Crystals for an extremal weight module over the quantized hyperbolic Kac-Moody algebra of rank 2

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Abstract

Let $\mathfrak{g}$ be a hyperbolic Kac-Moody algebra of rank 2, and set $\lambda := \Lambda_1 - \Lambda_2$, where Λ_1, Λ_2 are the fundamental weights. Denote by $V(\lambda)$ the extremal weight module of extremal weight λ with v_λ the extremal weight vector of weight λ, and by $\mathcal{B}(\lambda)$ the crystal basis of $V(\lambda)$ with u_λ the element corresponding to v_λ. We prove that (i) (the crystal graph of) $\mathcal{B}(\lambda)$ is connected, (ii) the subset $\mathcal{B}(\lambda)_\mu$ of elements of weight μ in $\mathcal{B}(\lambda)$ is a finite set for every integral weight μ, and $\mathcal{B}(\lambda)_\lambda = \{u_\lambda\}$, (iii) every extremal element in $\mathcal{B}(\lambda)$ is contained in the Weyl group orbit of u_λ. Also, we prove that the crystal $\mathbb{B}(\lambda)$ of all Lakshmibai-Seshadri paths of shape λ is connected, and give an explicit description of Lakshmibai-Seshadri paths of shape λ.

1 Introduction.

In this paper, we study the structure of the crystal basis of the extremal weight module $V(\lambda)$ of extremal weight $\lambda := \Lambda_1 - \Lambda_2$ over the quantized universal enveloping algebra associated to a hyperbolic Kac-Moody algebra of rank 2, where Λ_1, Λ_2 are the fundamental weights. Also, we study the structure of the crystal $\mathbb{B}(\lambda)$ of all Lakshmibai-Seshadri (LS for short) paths of shape λ, and give an explicit description of LS paths of shape λ.

Let us explain the background and motivation. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over $\mathbb{C}$ with P the integral weight lattice, $U_q(\mathfrak{g})$ the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated to $\mathfrak{g}$, with E_i and $F_i, i \in I$, the Chevalley generators corresponding to α_i and $-\alpha_i$, respectively. And let $\{\alpha_i\}_{i\in I}$ the set of the simple roots of $\mathfrak{g}$. The extremal weight module $V(\mu)$ of extremal weight $\mu \in P$ is the integrable $U_q(\mathfrak{g})$-module generated by a single element v_μ with the defining relation that v_μ is an extremal weight vector of weight μ; this module was introduced by Kashiwara [K2, Proposition 8.2.2] as a natural generalization of integrable highest (or lowest) weight modules. He also proved that $V(\mu)$ has a crystal basis $\mathcal{B}(\mu)$. We know from [K2, Proposition 8.2.2 (iv) and (v)] that $V(\mu) \cong V(w\mu)$ as $U_q(\mathfrak{g})$-modules, and $\mathcal{B}(\mu) \cong \mathcal{B}(w\mu)$ as crystals for all $\mu \in P$ and $w \in W$, where W is the Weyl group of $\mathfrak{g}$. Also, we know from the comment at the end of [K2, §8.2] that if $\mu \in P$ is dominant (resp., antidominant), then $V(\mu)$ is isomorphic to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight μ, and $\mathcal{B}(\mu)$ is isomorphic to its crystal basis. So, we are interested in those $\mu \in P$ such that

$$\text{any element of } W\mu \text{ is neither dominant nor antidominant.} \tag{1.1}$$

If $\mathfrak{g}$ is of finite type, then there is no $\mu \in P$ satisfying the condition (1.1); it is well-known that $W\mu$ contains a (unique) dominant integral weight for every $\mu \in P$. Assume that $\mathfrak{g}$ is of affine type, and let c be the canonical central element of $\mathfrak{g}$. Then, $\mu \in P$ satisfies the condition (1.1) if and only if μ is level-zero, that is, ($\mu \neq 0$, and) $\langle \mu, c\rangle = 0$. In [K5] and [BN], they deeply studied the basic structure of $V(\mu)$ and $\mathcal{B}(\mu)$ for level-zero $\mu \in P$. Using results in [K5] and [BN], Naito and Sagaki proved in [NS1] and [NS2] that if μ is a positive integer multiple of a level-zero fundamental weight, then the crystal basis $\mathcal{B}(\mu)$ is isomorphic to the crystal $\mathbb{B}(\mu)$ of LS paths of shape μ. After that, in [INS], they introduced semi-infinite LS paths in terms of the semi-infinite Bruhat order on the affine Weyl group, and proved that for level-zero dominant $\mu \in P$, the crystal basis $\mathcal{B}(\mu)$ is isomorphic to the crystal of semi-infinite LS paths of shape μ. Thus, in the finite and affine cases, we have already finished the basic study of the structures of $V(\mu)$ and $\mathcal{B}(\mu)$, and had combinatorial realization for $\mathcal{B}(\mu)$.

In this paper, we consider the case where $\mathfrak{g} = \mathfrak{g}(A)$ is a hyperbolic Kac-Moody algebra of rank 2 with Cartan matrix

$$A = \begin{pmatrix} 2 & -a_1 \\ -a_2 & 2 \end{pmatrix}, \text{ where } a_1, a_2 \in \mathbb{Z}_{>0},\ a_1a_2 > 4.$$

In Proposition 3.2, it will be proved that $\lambda = \Lambda_1 - \Lambda_2$ satisfies the condition (1.1) if $a_1, a_2 \geq 2$. We will prove the following theorems and corollary.

Theorem 1.1 (= Theorem 3.5)**.** The crystal graph of $\mathcal{B}(\lambda)$ is connected.

Corollary 1.2 (= Corollary 3.6)**.** For every $\mu \in P$, the subset $\mathcal{B}(\lambda)_\mu$ of elements of weight μ in $\mathcal{B}(\lambda)$ is a finite set. In particular, $\mathcal{B}(\lambda)_\lambda = \{u_\lambda\}$, where u_λ is the element of $\mathcal{B}(\lambda)$ corresponding to the extremal weight vector $v_\lambda \in V(\lambda)$.

Since $\mathcal{B}(\lambda)$ is a normal crystal, $\mathcal{B}(\lambda)$ has a canonical action S_w $(w \in W)$ of the Weyl group W (see §2.2). Then, $u_\lambda \in \mathcal{B}(\lambda)$ is an extremal element of weight λ.

Theorem 1.3 (= Theorem 3.7)**.** (1) Let $x, y \in W$. Then, $S_x u_\lambda = S_y u_\lambda$ if and only if $x\lambda = y\lambda$.
(2) If $b \in \mathcal{B}(\lambda)$ is extremal, then there exists $w \in W$ such that $b = S_w u_\lambda$.

Theorem 1.4 (= Theorem 3.8)**.** The crystal graph of $\mathbb{B}(\lambda)$ is connected.

Theorem 1.5 (= Theorem 3.9)**.** Assume that $a_1, a_2 \geq 2$. An LS path of shape $\lambda = \Lambda_1 - \Lambda_2$ is either of the form (i) or (ii):
(i) $(x_{m+s-1}\lambda, \dots, x_{m+1}\lambda, x_m\lambda; \sigma_0, \sigma_1, \dots, \sigma_s)$, where $m \geq 0, s \geq 1$, and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ satisfy the condition that $p_{m+s-u}\sigma_u \in \mathbb{Z}$ for $1 \leq u \leq s-1$.
(ii) $(y_{m-s+1}\lambda, \dots, y_{m-1}\lambda, y_m\lambda; \delta_0, \delta_1, \dots, \delta_s)$, where $m \geq s-1, s \geq 1$, and $0 = \delta_0 < \delta_1 < \cdots < \delta_s = 1$ satisfy the condition that $q_{m-s+u+1}\delta_u \in \mathbb{Z}$ for $1 \leq u \leq s-1$.
Here, the elements $x_m, y_m \in W, m \geq 0$, are defined in (3.2), (3.3), and the sequences $\{p_m\}_{m\geq 0}$ and $\{q_m\}_{m\geq 0}$ are defined in (3.6), (3.7).

As an application of the theorems and corollary above, we prove in [SY, Theorem 3.6] that the crystal basis $\mathcal{B}(\lambda)$ of the extremal weight module $V(\lambda)$ of extremal weight λ is isomorphic to the crystal $\mathbb{B}(\lambda)$ of all LS paths of shape λ.

This paper is organized as follows. In Section 2, we fix our notation, and recall the definitions and basic properties of extremal weight modules and their crystal bases. Also, we recall the definition of LS paths. In Subsections 3.1 and 3.2, we introduce some extra notation for the case that $\mathfrak{g}$ is a hyperbolic Kac-Moody algebra of rank 2, and show that $\lambda = \Lambda_1 - \Lambda_2$ satisfies the condition (1.1) if $a_1, a_2 \geq 2$. In Subsection 3.3, we state our main results (the four theorems and one corollary above). Subsections 3.4, 3.5, 3.6, and 3.7 are devoted to proofs of Theorems 3.5, 3.7, 3.8, and 3.9, respectively. In Appendix, we give an explicit description of the root operators.

2 Preliminaries.

2.1 Kac-Moody algebras.

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix with I the finite index set. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody algebra associated to A over $\mathbb{C}$. Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ the set of simple roots, and $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ the set of simple coroots. We set $Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. Denote by $W = \langle r_i \mid i \in I \rangle$ the Weyl group of $\mathfrak{g}$, where r_i is the simple reflection in α_i for $i \in I$. Let $\Lambda_i \in \mathfrak{h}^*, i \in I$, be the fundamental weights for $\mathfrak{g}$, i.e., $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for $i, j \in I$, and set $P := \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$. Let $P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i$ the set of dominant integral weights, and $-P^+ := \sum_{i \in I} \mathbb{Z}_{\leq 0} \Lambda_i$ the set of antidominant integral weights.

2.2 Crystal bases and crystals.

For the details on crystal bases and crystals, see [K3] and [HK]. Let $\mathcal{B}$ be a crystal, and let $\tilde{e}_i$ and $\tilde{f}_i$, $i \in I$, be the Kashiwara operators for $\mathcal{B}$. For $b \in \mathcal{B}$ and $i \in I$, we set $\tilde{e}_i^{\max} b := \tilde{e}_i^{\varepsilon_i(b)} b$ if $\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n b \neq \mathbf{0}\}$, where $\mathbf{0}$ is an extra element not contained in any crystal. Similarly, we set $\tilde{f}_i^{\max} b := \tilde{f}_i^{\varphi_i(b)} b$ if $\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n b \neq \mathbf{0}\}$.

Definition 2.1 (see [K2, page 389] and [K3, page 182])**.** A crystal $\mathcal{B}$ is said to be normal if it satisfies the following condition (N) for every $J \subset I$ such that the Levi subalgebra $\mathfrak{g}_J$ of $\mathfrak{g}$ corresponding to J is finite-dimensional:

(N) if we regard $\mathcal{B}$ as a crystal for $U_q(\mathfrak{g}_J)$ by restriction, then it is isomorphic to the crystal basis of a finite-dimensional $U_q(\mathfrak{g}_J)$-module.

Remark 2.2. If $\mathcal{B}$ is a normal crystal, then $\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n b \neq \mathbf{0}\}$ and $\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n b \neq \mathbf{0}\}$ for all $b \in \mathcal{B}$ and $i \in I$.

We know from [K2, §7] (see also [K3, Theorem 11.1]) that a normal crystal $\mathcal{B}$ has an action of the Weyl group W as follows. For $i \in I$ and $b \in \mathcal{B}$, we set

$$S_i b := \begin{cases} \tilde{f}_i^{\langle \mathrm{wt}(b), \alpha_i^\vee \rangle} b & \text{if } \langle \mathrm{wt}(b), \alpha_i^\vee \rangle \geq 0, \\ \tilde{e}_i^{-\langle \mathrm{wt}(b), \alpha_i^\vee \rangle} b & \text{if } \langle \mathrm{wt}(b), \alpha_i^\vee \rangle \leq 0. \end{cases}$$

Then, for $w \in W$, we set $S_w := S_{i_1} \cdots S_{i_k}$ if $w = r_{i_1} \cdots r_{i_k}$. Notice that $\mathrm{wt}(S_w b) = w\mathrm{wt}(b)$ for $w \in W$ and $b \in \mathcal{B}$.

An element b of a normal crystal $\mathcal{B}$ is said to be extremal if for each $w \in W$ and $i \in I$,

$$\begin{cases} \tilde{e}_i(S_w b) = \mathbf{0} & \text{if } \langle \mathrm{wt}(S_w b), \alpha_i^\vee \rangle \geq 0, \\ \tilde{f}_i(S_w b) = \mathbf{0} & \text{if } \langle \mathrm{wt}(S_w b), \alpha_i^\vee \rangle \leq 0. \end{cases}$$

Now, let $\mathcal{B}(\infty)$ (resp., $\mathcal{B}(-\infty)$) be the crystal basis of the negative part $U_q^-(\mathfrak{g})$ (resp., the positive part $U_q^+(\mathfrak{g})$) of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{C}(q)$ associated to $\mathfrak{g}$; for a realization of $\mathcal{B}(\infty)$, see Appendix B. Let $u_\infty \in \mathcal{B}(\infty)$ (resp., $u_{-\infty} \in \mathcal{B}(-\infty)$) be the element corresponding to $1 \in U_q^-(\mathfrak{g})$ (resp., $1 \in U_q^+(\mathfrak{g})$). Denote by $\tilde{e}_i$ and $\tilde{f}_i$, $i \in I$, the raising and lowering Kashiwara operators on $\mathcal{B}(\pm\infty)$, respectively. For $i \in I$, we define $\varepsilon_i, \varphi_i : \mathcal{B}(\infty) \to \mathbb{Z}$ and $\varepsilon_i, \varphi_i : \mathcal{B}(-\infty) \to \mathbb{Z}$ by

$$\varepsilon_i(b) := \max\{n \geq 0 \mid \tilde{e}_i^n b \neq \mathbf{0}\},\ \varphi_i(b) := \varepsilon_i(b) + \langle \mathrm{wt}(b), \alpha_i^\vee \rangle \text{ for } b \in \mathcal{B}(\infty),$$
$$\varphi_i(b) := \max\{n \geq 0 \mid \tilde{f}_i^n b \neq \mathbf{0}\},\ \varepsilon_i(b) := \varphi_i(b) - \langle \mathrm{wt}(b), \alpha_i^\vee \rangle \text{ for } b \in \mathcal{B}(-\infty),$$

respectively. Denote by $* : \mathcal{B}(\pm\infty) \to \mathcal{B}(\pm\infty)$ the $*$-operator on $\mathcal{B}(\pm\infty)$, which is induced from a $\mathbb{C}(q)$-algebra antiautomorphism $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ (see [K1, Theorem 2.1.1] and [K3, §8.3]). We see that $\mathrm{wt}(b^*) = \mathrm{wt}(b)$ for all $b \in \mathcal{B}(\pm\infty)$. The next lemma follows immediately from the fact that $\tilde{f}_i^k u_\infty$ (resp., $\tilde{e}_i^k u_{-\infty}$) is a unique element of weight $-k\alpha_i$ (resp., $k\alpha_i$) in $\mathcal{B}(\infty)$ (resp., $\mathcal{B}(-\infty)$).

Lemma 2.3. We have $(\tilde{f}_i^k u_\infty)^* = \tilde{f}_i^k u_\infty$ for all $k \in \mathbb{Z}_{\geq 0}$ and $i \in I$. Similarly, we have $(\tilde{e}_i^k u_{-\infty})^* = \tilde{e}_i^k u_{-\infty}$ for all $k \in \mathbb{Z}_{\geq 0}$ and $i \in I$.

For $\mu \in P$, denote by $\mathcal{T}_\mu = \{t_\mu\}$ the crystal consisting of a single element t_μ such that

$$\begin{cases} \mathrm{wt}(t_\mu) = \mu, & \\ \tilde{e}_i t_\mu = \tilde{f}_i t_\mu = \mathbf{0} & \text{for } \ i \in I, \\ \varepsilon_i(t_\mu) = \varphi_i(t_\mu) = -\infty & \text{for } \ i \in I. \end{cases}$$

2.3 Crystal bases of extremal weight modules.

Let $\mu \in P$ be an arbitrary integral weight. The extremal weight module $V(\mu)$ of extremal weight μ is, by definition, the integrable $U_q(\mathfrak{g})$-module generated by a single element v_μ with the defining relation that v_μ is an "extremal weight vector" of weight μ; recall from [K5, §3.1] that v_μ is an extremal weight vector of weight μ if (v_μ is a weight vector of weight μ and) there exists a family $\{v_w\}_{w \in W}$ of weight vectors in $V(\mu)$ such that $v_{\mathrm{id}} = v_\mu$, and such that for every $i \in I$ and $w \in W$ with $n := \langle w\mu, \alpha_i^\vee \rangle \geq 0$ (resp., ≤ 0), the equalities $E_i v_w = 0$ and $F_i^{(n)} v_w = v_{r_i w}$ (resp., $F_i v_w = 0$ and $E_i^{(-n)} v_w = v_{r_i w}$) hold, where for $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$, $E_i^{(m)}$ and $F_i^{(m)}$ are the m-th divided powers of E_i and F_i, respectively. Note that the weight of v_w is $w\mu$. We know from [K2, Proposition 8.2.2] that $V(\mu)$ has a crystal basis $\mathcal{B}(\mu)$.

Remark 2.4. We see from [K2, Proposition 8.2.2 (iv) and (v)] that $V(\mu) \cong V(w\mu)$ as $U_q(\mathfrak{g})$-modules, and $\mathcal{B}(\mu) \cong \mathcal{B}(w\mu)$ as crystals for all $\mu \in P$ and $w \in W$. Also, we know from the comment at the end of [K2, §8.2] that if $\mu \in P$ is dominant (resp., antidominant), then $V(\mu)$ is isomorphic, as a $U_q(\mathfrak{g})$-module, to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight μ, and $\mathcal{B}(\mu)$ is isomorphic, as a crystal, to its crystal basis. So, we are interested in those $\mu \in P$ such that

$$\text{any element of } \ W\mu \ \text{ is neither dominant nor antidominant.} \tag{2.1}$$

Now, the crystal basis $\mathcal{B}(\mu)$ can be realized (as a crystal) as follows. We set

$$\mathcal{B} := \bigsqcup_{\mu \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty);$$

in fact, $\mathcal{B}$ is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\tilde{U}_q(\mathfrak{g}))$ of the modified quantized universal enveloping algebra $\tilde{U}_q(\mathfrak{g})$ associated to $\mathfrak{g}$ (see [K2, Theorem 3.1.1]). Denote by $* : \mathcal{B} \to \mathcal{B}$

the $*$-operation on $\mathcal{B}$, which is induced from a $\mathbb{C}(q)$-algebra antiautomorphism $* : \tilde{U}_q(\mathfrak{g}) \to \tilde{U}_q(\mathfrak{g})$ (see [K2, Theorem 4.3.2]); we know from [K2, Corollary 4.3.3] that for $b_1 \in \mathcal{B}(\infty), b_2 \in \mathcal{B}(-\infty), \mu \in P$,

$$(b_1 \otimes t_\mu \otimes b_2)^* = b_1^* \otimes t_{-\mu-\mathrm{wt}(b_1)-\mathrm{wt}(b_2)} \otimes b_2^*. \tag{2.2}$$

Because $\mathcal{B}$ is a normal crystal by [K2, §2.1 and Theorem 3.1.1], $\mathcal{B}$ has the action of the Weyl group W as mentioned in §2.2. We know the following proposition from [K2, Proposition 8.2.2 (and Theorem 3.1.1)].

Proposition 2.5. For $\mu \in P$, the subset

$$\{b \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \mid b^* \text{ is extremal}\} \tag{2.3}$$

is a subcrystal of $\mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$, and is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\mu)$ of the extremal weight module $V(\mu)$ of extremal weight μ.

In the following, we identify the crystal basis $\mathcal{B}(\mu)$ with the subcrystal (2.3) of $\mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$. We set

$$u_\mu := u_\infty \otimes t_\mu \otimes u_{-\infty} \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \tag{2.4}$$

Then, u_μ is an extremal element of weight μ contained in $\mathcal{B}(\mu)$, which corresponds to the extremal weight vector $v_\mu \in V(\mu)$.

2.4 Lakshmibai-Seshadri paths.

Let us recall the definition of Lakshmibai-Seshadri paths (LS paths for short) from [L2, §4] (see also [Y, §2.2]). Let $\mu \in P$ be an arbitrary integral weight.

Definition 2.6. For $\nu, \nu' \in W\mu$, we write $\nu \geq \nu'$ if there exist a sequence $\nu = \xi_0, \xi_1, \dots, \xi_p = \nu'$ of elements in $W\mu$ and a sequence $\beta_1, \dots, \beta_p$ of positive real roots such that $\xi_q = r_{\beta_q}\xi_{q-1}$ and $\langle \xi_{q-1}, \beta_q^\vee \rangle < 0$ for each $q = 1, 2, \dots, p$, where for a positive real root β, $r_\beta \in W$ denotes the reflection in β, and $\beta^\vee$ denotes the dual root of β. If $\nu \geq \nu'$, then we define $\mathrm{dist}(\nu, \nu')$ to be the maximal length p of all possible such sequences $\nu = \xi_0, \xi_1, \dots, \xi_p = \nu'$ for (ν, ν').

Definition 2.7. Let $\nu, \nu' \in W\mu$ with $\nu > \nu'$, and let $0 < \sigma < 1$ be a rational number. A σ-chain for (ν, ν') is, by definition, a sequence $\nu = \xi_0 > \xi_1 > \cdots > \xi_p = \nu'$ of elements in $W\mu$ such that $\mathrm{dist}(\xi_{q-1}, \xi_q) = 1$ and $\sigma\langle \xi_{q-1}, \beta_q^\vee \rangle \in \mathbb{Z}_{<0}$ for all $q = 1, 2, \dots, p$, where β_q is the positive real root corresponding to $\xi_{q-1} > \xi_q$ with $\mathrm{dist}(\xi_{q-1}, \xi_q) = 1$.

Definition 2.8. An LS path of shape μ is a pair $\pi = (\underline{\nu}\,; \underline{\sigma})$ of a sequence $\underline{\nu} : \nu_1 > \nu_2 > \cdots > \nu_s$ of elements in $W\mu$ and a sequence $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers satisfying the condition that there exists a σ_u-chain for (ν_u, ν_{u+1}) for all $u = 1, 2, \dots, s-1$.

Denote by $\mathbb{B}(\mu)$ the set of LS paths of shape μ. We identify $\pi = (\underline{\nu}\,; \underline{\sigma}) \in \mathbb{B}(\mu)$ (as in Definition 2.8) with the following piecewise-linear continuous map $\pi : [0, 1] \to \mathbb{R} \otimes_{\mathbb{Z}} P$:

$$\pi(t) = \sum_{k=1}^{u-1} (\sigma_k - \sigma_{k-1})\nu_k + (t - \sigma_{u-1})\nu_u \quad \text{for} \quad \sigma_{u-1} \leq t \leq \sigma_u,\ 1 \leq u \leq s.$$

Remark 2.9. We see from the definition of LS paths that $\pi_\nu := (\nu; 0, 1) \in \mathbb{B}(\mu)$ for every $\nu \in W\mu$, which corresponds to the straight line $\pi_\nu(t) = t\nu$ for $t \in [0, 1]$.

Now, we endow $\mathbb{B}(\mu)$ with a crystal structure as follows. First, we define $\mathrm{wt}(\pi) := \pi(1)$ for $\pi \in \mathbb{B}(\mu)$; we know from [L2, Lemma 4.5] that $\pi(1) \in P$. Next, for $\pi \in \mathbb{B}(\mu)$ and $i \in I$, we define

$$H_i^\pi(t) := \langle \pi(t), \alpha_i^\vee \rangle \text{ for } t \in [0,1], \qquad m_i^\pi := \min\{H_i^\pi(t) \mid t \in [0,1]\}. \tag{2.5}$$

We know from [L2, Lemma 4.5] that

$$\text{all local minimal values of } H_i^\pi(t) \text{ are integers;} \tag{2.6}$$

in particular, m_i^π is a nonpositive integer, and $H_i^\pi(1) - m_i^\pi$ is a nonnegative integer. We define $\tilde{e}_i\pi$ as follows: If $m_i^\pi = 0$, then we set $\tilde{e}_i\pi := \mathbf{0}$. If $m_i^\pi \leq -1$, then we set

$$\begin{aligned} t_1 &:= \min\{t \in [0,1] \mid H_i^\pi(t) = m_i^\pi\}, \\ t_0 &:= \max\{t \in [0,t_1] \mid H_i^\pi(t) = m_i^\pi + 1\}; \end{aligned} \tag{2.7}$$

we see by (2.6) that

$$H_i^\pi(t) \text{ is strictly decreasing on } [t_0, t_1]. \tag{2.8}$$

We define

$$(\tilde{e}_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0, \\ r_i(\pi(t) - \pi(t_0)) + \pi(t_0) & \text{if } t_0 \leq t \leq t_1, \\ \pi(t) + \alpha_i & \text{if } t_1 \leq t \leq 1; \end{cases}$$

we know from [L2, §4] that $\tilde{e}_i\pi \in \mathbb{B}(\lambda)$. Similarly, we define $\tilde{f}_i\pi$ as follows: If $H_i^\pi(1) - m_i^\pi = 0$, then we set $\tilde{f}_i\pi := \mathbf{0}$. If $H_i^\pi(1) - m_i^\pi \geq 1$, then we set

$$\begin{aligned} t_0 &:= \max\{t \in [0,1] \mid H_i^\pi(t) = m_i^\pi\}, \\ t_1 &:= \min\{t \in [t_0,1] \mid H_i^\pi(t) = m_i^\pi + 1\}; \end{aligned} \tag{2.9}$$

we see by (2.6) that

$$H_i^\pi(t) \text{ is strictly increasing on } [t_0, t_1]. \tag{2.10}$$

We define

$$(\tilde{f}_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0, \\ r_i(\pi(t) - \pi(t_0)) + \pi(t_0) & \text{if } t_0 \leq t \leq t_1, \\ \pi(t) - \alpha_i & \text{if } t_1 \leq t \leq 1; \end{cases}$$

we know from [L2, §4] that $\tilde{f}_i\pi \in \mathbb{B}(\mu)$. We set $\tilde{e}_i\mathbf{0} = \tilde{f}_i\mathbf{0} := \mathbf{0}$ for $i \in I$. Finally, for $\pi \in \mathbb{B}(\mu)$ and $i \in I$, we set

$$\varepsilon_i(\pi) := \max\{n \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^n\pi \neq \mathbf{0}\}, \qquad \varphi_i(\pi) := \max\{n \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^n\pi \neq \mathbf{0}\}.$$

Theorem 2.10 ([L2, §2 and §4])**.** The set $\mathbb{B}(\mu)$, together with the maps $\mathrm{wt} : \mathbb{B}(\mu) \to P$, $\tilde{e}_i, \tilde{f}_i : \mathbb{B}(\mu) \cup \{\mathbf{0}\} \to \mathbb{B}(\mu) \cup \{\mathbf{0}\}, i \in I$, and $\varepsilon_i, \varphi_i : \mathbb{B}(\mu) \to \mathbb{Z}_{\geq 0}, i \in I$, becomes a crystal.

Remark 2.11. We see from the definition of LS paths that $\mathbb{B}(w\mu) = \mathbb{B}(\mu)$ for all $w \in W$ and $\mu \in P$. Also, we know from [K4] and [J] that if $\mu \in P$ is dominant (resp., antidominant), then $\mathbb{B}(\mu)$ is isomorphic, as a crystal, to the crystal basis of the integrable highest (resp., lowest) weight module of highest (resp., lowest) μ. Hence, also for $\mathbb{B}(\mu)$, we are interested in those $\mu \in P$ satisfying the condition (2.1).

For $\pi = (\nu_1,\ \nu_2,\ \ldots,\ \nu_s;\ \sigma_0,\ \sigma_1,\ \ldots,\ \sigma_s) \in \mathbb{B}(\lambda)$, we set $\iota(\pi) := \nu_1$ and $\kappa(\pi) := \nu_s$.

Lemma 2.12 ([L1, Proposition 4.2], [L2, Proposition 4.7])**.** Let $\pi \in \mathbb{B}(\lambda)$, and $i \in I$. If $\langle \kappa(\pi),\ \alpha_i^{\vee} \rangle > 0$, then $\kappa(\tilde{f}_i^{\max}\pi) = r_i\kappa(\pi)$. If $\langle \iota(\pi),\ \alpha_i^{\vee} \rangle < 0$, then $\iota(\tilde{e}_i^{\max}\pi) = r_i\iota(\pi)$.

Since $\mathbb{B}(\mu)$ is a normal crystal, $\mathbb{B}(\mu)$ has the action of the Weyl group W as mentioned in §2.2 (see also [L2, Theorem 8.1]). We can easily show the next lemma by induction on the length of $w \in W$.

Lemma 2.13. For $w \in W$, we have $S_w\pi_\mu = \pi_{w\mu}$. In particular, π_μ is an extremal element of weight μ.

3 Main results.

3.1 Hyperbolic Kac-Moody algebra of rank 2.

Form this section, we assume that

$$A = \begin{pmatrix} 2 & -a_1 \\ -a_2 & 2 \end{pmatrix}, \text{ where } a_1, a_2 \in \mathbb{Z}_{>0} \text{ and } a_1a_2 > 4, \tag{3.1}$$

with $I = \{1, 2\}$. Note that $W = \{x_m, y_m \mid m \in \mathbb{Z}_{\geq 0}\}$, where

$$x_m := \begin{cases} (r_2r_1)^k & \text{if } m = 2k \text{ with } k \in \mathbb{Z}_{\geq 0}, \\ r_1(r_2r_1)^k & \text{if } m = 2k+1 \text{ with } k \in \mathbb{Z}_{\geq 0}. \end{cases} \tag{3.2}$$

$$y_m := \begin{cases} (r_1r_2)^k & \text{if } m = 2k \text{ with } k \in \mathbb{Z}_{\geq 0}, \\ r_2(r_1r_2)^k & \text{if } m = 2k+1 \text{ with } k \in \mathbb{Z}_{\geq 0}. \end{cases} \tag{3.3}$$

Let Δ_{re}^+ denote the set of positive real roots. We see that

$$\Delta_{\mathrm{re}}^+ = \left\{x_l\alpha_2,\ y_{l+1}\alpha_1 \mid l \in \mathbb{Z}_{\mathrm{even}\geq 0}\right\} \sqcup \left\{y_l\alpha_1, x_{l+1}\alpha_2 \mid l \in \mathbb{Z}_{\mathrm{even}\geq 0}\right\}, \tag{3.4}$$

where $\mathbb{Z}_{\mathrm{even}\geq 0}$ denotes the set of even nonnegative integers.

Remark 3.1. In fact, we know from [Kac, Exercise 5.25] that

$$\Delta_{\mathrm{re}}^+ = \{c_j\alpha_1 + d_{j+1}\alpha_2 \text{ and } c_{j+1}\alpha_1 + d_j\alpha_2 \mid j \geq 0\},$$

where the sequences $\{c_j\}_{j\geq 0}$ and $\{d_j\}_{j\geq 0}$ are defined by

$$c_0 = d_0 = 0,\ d_1 = c_1 = 1,\ \text{and } \begin{cases} c_{j+2} = a_1d_{j+1} - c_j, \\ d_{j+2} = a_2c_{j+1} - d_j. \end{cases}$$

Recall that $P = \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\Lambda_2$, $P^+ = \mathbb{Z}_{\geq 0}\Lambda_1 + \mathbb{Z}_{\geq 0}\Lambda_2 \subset P$, and $-P^+ = \mathbb{Z}_{\leq 0}\Lambda_1 + \mathbb{Z}_{\leq 0}\Lambda_2 \subset P$.

3.2 An integral weight whose Weyl group orbit does not intersect with neither P^+ nor $-P^+$.

We set

$$\lambda := \Lambda_1 - \Lambda_2. \tag{3.5}$$

Proposition 3.2. Assume that $a_1, a_2 \geq 2$. The weight $\lambda = \Lambda_1 - \Lambda_2$ satisfies that $W\lambda \cap (P^+ \cup (-P^+)) = \emptyset$ (see (2.1)).

Remark 3.3. If $a_1 = 1$, then we have $y_1(\Lambda_1 - \Lambda_2) = r_2(\Lambda_1 - \Lambda_2) = \Lambda_2 \in P^+$. If $a_2 = 1$, then we have $x_1(\Lambda_1 - \Lambda_2) = r_1(\Lambda_1 - \Lambda_2) = -\Lambda_1 \in -P^+$.

Assume that $a_1, a_2 \geq 2$. We define $\{p_m\}_{m \in \mathbb{Z}_{\geq 0}}$ and $\{q_m\}_{m \in \mathbb{Z}_{\geq 0}}$ by:

$$p_0 = p_1 = 1 \text{ and } p_{m+2} = \begin{cases} a_2 p_{m+1} - p_m & \text{(if } m \text{ is even),} \\ a_1 p_{m+1} - p_m & \text{(if } m \text{ is odd),} \end{cases} \tag{3.6}$$

$$q_0 = q_1 = 1 \text{ and } q_{m+2} = \begin{cases} a_1 q_{m+1} - q_m & \text{(if } m \text{ is even),} \\ a_2 q_{m+1} - q_m & \text{(if } m \text{ is odd).} \end{cases} \tag{3.7}$$

Then we see that $1 = p_0 = p_1 \leq p_2 < p_3 < \cdots$ and $1 = q_0 = q_1 \leq q_2 < q_3 < \cdots$; note that $p_1 = p_2$ if and only if $a_2 = 2$, and $q_1 = q_2$ if and only if $a_1 = 2$. Proposition 3.2 follows immediately from the following lemmas and the fact that $W = \{x_m, y_m \mid m \in \mathbb{Z}_{\geq 0}\}$ (see (3.2) and (3.3)).

Lemma 3.4. Assume that $a_1, a_2 \geq 2$. For $m \in \mathbb{Z}_{\geq 0}$,

$$x_m \lambda = \begin{cases} p_{m+1}\Lambda_1 - p_m \Lambda_2 & \text{if } m \text{ is even,} \\ -p_m \Lambda_1 + p_{m+1}\Lambda_2 & \text{if } m \text{ is odd,} \end{cases} \tag{3.8}$$

$$y_m \lambda = \begin{cases} q_m \Lambda_1 - q_{m+1}\Lambda_2 & \text{if } m \text{ is even,} \\ -q_{m+1}\Lambda_1 + q_m \Lambda_2 & \text{if } m \text{ is odd.} \end{cases} \tag{3.9}$$

Proof. We give a proof only for (3.8); the proof for (3.9) is similar. We show (3.8) by induction on m. If $m = 0$ or $m = 1$, then (3.8) is obvious. Assume that $m > 1$. If m is even, then

$$x_{m+1}\lambda = r_1(x_m\lambda) = r_1(p_{m+1}\Lambda_1 - p_m\Lambda_2) = -p_{m+1}\Lambda_1 + (a_2 p_{m+1} - p_m)\Lambda_2.$$

Since m is even, we have $a_2 p_{m+1} - p_m = p_{m+2}$ by the definition (3.6). Therefore, we obtain $x_{m+1}\lambda = -p_{m+1}\Lambda_1 + p_{m+2}\Lambda_2$, as desired. If m is odd, then

$$x_{m+1}\lambda = r_2(x_m\lambda) = r_2(-p_m\Lambda_1 + p_{m+1}\Lambda_2) = (a_1 p_{m+1} - p_m)\Lambda_1 - p_{m+1}\Lambda_2.$$

Since m is odd, we have $a_1 p_{m+1} - p_m = p_{m+2}$ by the definition (3.7). Therefore, we obtain $x_{m+1}\lambda = p_{m+2}\Lambda_1 - p_{m+1}\Lambda_2$, as desired. $\square$

3.3 Main Theorems and corollary.

Theorem 3.5 (will be proved in §3.4)**.** For each $b \in \mathcal{B}(\lambda)$, there exist $i_1, \dots, i_k \in I$ such that $b = \tilde{f}_{i_k} \cdots \tilde{f}_{i_1} u_\lambda$ or $b = \tilde{e}_{i_k} \cdots \tilde{e}_{i_1} u_\lambda$. In particular, the crystal graph of $\mathcal{B}(\lambda)$ is connected.

The next corollary follows immediately from Theorem 3.5.

Corollary 3.6. For every $\mu \in P$, the subset $\mathcal{B}(\lambda)_\mu$ of elements of weight μ in $\mathcal{B}(\lambda)$ is a finite set. In particular, $\mathcal{B}(\lambda)_\lambda = \{u_\lambda\}$.

Recall from the comment after (2.4) that u_λ is an extremal element of weight λ.

Theorem 3.7 (will be proved in §3.5)**.** (1) Let $x, y \in W$. Then, $S_x u_\lambda = S_y u_\lambda$ if and only if $x\lambda = y\lambda$.

(2) If $b \in \mathcal{B}(\lambda)$ is extremal, then there exists $w \in W$ such that $b = S_w u_\lambda$.

Theorem 3.8 (will be proved in §3.6)**.** The crystal graph of $\mathbb{B}(\lambda)$ is connected.

Theorem 3.9 (will be proved in §3.7)**.** Assume that $a_1, a_2 \geq 2$. An LS path π of shape λ is either of the form (i) or (ii):
(i) $(x_{m+s-1}\lambda, \dots, x_{m+1}\lambda, x_m\lambda; \sigma_0, \sigma_1, \dots, \sigma_s)$, where $m \geq 0, s \geq 1$, and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ satisfy the condition that $p_{m+s-u}\sigma_u \in \mathbb{Z}$ for $1 \leq u \leq s-1$.
(ii) $(y_{m-s+1}\lambda, \dots, y_{m-1}\lambda, y_m\lambda; \delta_0, \delta_1, \dots, \delta_s)$, where $m \geq s-1, s \geq 1$, and $0 = \delta_0 < \delta_1 < \cdots < \delta_s = 1$ satisfy the condition that $q_{m-s+u+1}\delta_u \in \mathbb{Z}$ for $1 \leq u \leq s-1$.

3.4 Proof of Theorem 3.5.

Lemma 3.10. Let $i \in I$ and $b \in \mathcal{B}(\lambda)$ be such that $\tilde{e}_i b \neq \mathbf{0}$. If b is of the form : $b = b_1 \otimes t_\lambda \otimes u_{-\infty}$ with $b_1 \neq u_\infty$, then $\tilde{e}_i b = \tilde{e}_i b_1 \otimes t_\lambda \otimes u_{-\infty}$.

Proof. Suppose, for a contradiction, that $\tilde{e}_i b = b_1 \otimes t_\lambda \otimes \tilde{e}_i u_{-\infty}$. By (2.2) and Lemma 2.3, we see that

$$(\tilde{e}_i b)^* = b_1^* \otimes t_{-\lambda-\mathrm{wt}(b_1)-\alpha_i} \otimes \tilde{e}_i u_{-\infty}. \tag{3.10}$$

Notice that $\tilde{f}_i(\tilde{e}_i b)^* \neq \mathbf{0}$ by the tensor product rule of crystals. Since $\tilde{e}_i b \in \mathcal{B}(\lambda)$, it follows that $(\tilde{e}_i b)^*$ is an extremal element of weight $-\lambda$. Hence we see that $\tilde{f}_1(\tilde{e}_1 b)^* = \mathbf{0}$, which implies that $i = 2$, and $(\tilde{e}_2 b)^* = b_1^* \otimes t_{-\lambda-\mathrm{wt}(b_1)-\alpha_2} \otimes \tilde{e}_2 u_{-\infty}$. Since $(\tilde{e}_2 b)^*$ is an extremal element of weight $-\lambda$, we see that $\tilde{e}_1^2(\tilde{e}_2 b)^* = \tilde{e}_1 S_1(\tilde{e}_2 b)^* = \mathbf{0}$ and $\tilde{e}_2(\tilde{e}_2 b)^* = \mathbf{0}$. By these equalities and the tensor product rule of crystals, we have $\varepsilon_1(b_1^*) \leq \varepsilon_1((\tilde{e}_2 b)^*) = 1$ and $\varepsilon_2(b_1^*) \leq \varepsilon_2((\tilde{e}_2 b)^*) = 0$. Thus we get $\varepsilon_2(b_1^*) = 0$. Moreover, since $b_1 \neq u_\infty$ by assumption, we obtain $\varepsilon_1(b_1^*) = 1$. Thus, b_1^* is of the form : $b_1^* = \tilde{f}_1 b_1'$ for some $b_1' \in \mathcal{B}(\infty)$ such that $\tilde{e}_1 b_1' = \mathbf{0}$. Because $\tilde{e}_1^2(\tilde{e}_2 b)^* = \tilde{e}_1 S_1(\tilde{e}_2 b)^* = \mathbf{0}$ as seen above, we see by the tensor product rule of crystals that

$$S_1(\tilde{e}_2 b)^* = \tilde{e}_1(\tilde{e}_2 b)^* = \tilde{e}_1 b_1^* \otimes t_{-\lambda-\mathrm{wt}(b_1)-\alpha_2} \otimes \tilde{e}_2 u_{-\infty} = b_1' \otimes t_{-\lambda-\mathrm{wt}(b_1)-\alpha_2} \otimes \tilde{e}_2 u_{-\infty};$$

note that $\tilde{f}_2 S_1(\tilde{e}_2 b)^* \neq \mathbf{0}$ by the tensor product rule of crystals. However, since $S_1(\tilde{e}_2 b)^*$ is an extremal element of weight $-r_1\lambda$, and since $\langle -r_1\lambda, \alpha_2^\vee \rangle = -a_2 + 1 \leq 0$, we have $\tilde{f}_2(S_1(\tilde{e}_2 b)^*) = \mathbf{0}$, which is a contradiction. Thus we have proved the lemma. $\square$

The proof of the next lemma is similar to the proof of Lemma 3.10.

Lemma 3.11. Let $i \in I$ and $b \in \mathcal{B}(\lambda)$ be such that $\tilde{f}_i b \neq \mathbf{0}$. If b is of the form : $b = u_\infty \otimes t_\lambda \otimes b_2$ with $b_2 \neq u_{-\infty}$, then $\tilde{f}_i b = u_\infty \otimes t_\lambda \otimes \tilde{f}_i b_2$.

Proposition 3.12. It holds that $\mathcal{B}(\lambda) \subset (\mathcal{B}(\infty) \otimes t_\lambda \otimes u_{-\infty}) \cup (u_\infty \otimes t_\lambda \otimes \mathcal{B}(-\infty))$.

Proof. By Lemmas 3.10 and 3.11, the subset

$$\mathcal{B}(\lambda) \cap \big((\mathcal{B}(\lambda) \otimes t_\lambda \otimes u_{-\infty}) \cup (u_\infty \otimes t_\lambda \otimes \mathcal{B}(\lambda))\big) =: \mathcal{C}$$

is a subcrystal of $\mathcal{B}(\lambda)$. Therefore it suffices to show that every element $b \in \mathcal{B}(\lambda)$ is connected to an element of $\mathcal{C}$. Write b as $b = b_1 \otimes t_\lambda \otimes b_2$ with $b_1 \in \mathcal{B}(\infty)$ and $b_2 \in \mathcal{B}(-\infty)$. It is known that there exist $i_1, \dots, i_k \in I$ such that $\tilde{f}_{i_k}^{\max} \cdots \tilde{f}_{i_1}^{\max} b_2 = u_{-\infty}$. Then we see by the tensor product rule of crystals that $\tilde{f}_{i_k}^{\max} \cdots \tilde{f}_{i_1}^{\max} b = b_1' \otimes t_\lambda \otimes u_{-\infty}$ for some $b_1' \in \mathcal{B}(\infty)$, which implies that $\tilde{f}_{i_k}^{\max} \cdots \tilde{f}_{i_1}^{\max} b \in \mathcal{C}$. Thus we have proved the proposition. □

Recall that $-\mathrm{wt}(b_1) \in Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ for every $b_1 \in \mathcal{B}(\infty)$; for $\alpha = \sum_{i \in I} k_i \alpha_i \in Q_+$, we set $|\pm\alpha| := \sum_{i \in I} k_i \in \mathbb{Z}_{\geq 0}$.

Proof of Theorem 3.5. By Proposition 3.12, $b \in \mathcal{B}(\lambda)$ is either of the following forms: $b = b_1 \otimes t_\lambda \otimes u_{-\infty}$ for some $b_1 \in \mathcal{B}(\infty)$, or $b = u_\infty \otimes t_\lambda \otimes b_2$ for some $b_2 \in \mathcal{B}(-\infty)$. We show by induction on $|\mathrm{wt}(b_1)|$ that if $b \in \mathcal{B}(\lambda)$ is the form $b = b_1 \otimes t_\lambda \otimes u_{-\infty}$ for some $b_1 \in \mathcal{B}(\infty)$, then $b = \tilde{f}_{i_k} \cdots \tilde{f}_{i_1} u_\lambda$ for some $i_1, \dots, i_k \in I$. If $|\mathrm{wt}(b_1)| = 0$, then the assertion is obvious since $b_1 = u_\infty$, and hence $b = u_\lambda$. Assume that $|\mathrm{wt}(b_1)| \geq 1$. Since $b_1 \neq u_\infty$, there exists $i \in I$ such that $\tilde{e}_i b_1 \neq \mathbf{0}$; we see by the tensor product rule of crystals that $\tilde{e}_i b \neq \mathbf{0}$. Moreover, we deduce by Lemma 3.10 that

$$\tilde{e}_i b = \tilde{e}_i(b_1 \otimes t_\lambda \otimes u_{-\lambda}) = \tilde{e}_i b_1 \otimes t_\lambda \otimes u_{-\infty}.$$

Since $|\mathrm{wt}(\tilde{e}_i b_1)| = k - 1$, it follows by the induction hypothesis that there exist $i_1, \dots, i_k \in I$ such that $\tilde{e}_i b = \tilde{f}_{i_k} \cdots \tilde{f}_{i_1} u_\lambda$. Then we obtain $b = \tilde{f}_i \tilde{f}_{i_k} \cdots \tilde{f}_{i_1} u_\lambda$.

Similarly, we show by induction on $|\mathrm{wt}(b_2)|$ that if $b \in \mathcal{B}(\lambda)$ is the form $b = u_\infty \otimes t_\lambda \otimes b_2$ for some $b_2 \in \mathcal{B}(-\infty)$, then $b = \tilde{e}_{i_k} \cdots \tilde{e}_{i_1} u_\lambda$ for some $i_1, \dots, i_k \in I$. Thus we have proved Theorem 3.5. □

3.5 Proof of Theorem 3.7.

First, we show part (1) of Theorem 3.7. The “only if ” part is obvious. To show the “if” part, assume that $x\lambda = y\lambda$ for $x, y \in W$. Then the weight of the element $S_{x^{-1}} S_y u_\lambda$ is equal to $x^{-1} y \lambda = \lambda$. Therefore, by Corollary 3.6, we obtain $S_{x^{-1}} S_y u_\lambda = u_\lambda$, and hence $S_x u_\lambda = S_y u_\lambda$, as desired.

Next, we show part (2) of Theorem 3.7. Let $b \in \mathcal{B}(\lambda)$ be an extremal element. By Proposition 3.12, b is either of the following forms: $b = b_1 \otimes t_\lambda \otimes u_{-\infty}$ for some $b_1 \in \mathcal{B}(\infty)$ or $b = u_\infty \otimes t_\lambda \otimes b_2$ for some $b_2 \in \mathcal{B}(-\infty)$. We show by induction on $|\mathrm{wt}(b_1)|$ that if an extremal element $b \in \mathcal{B}(\lambda)$ is of the form : $b = b_1 \otimes t_\lambda \otimes u_{-\infty}$ for some $b_1 \in \mathcal{B}(\infty)$, then there exists $w \in W$ such that $b = S_w u_\lambda$. If $|\mathrm{wt}(b_1)| = 0$, then the assertion is obvious since $b_1 = u_\infty$, and hence $b = u_\lambda$. Assume that $|\mathrm{wt}(b_1)| > 0$. There exists $i \in I$ such that $\tilde{e}_i b_1 \neq \mathbf{0}$; notice that $\tilde{e}_i b \neq \mathbf{0}$ by the tensor product rule of crystals. Since b is an extremal element by assumption, we see that $\varphi_i(b) = 0$ and $\langle \mathrm{wt}(b), \alpha_i^\vee \rangle = -\varepsilon_i(b) \le -1$; we set $n := -\langle \mathrm{wt}(b), \alpha_i^\vee \rangle \ge 1$. We deduce that

$$\begin{aligned} S_i b &= \tilde{e}_i^n (b_1 \otimes t_\lambda \otimes u_{-\infty}) = \tilde{e}_i^{n-1}(\tilde{e}_i b_1 \otimes t_\lambda \otimes u_{-\infty}) \text{ by Lemma 3.10} \\ &= (\tilde{e}_i^n b_1) \otimes t_\lambda \otimes u_{-\infty} \text{ by the tensor product rule of crystals.} \end{aligned}$$

Because $S_i b$ is also an extremal element in $\mathcal{B}(\lambda)$, and $|\mathrm{wt}(\tilde{e}_i^n b_1)| < |\mathrm{wt}(b_1)|$, it follows from the induction hypothesis that $S_i b = S_w u_\lambda$ for some $w \in W$. Thus we get $b = S_i S_w u_\lambda = S_{r_i w} u_\lambda$.

Similarly, we can show by induction on $|\mathrm{wt}(b_2)|$ that if an extremal element $b \in \mathcal{B}(\lambda)$ is of the form: $b = u_\infty \otimes t_\lambda \otimes b_2$ for some $b_2 \in \mathcal{B}(-\infty)$, then there exists $w \in W$ such that $b = S_w u_\lambda$. Thus we have proved Theorem 3.7.

3.6 Proof of Theorem 3.8.

If $a_1 = 1$ or $a_2 = 1$, then $\mathbb{B}(\lambda)$ is connected by Remark 3.3 and (2.1). So, we assume that $a_1, a_2 \ge 2$. In order to prove Theorem 3.8 in this case, we need some lemmas.

Lemma 3.13. Let $m \in \mathbb{Z}_{\ge 0}$, and $\beta \in \Delta_{\mathrm{re}}^+$.
(1) Assume that m is even. Then, $\langle x_m\lambda,\ \beta^\vee \rangle \in \mathbb{Z}_{<0}$ if and only if $\beta = x_l\alpha_2$ or $y_{l+1}\alpha_1$ for some $l \in \mathbb{Z}_{\mathrm{even}\ge 0}$.
(2) Assume that m is odd. Then, $\langle x_m\lambda,\ \beta^\vee \rangle \in \mathbb{Z}_{<0}$ if and only if $\beta = y_l\alpha_1$ or $x_{l+1}\alpha_2$ for some $l \in \mathbb{Z}_{\mathrm{even}\ge 0}$.

Proof. We give a proof only for part (1); the proof for part (2) is similar. First we show the "if" part of part (1). Let $l \in \mathbb{Z}_{\mathrm{even}\ge 0}$. We have $\langle x_m\lambda, x_l\alpha_2^\vee \rangle = \langle x_l^{-1}x_m\lambda,\ \alpha_2^\vee \rangle$. Here, if $m \ge l$ (resp., $m \le l$), then $x_l^{-1}x_m$ is equal to x_{m-l} (resp., y_{l-m}). Therefore, by (3.8), we have $\langle x_m\lambda, x_l\alpha_2^\vee \rangle = -p_{m-l} \in \mathbb{Z}_{<0}$ (resp., $= -q_{l-m+1} \in \mathbb{Z}_{<0}$). Similarly, we can show that $\langle x_m\lambda, y_{l+1}\alpha_1^\vee \rangle < 0$.

Next, we show that the "only if " part of part (1); by (3.4), it suffices to show that if $\beta = x_{l+1}\alpha_2$ or $y_l\alpha_1$ for $l \in \mathbb{Z}_{\mathrm{even}\ge 0}$, then $\langle x_m\lambda, \beta^\vee \rangle > 0$. We have $\langle x_m\lambda, x_{l+1}\alpha_2^\vee \rangle = \langle x_{l+1}^{-1}x_m\lambda, \alpha_2^\vee \rangle = \langle x_{m+l+1}\lambda, \alpha_2^\vee \rangle$. By (3.8), we have $\langle x_m\lambda, x_{l+1}\alpha_2^\vee \rangle = p_{m+l+2} > 0$. Similarly, we can show that $\langle x_m\lambda, y_l\alpha_1^\vee \rangle > 0$. This completes the proof of the lemma. □

The next lemma can be shown in exactly the same way as Lemma 3.13.

Lemma 3.14. Let $m \in \mathbb{Z}_{\ge 0}$ and $\beta \in \Delta_{\mathrm{re}}^+$.
(1) Assume that m is even. Then, $\langle y_m\lambda,\ \beta^\vee \rangle \in \mathbb{Z}_{<0}$ if and only if $\beta = x_l\alpha_2$ or $y_{l+1}\alpha_1$ for some $l \in \mathbb{Z}_{\mathrm{even}\ge 0}$.
(2) Assume that m is odd. Then, $\langle y_m\lambda,\ \beta^\vee \rangle \in \mathbb{Z}_{<0}$ if and only if $\beta = y_l\alpha_1$ or $x_{l+1}\alpha_2$ for some $l \in \mathbb{Z}_{\mathrm{even}\ge 0}$.

Lemma 3.15. (1) For $m \in \mathbb{Z}_{\ge 1}$, we have $x_m\lambda > x_{m-1}\lambda$ with $\mathrm{dist}(x_m\lambda, x_{m-1}\lambda) = 1$. And $r_i x_m\lambda = x_{m-1}\lambda$, where $i = 2$ if m is even, and $i = 1$ if m is odd.
(2) For $m \in \mathbb{Z}_{\ge 1}$, we have $y_{m-1}\lambda > y_m\lambda$ with $\mathrm{dist}(y_{m-1}\lambda, y_m\lambda) = 1$. And $r_j y_m\lambda = y_{m-1}\lambda$, where $i = 1$ if m is even, and $i = 2$ if m is odd.

Proof. We give a proof only for part (1); the proof for part (2) is similar. We see from Lemma 3.13 that $\langle x_m\lambda,\ \alpha_i^\vee \rangle < 0$. Therefore, we obtain that $x_m\lambda > r_i x_m\lambda = x_{m-1}\lambda$. Since $\langle x_{m-1}\lambda,\ \alpha_i^\vee \rangle > 0$, we see by [L2, Lemma 4.1] that $\mathrm{dist}(r_i x_m\lambda, x_{m-1}\lambda) = \mathrm{dist}(x_m\lambda, x_{m-1}\lambda) - 1$. Since $\mathrm{dist}(r_i x_m\lambda, x_{m-1}\lambda) = \mathrm{dist}(x_{m-1}\lambda, x_{m-1}\lambda) = 0$, we obtain $\mathrm{dist}(x_m\lambda, x_{m-1}\lambda) = 1$, as desired. □

Proposition 3.16. The Hasse diagram of $W\lambda$ is

$$\cdots \xleftarrow{\alpha_1} x_2\lambda \xleftarrow{\alpha_2} x_1\lambda \xleftarrow{\alpha_1} x_0\lambda = \lambda = y_0\lambda \xleftarrow{\alpha_2} y_1\lambda \xleftarrow{\alpha_1} y_2\lambda \xleftarrow{\alpha_1} \cdots .$$

Proof. Let $\mu, \nu \in W\lambda$ be such that $\mu > \nu$ with $\mathrm{dist}(\mu, \nu) = 1$, and let $\beta \in \Delta^+_{\mathrm{re}}$ be the (unique) positive real root such that $\nu = r_\beta \mu$; by Lemma 3.15, it suffices to show that $\beta = \alpha_1$ or α_2. By Lemma 3.13, if $\mu = x_m\lambda$ and m is even, then $\beta = x_l\alpha_2$ or $y_{l+1}\alpha_1$ for some $l \in \mathbb{Z}_{\mathrm{even}\geq 0}$. Assume that $\beta = x_l\alpha_2$ for some $l \in \mathbb{Z}_{\mathrm{even}\geq 0}$; note that $r_\beta = (r_2r_1)^{\frac{l}{2}} r_2 (r_1r_2)^{\frac{l}{2}}$. We see from Lemma 3.15 that there exist a directed path

$$\mu = x_m\lambda \xleftarrow{\alpha_2} x_{m-1}\lambda \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_2} r_\beta\mu = \nu$$

of length $2l + 1$ from μ to ν in the Hasse diagram of $W\lambda$. Because $\mathrm{dist}(\mu, \nu) = 1$ by assumption, we obtain $l = 0$, and hence $\beta = \alpha_2$. Assume that $\beta = y_{l+1}\alpha_1$ for some $l \in \mathbb{Z}_{\mathrm{even}\geq 0}$; note that $r_2(r_1r_2)^{\frac{l}{2}} r_1 (r_2r_1)^{\frac{l}{2}} r_2$. By the same reasoning as above, there exists a direct path of length $2l+3 > 1$ from μ to ν in the Hasse diagram of $W\lambda$. However, this contradicts the assumption that $\mathrm{dist}(\mu, \nu) = 1$. Similarly, we can show that if $\mu = x_m\lambda$ and m is odd, then $\beta = \alpha_1$. Also, we can show the assertion for the case that $\mu = y_m\lambda$ in exactly the same way as above. This completes the proof of the proposition. □

Lemma 3.17. For any rational number $0 < \sigma < 1$ and any $\mu, \nu \in W\lambda$ such that $\mu > \nu$, there does not exist a σ-chain $\mu = \mu_0 > \cdots > \mu_r = \nu$ for (μ, ν) such that $\mu_k = \lambda$ for some $0 \leq k \leq r$.

Proof. Suppose that $\mu_k = \lambda$ for some $0 \leq k \leq r$. Note that $r \geq 1$ since $\mu > \nu$. If $k < r$ (resp., $k > 0$), then it follows from Proposition 3.16 that $\mu_{k+1} = r_2\lambda$ (resp., $\mu_{k-1} = r_1\lambda$) since $\mathrm{dist}(\mu_k, \mu_{k+1}) = 1$ (resp., $\mathrm{dist}(\mu_{k-1}, \mu_k) = 1$) by the assumption of the σ-chain. Thus, we obtain $\sigma = -\sigma\langle\lambda, \alpha_2^\vee\rangle \in \mathbb{Z}$ (resp., $\sigma = \sigma\langle\lambda, \alpha_1^\vee\rangle \in \mathbb{Z}$), which contradicts the assumption $0 < \sigma < 1$. If $k = 0$ or $k = r$, it is clear that $\sigma = -\sigma\langle\lambda, \alpha_2^\vee\rangle \in \mathbb{Z}$ or $\sigma = -\sigma\langle r_1\lambda, \alpha_1^\vee\rangle \in \mathbb{Z}$ by Proposition 3.16. This also contradicts the assumption. Thus, the lemma has been proved. □

The next proposition follows immediately from Lemma 3.17 and the definition of LS paths.

Proposition 3.18. Let $\pi = (\nu_1, \ldots, \nu_s ; \sigma_0, \ldots, \sigma_s) \in \mathbb{B}(\lambda)$. If $\nu_u = \lambda$ for some $1 \leq u \leq s$, then $s = 1$ and $\pi = (\lambda ; 0, 1)$.

Proof of Theorem 3.8. We show that every $\pi \in \mathbb{B}(\lambda)$ is connected to $(\lambda; 0, 1) \in \mathbb{B}(\lambda)$ in the crystal graph of $\mathbb{B}(\lambda)$. Assume first that $\iota(\pi) = x_m\lambda$ for some $m \in \mathbb{Z}_{\geq 0}$. We show by induction on m that π is connected to $(\lambda; 0, 1)$. If $m = 0$, then the assertion follows immediately from Proposition 3.18. Assume that $m > 0$. Define

$$i := \begin{cases} 2 & (\text{if } m \text{ is even}), \\ 1 & (\text{if } m \text{ is odd}); \end{cases}$$

note that $\langle x_m\lambda, \alpha_i^\vee\rangle < 0$ and $r_ix_m\lambda = x_{m-1}\lambda$ (see Lemma 3.15). By Lemma 2.12, $\iota(\tilde{e}_i^{\max}\pi) = r_i\iota(\pi) = r_ix_m\lambda = x_{m-1}\lambda$. By the induction hypothesis, $\tilde{e}_i^{\max}\pi$ is connected to $(\lambda ; 0, 1)$, and hence so is π.

Assume next that $\iota(\pi) = y_m\lambda$ for some $m \in \mathbb{Z}_{\geq 0}$. Since $\kappa(\pi) \leq \iota(\pi)$ by the definition of an LS path, we see by Proposition 3.16 that $\kappa(\pi) = y_k\lambda$ for some $k \geq m$. Hence it suffices to show that if $\pi \in \mathbb{B}(\lambda)$ satisfies that $\kappa(\pi) = y_k\lambda$ for some $k \in \mathbb{Z}_{\geq 0}$, then π is connected to $(\lambda;\ 0,\ 1)$. If $k = 0$, then the assertion follows immediately from Proposition 3.18. Assume that $k > 0$. Define

$$j := \begin{cases} 1 & \text{(if } k \text{ is even)}, \\ 2 & \text{(if } k \text{ is odd)}; \end{cases}$$

note that $\langle y_k\lambda, \alpha_j^\vee \rangle > 0$ and $r_j y_k \lambda = y_{k-1}\lambda$. By Lemma 2.12, $\kappa(\tilde{f}_j^{\max}\pi) = r_j\kappa(\pi) = r_j y_k\lambda = y_{k-1}\lambda$. By the induction hypothesis, $\tilde{f}_j^{\max}\pi$ is connected to $(\lambda\ ;\ 0,\ 1)$, and hence so is π. Thus, we have proved Theorem 3.8. $\square$

3.7 Proof of Theorem 3.9.

Throughout this subsection, we assume that $a_1, a_2 \neq 1$ in (3.1). Recall that the sequences $\{p_m\}_{m \in \mathbb{Z}_{\geq 0}}$ and $\{q_m\}_{m \in \mathbb{Z}_{\geq 0}}$ are defined in (3.6) and (3.7), respectively.

Lemma 3.19. For each $k \geq 0$, the numbers p_k and p_{k+1} are relatively prime. Also, the numbers q_k and q_{k+1} are relatively prime.

Proof. We give a proof only for p_k and p_{k+1}; the proof for q_k and q_{k+1} is similar. Suppose that the assertion is false, and let m be the minimum $k \geq 0$ such that p_k and p_{k+1} have a common divisor greater than 1. Let $d \in \mathbb{Z}_{>1}$ be a common divisor of p_m and p_{m+1}. Since

$$\begin{cases} p_{m+1} = a_2 p_m - p_{m-1} & \text{(if } m \text{ is even)}, \\ p_{m+1} = a_1 p_m - p_{m-1} & \text{(if } m \text{ is odd)}, \end{cases}$$

we can deduce that p_m and p_{m-1} have the same common divisor d, which contradicts the minimality of m. Thus, we have proved the lemma. $\square$

Lemma 3.20. Let $0 < \sigma < 1$ be a rational number, and let $\mu, \nu \in W\lambda$ be such that $\mu > \nu$. If $\mu = \mu_0 > \mu_1 > \cdots > \mu_t = \nu$ is a σ-chain for (μ, ν), then $t = 1$.

Proof. Suppose that $t \geq 2$. Assume first that $\mu_0 = x_m\lambda$; by Lemma 3.17, we have $m \geq 3$. Since $\mathrm{dist}(\mu_0, \mu_1) = \mathrm{dist}(\mu_1, \mu_2) = 1$ by the definition of a σ-chain, we see by Proposition 3.16 that $\mu_1 = x_{m-1}\lambda$ and $\mu_2 = x_{m-2}\lambda$. Take $i, j \in \{1, 2\}$ such that $\mu_1 = r_i\mu_0$ and $\mu_2 = r_j\mu_1$. Then, by Lemma 3.4,

$$\langle \mu_0, \alpha_i^\vee \rangle = \langle x_m\lambda, \alpha_i^\vee \rangle = -p_m, \quad \langle \mu_1, \alpha_j^\vee \rangle = \langle x_{m-1}\lambda, \alpha_j^\vee \rangle = -p_{m-1}.$$

Since $-p_m$ and $-p_{m-1}$ are relatively prime (see Lemma 3.19), there does not exist a $0 < \sigma < 1$ satisfying the condition that both $-\sigma p_m$ and $-\sigma p_{m-1}$ are integers. This contradicts our assumption that $\mu = \mu_0 > \mu_1 > \cdots > \mu_t = \nu$ is a σ-chain for (μ, ν). Similarly, we can get a contradiction also in the case of $\mu_0 = y_m\lambda$ for some $m \in \mathbb{Z}_{\geq 1}$. Thus, we have proved Lemma 3.20. $\square$

Proof of Theorem 3.9. Let $\pi = (\nu_1, \ldots, \nu_s; \sigma_0, \ldots, \sigma_s) \in \mathbb{B}(\lambda)$. Assume first that $\nu_s = x_m\lambda$ for some $m \geq 0$. Since $\nu_1 > \nu_2 > \cdots > \nu_s = x_m\lambda$ by the definition of an LS path, we see by Proposition 3.16 that

$$\nu_1 = x_{k_1}\lambda,\ \nu_2 = x_{k_2}\lambda,\ \ldots,\ \nu_{s-1} = x_{k_{s-1}}\lambda$$

for some $k_1 > k_2 > \cdots > k_{s-1} > m$. Here we recall that there exists a σ_{s-1}-chain for $(\nu_{s-1}, \nu_s) = (x_{k_{s-1}}\lambda, x_m\lambda)$ by the definition of an LS path. By Lemma 3.20, we see that the length of this σ_{s-1}-chain is equal to 1, which implies that $\mathrm{dist}(\nu_{s-1}, \nu_s) = \mathrm{dist}(x_{k_s-1}\lambda, x_m\lambda) = 1$. Hence it follows from Proposition 3.16 that $k_{s-1} = m+1$. Take $i \in I$ such that $x_m\lambda = r_i x_{m+1}\lambda$. Then, by the definition of a σ_{s-1}-chain, we have $\sigma_{s-1}\langle x_{m+1}\lambda, \alpha_i^\vee\rangle \in \mathbb{Z}$. Since $\langle x_{m+1}\lambda, \alpha_i^\vee\rangle = -p_{m+1}$ by (3.8), we obtain $p_{m+1}\sigma_{s-1} \in \mathbb{Z}$. By repeating this argument, we deduce that $k_u = m+s-u$ and $p_{m+s-u}\sigma_u \in \mathbb{Z}$ for every $1 \le u \le s-1$. Hence, π is of the form (i).

Assume next that $\pi = (\nu_1, \dots, \nu_s; \delta_0, \dots, \delta_s) \in \mathbb{B}(\lambda)$ and $\nu_s = y_m\lambda$ for some $m \ge 0$. Suppose that ($s \ge 2$ and) there exists $1 \le u \le s-1$ such that $\nu_{u+1} = y_k\lambda$ for some $k \ge 0$, but $\nu_u = x_l\lambda$ for some $l \ge 0$. By the definition of an LS path, there exists a δ_u-chain for (ν_u, ν_{u+1}) $(1 \le u \le s-1)$. Then, by Lemma 3.20, the length of this δ_u-chain is equal to 1, which implies that $\mathrm{dist}(\nu_u, \nu_{u+1}) = \mathrm{dist}(x_l\lambda, y_k\lambda) = 1$. By the Hasse diagram in Proposition 3.16, we see that $(l, k) = (1, 0)$ or $(0, 1)$ Since $x_0\lambda = y_0\lambda = \lambda$, it follows form Proposition 3.18 that $s = 1$, which contradicts $s \ge 2$. Therefore, we conclude that

$$\nu_1 = y_{k_1}\lambda,\ \nu_2 = y_{k_2}\lambda,\ \dots,\ \nu_s = y_{k_s}\lambda = y_m\lambda,$$

where $0 \le k_1 < k_2 < \cdots < k_{s-1} < k_s = m$. By the same argument as above, we deduce that $k_u = m - s + u$ and $q_{m-s+u+1}\delta_u \in \mathbb{Z}$. Hence, π is of the form (ii). This completes the proof of Theorem 3.9. $\square$

Appendices

A Explicit description of the root operators.

As an application of Theorem 3.9, we give an explicit description of the root operators $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 1, 2$. First, let $\pi \in \mathbb{B}(\lambda)$ be of the form (i) in Theorem 3.9. We set

$$C_u^{(1)} := \sum_{k=m+s-u}^{m+s-1} (\sigma_{m+s-k} - \sigma_{m+s-k-1})(-1)^k p_{k+\xi_k},$$
$$C_u^{(2)} := \sum_{k=m+s-u}^{m+s-1} (\sigma_{m+s-k} - \sigma_{m+s-k-1})(-1)^{k+1} p_{k+\xi_{k+1}},$$

where $\{p_m\}_{m\in\mathbb{Z}_{\ge 0}}$ is defined as (3.6), and for $k \in \mathbb{Z}_{\ge 0}$,

$$\xi_k := \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Then,

$$\mathrm{wt}(\pi) = C_s^{(1)}\Lambda_1 + C_s^{(2)}\Lambda_2.$$

Note that $\pm\langle x_u\lambda, \alpha_i^\vee\rangle > 0$ if and only if $\mp\langle x_{u+1}\lambda, \alpha_i^\vee\rangle > 0$ for each $u \in \mathbb{Z}_{\ge 0}$. Thus we see (cf. (2.5)) that

$$m_i^\pi = \min\{C_u^{(i)} \mid 0 \le u \le s\}.$$

Let us give an explicit description of $\tilde{f}_i\pi$. We set

$$u_0 := \max\{0 \le u \le s \mid C_u^{(i)} = m_i^\pi\};$$

if $u_0 = s$, then $\tilde{f}_i\pi = \mathbf{0}$. Assume that $0 \le u_0 \le s-1$; we see that σ_{u_0} is equal to t_0 in (2.9). By fact (2.10), we deduce that t_1 in (2.9) is equal to

$$\sigma'_{u_0} := \begin{cases} \sigma_{u_0} + \dfrac{1}{p_{m+s-u_0-1+\xi_{m+s-u_0-1}}} & \text{if } i = 1, \\ \\ \sigma_{u_0} + \dfrac{1}{p_{m+s-u_0-1+\xi_{m+s-u_0}}} & \text{if } i = 2, \end{cases}$$

which satisfies $\sigma_{u_0} < \sigma'_{u_0} \le \sigma_{u_0+1}$; notice that if $\sigma'_{u_0} = \sigma_{u_0+1}$, then $u_0 = s-1$, and hence $\sigma'_{u_0} = \sigma_s = 1$. We have

$$\tilde{f}_i\pi = \begin{cases} (x_{m+s}\lambda, x_{m+s-1}\lambda, \dots, x_m\lambda; \sigma_0, \sigma'_0, \sigma_1, \dots, \sigma_s) & \text{if } u_0 = 0 \text{ and } \sigma'_{u_0} < \sigma_{u_0+1}, \\ (r_i x_m\lambda; 0, 1) & \text{if } u_0 = 0 \text{ and } \sigma'_{u_0} = \sigma_{u_0+1}, \\ (x_{m+s-1}\lambda, \dots, x_m\lambda; \sigma_0, \dots, \sigma_{u_0-1}, \sigma'_{u_0}, \sigma_{u_0+1}, \dots, \sigma_s) & \text{if } u_0 \ge 1 \text{ and } \sigma'_{u_0} < \sigma_{u_0+1}, \\ (x_{m+s-1}\lambda, \dots, x_{m-1}\lambda; \sigma_0, \dots, \sigma_{s-2}, \sigma_s) & \text{if } u_0 \ge 1 \text{ and } \sigma'_{u_0} = \sigma_{u_0+1}. \end{cases} \tag{A.1}$$

Similarly, we give an explicit description of $\tilde{e}_i\pi$ as follows. We set

$$u_1 := \min\{0 \le u \le s \mid C_u^{(i)} = m_i^\pi\};$$

if $u_1 = 0$, then $\tilde{e}_i\pi = \mathbf{0}$. Assume that $1 \le u_1 \le s$; we see that σ_{u_1} is equal to t_1 in (2.7). By fact (2.8), we deduce that t_1 in (2.7) is equal to

$$\sigma'_{u_1} := \begin{cases} \sigma_{u_1} - \dfrac{1}{p_{m+s-u_1+\xi_{m+s-u_1}}} & \text{if } i = 1, \\ \\ \sigma_{u_1} - \dfrac{1}{p_{m+s-u_1+\xi_{m+s-u_1+1}}} & \text{if } i = 2, \end{cases}$$

which satisfies $\sigma_{u_1-1} \le \sigma'_{u_1} \le \sigma_{u_1}$; notice that if $\sigma'_{u_1} = \sigma_{u_1-1}$, then $u_1 = 1$ and hence $\sigma'_{u_1} = \sigma_0 = 0$. We have

$$\tilde{e}_i\pi = \begin{cases} (x_{m+s-1}\lambda, \dots, x_m\lambda, x_{m-1}\lambda; \sigma_0, \dots, \sigma_{s-1}, \sigma'_s, \sigma_s) & \text{if } u_1 = s \text{ and } \sigma_{u_1-1} < \sigma'_{u_1}, \\ (r_i x_m\lambda; 0, 1) & \text{if } u_1 = s \text{ and } \sigma_{u_1-1} = \sigma'_{u_1}, \\ (x_{m+s-1}\lambda, \dots, x_m\lambda; \sigma_0, \dots, \sigma_{u_1-1}, \sigma'_{u_1}, \sigma_{u_1+1}, \dots, \sigma_s) & \text{if } u_1 \le s-1,\ \sigma_{u_1-1} < \sigma'_{u_1}, \\ (x_{m+s-2}\lambda, \dots, x_m\lambda; \sigma_0, \sigma_2, \dots, \sigma_s) & \text{if } u_1 \le s-1,\ \sigma_{u_1-1} = \sigma'_{u_1}, \end{cases} \tag{A.2}$$

where we understand $x_{-1}\lambda = y_1\lambda$.

Example A.1. Assume that

$$\pi = (r_2r_1r_2r_1\lambda, r_1r_2r_1\lambda, r_2r_1\lambda; 0, \frac{1}{p_4}, \frac{2}{p_3}, 1) \in \mathbb{B}(\lambda),$$

which is an element of the form in Theorem 3.9 (i) with $m = 2$ and $s = 3$. Let us compute $\tilde{f}_i\pi$ for $i = 1, 2$, using formula (A.1). If $i = 1$, then we have $u_0 = 2, \sigma'_{u_0} = \frac{3}{p_3}$ if $a_1 = 2$ and $u_0 = 0, \sigma'_{u_0} = \frac{1}{p_5}$ if $a_1 \geq 3$; note that $\frac{3}{p_3} = 1$ if $b = 3$. Thus,

$$\tilde{f}_1\pi = \begin{cases} (r_2r_1r_2r_1\lambda, r_1r_2r_1\lambda; 0, \dfrac{1}{p_4}, 1) & \text{if } a_1 = 2, a_2 = 3, \\ (r_2r_1r_2r_1\lambda, r_1r_2r_1\lambda, r_2r_1\lambda; 0, \dfrac{1}{p_4}, \dfrac{3}{p_3}, 1) & \text{if } a_1 = 2, a_2 > 3, \\ (r_1r_2r_1r_2r_1\lambda, r_2r_1r_2r_1\lambda, r_1r_2r_1\lambda, r_2r_1\lambda; 0, \dfrac{1}{p_5}, \dfrac{1}{p_4}, \dfrac{2}{p_3}, 1) & \text{if } a_1 \geq 3; \end{cases}$$

remark that if $a_1 = 2$, then $a_2 > 3$. If $i = 2$, then we have $u_0 = 1, \sigma'_{u_0} = \frac{2}{p_4}$. Thus,

$$\tilde{f}_2\pi = (r_2r_1r_2r_1\lambda, r_1r_2r_1\lambda, r_2r_1\lambda; 0, \frac{2}{p_4}, \frac{2}{p_3}, 1).$$

Next, let $\pi \in \mathbb{B}(\lambda)$ be of the form (ii) in Theorem 3.9. By a similar argument to above, we have the following explicit descriptions of $\tilde{f}_i\pi$ and $\tilde{e}_i\pi$. We set

$$D_v^{(1)} := \sum_{k=m-s+1}^{m-s+v} (\delta_{k-m+s} - \delta_{k-m+s-1})(-1)^k q_{k+\xi_{k+1}},$$

$$D_v^{(2)} := \sum_{k=m-s+1}^{m-s+v} (\delta_{k-m+s} - \delta_{k-m+s-1}(-1)^{k+1} q_{k+\xi_k},$$

where $\{q_m\}_{m \in \mathbb{Z}_{\geq 0}}$ is define as (3.7). Then

$$\mathrm{wt}(\pi) = D_s^{(1)}\Lambda_1 + D_s^{(2)}\Lambda_2.$$

We have

$$m_i^\pi = \min\{D_v^{(i)} \mid 0 \leq v \leq s\}.$$

Let us give an explicit description of $\tilde{f}_i\pi$. We set

$$v_0 := \max\{0 \leq v \leq s \mid D_v^{(i)} = m_i^\pi\};$$

if $v_0 = s$, then $\tilde{f}_i\pi = \mathbf{0}$. Assume that $0 \leq v_0 \leq s-1$. we set

$$\delta'_{v_0} := \begin{cases} \delta_{v_0} + \dfrac{1}{p_{m-s+v_0+1+\xi_{m-s+v_0+2}}} & \text{if } i = 1, \\ \delta_{v_0} + \dfrac{1}{p_{m-s+v_0+1+\xi_{m-s+v_0+1}}} & \text{if } i = 2. \end{cases}$$

We have

$$\tilde{f}_i\pi = \begin{cases} (y_{m-s}\lambda, y_{m-s+1}\lambda, \ldots, y_m\lambda; \delta_0, \delta'_0, \delta_1, \ldots, \delta_s) & \text{if } v_0 = 0 \text{ and } \delta'_{v_0} < \delta_{v_0+1}, \\ (r_iy_m\lambda; 0, 1) & \text{if } v_0 = 0 \text{ and } \delta'_{v_0} = \delta_{v_0+1}, \\ (y_{m-s+1}\lambda, \ldots, y_m\lambda; \delta_0, \ldots, \delta_{v_0-1}, \delta'_{v_0}, \delta_{v_0+1}, \ldots, \delta_s) & \text{if } v_0 \geq 1 \text{ and } \delta'_{v_0} < \delta_{v_0+1}, \\ (y_{m-s+1}\lambda, \ldots, y_{m-1}\lambda; \delta_0, \ldots, \delta_{s-2}, \delta_s) & \text{if } v_0 \geq 1 \text{ and } \delta'_{u_0} = \delta_{v_0+1}. \end{cases} \tag{A.3}$$

where we understand $y_{-1}\lambda = x_1\lambda$.

Similarly, we give an explicit description of $\tilde{e}_i\pi$ as follows. We set

$$v_1 := \min\{0 \le v \le s \mid D_v^{(i)} = m_i^\pi\};$$

if $v_1 = 0$, then $\tilde{e}_i\pi = \mathbf{0}$. Assume that $1 \le v_1 \le s$. We set

$$\delta'_{v_1} := \begin{cases} \delta_{v_1} - \dfrac{1}{q_{m-s+v_1+\xi_{m-s+v_1+1}}} & \text{if } i = 1, \\ \delta_{v_1} - \dfrac{1}{q_{m-s+v_1+\xi_{m-s+v_1}}} & \text{if } i = 2. \end{cases}$$

We have

$$\tilde{e}_i\pi = \begin{cases} (y_{m-s+1}\lambda, \dots, y_m\lambda, y_{m+1}\lambda; \delta_0, \dots, \delta_{s-1}, \delta'_s, \delta_s) & \text{if } v_1 = s \text{ and } \delta_{v_1-1} < \delta'_{v_1}, \\ (r_i y_m\lambda; 0, 1) & \text{if } v_1 = s \text{ and } \delta_{v_1-1} = \delta'_{v_1}, \\ (y_{m-s+1}\lambda, \dots, y_m\lambda; \delta_0, \dots, \delta_{v_1-1}, \delta'_{v_1}, \delta_{v_1+1}, \dots, \delta_s) & \text{if } v_1 \le s-1,\ \delta_{v_1-1} < \delta'_{v_1}, \\ (y_{m-s+2}\lambda, \dots, y_m\lambda; \delta_0, \delta_2, \dots, \delta_s) & \text{if } v_1 \le s-1,\ \delta_{v_1-1} = \delta'_{v_1}. \end{cases} \tag{A.4}$$

B Realization of $\mathcal{B}(\infty)$.

Here we recall a realization of $\mathcal{B}(\infty)$ from [K1, §2.2] and [NZ, §2.4]. We set

$$\mathbb{Z}^\infty := \{(\dots, x_k, \dots, x_2, x_1) \mid x_k \in \mathbb{Z} \text{ and } x_k = 0 \text{ for } k \gg 0\};$$

Also, we fix an infinite sequence of $\iota = (\dots, i_k, \dots, i_2, i_1)$ of elements of I such that

$$i_k \ne i_{k+1} \text{ for any } k \ge 1, \text{ and } \sharp\{k \mid i_k = i\} = \infty \text{ for each } i \in I.$$

The crystal structure on $\mathbb{Z}^\infty$ corresponding to ι is defined as follows. Let $\vec{x} = (\dots, x_k, \dots, x_2, x_1) \in \mathbb{Z}^\infty$. For $k \ge 1$, we set

$$\sigma_k(\vec{x}) := x_k + \sum_{j>k} \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle x_j;$$

since $x_j = 0$ for $j \gg 0$, we see that $\sigma_k(\vec{x})$ is well-defined, and $\sigma_k(\vec{x}) = 0$ for $k \gg 0$. For $i \in I$, set $\sigma^{(i)}(\vec{x}) := \max\{\sigma_k(\vec{x}) \mid k \ge 1, i_k = i\}$, and

$$M^{(i)} = M^{(i)}(\vec{x}) := \{k \ge 1 \mid i_k = i, \sigma_k(\vec{x}) = \sigma^{(i)}(\vec{x})\}.$$

Observe that $\sigma^{(i)}(\vec{x}) \ge 0$, and that $M^{(i)} = M^{(i)}(\vec{x})$ is a finite set if and only if $\sigma^{(i)}(\vec{x}) > 0$. Now we define maps $\tilde{e}_i : \mathbb{Z}^\infty \to \mathbb{Z}^\infty \sqcup \{\mathbf{0}\}$ and $\tilde{f}_i : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ by

$$\tilde{f}_i(\vec{x}) := (\dots, x'_k, \dots, x'_2, x'_1), \text{ with } x'_k := x_k + \delta_{k, \min M^{(i)}};$$

$$\tilde{e}_i(\vec{x}) := \begin{cases} (\dots, x'_k, \dots, x'_2, x'_1), \text{ with } x'_k := x_k - \delta_{k, \max M^{(i)}} & \text{if } \sigma^{(i)}(\vec{x}) > 0, \\ \mathbf{0} & \text{if } \sigma^{(i)}(\vec{x}) \le 0. \end{cases}$$

Also, we define the weight map $\mathrm{wt} : \mathbb{Z}^\infty \to P$ and $\varepsilon_i, \varphi_i : \mathbb{Z}^\infty \to \mathbb{Z}$ by

$$\mathrm{wt}(\vec{x}) := -\sum_{i=1}^{\infty} x_j \alpha_{i_j},$$
$$\varepsilon_i(\vec{x}) := \sigma^{(i)}(\vec{x}), \qquad \varphi_i(\vec{x}) := \langle \mathrm{wt}(\vec{x}), \alpha_i^\vee \rangle + \varepsilon_i(\vec{x}).$$

Theorem B.1 ([K1, §2.2] and [NZ, §2.4])**.** The set $\mathbb{Z}^\infty$, together with the maps above, becomes a crystal; we denote this crystal by $\mathbb{Z}^\infty_\iota$. Moreover, the connected component of $\mathbb{Z}^\infty_\iota$ containing $(\ldots, 0, \ldots, 0, 0)$ is isomorphic, as a crystal, to $\mathcal{B}(\infty)$.

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