4.1.3}$$

this Hamiltonian is written as

$$H(y) = \hbar(\hat{a}^\dagger - \bar{z})(\hat{a} - z) + \frac{\hbar}{2}, \tag{4.1.4}$$

where $z = (2\hbar)^{-\frac{1}{2}}(y^1 + iy^2)$. This form means that the ground states of $H(y)$ are given by the eigenstates of the annihilation operator $\hat{a}$ with eigenvalue $z$:,

$$\hat{a}\,|z\rangle = z\,|z\rangle\,. \tag{4.1.5}$$

This is nothing but the usual definition of the canonical coherent states. $|z\rangle$ are labeled by the points on $\mathbf{R}^2$, and become the simultaneous eigenstates of $\hat{x}^i$ in the classical limit because we have

$$\lim_{\hbar\to 0} \langle z|H(y)|z\rangle = 0 \tag{4.1.6}$$

which means that

$$\lim_{\hbar\to 0} \|(\hat{x}^1 - y^1)\,|z\rangle\,\| = \lim_{\hbar\to 0} \|(\hat{x}^2 - y^2)\,|z\rangle\,\| = 0. \tag{4.1.7}$$

Note that we can generalize the definition of the Hamiltonian (4.1.2) to the case for general configurations of matrices straightforwardly although it is difficult to construct the creation-annihilation operators for such case.

#### Construction of classical geometry for given matrices

We generalize the above discussion to the configuration of given matrices $X^\mu$. First, we define the Hamiltonian for $X^\mu$ by the same form as (4.1.2) as [30]

$$H(y) = \frac{1}{2}\sum_{\mu=1}^{d}(X^\mu - y^\mu \mathbf{1}_N)^2, \tag{4.1.8}$$

where $y^\mu$ are $d$-dimensional real parameters and $\mathbf{1}_N$ is the $N\times N$ identity matrix. Since $H(y)$ is just a Hermitian matrix of finite size, we can always solve the eigenvalue equation of $H(y)$ and find the spectrum of it at each point $y \in \mathbf{R}^d$. Let $|n, y\rangle$ and $E_n(y)$ are the $n$-th eigenstates and eigenvalues of $H(y)$, respectively:

$$H(y)\,|n, y\rangle = E_n(y)\,|n, y\rangle\,, \quad n = 0, 1, \ldots, N-1. \tag{4.1.9}$$

We assume that the eigenvalues are ordered as $E_0(y) \leq E_1(y) \leq \cdots \leq E_{N-1}(y)$ and the eigenstates are normalized as $\langle n, y|m, y\rangle = \delta_{nm}$.

We focus on the ground states $|0, y\rangle$. In the case of (4.1.2), the ground states satisfy (4.1.6) at all points on $\mathbf{R}^2$. It means that there exist the canonical coherent states, namely approximate simultaneous eigenstates of $\hat{x}^i$, at all points on $\mathbf{R}^2$. In the general case of now, however, there do not exist such states at all points on $\mathbf{R}^d$. We therefore call $|0, y\rangle$ "coherent states" at point $y$ if and only if the following condition is satisfied:

$$f(y) := \lim_{N\to\infty} \langle 0, y|H(y)|0, y\rangle = \lim_{N\to\infty} E_0(y) = 0. \tag{4.1.10}$$

This condition is a generalization of (4.1.6) and assures that the states become the simultaneous eigenstates of $X^i$.

We define a space for $X^i$, which we denote by $\mathcal{M}$, as the subspace of $\mathbf{R}^d$ on which the coherent states exist:

$$\mathcal{M} = \{y \in \mathbf{R}^d \mid f(y) = 0\}, \tag{4.1.11}$$

We call $\mathcal{M}$ the classical space of $X^\mu$ by analogy with the relation between the classical phase space and the quantum system. Note that $\mathcal{M}$ is not necessarily to be smooth surface. Depending on given matrices, $\mathcal{M}$ may be not only a set of discrete points but also the empty set. In order to have non empty set, the existence of the simultaneous eigenstates of $X^\mu$ is essential. We emphasize that this situation is due to the interpretation of matrices as coordinate operators in this formulation.

Although the above method using Hamiltonian based on the analogy with the quantization of classical space, it is also related to the context of string theory. Let us recall that the matrix model is originally given as the effective action of $N$ D0-branes, and the matrices $X^\mu$ are bosonic fields of these D0-branes. In this context, the classical space of $X^\mu$ correspond to the surface as a bound state which is formed by $N$ D0-branes. In order to find the shape of the surface, we add another D0-brane as a prove to the system. If the probe interact with $N$ D0-branes, there are open strings stretched between the probe and any one of $N$ D0-branes. Since the length of the open string can be zero only if the probe is on the surface, we can find the shape of the surface by looking at the massless modes of the open string. The massless modes of open string correspond to the zero modes of a matrix Dirac-type operator defined by [31]

$$\not{D}(y) = \delta_{\mu\nu}\Gamma^\mu \otimes (X^\nu - y^\nu \mathbf{1}_N), \tag{4.1.12}$$

where $\Gamma^\mu$ are $d$-dimensional gamma matrices. Note that the Hamiltonian is related to this Dirac operator by

$$\not{D}^2(y) = \mathbf{1}_{2^{[d/2]}} \otimes 2H(y) + \frac{1}{4}[\Gamma^\mu, \Gamma^\nu] \otimes [X^\mu, X^\nu]\,. \tag{4.1.13}$$

where the symbol $[d/2]$ denote the maximal integer less than or equal to $d/2$. This relation means that the zero modes of $H(y)$ correspond to the zero mode of $\not{D}(y)$ if the given matrices commute each other in the large-$N$ limit.

## 4.2. Some examples

#### Example of $S^2$

We consider the case that the given matrices are

$$X^i = \frac{2r}{\sqrt{N^2-1}} L^i, \quad i = 1, 2, 3, \tag{4.2.1}$$

where $L^i$ are the $N$-dimensional irreducible representation of SU(2) generators. Those matrices give a matrix regularization of $S^2$ of radius $r$ embedded in $\mathbf{R}^3$ as we saw in the section 3.2. Using the algebraic identity (3.3.7), we find that the Hamiltonian for (4.2.1) is given by [30]

$$H(y) = \frac{r^2 + |y|^2}{2} - \frac{2ry^i L_i}{\sqrt{N^2-1}}. \tag{4.2.2}$$

In order to find the ground states of this Hamiltonian, we consider the specific rotation $R \in \mathrm{SO}(3)$ that brings the position vector of a point $y \in \mathbf{R}^3$ to the vector in the direction of the north pole, $y^{(0)} = (0, 0, |y|)$ where $|y| = \sqrt{y_i y^i}$:

$$y^{(0)i} = R^i{}_j y^j, \quad R = \begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix}, \tag{4.2.3}$$

where $\theta$ and $\varphi$ are polar coordinates of the position vector of the point $y \in \mathbf{R}^3$. The spin $J = (N-1)/2$ representation of this rotation, which satisfies the relation (3.3.8), is given by

$$U(\theta, \varphi) = e^{-i\varphi L^3} e^{-i\theta L^2} e^{i\varphi L^3} = e^{\frac{1}{2}\theta e^{i\varphi} L^- - \frac{1}{2}\theta e^{-i\varphi} L^+}, \tag{4.2.4}$$

where $L^\pm = L^1 \pm iL^2$ are the ladder operators. Using this unitary operator, the Hamiltonian is diagonalized as

$$H(y) = U(\theta, \varphi) \left( \frac{r^2 + |y|^2}{2} - \frac{2r|y|L_3}{\sqrt{N^2-1}} \right) U^\dagger(\theta, \varphi). \tag{4.2.5}$$

It easy to find that the lowest eigenvalue of this Hamiltonian is given by

$$E_0(y) = \frac{r^2 + |y|^2}{2} - \frac{2rJ|y|}{\sqrt{N^2-1}} \tag{4.2.6}$$

We focus on the points such that the condition (4.1.10) is satisfied. Since we have

$$f(y) = \lim_{N\to\infty} E_0(y) = \frac{1}{2}(r - |y|)^2, \tag{4.2.7}$$

we find that the classical space $\mathcal{M}$ of (4.2.1) is indeed $S^2$ of radius $r$.

**Example of $S^4$**

Let us consider the case that the given matrices are

$$X_\mu = \frac{1}{n} G_\mu^{(n)} \quad \mu \in \{1, 2, \ldots, 5\}, \tag{4.2.8}$$

which are correspond to the matrix configuration of a spherical longitudinal M5-brane. We choose the representation of $\Gamma_\mu$ as

$$\begin{aligned}
\Gamma_i &= \sigma_2 \otimes \sigma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \quad i \in \{1, 2, 3\}, \\
\Gamma_4 &= \sigma_1 \otimes \mathbb{1}_2 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \\
\Gamma_5 &= \sigma_3 \otimes \mathbb{1}_2 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},
\end{aligned} \tag{4.2.9}$$

where $\sigma_i$ are the Pauli matrices.

The Hamiltonian for the matrices is given by

$$H(y) = \frac{1}{2}(1 + |y|^2) - y_\mu X^\mu + O(1/n), \tag{4.2.10}$$

where $|y| = y_\mu y^\mu$. In order to find the spectrum of this Hamiltonian, we consider the specific $SO(5)$ rotation matrix $\Lambda$ that brings the vector in the direction of the pole $(0, 0, 0, 0, |y|)$ to the position vector of a point $y \in \mathbf{R}^5$: $y_\mu \Lambda^\mu{}_\nu = |y| \delta_{\nu 5}$. For this rotation there exists a corresponding unitary operator $U$ which satisfies

$$U^\dagger \Gamma_\mu U = \Lambda_\mu{}^\nu \Gamma_\nu. \tag{4.2.11}$$

The explicit form of $U$ is given by

$$U = \exp\left(-\frac{\chi}{2}\Gamma_{21}\right) \exp\left(-\frac{\psi}{2}\Gamma_{32}\right) \exp\left(-\frac{\phi}{2}\Gamma_{43}\right) \exp\left(-\frac{\theta}{2}\Gamma_{54}\right), \tag{4.2.12}$$

where $(\theta, \phi, \psi, \chi)$ are the five-dimensional polar coordinates and $\Gamma_{\mu\nu} = \frac{1}{2}[\Gamma_\mu, \Gamma_\nu]$ are the spin representation of the $SO(5)$ generators. The relation (4.2.11) means that the $n$-fold tensor product of $U$ satisfies

$$U^{\dagger \otimes n} (y_\mu X^\mu) U^{\otimes n} = |y| X_5. \tag{4.2.13}$$

Using this relation we can diagonalize the Hamiltonian as

$$H(y) = U^{\otimes n} \left[ \frac{1}{2}(1 + |y|^2) - |y| X_5 + O(1/n) \right] U^{\dagger \otimes n}. \tag{4.2.14}$$

Then we can easily find the ground states of $H(y)$ as

$$|0, y\rangle = U^{\otimes n} \, |\frac{n}{2}, \frac{n}{2} - \alpha\rangle \quad \alpha \in \{0, 1, \ldots, n\}, \tag{4.2.15}$$

where $|\frac{n}{2}, \frac{n}{2} - \alpha\rangle$ are the highest weight states of $X_5$ with degeneracy $n + 1$:

$$|\frac{n}{2}, \frac{n}{2} - \alpha\rangle = \mathrm{Sym}(\, |\uparrow\rangle^{\otimes(n-\alpha)} \otimes \, |\downarrow\rangle^{\otimes \alpha}), \quad |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}. \tag{4.2.16}$$

The eigenvalue of the ground states is

$$E_0(y) = \frac{1}{2}(1 - |y|^2) + O(1/n). \tag{4.2.17}$$

In the classical limit the zeros of $E_0(y)$ are points such that $|y| = 1$, and so the classical space is actually unit four sphere.

# 5. Geometric structures from matrices

In the previous section, we introduce a method to construct the associated classical space $\mathcal{M}$ from given matrices $X^i$. The method gives correct classical spaces to not only the example of the matrix regularization for $S^2$ and $T^2$ but also the example of the configuration of spherical longitudinal 5-brane.

We are interested in matrices that correspond to smooth surfaces like membrane. So, from now on we always assume that the classical space of given matrices form a smooth manifold.

## 5.1. Fiber bundles and one-form connection

We assume that the lowest eigenvalue $E_0(y)$ of $H(y)$ is $\mathcal{N}$-fold degenerate, and also the degree of degeneracy is uniform on $\mathcal{M}$, i.e. $\mathcal{N}$ does not depend on $\forall y \in \mathcal{M}$. In this case, at each points $y \in \mathcal{M}$, there exist the orthonormal coherent states $|a, y\rangle$ satisfying

$$H(y)\,|a, y\rangle = E_0(y)\,|a, y\rangle\,, \quad a = 0, 1, \ldots, \mathcal{N} - 1. \tag{5.1.1}$$

Note that in general the coherent states are not single valued at all points $y \in \mathcal{M}$. We can only define single valued coherent states locally on $\mathcal{M}$. The degeneracy subspace of $\mathcal{H}$ is defined by

$$\mathcal{H}_0(y) = \mathrm{Span}\,\{\,|a, y\rangle \mid a = 0, 1, \ldots, \mathcal{N} - 1\} \subset \mathcal{H}. \tag{5.1.2}$$

This subspace is uniquely determined by the eigenvalue equation of $H(y)$. The coherent states $|a, y\rangle$ provide an orthonormal basis for $\mathcal{H}_0(y)$.

Based on the geometric interpretation of Berry's phase [32][33] (see also ), we can construct bundle structures over the classical space $\mathcal{M}$. We first construct a $U(\mathcal{N})$ vector bundle $E$:

$$(E, \mathcal{M}, \breve{\pi}, U(\mathcal{N})). \tag{5.1.3}$$

The fiber of $E$ over a point $y \in \mathcal{M}$ is defined to be the degenerate subspace $\mathcal{H}_0(y)$. The typical fiber is $\mathbf{C}^{\mathcal{N}}$. The projection map $\breve{\pi} : E \to \mathcal{M}$ is defined by setting $\breve{\pi}(\,|\psi, y\rangle) = y$ if and only if $\breve{\pi}(\,|\psi, y\rangle) \in \mathcal{H}_0(y)$. Let $\{U_\alpha\}$ be a collection of the open subsets of $\mathcal{M}$. Since the typical fiber

$\mathbf{C}^{\mathcal{N}}$ is isomorphic to the fibers $\mathcal{H}_0(y)$, there exist an isomorphism from $\mathcal{H}_0(y)$ to $\mathbf{C}^{\mathcal{N}}$ for each open subsets $U_\alpha$,

$$\Phi_\alpha(y) : \mathcal{H}_0(y) \longrightarrow \{y\} \times \mathbf{C}^{\mathcal{N}} \equiv \mathbf{C}^{\mathcal{N}}, \quad y \in U_\alpha. \tag{5.1.4}$$

This also means that we can construct a diffeomorphisms $\Phi_a : \breve{\pi}^{-1}(U_\alpha) \to U_\alpha \times \mathbf{C}^{\mathcal{N}}$ using $\Phi_\alpha(y)$, and therefore there exist a local trivialization $(U_\alpha, \phi_\alpha)$. Concretely, we can construct an isomorphism $\Phi_\alpha(y)$ for each $U_\alpha$ using the coherent states $|a, y\rangle$ by setting

$$\Phi_\alpha(y)(\,|a, y\rangle) = e_a, \quad a = 0, 1, \ldots, \mathcal{N} - 1, \tag{5.1.5}$$

where $\{e_a\}$ is a complete orthonormal basis for $\mathbf{C}^{\mathcal{N}}$. As we mentioned at the beginning of this section, the single-valued eigenstates $|a, y\rangle$ are only defined locally, namely on $U_\alpha$. Let us consider two intersecting open subsets $U_\alpha$ and $U_\beta$. By the relation (5.1.5), we can construct the isomorphisms $\Phi_\alpha(y)$ and $\Phi_\beta(y)$ which are associated by the single-valued coherent states $|a, y\rangle$ and $|a, y\rangle'$ respectively. For $\forall y \in U_\alpha \cap U_\beta$, both $|a, y\rangle$ and $|a, y\rangle'$ satisfy the same equation (5.1.1) and form an orthonormal basis for $\mathcal{H}_0(y)$. Therefore, they are related by unitary transformations,

$$|a, y\rangle' = \sum_{b=0}^{\mathcal{N}-1} (y)\, |b, y\rangle\ \mathcal{U}_{ba}(y), \quad \forall y \in U_\alpha, \tag{5.1.6}$$

where $\mathcal{U}_{ab}(y)$ are matrix element of an $\mathcal{N} \times \mathcal{N}$ unitary matrix $\mathcal{U}(y) \in U(\mathcal{N})$. This means that the structure group is $U(\mathcal{N})$ which is formed by the transition functions $\breve{G}_{\alpha\beta} := \Phi_\alpha(y) \circ \Phi_\beta^{-1}(y)$.

We now introduce the notion of Berry's connection. The Berry's connection is a matrix-valued one-form $\breve{A}$ in terms of its matrix elements defined by

$$\breve{A}^{ab}(y) := -i \langle a, y|d|b, y\rangle\,. \tag{5.1.7}$$

which is defined on $U_\alpha$. By the definition, the components of the one-form $\breve{A}$ are Hermitian matrices:

$$(\breve{A}^{ab}(y))^* = i(d\,\langle b, y|)\,|a, y\rangle = -i\,\langle b, y|d|a, y\rangle = \breve{A}^{ab}(y), \tag{5.1.8}$$

where note that $\partial_\mu(\langle a, y|b, y\rangle) = 0$. Thus $\breve{A}$ is a Lie algebra-valued one-form whose components $\breve{A}_\mu$ belong to the Lie algebra $\mathfrak{u}(\mathcal{N})$ of $U(\mathcal{N})$. In addition, under the unitary transformation (5.1.6), $\breve{A}$ transforms as

$$\begin{aligned}
\breve{A}^{ab}(y) \longrightarrow \breve{A}'^{ab}(y) &= -i\,\langle a, y|'\, d\,|b, y\rangle' \\
&= -i \sum_{c,d=0}^{\mathcal{N}-1} \langle c, y|\ \mathcal{U}^*_{ca}(y)\, d\,(\mathcal{U}_{db}(y)\,|d, y\rangle) \\
&= -i \sum_{c,d=0}^{\mathcal{N}-1} \mathcal{U}^*_{ca}(y)\,\langle c, y|d|d, y\rangle\ \mathcal{U}_{db}(y) - i \sum_{c=0}^{\mathcal{N}-1} \mathcal{U}^*_{ca}(y)\, d\,\mathcal{U}_{cb}(y) \\
&= (\,\mathcal{U}^\dagger(y)\,\breve{A}\,\mathcal{U}(y))^{ab} - i(\mathcal{U}^\dagger(y)\, d\,\mathcal{U}(y))^{ab}.
\end{aligned} \tag{5.1.9}$$

This is very similar to the transformation of local connections on principle fiber bundles.

The $U(\mathcal{N})$ vector bundle $E$, which we constructed above, is actually interpreted as an associated vector bundle to a principle fiber bundle $\mathcal{E}$ over $\mathcal{M}$:

$$(\mathcal{E}, \mathcal{M}, \pi, G) \tag{5.1.10}$$

Let $(\rho, V)$ be an $\mathcal{N}$-dimensional unitary representation of the structure group $G$. The typical fiber of $E$ is the representation space $V = \mathbf{C}^{\mathcal{N}}$, and also the structure group of $E$ is the representation $\rho(G) = U(\mathcal{N})$. The representation $(\rho, V)$ induces a push-forward map $\rho_* : \mathfrak{g} \longrightarrow \mathfrak{u}(\mathcal{N})$ where $\mathfrak{g}$ and $\mathfrak{u}(\mathcal{N})$ are the Lie algebra of $G$ and $U(\mathcal{N})$ respectively. This map gives a representation of $\mathfrak{g}$. We can identify the Berry's connection $\check{A}$, which is defined by (5.1.7), with the representation of a local connection one-form $A$ on $\mathcal{E}$ by

$$\check{A}(y) = \rho_*(A(y)). \tag{5.1.11}$$

Note that we can always evaluate Berry's connection (5.1.7) when we are given matrices concretely. Through the Berry's connection we can find the information about the bundle structure of $\mathcal{M}$.

**Example:** $S^2$

The coherent states of the Hamiltonian (4.2.2) is given by

$$|0, \theta, \varphi\rangle := U(\theta, \varphi) |JJ\rangle , \tag{5.1.12}$$

where $U(\theta, \phi)$ is a unitary operator defined in (4.2.4) and the states $|JJ\rangle$ is the highest state of the standard basis for spin $J$ representation with $N = 2J + 1$. Since this is just a single state, the lowest eigenvalue in this case is not degenerate. The coherent state (5.1.12) is a single-valued at all points on $S^2$ except for the south pole $\theta = \pi$. At the south pole, (5.1.12) becomes

$$\begin{aligned} |0, \pi, \varphi\rangle &= e^{-i\varphi L_3} e^{-i\pi L_2} e^{i\phi L_3} |JJ\rangle \\ &= e^{-i\pi L_2} e^{2i\varphi L_3} |JJ\rangle \\ &= e^{2iJ\varphi} e^{-i\pi L_2} |JJ\rangle , \end{aligned} \tag{5.1.13}$$

where in the second line we used the Backer-Campbell-Hausdorff formula $e^{-i\pi L_2} e^{i\varphi L_3} e^{i\pi L_2} = e^{-i\varphi L_3}$. We therefore find that there are different normalized states for $\varphi$ varying in the range $0 \leq \varphi < 2\pi$ at the south pole. The single-valued coherent state is well-defined around the north pole but not at the south pole. The coherent state, which is well defined around south pole, is given by a phase transformation,

$$|0, \theta, \varphi\rangle' = e^{-2iJ\varphi} |0, \theta, \varphi\rangle = e^{-i\varphi L_3} e^{-i\pi L_2} e^{-i\phi L_3} |JJ\rangle . \tag{5.1.14}$$

The unitary transformation $U'(\theta,\varphi) := e^{-i\varphi L_3}e^{-i\pi L_2}e^{-i\phi L_3}$ corresponds to the rotation that the position vector of a point $y \in \mathbf{R}^3$ to the vector in the direction of the south pole. At the south pole, $|0,\theta,\varphi\rangle'$ becomes

$$|0,\theta,\varphi\rangle' = e^{-i\pi L_2}\,|JJ\rangle\,, \tag{5.1.15}$$

which is actually a single state.

We evaluate the Berry's connection for this case, on $S^2$ except at the south pole,

$$A(\theta,\phi) = -i\,\langle 0,\theta,\varphi|d|0,\theta,\varphi\rangle = -i\,\langle JJ|U^\dagger(\theta,\varphi)dU(\theta,\varphi)|JJ\rangle\,. \tag{5.1.16}$$

Because we can calculate as

$$\begin{aligned}
\partial_r U &= 0,\\
\partial_\theta U &= U\frac{1}{2}(e^{i\varphi}L^- - e^{-i\varphi}L^+),\\
\partial_\varphi U &= Ui\left\{(1-\cos\theta)L^3 + \frac{1}{2}\sin\theta(e^{i\varphi}L^- + e^{-i\varphi}L^+)\right\},
\end{aligned} \tag{5.1.17}$$

we have

$$\begin{aligned}
dU(\theta,\varphi)\,|J,J\rangle = &\frac{\sqrt{2J}}{2}e^{i\varphi}U(\theta,\varphi)\,|JJ-1\rangle\,d\theta,\\
&+i\left\{J(1-\cos\theta)U(\theta,\varphi)\,|JJ\rangle + \frac{\sqrt{2J}}{2}\sin\theta e^{i\varphi}U(\theta,\varphi)\,|JJ-1\rangle\right\}d\varphi.
\end{aligned} \tag{5.1.18}$$

Berry's connection (5.1.16) therefore is

$$A = J(1-\cos\theta)d\varphi, \quad \theta \neq \pi. \tag{5.1.19}$$

In terms of unit vectors $\boldsymbol{e}_\phi$ along $\varphi$ direction, this is also expressed as

$$\boldsymbol{A} = J\frac{1-\cos\theta}{|y|\sin\theta}\boldsymbol{e}_\phi, \quad \theta \neq \pi. \tag{5.1.20}$$

This is nothing but the Wu-Yang monopole.

**Example: fuzzy $S^4$**

The eigenvalue of the Hamiltonian for fuzzy $S^4$ is $n$-fold degenerate. The coherent states are given by

$$|0,y\rangle = U^{\otimes n}\,|JJ-\alpha\rangle \tag{5.1.21}$$

where $J = n/2$. We evaluate the Berry's connection for this case,

$$A = -i\,\langle 0,y|d|0,y\rangle = -i\,\langle JJ-\alpha|U^{\otimes n}dU^{\otimes n}|JJ-\alpha\rangle\,. \tag{5.1.22}$$

In order to evaluate the factor $U^{\otimes n} dU^{\otimes n}$, we evaluate the chiral projection operators

$$P_\pm = \frac{1}{2}(\mathbb{1}_4 \pm \Gamma_5) = \frac{1}{2}(\mathbb{1}_2 \pm \sigma_3) \otimes \mathbb{1}_2. \tag{5.1.23}$$

By the projection $P_+$ the factor $U^\dagger dU$ is restricted to

$$P_+ U^\dagger dU P_+ = P_+ \left( \sum_{\mu<\nu} c_{\mu\nu} \Gamma_{\mu\nu} \right) P_+ = \frac{1}{2}(\mathbb{1}_2 + \sigma_3) \otimes i \left( \sum_{i<j} c_{ij} \epsilon_{ijk} \sigma_k - \sum_i c_{i4} \sigma_i \right). \tag{5.1.24}$$

In the first line, we expand $U^\dagger dU$ in terms of the generator $\Gamma_{\mu\nu}$, and the expansion coefficients $c_{\mu\nu}$ will be explicitly given by (4.2.12) lately. In the last line, we use the fact that $P_+ \Gamma_{\mu 5} P_+ = 0$ for $\mu \neq 5$ and the explicit form of the gamma matrices. Furthermore, we can rewrite the last line in terms of the irreducible spin-$\frac{1}{2}$ representation of $SU(2)$ generators, $S_3 = \frac{1}{2}\sigma_3$ and $S_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, as

$$P_+ U^\dagger dU P_+ = \frac{1}{2}(\mathbb{1}_2 + \sigma_3) \otimes (2C_3 S_3 + C_+ S_+ + C_- S_-), \tag{5.1.25}$$

where

$$C_3 = i(c_{12} - c_{34}), \quad C_+ = -C_-^* = -(c_{13} + c_{24}) + i(c_{23} - c_{14}). \tag{5.1.26}$$

Notice that $\frac{1}{2}(\mathbb{1}_2 + \sigma_3) \otimes S_i$ are the reducible but indecomposable spin-$\frac{1}{2}$ representation of $SU(2)$ generators. We define the symmetric $n$-fold tensor product representation of such representation as

$$\rho_*^{(n)}(S_i) = \left( \frac{1}{2}(\mathbb{1}_2 + \sigma_3) \otimes S_i \right) \otimes P_+ \otimes \cdots \otimes P_+ + \cdots + P_+ \otimes \cdots \otimes P_+ \otimes \left( \frac{1}{2}(\mathbb{1}_2 + \sigma_3) \otimes S_i \right). \tag{5.1.27}$$

By the definition (4.2.16), we see that $|J, J-\alpha\rangle$ are the eigenstates of $\rho_*^{(n)}(S_3)$ with heighest weight $J$:

$$\begin{aligned} \rho_*^{(n)}(S_3) \, |J, J-\alpha\rangle &= (J-\alpha) \, |J, J-\alpha\rangle, \\ \rho_*^{(n)}(S_+) \, |J, J-\alpha\rangle &= \sqrt{\alpha(n-\alpha+1)} \, |J, J-\alpha+1\rangle, \\ \rho_*^{(n)}(S_-) \, |J, J-\alpha\rangle &= \sqrt{(n-\alpha)(\alpha+1)} \, |J, J-\alpha-1\rangle \end{aligned} \tag{5.1.28}$$

Using the fact that $P_+^{\otimes n} \, |J, J-\alpha\rangle = |J, J-\alpha\rangle$ and the expression

$$P_+^{\otimes n} U^{\dagger \otimes n} d(U^{\otimes n}) P_+^{\otimes n} = 2C_3 \rho_*^{(n)}(S_3) + C_+ \rho_*^{(n)}(S_+) + C_- \rho_*^{(n)}(S_-), \tag{5.1.29}$$

we can find

$$\begin{aligned} i\check{A}^{ab} &= [\rho_*^{(n)}(2C_3 S_3 + C_+ S_+ + C_- S_-)]^{ab} \\ &= [\rho_*^{(n)}(2C_3 S_3 + 2C_2 S_1 + 2C_1 S_2)]^{ab} \end{aligned} \tag{5.1.30}$$

Introducing the notation: $\check{A}^{ab} = \sum_i A_i \rho_*^{(n)}(S_i)$, then

$$\begin{aligned}
iA_1 &= 2iC_2 = 2i(c_{23} - c_{14}) = i(d\psi \cos\theta - d\chi \sin\psi \sin\phi \cos\theta), \\
iA_2 &= 2iC_1 = -2i(c_{13} + c_{24}) = -i(d\chi \sin\psi \cos\phi + d\psi \sin\phi \cos\theta), \\
iA_3 &= 2C_3 = 2i(c_{12} - c_{34}) = i(d\chi \cos\psi - d\phi \cos\theta),
\end{aligned} \tag{5.1.31}$$

The field strength is given by $F = \sum_i F_i \rho_*^{(n)}(S_i)$ where

$$F_i = dA_i - \frac{1}{2}\epsilon_{ijk} A_i \wedge A_j. \tag{5.1.32}$$

We can calculate $F_i$ as

$$\begin{aligned}
F^1 &= -\sin^2\theta \sin\phi \, d\phi \wedge d\psi + \sin\theta \sin\phi \sin\psi \, d\theta \wedge d\chi, \\
F^2 &= \sin^2\theta \sin\phi \sin\psi \, d\phi \wedge d\chi + \sin\phi \sin\theta \, d\theta \wedge d\psi, \\
F^3 &= -\sin^2\theta \sin^2\phi \sin\psi \, d\psi \wedge d\chi + \sin\theta \, d\theta \wedge d\phi.
\end{aligned} \tag{5.1.33}$$

These are antiself-dual. We also get

$$F_1 \wedge F_1 + F_2 \wedge F_2 + F_2 \wedge F_2 = -6 \sin^3\theta \sin^2\phi \sin\psi \, d\theta \wedge d\phi \wedge d\psi \wedge d\chi. \tag{5.1.34}$$

This configuration is known as the Yang monopole on $S^4$. The second Chern class is given by

$$\frac{1}{8\pi^2} \int \mathrm{Tr}(F \wedge F) = N. \tag{5.1.35}$$

## 5.2. Riemannian metric

We construct a $N \times N$ density matrix using the coherent states $|a, y\rangle$ such that the invariance under the gauge transformation (5.1.6) is manifest. The gauge transformations are just fiber-coordinates transformations, and the classical space which is defined as a set of zeros of $E_0(y)$ is clearly invariant under such transformation. So, focusing gauge invariant objects is very natural to find geometric structures on $\mathcal{M}$. Such gauge invariant density matrix must consist of all the coherent states as

$$\Lambda(y) = \sum_{a=0}^{N-1} p_a(y) \, |a, y\rangle\langle a, y| \, , \tag{5.2.1}$$

where the weight $p_a(y)$ satisfy $0 < p_a(y) < 1$ and $\sum_a p_a(y) = 1$. This density matrix is also defined only locally as well as the coherent states. We fix the point $y$ in the following. Under the gauge transformation, $\Lambda(y)$ is transformed as

$$\Lambda(y) \longrightarrow \sum_{a,b,\gamma=0}^{N-1} p_a(y) \, \mathcal{U}_{ab}(y) \, \mathcal{U}^*_{ac}(y) \, |b, y\rangle\langle c, y| \, . \tag{5.2.2}$$

This means that $\Lambda(y)$ is invariant if and only if the following relation holds for any $k \times k$ unitary matrix $U$:

$$\sum_{a=0}^{\mathcal{N}-1} p_a(y)\,\mathcal{U}_{ab}(y)\,\mathcal{U}^*_{ac}(y) = p_b(y)\delta_{cb}. \tag{5.2.3}$$

Multiplying the both sides of this relation by $\mathcal{U}_{dc}(y)$ and summing with respect to $c$, we get

$$p_d(y)\,\mathcal{U}_{db}(y) = p_b(y)\,\mathcal{U}_{db}(y). \tag{5.2.4}$$

If $d = b$, this relation trivially holds. If $d \neq b$, by choosing the unitary matrix such that $U_{db} \neq 0$ we get $p_d(y) = p_b(y)$. Since we can same discussion for all $d$ and $b$, we find

$$p_0(y) = p_1(y) = \cdots = p_{\mathcal{N}-1}(y) := p(y). \tag{5.2.5}$$

Combining this result with the condition $\sum_a p_a(y) = 1$, we get $p(y) = 1/\mathcal{N}$. The invariance under gauge transformation therefore uniquely determine the weight of the density matrix:

$$\Lambda(y) = \frac{1}{\mathcal{N}} \sum_{a=0}^{\mathcal{N}-1} |a, y\rangle\langle a, y| \tag{5.2.6}$$

This is a projection to the degeneracy subspace $\mathcal{H}_0(y)$.

#### Information metric

Note that we can interpret the density matrix (5.2.6) as a mapping from $\mathbf{R}^D$ to the space of all $N \times N$ density matrices:

$$\Lambda : y \in \mathbf{R}^D \longrightarrow \{\text{Set of all } N \times N \text{ density matrix}\} := \mathcal{D}. \tag{5.2.7}$$

If we restrict $\Lambda$ to the points on $\mathcal{M}$, it gives a smooth mapping from $\mathcal{M}$ to $\mathcal{D}$ since we assume that $|0, y\rangle$ are differentiable. Furthermore, we can show that $\Lambda$ and its differential $d\rho$ are injective as shown in the appendix B.2. This means that $\Lambda$ is a embedding from $\mathcal{M}$ to the space of all $N \times N$ density matrix. The viewpoint of $\mathcal{M}$ as a submanifold of $\mathcal{D}$ tell us a new insight. On $\mathcal{D}$, there is a natural Riemannian metric, which is called information metric, defined by the form of

$$ds^2 = \frac{1}{2}\mathrm{Tr}(Gd\Lambda), \quad d\Lambda = \Lambda G + G\Lambda. \tag{5.2.8}$$

where the trace is taken over the Hilbert space $\mathcal{H}$ on which density matrices acts. Using the embedding $\Lambda : \mathcal{M} \to \mathcal{D}$, we can define the pullback of the information metric which gives a Riemannian metric on $\mathcal{M}$. Since $\Lambda(y)$ defined by (5.2.6) satisfy $\rho^2 = \rho/\mathcal{N}$, we find

$$G(y) = \mathcal{N} d\Lambda(y). \tag{5.2.9}$$

The pullback of the information induced by $\Lambda$ is

$$ds^2 = \frac{\mathcal{N}}{2}\mathrm{Tr}(d\Lambda(y))^2. \tag{5.2.10}$$

In terms of coherent states, this can be also expressed as

$$ds^2 = \frac{1}{\mathcal{N}}\left(\sum_{a=0}^{\mathcal{N}-1} \|d\,|a,y\rangle\,\|^2 - \sum_{a,b=0}^{\mathcal{N}-1} |\,\langle a,y|d|b,y\rangle\,|^2\right). \tag{5.2.11}$$

**Example: fuzzy $S^2$**

Since $\mathcal{N} = 1$ in this case, the pullback of the information metric is

$$ds^2 = \|dU(\theta,\varphi)\,|J,J\rangle\,\|^2 - |\,\langle JJ|U^\dagger(\theta,\varphi)dU(\theta,\varphi)|JJ\rangle\,|^2, \tag{5.2.12}$$

The second term is nothing but the square of the Berry's connection, we find

$$-|\,\langle JJ|U^\dagger(\theta,\varphi)dU(\theta,\varphi)|JJ\rangle\,|^2 = -J^2(1-\cos\theta)^2(d\varphi)^2. \tag{5.2.13}$$

On the other hand, using the relation (5.1.18), we can calculate the first term in (5.2.12) as

$$\|dU(\theta,\varphi)\,|J,J\rangle\,\|^2 = \frac{J}{2}(d\theta)^2 + J^2(1-\cos\theta)^2(d\varphi)^2 + \frac{J}{2}\sin^2\theta(d\varphi)^2. \tag{5.2.14}$$

We therefore get

$$ds^2 = \frac{J}{2}((d\theta)^2 + \sin^2\theta(d\varphi)^2). \tag{5.2.15}$$

**Example: fuzzy $S^4$**

In the case for the fuzzy $S^4$ the information metric is

$$d^2s = \frac{4}{(n+1)}\left(-\sum_{\alpha,\beta}\left|\langle\frac{n}{2},\frac{n}{2}-\alpha|U^{\dagger\otimes n}d(U^{\otimes n})|\frac{n}{2},\frac{n}{2}-\beta\rangle\right|^2 + \sum_{\alpha}\langle\frac{n}{2},\frac{n}{2}-\alpha|\{d(U^{\dagger\otimes n})\}d(U^{\otimes n})|\frac{n}{2},\frac{n}{2}-\alpha\rangle\right). \tag{5.2.16}$$

we can evaluate the first term in (5.2.16) as

$$\begin{aligned}
&-\sum_{\alpha,\beta}\left|\langle\frac{n}{2},\frac{n}{2}-\alpha|U^{\dagger\otimes n}d(U^{\otimes n})|\frac{n}{2},\frac{n}{2}-\beta\rangle\right|^2 \\
&= -\sum_{\alpha,\beta}\left|\delta_{\alpha\beta}(n-2\alpha)C_3 + \delta_{\alpha(\beta+1)}\sqrt{\alpha(n-\alpha+1)}C_+ + \delta_{\alpha(\beta-1)}\sqrt{(n-\alpha)(\alpha+1)}C_-\right|^2 \\
&= -\sum_{\alpha}\left\{(n-2\alpha)^2|C_3|^2 + \alpha(n-\alpha+1)|C_+|^2 + (n-\alpha)(\alpha+1)|C_-|^2\right\} \\
&= -\sum_{\alpha}\left\{(n-2\alpha)^2|C_3|^2 + (2n\alpha-2\alpha^2+n)|C_+|^2\right\}.
\end{aligned} \tag{5.2.17}$$

On the other hand, by the projection $P_+$ the factor $(dU^\dagger)dU = -(U^\dagger dU)^2$ is restricted to

$$\begin{aligned} P_+(dU^\dagger)dUP_+ &= -P_+U^\dagger dU(P_+ + P_-)U^\dagger dUP_+ \\ &= -(P_+U^\dagger dUP_+)^2 - (P_+U^\dagger dUP_-)(P_-U^\dagger dUP_+). \end{aligned} \tag{5.2.18}$$

The second term in the last line is evaluated as

$$(P_+U^\dagger dUP_-)(P_-U^\dagger dUP_+) = \left(\sum_i c_{i5}\Gamma_{i5} + c_{45}\Gamma_{45}\right)^2 P_+ = -\left(c_{15}^2 + c_{25}^2 + c_{35}^2 + c_{45}^2\right)P_+, \tag{5.2.19}$$

so we get

$$P_+(dU^\dagger)dUP_+ + (P_+U^\dagger dUP_+)^2 = C_I P_+ \tag{5.2.20}$$

where

$$C_I = c_{15}^2 + c_{25}^2 + c_{35}^2 + c_{45}^2. \tag{5.2.21}$$

To evaluate the second term in (5.2.16) we see that

$$\begin{aligned} &P_+^{\otimes n}\{d(U^{\dagger\otimes n})\}d(U^{\otimes n})P_+^{\otimes n} \\ &\quad = \left[P_+(dU^\dagger)dUP_+ \otimes P_+ \otimes \cdots \otimes P_+ + \cdots\right] - \left[P_+U^\dagger dUP_+ \otimes P_+U^\dagger dUP_+ \otimes P_+ \otimes \cdots \otimes P_+ + \cdots\right] \\ &\quad = \left[\left\{P_+(dU^\dagger)dUP_+ + (P_+U^\dagger dUP_+)^2\right\} \otimes P_+ \otimes \cdots \otimes P_+ + \cdots\right] - \left(P_+^{\otimes n}U^{\dagger\otimes n}d(U^{\otimes n})P_+^{\otimes n}\right)^2. \end{aligned} \tag{5.2.22}$$

This means the expression

$$\begin{aligned} P_+^{\otimes n}\{d(U^{\dagger\otimes n})\}d(U^{\otimes n})P_+^{\otimes n} &= nC_I P_+^{\otimes n} - (2C_3S_3^{(n)} + C_+S_+^{(n)} + C_-S_-^{(n)})^2 \\ &= nC_I P_+^{\otimes n} - 4C_3^2(S_3^{(n)})^2 - 2C_+C_-\{(\boldsymbol{S}^{(n)})^2 - (S_3^{(n)})^2\}, \end{aligned} \tag{5.2.23}$$

where $(\boldsymbol{S}^{(n)})^2 = \sum_i (S_i^{(n)})^2$. Using this expression we can evaluate the second term in (5.2.16) as

$$\begin{aligned} &\sum_\alpha \langle \frac{n}{2}, \frac{n}{2} - \alpha|\{d(U^{\dagger\otimes n})\}d(U^{\otimes n})|\frac{n}{2}, \frac{n}{2} - \alpha\rangle \\ &\quad = \sum_\alpha \left[nC_I - (n-2\alpha)^2C_3^2 - 2\left\{\frac{n}{2}\left(\frac{n}{2}+1\right) - \left(\frac{n}{2}-\alpha\right)^2\right\}C_+C_-\right] \\ &\quad = n(n+1)C_I + \sum_\alpha \left\{(n-2\alpha)^2|C_3|^2 + (2n\alpha - 2\alpha^2 + n)|C_+|^2\right\}. \end{aligned} \tag{5.2.24}$$

In the last line we use the fact that $C_3$ is a pure imaginary and $C_+ = -C_-^*$. Thus we can evaluate the metric (5.2.16) as

$$ds^2 = 4nC_I. \tag{5.2.25}$$

Finally, from (4.2.12) we directly derive the eplicit form of the factor $U^\dagger dU$ as

$$\begin{aligned} U^\dagger dU = & \frac{1}{2} d\theta \Gamma_{45} + \frac{1}{2} d\phi(\cos\theta\Gamma_{34} + \sin\theta\Gamma_{35}) \\ & + \frac{1}{2} d\psi\{\cos\phi\Gamma_{23} + \sin\phi(\cos\theta\Gamma_{24} + \sin\theta\Gamma_{25})\} \\ & + \frac{1}{2} d\chi[\cos\psi\Gamma_{12} + \sin\psi\{\cos\phi\Gamma_{13} + \sin\phi(\cos\theta\Gamma_{14} + \sin\theta\Gamma_{15})\}]. \end{aligned} \tag{5.2.26}$$

This means

$$c_{15} = \frac{1}{2} d\chi \sin\psi \sin\phi \sin\theta, \quad c_{25} = \frac{1}{2} d\psi \sin\phi \sin\theta, \quad c_{35} = \frac{1}{2} d\phi \sin\theta, \quad c_{45} = \frac{1}{2} d\theta, \tag{5.2.27}$$

and so we get

$$ds^2 = n\Big(d\theta^2 + \sin^2\theta\, d\phi^2 + \sin^2\theta \sin^2\phi\, d\psi^2 + \sin^2\theta \sin^2\phi \sin^2\psi\, d\chi^2\Big). \tag{5.2.28}$$

This is actually natural Riemannian metric on the unit four sphere.

## 5.3. Kähler structure on membrane

In the previous sections, we see that we can construct a bundle structure and Riemann structure on $\mathcal{M}$ in terms of given matrices. In the construction, what we have assumed are the commutativity of given matrices in the large $N$ limit and differentiability of the eigenvalue of the coherent states of $H(y)$ for $X^i$. The first condition not only ensures that the $\mathcal{M}$ is not empty set, but also that the zero modes of the Dirac operator corresponds to that of $H(y)$. The second condition means that $\mathcal{M}$ is a differentiable manifold. In this section, we further assume the following conditions:

(i). The given matrices satisfy the commutation relation

$$[X^\mu, X^\nu] = i\hbar(N) W^{\mu\nu}(X) + \cdots, \tag{5.3.1}$$

where $W^{\mu\nu}(X)$ is a polynomial of $X^\mu$ such that its definition (degree and coefficients) does not depend on $N$.

(ii). The eigenvalue of the coherent states of $H(y)$ is not degenerate.

In order to understand the meanings of these conditions, let us focus on the context of membranes. Let $x^i$ are embeddings from a membrane $\Sigma$ to $\mathbf{R}^d$, and we assume that the induced Poisson tensor is given by

$$W^{ij}(x) = w^{ab} \partial_a x^i \partial_b x^j = \{x^i, x^j\}. \tag{5.3.2}$$

Defining $T_N(x^i) = X^i$ using given matrices, the first condition (i) means that the given matrices satisfy

$$\begin{aligned}
&\lim_{N\to\infty} \|T_N(x^i x^j) - T_N(x^i)T_N(x^j)\| = 0, \\
&\lim_{N\to\infty} \|\left[T_N(x^i), T_N(x^j)\right] - i\hbar(N)T_N(\{x^i, x^j\})\| = 0.
\end{aligned} \tag{5.3.3}$$

We recall that these conditions are essential which Hermitian matrices should satisfy in the approximation of the Poisson algebra by $\mathfrak{u}(N)$. Therefore, (i) means that the given matrices approximate the Poisson algebra of a membrane whose induced Poisson tensor is $W^{ij}$.

The second condition (ii) means that $\mathcal{M}$ is "single" membrane. For example, we consider a configuration of matrices $X^i$ which associate a single membrane like $S^2$ or $T^2$. Let $|0, y\rangle$ be the coherent states of the Hamiltonian $H(y)$ for $X^i$. We consider new configuration of matrices $\tilde{X}^i$ whose diagonal blocs are $X^i$:

$$\tilde{X}^i = \begin{pmatrix} X^i & O \\ O & X^i \end{pmatrix}. \tag{5.3.4}$$

This configuration corresponds to a double membrane. The Hamiltonian for the new configuration $\tilde{X}^i$ is of the form

$$\tilde{H}(y) = \begin{pmatrix} H(y) & O \\ O & H(y) \end{pmatrix}. \tag{5.3.5}$$

The coherent states of $\tilde{H}(y)$ is clearly given by

$$\begin{pmatrix} |0, y\rangle \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ |0, y\rangle \end{pmatrix}, \tag{5.3.6}$$

and so the eigenvalue of the ground states of $\tilde{H}(y)$ is degenerate.

Under above conditions, we show that the classical space possesses a Kähler structure [10]. More concretely, we will construct tensors on $\mathcal{M}$,

$$\omega_{\mu\nu}, \; J^\mu{}_\nu, \; g_{\mu\nu}, \tag{5.3.7}$$

which are components of a symplectic form $\omega$, complex structure $J$ and Riemann metric $g$, respectively. Also we will show that these structures satisfy the compatible condition $\omega(u, Jv) = g(u, v)$ where $u, v \in T\mathcal{M}$, an therefore the triplet $(\omega, J, g)$ form a Kähler structure on $\mathcal{M}$. Note that the indexes $\mu, \nu$ runs from 1 to $d$ which is the dimension of not $\mathcal{M}$ but $\mathbf{R}^d$. Therefore, we also need to check that (5.3.7) are tangent tensors on $\mathcal{M}$. There is a useful operator, which projects $d$-dimensional vector $B^\mu(y)$ onto its tangent components along $\mathcal{M}$, defined by

$$P^\mu{}_\nu(y) = \delta^\mu_\nu - \partial^\mu \partial_\nu f(y), \tag{5.3.8}$$

where $f(y)$ is the function defined by (4.1.10). If $B^\mu$ is a tangent vector on $\in \mathcal{M}$, it satisfies

$$P^\mu{}_\nu(y)B^\nu(y) = B^\nu(y), \tag{5.3.9}$$

for $y \in \mathcal{M}$.

Berry's connection and information metric are regarded as symplectic and Riemannian structures, respectively. In particular, since compact orientable surface always possesses a Kähler structure, Kähler structure is a fundamental structure on membranes.

#### Poisson structure

We first show that the polynomial $W^{\mu\nu}$ in (5.3.1) gives a Poisson tensor on $\mathcal{M}$. We define a $d$-dimensional real antisymmetric tensor by

$$W^{\mu\nu}(y) = -i \lim_{N\to\infty} \frac{1}{\hbar_N} \langle 0, y| [X^\mu, X^\nu] |0, y\rangle . \tag{5.3.10}$$

For $y \in \mathcal{M}$, we can see that

$$P^\mu{}_\nu(y)W^{\mu\nu}(y) = W^{\mu\rho}(y), \tag{5.3.11}$$

which means $W^{\mu\nu}$ is a bivector on $\mathcal{M}$. Also we can show that $W^{\mu\nu}(y)$ satisfy the Jacobi identity $y \in \mathcal{M}$, and therefore (5.3.10) gives a Poisson tensor on $\mathcal{M}$.

#### Symplectic structure

We have been saw that Berry's connection gives a gauge connection $A(y)$ on a principle fiber bundle over $\mathcal{M}$. In particular, in the case that $E_0(y)$ is not degenerate, the connection is a U(1) geuge connection, namely symplectic potential. Then, the curvature for U(1) gauge connection is a closed non-degenerate two-form, namely symplectic form. Let us define the covariant derivative using Berry's connection as

$$D_\mu = \partial_\mu - iA_\mu = \partial_\mu - \langle 0, y|\partial_\mu|0, y\rangle . \tag{5.3.12}$$

The components of the curvature are

$$F_{\mu\nu}(y) = i[D_\mu, D_\nu] . \tag{5.3.13}$$

Then, we construct a symplectic form on $\mathcal{M}$ as the normalized curvature,

$$\omega_{\mu\nu}(y) = -\lim_{N\to\infty} \hbar_N F_{\mu\nu}(y). \tag{5.3.14}$$

Note that for $y \in \mathcal{M}$ this satisfy

$$W^{\mu\rho}(y)\omega_{\rho\nu}(y) = P^\mu{}_\nu(y). \tag{5.3.15}$$

#### Complex structure

We define $d \times d$ matrix by

$$J(y) = \frac{1}{\sqrt{W(y)W^T(y)}} W(y), \tag{5.3.16}$$

where $W(y)$ is the $d \times d$ matrix whose components are given by $W^{\mu\nu(y)}$. The product in (5.3.16) is just the matrix product. The factor $1/\sqrt{WW^T}$ is defined as follows: Since $W$ is non-degenerate on the space of tangent vectors from (5.3.15), there are is a basis such that $W$ becomes the canonical form,

$$\begin{aligned} \{P^\mu{}_\nu\} &= V \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} V^{-1}, \\ \{W^\mu{}_\nu\} &= V \begin{pmatrix} 0 & \\ & \tilde{W} \end{pmatrix} V^{-1}, \end{aligned} \tag{5.3.17}$$

where $V$ is a transition matrix, and the up-left and bottom-right blocs act on the space of the normal and the tangent vectors with dimensions $d - \dim \mathcal{M}$ and $\dim \mathcal{M}$, respectively. Here, $\tilde{W}$ is a non-degenerate real antisymmetric matrix, which consists of $2 \times 2$ diagonal blocks. The factor $1/\sqrt{WW^T}$ is defined as

$$\frac{1}{\sqrt{WW^T}} = V \begin{pmatrix} 0 & \\ & \frac{1}{\sqrt{\tilde{W}\tilde{W}^T}} \end{pmatrix} V^{-1}. \tag{5.3.18}$$

The definition of $J$ implies that $J$ satisfy, for $y \in \mathcal{M}$,

$$P(y)J(y) = J(y)P(y) = J(y), \quad J^2(y) = -P(y). \tag{5.3.19}$$

These conditions guarantee that $J$ defines an almost complex structure on $\mathcal{M}$.

Note that the by differentiating the first relation, we get

$$\partial_\rho P^\mu{}_\nu + [\Gamma_\rho, P]^\mu{}_\nu = 0, \quad \Gamma_\rho := V\partial_\rho V^{-1}, \tag{5.3.20}$$

where the derivative is restricted to tangential directions on $\mathcal{M}$. Defining a covariant derivatives as

$$\nabla_\rho B^\mu = \partial_\rho B^\mu + (\Gamma_\rho)^\mu{}_\nu B^\nu, \quad \nabla_\rho B_\mu = \partial_\rho B_\mu - (\Gamma_\rho)^\nu{}_\mu B_\nu, \tag{5.3.21}$$

for a $d$-dimensional tensor $B_\mu$. In terms of this covariant derivatives, the (5.3.20) is expressed as $\nabla P = 0$. This equation implies that $\nabla$ is a covariant derivative on $\mathcal{M}$ which preserve inner product with respect to $P$. From the second relation in (5.3.17) and (5.3.18), we have

$$\nabla_\rho J^\mu{}_\nu = 0. \tag{5.3.22}$$

This is nothing but the integrability of $J$, and therefore $J$ gives a complex structure on $\mathcal{M}$.

#### Riemann structure

In terms of the Poisson tensor, the Riemann metric given by the pullback of the information metric is written by

$$g = \frac{1}{\sqrt{WW^T}}. \tag{5.3.23}$$

Because we construct the complex structure $J$ to be compatible with $\omega$ and $g$, the triplet $(\omega, J, g)$ gives a Kähler structure on $\mathcal{M}$.

# 6. Conclusion

The main purpose of our study is to understand matrix models, which are expected to give non-perturbative formulations of string and M- theories. In order to understand the matrix-model formulations of string and M-theories, one needs to understand how the fundamental objects in string and M- theories are described in terms of matrix variables. In this thesis, we studied the relation between matrix configurations in the matrix models and smooth brane geometries in string and M- theories.

To relate the matrix configurations to the brane geometries, we introduced a method based on an analogy with the canonical quan- tization of classical mechanics. This method makes it possible to define a manifold, which corresponds to the worldvolume of a brane, from a given configurations of Hermitian matrices in the matrix models. This method works not only for M2-branes but also for other types of branes, for which the well-known method of the matrix regularization can not always be defined.

We developed a theory of differential geometry of the emergent brane geomtry. In particular, we proposed that Berry's connection and the information metric can be used to characterize the brane geometry. The information metric provides a Riemannian structure on the brane worldvolume, while Berry's conneciton captures the bundle structure of gauge fields. We explicitly computed these geometric structures for fuzzy $S^2$ and fuzzy $S^4$. For these examples, we found that the information metric is given by the ordinary round metrics and Berry's connection takes topologically nontrivial cofigurations such as the Wu-Yang monopole on $S^2$ and the Yang monopole on $S^4$. Our construction of these geometric objects works in very general framework and can be applied to any matrix models for string and M- theories.

Finally, we focused on the context of M-theory and considered a situation that the emergent geometry corresponds to a single closed orientable membrane. In this case, since the theory on the membrane has a natural Kähler structure on the worldvolume, there must exists a correpoding geometric structure on the emergent brane geometry. We showed that in addition to Berry's connection and the information metric, a complex structure can also be defined from the given matrix configuraitons. Furthremore, we also proved that these geometric structures form a compatible triplet of the Kähler structure. This result shows that the geometric objects proposed in this thesis correspond to the fundamental geometric structures on the brane worldvolume.

We consider that our results are also relevant to the recent proposals [36][37] on the emergent gravity in the matrix models. Since the metric structure is the central object in formulating gravitational theories, we expect that the information metric will play an important role in understanding these proposals.

Another interesting future direction is to understand the Seiberg-Witten map [38], which relates noncommutative gauge fields with ordinary commutative gauge fields on D-branes. In our setup, fluctuations of matrices correspond to the noncommutative gauge fields, while Berry's connection corresponds to the commutatve gauge fields. Our formulation may provide an explicit realization of the Seiberg-Witten map. The Seiberg-witten map has been constructed only for some symmetric spaces such as $R^{2n}$ or $S^2$. On the other hand, our formulation is not limited to these spaces and hence may provide a more general construction of the Seiberg-Witten map.

# A. Membrane theory

## A.1. Bosonic membrane theory

In this section, we review the theory of a classical relativistic bosonic membrane moving in flat $d$-dimensional Minkowski space. This analysis is very similar to the theory of a classical relativistic bosonic string except the difference of the dimension between membranes and strings. However, off course, the only one difference leads completely different treatment of the theory, and unfortunately the membrane theory seems to be difficult to deal with unlike the string theory.

A string moving in $(d-1)$-dimensional space sweeps out a trajectory described by a surface in $d$-dimensional space-time, called world-sheet. Similarly, a dynamical membrane moving in $(d-1)$-dimensional space sweeps out a 3-dimensional hypersurface in $d$-dimensional space-time, called *world-volume*. The world-volume of the membrane is interpreted as a 3-dimensional submanifold $\mathcal{V}$ of $d$-dimensional Minkowski space, and then we can describe the motion of the membrane by the embedding $X : \mathcal{V} \to \mathbf{R}^{d-1,1}$. Let $\sigma^\alpha, \alpha \in \{1, 2, 3\}$, are local coordinates on $\mathcal{V}$. We will also use the notation $\tau = \sigma^0$ and indices $a, b, \ldots \in \{1, 2\}$ to describe "spatial" coordinates $\sigma^a$ on the world-volume. In such a coordinate system, the motion of the membrane moving in the space-time is described by a set of $d$ embedding functions $X^\mu(\sigma^0, \sigma^1, \sigma^2)$.

Let us consider the classical action for a membrane. We can determine the form by the same argument for Nambu-Goto action in string theory. That is, the action is given by a local quantity written in terms of $X^\mu$ that is invariant under the Poincaré transformation in the space-time and the diffeomorphism on $\mathcal{V}$. The simplest such quantity is the volume swept out by the membrane:

$$S = -T \int d^3\sigma \sqrt{-\mathrm{det} h_{\alpha\beta}}\,, \tag{A.1.1}$$

where $T$ is a constant with a dimension of (mass)$^3$ which can be interpreted as the membrane tension, and

$$h_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X_\mu\,, \tag{A.1.2}$$

is a metric on $\mathcal{V}$ induced by the flat metric diag$(- + + \cdots +)$ on $\mathbf{R}^{d-1,1}$. The minus sign of the action guarantee that the action can have the minimum value. Using the Planck length $l_p$ which is the only dimensional physical constant in this theory, we can express the constant as $T = 1/(2\pi)^2 l_p^3$.

As the case of string theory, through introducing a auxiliary metric $\gamma_{\alpha\beta}$ on the world-volume, we can rewrite the action (A.1.1) into the polynomial form which leads to the same classical equations of motion,

$$S = -\frac{T}{2}\int d^3\sigma\sqrt{-\gamma}\left(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - 1\right), \tag{A.1.3}$$

where $\gamma \equiv \det\gamma_{\alpha\beta}$. Using $\delta\gamma = -\gamma\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta}$, we find the equation of motion for auxiliary metric $\gamma_{\alpha\beta}$,

$$h_{\alpha\beta} = \frac{1}{2}\gamma_{\alpha\beta}\left(\gamma^{\delta\eta}h_{\delta\eta} - 1\right). \tag{A.1.4}$$

By contracting $\gamma^{\alpha\beta}$ to the both sides, we get

$$\gamma^{\alpha\beta}h_{\alpha\beta} = \frac{3}{2}\left(\gamma^{\alpha\beta}h_{\alpha\beta} - 1\right). \tag{A.1.5}$$

This means $\gamma^{\alpha\beta}h_{\alpha\beta} = 3$, and hence the solution of the equation (A.1.4) is

$$\gamma_{\alpha\beta} = h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu\,. \tag{A.1.6}$$

Using the equation of motion for auxiliary metric $\gamma_{\alpha\beta}$, we easily find that the action (A.1.3) actually equivalent to the original action (A.1.1). The equation of motion for $X^\mu$ is given by

$$\partial_\alpha(\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu) = 0\,. \tag{A.1.7}$$

The action (A.1.3) is analogue of the Polyakov action for the bosonic string. However, there is a great difference between them; The action (A.1.3) does not have a scale symmetry. The second term in the action (A.1.3), which is like a cosmological constant, is not scale invariant. Note that although the first term also break the Weyl symmetry, it can be "scale" invariant if we transform not only $\gamma_{\alpha\beta}$ but also $X^\mu$. Therefore, in the action (A.1.3), the only second term essentially break the scale symmetry. The absence of the scale symmetry means that unlike the string theory the membrane theory can not gain various benefit from the scale symmetry.

To simplify the analysis of the action (A.1.1) (or (A.1.3)), we assume that $\mathcal{V}$ is the form of $\mathbf{R}\times\Sigma$, where $\Sigma$ is a Riemann surface of fixed topology. This assumption does not really restrict the situation because Riemman surfaces, namely, closed and oriented 2-dimensional manifold, is very general surface. In this case, the metric on $\mathcal{V}$ is written as

$$d^2s = h_{00}(\sigma)(d\sigma^0)^2 + h_{ab}(\sigma)d\sigma^a d\sigma^b, \tag{A.1.8}$$

where $h_{ab}$ is a metric on $\Sigma$. The dependence of $h_{ab}$ on the "time" coordinate $\sigma^0$ means that the form of the membrane can change in the process of time evolution.

In addition, we gauge-fix the metric $h_{\alpha\beta}$ using the symmetries of the theory. However, unlike in the case of the classical string, where there are three components of the metric and three continuous symmetries (two diffeomorphism symmetries and the Weyl symmetry), in the case of the membrane we have six components of the metric and only three diffeomorphism symmetries. Thus, we can not fix all the components of the metric. Using one diffeomorphism symmetry, we fix a component $h_{00}$ as

$$h_{00} = -\frac{4}{\nu^2}\bar{h} \equiv -\frac{4}{\nu^2}\mathrm{det}h_{ab}\,, \tag{A.1.9}$$

where $\nu$ is an arbitrary constant which is chosen later. In this situation, we find

$$\sqrt{-\mathrm{det}h_{\alpha\beta}} = \sqrt{-h_{00}\cdot \mathrm{det}h_{ab}} = -\frac{\nu}{2}h_{00} = \frac{\nu}{4}\left(-h_{00} + \frac{4}{\nu^2}\bar{h}\right)\,. \tag{A.1.10}$$

Hence, the action (A.1.1) is rewritten as

$$S = \frac{T\nu}{4}\int d^3\sigma\left(\dot{X}^\mu\dot{X}_\mu - \frac{4}{\nu^2}\bar{h}\right)\,. \tag{A.1.11}$$

On the 2-dimensional surface, we can naturally introduce a Poisson bracket. Because the action (A.1.11) still have diffeomorphism symmetry for $\sigma^a$, we can fix the form of the Poisson bracket on $\Sigma$ at each $\tau$ as

$$\{f, g\} = \epsilon^{ab}\partial_a f\partial_b g\,, \tag{A.1.12}$$

where $f, g \in C(\Sigma)$. In terms of this Poisson bracket, the factor $\bar{h}$ is expressed as

$$\bar{h} = \epsilon^{ab}h_{a1}h_{b2} = \frac{1}{2}\epsilon^{ab}\epsilon^{cd}h_{ac}h_{bd} = \frac{1}{2}\left(\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu\right)\left(\epsilon^{cd}\partial_c X_\mu\partial_d X_\nu\right) = \frac{1}{2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}\,, \tag{A.1.13}$$

and hence the membrane action becomes

$$S = \frac{T\nu}{4}\int d^3\sigma\left(\dot{X}^\mu\dot{X}_\mu - \frac{2}{\nu^2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}\right)\,. \tag{A.1.14}$$

**Light-front bosonic membrane**

We consider the membrane theory in light-front coordinates

$$X^\pm = \left(X^0 \pm X^{d-1}\right)/\sqrt{2}\,. \tag{A.1.15}$$

In this coordinates, the flat metric diag$(- + + \cdots +)$ on $\mathbb{R}^{d-1,1}$ becomes

$$d^2s = -2dX^+dX^- + dX^i dX_i\,. \tag{A.1.16}$$

Using the Poincaré symmetry, we choose the light-front gauge

$$X^+(\tau, \sigma^1, \sigma^2) = \tau\,. \tag{A.1.17}$$

In this gauge, the action (A.1.14) is rewritten as

$$S = \frac{T\nu}{4}\int d^3\sigma \left(-2\dot{X}^- + \dot{X}^i\dot{X}^i - \frac{2}{\nu^2}\{X^i, X^j\}\{X_i, X_j\}\right) . \tag{A.1.18}$$

Note that $X_\pm = -X^\mp$ and $\partial_a X^+ = -\partial_a X_- = \partial_a \tau = 0$.

We can also go to a Hamiltonian formalism by computing the canonically conjugate momentum densities. The Lagrange density of the theory is

$$\mathcal{L} = \frac{T\nu}{4}\left(-2\dot{X}^- + \dot{X}^i\dot{X}^i - \frac{2}{\nu^2}\{X^i, X^j\}\{X_i, X_j\}\right) . \tag{A.1.19}$$

It is easy to find that each of the momentum density is given by

$$P^+ = \frac{\partial\mathcal{L}}{\partial\dot{X}_+} = -\frac{\partial\mathcal{L}}{\partial\dot{X}^-} = \frac{T\nu}{2}, \tag{A.1.20}$$

$$P^- = \frac{\partial\mathcal{L}}{\partial\dot{X}_-} = -\frac{\partial\mathcal{L}}{\partial\dot{X}^+} = 0, \tag{A.1.21}$$

$$P^i = \frac{\partial\mathcal{L}}{\partial\dot{X}_i} = \frac{T\nu}{2}\dot{X}^i . \tag{A.1.22}$$

Hence, the Hamiltonian of the theory is given by

$$\begin{aligned} H &= \int d^2\sigma \left(-P^+\dot{X}^- - P^-\dot{X}^+ + P_i\dot{X}^i - \mathcal{L}\right) , \\ &= \frac{\nu T}{4}\int d^2\sigma \left(\dot{X}^i\dot{X}^i + \frac{2}{\nu^2}\{X^i, X^j\}\{X_i, X_j\}\right) . \end{aligned} \tag{A.1.23}$$

## A.2. Supermembrane theory

In the case of string theory, there are two approaches to formulation of superstring: Neveu-Schwarz-Ramond (NSR) approach introducing world-sheet fermions and Green-Schwarz introducing space-time fermions. In the the NSR formulation, the world-sheet supersymmetry is manifest, but it is difficult to show the target space supersymmetry of the theory explicitly. On the other hand, in the Green-Schwarz formulation, the target space supersymmetry, but the theory does not have a world-sheet supersymmetry because there are no world-sheet fermions. In the Green-Schwarz formulation, however, the theory has not only a global suparsymmetry but a local superesymetry for space-time dimension $d = 3, 4, 6, 10$. Such local supersymmetry, which is called $\kappa$ symmetry, restricts the dimension of the theory even in the classical level.

In the case of membrane theory, the way of realizing the world-volume superesymmetry has not known so far. On other hand, Green-Schwarz formulation of superemembrane have been found [35]. Similarly the case of string, the existence of $\kappa$ symmetry restrict the space-time dimension of the membrane theory to $d = 4, 5, 7, 11$. These theories have 2,4,8,16 supercharges respectively.

Although the construction in [Bergshoeff-Sezgin-Townsend] formulates the supermembrane in a general curved space-time, we focus on the case for supermembrane in flat space which is related to BFSS matrix models.

On the analogy of the Green-Schwarz formulation of superstring in flat space, we can get the supermembrane action by replacing the term $\partial_\alpha X^\mu$ in the bosonic membrane action with

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu + \bar{\theta}\Gamma^\mu\partial_\alpha\theta, \tag{A.2.1}$$

where $\theta$ is a 16-components Majorana spinors of SO(9). The supermembrane action in flat space is

$$\begin{aligned} S = -\frac{T}{2}\int d^3\sigma \Bigg[ & \sqrt{-\gamma}(\gamma^{\alpha\beta}\Pi^\mu_\alpha\Pi^\nu_\beta\eta_{\mu\nu}) - 1) \\ & - \epsilon^{\alpha\beta\gamma}\bar{\theta}\Gamma_{\mu\nu}\partial_\gamma\theta\left\{\frac{1}{2}\partial_\alpha X^\mu(\partial_\beta + \bar{\theta}\Gamma^\nu\partial_\beta\theta) + \frac{1}{6}(\bar{\theta}\Gamma^\mu\partial_\alpha\theta)(\bar{\theta}\Gamma^\nu\partial_\beta\theta)\right\}\Bigg]. \end{aligned} \tag{A.2.2}$$

The extra terms are added to hold $\kappa$ symmetry. This action has the target space supersymmetry

$$\begin{aligned} \delta\theta &= \epsilon, \\ \delta X^m u &= -\bar{\epsilon}\Gamma^\mu\theta. \end{aligned} \tag{A.2.3}$$

$$\Gamma = \frac{\epsilon^{ijk}}{6\sqrt{-g}}\Pi^\mu_i\Pi^\nu_j\Pi^\rho_k\Gamma_{\mu\nu\rho} \tag{A.2.4}$$

$\kappa$ symmetry is

$$\theta = (1-\gamma)\kappa, \quad \delta X^\mu = \bar{\kappa}(1-\Gamma)\Gamma^\mu\theta \tag{A.2.5}$$

**Instability of membrane**

We consider a system of single bosonic membrane with a constant tension $T$. The energy of the system is given by the energy of the membrane, namely the area of the membrane times $T$. Let us imagine a part of the membrane stretching and having a long narrow spike. If we roughly regard the spike as a cylinder with a radius $r$ and length $L$, the energy to have such spike is $2\pi rLT$. For a small radius such that $r \ll 1/TL$, the length $L$ can be very large without increasing the energy. It means that membranes have very long spikes with just a little energy cost. Since the energy always fluctuates in quantum theory, quantum membranes are no longer an local objects in space-time. More higher dimensional objects also have same problem. On the other hand, quantum strings does not have such problem. In order to have spikes, strings always need the energy proportional to the length of the spike, and there is no other parameter capable of controlling the energy as the radius in the case of membranes.

# B. Detail calculations

## B.1. Derivation of equation (3.2.9)

First, we show that

$$\begin{aligned}
\lim_{N\to\infty} \|\hat{f}_N \hat{h}_N - T_N(fh)\| &= \lim_{N\to\infty} \|\hat{f}_N \hat{h}_N - \hat{f}_N T_N(h) + \hat{f} T_N(h) - T_N(fh)\| \\
&\leq \lim_{N\to\infty} \Big( \|\hat{f}_N\| \|\hat{h}_N - T_N(h)\| \\
&\quad + \|\hat{f} T_N(h) - T_N(f) T_N(h) + t_N(f) T_N(h) - T_N(fh)\| \Big) \\
&\leq \lim_{N\to\infty} \Big( \|\hat{f}_N\| \|\hat{h}_N - T_N(h)\| \\
&\quad + \|\hat{f}_N - T_N(f)\| \|T_N(h)\| + \|T_N(f) T_N(h) - T_N(fh)\| \Big) \\
&= 0.
\end{aligned} \tag{B.1.1}$$

In the second and third line, we use the triangle inequality and the last line follows from the condition (3.2.6). Note that $\|\hat{f}_N\|$ is bounded by (3.2.5) since the definition of $\hat{f}_N$ means

$$\lim_{N\to\infty} (\|\hat{f}_N\| - \|T_N(f)\|) \leq \lim_{N\to\infty} \|\hat{f}_N - T_N(f)\| = 0, \tag{B.1.2}$$

and so $\lim_{N\to\infty} \|\hat{f}_N\| = \lim_{N\to\infty} \|T_N(f)\|$. Then, we derive (3.2.9) from (B.1.1) by computing

$$\begin{aligned}
\lim_{N\to\infty} 2\pi\hbar_N \mathrm{Tr} \hat{f}_N \hat{h}_N &= \lim_{N\to\infty} 2\pi\hbar_N \mathrm{Tr}(\hat{f}_N \hat{h}_N - T_N(fh) + T_N(fh)) \\
&= \lim_{N\to\infty} 2\pi\hbar_N \mathrm{Tr} T_N(fh) \\
&= \int_\Sigma f h \, \omega.
\end{aligned} \tag{B.1.3}$$

The last line follows from the condition (3.2.8).

## B.2. Proof of embedding

We prove that $\Lambda$ gives a embedding in the context of Dirac operator because the zero modes of Dirac operator are correspond to that of the Hamiltonian in the large-$N$ limit. We first show that

$\Lambda : y \in \mathcal{M} \to \mathbf{C}P^N$ is injective. Let $y, y \in \mathcal{M}$ and $y \neq y'$. It follows that

$$
\begin{aligned}
\not{D}(y)\Lambda(y) &= \Gamma^\mu \otimes (X_\mu - y_\mu)\Lambda(y) = 0, \\
\not{D}(y')\Lambda(y') &= \Gamma^\mu \otimes (X_\mu - y'_\mu)\Lambda(y') = 0.
\end{aligned}
\tag{B.2.1}
$$

Similarly, $\Lambda(y)\overleftarrow{\not{D}}(y) = \Lambda(y')\overleftarrow{\not{D}}(y') = 0$ also follows. If $\Lambda(y) = \Lambda(y')$, we find

$$
(\Gamma^\mu \otimes 1_N)(y_\mu - y'_\mu)\Lambda(y) = \Lambda(y)(\Gamma^\mu \otimes 1_N)(y_\mu - y'_\mu) = 0.
\tag{B.2.2}
$$

Combining these relation, we get

$$
\begin{aligned}
0 &= \Lambda(y)(\Gamma^\mu \otimes 1_N)(y_\mu - y'_\mu)(\Gamma^\nu \otimes 1_N)(y_\nu - y'_\nu)\Lambda(y) \\
&= (y_\mu - y'_\mu)^2\Lambda^2(y).
\end{aligned}
\tag{B.2.3}
$$

Since $\Lambda^2(y) = \frac{1}{k}\Lambda(y)$, this relation means that $\Lambda(y)$ is zero matrix, but it is inconsistent. Therefore, we conclude that $\Lambda(y) \neq \Lambda(y')$ if $y \neq y'$, and so $\Lambda$ is injective.

We next show that the differential $d\Lambda : T\mathcal{M} \to T\mathbf{C}P^N$ is also injective. Let $c^\mu(y)\partial_\mu$ be a tangent vector field on $\mathcal{M}$. The injectivity of $d\Lambda$ equivalent to that if $c^\mu\partial_\mu\Lambda = 0$, $c^\mu$ is vanishing. We assume that $c^\mu\partial_\mu\Lambda = 0$ on $\mathcal{M}$. Then, it follows that

$$
\begin{aligned}
0 &= c^\mu(\partial_\mu\Lambda(y))\,|a, y\rangle \\
&= \frac{c^\mu}{k}\left(\partial_\mu \sum_{b=0}^{k} |b, y\rangle\langle b, y|\right)|a, y\rangle \\
&= \frac{c^\mu}{k}\left(1 - \sum_{b=0}^{k} |b, y\rangle\langle b, y|\right)\partial_\mu\,|a, y\rangle \\
&= \frac{c^\mu}{k}\sum_{n=k+1} |n, y\rangle\langle n, y|\,\partial_\mu\,|a, y\rangle\,.
\end{aligned}
\tag{B.2.4}
$$

Since $\{\,|n, y\rangle\}$ is a orthonormal basis, we find that

$$
c^\mu\,\langle n, y|\partial_\mu|a, y\rangle = 0, \quad n = k+1, k+2, \ldots, N.
\tag{B.2.5}
$$

This relation means that

$$
c^\mu\,\langle n, y|\Gamma_\mu|a, y\rangle = 0, \quad n = k+1, k+2, \ldots, N,
\tag{B.2.6}
$$

because it holds that

$$
0 = c^\mu\partial_\mu(\not{D}(y)\,|a, y\rangle) = -c^\mu\Gamma_\mu\,|a, y\rangle + \not{D}(y)c^\mu\partial_\mu\,|a, y\rangle\,,
\tag{B.2.7}
$$

so $c^\mu\,\langle n, y|\partial_\mu|a, y\rangle = c^\mu\,\langle n, y|\Gamma_\mu|a, y\rangle/E_n$ for $n = k+1, \ldots, N$. We also find that

$$
c^\mu\,\langle a, y|\Gamma_\mu|b, y\rangle = 0.
\tag{B.2.8}
$$

Combining these relation, we get

$$c^\mu \Gamma_\mu |a, y\rangle = 0. \tag{B.2.9}$$

Using this, we can calculate as

$$0 = \| c^\mu \Gamma_\mu |a, y\rangle \|^2 = c^\mu c^\nu \langle a, y|\Gamma_\mu \Gamma_\nu|b, y\rangle = |c|^2. \tag{B.2.10}$$

This show that $c^\mu = 0$, so we conclude that $d\Lambda$ is injective.

# References


[1] E. Witten, “String theory dynamics in various dimensions,” Nucl. Phys. B **443**, 85 (1995) [hep-th/9503124].

[2] J. Polchinski, S. Chaudhuri and C. V. Johnson, “Notes on D-branes,” hep-th/9602052.

[3] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A Conjecture,” Phys. Rev. D **55**, 5112 (1997) [hep-th/9610043].

[4] D. N. Kabat and P. Pouliot, “A Comment on zero-brane quantum mechanics,” Phys. Rev. Lett. **77**, 1004 (1996) [hep-th/9603127].

[5] M. R. Douglas, D. N. Kabat, P. Pouliot and S. H. Shenker, “D-branes and short distances in string theory,” Nucl. Phys. B **485**, 85 (1997) [hep-th/9608024].

[6] B. de Wit, J. Hoppe and H. Nicolai, “On the Quantum Mechanics of Supermembranes,” Nucl. Phys. B **305**, 545 (1988).

[7] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A Large N reduced model as superstring,” Nucl. Phys. B **498**, 467 (1997) [hep-th/9612115].

[8] A. Connes. “Noncommutative geometry” Academic Press, New York, 1994

[9] A. Connes, M. R. Douglas and A. S. Schwarz, “Noncommutative geometry and matrix theory: Compactification on tori,” JHEP **9802**, 003 (1998) [hep-th/9711162].

[10] G. Ishiki, T. Matsumoto and H. Muraki, “Kähler structure in the commutative limit of matrix geometry,” JHEP **1608**, 042 (2016) [arXiv:1603.09146 [hep-th]].

[11] J. Dai, R. G. Leigh and J. Polchinski, “New Connections Between String Theories,” Mod. Phys. Lett. A **4**, 2073 (1989).

[12] J. Polchinski, “Tasi lectures on D-branes,” hep-th/9611050.

[13] M. Born and L. Infeld, “Foundations of the new field theory,” Proc. Roy. Soc. Lond. A **144**, no. 852, 425 (1934).

[14] E. S. Fradkin and A. A. Tseytlin, "Nonlinear Electrodynamics from Quantized Strings," Phys. Lett. **163B**, 123 (1985).

[15] A. Abouelsaood, C. G. Callan, Jr., C. R. Nappi and S. A. Yost, "Open Strings in Background Gauge Fields," Nucl. Phys. B **280**, 599 (1987).

[16] R. G. Leigh, "Dirac-Born-Infeld Action from Dirichlet Sigma Model," Mod. Phys. Lett. A **4**, 2767 (1989).

[17] E. Witten, "Bound states of strings and p-branes," Nucl. Phys. B **460**, 335 (1996) [hep-th/9510135].

[18] L. Susskind, "Another conjecture about M(atrix) theory," hep-th/9704080.

[19] N. Seiberg, "Why is the matrix model correct?," Phys. Rev. Lett. **79**, 3577 (1997) [hep-th/9710009].

[20] W. Taylor, "M(atrix) theory: Matrix quantum mechanics as a fundamental theory," Rev. Mod. Phys. **73**, 419 (2001) [hep-th/0101126].

[21] B. de Wit, J. Hoppe and H. Nicolai, "On the Quantum Mechanics of Supermembranes," Nucl. Phys. B **305**, 545 (1988).

[22] P. K. Townsend, "D-branes from M-branes," Phys. Lett. B **373**, 68 (1996) [hep-th/9512062].

[23] J. Arnlind, J. Hoppe and G. Huisken, "Multi-linear formulation of differential geometry and matrix regularizations," J. Diff. Geom. **91**, no. 1, 1 (2012) [arXiv:1009.4779 [math.DG]].

[24] J. Madore, "The Fuzzy sphere," Class. Quant. Grav. **9**, 69 (1992).

[25] D. B. Fairlie and C. K. Zachos, "Infinite Dimensional Algebras, Sine Brackets and SU(Infinity)," Phys. Lett. B **224**, 101 (1989).

[26] D. B. Fairlie, P. Fletcher and C. K. Zachos, "Infinite Dimensional Algebras and a Trigonometric Basis for the Classical Lie Algebras," J. Math. Phys. **31**, 1088 (1990).

[27] O. J. Ganor, S. Ramgoolam and W. Taylor, "Branes, fluxes and duality in M(atrix) theory," Nucl. Phys. B **492**, 191 (1997) [hep-th/9611202].

[28] T. Banks, N. Seiberg and S. H. Shenker, "Branes from matrices," Nucl. Phys. B **490**, 91 (1997) [hep-th/9612157].

[29] W. Taylor, "The M(atrix) model of M theory," NATO Sci. Ser. C **556**, 91 (2000) [hep-th/0002016].

[30] G. Ishiki, "Matrix Geometry and Coherent States," Phys. Rev. D **92**, no. 4, 046009 (2015) [arXiv:1503.01230 [hep-th]].

[31] D. Berenstein and E. Dzienkowski, "Matrix embeddings on flat $R^3$ and the geometry of membranes," Phys. Rev. D **86**, 086001 (2012) [arXiv:1204.2788 [hep-th]].

[32] M. V. Berry, "Quantal phase factors accompanying adiabatic changes," Proc. Roy. Soc. Lond. A **392**, 45 (1984).

[33] B. Simon, "Holonomy, the quantum adiabatic theorem, and Berry's phase," Phys. Rev. Lett. **51**, 2167 (1983).

[34] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu, J. Zwanziger "The Geometric Phase in Quantum Systems, Foundations, Mathematical Concepts, and Applications in Molecular and Condensed Matter Physics", Springer, Berlin, Heidelberg, 2003

[35] E. Bergshoeff, E. Sezgin and P. K. Townsend, "Properties of the Eleven-Dimensional Super Membrane Theory," Annals Phys. **185**, 330 (1988).

[36] H. Steinacker, "Emergent Geometry and Gravity from Matrix Models: an Introduction," Class. Quant. Grav. **27**, 133001 (2010) [arXiv:1003.4134 [hep-th]].

[37] M. Hanada, H. Kawai and Y. Kimura, "Describing curved spaces by matrices," Prog. Theor. Phys. **114**, 1295 (2006) [hep-th/0508211].

[38] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999) [hep-th/9908142].