Spectral theory for interior transmission eigenvalue problems on Riemannian manifolds

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Abstract

In this thesis, we study two cases of interior transmission eigenvalue problem on two compact Riemannian manifolds with common smooth boundary. In particular, we focus on the distribution of the corresponding interior transmission eigenvalues.

First case is a locally anisotropic interior transmission eigenvalue problem. Our first result is the discreteness of the set of the corresponding eigenvalues. Moreover, we also give the eigenvalue free region. In order to prove this, we employ the so-called T-coercive method.

Second case is a locally isotropic interior transmission eigenvalue problem. Our second main result is the set of the corresponding eigenvalues forms a discrete set and the existence of infinitely many this eigenvalues. We also mention its Weyl type lower bound.

Our results in this thesis appear in [26] and [22].

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Notation

In this thesis, we use the following basic symbols:

$\mathbf{Z}_{>0}$	:	the set of positive integers
$\mathbf{Z}_{\geq 0}$	:	the set of non-negative integers
$\mathbf{R}$	:	the set of real numbers
$\mathbf{R}_{>0}$	:	the set of positive real numbers
$\mathbf{R}_{\geq 0}$	:	the set of non-negative real numbers
$\mathbf{R}^d$	:	d-dimensional Euclidean space
$[a,b]$	:	closed interval $\{x \in \mathbf{R} \mid a \leq x \leq b\}$
(a,b)	:	open interval $\{x \in \mathbf{R} \mid a < x < b\}$
$\mathbf{C}$	:	the set of complex numbers
$\mathbf{C}^d$	:	d-dimensional complex space
$\mathrm{Re}\, z$	:	the real part of $z \in \mathbf{C}$
$\mathrm{Im}\, z$	:	the imaginary part of $z \in \mathbf{C}$
$\overline{z}$	:	the complex conjugate of $z \in \mathbf{C}$
$(\cdot,\cdot)$	:	an inner product on $\mathbf{C}^d$
$\lvert\cdot\rvert$	:	the norm on $\mathbf{C}^d$ denoted by $\lvert\boldsymbol{z}\rvert = \sqrt{(\boldsymbol{z},\boldsymbol{z})}$ for $\boldsymbol{z} \in \mathbf{C}^d$

Let Ω be a domain, i.e., an open connected subset of $\mathbf{R}^d$. We consider the space of Lebesgue measurable functions u on Ω such that

$$\|u\|_{L^\infty(\Omega)} = \inf\{C \geq 0 \mid |u(x)| \leq C \text{ a.e., } x \in \Omega\} < \infty.$$

This space is denoted by $L^\infty(\Omega)$ and $\|\cdot\|_{L^\infty(\Omega)}$ is called $L^\infty(\Omega)$-norm. The space $L^\infty(\Omega)$ is a Banach space with the norm $\|\cdot\|_{L^\infty(\Omega)}$. We denote by $(L^\infty(\Omega))^{d\times d}$ the space of $d\times d$-matrix valued functions with $L^\infty(\Omega)$ entries. We also consider the space of Lebesgue measurable functions u on Ω such that

$$\|u\|_{L^2(\Omega)} = \left\{\int_\Omega |u(x)|^2 dx\right\}^{1/2} < \infty.$$

This space is denoted by $L^2(\Omega)$ and $\|\cdot\|_{L^2(\Omega)}$ is called $L^2(\Omega)$-norm. The space $L^2(M)$ is a Hilbert space with the $L^2(\Omega)$-inner product

$$(u,v)_{L^2(\Omega)} = \int_\Omega u(x)\overline{v(x)}dx \quad \text{for} \quad u,v \in L^2(\Omega).$$

A d-dimensional vector $\alpha = (\alpha_1,\ldots,\alpha_d)$ with non-negative integer coordinates is called a *multi-index*. Put $|\alpha| = \alpha_1 + \cdots + \alpha_d$. For $\partial_i = \partial/\partial x_i\,(i=1,\ldots,d)$, we write $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$. Let $\overline{\Omega}$ be the closure of Ω. We denote by $C(\Omega)$ and $C(\overline{\Omega})$ the space of continuous functions on Ω and $\overline{\Omega}$, respectively. For any non-negative integer k, let $C^k(\Omega)$ be the space of functions u which, together with all their partial derivatives $\partial^\alpha u$ of orders $|\alpha| \leq k$, are continuous on Ω.

Let $C^k(\overline{\Omega})$ be the space of functions $u \in C^k(\Omega)$ which, together with all their partial derivatives $\partial^\alpha u$ of orders $|\alpha| \le k$, have continuous extensions to $\overline{\Omega}$. We denote

$$C^0(\Omega) = C(\Omega), \quad C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega), \quad C^0(\overline{\Omega}) = C(\overline{\Omega}), \quad C^\infty(\overline{\Omega}) = \bigcap_{k=0}^{\infty} C^k(\overline{\Omega}).$$

We denote the $C^k(\overline{\Omega})$-norm $\| \cdot \|_{C^k(\overline{\Omega})}$ by

$$\|u\|_{C^k(\overline{\Omega})} = \max_{|\alpha| \le k} \max_{x \in \overline{\Omega}} |\partial^\alpha u(x)| \quad \text{for} \quad u \in C^k(\overline{\Omega}).$$

For any non-negative integer k, if Ω is bounded, the space $C^k(\overline{\Omega})$ is a Banach space with the norm $\| \cdot \|_{C^k(\overline{\Omega})}$.

For $u \in C^0(\Omega)$, a closure of the set $\{x \in \Omega \mid u(x) \neq 0\}$ in Ω is called a *support* of u. We denote the support of u by $\operatorname{supp} u$. Let $C_0^\infty(\Omega)$ be a set of functions $u \in C^\infty(\Omega)$ such that $\operatorname{supp} u$ is a compact subset of Ω.

Now, we define a convergence in the space $C_0^\infty(\Omega)$. A sequence $\varphi_n \in C_0^\infty(\Omega)$ converges to $\varphi \in C_0^\infty(\Omega)$ if there exists a compact set $K \subset \Omega$ such that

$$\operatorname{supp} \varphi_n \subset K \quad \text{and} \quad \|\varphi_n - \varphi\|_{C^k(\overline{\Omega})} \to 0 \quad \text{as} \quad n \to \infty$$

for all $k \in \mathbf{Z}_{\ge 0}$. We denote by $\mathcal{D}(\Omega)$ the linear space $C_0^\infty(\overline{\Omega})$ with such convergence.

A linear functional u on $\mathcal{D}(\Omega)$ is called a *distribution* on Ω if the convergence $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ implies that $\langle u, \varphi_n \rangle \to \langle u, \varphi \rangle$. We denote the space of distributions by $\mathcal{D}'(\Omega)$.

The derivative $\mathcal{D}^\alpha u$ of $u \in \mathcal{D}'(\Omega)$ is also a distribution on Ω and defined by

$$\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle \quad \text{for all} \quad \varphi \in \mathcal{D}(\Omega).$$

Let Ω be a bounded domain of $\mathbf{R}^d$. For any non-negative integer m, the Sobolev space $H^m(\Omega)$ is the space of $u \in \mathcal{D}'(\Omega)$ such that $\mathcal{D}^\alpha u \in L^2(\Omega)$ for $|\alpha| \le m$ with the $H^m(\Omega)$-norm

$$\|u\|_{H^m(\Omega)} = \left\{ \sum_{|\alpha| \le m} \|\partial^\alpha u\|_{L^2(\Omega)}^2 \right\}^{1/2} \quad \text{for} \quad m \ge 1, \quad \|u\|_{H^0(\Omega)} = \|u\|_{L^2(\Omega)}.$$

Let $s = m + \sigma$, where m is a non-negative integer and $0 < \sigma < 1$. The Sobolev space $H^s(\Omega)$ is the space of functions $u \in H^m(\Omega)$ such that

$$\|u\|_{H^s(\Omega)} = \left\{ \|u\|_{H^m(\Omega)}^2 + \sum_{|\alpha| = m} \iint_{\Omega \times \Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{d+2\sigma}} dxdy \right\}^{1/2} < \infty.$$

The Sobolev space $H^s(\Omega)\,(s \ge 0)$ is a Hilbert space with the $H^s(\Omega)$-norm $\| \cdot \|_{H^s(\Omega)}$.

For any positive integer m, let $H_0^m(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ by $\| \cdot \|_{H^m(\Omega)}$.

Contents

Part I

Introduction

Chapter 1

Non-scattering energy and interior transmission eigenvalue

In this thesis, we study the interior transmission eigenvalue problem (the **ITE** problem for short) on two compact Riemannian manifolds with common smooth boundary. The ITE problem arises from scattering theory, in particular, from non-scattering phenomena (see e.g., Vesalainen [31], [32] for quantum and acoustic scattering).

In this chapter, let us recall some basic notions of scattering theory and non-scattering phenomena in Euclidean space. We also state the ITE problem on a compact subset of the Euclidean space. Moreover, we introduce some preceding studies corresponding to the distribution of the eigenvalues for the ITE problem.

We now consider the case of time-harmonic acoustic scattering problem on d-dimensional Euclidean space $\mathbf{R}^d$ for $d \geq 2$ with compactly supported penetrable inhomogeneous medium. We assume that there exists a bounded domain $D \subset \mathbf{R}^d$ with smooth boundary ∂D such that the support of the penetrable inhomogeneous medium is given by D. We denote by ν the outward normal vector to the boundary ∂D and by I_d the $d \times d$ identity matrix. We introduce two functions in $\mathbf{R}^d$ represented by a $d \times d$ matrix valued function A with bounded entries such that $A(x) \equiv I_d$ outside D and by a bounded scalar valued function n such that $n(x) \equiv 1$ outside D.

We deal with a stationary acoustic total wave u satisfying the perturbed Helmholtz equation

$$(-\nabla \cdot A\nabla - k^2 n)u = 0 \quad \text{in} \quad \mathbf{R}^d \tag{1.1}$$

where $k > 0$ is the wave number, $\nabla\cdot$ and ∇ are the divergence operator and gradient operator on $\mathbf{R}^d$, respectively. In addition, we assume that u satisfies

$$(u)^+ = (u)^-, \quad (\partial_\nu u)^+ = (\partial_{\nu_A} u)^- \quad \text{on} \quad \partial D.$$

Here, for a generic function ϕ on $\mathbf{R}^d$, we denote

$$(\phi)^\pm(x) := \lim_{h \to 0} \phi(x \pm h\nu(x)), \quad h > 0, \ x \in \partial D$$

and

$$\partial_{\nu_A}\phi := (A\nabla\phi, \nu), \quad \partial_\nu \phi := \partial_{\nu_{I_d}}\phi.$$

Then we consider that a solution to (1.1) is written in the form

$$u = u^i + u^s.$$

Here, u^i is an incident wave satisfying the free Helmholtz equation

$$(-\Delta - k^2)u^i = 0 \quad \text{in} \quad \mathbf{R}^d \tag{1.2}$$

and u^s is the corresponding scattered wave satisfying

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{1.3}$$

where (1.3) is assumed to hold uniformly in $\hat{x} = x/|x|$. Here, Δ is the Laplacian on $\mathbf{R}^d$ and $r = |x| := (x_1^2 + \cdots + x_d^2)^{1/2}$ for $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$. The condition (1.3) is called the *Sommerfeld radiation condition* which guarantees that the scattered wave is outgoing. Now let $u^i = u^i(x)$ be a plane wave $e^{ik(x,\omega)}$ with an incident direction ω in the $(d-1)$-dimensional sphere $S^{d-1} = \{x \in \mathbf{R}^d \mid |x| = 1\}$ and a fixed positive wave number k (or a fixed positive energy k^2). Using (1.1), (1.2) and the definition of the functions A and n, we can easily show that the corresponding scattered wave u^s satisfies the Helmholtz equation

$$(-\Delta - k^2)u^s = 0 \quad \text{in} \quad \mathbf{R}^3 \setminus D.$$

Such a solution u^s satisfying the Sommerfeld radiation condition (1.3) has the asymptotic behavior of an outgoing spherical wave

$$u^s(x) = C(k)\frac{e^{ikr}}{r^{\frac{d-1}{2}}}a(k;\omega,\hat{x}) + o\left(\frac{1}{r^{\frac{d-1}{2}}}\right) \quad \text{as} \quad r \to \infty$$

for some positive constant $C(k)$ depending on k (see e.g., [11] Theorem 2.6). Here, $\hat{x}$ is the scattered direction of u^s and the function $a(k;\omega,\hat{x})$ is called the scattering amplitude. Let $\hat{F}(k)$ be the integral operator on the space of square integrable functions on S^{d-1} with the integral kernel $a(k;\omega,\hat{x})$, more precisely

$$(\hat{F}(k)\phi)(\hat{x}) = \int_{S^{d-1}} a(k;\omega,\hat{x})\phi(\omega)dS^{d-1}$$

where ϕ is a square integrable function on S^{d-1} and the symbol dS^{d-1} denotes the surface element on S^{d-1}. Then the S-matrix is given by $\hat{S}(k) = 1 - 2\pi i\hat{F}(k)$. If one is an eigenvalue of $\hat{S}(k)$ for $k > 0$, then k is called a *non-scattering wave number* (or k^2 is called a *non-scattering energy*). We denote the set of all non-scattering wave numbers by σ_N. For $k \in \sigma_N$, the scattering amplitude of the corresponding scattered wave $u^s = u^s(k;x)$ vanishes. Then u^s also vanishes outside D from the Rellich type uniqueness theorem (see e.g., [24], [30]). Hence, if k is a non-scattering wave number, there exists a non-trivial solution of the boundary value problem for a system of Helmholtz equations for u^i and u of the form

$$(-\Delta - k^2)u^i = 0 \quad \text{in} \quad \mathbf{R}^d; \tag{1.4}$$

$$(-\nabla \cdot A\nabla - k^2 n)u = 0 \quad \text{in} \quad D; \tag{1.5}$$

$$u^i - u = 0 \quad \text{on} \quad \partial D; \tag{1.6}$$

$$\partial_\nu u^i - \partial_{\nu_A} u = 0 \quad \text{on} \quad \partial D. \tag{1.7}$$

Conversely, we suppose that (1.4)–(1.7) depending on a positive constant k has a non-trivial solution. Putting $u = u^i$ outside D, we can extend u as a solution of (1.1). Letting $u^s = u - u^i$, we can show that the scattering amplitude corresponding to u^s identically vanishes. Hence, k is also a non-scattering wave number. Therefore, k is a non-scattering wave number if and only if there exists a nontrivial solution of the boundary value problem (1.4)–(1.7).

In order to study the spectral properties of non-scattering wave numbers, we consider the boundary value problem for a system of Helmholtz equations for unknown functions v and w of the form

$$(-\Delta - k^2)v = 0 \quad \text{in} \quad D; \tag{1.8}$$
$$(-\nabla \cdot A\nabla - k^2 n)w = 0 \quad \text{in} \quad D; \tag{1.9}$$
$$v - w = 0 \quad \text{on} \quad \partial D; \tag{1.10}$$
$$\partial_\nu v - \partial_{\nu_A} w = 0 \quad \text{on} \quad \partial D. \tag{1.11}$$

The above boundary value problem is called the *interior transmission eigenvalue problem* (the **ITE** problem for short). If there exists a non-trivial solution of the ITE problem (1.8)–(1.11) for some $k \in \mathbf{C}$, we call such a complex number k an *interior transmission eigenvalue* (an **ITE** for short). We denote the set of all ITEs by σ_I. We note that the ITE problem (1.8)–(1.11) is an eigenvalue problem for a non-selfadjoint operator. Therefore, ITEs do not necessarily belong to $\mathbf{R}$. Also note that from the definition of σ_N and σ_I, the inclusion relation $\sigma_N \subset \sigma_I$ holds.

We are interested in detailed properties of non-scattering wave numbers. However, it is difficult to directly deal with the S-matrix having a one eigenvalue or the boundary value problem (1.4)–(1.7). Currently, there are only a few results in some special case as follows.

Case.1. Spherically symmetric media. The ITE problem was first studied by Colton and Monk [12]. In particular, they dealt with the ITE problem on the unit ball. Let $B = \{x \in \mathbf{R}^d \mid |x| < 1\}$ and n_B be a smooth function on $[0, \infty)$ such that $n_B(r) \not\equiv 1$ on $[0, 1)$ and $n_B(r) \equiv 1$ on $[1, \infty)$. They considered the ITE problem on B with $A(x) \equiv I_d$ and $n(x) = n_B(r)$ of the form

$$(-\Delta - k^2)v = 0 \quad \text{in} \quad B; \tag{1.12}$$
$$(-\Delta - k^2 n_B(r))w = 0 \quad \text{in} \quad B; \tag{1.13}$$
$$v - w = 0 \quad \text{on} \quad S^{d-1}; \tag{1.14}$$
$$\frac{\partial v}{\partial r} - \frac{\partial w}{\partial r} = 0 \quad \text{on} \quad S^{d-1}. \tag{1.15}$$

In this case, Colton and Monk proved that the relation $\sigma_N = \sigma_I$ holds. Indeed, we assume that k is an ITE for the ITE problem (1.12)–(1.15). Let a pair of functions (v, w) be a solution of the ITE problem (1.12)–(1.15) with $k \in \sigma_I$. Using the spherically harmonics and the spherically Bessel functions, we can show that the function $v = v(k)$ can be extended outside B as a solution of (1.4). Hence, we can reduce the ITE problem (1.12)–(1.15) to the boundary value problem (1.4)–(1.7) with $A(x) \equiv I_d$, $n(x) = n_B(r)$ and $D = B$. Therefore, we can conclude that k is a non-scattering wave number.

Case.2. Corner scatterer. Blåsten, Päivärinta and Sylvester [2] dealt with the ITE problem on a rectangle. Let R be a d-dimensional rectangle, χ be a characteristic function of R and ϕ be a smooth function on $\mathbf{R}^d$ such that $\phi \neq 0$ on a corner of R. We put $n_R = \chi_R \phi + 1$. They considered the ITE problem on R with $A(x) \equiv I_d$ and $n = n_R$ of the form

$$(-\Delta - k^2)v = 0 \quad \text{in} \quad R; \tag{1.16}$$
$$(-\Delta - k^2 n_R)w = 0 \quad \text{in} \quad R; \tag{1.17}$$
$$v - w = 0 \quad \text{on} \quad \partial R; \tag{1.18}$$
$$\partial_\nu v - \partial_\nu w = 0 \quad \text{on} \quad \partial R. \tag{1.19}$$

In this case, Blåsten, Päivärinta and Sylvester proved that $\sigma_I \setminus \sigma_N \neq \emptyset$ holds.

From the above, we understand that ITEs relate to non-scattering wave numbers. Hence, as the first step, we will focus on the distribution of ITEs.

Chapter 2

Distribution of interior transmission eigenvalues

The two functions A and n is appeared in the ITE problem (1.8)–(1.11). Now we study the two particular cases of ITE problems as follows.

Definition 2.0.1. If A is identically equal to I_d (resp. is not identically equal to I_d), the boundary value problem (1.8)–(1.11) is called the *ITE problem for isotropic media* (resp. is called the *ITE problem for anisotropic media*).

The purpose of the following section is to provide a survey of the preceding studies of these ITE problems which employs different type of mathematical techniques.

2.1 The interior transmission eigenvalue problem for isotropic media

We consider the ITE problem for isotropic media, more precisely we find $(v, w) \in L^2(D) \times L^2(D)$ such that $v - w \in H_0^2(D)$ satisfying

$$(-\Delta - k^2)v = 0 \quad \text{in} \quad D; \tag{2.1}$$

$$(-\Delta - k^2 n)w = 0 \quad \text{in} \quad D; \tag{2.2}$$

$$v - w = 0 \quad \text{on} \quad \partial D; \tag{2.3}$$

$$\partial_\nu v - \partial_\nu w = 0 \quad \text{on} \quad \partial D. \tag{2.4}$$

In this section, if there exists a non-trivial solution $(v, w) \in L^2(D) \times L^2(D)$ of the ITE problem (2.1)–(2.4) satisfying $v - w \in H_0^2(D)$ for some $k \in \mathbf{C}$, we call such a complex number k an ITE.

For the particular case of a spherically stratified medium, the following result of the distribution of ITEs is well-known (see e.g., Theorem 3.1 in [9]).

Theorem 2.1.2. *Assume that $n_B \in C^2([0,1])$, $\mathrm{Im}\,(n_B(r)) = 0$ and either $n_B(1) \neq 1$ or $n_B(1) = 1$ and $\int_0^1 \sqrt{n_B(\rho)}d\rho \neq 1$. Then there exists an infinite discrete set of transmission for the ITE problem (1.12)–(1.15). Furthermore the set of all transmission eigenvalues is discrete.*

The existence of ITEs for non-spherically stratified media remained open problem until Sylvester and Päivärinta [23]. They proved the existence of at least one ITEs. Since [23], the existence of ITEs for general case has been widely studied. The existence of infinitely many ITEs for non-spherically stratified media was proven in [7] under certain assumptions on n as follows.

Theorem 2.1.3 (Cakoni-Gintides-Haddar [7], Theorem 2.5)**.** *Let $n \in L^\infty(D)$ satisfy either one of the following assumptions :*

(1) $1+\alpha \le \inf_D(n) \le n(x) \le \sup_D(n) < \infty \quad x \in D$

(2) $0 < \inf_D(n) \le n(x) \le \sup_D(n) < 1-\beta \quad x \in D$

for some constant $\alpha, \beta > 0$. Then there exists an infinite discrete set of real ITEs with only possible accumulation point at $+\infty$.

To prove this, they used the *variational form method.* We assume that $\mathrm{Im}\,(n) = 0$ and that $n-1$ does not change sign and is bounded away from zero inside D. Put $\lambda := k^2$. Then we rewrite the ITE problem (2.1)–(2.4) as a forth order equation of the form

$$(\Delta + \lambda n)\frac{1}{n-1}(\Delta+\lambda)u = 0 \quad \text{for} \quad u = w - v \in H_0^2(D) \tag{2.5}$$

which in variational form, after integration by parts, is formulated as finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}(\Delta+\lambda)u(\Delta+\lambda n)\overline{v}dx = 0 \quad \text{for all} \quad v \in H_0^2(D). \tag{2.6}$$

Using the Riesz representation theorem, we define the bounded linear self-adjoint operators $\mathcal{A}(\lambda) : H_0^2(D) \to H_0^2(D)$ and $\mathcal{B} : H_0^2(D) \to H_0^2(D)$ by

$$(\mathcal{A}(\lambda)u, v)_{H^2(D)} = \int_D \frac{1}{n-1}(\Delta+\lambda)u(\Delta+\lambda)\overline{v}dx + \lambda^2 \int_D u\overline{v}dx$$

and

$$(\mathcal{B}u, v) = \int_D (\nabla u, \nabla v)dx$$

for all $u, v \in H_0^2(D)$, respectively. Summarizing the above argument, we obtain that k is an ITE if and only if the kernel of the operator $\mathcal{A}(\lambda) - \lambda\mathcal{B}$ has non-trivial kernel. In [7], to prove the existence of an infinite countable set of ITEs, Cakoni, Gintides and Haddar dealt with the generalized min-max principle for the operators $\mathcal{A}(\lambda)$ and $\mathcal{B}$ (see e.g. [8], [7]). This argument does not necessarily need the regularity of the function n. Hence, it is sufficient to assume that n is in $L^\infty(D)$. These method is called the *variational method.*

In Theorem 2.1.3, $n-1$ is either positive or negative and bounded away form zero inside D. However, Sylvester [28] proved the discreteness of ITEs under more relaxed assumptions on n such that $n-1$ or $1-n$ is positive only in a neighborhood of ∂D as follows.

Theorem 2.1.4 (Sylvester [28], Theorem 1.2)**.** *Assume that there exist constants $\theta \in (-\pi/2, \pi/2)$ and $n_*, n^* \in \mathbf{R}$ with $1 < n_* \le n^*$ such that*

(1) $\mathrm{Re}\left(e^{i\theta}(n(x)-1)\right) > n_* - 1$ *in some neighborhood of ∂D, or that $n(x)$ is real valued in all of D, and satisfies $n(x) \le 2 - n_*$ in some neighborhood of ∂D;*

(2) $|n(x) - 1| < n^* - 1$ *in all of D;*

(3) $\mathrm{Re}\,(n(x)) \ge \delta > 0$ *in all of D.*

Then there exists a (possibly empty) discrete set of ITEs.

On the other hand, Lakshtanov and Vainberg [21] proved that there exists an infinite set of ITEs under assumptions on n only on ∂D, more precisely

$$n(x) - 1 \not\equiv 0 \quad \text{for} \quad x \in \partial D, \tag{2.7}$$

or

$$n(x) - 1 \equiv 0, \quad \partial_\nu n(x) \not\equiv 0 \quad \text{for} \quad x \in \partial D. \tag{2.8}$$

They also proved a result on the Weyl type lower bound on counting function of ITEs as follows.

Theorem 2.1.5 (Lakshtanov-Vainberg [21], Theorem 1.3 and Theorem 1.4)**.** *We assume that n is real valued in all of $\overline{D}$. Let one of the conditions* (2.7) *or* (2.8) *holds. There exists a discrete set of ITEs with only possible accumulation point at infinity. Moreover, the set of positive ITEs is infinite and these counting function N_T has the following lower estimate*

$$N_T(\lambda) \geq C\lambda^{d/2} + O(\lambda^{(d-1)/2}) \quad \textit{as} \quad \lambda \to \infty.$$

Here $\lambda = k^2$ and C depend only on n and D.

They characterized an ITE by the Dirichlet-to-Neumann operators for the equations (2.1) and (2.2) and analyzed these operators by using pseudo-differential calculus. These method called the *Dirichlet-to-Neumann map method*. More precisely, we introduce this Dirichlet-to-Neumann map method in Part IV. We extend this method to the case of ITE problem on compact manifolds corresponding to the ITE problem for isotropic media.

2.2 The interior transmission eigenvalue problem for anisotropic media

In this section, we discuss the ITE problem for anisotropic media, i.e., the ITE problem (1.8)–(1.11) with $A(x) \not\equiv I_d$. In the case $n(x) \equiv 1$, letting $N(x) = A(x)^{-1}$, we can rewrite the ITE problem (1.8)–(1.11) as the ITE problem for vector valued functions $\mathbf{v} = \nabla v$ and $\mathbf{w} = \nabla w$ of the form

$$\nabla(\nabla \cdot \mathbf{v}) + k^2\mathbf{v} = 0 \quad \text{in} \quad D; \tag{2.9}$$

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 N\mathbf{w} = 0 \quad \text{in} \quad D; \tag{2.10}$$

$$(\mathbf{v}, \nu) - (\mathbf{w}, \nu) = 0 \quad \text{on} \quad \partial D; \tag{2.11}$$

$$\nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{w} = 0 \quad \text{on} \quad \partial D. \tag{2.12}$$

(2.9)–(2.12) is similar to the ITE problem for isotropic media. Therefore, using similar approach to the analysis of the ITE problem for isotropic media, we can obtain similar results of the distribution of ITEs (see e.g. [6])

In the case $n(x) \not\equiv 1$, we employ a different approach from the ITE problem for isotropic media or for anisotropic media with $n(x) \equiv 1$. The discreteness of ITEs was proven in [13] under some assumptions on A and n only on a neighborhood of ∂D. They were also given a result on the location of transmission eigenvalues. $\mathcal{V}$ denotes a neighborhood of ∂D inside D. We set

$$A_* := \inf_{x\in\mathcal{V}} \inf_{\xi\in S^{d-1}} (A(x)\xi, \xi) > 0, \quad A^* := \sup_{x\in\mathcal{V}} \sup_{\xi\in S^{d-1}} (A(x)\xi, \xi) < \infty,$$

$$n_* := \inf_{x\in\mathcal{V}} n(x) > 0, \quad n^* := \sup_{x\in\mathcal{V}} n(x) < \infty.$$

Theorem 2.2.6 (Bonnet-Ben Dhia-Chesnel-Haddar [13], Theorem 4.2 and Theorem 5.1)**.** *Assume that either*

$$A(x) \leq A^* I_d < I_d \quad and \quad n(x) \leq n^* < 1 \quad a.e. \quad x \in \mathcal{V},$$

or

$$A(x) \geq A_* I_d > I_d \quad and \quad n(x) \geq n_* > 1 \quad a.e. \quad x \in \mathcal{V}.$$

Then the set of transmission eigenvalues is discrete in $\mathbf{C}$*. Moreover, there exist two positive constants* ρ *and* δ *such that if* $k \in \mathbf{C}$ *satisfies* $|k| > \rho$ *and* $|\mathrm{Re}\, k| < \delta |\mathrm{Im}\, k|$*, then* k *is not an ITE.*

They rewritten the ITE problem (1.8)–(1.11) for anisotropic media as a variational form which is different from (2.6). However, a sesquilinear form appeared in this variational form has non-ellipticity. Using an isomorphism T, we can avoid this difficulty. Such a method is called the T*-coercivity method.* More precisely, we introduce this T-coercivity method in Part III. We extend this method to the case of ITE problem on compact manifolds corresponding to the ITE problem for anisotropic media.

On the other hand, the existence of ITEs was proven in [10].

Theorem 2.2.7 (Cakoni-Kirsch [10], Theorem 4.8)**.** *Assume that either* $A_* > 1$ *and* $n^* < 1$*, or* $A^* < 1$ *and* $n_* > 1$*. Then there exists an infinite discrete set of real ITEs with only possible accumulation point at* $+\infty$*.*

Part II

Preliminaries

Chapter 3

Notation of manifold and function spaces

For $d \geq 2$, let M be a d-dimensional compact, connected and oriented Riemannian manifold endowed with a smooth Riemannian metric g and with a smooth boundary ∂M.

For a local coordinates $x = (x_1, \ldots, x_d) \in M$ and a function f defined on a neighborhood of x, let

$$\frac{\partial f}{\partial x_i}(x) \quad (i = 1, \ldots d)$$

be a directional derivative along x_i at x. We define a operator $(\partial_i)_x \, (i = 1, \ldots, d)$ by

$$(\partial_i)_x : f \mapsto \frac{\partial f}{\partial x_i}(x).$$

The vector space spanned by $(\partial_1)_x, \ldots, (\partial_d)_x$ is called a tangent space of M at x and is denoted by T_xM. An element of T_xM is called a tangent vector of M at $x \in M$. Hence, we write tangent vectors X_x on T_xM as $X_x = \sum_{i=1}^{d} X_i(x)(\partial_i)_x$. Here, $X_1, \ldots, X_d$ are smooth functions on M. We denote the inner product and the norm on T_xM by

$$(X_x, Y_x)_g = \sum_{i,j=1}^{d} g_{ij}(x) X_i(x) \overline{Y_j(x)}, \quad |X_x|_g = \sqrt{(X_x, X_x)_g},$$

for $X_x, Y_x \in T_x(M)$ and smooth functions $X_i, Y_i \, (i = 1, \ldots, d)$, respectively. We call $TM = \cup_{x \in M} T_xM$ the tangent bundle of M. A vector field X on M is defined by assigning each point $x \in M$ to the tangent vector $X_x \in T_xM$ as

$$X : M \ni x \mapsto \{X_x\}_{x \in M} \in TM.$$

The space of all smooth vector fields is denoted by $\Gamma(TM)$. We define the vector field $\partial_i \, (i = 1, \ldots, d)$ by

$$\partial_i : M \ni x \mapsto \{(\partial_i)_x\}_{x \in M} \in TM.$$

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, we write $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. For $\xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$, we use the similar manner.

For $x \in M$, the dual space T_x^*M of T_xM is called a cotangent space of M at $x \in M$ and its elements are called cotangent vectors. We call $T^*M = \cup_{x \in M} T_x^*M$ the cotangent bundle on M.

A differential 1-form ω on M is defined by assigning each point $x \in M$ to the cotangent vector $\omega_x \in T_x^* M$ as

$$\omega : M \ni x \mapsto \{\omega_x\}_{x \in M} \in T^* M.$$

The space of all smooth differential 1-forms is denoted by $\Omega^1(M)$. Let $(dx_1)_x, \ldots, (dx_d)_x$ be a dual basis of $(\partial_1)_x, \ldots, (\partial_d)_x$. We define the differential 1-form dx_i $(i = 1, \ldots, d)$ by

$$dx_i : M \ni x \mapsto \{(dx_i)_x\}_{x \in M}.$$

For $\omega_1, \omega_2 \in \Omega^1(M)$, the exterior product of differential forms is defined by

$$(\omega_1 \wedge \omega_2)(X_1, X_2) = \omega_1(X_1)\omega_2(X_2) - \omega_2(X_1)\omega_1(X_2)$$

where X_1, X_2 are arbitrary vector fields on M.

We fix local coordinates $x = (x_1, \ldots, x_d)$ of M. We regard $g = g(x)$ as a positive-definite symmetric $d \times d$-matrix valued function and we write $g(x) = (g_{ij}(x))_{i,j=1}^d$. We denote the inverse matrix of $g(x)$ by $g(x)^{-1} = (g^{ij}(x))_{i,j=1}^d$. The determinant of $g(x)$ and the volume element on M are denoted by $G(x)$ and $dV_g := \sqrt{G} dx = \sqrt{G} dx_1 \wedge \cdots \wedge dx_d$, respectively. A symbol dS and dS_g denote the surface elements on ∂M induced by dx and dV_g, respectively.

The space of all infinitely differentiable functions on M, $\overline{M}$ and ∂M are denoted by $C^\infty(M)$, $C^\infty(\overline{M})$ and $C^\infty(\partial M)$, repectively. Let $\Delta_g : C^\infty(M) \to C^\infty(M)$ and $\nabla_g : C^\infty(M) \to \Gamma(TM)$ be the Laplace-Beltrami operator and the gradient operator on M, respectively. In local coordinates on M, those operators are written in the form

$$\Delta_g u = G^{-1/2} \sum_{i,j=1}^d \partial_i (g^{ij} G^{1/2} \partial_j u), \quad (\nabla_g u)_x = \sum_{i,j=1}^d g^{ij} (\partial_i u)(\partial_j)_x$$

for all $u \in C^\infty(M)$, respectively. Here, $(\nabla_g u)_x$ denotes the corresponding tangent vector in $T_x M$.

For measurable functions u on M and f on ∂M, we define

$$\|u\|_{L^\infty(M)} = \inf\{C_1 \geq 0 \mid |u(x)| \leq C_1 \text{ a.e.}, x \in M\},$$
$$\|f\|_{L^\infty(\partial M)} = \inf\{C_2 \geq 0 \mid |f(x)| \leq C_2 \text{ a.e.}, x \in \partial M\},$$

respectively. We define $L^\infty(M)$ and $L^\infty(\partial M)$ by the space of all measurable functions u on M such that $\|u\|_{L^\infty(M)} < \infty$ and the space of all measurable functions f on ∂M such that $\|f\|_{L^\infty(\partial M)} < \infty$, respectively. We denote the $L^2(M)$-inner product and the $L^2(M)$-norm on $C^\infty(M)$ and the $L^2(\partial M)$-inner product and the $L^2(\partial M)$-norm on $C^\infty(\partial M)$ by

$$(u, v)_M = \int_M u \overline{v} dV_g, \quad \|u\|_M = \sqrt{(u, u)_M}, \quad u, v \in C^\infty(M),$$
$$(f, g)_{\partial M} = \int_{\partial M} f \overline{g} dS, \quad \|f\|_{\partial M} = \sqrt{(f, f)_{\partial M}}, \quad f, g \in C^\infty(\partial M),$$

respectively. Then the completion of $C^\infty(M)$ by $\|\cdot\|_M$ and the completion of $C^\infty(\partial M)$ by $\|\cdot\|_{\partial M}$ are denoted by $L^2(M)$ and $L^2(\partial M)$, respectively. For a strictly positive function $\eta \in L^\infty(M)$, we denote the $L^2(M, \eta)$-inner product and the $L^2(M, \eta)$-norm on $C^\infty(M)$ by

$$(u, v)_{L^2(M,\eta)} = (\eta u, v)_M, \quad \|u\|_{L^2(M,\eta)} = \sqrt{(u, u)_{L^2(M,\eta)}}, \quad u, v \in C^\infty(M),$$

respectively. Then the completion of $C^\infty(M)$ by $\|\cdot\|_{L^2(M,\eta)}$ is denoted by $L^2(M,\eta)$. We denote the $L^2(TM)$-inner product and the $L^2(TM)$-norm on $\Gamma(TM)$ by

$$\begin{aligned}(X,Y)_{TM} &= \int_M (X_x, Y_x)_g dV_g, \\ \|X\|_{TM} &= \sqrt{(X,X)_{TM}},\end{aligned} \qquad X, Y \in \Gamma(TM),$$

respectively. Then the completion of $\Gamma(TM)$ by $\|\cdot\|_{TM}$ is denoted by $L^2(TM)$. We denote the $H^1(M)$-inner product and the $H^1(M)$-norm on $C^\infty(M)$ by

$$\begin{aligned}(u,v)_{H^1(M)} &= (\nabla_g u, \nabla_g v)_{TM} + (u,v)_M, \\ \|u\|_{H^1(M)} &= \sqrt{(u,u)_{H^1(M)}},\end{aligned} \qquad u, v \in C^\infty(M),$$

respectively. Then the completion of $C^\infty(M)$ by $\|\cdot\|_{H^1(M)}$ is denoted by $H^1(M)$. We denote the Christoffel symbol ${}_g\Gamma^k_{ij}$ by

$${}_g\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{d} g^{kl}\left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right).$$

For $u \in C^\infty(M)$, $\nabla_g^2 u$ denotes the 2nd covariant derivative of u and the components of $\nabla_g^2 u$ in local coordinates are given by

$$(\nabla_g^2 u)_{ij} = \partial_i \partial_j u - \sum_{k=1}^{d} {}_g\Gamma^k_{ij}\partial_k u.$$

Put

$$|\nabla_g^2 u|^2 = \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}\sum_{l=1}^{d} g^{ij}g^{kl}(\nabla_g^2 u)_{ik}(\nabla_g^2 u)_{jl}.$$

We denote the $H^2(M)$-norm on $C^\infty(M)$ by

$$\|u\|_{H^2(M)} = \left(\int_M |\nabla_g^2 u|^2 dV_g + \|u\|^2_{H^1(M)}\right)^{1/2}.$$

Then the completion of $C^\infty(M)$ by $\|\cdot\|_{H^2(M)}$ is denoted by $H^2(M)$.

Let N be a positive integer. For each finite open covering $\{U_j\}_{j=1}^N$ of ∂M, there exists a partition of unity $\{\rho_j\}_{j=1}^N$ such that $\rho_j \in C^\infty(\partial M)$, $0 \le \rho_j \le 1$ and $\mathrm{supp}(\rho_j) \subset U_j$ for $j = 1, 2, \ldots, N$ and $\sum_{j=1}^N \rho_j \equiv 1$ on ∂M. Let $\varphi_j : U_j \ni x \mapsto \varphi_j(x) = y' = (y_1, \ldots, y_{d-1}) \in \mathbf{R}^{d-1}$ be a diffeomorphism from U_j onto $V_j := \varphi(U_j)$ such that $\varphi_j(U_j \cap \partial M) \subset \{y = (y', y_d) \mid y_d = 0\}$.

For $s \ge 0$, let $H^s(\partial M)$ be a Sobolev space of functions u such that, in local coodinates $y = (y_1, \ldots, y_{d-1})$, we have $\rho_j u \in H^s(\varphi_j(U_j \cap \partial M))$. The norm in $H^s(\partial M)$ is given by

$$\|u\|_{H^s(\partial M)} = \left\{\sum_{j=1}^{N} \|\rho_j u\|^2_{H^s(\varphi_j(U_j\cap\partial M))}\right\}^{1/2}.$$

The space $H^s(\partial M)$ is a Hilbert space with the $H^s(\partial M)$-norm.

We denote the outward normal derivative on ∂M with respect to g by ∂_ν. We define the trace map

$$\gamma_0, \gamma_1 : C^\infty(\overline{M}) \to C^\infty(\partial M)$$

by the formula

$$\gamma_0 u = u|_{\partial M}, \quad \gamma_1 u = \partial_\nu u|_{\partial M}.$$

Then the trace map $\gamma_0, \gamma_1 : C^\infty(\overline{M}) \to C^\infty(\partial M)$ extend uniquely to continuous linear maps

$$\gamma_0 : H^2(M) \to H^{3/2}(\partial M), \quad \gamma_1 : H^2(M) \to H^{1/2}(\partial M).$$

For the sake of simplicity, we often simply write $\gamma_0 u$ and $\gamma_1 u$ as u and $\partial_\nu u$ on ∂M, respectively.

Part III

A locally anisotropic interior transmission eigenvalue problem

Chapter 4

A locally anisotropic interior transmission eigenvalue problem

4.1 Our setting and main results I

To begin with, let us explain our setting in this part. For $d \geq 2$, let M_1 and M_2 be d-dimensional connected and compact smooth oriented manifolds endowed with Riemannian metrics g_1 and g_2 and with smooth boundaries ∂M_1 and ∂M_2, respectively. Throughout this thesis, we assume that

$$\begin{aligned} &\cdot\ M_1 \text{ and } M_2 \text{ have a common boundary } \Gamma := \partial M_1 = \partial M_2. \\ &\cdot\ \Gamma \text{ is a disjoint union of a finite number of connected and closed compon-} \\ &\ \ \text{ents } \Gamma_1, \ldots, \Gamma_N, \text{ namely } \Gamma = \amalg_{j=1}^{N} \Gamma_j. \end{aligned} \tag{A}$$

In addition, we assume that

$$\begin{aligned} &\cdot\ \text{Let } \Sigma := M_1 \cap M_2. \text{ Then there exist connected neighborhoods } \Sigma_j \text{ of} \\ &\ \ \Gamma_j (1 \leq j \leq N) \text{ such that } \Sigma \text{ is written as the disjoint union of } \Sigma_1, \ldots, \Sigma_N, \\ &\ \ \text{namely, } \Sigma = \amalg_{j=1}^{N} \Sigma_j \text{ (see Figure 4.1).} \end{aligned} \tag{A-1}$$

and

$$g_1(x) \neq g_2(x) \quad \text{for some} \quad x \in \Sigma. \tag{A-2}$$

Here, we note that we do not necessarily assume that M_1 and M_2 are diffeomorphic.

In this section, we assume (A), (A-1) and (A-2). For functions $n_l \in L^\infty(M_l)\,(l = 1, 2)$ and $\zeta \in L^\infty(\Gamma)$ and for $k \in \mathbf{C}$, we consider a boundary value problem for a system of Helmholtz equations for unknown functions u_1 and u_2 of the form

$$(-\Delta_{g_1} - k^2 n_1) u_1 = 0 \quad \text{in} \quad M_1; \tag{4.1}$$

$$(-\Delta_{g_2} - k^2 n_2) u_2 = 0 \quad \text{in} \quad M_2; \tag{4.2}$$

$$u_1 - u_2 = 0 \quad \text{on} \quad \Gamma; \tag{4.3}$$

$$\sqrt{G_1} \partial_{\nu,1} u_1 - \sqrt{G_2} \partial_{\nu,2} u_2 = \zeta u_1 \quad \text{on} \quad \Gamma. \tag{4.4}$$

Here, in the above, $\partial_{\nu,1}$ and $\partial_{\nu,2}$ denote the outward normal derivatives on Γ with respect to g_1 and g_2, respectively. We call the above boundary value problem a *locally anisotropic interior transmission eigenvalue problem.*

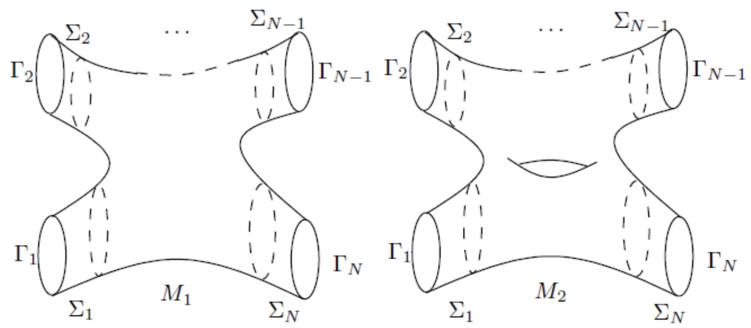

Figure 4.1: Examples of M_1 and M_2 with common boundary $\Gamma = \amalg_{j=1}^N \Gamma_j$.

Remark 4.1.8. In scattering theory, the above functions $n_l\,(l = 1, 2)$ and ζ are called a *refractive index* and a *conductive boundary parameter*, respectively. Usually, we assume that n_1 and n_2 are real valued functions and that ζ is a purely imaginary valued function. For the details, see [5]. However, in this thesis, we allow n_1, n_2 and ζ to be complex valued functions.

We put

$$\mathbf{H} := H^1(M_1) \times H^1(M_2).$$

Then $\mathbf{H}$ is a Hilbert space equipped with the inner product $(\cdot,\cdot)_{\mathbf{H}} := (\cdot,\cdot)_{H^1(M_1)} + (\cdot,\cdot)_{H^1(M_2)}$ and the norm $\|\cdot\|_{\mathbf{H}} := (\cdot,\cdot)_{\mathbf{H}}^{1/2}$. Now let us go into the definition of an ITE for the locally anisotropic ITE problem.

Definition 4.1.9. If there exists a non-trivial solution $(u_1, u_2) \in \mathbf{H}$ of the locally anisotropic ITE problem (4.1)–(4.4) for some $k \in \mathbf{C}$, we call such a complex number k a *locally anisotropic interior transmission eigenvalue.*

Definition 4.1.10.

- We denote the set of locally anisotropic ITEs by $\sigma_{a,I}$.
- A pair of functions $(u_1, u_2) \in \mathbf{H}$ is called a *locally anisotropic interior transmission eigenfunction* associated with $k \in \sigma_{a,I}$, if (u_1, u_2) satisfies the locally anisotropic ITE problem (4.1)-(4.4) corresponding to k.
- The dimension of the space spanned by all locally anisotropic interior transmission eigenfunctions (u_1, u_2) associated with $k \in \sigma_{a,I}$ is called the multiplicity of k.

Our first main result in this chapter is stated as follows.

Theorem 4.1.11. *Suppose (A) and (A-1). Let $n_l \in L^\infty(M_l)\,(l = 1, 2)$ and $\zeta \in L^\infty(\Gamma)$ be complex valued functions. We assume that g_1 and g_2 satisfy*

$$\frac{g_2}{\sqrt{G_2}} \le c\,\frac{g_1}{\sqrt{G_1}} \quad \text{on} \quad \Sigma \tag{4.5}$$

for some constant $0 < c < 1$. Then there exists a constant $\zeta_0 > 0$ such that for ζ with $\mathrm{Re}\,\zeta \ge -\zeta_0$, the set $\sigma_{a,I}$ of locally anisotropic ITEs is a discrete subset of $\mathbf{C}$. The point at infinity is the only possible accumulation point of $\sigma_{a,I}$. Furthermore, the multiplicity of each locally anisotropic ITE is finite.

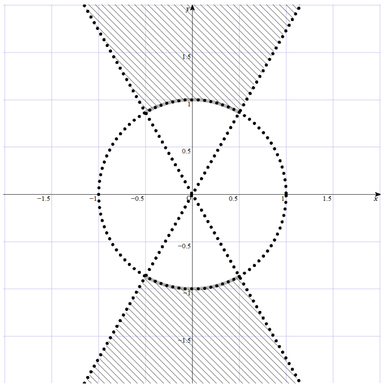

Figure 4.2: An example of $N(r,\theta)\,(r=1,\theta=\pi/3)$.

Remark 4.1.12. The condition (4.5) on g_1 and g_2 implies that g_1 and g_1 satisfy (A-2), namely, the boundary value problem (4.1)–(4.4) is the locally anisotropic ITE problem.

For $r,\theta>0$, we put

$$N(r,\theta) := \{k \in \mathbf{C} \mid |k| > r \text{ and } |\mathrm{Im}\,k| > (\tan\theta)|\mathrm{Re}\,k|\}$$

(see Figure 4.2). Then our second main in this section result is given by the following.

Theorem 4.1.13. *Suppose (A) and (A-1). Let* $n_l \in L^\infty(M_l)\,(l=1,2)$ *and* $\zeta \in L^\infty(\Gamma)$ *be complex valued functions. We assume that* $\mathrm{Re}\,n_1$ *and* $\mathrm{Re}\,n_2$ *are strictly positive functions. We also assume that* g_1 *and* g_2 *satisfy (4.5) and* n_1 *and* n_2 *satisfy*

$$\sup_\Sigma\left(\sqrt{G_1}(\mathrm{Re}\,n_1)\right) < \inf_\Sigma\left(\sqrt{G_2}(\mathrm{Re}\,n_2)\right). \tag{4.6}$$

Then there exist positive constants r,θ,ϵ_0 *and* ζ_0 *such that there are no locally anisotropic ITEs in the region* $N(r,\theta)$ *for* n_1 *with* $|\mathrm{Im}\,n_1| < \epsilon_0$ *in* Σ *and for* ζ *with* $\mathrm{Re}\,\zeta \ge -\zeta_0$ *on* Γ.

In [13], by using analytic Fredholm theorem (see e.g., Theorem 1 in [3]), Bonnet-Ben Dhia, Chesnel and Haddar proved the discreteness of $\sigma_{a,I}$. In our setting, instead of analytic Fredholm theorem, we use the theory of compact operators to simplify their argument. As a result, we are able to remove their assumption which is essential to use analytic Fredholm theorem. Furthermore, we note that in this thesis, we introduce a new function ζ called a boundary conductive parameter in the ITE problem (4.1)–(4.4). This parameter ζ plays an important role in scattering problem with conductive transmission condition. In this sense, we can say that our problem is a slightly more generalized version of the original ITE problem.

4.2 T-coercivity method

In order to prove the discreteness of locally anisotropic ITEs, we employ the T-coercivity method (see for example [13], [14]). Let

$$\mathbf{H}_0 := \{(u_1,u_2) \in \mathbf{H} \mid u_1 = u_2 \quad \text{on} \quad \Gamma\}.$$

Let ∇_{g_1} and ∇_{g_2} be the gradient operators on (M_1,g_1) and on (M_2,g_2), respectively. We define a sesquilinear form $A_k[\cdot,\cdot]$ on $\mathbf{H}_0 \times \mathbf{H}_0$ by

$$A_k[(u_1,u_2),(v_1,v_2)] := (\nabla_{g_1}u_1, \nabla_{g_1}v_1)_{TM_1} - (\nabla_{g_2}u_2, \nabla_{g_2}v_2)_{TM_2}$$

$$- k^2 \left((n_1 u_1, v_1)_{M_1} - (n_2 u_2, v_2)_{M_2} \right) - (\zeta u_1, v_1)_\Gamma$$

for all (u_1, u_2), $(v_1, v_2) \in \mathbf{H}_0$. We can easily show that the locally anisotropic ITE problem (4.1)–(4.4) has a non-trivial solution $(u_1, u_2) \in \mathbf{H}$ if and only if the variational problem of the form

$$A_k[(u_1, u_2), (v_1, v_2)] = 0 \quad \text{for all} \quad (v_1, v_2) \in \mathbf{H}_0$$

has a non-trivial solution $(u_1, u_2) \in \mathbf{H}_0$. We define an operator T on $\mathbf{H}_0$ by

$$T(u_1, u_2) = (u_1 - 2\chi u_2, -u_2) \tag{4.7}$$

for $(u_1, u_2) \in \mathbf{H}_0$. Here, χ is a smooth cut-off function on M_2 such that $\chi = 1$ in a small neighborhood of Γ with support in $\Sigma \cap M_2$ and $0 \leq \chi(x) \leq 1$ for all $x \in M_2$. Let $I_{\mathbf{H}}$ be the identity operator on $\mathbf{H}$. Since $T^2 = I_{\mathbf{H}}$, T is an isomorphism on $\mathbf{H}_0$. By using this isomorphism, we define a sesquilinear form $A_k^T[\cdot, \cdot]$ on $\mathbf{H}_0 \times \mathbf{H}_0$ by

$$A_k^T[(u_1, u_2), (v_1, v_2)] := A_k[(u_1, u_2), T(v_1, v_2)]$$

for all $(u_1, u_2), (v_1, v_2) \in \mathbf{H}_0$. We can easily show that the above sesquilinear form $A_k^T[\cdot, \cdot]$ is non-degenerate and bounded on $\mathbf{H}_0 \times \mathbf{H}_0$. Hence, applying the first representation theorem (see e.g., Theorem 2.1 in [18]) or the Riesz representation theorem to the sesquilinear form $A_k^T[\cdot, \cdot]$, we find that there exists a bounded linear operator $\mathscr{A}^T(k)$ on $\mathbf{H}_0$ such that

$$A_k^T[(u_1, u_2), (v_1, v_2)] = (\mathscr{A}^T(k)(u_1, u_2), (v_1, v_2))_{\mathbf{H}}$$

for all $(u_1, u_2), (v_1, v_2) \in \mathbf{H}_0$. Summarizing the above argument, we obtain the following proposition.

Proposition 4.2.14. *A point $k \in \mathbf{C}$ is a locally anisotropic ITE if and only if the operator $\mathscr{A}^T(k)$ on $\mathbf{H}_0$ has a non-trivial kernel. In this case, each element of the kernel of $\mathscr{A}^T(k)$ is interior transmission eigenfunction associated with $k \in \sigma_{a,I}$. The multiplicity of $k \in \sigma_{a,I}$ coincides with the dimension of the kernel of $\mathscr{A}^T(k)$.*

Now, let us introduce the notion of *strictly coercive* for a bounded linear operator.

Definition 4.2.15. Let H be a Hilbert space equipped with inner product $(\cdot, \cdot)_H$ and norm $\| \cdot \|_H = \sqrt{(\cdot, \cdot)_H}$. A bounded linear operator $B : H \to H$ is said to be *strictly coercive* if there exists a constant $C > 0$ such that

$$\mathrm{Re}\,(B\varphi, \varphi)_H \geq C \|\varphi\|_H^2$$

for all $\varphi \in H$.

The following theorem is well-known as the Lax–Milgram theorem (see e.g., Theorem 13.23 in [19]).

Theorem 4.2.16. *In a Hilbert space H, a strictly coercive bounded linear operator $B : H \to H$ has a bounded inverse.*

Let $\kappa \in \mathbf{R} \setminus \{0\}$ and $\epsilon, \delta > 0$ be constants such that

$$\epsilon^* := \sup_\Sigma (\sqrt{G_1})\epsilon < \inf_\Sigma (\sqrt{G_2})\delta =: \delta_*.$$

We define a sesquilinear form $A_{i\kappa,\epsilon,\delta}[\cdot,\cdot]$ on $\mathbf{H}_0 \times \mathbf{H}_0$ by

$$\begin{aligned} A_{i\kappa,\epsilon,\delta}[(u_1,u_2),(v_1,v_2)] := &(\nabla_{g_1}u_1, \nabla_{g_1}v_1)_{TM_1} - (\nabla_{g_2}u_2, \nabla_{g_2}v_2)_{TM_2} \\ &+ \kappa^2 \left((\epsilon u_1, v_1)_{M_1} - (\delta u_2, v_2)_{M_2}\right) - (\zeta u_1, v_1)_\Gamma \end{aligned}$$

for all $(u_1,u_2),(v_1,v_2) \in \mathbf{H}_0$. In addition, we define a bounded operator $\mathscr{I}_{\kappa,\delta,\epsilon}$ on $\mathbf{H}_0$ by

$$(\mathscr{I}_{\kappa,\epsilon,\delta}(u_1,u_2),(v_1,v_2))_{\mathbf{H}} := A_{i\kappa,\epsilon,\delta}[(u_1,u_2),T(v_1,v_2)]$$

for all $(u_1,u_2),(v_1,v_2) \in \mathbf{H}_0$.

Now in order to reduce the locally anisotropic ITE problem (4.1)–(4.4) to the eigenvalue problem for a certain compact operator, we state the following key lemma.

Lemma 4.2.17. *Let $n_l \in L^\infty(M_l)\,(l=1,2)$ and $\zeta \in L^\infty(\Gamma)$ be complex valued functions. We assume that g_1 and g_2 satisfy (4.5). Then there exist a point $\zeta_0 > 0$ and a constant $C > 0$ such that for ζ with $\mathrm{Re}\,\zeta \geq -\zeta_0$, the inequality*

$$\mathrm{Re}\,(\mathscr{I}_{\kappa,\epsilon,\delta}(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \geq C\|(u_1,u_2)\|^2_{\mathbf{H}}, \quad (u_1,u_2) \in \mathbf{H}_0 \tag{4.8}$$

holds.

Proof. We have the equality

$$\begin{aligned} &\mathrm{Re}\,(\mathscr{I}_{\kappa,\epsilon,\delta}(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \\ =&\int_{M_1\setminus\Sigma} |\nabla_{g_1}u_1|^2_{g_1} dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2}u_2|^2_{g_2} dV_{g_2} + \kappa^2(\epsilon\|u_1\|^2_{M_1\setminus\Sigma} + \delta\|u_2\|^2_{M_2\setminus\Sigma}) \\ &+ \int_\Sigma |\nabla_{g_1}u_1|^2_{g_1} dV_{g_1} + \int_\Sigma |\nabla_{g_2}u_2|^2_{g_2} dV_{g_2} + \kappa^2\left(\epsilon\int_\Sigma |u_1|^2 dV_{g_1} + \int_\Sigma \delta|u_2|^2 dV_{g_2}\right) \\ &- 2\mathrm{Re}\,(\nabla_{g_1}u_1, \nabla_{g_1}(\chi u_2))_{TM_1} - 2\kappa^2\epsilon\mathrm{Re}\,(u_1,\chi u_2)_{M_1} + \mathrm{Re}\,(\zeta u_1,u_1)_\Gamma \end{aligned} \tag{4.9}$$

for all $(u_1,u_2) \in \mathbf{H}_0$. Using Young's inequality and (4.5), we have

$$\begin{aligned} &2\mathrm{Re}\,(\nabla_{g_1}u_1, \nabla_{g_1}(\chi u_2))_{TM_1} \\ \leq&(\alpha+\beta)\int_\Sigma |\nabla_{g_1}u_1|^2_{g_1} dV_{g_1} + \alpha^{-1}\int_\Sigma |\nabla_{g_1}u_2|^2_{g_1} dV_{g_1} \\ &+ \beta^{-1}\int_\Sigma |\nabla_{g_1}\chi|^2_{g_1} |u_2|^2 dV_{g_1} \\ \leq&(\alpha+\beta)\int_\Sigma |\nabla_{g_1}u_1|^2_{g_1} dV_{g_1} + c\alpha^{-1}\int_\Sigma |\nabla_{g_2}u_2|^2_{g_2} dV_{g_2} \\ &+ \beta^{-1}\sup_\Sigma\left(|\nabla_{g_1}\chi|^2_{g_1}\sqrt{\frac{G_1}{G_2}}\right)\int_\Sigma |u_2|^2 dV_{g_2} \end{aligned} \tag{4.10}$$

and

$$2\kappa^2\epsilon\mathrm{Re}\,(u_1,\chi u_2)_{M_1} \leq \kappa^2\epsilon\gamma\int_\Sigma |u_1|^2 dV_{g_1} + \kappa^2\int_\Sigma \frac{1}{\sqrt{G_2}}\gamma^{-1}\sqrt{G_1}\epsilon|u_2|^2 dV_{g_2} \tag{4.11}$$

for all $\alpha, \beta, \gamma > 0$. Plugging (4.10) and (4.11) into (4.9), we obtain

$$
\begin{aligned}
&\mathrm{Re}\,(\mathscr{I}_{\kappa,\epsilon,\delta}(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \\
&\geq \int_{M_1\setminus\Sigma} |\nabla_{g_1} u_1|_{g_1}^2 dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2} u_2|_{g_2}^2 dV_{g_2} + \kappa^2(\epsilon\|u_1\|_{M_1\setminus\Sigma}^2 + \delta\|u_2\|_{M_2\setminus\Sigma}^2) \\
&\quad + (1-\alpha-\beta)\int_\Sigma |\nabla_{g_1} u_1|_{g_1}^2 dV_{g_1} + (1-c\alpha^{-1})\int_\Sigma |\nabla_{g_2} u_2|_{g_2}^2 dV_{g_2} \\
&\quad + \kappa^2\epsilon(1-\gamma)\int_\Sigma |u_1|^2 dV_{g_1} \\
&\quad + \kappa^2\int_\Sigma \frac{1}{\sqrt{G_2}}(\delta_* - \gamma^{-1}\epsilon^*)|u_2|^2 dV_{g_2} - \beta^{-1}\sup_\Sigma\left(|\nabla_{g_1}\chi|_{g_1}^2\sqrt{\frac{G_1}{G_2}}\right)\int_\Sigma |u_2|^2 dV_{g_2} \\
&\quad - \zeta_0\|u_1\|_\Gamma^2.
\end{aligned}
$$

Taking γ such that $\epsilon^*/\delta_* < \gamma < 1$, we have

$$
\begin{aligned}
&\mathrm{Re}\,(\mathscr{I}_{\kappa,\epsilon,\delta}(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \\
&\geq \int_{M_1\setminus\Sigma} |\nabla_{g_1} u_1|_{g_1}^2 dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2} u_2|_{g_2}^2 dV_{g_2} + \kappa^2(\epsilon\|u_1\|_{M_1\setminus\Sigma}^2 + \delta\|u_2\|_{M_2\setminus\Sigma}^2) \\
&\quad + (1-\alpha-\beta)\int_\Sigma |\nabla_{g_1} u_1|_{g_1}^2 dV_{g_1} + (1-c\alpha^{-1})\int_\Sigma |\nabla_{g_2} u_2|_{g_2}^2 dV_{g_2} \\
&\quad + \kappa^2\epsilon(1-\gamma)\int_\Sigma |u_1|^2 dV_{g_1} + \left(\kappa^2 c_1(\delta_* - \gamma^{-1}\epsilon^*) - c_2\beta^{-1}\right)\int_\Sigma |u_2|^2 dV_{g_2} - \zeta_0\|u_1\|_\Gamma^2
\end{aligned}
$$

for some $c_1, c_2 > 0$. Using the trace theorem, we obtain

$$
\|u_1\|_\Gamma \leq c_3\|u_1\|_{H^1(M_1)} \tag{4.12}
$$

for some $c_3 > 0$. By taking α, β such that

$$
c < \alpha < 1, \quad 0 < \beta < 1 - \alpha
$$

and using (4.12), letting $|\kappa| > 0$ large enough and $\zeta_0 > 0$ small enough, more precisely taking

$$
\kappa^2 > \frac{c_2\beta^{-1}}{c_1(\delta_* - \gamma^{-1}\epsilon^*)}, \quad 0 < \zeta_0 < c_3^{-1}\min\{1-\alpha-\beta, \kappa^2\epsilon(1-\gamma)\},
$$

we can easily show that there exists a constant $C > 0$ such that the inequality (4.8) holds. □

Remark 4.2.18. For example, we take

$$
\alpha = \frac{c+1}{2}, \quad \beta = \frac{1-c}{4}, \quad \gamma = \frac{\delta_* + \epsilon^*}{2\delta_*}, \quad \zeta_0 = \frac{1-c}{8c_3^2}
$$

and

$$
\kappa^2 = \max\left\{\frac{2\delta_*}{\epsilon(\delta_* - \epsilon^*)}, \frac{1}{\delta}, \frac{\delta_* + \epsilon^*}{c_1\delta_*(\delta_* - \epsilon^*)}\left(1 + \frac{4c_2}{1-c}\right)\right\}.
$$

Then the constant $C > 0$ appeared in (4.8) is equal to $(1-c)/8$.

Remark 4.2.19. As stated above, using the isomorphism T given by (4.7), we can avoid the difficulty arising from the non-ellipticity of the sesquilinear form $A_k[\cdot,\cdot]$. Such a method is called the T-coercivity method. This method was first introduced by Bonnet-Ben Dhia, Ciarlet and Zwölf [14]. Using the idea of T-coercivity, they proved that the electromagnetic wave transmission problem is well-posed when dielectric constant changes its sign.

Using the above lemma, we can write $\mathscr{A}^T(k)$ as the sum of an isomorphism and a compact operator as follows.

Proposition 4.2.20. *Let $n_l \in L^\infty(M_l)\,(l = 1, 2)$ and $\zeta \in L^\infty(\Gamma)$ be complex valued functions. We assume that g_1 and g_2 satisfy (4.5). Then there exists a point $\zeta_0 > 0$ such that for ζ with $\operatorname{Re}\zeta \geq -\zeta_0$ and for all $k \in \mathbf{C}$, the operator $\mathscr{A}^T(k)$ is written in the form $\mathscr{A}^T(k) = \mathscr{I} + \mathscr{K}$ where $\mathscr{I}$ is an isomorphism on $\mathbf{H}_0$ and $\mathscr{K}$ is a compact operator on $\mathbf{H}_0$. As a result, $\mathscr{A}^T(k)$ is a Fredholm operator on $\mathbf{H}_0$ for all $k \in \mathbf{C}$*

Proof. By Lemma 4.2.17, the inequality (4.8) holds. Applying Theorem 4.2.16 to the bounded linear operator $\mathscr{I}_{\kappa,\epsilon,\delta}$, we find that $\mathscr{I}_{\kappa,\epsilon,\delta}$ is an isomorphism on $\mathbf{H}_0$. Recall that $\mathscr{A}^T(k)$ and $\mathscr{I}_{\kappa,\epsilon,\delta}$ are written as

$$\begin{aligned}&(\mathscr{A}^T(k)(u_1, u_2), (v_1, v_2))_{\mathbf{H}}\\ =&(\nabla_{g_1} u_1, \nabla_{g_1} v_1)_{TM_1} + (\nabla_{g_2} u_2, \nabla_{g_2} v_2)_{TM_2} - 2(\nabla_{g_1} u_1, \nabla_{g_1}(\chi v_2))_{TM_1}\\ &- k^2\left((n_1 u_1, v_1)_{M_1} + (n_2 u_2, v_2)_{M_2} - 2(n_1 u_1, \chi v_2)_{M_1}\right) - (\zeta u_1, v_1)_\Gamma\end{aligned}$$

and

$$\begin{aligned}&(\mathscr{I}_{\kappa,\epsilon,\delta}(u_1, u_2), (v_1, v_2))_{\mathbf{H}}\\ =&(\nabla_{g_1} u_1, \nabla_{g_1} v_1)_{TM_1} + (\nabla_{g_2} u_2, \nabla_{g_2} v_2)_{TM_2} - 2(\nabla_{g_1} u_1, \nabla_{g_1}(\chi v_2))_{TM_1}\\ &+ \kappa^2\left((\epsilon u_1, v_1)_{M_1} + (\delta u_2, v_2)_{M_2} - 2(\epsilon u_1, \chi v_2)_{M_1}\right) - (\zeta u_1, v_1)_\Gamma\end{aligned}$$

for (u_1, u_2), $(v_1, v_2) \in \mathbf{H}_0$, respectively. We put $\mathscr{K} := \mathscr{A}^T(k) - \mathscr{I}_{\kappa,\epsilon,\delta}$. Then the operator $\mathscr{K}$ satisfies

$$\begin{aligned}&(\mathscr{K}(u_1, u_2), (v_1, v_2))_{\mathbf{H}}\\ =&- k^2\left((n_1 u_1, v_1)_{M_1} + (n_2 u_2, v_2)_{M_2} - 2(n_1 u_1, \chi v_2)_{M_1}\right)\\ &- \kappa^2\left((\epsilon u_1, v_1)_{M_1} + (\delta u_2, v_2)_{M_2} - 2(\epsilon u_1, \chi v_2)_{M_1}\right)\end{aligned}$$

for all (u_1, u_2), $(v_1, v_2) \in \mathbf{H}_0$. Therefore, the inequality

$$|(\mathscr{K}(u_1, u_2), (v_1, v_2))_{\mathbf{H}}| \leq C\|(u_1, u_2)\|_{L^2(M_1)\times L^2(M_2)}\|(v_1, v_2)\|_{\mathbf{H}}$$

holds for some constant $C > 0$ depending on k. Here, $\|\cdot\|_{L^2(M_1)\times L^2(M_2)}$ is a norm of the Hilbert space $L^2(M_1) \times L^2(M_2)$ and denotes

$$\|(u_1, u_2)\|_{L^2(M_1)\times L^2(M_2)} = (\|u_1\|^2_{M_1} + \|u_2\|^2_{M_2})^{1/2}$$

for $(u_1, u_2) \in L^2(M_1) \times L^2(M_2)$. The above inequality is equivalent to

$$\|\mathscr{K}(u_1, u_2)\|_{\mathbf{H}} \leq C\|(u_1, u_2)\|_{L^2(M_1)\times L^2(M_2)} \tag{4.13}$$

for all $(u_1, u_2) \in \mathbf{H}_0$. By the Rellich–Kondrashov theorem (see e.g., Theorem 6.3 in [1]), a bounded sequence in $\mathbf{H}_0$ has a Cauchy subsequence in $L^2(M_1) \times L^2(M_2)$. Let $\{(u_{1n}, u_{2n})\}_{n=1}^\infty$ be such a subsequence. Using the inequality (4.13), we have

$$\|\mathscr{K}(u_{1n}, u_{2n}) - \mathscr{K}(u_{1m}, u_{2m})\|_{\mathbf{H}} \leq C\|(u_{1n}, u_{2n}) - (u_{1m}, u_{2m})\|_{L^2(M_1)\times L^2(M_1)}.$$

This means that $\{\mathscr{K}(u_{1n}, u_{2n})\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbf{H}_0$. Thus, $\mathscr{K}$ is a compact operator on $\mathbf{H}_0$. If we take $\mathscr{I} = \mathscr{I}_{\kappa,\epsilon,\delta}$, then we have $\mathscr{A}^T(k) = \mathscr{I} + \mathscr{K}$, which proves the assertion. □

4.3 Proof of Theorem 4.1.11 and Theorem 4.1.13

First, we prove Theorem 4.1.11.

Proof of Theorem 4.1.11. Let us define two operators $\mathscr{F}$ and $\mathscr{G}_{\kappa,\epsilon,\delta}$ on $\mathbf{H}_0$ by

$$(\mathscr{F}(u_1,u_2),(v_1,v_2))_{\mathbf{H}} = (n_1u_1,v_1)_{M_1} + (n_2u_2,v_2)_{M_2} - 2(n_1u_1,\chi v_2)_{M_1}$$

and

$$(\mathscr{G}_{\kappa,\epsilon,\delta}(u_1,u_2),(v_1,v_2))_{\mathbf{H}} = \kappa^2\left((\epsilon u_1,v_1)_{M_1} + (\delta u_2,v_2)_{M_2} - 2(\epsilon u_1,\chi v_2)_{M_1}\right)$$

for all $(u_1,u_2),(v_1,v_2)\in\mathbf{H}_0$, respectively. By the same argument as in the proof of Proposition 4.2.20, we can show that $\mathscr{F}$ and $\mathscr{G}_{\kappa,\epsilon,\delta}$ are also compact operators on $\mathbf{H}_0$. Using these operators, we rewrite $\mathscr{A}^T(k)$ as

$$\mathscr{I}_{\kappa,\epsilon,\delta} - k^2\mathscr{F} - \mathscr{G}_{\kappa,\epsilon,\delta}.$$

Let us take $\epsilon,\delta>0$ such that $\sup_\Sigma(\sqrt{G_1})\epsilon < \inf_\Sigma(\sqrt{G_2})\delta$. Next, we choose ϵ and δ small enough such that $\|\mathscr{I}^{-1}_{\kappa,\epsilon,\delta}\mathscr{G}_{\kappa,\epsilon,\delta}\|_{\mathbf{H}_0\to\mathbf{H}_0} < 1$. Here, $\|\cdot\|_{\mathbf{H}_0\to\mathbf{H}_0}$ denotes the operator norm for bounded linear operators on $\mathbf{H}_0$. Then we can easily show that $I_{\mathbf{H}} - \mathscr{I}^{-1}_{\kappa,\epsilon,\delta}\mathscr{G}_{\kappa,\epsilon,\delta}$ is a bijection on $\mathbf{H}_0$ and has a bounded inverse. Therefore, a locally anisotropic interior transmission eigenfunction $(u_1,u_2)\in\mathbf{H}_0$ associated with $k\in\sigma_{a,I}$ satisfies

$$\begin{aligned} 0 &= \mathscr{I}^{-1}_{\kappa,\epsilon,\delta}\mathscr{A}^T(k)(u_1,u_2) \\ &= (I_{\mathbf{H}} - \mathscr{I}^{-1}_{\kappa,\epsilon,\delta}\mathscr{G}_{\kappa,\epsilon,\delta})(u_1,u_2) - k^2\mathscr{I}^{-1}_{\kappa,\epsilon,\delta}\mathscr{F}(u_1,u_2). \end{aligned} \tag{4.14}$$

Put $\mathscr{B} = (I_{\mathbf{H}} - \mathscr{I}^{-1}_{\kappa,\epsilon,\delta}\mathscr{G}_{\kappa,\epsilon,\delta})^{-1}\mathscr{I}^{-1}_{\kappa,\epsilon,\delta}$. Obviously, $\mathscr{B}$ is a bounded operator on $\mathbf{H}_0$ and is independent of k. Thus, $\mathscr{B}\mathscr{F}$ is also a compact operator on $\mathbf{H}_0$. Moreover, it follows easily from (4.14) that

$$\mathscr{B}\mathscr{F}(u_1,u_2) = k^{-2}(u_1,u_2)$$

for all $(u_1,u_2)\in\mathbf{H}_0\setminus\{(0,0)\}$. As a conclusion, $(u_1,u_2)\in\mathbf{H}_0$ is a locally anisotropic interior transmission eigenfunction associated with $k\in\sigma_{a,I}\setminus\{0\}$ if and only if $k^{-2}\in\mathbf{C}$ is an eigenvalue of the compact operator $\mathscr{B}\mathscr{F}$ on $\mathbf{H}_0$ and $(u_1,u_2)\in\mathbf{H}_0$ is the corresponding eigenfunction associated with k^{-2}. As is well-known in the theory of compact operators, 0 is the only possible accumulation point of eigenvalues of a compact operator. Therefore, we obtain the assertion of Theorem 4.1.11. □

Next, we prove Theorem 4.1.13.

Proof of Theorem 4.1.13. It is sufficient to prove that there exist constants $r>0$ and $\theta\in(0,\pi/2]$ such that for all $k\in N(r,\theta)$ and for some constant $C>0$, the inequality

$$\operatorname{Re}\left(\mathscr{A}^T(k)(u_1,u_2),(u_1,u_2)\right) \geq C\|(u_1,u_2)\|^2_{\mathbf{H}}, \quad (u_1,u_2)\in\mathbf{H}_0 \tag{4.15}$$

holds. Indeed, applying Theorem 4.2.16 to the bounded linear operator $\mathscr{A}^T(k)$, we find that for $k\in N(r,\theta)$, $\mathscr{A}^T(k)$ is an isomorphism on $\mathbf{H}_0$ and has a trivial kernel. Hence, such a complex number k is not a locally anisotropic ITE.

We put

$$n_1^* := \sup_\Sigma\left(\sqrt{G_1}(\operatorname{Re} n_1)\right), \quad n_{2*} := \inf_\Sigma\left(\sqrt{G_2}(\operatorname{Re} n_2)\right).$$

We assume that n_1 satisfies

$$|\mathrm{Im}\, n_1| < \epsilon_0 \quad \text{in} \quad \Sigma$$

for some constant $\epsilon_0 > 0$. Then we derive the estimate

$$\begin{aligned}
&2\mathrm{Re}\,(n_1 u_1, \chi u_2)_{M_1} \\
\leq &\gamma \int_\Sigma (\mathrm{Re}\, n_1)|u_1|^2 dV_{g_1} + \int_\Sigma \frac{1}{\sqrt{G_2}} \gamma^{-1} (\sqrt{G_1} \mathrm{Re}\, n_1)|u_2|^2 dV_{g_2} \\
&+ \epsilon_0 \int_\Sigma |u_1|^2 dV_{g_1} + \epsilon_0 \sup_\Sigma \left(\sqrt{\frac{G_1}{G_2}} \right) \int_\Sigma |u_2|^2 dV_{g_2}
\end{aligned} \tag{4.16}$$

for all $\gamma > 0$. Let $\rho \in \mathbf{R} \setminus \{0\}$. Using (4.5), (4.10) and (4.16), we obtain

$$\begin{aligned}
&\mathrm{Re}\,(\mathscr{A}^T(i\rho)(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \\
\geq &\int_{M_1\setminus\Sigma} |\nabla_{g_1} u_1|^2_{g_1} dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2} u_2|^2_{g_2} dV_{g_2} \\
&+ \rho^2 \left(\inf_{M_1\setminus\Sigma} (\mathrm{Re}\, n_1) \|u_1\|^2_{M_1\setminus\Sigma} + \inf_{M_2\setminus\Sigma} (\mathrm{Re}\, n_2) \|u_2\|^2_{M_2\setminus\Sigma} \right) \\
&+ (1-\alpha-\beta) \int_\Sigma |\nabla_{g_1} u_1|^2_{g_1} dV_{g_1} + (1 - c\alpha^{-1}) \int_\Sigma |\nabla_{g_2} u_2|^2_{g_2} dV_{g_2} \\
&+ \rho^2 \int_\Sigma (1-\gamma)(\mathrm{Re}\, n_1)|u_1|^2 dV_{g_1} - \rho^2 \epsilon_0 \int_\Sigma |u_1|^2 dV_{g_1} \\
&+ \rho^2 \int_\Sigma \frac{1}{\sqrt{G_2}} (n_{2*} - \gamma^{-1} n_1^*)|u_2|^2 dV_{g_2} - \rho^2 \epsilon_0 \sup_\Sigma \left(\sqrt{\frac{G_1}{G_2}} \right) \int_\Sigma |u_2|^2 dV_{g_2} \\
&- \beta^{-1} \sup_\Sigma \left(|\nabla_{g_1} \chi|^2_{g_1} \sqrt{\frac{G_1}{G_2}} \right) \int_\Sigma |u_2|^2 dV_{g_2} - \zeta_0 \|u_1\|^2_\Gamma.
\end{aligned}$$

for all α, β, $\gamma > 0$. Taking γ such that $n_1^*/n_{2*} < \gamma < 1$, we have

$$\begin{aligned}
&\mathrm{Re}\,(\mathscr{A}^T(i\rho)(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \\
\geq &\int_{M_1\setminus\Sigma} |\nabla_{g_1} u_1|^2_{g_1} dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2} u_2|^2_{g_2} dV_{g_2} \\
&+ \rho^2 \left(\inf_{M_1\setminus\Sigma} (\mathrm{Re}\, n_1) \|u_1\|^2_{M_1\setminus\Sigma} + \inf_{M_2\setminus\Sigma} (\mathrm{Re}\, n_2) \|u_2\|^2_{M_2\setminus\Sigma} \right) \\
&+ (1-\alpha-\beta) \int_\Sigma |\nabla_{g_1} u_1|^2_{g_1} dV_{g_1} + (1 - c\alpha^{-1}) \int_\Sigma |\nabla_{g_2} u_2|^2_{g_2} dV_{g_2} \\
&+ \rho^2 \left((1-\gamma) \inf_\Sigma (\mathrm{Re}\, n_1) - \epsilon_0 \right) \int_\Sigma |u_1|^2 dV_{g_1} \\
&+ \left(\rho^2 \left(c_1 (n_{2*} - \gamma^{-1} n_1^*) - c_4 \epsilon_0 \right) - c_2 \beta^{-1} \right) \int_\Sigma |u_2|^2 dV_{g_2} - \zeta_0 \|u_1\|^2_\Gamma
\end{aligned}$$

for some $c_1, c_2, c_4 > 0$. Using the same argument as in the proof of Lemma 4.2.17, for a suitable choice of constants α, β, γ and a small constant $\epsilon_0 > 0$ and a large constant $r > 0$ and letting $|\rho| > r$, we have

$$\begin{aligned}
&\mathrm{Re}\,(\mathscr{A}^T(i\rho)(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \\
\geq &C_1 (\|\nabla_{g_1} u_1\|^2_{TM_1} + \|\nabla_{g_2} u_2\|^2_{TM_2}) + C_2 \rho^2 (\|u_1\|^2_{M_1} + \|u_2\|^2_{M_2}) - \zeta_0 \|u_1\|^2_\Gamma
\end{aligned} \tag{4.17}$$

for some constants $C_1, C_2 > 0$. On the other hand, taking $k = i\rho e^{i\varphi}$ with $0 \leq \varphi < \pi/2$, we find that there exists a constant $C_3 > 0$ such that

$$\begin{aligned} &\mathrm{Re}\,((\mathscr{A}^T(i\rho) - \mathscr{A}^T(k))(u_1, u_2), (u_1, u_2))_{\mathbf{H}} \\ \leq &C_3\rho^2|1 - e^{2i\varphi}|(\|u_1\|^2_{M_1} + \|u_2\|^2_{M_2}) \end{aligned} \tag{4.18}$$

for all $(u_1, u_2) \in \mathbf{H}_0$. Combining (4.17) with (4.18), we obtain

$$\begin{aligned} &\mathrm{Re}\,(\mathscr{A}^T(k)(u_1, u_2), (u_1, u_2))_{\mathbf{H}} \\ \geq &C_1(\|\nabla_{g_1} u_1\|^2_{TM_1} + \|\nabla_{g_2} u_2\|^2_{TM_2}) \\ &+ (C_2 - C_3|1 - e^{2i\varphi}|)\rho^2(\|u_1\|^2_{M_1} + \|u_2\|^2_{M_2}) - \zeta_0\|u_1\|^2_{\Gamma} \end{aligned}$$

for all $(u_1, u_2) \in \mathbf{H}_0$. By choosing φ, $\zeta_0 > 0$ small enough and using (4.12), we have

$$\mathrm{Re}\,(\mathscr{A}^T(k)(u_1, u_2), (u_1, u_2))_{\mathbf{H}} \geq C\|(u_1, u_2)\|^2_{\mathbf{H}}$$

for some constant $C > 0$. We put $\theta := \pi/2 - \varphi$. Then for all $k \in N(r, \theta)$, the inequality (4.15) holds. Therefore, we obtain the assertion of the Theorem 4.1.13. □

Part IV

A locally isotropic interior transmission eigenvalue problem

Chapter 5

A locally isotropic interior transmission eigenvalue problem

5.1 Our setting and main results II

For $d \geq 2$, we consider two d-dimensional connected and compact smooth oriented Riemannian manifolds (M_1, g_1) and (M_2, g_2) which satisfy the assumption (A). In addition, we assume that g_1 and g_2 satisfy

$$g_1 = g_2 \quad \text{on} \quad \Gamma. \tag{I-1}$$

We also note that we need our geometric assumptions only in some small neighborhoods of the boundary Γ, in particular, we do not assume that M_1 and M_2 are diffeomorphic outside of these neighborhoods.

In this section, we assume (A) and (I-1). For strictly positive functions $n_l \in C^\infty(\overline{M_l})$ $(l = 1, 2)$ and $\zeta \in C^\infty(\Gamma)$ and for $\lambda \in \mathbf{C}$, we consider a boundary value problem for a system of Helmholtz equations for unknown functions u_1 and u_2 of the form

$$(-\Delta_{g_1} - \lambda n_1)u_1 = 0 \quad \text{in} \quad M_1; \tag{5.1}$$

$$(-\Delta_{g_2} - \lambda n_2)u_2 = 0 \quad \text{in} \quad M_2; \tag{5.2}$$

$$u_1 - u_2 = 0 \quad \text{on} \quad \Gamma; \tag{5.3}$$

$$\partial_{\nu,1}u_1 - \partial_{\nu,2}u_2 = \zeta u_1 \quad \text{on} \quad \Gamma. \tag{5.4}$$

We call the above boundary value problem a *locally isotropic interior transmission eigenvalue problem.* For ζ, this thesis covers the following two cases : (i) $\zeta = 0$ on Γ, or (ii) ζ is strictly positive or strictly negative on every component of Γ. Now let us go into the definition of an interior transmission eigenvalue for the locally isotropic ITE problem.

Definition 5.1.21. If there exists a non-trivial solution $(u_1, u_2) \in H^2(M_1) \times H^2(M_2)$ of the locally isotropic ITE problem (5.1)–(5.4) for some $\lambda \in \mathbf{C}$, we call such a complex number λ a *locally isotropic interior transmission eigenvalue.*

Definition 5.1.22.

- We denote the set of locally isotropic ITEs by $\sigma_{i,I}$.
- A pair of functions $(u_1, u_2) \in H^2(M_1) \times H^2(M_2)$ is called a *locally isotropic interior transmission eigenfunction* associated with $\lambda \in \sigma_{i,I}$, if (u_1, u_2) satisfies the locally isotropic ITE problem (5.1)–(5.4) corresponding to λ.

· The dimension of the space spanned by all locally isotropic interior transmission eigenfunctions (u_1, u_2) associated with $\lambda \in \sigma_{i,I}$ is called the multiplicity of λ.

We take an arbitrary closed sector S_0 centered at the origin such that $S_0 \cap \mathbf{R}_{>0} = \emptyset$.

Our first main result in this chapter is stated as follows.

Theorem 5.1.23. *Suppose (A) and (I-1). We assume that either*

$$\partial_{\nu,1}^m g_1^{ij} = \partial_{\nu,2}^m g_2^{ij} \quad \textit{for} \quad m \leq 2 \quad \textit{and} \quad n_1 \neq n_2 \quad \textit{on} \quad \Gamma \tag{I-2-1}$$

or

$$\partial_{\nu,1}^m g_1^{ij} = \partial_{\nu,2}^m g_2^{ij} \quad \textit{for} \quad m \leq 3 \quad \textit{and} \quad n_1 = n_2,\ \partial_{\nu,1} n_1 \neq \partial_{\nu,2} n_2 \quad \textit{on} \quad \Gamma. \tag{I-2-2}$$

The set of locally isotropic ITEs consists of a discrete subset of $\mathbf{C}$ *with the only possible accumulation points at 0 and infinity. There exist at most finitely many locally isotropic ITEs in* $S_0 \cap \{\lambda \in \mathbf{C} \mid |\lambda| \geq 1\}$.

For a small constant $\alpha > 0$, we define the counting function of locally isotropic ITEs with multiplicities taken into account by

$$N_T = \#\{j \mid \alpha < \lambda_j^T \leq \lambda\}$$

where $\lambda_1^T \leq \lambda_2^T \leq \cdots$ are locally isotropic ITEs included in (α, ∞). Let $\mathcal{O}_l(x) = \{\xi \in \mathbf{R}^d \mid \sum_{i,j=1}^d g_l^{ij}(x)\xi_i\xi_j \leq n_l(x)\}$ for $x \in M_l$ and

$$V_l = (2\pi)^{-d} \int_{M_l} \int_{\mathcal{O}_l} d\xi dV_l.$$

Our second main result in this chapter is stated as follows.

Theorem 5.1.24. *Suppose (A) and (I-1). We assume that*

$$\textit{If } \zeta \neq 0, \textit{ then } -\zeta \textit{ do not change its sign on whole of } \Gamma \textit{ and this sign is denoted by } \gamma_0. \tag{I-3-0}$$

$$\textit{If } \zeta = 0 \textit{ and suppose (I-2-1), then } n_2 - n_1 \textit{ do not change its sign on whole of } \Gamma \textit{ and this sign is denoted by } \gamma_1. \tag{I-3-1}$$

$$\textit{If } \zeta = 0 \textit{ and suppose (I-2-2), then } \partial_{\nu,1} n_1 - \partial_{\nu,2} n_2 \textit{ do not change its sign on whole of } \Gamma \textit{ and this sign is denoted by } \gamma_2. \tag{I-3-2}$$

For each (I-3-n), let $\gamma = \gamma_n$. *If* $\gamma(V_1 - V_2) > 0$, $N_T(\lambda)$ *satisfies asymptotically as* $\lambda \to \infty$

$$N_T(\lambda) \geq \gamma(V_1 - V_2)\lambda^{d/2} + O(\lambda^{(d-1)/2}).$$

5.2 Dirichlet-to-Neumann map

5.2.1 Dirichlet-to-Neumann map

Let (M, g) be a d-dimensional connected and compact smooth oriented Riemannian manifold with smooth boundary ∂M. For functions $n \in C^\infty(\overline{M})$ and $f \in H^{3/2}(\partial M)$, we consider the following Dirichlet boundary value problem for unknown function $u \in H^2(M)$ of the form

$$\begin{aligned} (-\Delta_g - \lambda n)u = 0 \quad &\text{in} \quad M; \\ u = f \quad &\text{on} \quad \partial M. \end{aligned} \tag{5.5}$$

We define the Dirichlet-to-Neumann (the **D-N** map for short) $\Lambda(\lambda)$ by

$$\Lambda(\lambda)f = \partial_\nu u \quad \text{on} \quad \partial M, \tag{5.6}$$

where u is a solution of (5.5).

In the following, we call λ a Dirichlet eigenvalue if there exists a non-trivial solution of the equation

$$\begin{aligned} (-\Delta_g - \lambda n)u = 0 \quad &\text{in} \quad M; \\ u = 0 \quad &\text{on} \quad \partial M. \end{aligned} \tag{5.7}$$

In fact, (5.7) is equivalent to

$$\begin{aligned} (-n^{-1}\Delta_g - \lambda)u = 0 \quad &\text{in} \quad M; \\ u = 0 \quad &\text{on} \quad \partial M, \end{aligned} \tag{5.8}$$

which is an eigenvalue problem of the second-order self-adjoint elliptic operator $L = -n^{-1}\Delta_g$ in $L^2(M, n)$ with the Dirichlet boundary condition. Then its eigenvalues form an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, satisfying the Weyl's asymptotics which we derive in §5.3. The corresponding eigenfunctions ϕ_j $(j = 1, 2, \ldots)$ can be chosen so that $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis in $L^2(M, n)$. We denote the set of Dirichlet eigenvalues by $\sigma_D := \{\lambda_j\}_{j=1}^\infty$. For $\lambda \notin \sigma_D$, the D-N map $\Lambda(\lambda)$ is well-defined and extends uniquely as a continuous operator $\Lambda(\lambda) : H^{3/2}(\partial M) \to H^{1/2}(\partial M)$.

Let $\mathcal{E}_j \subset \mathbf{Z}_{>0}$ such that $\amalg_{j=1}^\infty \mathcal{E}_j = \mathbf{Z}_{>0}$, and i_1 and i_2 belong to the same set $\mathcal{E}_j$ $(j = 1, 2, \ldots)$ if and only if $\lambda_{i_1} = \lambda_{i_2}$.

Proposition 5.2.25. *$\Lambda(\lambda)$ is meromorphic with respect to $\lambda \in \mathbf{C}$ and has first order poles at each $\lambda \in \sigma_D$. Moreover, $\Lambda(\lambda)$ has the following representations.*

(1) *For $x \in \partial M$ and $f \in H^{3/2}(\partial M)$, we have*

$$\Lambda(\lambda)f(x) = -\sum_{j=1}^\infty \int_{\partial M} \frac{\partial_{\nu(x)}\phi_j(x)\, \partial_{\nu(y)}\phi_j(y)}{\lambda_j - \lambda} f(y) dS_g(y). \tag{5.9}$$

(2) *In a neighborhood of an arbitrary fixed point $\lambda_j \in \sigma_D$ for $j = 1, 2, \ldots$, we have*

$$\Lambda(\lambda) = \frac{Q_j}{\lambda_j - \lambda} + H_j(\lambda), \tag{5.10}$$

where Q_j is the residue of $\Lambda(\lambda)$ at $\lambda = \lambda_j$ given by

$$Q_j f = -\sum_{i \in \mathcal{E}_j} \left(\int_{\partial M} \partial_{\nu(y)}\phi_i(y)\, f(y) dS_g(y) \right) \partial_\nu \phi_i, \tag{5.11}$$

and $H_j(\lambda) : H^{3/2}(\partial M) \to H^{1/2}(\partial M)$ is analytic in a neighborhood of λ_j.

Proof. We follow the argument of §4.1.12 in [17]. Let $E \in H^2(M)$ be an extension of f into M satisfying $E\big|_{\partial M} = f$ and $\|E\|_{H^2(M)} \leq C\|f\|_{H^{3/2}(\partial M)}$ for some constants $C > 0$. Then we have

$$(-n^{-1}\Delta_g - \lambda)(u - E) = -(-n^{-1}\Delta_g - \lambda)E$$

where u is a solution of (5.5). Since $R(\lambda) := (-n^{-1}\Delta_g - \lambda)^{-1}$ is a meromorphic operator valued function with first order poles only at $\lambda_j \in \sigma_D$, $u = E - R(\lambda)(-n^{-1}\Delta_g - \lambda)E$ is also a meromorphic $H^2(M)$-valued function with first order poles only at $\lambda_j \in \sigma_D$.

Next we prove (5.9). Integrating by parts, we compute the Fourier coefficients of u with respect to the real-valued eigenfunction ϕ_j :

$$(u, \phi_i)_{L^2(M,n)} = -\int_{\partial M} \frac{\partial_{\nu(y)}\phi_i(y)}{\lambda_i - \lambda} f(y) dS_g(y). \tag{5.12}$$

From this formula and the outward normal derivative of u, $\Lambda(\lambda)$ satisfies (5.9).

Finally we verify (5.10) and (5.11). Let $P_j : L^2(M,n) \to L^2(M,n)$ be the projection to the eigenspace corresponding to $\lambda_j \in \sigma_D$ i.e.,

$$P_j v = \sum_{i\in\mathcal{E}_j} (v, \phi_i)_{L^2(M,n)} \phi_i \quad \text{for} \quad v \in L^2(M,n).$$

In view of (5.12), we have

$$P_j u = -\frac{1}{\lambda_j - \lambda} \sum_{i\in\mathcal{E}_j} \left(\int_{\partial M} \partial_{\nu(y)}\phi_i(y)\, f(y) dS_g(y) \right) \phi_i,$$

and this implies (5.11). Moreover,

$$(1 - P_j)u = -\sum_{i\in\mathbf{Z}_{>0}\setminus\mathcal{E}_j} \frac{1}{\lambda_i - \lambda} \left(\int_{\partial M} \partial_{\nu(y)}\phi_i(y)\, f(y) dS_g(y) \right) \phi_i,$$

is analytic with respect to λ in a neighborhood of λ_j. Putting $H_j(\lambda)f = \partial_\nu((1 - P_j)u)$ on ∂M, we obatain the assertion of Proposition. □

Remark 5.2.26. The formula (5.11) means that the range of Q_j is a finite dimensional subspace spanned by $\partial_\nu\phi_i$ for $i \in \mathcal{E}_j$. Note that $\partial_\nu\phi_i$ for all $i \in \mathcal{E}_j$ are linear independent since $\{\phi_i\}_{i=1}^\infty$ is the orthogonal basis in $L^2(M,n)$. Hence $\dim \operatorname{Ran} Q_j$ coincides with the multiplicity of $\lambda_j \in \sigma_D$. We can see that the integral kernel of Q_j given by

$$-\sum_{i\in\mathcal{E}_j} \partial_{\nu(x)}\phi_i(x)\, \partial_{\nu(y)}\phi_i(y)$$

is smooth in $(x, y) \in \partial M \times \partial M$ by the regularity of Dirichlet eigenfunctions.

Let $\lambda_j \in \sigma_D$. We define $E(\lambda_j) \subset H^2(M)$ as the eigenspace associated by λ_j, and $B(\lambda_j)$ as the subspace of $H^{3/2}(\partial M)$ spanned by $\partial_\nu\phi_i$ for all $i \in \mathcal{E}_j$. We denote $E(\lambda_j)^c$ and $B(\lambda_j)^c$ as their orthogonal complements in $L^2(M)$ and $L^2(\partial M)$, respectively.

Lemma 5.2.27. *Let $\lambda_j \in \sigma_D$. Then the equation* (5.5) *with $\lambda = \lambda_j$ and $f \in H^{3/2}(\partial M)$ has a non trivial solution in $H^2(M)$ if and only if $f \in B(\lambda_j)^c$.*

Proof. If f is orthogonal to $\partial_\nu\phi_i$ for all $i \in \mathcal{E}_j$, there exist general solutions in $H^2(M)$ of the equation (5.5) with $\lambda = \lambda_j$ of the form

$$u = -\sum_{i\in\mathbf{Z}_{>0}\setminus\mathcal{E}_j} \frac{1}{\lambda_i - \lambda} \left(\int_{\partial M} \partial_{\nu(y)}\phi_i(y)\, f(y) dS_g(y) \right) \phi_i + \sum_{i\in\mathcal{E}_j} c_i\phi_i, \tag{5.13}$$

for any $c_i \in \mathbf{C}\,(i \in \mathcal{E}_j)$. Conversely, if $u \in H^2(M)$ is a non trivial solution of (5.5) with $\lambda = \lambda_j$, using Green's formula, we have

$$\int_M (\Delta_g u\, \phi_i - u\, \Delta_g \phi_i)\, dV_g = -\int_{\partial M} f\, \partial_\nu \phi_i dS_g,$$

for all $i \in \mathcal{E}_j$. Since ϕ_i is an eigenfunction associated with λ_i for all $i \in \mathcal{E}_j$, the left-hand side is equal to zero. Then f is orthogonal to $\partial_\nu \phi_i$ for all $i \in \mathcal{E}_j$. □

The above lemma implies a unique solvability in a subspace as follows.

Corollary 5.2.28. *Let $\lambda_j \in \sigma_D$. For any $f \in B(\lambda_j)^c$, there exists a unique solution $u \in E(\lambda_j)^c \cap H^2(M)$ of the equation* (5.5) *with $\lambda = \lambda_j$ represented by*

$$u = -\sum_{i \in \mathbf{Z}_{>0} \setminus \mathcal{E}_j} \frac{1}{\lambda_i - \lambda} \left(\int_{\partial M} \partial_{\nu(y)} \phi_i(y)\, f(y) dS_g(y) \right) \phi_i. \tag{5.14}$$

Proof. We have only to check the uniqueness. This is trivial since the equation (5.7) with $\lambda = \lambda_j$ has only the trivial solution in $E(\lambda_j)^c \cap H^2(M)$. □

Let $n_l \in C^\infty(\overline{M_l})\,(l = 1, 2)$ and $f \in H^{3/2}(\Gamma)$. For the Dirichlet boundary value problem of the form

$$\begin{aligned} (-\Delta_{g_l} - \lambda n_l) u_l = 0 \quad &\text{in} \quad M_l; \\ u_l = f \quad &\text{on} \quad \Gamma, \end{aligned} \tag{5.15}$$

we define the D-N map $\Lambda_l(\lambda)$ by

$$\Lambda_l(\lambda) f = \partial_{\nu,l} u_l \quad \text{on} \quad \Gamma.$$

We also denote the set of corresponding Dirichlet eigenvalues by $\sigma_{D,l}\,(l = 1, 2)$ and the corresponding Dirichlet eigenfunctions by $\phi_{l,j}\,(j = 1, 2, \ldots)$, repectively. The residue of $\Lambda_l(\lambda)\,(l = 1, 2)$ at $\lambda = \lambda_{l,j} \in \sigma_{D,l}$ is denoted by $Q_{l,j}$ and the corresponding analytic part is given by $H_{l,j}(\lambda)$. For $\lambda_{l,j} \in \sigma_{D,l}\,(l = 1, 2)$, we use the similar notations $\mathcal{E}_{l,j}, E_l(\lambda_{l,j}), B_l(\lambda_{l,j}), E_l(\lambda_{l,j})^c$ and $B_l(\lambda_{l,j})^c$ to the above.

As has been in Propositions 5.2.25, $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is also meromorphic with respect to $\lambda \in \mathbf{C}$ and has first order poles on $\sigma_{D,1} \cup \sigma_{D,2}$. We denote the set of poles of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ by σ_P. In a neighborhood of an arbitrary fixed point $\lambda_j \in \sigma_P$ for $j = 1, 2, \ldots$, we have

$$\Lambda_1(\lambda) - \Lambda_2(\lambda) = \frac{Q_{0,j}}{\lambda_j - \lambda} + H_{0,j}(\lambda), \tag{5.16}$$

where $Q_{0,j}$ is the residue of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ at $\lambda = \lambda_j$ and $H_{0,j}$ is the corresponding analytic part. $Q_{0,j}$ and $H_{0,j}(\lambda)\,(j = 1, 2, \ldots)$ have same properties of $Q_{l,j}$ and $H_l(\lambda)\,(l = 1, 2)$, respectively. In the following, we define the kernel of $\Lambda_1(\lambda) - \Lambda_2(\lambda) - \zeta$ by

$$\begin{aligned} &\ker(\Lambda_1(\lambda) - \Lambda_2(\lambda) - \zeta) \\ &= \begin{cases} \{f \in H^{3/2}(\Gamma) \mid (\Lambda_1(\lambda) - \Lambda_2(\lambda) - \zeta) f = 0\} & \text{if} \quad \lambda \notin \sigma_P, \\ \{f \in H^{3/2}(\Gamma) \mid Q_{0,j} f = (H_{0,j}(\lambda_j) - \zeta)\, f = 0\} & \text{if} \quad \lambda = \lambda_j \in \sigma_P. \end{cases} \end{aligned} \tag{5.17}$$

Now we can state the relation between locally isotropic ITEs and the D-N map as follows.

Lemma 5.2.29. (1) *Suppose $\lambda_j \notin \sigma_{D,1} \cap \sigma_{D,2}$. Then $\lambda_j \in \mathbf{C}$ is a locally isotropic ITE if and only if $\ker(\Lambda_1(\lambda_j) - \Lambda_2(\lambda_j) - \zeta) \neq \{0\}$. The multiplicity of λ_j coincides with $\dim(\ker(\Lambda_1(\lambda_j) - \Lambda_2(\lambda_j) - \zeta))$.*

(2) *Suppose* $\lambda_j \in \sigma_{D,1} \cap \sigma_{D,2}$. *Then* $\lambda_j \in \mathbf{R}$ *is a locally isotropic ITE if and only if* $\ker(\Lambda_1(\lambda_j) - \Lambda_2(\lambda_j) - \zeta) \neq \{0\}$ *or the ranges of* $Q_{1,j}$ *and* $Q_{2,j}$ *have a non trivial intersection. The multiplicity of* λ_j *coincides with the sum of* $\dim(\ker(\Lambda_1(\lambda_j) - \Lambda_2(\lambda_j) - \zeta))$ *and the dimension of the above intersection.*

Proof. We first prove the assertion (1). When $\lambda_j \notin \sigma_{D,1} \cup \sigma_{D,2}$, this lemma is a direct consequence of the definition of locally isotropic ITEs. We have only to show for $\lambda_j \in \sigma_{D,1} \backslash \sigma_{D,2}$. For $0 \neq f \in \ker(\Lambda_1(\lambda_j) - \Lambda_2(\lambda_j) - \zeta)$, we have $Q_{1,j} f = (H_{1,j}(\lambda_j) - \Lambda_2(\lambda_j) - \zeta) f = 0$. From $Q_{1,j} f = 0$ and (5.11), we have $f \in B_1(\lambda_j)^c$. By Lemma 5.2.27, the Dirichlet boundary value problem of the form

$$\begin{aligned} (-\Delta_{g_1} - \lambda_j n_1) u_1 &= 0 \quad \text{in} \quad M_1; \\ u_1 &= f \quad \text{on} \quad \Gamma, \end{aligned} \tag{5.18}$$

has a non trivial solution. On the other hand, using the equality $\Lambda_2(\lambda_j) f = (H_{1,j}(\lambda_j) - \zeta) f$, we obtain the boundary value problem of the form

$$\begin{aligned} (-\Delta_{g_2} - \lambda_j n_2) u_2 &= 0 \quad \text{in} \quad M_2; \\ u_2 &= f \quad \text{on} \quad \Gamma; \\ \partial_{\nu,2} u_2 &= (H_{1,j}(\lambda_j) - \zeta) f \quad \text{on} \quad \Gamma. \end{aligned} \tag{5.19}$$

Summarizing (5.18), (5.19) and $\partial_{\nu,1} u_1 = H_{1,j}(\lambda_j) f$, we conclude that λ_j is a locally isotropic ITE. Conversely, if $\lambda_j \in \sigma_{D,1} \setminus \sigma_{D,2}$ is a locally isotropic ITE, the equation (5.15) for $l = 1$ and $\lambda = \lambda_j$ with the condition $u_1 = f \neq 0$ on Γ must have a non trivial solution. From Lemma 5.2.27, we have $f \in B_1(\lambda_j)^c$. In view of (5.11), this implies $Q_{1,j} f = 0$. From the definition of $Q_{1,j}$ and $H_{1,j}(\lambda)$, we obtain $\partial_{\nu,1} u_1 = H_{1,j}(\lambda_j) f$. Hence, the boundary condition $\partial_{\nu,1} u_1 - \partial_{\nu,2} u_2 = \zeta u_1$ implies that $(H_{1,j}(\lambda_j) - \Lambda_2(\lambda_j) - \zeta) f = 0$. Therefore, we obtain $f \in \ker(\Lambda_1(\lambda_j) - \Lambda_2(\lambda_j) - \zeta)$. We have proven the assertion (1).

For the assertion (2), we have only to show the latter case. We note that in this case, $\zeta = 0$ on Γ. Let $\lambda_j \in \sigma_{D,1} \cap \sigma_{D,2}$. In fact, if there exists a non trivial solution $(u_1, u_2) \in E_1(\lambda_j) \times E_2(\lambda_j)$ of the locally isotropic ITE problem of the form

$$\begin{aligned} (-\Delta_{g_1} - \lambda_j n_1) u_1 &= 0 \quad \text{in} \quad M_1; \\ (-\Delta_{g_2} - \lambda_j n_2) u_2 &= 0 \quad \text{in} \quad M_2; \\ u_1 = u_2 &= 0 \quad \text{on} \quad \Gamma; \\ \partial_{\nu,1} u_1 &= \partial_{\nu,2} u_2 \quad \text{on} \quad \Gamma, \end{aligned}$$

then we have that the ranges of $Q_{1,j}$ and $Q_{2,j}$ have a non trivial intersection. Conversely, if the ranges of $Q_{1,j}$ and $Q_{2,j}$ have a non trivial intersection, then there exist numbers i_1 and i_2 such that the eigenfunctions ϕ_{1,i_1} associated with $\lambda_{i_1} \in \sigma_{D,1}$ with $i_1 \in \mathcal{E}_{1,j}$ and ϕ_{2,i_2} associated with $\lambda_{i_2} \in \sigma_{D,2}$ with $i_2 \in \mathcal{E}_{2,j}$ satisfy that $\partial_{\nu,1} \phi_{1,i_1} = \partial_{\nu,2} \phi_{2,i_2}$ on Γ. Therefore, $\lambda_j = \lambda_{i_1} = \lambda_{i_2}$ is a locally isotropic ITE. □

Remark 5.2.30. Let λ_j be a locally isotropic ITE. In [21], the authors call λ_j *singular ITE* if λ_j satisfies the latter condition in the assertion (2) of Lemma 5.2.29.

5.2.2 Local regularizer

Now let us compute the symbol of the D-N map. Here we construct the local regularizer for the equation (5.5). As in [20], we follow the argument of §2 in [21], slightly modifying it for our case.

In the following, we assume that the equation (5.5) is uniquely solvable in $H^2(M)$ or a suitable subspace of $L^2(M)$.

We take a point $x^{(0)} \in \partial M$ and fix it. Let $V \subset \partial M$ be a sufficiently small neighborhood of $x^{(0)}$ in ∂M. There exists a small open domain $U \subset M$ such that $\overline{U} \cap \Gamma = V$ and U is diffeomorphic to an open domain $\Omega \subset \mathbf{R}^d_+ := \{y = (y_1, \ldots, y_{d-1}, y_d) \mid y_d \geq 0\}$.

We introduce local coordinates $y = (y_1, \ldots, y_{d-1}, y_d)$ in Ω with the center $x^{(0)} \in V$ such that

$$
\begin{aligned}
&\cdot\, x^{(0)} = 0 \in \mathbf{R}^d_+.\\
&\cdot\, \Omega \text{ is given by the upper half unit ball } \{y \in \mathbf{R}^d_+ \mid |y| < 1,\ y_d > 0\}.\\
&\cdot\, \text{the subset } \partial\Omega^0 := \{y \in \overline{\Omega} \mid y_d = 0\} \text{ is diffeomorphic to } V.\\
&\cdot\, y_d \text{ is the distance between a point } y = (y_1, \ldots, y_{d-1}, y_d) \in \Omega \text{ and } \partial\Omega^0.
\end{aligned}
\tag{5.20}
$$

Therefore, we have

$$
(g^{ij}(y))_{i,j=1}^d = \begin{bmatrix} \widetilde{g}(y') & \widetilde{p}(y) \\ {}^t\widetilde{p}(y) & 1 \end{bmatrix}
$$

for $y = (y', y_d) = (y_1, \ldots, y_{d-1}, y_d)$ in U where $\widetilde{g}(y') = (\widetilde{g}^{ij}(y'))_{i,j=1}^{d-1}$ is a smooth, positive-definite and symmetric $(d-1) \times (d-1)$-matrix valued function and $\widetilde{p}(y) = {}^t(\widetilde{p}_1(y), \ldots, \widetilde{p}_{d-1}(y))$ is a $(d-1)$-dimensional vector valued function satisfying $\widetilde{p}_i(y)\big|_{y_d=0} = 0$.

A function $F(y', y_d, \xi', \xi_d)$ with $(y', y_d), (\xi', \xi_d) \in \mathbf{R}^d$ is homogeneous of the generalized degree s if F satisfies

$$
F(t^{-1}y', t^{-1}y_d, t\xi', t\xi_d) = t^s F(y', y_d, \xi', \xi_d), \tag{5.21}
$$

for any $t > 0$. For $F(y_d, \xi')$, we define the homogeneity by the similar manner.

For $n \in C^\infty(\overline{M})$, taking the y-coordinate as above, we can rewrite $D := -\Delta_g - \lambda n$ as

$$
D = -\partial_d^2 - \sum_{i,j=1}^{d-1} \widetilde{g}^{ij}(y')\partial_i\partial_j - 2\sum_{i=1}^{d} \widetilde{p}_i(y)\partial_i\partial_d - \sum_{i=1}^{d} \widetilde{h}_i(y)\partial_i - \lambda n(y) \tag{5.22}
$$

in U with $\widetilde{h}_i(y) = (\sqrt{G})^{-1}\sum_{j=1}^d \partial_j(\sqrt{G}\, g^{ij})$.

The symbol of D is given by

$$
D(\lambda; y', y_d, \xi', \xi_d) = \xi_d^2 + \sum_{i,j=1}^{d-1} \widetilde{g}^{ij}(y')\xi_i\xi_j + 2\sum_{i=1}^{d} \widetilde{p}_i(y)\xi_i\xi_d - i\sum_{i=1}^{d-1} \widetilde{h}_i(y)\xi_i - \lambda n(y). \tag{5.23}
$$

In the following, let $N > 0$ be a sufficiently large integer. Now we take $z = (z', 0) \in \partial\Omega^0$ arbitrarily and fix it. Using the Taylor series of $\widetilde{g}^{ij}(y')$, $\widetilde{p}_i(y)$, $\widetilde{h}_i(y)$ and $n(y)$ with respect to y centered at $z = (z', 0) \in \partial\Omega^0$, we can expand the symbol $D(\lambda; y', y_d, \xi', \xi_d)$ of D as the sum of following terms :

$$
D_0(z'; \xi', \xi_d) = \xi_d^2 + \sum_{i,j=1}^{d-1} \widetilde{g}^{ij}(z')\xi_i\xi_j, \tag{5.24}
$$

$$
\begin{aligned}
D_1(z'; y' - z', y_d, \xi', \xi_d) = &\sum_{i,j=1}^{d-1} (\nabla_{y'}\widetilde{g}^{ij})(z') \cdot (y' - z')\xi_i\xi_j - i\sum_{i=1}^{d} \widetilde{h}_i(z', 0)\xi_i \\
&+ 2\sum_{i=1}^{d-1} \left\{(\nabla_{y'}\widetilde{p}_i)(z', 0) \cdot (y' - z') + (\partial_d\widetilde{p}_i)(z', 0)y_d\right\} \xi_i\xi_d,
\end{aligned}
\tag{5.25}
$$

and

$$
\begin{aligned}
&D_m(\lambda, z'; y'-z', y_d, \xi', \xi_d) \\
&= \sum_{i,j=1}^{d-1} \sum_{|\alpha'|=m} \frac{(\partial_{y'}^{\alpha'} \widetilde{g}^{ij})(z')}{\alpha'!}(y'-z')^{\alpha'} \xi_i \xi_j + 2\sum_{i=1}^{d} \sum_{|\alpha|=m} \frac{(\partial_y^{\alpha} \widetilde{p}_i)(z',0)}{\alpha!}(y'-z')^{\alpha'} y_d^{\alpha_d} \xi_i \xi_d \\
&\quad + i\sum_{i=1}^{d} \sum_{|\alpha|=m-1} \frac{(\partial_y^{\alpha} \widetilde{h}_i)(z',0)}{\alpha!}(y'-z')^{\alpha'} y_d^{\alpha_d} \xi_i - \lambda \sum_{|\alpha|=m-2} \frac{(\partial_y^{\alpha} n)(z',0)}{\alpha!}(y'-z')^{\alpha'} y_d^{\alpha_d},
\end{aligned}
\tag{5.26}
$$

for $2 \le m \le N$ with the remainder term $D'_{N+1}(\lambda, z'; y'-z', y_d, \xi', \xi_d)$ which has zero of order $N-1$ at $y'=0$ or $(y', y_d) = (0,0)$. Hence, we rewrite the sum of (5.24)–(5.26) and the remainder term as

$$
\begin{aligned}
D(\lambda; y', y_d, \xi', \xi_d) &= D_0(z'; \xi', \xi_d) + D_1(z'; y'-z', y_d, \xi', \xi_d) \\
&+ \sum_{m=2}^{N} D_m(\lambda, z'; y'-z', y_d, \xi', \xi_d) + D'_{N+1}(\lambda, z'; y'-z', y_d, \xi', \xi_d).
\end{aligned}
\tag{5.27}
$$

Then each $D_m(\lambda, z'; y'-z', y_d, \xi', \xi_d)$ is a homogeneous polynomial in $y'-z', y_d, \xi', \xi_d$ of generalized degree $2-m$. In particular, D_0 is the principal symbol of D. D'_{N+1} vanishes at $(z', 0)$ and the order of the zero is $N-1$.

In the following, we denote

$$
|\xi'|^2_{\partial M} := \sum_{i,j=1}^{d-1} \widetilde{g}^{ij}(y')\xi_i \xi_j. \tag{5.28}
$$

We define the following differential operators :

$$
\widetilde{D}_0 = D_0(z'; \xi', i\partial_d) = -\partial_d^2 + |\xi'|^2_{\partial M}, \tag{5.29}
$$

$$
\widetilde{D}_1 = D_1(z'; -i\partial_{\xi'}, y_d, \xi', i\partial_d), \tag{5.30}
$$

and

$$
\widetilde{D}_m = D_m(\lambda, z'; -i\partial_{\xi'}, y_d, \xi', i\partial_d), \quad m \ge 2. \tag{5.31}
$$

Proposition 5.2.31. *Let $F(y_d, \xi')$ be a smooth function and homogeneous of the generalized degree s with respect to y_d and ξ'. Then we have that $\widetilde{D}_m F$ is the homogeneous of the generalized degree $2-m+s$ with respect to y_d and ξ'.*

Proof. Note that $F(y_d, \xi') = |\xi'|^s F(|\xi'| y_d, \hat{\xi}')$. Then we can easily show that $\partial_d F$ and $\partial_{\xi_j} F$ are homogeneous of generalized degree $s+1$ and $s-1$, respectively. □

Now let us construct an approximate solution of (5.5).

Lemma 5.2.32. *Suppose $|\xi'|_{\partial M} \neq 0$. The boundary value problem for a system of second order ordinary differential equations of the form*

$$
\sum_{n=0}^{m} \widetilde{D}_n E_{m-n}(\lambda, z'; y_d, \xi') = 0, \quad m \ge 0; \tag{5.32}
$$

$$
E_0\big|_{y_d=0} = 1, \quad E_m\big|_{y_d=0} = 0, \quad m \ge 1, \tag{5.33}
$$

has a unique solution $\{E_m\}_{m=0,1,2,\ldots}$ satisfying that each E_m converges to zero as $y_d \to \infty$. In particular, we have $E_0(z'; y_d, \xi') = e^{-|\xi'|_{\partial M} y_d}$. Each solution E_m is smooth and homogeneous with respect to y_d and ξ' of generalized degree $-m$. (For $m \ge 2$, each E_m depends also on λ. We omit λ in the notation.)

Proof. Since $\widetilde{D}_0 = -\partial_d^2 + |\xi'|^2_{\partial M}$ and $E_0\big|_{y_d=0} = 0$, we have $E_0(z'; y_d, \xi') = e^{-|\xi'|_{\partial M} y_d}$. Obviously, E_0 is homogeneous of the generalized degree 0.

We assume that a function $p(y_d, \xi')$ decays exponentially as $y_d \to \infty$ and is homogeneous of the generalized degree s. Let us consider a solution v of the boundary value problem with known function $p(y_d, \xi')$ of the form

$$\begin{aligned} (-\partial_d^2 + |\xi'|^2_{\partial M})v = p \quad &\text{on} \quad (0, \infty); \\ v(0, \xi') = 0, \end{aligned} \tag{5.34}$$

satisfying that $v(y_d, \xi')$ converges to zero as $y_d \to \infty$. Extending v and p to be zero in $-\infty < y_d < 0$, we have

$$v(y_d, \xi') = \frac{1}{2|\xi'|_{\partial M}} \left(\int_0^{y_d} e^{-|\xi'|_{\partial M}(y_d - \eta)} p(\eta, \xi') d\eta + \int_{y_d}^{\infty} e^{-|\xi'|_{\partial M}(\eta - y_d)} p(\eta, \xi') d\eta \right).$$

Then, putting $\tau = t\eta$, we have

$$\begin{aligned} &v(t^{-1} y_d, t\xi') \\ &= \frac{t^{s-2}}{2|\xi'|_{\partial M}} \left(\int_0^{y_d} e^{-|\xi'|_{\partial M}(y_d - \tau)} p(\tau, \xi') d\tau + \int_{y_d}^{\infty} e^{-|\xi'|_{\partial M}(\tau - y_d)} p(\tau, \xi') d\tau \right) \\ &= t^{s-2} v(y_d, \xi'), \end{aligned}$$

which shows that v is homogeneous of the generalized degree $s - 2$ with respect to y_d and ξ'. In view of Proposition 5.2.31, we have $\widetilde{D}_1 E_0$ is homogeneous of the generalized degree 1. Therefore, we obtain E_1 is homogeneous of the generalized degree -1. Repeating the similar argument inductively, we can easily show that E_m is homogeneous of the generalized degree $-m$. □

Let $\beta(\xi') \in C^\infty(\mathbf{R}^{d-1})$ vanish in a small neighborhood of $\xi' = 0$, and be equal to one outside a large neighborhood of $\xi' = 0$. Taking $\psi \in H^{3/2}(\partial\Omega^0)$ with a compact support in $\partial\Omega^0$, we define for $y' \in \partial\Omega^0$

$$(Q_m \psi)(z'; y', y_d) = (2\pi)^{-(d-1)} \int e^{iy' \cdot \xi'} \beta(\xi') E_m(z'; y_d, \xi') \int e^{-iw' \cdot \xi'} \psi(w') dw' d\xi' \tag{5.35}$$

and we put

$$R_N = \sum_{m=0}^{N} Q_m. \tag{5.36}$$

Letting

$$\begin{aligned} q_m(z'; y', y_d) &= (2\pi)^{-(d-1)} \int e^{iy' \cdot \xi'} \beta(\xi') E_m(z'; y_d, \xi') d\xi', \\ r_N(z'; y' - w', y_d) &= \sum_{m=0}^{N} q_m(z'; y' - w', y_d) \end{aligned} \tag{5.37}$$

we have that q_m and r_N are distributions in $\mathcal{S}'$, and

$$(Q_m \psi)(z'; y', y_d) = \int q_m(z'; y' - w', y_d) \psi(w') dw', \tag{5.38}$$

$$(R_N \psi)(z'; y', y_d) = \int r_N(z'; y' - w', y_d) \psi(w') dw'. \tag{5.39}$$

We represent D appeared in (5.22) in the form

$$D = D_0(z'; i\partial_{y'}, i\partial_d) + D_1(z'; y'-z', y_d, i\partial_{y'}, i\partial_d) + \sum_{m=2}^{N} D_m(\lambda, z'; y'-z', y_d, i\partial_{y'}, i\partial_d) + D'_{N+1}(\lambda, z'; y'-z', y_d, i\partial_{y'}, i\partial_d).$$

In the following, we consider

$$Dr_N = \sum_{J=0}^{N} \sum_{l+m=J} D_l q_m + \sum_{J=N+1}^{2N} \sum_{l,m\le N, l+m=J} D_l q_m + D'_{N+1} r_N. \tag{5.40}$$

Lemma 5.2.33. *Let l, m and N be sufficiently large. We have $D_l q_m \in H^\gamma(\Omega)$ and $D'_{N+1} r_N \in H^{\gamma'}(\Omega)$ where $\gamma = O(l+m)$ and $\gamma' = O(N)$.*

Proof. Note that $D_l(\lambda, z'; y'-z', y_d, i\partial_{y'}, i\partial_d)$ and $D'_{N+1}(\lambda, z'; y'-z', y_d, i\partial_{y'}, i\partial_d)$ are operators which are given by sums of terms like $(y'-z')^{\alpha'} y_d^{\alpha_d} \partial_{y'}^{\beta'} \partial_d^{\beta_d}$ up to a smooth function with $-|\alpha'| - \alpha_d + |\beta'| + \beta_d = 2 - l$ or $2-(N+1)$ and $|\beta'| + \beta_d \le 2$. In view of Proposition 5.2.31, it is sufficient to show

$$(y')^{\alpha'} y_d^{\alpha_d} q_m(z; y', y_d) \in H^\gamma(\Omega), \tag{5.41}$$

since the derivative $\partial_{y'}^{\beta'} \partial_d^{\beta_d}$ is order zero, one or two.

Now we have

$$(y')^{\alpha'} y_d^{\alpha_d} q_m(z; y', y_d) = i^{|\alpha'|} (2\pi)^{-(d-1)} \int e^{iy'\cdot\xi'} \partial_{\xi'}^{\alpha'} \big(y_d^{\alpha_d} \beta(\xi') |\xi'|^{-m} E_m(z'; |\xi'| y_d, \hat{\xi}') \big) d\xi'.$$

Since $y_d^{\alpha_d} |\xi'|^{-m} E_m(z'; |\xi'| y_d, \hat{\xi}')$ is homogeneous of the generalized degree $-m-\alpha_d$, using Proposition 5.2.31, we have

$$\left| \partial_{\xi'}^{\alpha'} \big(y_d^{\alpha_d} \beta(\xi') E_m(z'; y_d, \xi') \big) \right| \le C_{m,\alpha} (1 + |\xi'|)^{-m-|\alpha'|-\alpha_d},$$

which implies (5.41). ☐

Theorem 5.2.34. *Let $N > 1$ be sufficiently large. We have that R_N is local regularizer for (5.5), i.e.,*

$$DR_N\psi \in H^s(\Omega), \quad R_N\psi\big|_{y_d=0} - \psi \in C^\infty(\partial\Omega^0), \tag{5.42}$$

for $\psi \in H^{3/2}(\partial\Omega^0)$ which has a compact support in $\partial\Omega^0$ and $s = O(N)$.

Proof. Note that

$$\begin{aligned} &D_l(\lambda, z; y'-z', y_d, i\partial_{y'}, i\partial_d) q_m(z'; y'-w', y_d) \\ &= (2\pi)^{-(d-1)} \int e^{i(y'-w')\cdot\xi'} \widetilde{D}_l\big(\beta(\xi') E_m(z'; y_d, \xi')\big) d\xi'. \end{aligned} \tag{5.43}$$

Using (5.32), we have

$$\sum_{J=0}^{N} \sum_{l+m=J} \widetilde{D}_l E_m(z'; y_d, \xi') = 0. \tag{5.44}$$

In view of Lemma 5.2.33 and (5.40), we have that (5.43) and (5.44) imply $DR_N\psi \in H^s(\Omega)$ for $s = O(N)$.

We obatin that

$$\begin{aligned}
&R_N\psi(y', y_d) - \psi(y') \\
&= (2\pi)^{-(d-1)} \iint e^{i(y'-w')\cdot\xi'} \left(\sum_{m=0}^{N} \beta(\xi') E_m(z'; y_d, \xi') - 1 \right) \psi(w') d\xi' dw' \\
&\to (2\pi)^{-(d-1)} \iint e^{i(y'-w')\cdot\xi'} \left(\beta(\xi') - 1 \right) \psi(w') d\xi' dw',
\end{aligned}$$

as $y_d \to 0$. Since $\beta(\xi') - 1 \in C_0^\infty(\mathbf{R}^{d-1})$, we have $R_N\psi\big|_{y_d=0} - \psi \in C^\infty(\partial\Omega^0)$. □

Remark 5.2.35. The formal sum

$$(R\psi)(z'; y', y_d) := \int \sum_{m=0}^{\infty} q_m(z'; y' - w', y_d)\psi(w')dw',$$

is a singular integro-differential operator (see [29]). In general, a linear operator P on a d-dimensional compact manifold M is a singular integro-differential operator of order l if there exist homogeneous functions $p_j(x, \xi) \in C^\infty(M, \mathbf{R}^d \setminus \{0\})$ in ξ with homogeneous degree $l - j$ such that for a function u with support in a local coordinate neighborhood $U \subset M$,

$$Pu(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} \beta(\xi) \sum_{j=0}^{N} p_j(x, \xi)u(y)dyd\xi + T_{N+1}u \quad \text{for} \quad x \in U$$

where $\beta \in C^\infty(\mathbf{R}^d)$ is an arbitrary function which satisfies $\beta(\xi) = 0$ for $|\xi| \leq 1$ and $\beta(\xi) = 1$ for $|\xi| \geq 2$, and T_{N+1} is an operator which increases the smoothness i.e. $H^s(M) \to H^{s+O(N)}(M)$ for any $s \in \mathbf{R}$. The principal symbol of P is $p_0(x, \xi)$ and the full symbol of P is the formal sum $\sum_{j=0}^{\infty} p_j(x, \xi)$. Then the ellipticity of P is defined by $p_0(x, \xi) \neq 0$ for all $\xi \neq 0$. This implies that we can construct the parametrix of P (see [16]). Therefore, if P is an elliptic singular integro-differential operator, P is Fredholm.

Since we have $\partial_\nu = -\partial_d$ in y-coordinates, we can easily show the following fact. As a consequence of Corollary 5.2.28 and Theorem 5.2.34. See also Lemma 11 and Theorem 14 in [29].

Corollary 5.2.36. (1) *When $\lambda \notin \sigma_D$, $\Lambda(\lambda)$ is a singular integro-differential operator on $H^{3/2}(\partial M)$ with the full symbol given by the asymptotic series of the form*

$$\Lambda(\lambda; y', \xi') = -\sum_{m=0}^{\infty} \partial_d E_m(y'; y_d, \xi')\Big|_{y_d=0} \quad \textit{for} \quad y' \in \partial\Omega^0. \tag{5.45}$$

(2) *When $\lambda_j \in \sigma_D$, the regular part $H_j(\lambda)$ of $\Lambda(\lambda)$ at λ_j is a singular integro-differential operator on $B(\lambda_j)^c$ with the full symbol given by* (5.45).

5.2.3 Principal symbol of the D-N map

On the Riemannian manifolds (M_1, g_1) and (M_2, g_2), we consider the similar argument to the above. We take a point $x^{(0)} \in \Gamma$ and fix it. Let $V \subset \Gamma$ and $U_l \subset M_l \, (l = 1, 2)$ be a sufficiently small neighborhood of $x^{(0)}$ in Γ and a small open domain such that $\overline{U_l} \cap \Gamma = V$

and U_l is diffeomorphic to an open domain $\Omega \subset \mathbf{H}^d$, respcetively. We take the similar local coordinates $y = (y', y_d) = (y_1, \ldots, y_{d-1}, y_d)$ in Ω to (5.20). We use the similar notations $\widetilde{g}_l(y') = (\widetilde{g}_l^{ij}(y'))_{i,j=1}^{d-1}, \widetilde{p}_l(y) = {}^t(\widetilde{p}_{l,1}(y), \ldots, \widetilde{p}_{l,d-1}(y))$. In view of the assumption (I-1), we have in y-coordinates that $\widetilde{g}_1^{ij}(y') = \widetilde{g}_2^{ij}(y')$, $\widetilde{p}_{1,i}(y)|_{y_d=0} = \widetilde{p}_{2,i}(y)|_{y_d=0} = 0$.

For $n_l \in C^\infty(\overline{M_l})$ $(l = 1, 2)$, the symbol of $D_l = -\Delta_{g_l} - \lambda n_l$ is denoted by $D_l(\lambda; y', y_d, \xi', \xi_d)$. Using the Taylor series of the coefficients of the polynomial $D_l(\lambda; y', y_d, \xi', \xi_d)$ in $\xi = (\xi', \xi_d)$ of degree 2 with respect to y centered at $z' = (z', 0) \in \partial\Omega_0$, we can expand $D_l(\lambda, z'; y', y_d, \xi', \xi_d)$ as the sum of the form

$$D_l(\lambda, z'; y', y_d, \xi', \xi_d) = \sum_{m=0}^{N} D_{l,m}(\lambda, z'; y' - z', y_d, \xi', \xi_d) + D'_{N+1}(\lambda, z'; y' - z', y_d, \xi', \xi_d)$$

for large integer N. Here, each $D_{l,m}(\lambda, z'; y', y_d, \xi', \xi_d)$ $(l = 1, 2)$ is a homogeneous polynomial in y', y_d, ξ', ξ_d of generalized degree $2 - m$ and the remainder term $D'_{l,N+1}(\lambda, z'; y' - z', y_d, \xi', \xi_d)$ has zero of order $N - 1$ at $y' = 0$ or $(y', y_d) = (0, 0)$.

Let $|\xi'|_\Gamma^2 := \sum_{i,j=1}^{d-1} \widetilde{g}^{ij}(y')\xi_i\xi_j$. We also define the operators $\widetilde{D}_{l,m}$ $(m = 0, 1, 2, \ldots)$ by

$$\widetilde{D}_{l,m} = D_{l,m}(\lambda, z'; -i\partial_{\xi'}, y_d, \xi', i\partial_d)$$

In particular, $\widetilde{D}_{l,0}$ is represented as $-\partial_d^2 + |\xi'|_\Gamma^2$ and $\widetilde{D}_{l,1}$ is independent of λ. The sequence $\{E_{l,m}\}_{m=0,1,2,\ldots}$ is defined by the solution of the boundary value problems of the form

$$\begin{aligned} &\sum_{n=0}^{m} \widetilde{D}_{l,n} E_{l,m-n}(\lambda, z'; y_d, \xi') = 0, \quad m \geq 0; \\ &E_{l,0}\big|_{y_d=0} = 1, \quad E_{l,m}|_{y_d=0} = 0, \quad m \geq 1, \end{aligned} \tag{5.46}$$

with the additional condition such that $E_{l,m}$ converges to zero as $y_d \to \infty$ for $m \geq 0$.

We compute the principal symbol of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$. In y-coordinates, we can locally represent $\partial_{\nu,l}^m = (-1)^m \partial_d^m$ $(l = 1, 2)$. Under the assumptions (A) and (I-1), we additionally assume on Γ that the metrics g_1, g_2 and the functions n_1, n_2 satisfy either

$$\begin{aligned} &\cdot \text{ For all } x \in \Gamma, \partial_{\nu,1}^m g_1^{ij}(x) = \partial_{\nu,2}^m g_2^{ij}(x) \text{ for } m \leq 2 \text{ and } i, j = 1, \ldots, d. \\ &\cdot\ n_1(x) \neq n_2(x). \end{aligned} \tag{I-2-1}$$

or

$$\begin{aligned} &\cdot \text{ For all } x \in \Gamma, \partial_{\nu,1}^m g_1^{ij}(x) = \partial_{\nu,2}^m g_2^{ij}(x) \text{ for } m \leq 3, \text{ and } i, j = 1, \ldots, d. \\ &\cdot\ n_1(x) = n_2(x), \partial_{\nu,1} n_1(x) \neq \partial_{\nu,2} n_2(x). \end{aligned} \tag{I-2-2}$$

Note that, under the assumptions (I-1) with (I-2-1) or (I-2-2), we can see $\widetilde{D}_{1,m} = \widetilde{D}_{2,m}$ for $m \leq 1$ or $m \leq 2$, respectively.

When $\lambda = \lambda_j \in \sigma_P$, we define a subspace $B_0(\lambda_j)$ of $H^{3/2}(\Gamma)$ by $B_0(\lambda_j) = \widetilde{B}_1(\lambda_j) \cup \widetilde{B}_2(\lambda_j)$ where $\widetilde{B}_l(\lambda_j) = B_l(\lambda_j)$ if $\lambda_j \in \sigma_{D,l}$, and $\widetilde{B}_l(\lambda_j) = \emptyset$ if otherwise. We denote $B_0(\lambda_j)^c$ as the orthogonal complement of $B_0(\lambda_j)$ in $L^2(\Gamma)$.

When $\lambda = \lambda_j \in \sigma_P$, we call $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ Fredholm if its regular part $H_{0,j}(\lambda_j)$ is Fredholm.

Lemma 5.2.37. *In the following, we suppose $\lambda \neq 0$.*

(1) *Let $\lambda \notin \sigma_P$. For the case* (I-2-1), *we have $\Lambda_1(\lambda) - \Lambda_2(\lambda) : H^{3/2}(\Gamma) \to H^{5/2}(\Gamma)$ is an elliptic singular integro-differential operator with the principal symbol of the form*

$$-\frac{\lambda(n_1(x) - n_2(x))}{2|\xi'|_\Gamma} \quad \textit{for} \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1}. \tag{5.47}$$

(2) *Let* $\lambda \notin \sigma_P$. *For the case* (I-2-2)*, we have* $\Lambda_1(\lambda) - \Lambda_2(\lambda) : H^{3/2}(\Gamma) \to H^{7/2}(\Gamma)$ *is an elliptic singular integro-differential operator with the principal symbol of the form*

$$\frac{\lambda(\partial_{\nu,1} n_1(x) - \partial_{\nu,2} n_2(x))}{4|\xi'|_\Gamma^2} \quad \text{for} \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1}. \tag{5.48}$$

(3) *When* $\lambda \in \sigma_P$*, the regular part of* $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ *is singular integro-differential operator on* $B_0(\lambda)^c$ *with order* -1 *for* (I-2-1) *or* -2 *for* (I-2-2)*. Its principal symbol is given by* (5.47) *or* (5.48)*, respectively.*

(4) *For both of* (I-2-1) *or* (I-2-2)*,* $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ *is Fredholm for* $\lambda \in \mathbf{C} \setminus \{0\}$.

Proof. Let g_1, g_2 and n_1, n_2 satisfy (I-2-1). In y-coordinates, we have $\widetilde{D}_{1,j} = \widetilde{D}_{2,j}$ for $j = 0, 1$ and $\widetilde{D}_{1,2} - \widetilde{D}_{2,2} = -\lambda(n_1(y', 0) - n_2(y', 0))$. From (5.46), $E_{l,j}$ for $j = 0, 1, 2$ satisfy that $E_{1,0} = E_{2,0} = e^{-|\xi'|_\Gamma y_d}$, $E_{1,1} = E_{2,1}$ and

$$\begin{aligned} &(-\partial_d^2 + |\xi'|_\Gamma^2)(E_{1,2} - E_{2,2}) = \lambda(n_1(y', 0) - n_2(y', 0))e^{-|\xi'|_\Gamma y_d}, \\ &E_{1,2}\big|_{y_d=0} - E_{2,2}\big|_{y_d=0} = 0, \quad E_{1,2} - E_{2,2} \to 0 \quad \text{as} \quad y_d \to \infty \end{aligned} \tag{5.49}$$

respectively. A particular solution of (5.49) is given by

$$\frac{\lambda(n_1(y', 0) - n_2(y', 0))}{2|\xi'|_\Gamma} y_d e^{-|\xi'|_\Gamma y_d},$$

which vanishes at $y_d = 0$ and $y_d \to \infty$. Then we can take it as $E_{1,2} - E_{2,2}$, and $-\partial_d(E_{1,2} - E_{2,2})$ at $y_d = 0$ is the principal symbol of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$. In view of the assertion (1) in Corollary 5.2.36, we have the assertion (1).

Next we assume that g_1, g_2 and n_1, n_2 satisfy (I-2-2). As above, we have $\widetilde{D}_{1,j} = \widetilde{D}_{2,j}$ for $j = 0, 1, 2$, and $\widetilde{D}_{1,3} - \widetilde{D}_{2,3} = -\lambda(\partial_d n_1(y', 0) - \partial_d n_2(y', 0))y_d$. Then we also obtain that $E_{1,2} = E_{2,2}$ and

$$E_{1,3} - E_{2,3} = \frac{\lambda}{4}(\partial_d n_1(y', 0) - \partial_d n_2(y', 0)) \frac{y_d}{|\xi'|_\Gamma} \left(y_d + \frac{1}{|\xi'|_\Gamma} \right) e^{-|\xi'|_\Gamma y_d}.$$

Hence we obtain the assertion (2).

In view of Corollary 5.2.28 and the assertion (2) in Corollary 5.2.36, we can show the assertion (3) by the similar way.

The ellipticity of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ implies that $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is Fredholm for $\lambda \in \mathbf{C} \setminus \{0\}$. □

5.3 Interior transmission eigenvalues

5.3.1 Discreteness of the set of ITEs

For the proof of discreteness, we need to use the analytic Fredholm theory which was generalized by [4]. See also Appendix A in [27]. Let H_1 and H_2 are Hilbert spaces. We take a connected open domain $G \subset \mathbf{C}$. An operator valued function $A(z) : H_1 \to H_2$ for $z \in G$ is finitely meromorphic if the principal part of the Laurent series at a pole of $A(z)$ is a finite rank operator. In particular, $\Lambda_l(\lambda) : H^{3/2}(\Gamma) \to H^{1/2}(\Gamma)$ $(l = 1, 2)$ is finitely meromorphic in $\mathbf{C} \setminus \{0\}$ as has been seen in Proposition 5.2.25.

Theorem 5.3.38. *Suppose an operator valued function $A(z) : H_1 \to H_2$ for $z \in G$ is finitely meromorphic and Fredholm. If there exists its bounded inverse $A(z_0)^{-1} : H_2 \to H_1$ at a point $z_0 \in G$, then $z \mapsto A(z)^{-1}$ is finitely meromorphic and Fredholm in G.*

From the above theorem, if there exists a point $\lambda_0 \in \mathbf{C} \setminus \{0\}$ such that $\Lambda_1(\lambda_0) - \Lambda_2(\lambda_0)$ is invertible, $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is invertible for all $\lambda \in \mathbf{C} \setminus (\{0\} \cup S')$ where S' is a discrete subset of $\mathbf{C}$. Therefore, for the proof of the discreteness, we have only to show that $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is invertible for some $\lambda \in \mathbf{C} \setminus \{0\}$.

We expand the symbol of $D_l\,(l = 1, 2)$ centered at $(z', 0) \in \partial\Omega^0$ by the same manner in subsection 5.2.2. However, here we change the definition of homogeneous functions with generalized degree s by

$$F(t\kappa; t^{-1}y', t^{-1}y_d, t\xi', t\xi_d) = t^s F(\kappa; y', y_d, \xi', \xi_d) \quad t > 0, \quad \kappa = \sqrt{\lambda} \tag{5.50}$$

for $\lambda \in \mathbf{C} \setminus \{0\}$, taking a suitable branch of $\kappa = \sqrt{\lambda}$. We gather terms of the same generalized degree in the sense (5.50), and we denote the symbol in y-coordinates as

$$D_l(\kappa; y', y_d, \xi', \xi_d) = \sum_{m=0}^{N} \mathcal{D}_{l,m}(\kappa, z'; y' - z', y_d, \xi', \xi_d)$$

up to the remainder term where $\mathcal{D}_{l,m}(\kappa, z'; y'-z', y_d, \xi', \xi_d)$ homogeneous functions of generalized degree $2 - m$. In particular, putting $\widetilde{\mathcal{D}}^{(\lambda)}_{l,m} = \mathcal{D}_{l,m}(\kappa, z'; -i\partial_{\xi'}, y_d, \xi', i\partial_d)$, we have

$$\widetilde{\mathcal{D}}^{(\lambda)}_{l,0} = -\partial_d^2 + |\xi'|^2_\Gamma - \lambda n_l(z', 0), \tag{5.51}$$

$$\widetilde{\mathcal{D}}^{(\lambda)}_{l,1} = \widetilde{D}_{l,1} + \lambda \widetilde{B}^{(\lambda)}_{l,1} \tag{5.52}$$

where $\widetilde{D}_{l,1}$ is defined by (5.30) and

$$\widetilde{B}^{(\lambda)}_{l,1} = i\nabla_{y'} n_l(z', 0) \cdot \nabla_{\xi'} - y_d \partial_d n_l(z', 0).$$

We denote by $\{E^{(\lambda)}_{l,m}\}_{m \geq 0}$ the solution of

$$\begin{aligned} &\sum_{n=0}^{m} \widetilde{\mathcal{D}}^{(\lambda)}_{l,n} E^{(\lambda)}_{l,m-n}(z'; y_d, \xi') = 0, \quad m \geq 0, \\ &E^{(\lambda)}_{l,0}\big|_{y_d=0} = 1, \quad E^{(\lambda)}_{l,m}\big|_{y_d=0} = 0, \quad m \geq 1 \end{aligned} \tag{5.53}$$

with the additional condition such that $E^{(\lambda)}_{l,m} \to 0$ converges to zero as $y_d \to \infty$ for $m \geq 0$.

In order to apply the theory of parameter-dependent pseudo-differential operators to $\Lambda_1(\lambda) - \Lambda_2(\lambda)$, we recall its definition. Let N be a d-dimensional compact manifold without boundary. We call $p(x, \xi, \tau) \in C^\infty(N \times \mathbf{R}^d \times \mathbf{R}_{\geq 0})$ a *uniformly estimated polyhomogeneous symbol of order s and regularity r* if $p(x, \xi, \tau)$ satisfies

$$\begin{aligned} &|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^j p(x, \xi, \tau)| \\ &\leq C_{\alpha\beta j} \left(\langle \xi \rangle^{r-|\beta|} + (|\xi|^2 + \tau^2 + 1)^{(r-|\beta|)/2} \right) (|\xi|^2 + \tau^2 + 1)^{(s-r-j)/2} \end{aligned} \tag{5.54}$$

on $N \times \mathbf{R}^d \times \mathbf{R}_{\geq 0}$ for constants $C_{\alpha\beta j} > 0$, and $p(x, \xi, \tau)$ has the asymptotic expansion

$$p(x, \xi, \tau) \sim \sum_{l=0}^{\infty} p_{s-l}(x, \xi, \tau) \tag{5.55}$$

where $p_{s-l}(x,\xi,\tau)$ is homogeneous with generalized degree $s-l$ with respect to ξ,τ in the sense of

$$p_{s-l}(x,t\xi,t\tau) = t^{s-l}p_{s-l}(x,\xi,\tau) \quad \text{for} \quad t>0. \tag{5.56}$$

A pseudo-differential operator $P(\tau)$ on N with a uniformly estimated polyhomogeneous symbol $p(x,\xi,\tau)$ is said to be *uniformly parameter elliptic* if the principal symbol $p_d(x,\xi\tau)$ does not vanish when $|\xi|+\tau \neq 0$. For more information and general theory on parameter-dependent operators, one can refer Chapters 2 and 3 in [15].

Let us turn to $\Lambda_1(\lambda)-\Lambda_2(\lambda)$. For $\lambda \in \mathbf{C}\setminus\mathbf{R}_{\geq 0}$, we put $\sqrt{\lambda}=\tau e^{i\theta}$ with $\tau>0$ and $\theta \in \mathbf{R}$ such that $\theta \neq 0$ modulo π. In the following, we fix a suitable θ and put

$$R(\tau) = \tau^{-2}e^{-2i\theta}(\Lambda_1(\tau^2e^{2i\theta}) - \Lambda_2(\tau^2e^{2i\theta})). \tag{5.57}$$

Lemma 5.3.39. *Let $\lambda = \tau^2e^{2i\theta} \in \mathbf{C}\setminus\mathbf{R}_{\geq 0}$.*

(1) *We assume that (I-2-1) holds. Then $R(\tau)$ is uniformly parameter elliptic with order -1 and regularity ∞. Its principal symbol is*

$$\frac{-(n_1(x)-n_2(x))}{\sqrt{|\xi'|_\Gamma^2-\tau^2e^{2i\theta}n_1(x)}+\sqrt{|\xi'|_\Gamma^2-\tau^2e^{2i\theta}n_2(x)}} \quad \textit{for} \quad x\in\Gamma, \quad \xi'\in\mathbf{R}^{d-1}. \tag{5.58}$$

(2) *We assume that (I-2-2) holds. Then $R(\tau)$ is uniformly parameter elliptic with order -2 and regularity ∞. Its principal symbol is*

$$\frac{(\partial_{\nu_1}n_1(x)-\partial_{\nu_2}n_2(x))}{4(|\xi'|_\Gamma^2-\tau^2e^{2i\theta}n(x))} \quad \textit{for} \quad x\in\Gamma, \quad \xi'\in\mathbf{R}^{d-1} \tag{5.59}$$

where $n(x) := n_1(x) = n_2(x)$.

Proof. We fix an arbitrary point $(z',0)\in\partial\Omega^0$. Suppose that (I-2-1) holds. Obviously we have

$$E_{l,0}^{(\lambda)}(z';\xi',y_d) = \exp\left(-\sqrt{|\xi'|_\Gamma^2-\lambda n_l(z',0)}y_d\right). \tag{5.60}$$

Under the assumption, we also have $\widetilde{\mathcal{A}}_{1,0}^{(\lambda)} \neq \widetilde{\mathcal{A}}_{2,0}^{(\lambda)}$ so that $E_{1,0}^{(\lambda)} \neq E_{2,0}^{(\lambda)}$. Then the principal symbol $-\partial_d(E_{1,0}^{(\lambda)}-E_{2,0}^{(\lambda)})\big|_{y_d=0}$ of $\Lambda_1(\lambda)-\Lambda_2(\lambda)$ is given by

$$\frac{-\lambda(n_1(x)-n_2(x))}{\sqrt{|\xi'|_\Gamma^2-\lambda n_1(x)}+\sqrt{|\xi'|_\Gamma^2-\lambda n_2(x)}}. \tag{5.61}$$

This shows (5.58).

Let us consider the case (I-2-2). In view of $n_1=n_2(=n)$ on Γ, we have $\widetilde{\mathcal{A}}_{1,0}^{(\lambda)} = \widetilde{\mathcal{A}}_{2,0}^{(\lambda)}$ so that

$$E_{1,0}^{(\lambda)}(z';\xi',y_d) = E_{2,0}^{(\lambda)}(z';\xi',y_d) = \exp\left(-\sqrt{|\xi'|_\Gamma^2-\lambda n(z',0)}y_d\right).$$

Since we have assumed (I-1) and (I-2-2), we have

$$\widetilde{\mathcal{A}}_{1,1}^{(\lambda)} - \widetilde{\mathcal{A}}_{2,1}^{(\lambda)} = -\lambda(\partial_d n_1(z',0)-\partial_d n_2(z',0))y_d. \tag{5.62}$$

Then $E_{1,1}^{(\lambda)} - E_{2,1}^{(\lambda)}$ satisfies the equation

$$(-\partial_d^2 + |\xi'|_\Gamma^2 - \lambda n(z',0))(E_{1,1}^{(\lambda)} - E_{2,1}^{(\lambda)})$$
$$= \lambda(\partial_d n_1(z',0) - \partial_d n_2(z',0)) y_d \exp\left(-\sqrt{|\xi'|_\Gamma^2 - \lambda n(z',0)}\, y_d\right).$$

Precisely, we obtain

$$E_{1,1}^{(\lambda)}(z';\xi',y_d) - E_{2,1}^{(\lambda)}(z';\xi',y_d) = -\frac{\lambda}{4}(\partial_d n_1(z',0) - \partial_d n_2(z',0))$$
$$\times \left(\frac{y_d^2}{\sqrt{|\xi'|_\Gamma^2 - \lambda n(z',0)}} + \frac{y_d}{|\xi'|_\Gamma^2 - \lambda n(z',0)}\right) \exp\left(-\sqrt{|\xi'|_\Gamma^2 - \lambda n(z',0)}\, y_d\right).$$

Then the principal symbol $-\partial_d(E_{1,1}^{(\lambda)} - E_{2,1}^{(\lambda)})\big|_{y_d=0}$ of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is given by

$$\frac{\lambda(\partial_{\nu_1} n_1(x) - \partial_{\nu_2} n_2(x))}{4(|\xi'|_\Gamma^2 - \lambda n(x))}.$$

This shows (5.59). □

In view of Lemma 5.3.39, we can obtain a uniform estimate in τ of $R(\tau)$ and its inverse. In the following, we define the Hilbert space $H^{m,t}(\Gamma)$ for $t \geq 1$ by the norm

$$\|f\|_{H^{m,t}(\Gamma)}^2 = \|f\|_{H^m(\Gamma)}^2 + t^{2m}\|f\|_{L^2(\Gamma)}^2.$$

Lemma 5.3.40. *For sufficiently large $\tau > 0$, there exists $R(\tau)^{-1} : H^{m,\tau}(\Gamma) \to H^{m-s,\tau}(\Gamma)$ for any $m \in \mathbf{R}$ where $s = 1$ for (I-2-1) or $s = 2$ for (I-2-2).*

Proof. In view of Lemma 5.3.39, we can construct the parametrix of $R(\tau)$. The lemma is a direct consequence of Theorem 3.2.11 in [15]. □

Let us turn to the case $\zeta \neq 0$. In view of

$$\Lambda_1(\lambda) - \Lambda_2(\lambda) - \zeta = \zeta^{1/2}(\zeta^{-1/2}(\Lambda_1(\lambda) - \Lambda_2(\lambda))\zeta^{-1/2} - 1)\zeta^{1/2},$$

we put

$$K(\lambda) = \zeta^{-1/2}(\Lambda_1(\lambda) - \Lambda_2(\lambda))\zeta^{-1/2}. \tag{5.63}$$

Since $\zeta \in C^\infty(\Gamma)$ is strictly positive or strictly negative and $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ has a negative order, the operator $K(\lambda)$ is compact in $L^2(\Gamma)$ when λ is not a pole. Since $K(\lambda)$ is meromorphic with respect to λ, we have the following lemma. The proof is completely same of and 2.4 in [21]. Note that we will refer the above lemma again later.

Lemma 5.3.41. *Let $\{\kappa_j(\lambda)\}$ be the set of eigenvalues of $K(\lambda)$. Then every $\kappa_j(\lambda)$ is meromorphic with respect to λ. If λ_0 is a pole of $K(\lambda)$ and p is the rank of the residue of $K(\lambda)$ at λ_0, p eigenvalues and its eigenfunctions have a pole at λ_0. Moreover, $\mathrm{res}_{\lambda=\lambda_0}\kappa_j(\lambda)$ are eigenvalues of $\mathrm{res}_{\lambda=\lambda_0}K(\lambda)$.*

As a consequence, we have the following lemma.

Lemma 5.3.42. *There exist $\lambda \in \mathbf{C} \setminus \mathbf{R}_{\geq 0}$ such that $1 \notin \{\kappa_j(\lambda)\}$. In particular, $K(\lambda) - 1$ has the bounded inverse for some $\lambda \in \mathbf{C} \setminus \{0\}$.*

Proof. Note that the set $\mathcal{A} = \{\lambda \in \mathbf{C}\setminus\{0\}\ ;\ \lambda$ is not a pole of $K(\lambda)\}$ is a connected domain in $\mathbf{C}\setminus\{0\}$. Since $K(\lambda)$ is compact, $\{\kappa_j(\lambda)\}$ is the set of eigenvalues of finite multiplicities with the only possible accumulation point at 0.

We take a point $\lambda_1 \in \mathbf{C}\setminus\mathbf{R}_{\geq 0}$ such that $\kappa_j(\lambda_1) = \cdots = \kappa_{j+l-1}(\lambda_1) = 1$. In view of the discreteness of eigenvalues, there exists a small constant $\epsilon_0 > 0$ such that $|\kappa_m(\lambda_1) - 1| > \epsilon_0$ for $m \notin \{j, j+1, \ldots, j+l-1\}$. Taking a sufficiently small $\delta > 0$, we also have $|\kappa_m(\lambda) - 1| > \epsilon_0$ for $|\lambda - \lambda_1| < \delta$.

Suppose that there exists an eigenvalue $\kappa_{j'}(\lambda)$ with $j' \in \{j, j+1, \ldots, j+l-1\}$ such that $\kappa_{j'}(\lambda) = 1$ in $\{\lambda \in \mathbf{C} \mid |\lambda - \lambda_1| < \delta\}$. Since $\kappa_{j'}(\lambda)$ is analytic in $\mathcal{A}$, we have $\kappa_{j'}(\lambda) = 1$ in $\mathcal{A}$. We take a pole λ_0 of $\kappa_{j'}(\lambda)$. In a small neighborhood of λ_0, $\kappa_{j'}(\lambda)$ can be written by

$$\kappa_{j'}(\lambda) = \frac{\mathrm{res}_{\lambda=\lambda_0}\kappa_{j'}(\lambda)}{\lambda_0 - \lambda} + \widetilde{\kappa}_{j'}(\lambda),$$

where $\widetilde{\kappa}_{j'}(\lambda)$ is analytic in this neighborhood. However, we obtain

$$\mathrm{res}_{\lambda=\lambda_0}\kappa_{j'}(\lambda) = (\lambda_0 - \lambda)(1 - \widetilde{\kappa}_{j'}(\lambda)) \to 0,$$

as $\lambda \to \lambda_0$. This is a contradiction. $\square$

Now we have our first main theorem as a corollary of Theorem 5.3.38, Lemma 5.3.40 and Lemma 5.3.42. We take an arbitrary closed sector $\mathbf{S}_0$ centered at the origin such that $\mathbf{S}_0 \cap \mathbf{R}_{>0} = \emptyset$. We put $\mathbf{S}_0^e := \mathbf{S}_0 \cap \{\lambda \in \mathbf{C} \mid |\lambda| \geq 1\}$.

Theorem 5.3.43. *Suppose (A) and (I-1). We assume that either (I-2-1) or (I-2-2). The set of locally isotropic ITEs consists of a discrete subset of* $\mathbf{C}$ *with the only possible accumulation points at* 0 *and infinity. There exist at most finitely many ITEs in* $\mathbf{S}_0^e$.

Proof. Note that $\Lambda_1(\lambda) - \Lambda_2(\lambda) - \zeta$ is finitely meromorphic and Fredholm for $\lambda \in \mathbf{C}\setminus\{0\}$. Lemma 5.3.40 implies that the bounded inverse $(\Lambda_1(\lambda) - \Lambda_2(\lambda))^{-1}$ exists for $\lambda \in \mathbf{S}_0^e$ with sufficiently large $|\lambda|$. Lemma 5.3.42 implies that the bounded inverse $(\Lambda_1(\lambda) - \Lambda_2(\lambda) - \zeta)^{-1}$ exists for some $\lambda \in \mathbf{C}\setminus\mathbf{R}_{\geq 0}$. In view of Theorem 5.3.38, we obtain the theorem for both of the cases $\zeta = 0$ and $\zeta \neq 0$. $\square$

5.3.2 Weyl type estimate for interior transmission eigenvalues

In the following, we use Weyl's asymptotic behavior for Dirichlet eigenvalues of $-n_l^{-1}\Delta_{g_l}$ ($l = 1, 2$) on M_l. The following fact is a direct consequence of Theorem 1.2.1 in [25].

Theorem 5.3.44. *Let* $\mathcal{O}_l(x) = \{\xi \in \mathbf{R}^d \mid \sum_{i,j=1}^d g_l^{ij}(x)\xi_i\xi_j \leq n_l(x)\}$ *for each* $x \in M_l$ *and*

$$v(\mathcal{O}_l(x)) := \int_{\mathcal{O}_l(x)} d\xi,$$

be the volume of $\mathcal{O}_l(x)$ *associated by the Euclidean measure. Then* $N_l(\lambda) := \#\{j \mid \lambda_{l,j} \leq \lambda\}$ *satisfies as* $\lambda \to \infty$

$$N_l(\lambda) = V_l\lambda^{d/2} + O(\lambda^{(d-1)/2}), \quad V_l = (2\pi)^{-d}\int_{M_l} v(\mathcal{O}_l(x))dV_l. \tag{5.64}$$

Taking an arbitrary point $x^{(0)} \in \Gamma$, we take a small neighborhood $V \subset \Gamma$ of $x^{(0)}$ and a sufficiently small open domain Ω which is diffeomorphic to $U_1 \cong U_2$ such that $\overline{U_1} \cap \Gamma = \overline{U_2} \cap \Gamma = V$ as has been defend in the beginning of §2.2. Then, identifying $x \in V$ with the corresponding point on $\partial\Omega^0$, we have that

$$\gamma_\zeta(x) := -\mathrm{sgn}(\zeta(y)) \quad \text{for} \quad \zeta \neq 0, \tag{5.65}$$

and

$$\gamma_0(x) := \begin{cases} \mathrm{sgn}(n_2(y) - n_1(y)) & \text{for} \quad \text{(I-2-1)}, \\ \mathrm{sgn}(\partial_{\nu_1} n_1(y) - \partial_{\nu_2} n_2(y)) & \text{for} \quad \text{(I-2-2)}, \end{cases} \tag{5.66}$$

for $y \in \Omega$ are well-defined constant functions $\gamma_0(x) = 1$ or -1 and $\gamma_\zeta(x) = 1$ or -1 for $x \in V$, respectively. The functions $\gamma_0(x) = 1$ or -1 and $\gamma_\zeta(x) = 1$ or -1 can be naturally extended on every connected component of Γ, respectively. We also define the function γ on Γ by

$$\gamma = \begin{cases} \gamma_\zeta & \text{for} \quad \zeta \neq 0, \\ \gamma_0 & \text{for} \quad \zeta = 0. \end{cases} \tag{5.67}$$

Generally, the function γ can change its value for each connected component. However, let us impose the following third assumption for the proof of Theorem 5.3.51. In the following, we suppose (I-3) for all lemmas.

If $\zeta \neq 0$, then ζ does not change its sign on whole of Γ.
If $\zeta = 0$, then $n_2(x) - n_1(x)$ or $\partial_{\nu_1} n_1(x) - \partial_{\nu_2} n_2(x)$ do not change its sign on whole of Γ.
(I-3)

In particular, the function γ is constant 1 or -1 on Γ. In the following, we use an auxiliary operator defined by

$$B(\lambda) = \gamma D_\Gamma^{(1+s)/4} (\Lambda_1(\lambda) - \Lambda_2(\lambda) - \zeta) D_\Gamma^{(1+s)/4} \tag{5.68}$$

where $s = 0$ for $\zeta \neq 0$ or $s = 1$ for (I-2-1) or $s = 2$ for (I-2-2). Here D_Γ is given by $D_\Gamma = -\Delta_\Gamma + 1$ where Δ_Γ is the Laplace-Beltrami operator on Γ. Then $B(\lambda)$ is a first order singular integro-differential operator when λ is not a pole of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$.

Lemma 5.3.45. (1) *Suppose $\lambda_j \notin \sigma_{D,1} \cap \sigma_{D,2}$. Then $\lambda_j \in \mathbf{C}$ is a locally isotropic ITE if and only if* $\ker B(\lambda_j) \neq \{0\}$. *The multiplicity of λ_j coincides with* $\dim \ker B(\lambda_j)$.

(2) *Suppose $\lambda_j \in \sigma_{D,1} \cap \sigma_{D,2}$. Then $\lambda_j \in \mathbf{R}$ is a locally isotropic ITE if and only if* $\ker B(\lambda_j) \neq \{0\}$ *or the ranges of $\gamma D_\Gamma^{(1+s)/4} Q_{1,j} D_\Gamma^{(1+s)/4}$ and $\gamma D_\Gamma^{(1+s)/4} Q_{2,j} D_\Gamma^{(1+s)/4}$ have a non trivial intersection. The multiplicity of λ_j coincides with the sum of* $\dim \ker B(\lambda_j)$ *and the dimension of the above intersection.*

Proof. Since $-\Delta_\Gamma + 1$ is invertible, the lemma is a direct consequence of Lemma 5.2.29. □

Lemma 5.3.46. *Let $\lambda \notin \sigma_P$.*

(1) *For $\zeta \neq 0$, $B(\lambda)$ is a first order, symmetric and elliptic singular integro-differential operator. Its principal symbol is*

$$-\gamma \zeta(x) |\xi'|_\Gamma \quad \textit{for} \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1}. \tag{5.69}$$

(2) *For* $\zeta = 0$, $B(\lambda)$ *is a first order, symmetric and elliptic singular integro-differential operator. Its principal symbol is*

$$\frac{\lambda\gamma(n_2(x) - n_1(x))}{2}|\xi'|_\Gamma \quad \textit{for} \quad x \in \Gamma \quad \xi' \in \mathbf{R}^{d-1} \tag{5.70}$$

for (A-2-1), or

$$\frac{\lambda\gamma(\partial_{\nu_1} n_1(x) - \partial_{\nu_2} n_2(x))}{4}|\xi'|_\Gamma \quad \textit{for} \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1} \tag{5.71}$$

for (A-2-2).

(3) *For* $\lambda \in \mathbf{R}_{>0}$, *the spectrum of* $B(\lambda)$ *is discrete and consists of the set of real eigenvalues* $\{\mu_j(\lambda)\}_{j=1}^{\infty}$.

Proof. We have the first assertion by direct computation using Lemma 5.2.37. From the first assertion, we also see the second assertion. □

Since $B(\lambda)$ has a positive principal symbol and $B(\lambda)$ is meromorphic with respect to λ, we also have the following lemma. For the proof, see Lemmas 2.3 and 2.4 in [21]. Note that, in view of (5.10), we define the residue $\mathrm{res}_{\lambda=\lambda_0}\mu_j(\lambda)$ of $\mu_j(\lambda)$ at a pole λ_0 by the expansion

$$\mu_j(\lambda) = \frac{\mathrm{res}_{\lambda=\lambda_0}\mu_j(\lambda)}{\lambda_0 - \lambda} + \widetilde{\mu}_j(\lambda), \tag{5.72}$$

where $\widetilde{\mu}_j(\lambda)$ is analytic in a small neighborhood of λ_0.

Lemma 5.3.47. (1) *For each compact interval* $I \subset \mathbf{R}_{>0}$ *such that any pole of* $B(\lambda)$ *are not included in* I, *there exists a constant* $C(I) > 0$ *such that* $\mu_j(\lambda) \geq -C(I)$ *for* $\lambda \in I$, $j = 1, 2, \ldots$.

(2) *If* $B(\lambda)$ *is analytic in a neighborhood of* λ_0, *all eigenvalues* $\mu_j(\lambda)$ *are analytic in this neighborhood. If* λ_0 *is a pole of* $B(\lambda)$ *and* p *is the rank of the residue of* $B(\lambda)$ *at* λ_0, p *eigenvalues* $\mu_j(\lambda)$ *and its eigenfunctions have a pole at* λ_0. *Moreover,* $\mathrm{res}_{\lambda=\lambda_0}\mu_j(\lambda)$ *are eigenvalues of* $\mathrm{res}_{\lambda=\lambda_0}B(\lambda)$.

We choose a small constant $\alpha \in (0, \min\{\lambda_{1,1}, \lambda_{2,1}\})$. We define counting function with multiplicities taken into account :

$$N_T(\lambda) = \#\{j \mid \alpha < \lambda_j^T \leq \lambda\} \tag{5.73}$$

where $\lambda_1^T \leq \lambda_2^T \leq \cdots$ are ITEs included in (α, ∞).

Now we consider the relation between $\{\lambda_j^T\}$ and $\{\mu_j(\lambda)\}$ for $\lambda \in (\alpha, \infty)$. Roughly speaking, we can evaluate $N_T(\lambda)$ by the number of the singular ITEs and the number of λ satisfying $\mu_j(\lambda) = 0$. We put

$$N_-(\lambda) = \#\{j \mid \mu_j(\lambda) < 0\} \quad \text{for} \quad \lambda \notin \{\lambda_j^T\} \cup \{\lambda_{1,j}\} \cup \{\lambda_{2,j}\}. \tag{5.74}$$

Assume that λ' moves from α to ∞. Since $\mu_j(\lambda')$ is meromorphic with respect to $\lambda', N_-(\lambda')$ changes only when some $\mu_j(\lambda')$ pass through 0 or λ' passes through a pole of $B(\lambda')$. When λ' moves from α to $\lambda > \alpha$, we denote by $\mathcal{N}_0(\lambda)$ the change of $N_-(\lambda) - N_-(\alpha)$ due to the first case, and $\mathcal{N}_{-\infty}(\lambda)$ as the change due to the second case, i.e.,

$$N_-(\lambda) - N_-(\alpha) = \mathcal{N}_0(\lambda) + \mathcal{N}_{-\infty}(\lambda). \tag{5.75}$$

For a pole λ_0 of $B(\lambda)$, we put

$$\delta\mathcal{N}_{-\infty}(\lambda_0) = N_-(\lambda_0 + \epsilon) - N_-(\lambda_0 - \epsilon) \tag{5.76}$$

for any $\epsilon > 0$.

Lemma 5.3.48. *Let $\lambda_0 \in \mathbf{R}_{>0}$ be a pole of $B(\lambda)$. We have $\delta\mathcal{N}_{-\infty}(\lambda_0) = s_+(\lambda_0) - s_-(\lambda_0)$ for $s_\pm(\lambda_0) = \#\{j \mid \pm \mathrm{res}_{\lambda=\lambda_0}\mu_j(\lambda) > 0\}$.*

Proof. In view of Lemma 5.3.47, some eigenvalues $\mu_j(\lambda)$ have a pole at λ_0. If $\pm\mathrm{res}_{\lambda=\lambda_0}\mu_j(\lambda) > 0$, we have $\mu_j(\lambda) \to \mp\infty$ as $\lambda \to \lambda_0 + 0$ and $\mu_j(\lambda) \to \pm\infty$ as $\lambda \to \lambda_0 - 0$, respectively. Then the number of negative eigenvalues decreases for $\mathrm{res}_{\lambda=\lambda_0}\mu_j(\lambda) < 0$ and increases for $\mathrm{res}_{\lambda=\lambda_0}\mu_j(\lambda) > 0$ when λ passes through λ_0 from α. This implies the lemma. □

Lemma 5.3.49. *If $\lambda_j \in \mathbf{R}_{>0}$ is a pole of $\Lambda_l(\lambda_j)$ $(l = 1, 2)$, the residue $Q_{l,j}$ is negative.*

Proof. Recall that $B_l(\lambda_j)$ is the subspace of $L^2(\Gamma)$ spanned by $\partial_{\nu,l}\phi_{l,i}$ for $i \in \mathcal{E}_{l,j}$. In view of (5.11), we have for $0 \neq f \in B_k(\lambda_0)$

$$(Q_{l,j} f, f)_{L^2(\Gamma)} = -\sum_{i \in \mathcal{E}_{l,j}} |(\partial_{\nu,l}\phi_{l,i}, f)_{L^2(\Gamma)}|^2 < 0.$$

Then we have $Q_{l,j} < 0$. □

For $\lambda_j \in \sigma_{D,l}$ $(l = 1, 2)$, we put $m_l(\lambda_j) = \dim \mathrm{Ran}\, Q_{l,j}$. For $\lambda_j \in \sigma_{D,1} \cap \sigma_{D,2}$, $m(\lambda_j) = \dim(\mathrm{Ran}\, Q_{1,j} \cap \mathrm{Ran}\, Q_{2,j})$.

Lemma 5.3.50. *Let $\lambda_0 \in \mathbf{R}_{>0}$ be a pole of $B(\lambda)$.*

(1) *If $\lambda_0 \notin \sigma_{D,1} \cap \sigma_{D,2}$, we have $\delta\mathcal{N}_{-\infty}(\lambda_0) + \gamma(m_1(\lambda_0) - m_2(\lambda_0)) = 0$.*

(2) *If $\lambda_0 \in \sigma_{D,1} \cap \sigma_{D,2}$, we have $|\delta\mathcal{N}_{-\infty}(\lambda_0) + \gamma(m_1(\lambda_0) - m_2(\lambda_0))| \leq m(\lambda_0)$.*

Proof. First we prove the assertion (1). Suppose $\lambda_j \in \sigma_{D,1} \setminus \sigma_{D,2}$. We can expand $B(\lambda_j)$ in a small neighborhood of λ_j as

$$B(\lambda) = \frac{\gamma D_\Gamma^{(1+s)/4} Q_{1,j} D_\Gamma^{(1+s)/4}}{\lambda_j - \lambda} + \widetilde{H}_{1,j}(\lambda),$$

where $\widetilde{H}_{1,j}(\lambda) := \gamma D_\Gamma^{(1+s)/4}(H_{1,j}(\lambda) - \Lambda_2(\lambda) - \zeta) D_\Gamma^{(1+s)/4}$ is analytic. From Lemma 5.3.49, we have $Q_{1,j} < 0$ and also $D_\Gamma^{(1+s)/4} Q_{1,j} D_\Gamma^{(1+s)/4} < 0$ so that $D_\Gamma^{(1+s)/4} Q_{1,j} D_\Gamma^{(1+s)/4}$ has exactly $m_1(\lambda_j)$ strictly negative eigenvalues. Hence we have $\mathrm{sgn}(\mathrm{res}_{\lambda=\lambda_j}\mu_i(\lambda)) = -\gamma$. In view of the assertion (2) in Lemma 5.3.47, this means $s_+(\lambda_j) = 0$ and $s_-(\lambda_j) = m_1(\lambda_j)$ for $\gamma = 1$, or $s_+(\lambda_j) = m_1(\lambda_j)$ and $s_-(\lambda_j) = 0$ for $\gamma = -1$. Lemma 5.3.48 implies $\delta\mathcal{N}_{-\infty}(\lambda_j) = \gamma(m_2(\lambda_j) - m_1(\lambda_j))$ with $m_2(\lambda_j) = 0$. For the case $\lambda_j \in \sigma_{D,2} \setminus \sigma_{D,1}$, we can see the same formula with $m_1(\lambda_j) = 0$ by the similar way. Plugging these two cases, we obtain the assertion (1).

Let us prove the assertion (2). Suppose $\lambda_j = \lambda_{1,j(i_1)} = \lambda_{2,j(i_2)}$ for $\lambda_{1,j(i_1)} \in \sigma_{D,1}$ and $\lambda_{2,j(i_2)} \in \sigma_{D,2}$. Then we have the following representation in a small neighborhood of λ_j

$$B(\lambda) = \frac{\gamma \widetilde{Q}_{0,j}}{\lambda_j - \lambda} + \widetilde{H}_{0,j}(\lambda)$$

where $\widetilde{Q}_{0,j} = D_\Gamma^{(1+s)/4}(Q_{1,j(i_1)} - Q_{2,j(i_2)}) D_\Gamma^{(1+s)/4}$ and $\widetilde{H}_{0,j}(\lambda) = \gamma D_\Gamma^{(1+s)/4}(H_{1.j}(\lambda) - H_{2,j}(\lambda) - \zeta) D_\Gamma^{(1+s)/4}$. We see that $\widetilde{Q}_{0,j} < 0$ on $B_1(\lambda_{1,j(i_1)}) \cap B_2(\lambda_{2,j(i_2)})^\perp$ and $\widetilde{Q}_{0,j} > 0$ on $B_1(\lambda_{1,j(i_1)})^\perp \cap B_2(\lambda_{2,j(i_2)})$. If $\gamma = 1$, we have $m_2(\lambda_j) - m(\lambda_j) \leq s_+(\lambda_j) \leq m_2(\lambda_j)$ and $m_1(\lambda_j) - m(\lambda_j) \leq s_-(\lambda_j) \leq m_1(\lambda_j)$. If $\gamma = -1$, we have $m_1(\lambda_j) - m(\lambda_j) \leq s_+(\lambda_j) \leq m_1(\lambda_j)$ and $m_2(\lambda_j) - m(\lambda_j) \leq s_-(\lambda_j) \leq m_2(\lambda_j)$. These inequalities and Lemma 5.3.48 imply the assertion (2). □

Now we have arrived at our main result on the Weyl type lower bound for $N_T(\lambda)$.

Theorem 5.3.51. *Suppose* (A) *and* $(I\text{-}1)$*. We assume that either* $(I\text{-}2\text{-}1)$ *or* $(I\text{-}2\text{-}2)$*, and* $(I\text{-}3)$*. For large* $\lambda \in \mathbf{R}_{>0}$*, we have*

$$N_T(\lambda) \geq \gamma \sum_{\alpha<\lambda'\leq\lambda} (m_1(\lambda') - m_2(\lambda')) - N_-(\alpha) \tag{5.77}$$

where the summation is taken over poles $\lambda' \in (\alpha, \lambda]$ *of* $\Lambda_1(\lambda) - \Lambda_2(\lambda)$*. Moreover, if* $\gamma(V_1 - V_2) > 0$ *where* $V_1, V_2 > 0$ *are defined in (5.64),* $N_T(\lambda)$ *satisfies asymptotically as* $\lambda \to \infty$

$$N_T(\lambda) \geq \gamma(V_1 - V_2)\lambda^{d/2} + O(\lambda^{(d-1)/2}). \tag{5.78}$$

Proof. We prove for the case $\sigma_{D,1} \cap \sigma_{D,2} \neq \emptyset$. For $\sigma_{D,1} \cap \sigma_{D,2} = \emptyset$, the proof is similar and can be slightly simplified. Letting us recall that we call λ is a singular ITE when λ satisfies the latter condition of the assertion (2) of Lemma 5.2.29, we put

$$N_{sng}(\lambda) = \#\{\text{singular ITEs} \in (\alpha, \lambda] \subset \mathbf{R}_{>0}\}.$$

Here $N_{sng}(\lambda)$ counts the number of singular ITEs with multiplicities taken into account. Note that $\mathcal{N}_0(\lambda) + N_{sng}(\lambda) \leq N_T(\lambda)$ by the definition of $\mathcal{N}_0(\lambda)$ and Lemma 5.3.45. We take the summation of $|\delta\mathcal{N}_{-\infty}(\lambda') + \gamma(m_1(\lambda') - m_2(\lambda'))| \leq m(\lambda')$ in $(\alpha, \lambda]$. Then we have

$$\left| \mathcal{N}_{-\infty}(\lambda) + \gamma \sum_{\alpha<\lambda'\leq\lambda} (m_1(\lambda') - m_2(\lambda')) \right| \leq N_{sng}(\lambda).$$

See also Remark of Proposition 5.2.25. Plugging this inequality and (5.75), we have

$$N_-(\lambda) - N_-(\alpha) + \gamma \sum_{\alpha<\lambda'\leq\lambda} (m_1(\lambda') - m_2(\lambda')) \leq \mathcal{N}_0(\lambda) + N_{sng}(\lambda) \leq N_T(\lambda).$$

Since $N_-(\lambda) \geq 0$, we obtain (5.77).

The inequality (5.77) implies

$$N_T(\lambda) \geq \gamma(N_1(\lambda) - N_2(\lambda)) - N_-(\alpha).$$

The asymptotic estimate (5.78) is a direct consequence of this inequality and Theorem 5.3.44.

□

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