Construction of Gray code for a group based on semidirect-product structure and its application to groups of order 16

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Abstract

In a earlier paper[8], we considered Type 1 Gray maps and Type 2 Gray maps for groups of order 16, and we succeeded the Type 1 construction for all groups of order 16 and confirmed that we can construct Type 2 maps for several groups of order 16, but failed to construct such maps for other groups.

In another paper[9], we suggested a new design principle of Gray maps for groups and tried to apply it to several concrete groups. Though the trial made some success, the method is not very constructive.

Therefore, in this paper we try to design a more constructive method based on the semidirect-product structure of the target group.

1 Introduction

Coding theory is very important research subject which forms a base of the information and communication technology, because it is widely used as means of data communication, recode device such as CD, DVD and the computer disk etc., where the high reliability is necessary.

The study on coding theory began with the article that Shannon published in 1948, *A mathematical theory of communication*. Codes playing a key role of the studies are BCH code, Reed-Solomon code and Algebraic Geometry code etc., defined over finite fields. On the other hand, a code defined over ring of integers (mainly $\mathbb{Z}/4\mathbb{Z}$) is studied flourishingly.

The detection and the correction methods of errors that occurred in encoded information play a key role of the study, but these studies may not reflect the structure of the information before the encoding.

In 1947 Frank Gray devised the so called Gray code to have the character that the Hamming distance between adjacent codewords is one. It is expected that this code reduces the influence for error outbreak, if we assign the nearer codeword to the nearer piece of information, by investigation the structure of the information before the encoding.

Reza Sobhani [1] designed two classes of Gray maps called Type 1 Gray map and Type 2 Gray map, for finite p-groups. Both are constructed as extensions of a Gray map for a smaller group. Type 1 method constructs a code for a group from a code for a maximal subgroup of the target group naturally, but it doubles the length of the resulting code.

The Type 2 method in contrast, generally construct a shorter code than Type 1 that is just 1 bit longer than that for the based maximal subgroup. However, in our trial [8], among all the groups of order 16, only 6 groups allow a Type 2 extension from 3-bit Gray codes for groups of order 8.

So, we proposed a new design policy for an arbitrary finite group (not necessary to be a p-group) in [9]. Our idea to construct an n-bit Gray code for group G is to search in the group of affine permutation of degree n for a subgroup isomorphic to G with a suitable property. This method is different from both Type 1 and Type 2.

In [9], we showed that this method can reconstruct 4-bit Gray maps for G_2, G_3, G_7, G_8, G_9, G_{12} and G_{13} [1]. Also we showed that our method is effective to several non-p-groups of simple type, namely, $C_{2n}, C_{2n+1}, D_6, D_{10}$ and D_{12}. However, since our construction in [9] is somewhat ad hoc, We propose a more constructive method in this paper.

We believe the method can also contribute to constructing non-binary codes. However, in order to concentrate on binary codes here, we assume that the information is encoded in $\mathbb{Z}_2^n$, throughout this paper.

2 Preliminaries

2.1 Hamming-distance, Hamming-weight and Gray map

In this section we assume that G is a finite 2-group of order 2^m. We review some key definitions and a lemma on Gray maps in [1, 5].

Definition 1. For any two elements $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{Z}_2^n$, the *Hamming-distance* between $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$d(\mathbf{u}, \mathbf{v}) \stackrel{\text{def.}}{=} |\{i \mid 1 \le i \le n, u_i \ne v_i\}|.$$

The Hamming-distance is indeed a distance on $\mathbb{Z}_2^n$ [5].

[1] We follow Wild [2] for the name of groups of order 16. Refer to Remark 3 for each group G_i.

Definition 2. The *Hamming-weight* of an element $\mathbf{u} \in \mathbb{Z}_2^n$ is defined by

$$w(\mathbf{u}) \stackrel{\text{def.}}{=} |\{i \mid 1 \leq i \leq n, u_i \neq 0\}|.$$

Definition 3. A map $\phi : G \to \mathbb{Z}_2^n$ is said to be a *Gray map*, if it is an injection and

$$w(\phi(a^{-1}b)) = d(\phi(a), \phi(b))$$

holds for all a, b in G.[2]

Lemma 1. Let $\phi : G \to \mathbb{Z}_2^n$ be a Gray map. Then,

(1) For $g \in G$ we have $w(\phi(g)) = 0$ iff $g = e$, where e stands for the identity of G,

(2) For all g in G we have $w(\phi(g)) = w(\phi(g^{-1}))$,

(3) For all x, y in G we have $w(\phi(xy)) \leq w(\phi(x)) + w(\phi(y))$.

Proof: Assume that ϕ is a Gray map.

(1) $0 = w(\phi(g)) = w(\phi(e^{-1}g)) = d(\phi(e), \phi(g)) \Longleftrightarrow \phi(g) = \phi(e) \Longleftrightarrow g = e$,

(2) $w(\phi(g)) = w(\phi(e^{-1}g)) = d(\phi(e), \phi(g)) = d(\phi(g), \phi(e)) = w(\phi(g^{-1}e)) = w(\phi(g^{-1}))$,

(3) $w(\phi(g)) + w(\phi(h)) = d(\phi(g^{-1}), \phi(e)) + d(\phi(e), \phi(h)) \geq d(\phi(g^{-1}), \phi(h)) = w(\phi(gh))$.

We define map $d_\phi : G \times G \to \mathbb{N} \cup \{0\}$ by $d_\phi(a, b) = d(\phi(a), \phi(b))$. Then, d_ϕ is a distance on G clearly.

2.2 Cyclic extensions

For notational convenience, we use the standard presentation $\langle X \mid \Delta \rangle$ of groups by generator X and relation Δ [4].

For example, the cyclic group C_n of order n is represented as $\langle x \mid x^n = e \rangle$, the *Klein four group* $K_4 = C_2 \times C_2$ as $\langle x, y \mid x^2 = y^2 = e, xy = yx \rangle$, and $C_2^3 = C_2 \times C_2 \times C_2$ is represented as $\langle x, y, z \mid x^2 = y^2 = z^2 = e, yx = xy, zx = xz, yz = zy \rangle$. The direct product of C_4 and C_2 is represented as $\langle x, y \mid x^4 = y^2 = e, yx = xy \rangle$.

Since group $C_4 \times C_2$ appears frequently in this paper we denote it by K_8 as in [2]. Similarly, we denote the dihedral group $\langle x, y \mid x^n = y^2 = e, yx = x^{n-1}y \rangle$ of order 2n by D_{2n}, and the quaternion group $\langle x, y \mid x^4 = e, y^2 = x^2, yx = x^3y \rangle$ of order 8 by Q_8.

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[2] In Sobhani's definion of the Gray map [1], function d_ϕ is defined by $d_\phi(a, b) = w(\phi(ab^{-1}))$ and is required to be indeed a distance on G. For simplicity in our definition, map ϕ is required just to be an injection, accepting suggestion of a IPSJ referee.

Let N be a normal subgroup of G (in symbol $N \lhd G$). We denote by t_a the conjugation automorphism of N defined by element $a \in G$ (namely $t_a(x) \stackrel{\text{def.}}{=} axa^{-1}$ for element $x \in N$).

Suppose that $G/N \simeq C_n$ and pick any a in G such that the coset Na has order n in G/N. If we put $v = a^n$ and $\tau = t_a$, then $v \in N, \tau(v) = t_a(v) = aa^n a^{-1} = a^n = v$, and $\tau^n = t_a^n = t_{a^n} = t_v$.

Definition 4. A quadruple (N, n, τ, v) is said to be an *extension type* if N is a group, v is an element in N, and τ is an automorphism of N such that $\tau(v) = v$ and $\tau^n = t_v$.

Remark 1. An extension type determines the structure of group $G = \langle N, a \rangle$ uniquely.

Remark 2. The set $\mathrm{Aut}(G)$ of all automorphisms of a group G forms a group under composition of mappings. Let X generate G. Then each $\theta : G \to G$ in $\mathrm{Aut}(G)$ is determined by its values on X. In particular $\mathrm{Aut}(C_4)$, $\mathrm{Aut}(C_8)$, $\mathrm{Aut}(K_8)$ and $\mathrm{Aut}(D_8)$ consist of the following respective functions [2, 9]:

Aut(C_4) and Aut(C_8)$\simeq K_4$

$\mathrm{Aut}(C_4)$	effect on x	$\mathrm{Aut}(C_8)$	effect on x
φ_1	x	σ_1	x
φ_2	x^3	σ_2	x^3
		σ_3	x^5
		σ_4	x^7

Aut(K_8) $\simeq D_8$

$\mathrm{Aut}(K_8)$	effect on x	effect on y	order of automorphism
ψ_1	x	y	1
ψ_2	x^3y	x^2y	4
ψ_3	x^3	y	2
ψ_4	xy	x^2y	4
ψ_5	xy	y	2
ψ_6	x^3	x^2y	2
ψ_7	x^3y	y	2
ψ_8	x	x^2y	2

Aut$(D_8)\simeq D_8$

Aut(D_8)	effect on x	effect on y	order of automorphism
α_1	x	y	1
α_2	x	xy	4
α_3	x	x^2y	2
α_4	x	x^3y	4
α_5	x^3	y	2
α_6	x^3	xy	2
α_7	x^3	x^2y	2
α_8	x^3	x^3y	2

Group Aut(Q_8) is isomorphic to symmetric group S_4 and Group Aut(C_2^3) consists of $7 \times 6 \times 4 = 168$ elements.

Remark 3. In [2], Marcel Wild denotes the 14 groups of order 16 (besides the outsider $G_0 = C_2 \times C_2 \times C_2 \times C_2$) as follows (we add the last column to show extension types[3] of each group.):

$G_1 = C_8 \times C_2$	$(C_8, 2, \sigma_1, e)$, $(K_8, 2, \psi_1, x)$
$G_2 = C_8 \rtimes_{\sigma_2} C_2$	$(C_8, 2, \sigma_2, e)$, $(D_8, 2, \alpha_8, x^2)$, $(Q_8, 2, \beta_1, e)$
$G_3 = C_8 \rtimes_{\sigma_3} C_2$	$(C_8, 2, \sigma_3, e)$, $(K_8, 2, \psi_8, x)$
$G_4 = C_8 \rtimes_{\sigma_4} C_2$	$(C_8, 2, \sigma_4, e)$, $(D_8, 2, \alpha_6, e)$
$G_5 = Q_{16}$	$(C_8, 2, \sigma_4, x^4)$, $(Q_8, 2, \beta_1, x^2)$
$G_6 = C_{16}$	$(C_8, 2, \sigma_1, x)$
$G_7 = C_4 \times K_4$	$(K_8, 2, \psi_1, e)$, $(C_2^3, 2, \gamma_1, z)$, $(C_4, 4, \varphi_1, e)$
$G_8 = D_8 \times C_2$	$(K_8, 2, \psi_3, e)$, $(D_8, 2, \alpha_1, e)$, $(C_2^3, 2, \gamma_2, e)$
$G_9 = K_4 \rtimes_{\sigma} C_4$	$(K_8, 2, \psi_7, e)$, $(C_2^3, 2, \gamma_3, yz)$, $(K_4, 4, \sigma, e)$
$G_{10} = Q_8 \rtimes_{\tau_6} C_2$	$(K_8, 2, \psi_6, e)$, $(D_8, 2, \alpha_3, e)$, $(Q_8, 2, \beta_2, e)$
$G_{11} = Q_8 \times C_2$	$(K_8, 2, \psi_3, x^2)$, $(Q_8, 2, \beta_3, e)$
$G_{12} = C_4 \rtimes_{\varphi_2} C_4$	$(K_8, 2, \psi_5, x^2)$, $(C_4, 4, \varphi_2, e)$
$G_{13} = C_4 \times C_4$	$(K_8, 2, \psi_1, y)$, $(C_4, 4, \varphi_1, e)$,

[3] An extension type determines the group structure, but a group can have several extension types even if the base group is fixed. We select a few of specific extension types for the reason described later.

where the automorphisms of Q_8 and C_2^3 in the table above are as follows:

$$\begin{aligned}
\beta_1 : Q_8 \to Q_8 &\quad (x \mapsto x^3, y \mapsto xy),\\
\beta_2 : Q_8 \to Q_8 &\quad (x \mapsto x, y \mapsto x^2 y),\\
\beta_3 : Q_8 \to Q_8 &\quad (x \mapsto x, y \mapsto y),\\
\gamma_1 : C_2^3 \to C_2^3 &\quad (x \mapsto x, y \mapsto y, z \mapsto z),\\
\gamma_2 : C_2^3 \to C_2^3 &\quad (x \mapsto x, y \mapsto xy, z \mapsto z),\\
\gamma_3 : C_2^3 \to C_2^3 &\quad (x \mapsto x, y \mapsto xy, z \mapsto xz).
\end{aligned}$$

2.3 Type 1 Gray maps

In this subsection, we assume that H is a maximal subgroup of G with $[G : H] = 2$, and x is an arbitrary element in $G \setminus H$ and h is an arbitrary element in H. Type 1 Gray map for G is constructed as follows based on a Gray map for H.

Let us denote by $\mathbf{0}$ and $\mathbf{1}$ the vectors in $\mathbb{Z}_2^n$ whose components are all 0 and 1, respectively. Also we denote the usual concatenation of vectors by $(\ |\ )$. Suppose $\phi : H \to \mathbb{Z}_2^n$ is a Gray map and define the map $\hat{\phi} : G \to \mathbb{Z}_2^{2n}$ by $\hat{\phi}(h) = (\phi(h) \mid \phi(h))$ and $\hat{\phi}(xh) = (\phi(h) \mid \phi(h) + \mathbf{1})$ [1]. We can easily see that $w(\hat{\phi}(g)) = 2w(\phi(g))$ for $g \in H$ and $w(\hat{\phi}(g)) = n$ for $g \notin H$. So the proofs of the following lemmas and theorem are routines.

Lemma 2. For all $g \in G$ we have $w(\hat{\phi}(g)) = w(\hat{\phi}(g^{-1}))$.

Lemma 3. For all $a, b \in G$ we have $w(\hat{\phi}(ab)) \leq w(\hat{\phi}(a)) + w(\hat{\phi}(b))$.

Theorem 1. With notation as above, the map $\hat{\phi}$ is a Gray map.

Refer to [1] for the details[4]

Remark 4. In [8], we constructed Type 1 Gray maps for all groups $G_0, G_1, \ldots, G_{12}$ and G_{13} of order 16.

Since our recipe (describe in Section 5) failed to construct Gray maps for G_2, G_3, G_5 and G_6, we show construction examples of them.

(1) Type 1 Gray map for G_2

Let G be $G_2 = (C_8, 2, \sigma_2, e)$. Assume that $H = C_8 \leq G$ be the maximal subgroup of G. Let $\phi_1 : H \to \mathbb{Z}_2^4$ be the previously constructed Gray map for C_8[8]. Set $\phi_2 \overset{\text{def.}}{=} \hat{\phi}_1$, we have,

[4] The proof of Theorem 1 written in [1] contains a small error caused by the definition of distance d_ϕ, but it is not essential.

$$\begin{array}{lll}
\phi_2(e) & = (\phi_1(e) \mid \phi_1(e)) & = 00000000 \\
\phi_2(x) & = (\phi_1(x) \mid \phi_1(x)) & = 00110011 \\
\phi_2(x^2) & = (\phi_1(x^2) \mid \phi_1(x^2)) & = 01010101 \\
\phi_2(x^3) & = (\phi_1(x^3) \mid \phi_1(x^3)) & = 01100110 \\
\phi_2(x^4) & = (\phi_1(x^4) \mid \phi_1(x^4)) & = 11111111 \\
\phi_2(x^5) & = (\phi_1(x^5) \mid \phi_1(x^5)) & = 11001100 \\
\phi_2(x^6) & = (\phi_1(x^6) \mid \phi_1(x^6)) & = 10101010 \\
\phi_2(x^7) & = (\phi_1(x^7) \mid \phi_1(x^7)) & = 10011001 \\
\phi_2(a) & = (\phi_1(e) \mid \phi_1(e) + 1111) & = 00001111 \\
\phi_2(xa) & = (\phi_1(x) \mid \phi_1(x) + 1111) & = 00111100 \\
\phi_2(x^2a) & = (\phi_1(x^2) \mid \phi_1(x^2) + 1111) & = 01011010 \\
\phi_2(x^3a) & = (\phi_1(x^3) \mid \phi_1(x^3) + 1111) & = 01101001 \\
\phi_2(x^4a) & = (\phi_1(x^4) \mid \phi_1(x^4) + 1111) & = 11110000 \\
\phi_2(x^5a) & = (\phi_1(x^5) \mid \phi_1(x^5) + 1111) & = 11000011 \\
\phi_2(x^6a) & = (\phi_1(x^6) \mid \phi_1(x^6) + 1111) & = 10100101 \\
\phi_2(x^7a) & = (\phi_1(x^7) \mid \phi_1(x^7) + 1111) & = 10010110
\end{array}$$

(2) Type 1 Gray map for G_3, G_5 and G_6.

Let G be $G_3(C_8, 2, \sigma_3, e)$. Assume that $H = C_8 \leq G$ be the maximal subgroup of G. Let $\phi_1 : H \to \mathbb{Z}_2^4$ be the previously constructed Gray map for C_8[8]. Set $\phi_2 \overset{\text{def.}}{=} \hat{\phi}_1$, we have the same Gray map with G_2. Also we have the same Gray map with G_2 for G_3, G_5 and G_6.

2.4 Type 2 Gray maps

In this subsection, we assume that G is isomorphic to the semidirect product $K \rtimes_\psi H$ of two finite 2-groups K and H where $\psi : H \to \mathrm{Aut}(K)$ is the conjugation homomorphism, i.e. ψ_h is the automorphism on K defined by $\psi_h(k) = hkh^{-1}$. Suppose further that both H and K accept Gray maps $\theta_1 : H \to \mathbb{Z}_2^{n_1}$ and $\theta_2 : K \to \mathbb{Z}_2^{n_2}$, where θ_2 is compatible with ψ in the sense that for all $h \in H$ and $k \in K$

$$w(\theta_2(k)) = w(\theta_2(\psi_h(k))).$$

Every element $g \in G$ can be written uniquely in form kh by an element $k \in K$ and an element $h \in H$. Then, define map θ from G to $\mathbb{Z}_2^{n_1+n_2}$ as

$$\theta(g) = \theta(kh) = (\theta_2(k) \mid \theta_1(h)),$$

where we denote the usual concatenation of vectors by (|).

Theorem 2 (Sobhani [1]). The map θ defined above is a Gray map.

Proof: Let $a = kh, b = k'h'$ be elements of G. Then

$$\begin{aligned} w(\theta(a^{-1}b)) &= w(\theta(h^{-1}k^{-1}k'h')) = w(\theta(\psi_{h^{-1}}(k^{-1}k')h^{-1}h')) \\ &= w(\theta_2(\psi_{h^{-1}}(k^{-1}k')) \mid \theta_1(h^{-1}h')) \\ &= w(\theta_2(\psi_{h^{-1}}(k^{-1}k'))) + w(\theta_1(h^{-1}h')) \\ &= w(\theta_2(k^{-1}k')) + w(\theta_1(h^{-1}h')) \\ &= d(\theta_2(k), \theta_2(k')) + d(\theta_1(h), \theta_1(h')) \\ &= d((\theta_2(k) \mid \theta_1(h)), (\theta_2(k') \mid \theta_1(h'))) \\ &= d(\theta(kh), \theta(k'h')) = d(\theta(a), \theta(b)). \end{aligned}$$

Since θ_1 and θ_2 are injections, θ is clearly an injection.

Remark 5. In [8], we constructed Type 2 Gray maps for $G_0, G_7, G_8, G_9, G_{12}$ and G_{13}.

However, compatible map θ_2 may not exist and, even if one exists, it is not very easy to find.

3 Embedding to the group of affine permutations and the induced Gray map

In this section, we assume that G is an arbitrary finite group (not necessary to be a p-group). Cayley's theorem says that every finite group can be embedded in the symmetric group of degree $|G|$ as a subgroup.

Define the mapping $g : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ as $g(u) = uP + c$ for all u in $\mathbb{Z}_2^n$, where c is a fixed element in $\mathbb{Z}_2^n$ and P is a fixed permutation matrix of order n. (A permutation matrix of order n is a $n \times n$-matrix which has exactly one 1 in each row and column and whose other entries are all 0. As is well known, a permutation matrix represents just a replacement of coordinates of vectors.) Since the mapping g above is an affine transformation over $\mathbb{Z}_2^n$, we call a mapping of this form an *affine permutation* [5] of degree n.

Our ideas to construct a Gray map for an arbitrary group is realizing Cayley's theorem over the group of affine permutations, instead of the symmetric group. The key points are that the set of all the affine permutations forms a group with respect to composition as a transformation from $\mathbb{Z}_2^n$ to itself and every affine permutation is an isometry with respect to Hamming distance.

In fact, let $g(u) = uP + c$ and $h(u) = uQ + d$ (we denote them by $[P, c]$ and $[Q, d]$, respectively) be two affine permutations. Since

$$(h \circ g)u = (uP + c)Q + d = uPQ + cQ + d,$$

the composition $h \circ g = [Q, d] \circ [P, c]$ is denoted by $[PQ, cQ + d]$ and is itself an affine permutation since PQ is a permutation matrix again.

Moreover, it is easily verified that the identity permutation is $[E, \mathbf{0}]$ and the inverse permutation of $[P, c]$ is $[P^{-1}, cP^{-1}]$. Thus, the set of all the affine permutations of degree n forms a group, which we denote by $\mathcal{AP}(n)$.

Next, let us confirm that every affine permutation $g = [P, c]$ is an isometry. Since P is a permutation matrix and c is a constant vector, clearly from definition of Hamming-distance, for any u and v in $\mathbb{Z}_2^n$

$$d_H(g(u), g(v)) = d_H(uP + c, vP + c) = d_H(uP, vP) = d_H(u, v)$$

holds.

Suppose that G is isomorphic to a subgroup of $\mathcal{AP}(n)$. For simplicity, in what follows, we regard G as identical with the subgroup. Therefore, an element $g \in G$ can be written in form $[P, c]$ by a permutation matrix P and a constant $c \in \mathbb{Z}_2^n$. We call c the *code-part* of affine permutation $[P, c]$. The idea is that we employ the code-part c as the codeword for element $[P, c]$ in G.

Theorem 3. Let G be a subgroup of $\mathcal{AP}(n)$ and consider the function $\phi : G \to \mathbb{Z}_2^n$ that maps each element $[P, c] \in G$ to its code-part c. Then, ϕ is a Gray map, if and only if it is an injection.

Proof: Let $a = [P, c]$, $b = [Q, d]$. Then,

$$\begin{aligned} w(\phi(a^{-1}b)) &= w(\phi([P^{-1}, cP^{-1}][Q, d])) \\ &= w(\phi[QP^{-1}, dP^{-1} + cP^{-1}]) \\ &= w(dP^{-1} + cP^{-1}) = w(d + c) \\ &= d(c, d) = d(\phi(a), \phi(b)). \end{aligned}$$

Thus, in order to construct an n-bit Gray code for group G, we only need to search in the group of affine permutation of degree n for a subgroup isomorphic to G such that map ϕ is injective.

Remark 6. A permutation matrix is denoted by symbol P_π, where π is a permutation of n elements, namely P_π is the matrix in which the $(i, \pi(i))$ entries are 1 and all the other entries are 0. Henceforth, we mainly employs this notation for permutation matrices. Note that multiplying a row vector by P_π permutes the components of the vector in the following way:

$$(a_1, a_2, \ldots, a_n)P_\pi = (a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, \ldots, a_{\pi^{-1}(n)}),$$

and that $P_\pi^T = P_\pi^{-1} = P_{\pi^{-1}}$, so

$$(a_1, a_2, \ldots, a_n)P_\pi^T = (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}).$$

4 Extension of embedding based on semidirect-product structure

In this section we assume that G is isomorphic to the semidirect product $G \simeq K \rtimes_\psi H$ of a normal subgroup K and a subgroup H where ψ is the conjugation homomorphism. Suppose further that both K and H can be embedded to the group of affine permutations (described in Section 3), namely, there exist embeddings $\phi_K : K \to \mathcal{AP}(m)$, $\phi_H : H \to \mathcal{AP}(n)$. Assuming that $\phi_K(k) = [P_k, c_k]$ for $k \in K$ and $\phi_H(h) = [Q_h, d_h]$ for $h \in H$, we try to define an embedding $\phi_G : G \to \mathcal{AP}(m+n)$.

Any element g in G can be written in form kh by an element $k \in K$ and an element $h \in H$ uniquely. We want to embed $g = kh$ in form $\phi_G(kh) = \left[\begin{pmatrix} P_{kh} & O \\ O & Q_h \end{pmatrix}, (c_{kh} \mid d_h)\right]$, where P_{kh} is some permutation matrix of degree m. In particular, assume that $k \in K$ is embedded in form $\phi_G(ke) = \left[\begin{pmatrix} P_k & O \\ O & E \end{pmatrix}, (c_k \mid \mathbf{0})\right]$ as an element ke in G. Select an element $a \in G \setminus K$ and let us embed it in form $\phi_G(a) = \left[\begin{pmatrix} P_a & O \\ O & Q_h \end{pmatrix}, (c_a \mid d_h)\right]$ where a is written as kh by $k \in K$ and $h \in H$. Then, element $\psi_a(k) = aka^{-1}$ is embedded to

$$\left[\begin{pmatrix} P_a^{-1}P_kP_a & O \\ O & E \end{pmatrix}, c_aP_a^{-1}P_kP_a + c_kP_a + c_a \mid \mathbf{0})\right].$$

So, in order such an embedding to be successful, it is necessary that

$$P_a^{-1}P_kP_a = P_{aka^{-1}}, \tag{A}$$
$$c_aP_a^{-1}P_kP_a + c_kP_a + c_a = c_{aka^{-1}}. \tag{B}$$

If we put $c_a = \mathbf{0}$, then the latter condition (B) reduces to

$$c_kP_a = c_{aka^{-1}}. \tag{B'}$$

In this case, since P_a is a permutation, we have $w(c_k) = w(c_{aka^{-1}})$ and Theorem 2 guarantees that the embedding induces a Gray map. Therefore, a promising candidate for $\phi_G(a)$ is $\left[\begin{pmatrix} P_a & O \\ O & Q_h \end{pmatrix}, (\mathbf{0} \mid d_h)\right]$ with P_a satisfying conditions (A) and (B'). Moreover, if an element $g \in G$ have code part of form $(\mathbf{0} \mid d_h)$ and the coset Ka has order n in $H \simeq G/K$, then $\phi_G(a^n)$ is written as $\left[\begin{pmatrix} P_a{}^n & O \\ O & E \end{pmatrix}, (\mathbf{0} \mid \mathbf{0})\right]$. So, in order the code part to be injective, a must have order n also in G and $P_a{}^n$ must be E. Therefore, if we want to give a codeword of form $(\mathbf{0} \mid d_h)$ to element a, we can further limit the candidate a and P_a as described above.

5 A recipe of semidirect-product construction of Gray maps for groups of order 16

Guided by the previous section, here we describe a design method of Gray maps for groups of order 16 based on semidirect-product structure. Our recipe is as follows:

(1) If G has extension type $(K, 2, \tau, e)$ and K is embedded in $\mathcal{AP}(n)$ by ϕ_K, then:

(1-1) For any $k \in K$ define $\phi_G(k) = \left[\begin{pmatrix} P_k & O \\ O & 1 \end{pmatrix}, (c_k \mid 0)\right]$, where $\phi_K(k) = [P_k, c_k]$.

(1-2) Select an element a of order 2 in $G \setminus K$.

(1-3) Search for a permutation matrix P_a of degree n satisfying ${P_a}^2 = E$, (A), (B') and define $\phi_G(a) = \left[\begin{pmatrix} P_a & O \\ O & 1 \end{pmatrix}, (\mathbf{0} \mid 1)\right]$,

(1-4) Since the other values of ϕ_G are automatically determined, check if ϕ_G successfully embeds G to $\mathcal{AP}(n+1)$.

(2) If G has extension type $(K, 4, \tau, e)$ and K is embedded in $\mathcal{AP}(n)$ by ϕ_K, then:

(2-1) For any $k \in K$ define $\phi_G(k) = \left[\begin{pmatrix} P_k & O \\ O & E \end{pmatrix}, (c_k \mid 00)\right]$, where $\phi_K(k) = [P_k, c_k]$.

(2-2) Select an arbitrary element a of order 4 in $G \setminus K$.

(2-3) Search for a permutation matrix P_a of degree n satisfying ${P_a}^4 = E$, (A), (B') and define $\phi_G(a) = \left[\begin{pmatrix} P_a & O \\ O & P \end{pmatrix}, (\mathbf{0} \mid 10)\right]$, where P is permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(2-4) Since the other values of ϕ_G are automatically determined, check if ϕ_G successfully embeds G to $\mathcal{AP}(n+2)$.

6 Construction Examples of Gray map for groups of order 4 and order 8

In this section we show construction examples using the recipe. Since the above recipe is not applicable for C_4, C_8 and Q_8, we show construction examples for them by heuristic method.

For simplicity, in what follows, we denote $P_{\pi_i}^T$ by P_i.

6.1 Construction Examples based on the semidirect-product structure

(1) $K_4 = \langle x, y \mid x^2 = y^2 = e, xy = yx \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2] \rangle$, where $c_1 = 10, c_2 = 01, \pi_1$ and π_2 are the identity permutations. K_4 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 00] & = 00 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[E, 10] & = 10 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[E, 01] & = 01 \\
\phi(xy) & = \phi[P_2 P_1, 01 P_1 + c_1] & = \phi[E, 11] & = 11
\end{array}$$

(2) $K_8 = \langle x, y \mid x^4 = y^2 = e, xy = yx \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2] \rangle$, where $c_1 = 100, c_2 = 001, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right)$ and π_2 is the identity permutation. K_8 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 000] & = 000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 100] & = 100 \\
\phi(x^2) & = \phi[P_1^2, 100 P_1 + c_1] & = \phi[E, 110] & = 110 \\
\phi(x^3) & = \phi[P_1^3, 110 P_1 + c_1] & = \phi[P_1, 010] & = 010 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[E, 001] & = 001 \\
\phi(xy) & = \phi[P_2 P_1, 001 P_1 + c_1] & = \phi[P_1, 101] & = 101 \\
\phi(x^2 y) & = \phi[P_2 P_1^2, 101 P_1 + c_1] & = \phi[E, 111] & = 111 \\
\phi(x^3 y) & = \phi[P_2 P_1^3, 111 P_1 + c_1] & = \phi[P_1, 011] & = 011
\end{array}$$

(3) $D_8 = \langle x, y \mid x^4 = y^2 = e, xy = yx^3 \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2] \rangle$, where $c_1 = 100, c_2 = 001, \pi_1 = \pi_2 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right)$. D_8 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 000] & = 000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 100] & = 100 \\
\phi(x^2) & = \phi[P_1^2, 100 P_1 + c_1] & = \phi[E, 110] & = 110 \\
\phi(x^3) & = \phi[P_1^3, 110 P_1 + c_1] & = \phi[P_1, 010] & = 010 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[P_2, 001] & = 001 \\
\phi(xy) & = \phi[P_2 P_1, 001 P_1 + c_1] & = \phi[E, 101] & = 101 \\
\phi(x^2 y) & = \phi[P_2 P_1^2, 101 P_1 + c_1] & = \phi[P_1, 111] & = 111 \\
\phi(x^3 y) & = \phi[P_2 P_1^3, 111 P_1 + c_1] & = \phi[E, 011] & = 011
\end{array}$$

6.2 Construction Examples without using the recipe

(1) $C_4 = \langle x \mid x^4 = e \rangle \simeq \langle [P_\pi^T, c] \rangle$, where $c = 10$ and $\pi = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. C_4 has the following Gray map:

$$\begin{aligned}
\phi(e) &= \phi[E, \mathbf{0}] &&= \phi[E, 00] &&= 00\\
\phi(x) &= \phi[P, c] &&= \phi[P, 10] &&= 10\\
\phi(x^2) &= \phi[P^2, 10P + c] &&= \phi[E, 11] &&= 11\\
\phi(x^3) &= \phi[P^3, 11P + c] &&= \phi[P, 01] &&= 01
\end{aligned}$$

(2) $C_8 = \langle x \mid x^8 = e\rangle \simeq \langle [P_\pi^T, c]\rangle$, where $c = 1000$ and $\pi = \left(\begin{smallmatrix}1&2&3&4\\4&1&2&3\end{smallmatrix}\right)$. C_8 has the following Gray map:

$$\begin{aligned}
\phi(e) &= \phi[E, \mathbf{0}] &&= \phi[E, 0000] &&= 0000\\
\phi(x) &= \phi[P, c] &&= \phi[P, 1000] &&= 1000\\
\phi(x^2) &= \phi[P^2, 1000P + c] &&= \phi[P^2, 1100] &&= 1100\\
\phi(x^3) &= \phi[P^3, 1100P + c] &&= \phi[P^3, 1110] &&= 1110\\
\phi(x^4) &= \phi[P^4, 1110P + c] &&= \phi[E, 1111] &&= 1111\\
\phi(x^5) &= \phi[P^5, 1111P + c] &&= \phi[P, 0111] &&= 0111\\
\phi(x^6) &= \phi[P^6, 0111P + c] &&= \phi[P^2, 0011] &&= 0011\\
\phi(x^7) &= \phi[P^7, 0011P + c] &&= \phi[P^3, 0001] &&= 0001
\end{aligned}$$

(3) $Q_8 = \langle x, y \mid x^4 = e, x^2 = y^2, xy = yx^3\rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2]\rangle$, where $c_1 = 1100, c_2 = 0110, \pi_1 = \left(\begin{smallmatrix}1&2&3&4\\3&4&1&2\end{smallmatrix}\right)$, and $\pi_2 = \left(\begin{smallmatrix}1&2&3&4\\2&1&4&3\end{smallmatrix}\right)$. Q_8 has the following Gray map:

$$\begin{aligned}
\phi(e) &= \phi[E, \mathbf{0}] &&= \phi[E, 0000] &&= 0000\\
\phi(x) &= \phi[P_1, c_1] &&= \phi[P_1, 1100] &&= 1100\\
\phi(x^2) &= \phi[P_1^2, 1100P_1 + c_1] &&= \phi[E, 1111] &&= 1111\\
\phi(x^3) &= \phi[P_1^3, 1111P_1 + c_1] &&= \phi[P_1, 0011] &&= 0011\\
\phi(y) &= \phi[P_2, c_2] &&= \phi[P_2, 0110] &&= 0110\\
\phi(xy) &= \phi[P_2P_1, 0110P_1 + c_1] &&= \phi[P_2P_1, 0101] &&= 0101\\
\phi(x^2y) &= \phi[P_2P_1^2, 0101P_1 + c_1] &&= \phi[P_2, 1001] &&= 1001\\
\phi(x^3y) &= \phi[P_2P_1^3, 1001P_1 + c_1] &&= \phi[P_2P_1, 1010] &&= 1010
\end{aligned}$$

7 Construction Examples of Gray map for groups of order 16

Similarly, in this section we show construction examples using the recipe. Since the recipe are not applicable for G_2 and G_3, we show construction examples by heuristic method.

7.1 Construction Examples based on the semidirect-product structure

(1) $G_1 = \langle x, a \mid x^8 = a^2 = e, xa = ax \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2] \rangle$, where $c_1 = 10000, c_2 = 00001, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 3 & 5 \end{smallmatrix}\right)$ and π_2 is the identity permutation. G_1 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 00000] & = 00000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 10000] & = 10000 \\
\phi(x^2) & = \phi[P_1^2, 10000P_1 + c_1] & = \phi[P_1^2, 11000] & = 11000 \\
\phi(x^3) & = \phi[P_1^3, 11000P_1 + c_1] & = \phi[P_1^3, 11100] & = 11100 \\
\phi(x^4) & = \phi[P_1^4, 11100P_1 + c_1] & = \phi[E, 11110] & = 11110 \\
\phi(x^5) & = \phi[P_1^5, 11110P_1 + c_1] & = \phi[P_1, 01110] & = 01110 \\
\phi(x^6) & = \phi[P_1^6, 01110P_1 + c_1] & = \phi[P_1^2, 00110] & = 00110 \\
\phi(x^7) & = \phi[P_1^7, 00110P_1 + c_1] & = \phi[P_1^3, 00010] & = 00010 \\
\phi(a) & = \phi[P_2, c_2] & = \phi[E, 00001] & = 00001 \\
\phi(xa) & = \phi[P_2P_1, 00001P_1 + c_1] & = \phi[P_1, 10001] & = 10001 \\
\phi(x^2a) & = \phi[P_2P_1^2, 10001P_1 + c_1] & = \phi[P_1^2, 11001] & = 11001 \\
\phi(x^3a) & = \phi[P_2P_1^3, 11001P_1 + c_1] & = \phi[P_1^3, 11101] & = 11101 \\
\phi(x^4a) & = \phi[P_2P_1^4, 11101P_1 + c_1] & = \phi[E, 11111] & = 11111 \\
\phi(x^5a) & = \phi[P_2P_1^5, 11111P_1 + c_1] & = \phi[P_1, 01111] & = 01111 \\
\phi(x^6a) & = \phi[P_2P_1^6, 01111P_1 + c_1] & = \phi[P_1^2, 00111] & = 00111 \\
\phi(x^7a) & = \phi[P_2P_1^7, 00111P_1 + c_1] & = \phi[P_1^3, 00011] & = 00011
\end{array}$$

(2) $G_4 = \langle x, a \mid x^8 = a^2 = e, xa = ax^7 \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2] \rangle$, where $c_1 = 10000, c_2 = 00001, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 3 & 5 \end{smallmatrix}\right)$ and π_2=$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{smallmatrix}\right)$.

G_4 has the following Gray map:

$$
\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 00000] & = 00000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 10000] & = 10000 \\
\phi(x^2) & = \phi[P_1^2, 10000P_1 + c_1] & = \phi[P_1^2, 11000] & = 11000 \\
\phi(x^3) & = \phi[P_1^3, 11000P_1 + c_1] & = \phi[P_1^3, 11100] & = 11100 \\
\phi(x^4) & = \phi[P_1^4, 11100P_1 + c_1] & = \phi[E, 11110] & = 11110 \\
\phi(x^5) & = \phi[P_1^5, 11110P_1 + c_1] & = \phi[P_1, 01110] & = 01110 \\
\phi(x^6) & = \phi[P_1^6, 01110P_1 + c_1] & = \phi[P_1^2, 00110] & = 00110 \\
\phi(x^7) & = \phi[P_1^7, 00110P_1 + c_1] & = \phi[P_1^3, 00010] & = 00010 \\
\phi(a) & = \phi[P_2, c_2] & = \phi[E, 00001] & = 00001 \\
\phi(xa) & = \phi[P_2P_1, 00001P_1 + c_1] & = \phi[P_1, 10001] & = 10001 \\
\phi(x^2a) & = \phi[P_2P_1^2, 10001P_1 + c_1] & = \phi[P_1^2, 11001] & = 11001 \\
\phi(x^3a) & = \phi[P_2P_1^3, 11001P_1 + c_1] & = \phi[P_1^3, 11101] & = 11101 \\
\phi(x^4a) & = \phi[P_2P_1^4, 11101P_1 + c_1] & = \phi[E, 11111] & = 11111 \\
\phi(x^5a) & = \phi[P_2P_1^5, 11111P_1 + c_1] & = \phi[P_1, 01111] & = 01111 \\
\phi(x^6a) & = \phi[P_2P_1^6, 01111P_1 + c_1] & = \phi[P_1^2, 00111] & = 00111 \\
\phi(x^7a) & = \phi[P_2P_1^7, 00111P_1 + c_1] & = \phi[P_1^3, 00011] & = 00011
\end{array}
$$

(3) $G_7 = \langle x, y, a \mid x^4 = y^2 = a^2 = e, xy = yx, xa = ax, ya = ay\rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2], [P_{\pi_3}^T, c_3]\rangle$, where $c_1 = 1000, c_2 = 0010, c_3 = 0001, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{smallmatrix}\right)$ and π_2, π_3 are the identity permutations.

G_7 has the following Gray map:

$$
\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 1000] & = 1000 \\
\phi(x^2) & = \phi[P_1^2, 1000P_1 + c_1] & = \phi[E, 1100] & = 1100 \\
\phi(x^3) & = \phi[P_1^3, 1100P_1 + c_1] & = \phi[P_1, 0100] & = 0100 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[E, 0010] & = 0010 \\
\phi(xy) & = \phi[P_2P_1, 0010P_1 + c_1] & = \phi[P_1, 1010] & = 1010 \\
\phi(x^2y) & = \phi[P_2P_1^2, 1010P_1 + c_1] & = \phi[E, 1110] & = 1110 \\
\phi(x^3y) & = \phi[P_2P_1^3, 1110P_1 + c_1] & = \phi[P_1, 0110] & = 0110 \\
\phi(a) & = \phi[P_3, c_3] & = \phi[E, 0001] & = 0001 \\
\phi(xa) & = \phi[P_3P_1, 0001P_1 + c_1] & = \phi[P_1, 1001] & = 1001 \\
\phi(x^2a) & = \phi[P_3P_1^2, 1001P_1 + c_1] & = \phi[E, 1101] & = 1101 \\
\phi(x^3a) & = \phi[P_3P_1^3, 1101P_1 + c_1] & = \phi[P_1, 0101] & = 0101 \\
\phi(ya) & = \phi[P_3P_2, 0001P_2 + c_2] & = \phi[E, 0011] & = 0011 \\
\phi(xya) & = \phi[P_3P_2P_1, 0011P_1 + c_1] & = \phi[P_1, 1011] & = 1011 \\
\phi(x^2ya) & = \phi[P_3P_2P_1^2, 1011P_1 + c_1] & = \phi[E, 1111] & = 1111 \\
\phi(x^3ya) & = \phi[P_3P_2P_1^3, 1111P_1 + c_1] & = \phi[P_1, 0111] & = 0111
\end{array}
$$

(4) $G_8 = \langle x, y, a \mid x^4 = y^2 = a^2 = e, xy = yx, xa = ax^3, ya = ay\rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2], [P_{\pi_3}^T, c_3]\rangle$,

where $c_1 = 1000, c_2 = 0010, c_3 = 0001, \pi_1 = \pi_3 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{smallmatrix}\right)$ and π_2 is the identity permutation.

G_8 has the following Gray map:

$$
\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 1000] & = 1000 \\
\phi(x^2) & = \phi[P_1^2, 1000P_1 + c_1] & = \phi[E, 1100] & = 1100 \\
\phi(x^3) & = \phi[P_1^3, 1100P_1 + c_1] & = \phi[P_1, 0100] & = 0100 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[E, 0010] & = 0010 \\
\phi(xy) & = \phi[P_2P_1, 0010P_1 + c_1] & = \phi[P_1, 1010] & = 1010 \\
\phi(x^2y) & = \phi[P_2P_1^2, 1010P_1 + c_1] & = \phi[E, 1110] & = 1110 \\
\phi(x^3y) & = \phi[P_2P_1^3, 1110P_1 + c_1] & = \phi[P_1, 0110] & = 0110 \\
\phi(a) & = \phi[P_3, c_3] & = \phi[P_3, 0001] & = 0001 \\
\phi(xa) & = \phi[P_3P_1, 0001P_1 + c_1] & = \phi[P_1, 1001] & = 1001 \\
\phi(x^2a) & = \phi[P_3P_1^2, 1001P_1 + c_1] & = \phi[E, 1101] & = 1101 \\
\phi(x^3a) & = \phi[P_3P_1^3, 1101P_1 + c_1] & = \phi[P_1, 0101] & = 0101 \\
\phi(ya) & = \phi[P_3P_2, 0001P_2 + c_2] & = \phi[P_3, 0011] & = 0011 \\
\phi(xya) & = \phi[P_3P_2P_1, 0011P_1 + c_1] & = \phi[E, 1011] & = 1011 \\
\phi(x^2ya) & = \phi[P_3P_2P_1^2, 1011P_1 + c_1] & = \phi[P_3, 1111] & = 1111 \\
\phi(x^3ya) & = \phi[P_3P_2P_1^3, 1111P_1 + c_1] & = \phi[E, 0111] & = 0111
\end{array}
$$

(5) $G_8 = \langle x, y, a \mid x^4 = y^2 = a^2 = e, xy = yx^3, xa = ax, ya = ay\rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2], [P_{\pi_3}^T, c_3]\rangle$, where $c_1 = 1000, c_2 = 0010, c_3 = 0001, \pi_1 = \pi_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{smallmatrix}\right)$ and π_3 is the identity permutation.

G_8 has the following Gray map:

$$
\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 1000] & = 1000 \\
\phi(x^2) & = \phi[P_1^2, 1000P_1 + c_1] & = \phi[E, 1100] & = 1100 \\
\phi(x^3) & = \phi[P_1^3, 1100P_1 + c_1] & = \phi[P_1, 0100] & = 0100 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[P_2, 0010] & = 0010 \\
\phi(xy) & = \phi[P_2P_1, 0010P_1 + c_1] & = \phi[E, 1010] & = 1010 \\
\phi(x^2y) & = \phi[P_2P_1^2, 1010P_1 + c_1] & = \phi[P_2, 1110] & = 1110 \\
\phi(x^3y) & = \phi[P_2P_1^3, 1110P_1 + c_1] & = \phi[E, 0110] & = 0110 \\
\phi(a) & = \phi[P_3, c_3] & = \phi[E, 0001] & = 0001 \\
\phi(xa) & = \phi[P_3P_1, 0001P_1 + c_1] & = \phi[P_1, 1001] & = 1001 \\
\phi(x^2a) & = \phi[P_3P_1^2, 1001P_1 + c_1] & = \phi[E, 1101] & = 1101 \\
\phi(x^3a) & = \phi[P_3P_1^3, 1101P_1 + c_1] & = \phi[P_1, 0101] & = 0101 \\
\phi(ya) & = \phi[P_3P_2, 0001P_2 + c_2] & = \phi[P_2, 0011] & = 0011 \\
\phi(xya) & = \phi[P_3P_2P_1, 0011P_1 + c_1] & = \phi[E, 1011] & = 1011 \\
\phi(x^2ya) & = \phi[P_3P_2P_1^2, 1011P_1 + c_1] & = \phi[P_2, 1111] & = 1111 \\
\phi(x^3ya) & = \phi[P_3P_2P_1^3, 1111P_1 + c_1] & = \phi[E, 0111] & = 0111
\end{array}
$$

(6) $G_9 = \langle x, y, a \mid x^2 = y^2 = a^4 = e, xy = yx, ax = ya, ay = xa \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2], [P_{\pi_3}^T, c_3] \rangle$, where $c_1 = 1000, c_2 = 0100, c_3 = 0010, \pi_3 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{smallmatrix}\right)$ and π_1, π_2 are the identity permutations.

G_9 has the following Gray map:

$$
\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[E, 1000] & = 1000 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[E, 0100] & = 0100 \\
\phi(xy) & = \phi[P_2P_1, 0100P_1 + c_1] & = \phi[E, 1100] & = 1100 \\
\phi(a) & = \phi[P_3, c_3] & = \phi[P_3, 0010] & = 0010 \\
\phi(xa) & = \phi[P_3P_1, 0010P_1 + c_1] & = \phi[P_3, 1010] & = 1010 \\
\phi(ya) & = \phi[P_3P_2, 0010P_2 + c_2] & = \phi[P_3, 0110] & = 0110 \\
\phi(xya) & = \phi[P_3P_2P_1, 0110P_1 + c_1] & = \phi[P_3, 1110] & = 1110 \\
\phi(a^2) & = \phi[P_3^2, 0010P_3 + c_3] & = \phi[E, 0011] & = 0011 \\
\phi(xa^2) & = \phi[P_3^2P_1, 0011P_1 + c_1] & = \phi[E, 1011] & = 1011 \\
\phi(ya^2) & = \phi[P_3^2P_2, 0011P_2 + c_2] & = \phi[E, 0111] & = 0111 \\
\phi(xya^2) & = \phi[P_3^2P_2P_1, 0111P_1 + c_1] & = \phi[E, 1111] & = 1111 \\
\phi(a^3) & = \phi[P_3^3, 0011P_3 + c_3] & = \phi[P_3, 0001] & = 0001 \\
\phi(xa^3) & = \phi[P_3^3P_1, 0001P_1 + c_1] & = \phi[P_3, 1001] & = 1001 \\
\phi(ya^3) & = \phi[P_3^3P_2, 0001P_2 + c_2] & = \phi[P_3, 0101] & = 0101 \\
\phi(xya^3) & = \phi[P_3^3P_2P_1, 0101P_1 + c_1] & = \phi[P_3, 1101] & = 1101
\end{array}
$$

(7) $G_{10} = \langle x, y, a \mid x^4 = e, y^2 = x^2, xy = yx^3, xa = ax, ay = x^2ya \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2], [P_{\pi_3}^T, c_3] \rangle$,

where $c_1 = 11000, c_2 = 01100, c_3 = 00001, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{smallmatrix}\right)$ and $\pi_2 = \pi_3 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{smallmatrix}\right)$.

G_{10} has the following Gray map:

$$
\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 00000] & = 00000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 11000] & = 11000 \\
\phi(x^2) & = \phi[P_1^2, 11000P_1 + c_1] & = \phi[E, 11110] & = 11110 \\
\phi(x^3) & = \phi[P_1^3, 11110P_1 + c_1] & = \phi[P_1, 00110] & = 00110 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[P_2, 01100] & = 01100 \\
\phi(xy) & = \phi[P_2P_1, 01100P_1 + c_1] & = \phi[P_2P_1, 01010] & = 01010 \\
\phi(x^2y) & = \phi[P_2P_1^2, 01010P_1 + c_1] & = \phi[P_2, 10010] & = 10010 \\
\phi(x^3y) & = \phi[P_2P_1^3, 10010P_1 + c_1] & = \phi[P_2P_1, 10100] & = 10100 \\
\phi(a) & = \phi[P_3, c_3] & = \phi[P_3, 00001] & = 00001 \\
\phi(xa) & = \phi[P_3P_1, 00001P_1 + c_1] & = \phi[E, 11001] & = 11001 \\
\phi(x^2a) & = \phi[P_3P_1^2, 11001P_1 + c_1] & = \phi[P_3, 11111] & = 11111 \\
\phi(x^3a) & = \phi[P_3P_1^3, 11111P_1 + c_1] & = \phi[P_3P_1, 00111] & = 00111 \\
\phi(ya) & = \phi[P_3P_2, 00110P_2 + c_2] & = \phi[P_3P_2, 01101] & = 01101 \\
\phi(xya) & = \phi[P_3P_2P_1, 01101P_1 + c_1] & = \phi[P_1, 01011] & = 01011 \\
\phi(x^2ya) & = \phi[P_3P_2P_1^2, 01011P_1 + c_1] & = \phi[P_3P_2, 10011] & = 10011 \\
\phi(x^3ya) & = \phi[P_3P_2P_1^3, 10011P_1 + c_1] & = \phi[P_1, 10101] & = 10101
\end{array}
$$

(8) $G_{11} = \langle x, y, a \mid x^4 = e, y^2 = x^2, xy = yx^3, xa = ax, ay = ya \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2], [P_{\pi_3}^T, c_3] \rangle$, where $c_1 = 11000, c_2 = 01100, c_3 = 00001, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{smallmatrix}\right), \pi_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{smallmatrix}\right)$ and π_3 is the identity permutation.

G_{11} has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 00000] & = 00000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 11000] & = 11000 \\
\phi(x^2) & = \phi[P_1^2, 11000P_1 + c_1] & = \phi[E, 11110] & = 11110 \\
\phi(x^3) & = \phi[P_1^3, 11110P_1 + c_1] & = \phi[P_1, 00110] & = 00110 \\
\phi(y) & = \phi[P_2, c_2] & = \phi[P_2, 01100] & = 01100 \\
\phi(xy) & = \phi[P_2P_1, 01100P_1 + c_1] & = \phi[P_2P_1, 01010] & = 01010 \\
\phi(x^2y) & = \phi[P_2P_1^2, 01010P_1 + c_1] & = \phi[P_2, 10010] & = 10010 \\
\phi(x^3y) & = \phi[P_2P_1^3, 10010P_1 + c_1] & = \phi[P_2P_1, 10100] & = 10100 \\
\phi(a) & = \phi[P_3, c_3] & = \phi[E, 00001] & = 00001 \\
\phi(xa) & = \phi[P_3P_1, 00001P_1 + c_1] & = \phi[P_1, 11001] & = 11001 \\
\phi(x^2a) & = \phi[P_3P_1^2, 11001P_1 + c_1] & = \phi[E, 11111] & = 11111 \\
\phi(x^3a) & = \phi[P_3P_1^3, 11111P_1 + c_1] & = \phi[P_1, 00111] & = 00111 \\
\phi(ya) & = \phi[P_3P_2, 00001P_2 + c_2] & = \phi[P_2, 01101] & = 01101 \\
\phi(xya) & = \phi[P_3P_2P_1, 01101P_1 + c_1] & = \phi[P_2P_1, 01011] & = 01011 \\
\phi(x^2ya) & = \phi[P_3P_2P_1^2, 01011P_1 + c_1] & = \phi[P_2, 10011] & = 10011 \\
\phi(x^3ya) & = \phi[P_3P_2P_1^3, 10011P_1 + c_1] & = \phi[P_2P_1, 10101] & = 10101
\end{array}$$

(9) $G_{12} = \langle x, a \mid x^4 = a^4 = e, xa = ax^3 \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2] \rangle$, where $c_1 = 1000, c_2 = 0010, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{smallmatrix}\right), \pi_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{smallmatrix}\right)$.

G_{12} has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 1000] & = 1000 \\
\phi(x^2) & = \phi[P_1^2, 1000P_1 + c_1] & = \phi[E, 1100] & = 1100 \\
\phi(x^3) & = \phi[P_1^3, 1100P_1 + c_1] & = \phi[P_1, 0100] & = 0100 \\
\phi(a) & = \phi[P_2, c_2] & = \phi[P_2, 0010] & = 0010 \\
\phi(xa) & = \phi[P_2P_1, 0010P_1 + c_1] & = \phi[P_2P_1, 1010] & = 1010 \\
\phi(x^2a) & = \phi[P_2P_1^2, 1010P_1 + c_1] & = \phi[P_2, 1110] & = 1110 \\
\phi(x^3a) & = \phi[P_2P_1^3, 1110P_1 + c_1] & = \phi[P_2P_1, 0110] & = 0110 \\
\phi(a^2) & = \phi[P_2^2, 0010P_2 + c_2] & = \phi[E, 0011] & = 0011 \\
\phi(xa^2) & = \phi[P_2^2P_1, 0011P_1 + c_1] & = \phi[P_1, 1011] & = 1011 \\
\phi(x^2a^2) & = \phi[P_2^2P_1^2, 1011P_1 + c_1] & = \phi[E, 1111] & = 1111 \\
\phi(x^3a^2) & = \phi[P_2^2P_1^3, 1111P_1 + c_1] & = \phi[P_1, 0111] & = 0111 \\
\phi(a^3) & = \phi[P_2^3, 0011P_2 + c_2] & = \phi[P_2, 0001] & = 0001 \\
\phi(xa^3) & = \phi[P_2^3P_1, 0001P_1 + c_1] & = \phi[P_2P_1, 1001] & = 1001 \\
\phi(x^2a^3) & = \phi[P_2^3P_1^2, 1001P_1 + c_1] & = \phi[P_2, 1101] & = 1101 \\
\phi(x^3a^3) & = \phi[P_2^3P_1^3, 1101P_1 + c_1] & = \phi[P_2P_1, 0101] & = 0101
\end{array}$$

(10) $G_{13} = \langle x, a \mid x^4 = a^4 = e, xa = ax \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2] \rangle$, where $c_1 = 1000, c_2 = 0010, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{smallmatrix}\right)$, and $\pi_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{smallmatrix}\right)$.

G_{13} has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 1000] & = 1000 \\
\phi(x^2) & = \phi[P_1^2, 1000P_1 + c_1] & = \phi[E, 1100] & = 1100 \\
\phi(x^3) & = \phi[P_1^3, 1100P_1 + c_1] & = \phi[P_1, 0100] & = 0100 \\
\phi(a) & = \phi[P_2, c_2] & = \phi[P_2, 0010] & = 0010 \\
\phi(xa) & = \phi[P_2P_1, 0010P_1 + c_1] & = \phi[P_2P_1, 1010] & = 1010 \\
\phi(x^2a) & = \phi[P_2P_1^2, 1010P_1 + c_1] & = \phi[P_2, 1110] & = 1110 \\
\phi(x^3a) & = \phi[P_2P_1^3, 1110P_1 + c_1] & = \phi[P_2P_1, 0110] & = 0110 \\
\phi(a^2) & = \phi[P_2^2, 0010P_2 + c_2] & = \phi[E, 0011] & = 0011 \\
\phi(xa^2) & = \phi[P_2^2P_1, 0011P_1 + c_1] & = \phi[P_1, 1011] & = 1011 \\
\phi(x^2a^2) & = \phi[P_2^2P_1^2, 1011P_1 + c_1] & = \phi[E, 1111] & = 1111 \\
\phi(x^3a^2) & = \phi[P_2^2P_1^3, 1111P_1 + c_1] & = \phi[P_1, 0111] & = 0111 \\
\phi(a^3) & = \phi[P_2^3, 0011P_2 + c_2] & = \phi[P_2, 0001] & = 0001 \\
\phi(xa^3) & = \phi[P_2^3P_1, 0001P_1 + c_1] & = \phi[P_2P_1, 1001] & = 1001 \\
\phi(x^2a^3) & = \phi[P_2^3P_1^2, 1001P_1 + c_1] & = \phi[P_2, 1101] & = 1101 \\
\phi(x^3a^3) & = \phi[P_2^3P_1^3, 1101P_1 + c_1] & = \phi[P_2P_1, 0101] & = 0101
\end{array}$$

7.2 Construction Examples without using the recipe

(1) $G_2 = \langle x, a \mid x^8 = a^2 = e, xa = ax^3 \rangle \simeq \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2] \rangle$, where $c_1 = 0001, c_2 = 0010, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right)$ and $\pi_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{smallmatrix}\right)$.

G_2 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 0001] & = 0001 \\
\phi(x^2) & = \phi[P_1^2, 0001P_1 + c_1] & = \phi[P_1^2, 0011] & = 0011 \\
\phi(x^3) & = \phi[P_1^3, 0011P_1 + c_1] & = \phi[P_1^3, 0111] & = 0111 \\
\phi(x^4) & = \phi[P_1^4, 0111P_1 + c_1] & = \phi[E, 1111] & = 1111 \\
\phi(x^5) & = \phi[P_1^5, 1111P_1 + c_1] & = \phi[P_1, 1110] & = 1110 \\
\phi(x^6) & = \phi[P_1^6, 1110P_1 + c_1] & = \phi[P_1^2, 1100] & = 1100 \\
\phi(x^7) & = \phi[P_1^7, 1100P_1 + c_1] & = \phi[P_1^3, 1000] & = 1000 \\
\phi(a) & = \phi[P_2, c_2] & = \phi[P_2, 0010] & = 0010 \\
\phi(xa) & = \phi[P_2P_1, 0010P_1 + c_1] & = \phi[P_2P_1, 0101] & = 0101 \\
\phi(x^2a) & = \phi[P_2P_1^2, 0101P_1 + c_1] & = \phi[P_2P_1^2, 1011] & = 1011 \\
\phi(x^3a) & = \phi[P_2P_1^3, 1011P_1 + c_1] & = \phi[P_2P_1^3, 0110] & = 0110 \\
\phi(x^4a) & = \phi[P_2P_1^4, 0110P_1 + c_1] & = \phi[P_2, 1101] & = 1101 \\
\phi(x^5a) & = \phi[P_2P_1^5, 1101P_1 + c_1] & = \phi[P_2P_1, 1010] & = 1010 \\
\phi(x^6a) & = \phi[P_2P_1^6, 1010P_1 + c_1] & = \phi[P_2P_1^2, 0100] & = 0100 \\
\phi(x^7a) & = \phi[P_2P_1^7, 0100P_1 + c_1] & = \phi[P_2P_1^3, 1001] & = 1001
\end{array}$$

(2) $G_3 = \langle x, a \mid x^8 = a^2 = e, xa = ax^5 \rangle \simeq \langle [P^T_{\pi_1}, c_1], [P^T_{\pi_2}, c_2] \rangle$, where $c_1 = 0001, c_2 = 0101, \pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right)$ and π_2 is the identity permutation.

G_3 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 0001] & = 0001 \\
\phi(x^2) & = \phi[P_1^2, 0001P_1 + c_1] & = \phi[P_1^2, 0011] & = 0011 \\
\phi(x^3) & = \phi[P_1^3, 0011P_1 + c_1] & = \phi[P_1^3, 0111] & = 0111 \\
\phi(x^4) & = \phi[P_1^4, 0111P_1 + c_1] & = \phi[E, 1111] & = 1111 \\
\phi(x^5) & = \phi[P_1^5, 1111P_1 + c_1] & = \phi[P_1, 1110] & = 1110 \\
\phi(x^6) & = \phi[P_1^6, 1110P_1 + c_1] & = \phi[P_1^2, 1100] & = 1100 \\
\phi(x^7) & = \phi[P_1^7, 1100P_1 + c_1] & = \phi[P_1^3, 1000] & = 1000 \\
\phi(a) & = \phi[P_2, c_2] & = \phi[P_2, 0101] & = 0101 \\
\phi(xa) & = \phi[P_2P_1, 0101P_1 + c_1] & = \phi[P_2P_1, 1011] & = 1011 \\
\phi(x^2a) & = \phi[P_2P_1^2, 1011P_1 + c_1] & = \phi[P_2P_1^2, 0110] & = 0110 \\
\phi(x^3a) & = \phi[P_2P_1^3, 0110P_1 + c_1] & = \phi[P_2P_1^3, 1101] & = 1101 \\
\phi(x^4a) & = \phi[P_2P_1^4, 1101P_1 + c_1] & = \phi[P_2, 1010] & = 1010 \\
\phi(x^5a) & = \phi[P_2P_1^5, 1010P_1 + c_1] & = \phi[P_2P_1, 0100] & = 0100 \\
\phi(x^6a) & = \phi[P_2P_1^6, 0100P_1 + c_1] & = \phi[P_2P_1^2, 1001] & = 1001 \\
\phi(x^7a) & = \phi[P_2P_1^7, 1001P_1 + c_1] & = \phi[P_2P_1^3, 0010] & = 0010
\end{array}$$

8 Construction Examples of Gray maps for a group other than 2-group

In this section we show that our method can also construct Gray maps for several non-p-groups.

(1) $C_3 = \langle x \mid x^3 = e \rangle \cong \langle [P^T_\pi, c] \rangle$, where $c = 110$ and $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$. C_3 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 000] & = 000 \\
\phi(x) & = \phi[P, c] & = \phi[P, 110] & = 110 \\
\phi(x^2) & = \phi[P^2, 110P + c] & = \phi[P^2, 011] & = 011
\end{array}$$

(2) $C_5 = \langle x \mid x^5 = e \rangle \cong \langle [P^T_\pi, c] \rangle$, where $c = 11000$ and $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{smallmatrix}\right)$. C_5 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 00000] & = 00000 \\
\phi(x) & = \phi[P, c] & = \phi[P, 11000] & = 11000 \\
\phi(x^2) & = \phi[P^2, 11000P + c] & = \phi[P^2, 11110] & = 11110 \\
\phi(x^3) & = \phi[P^3, 11110P + c] & = \phi[P^3, 01111] & = 01111 \\
\phi(x^4) & = \phi[P^4, 01111P + c] & = \phi[P^4, 00011] & = 00011
\end{array}$$

(3) $C_6 = \langle x \mid x^6 = e\rangle \cong \langle [P_\pi^T, c]\rangle$, where $c = 100$ and $\pi = \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$. C_6 has the following Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 000] & = 000 \\
\phi(x) & = \phi[P, c] & = \phi[P, 100] & = 100 \\
\phi(x^2) & = \phi[P^2, 100P + c] & = \phi[P^2, 110] & = 110 \\
\phi(x^3) & = \phi[P^3, 110P + c] & = \phi[E, 111] & = 111 \\
\phi(x^4) & = \phi[P^4, 111P + c] & = \phi[P, 011] & = 011 \\
\phi(x^5) & = \phi[P^5, 011P + c] & = \phi[P^2, 001] & = 001
\end{array}$$

(4) For $n \in \mathbb{N}$, $C_{2n} = \langle x \mid x^{2n} = e\rangle \cong \langle [P_\pi^T, c]\rangle$, where $c = 10\ldots0$, and $\pi = \left(\begin{smallmatrix}1&2&3&\cdots&n\\n&1&2&\ldots&n-1\end{smallmatrix}\right)$. This gives a Gray map for C_{2n} over $\mathbb{Z}_2^n$.

(5) For $n \in \mathbb{N}$, $C_{2n+1} = \langle x \mid x^{2n+1} = e\rangle \cong \langle [P_\pi^T, c]\rangle$, where $c = 110\ldots0$ and $\pi = \left(\begin{smallmatrix}1&2&3&\ldots&2n+1\\2n&2n+1&1&\ldots&2n-1\end{smallmatrix}\right)$. This gives a Gray map for C_{2n+1} over $\mathbb{Z}_2^{2n+1}$.

(6) $D_6 = \langle x, y \mid x^3 = y^2 = e, xy = yx^2\rangle \cong \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2]\rangle$, where $c_1 = 011, c_2 = 010$, $\pi_1 = \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$, and π_2 is the identity permutation. D_6 has the following 3-bit Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 000] & = 000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 011] & = 011 \\
\phi(x^2) & = \phi[P_1^2, 011P_1 + c_1] & = \phi[P_1^2, 101] & = 101 \\
\phi(y) & = \phi[E, c_2] & = \phi[E, 010] & = 010 \\
\phi(xy) & = \phi[P_1, 010P_1 + c_1] & = \phi[P_1, 111] & = 111 \\
\phi(x^2y) & = \phi[P_1^2, 111P + c_1] & = \phi[P_1^2, 100] & = 100
\end{array}$$

(7) $D_{10} = \langle x, y \mid x^5 = y^2 = e, xy = yx^4\rangle \cong \langle [P_{\pi_1}^T, c_1], [P_{\pi_2}^T, c_2]\rangle$, where $c_1 = 00101, c_2 = 01101$, $\pi_1 = \left(\begin{smallmatrix}1&2&3&4&5\\2&3&4&5&1\end{smallmatrix}\right)$, and π_2 is the identity permutation. D_{10} has the following 5-bit Gray map:

$$\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 00000] & = 00000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 00101] & = 00101 \\
\phi(x^2) & = \phi[P_1^2, 00101P_1 + c_1] & = \phi[P_1^2, 01111] & = 01111 \\
\phi(x^3) & = \phi[P_1^3, 01111P_1 + c_1] & = \phi[P_1^3, 11011] & = 11011 \\
\phi(x^4) & = \phi[P_1^4, 11011P_1 + c_1] & = \phi[P_1^4, 10010] & = 10010 \\
\phi(y) & = \phi[E, c_2] & = \phi[E, 01101] & = 01101 \\
\phi(xy) & = \phi[P_1, 01101P_1 + c_1] & = \phi[P_1, 11111] & = 11111 \\
\phi(x^2y) & = \phi[P_1^2, 11111P + c_1] & = \phi[P_1^2, 11010] & = 11010 \\
\phi(x^3y) & = \phi[P_1^3, 11010P + c_1] & = \phi[P_1^3, 10000] & = 10000 \\
\phi(x^4y) & = \phi[P_1^4, 10000P + c_1] & = \phi[P_1^4, 00100] & = 00100
\end{array}$$

(8) $D_{12} = \langle x, y \mid x^6 = y^2 = e, xy = yx^5 \rangle \cong \langle [P^T_{\pi_1}, c_1], [P^T_{\pi_2}, c_2] \rangle$, where $c_1 = 0010, c_2 = 0111$, $\pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{smallmatrix}\right)$, and π_2 is the identity permutation. D_{12} has the following 4-bit Gray map:

$$
\begin{array}{llll}
\phi(e) & = \phi[E, \mathbf{0}] & = \phi[E, 0000] & = 0000 \\
\phi(x) & = \phi[P_1, c_1] & = \phi[P_1, 0010] & = 0010 \\
\phi(x^2) & = \phi[P_1^2, 0010P_1 + c_1] & = \phi[P_1^2, 0110] & = 0110 \\
\phi(x^3) & = \phi[P_1^3, 0110P_1 + c_1] & = \phi[E, 1110] & = 1110 \\
\phi(x^4) & = \phi[P_1^4, 1110P_1 + c_1] & = \phi[P_1, 1100] & = 1100 \\
\phi(x^5) & = \phi[P_1^5, 1100P_1 + c_1] & = \phi[P_1^2, 1000] & = 1000 \\
\phi(y) & = \phi[E, c_2] & = \phi[E, 0111] & = 0111 \\
\phi(xy) & = \phi[P_1, 0111P_1 + c_1] & = \phi[P_1, 1111] & = 1111 \\
\phi(x^2y) & = \phi[P_1^2, 1111P + c_1] & = \phi[P_1^2, 1101] & = 1101 \\
\phi(x^3y) & = \phi[P_1^3, 1101P + c_1] & = \phi[E, 1001] & = 1001 \\
\phi(x^4y) & = \phi[P_1^4, 1001P + c_1] & = \phi[P_1, 0001] & = 0001 \\
\phi(x^5y) & = \phi[P_1^5, 0001P + c_1] & = \phi[P_1^2, 0011] & = 0011
\end{array}
$$

9 Summary

We propose a constructive method to design Gray maps for groups of order 16 in this paper.

We have shown that our method can construct Gray maps for several groups of order 16, namely, G_1, G_4, G_7, G_8, G_9, G_{10}, G_{11}, G_{12} and G_{13}. This method required less time and effort to design a Gray map than that in the previous paper [9].

However, our recipe failed to construct Gray maps for $G_5 = Q_{16}$ and $G_6 = C_{16}$ because the groups do not have extension type of form $(K, 2, \tau, e)$ and so does it for $G_2 = (C_8, 2, \sigma_2, e)$ and $G_3 = (C_8, 2, \sigma_3, e)$ because $w(c_x) \neq w(c_{\sigma_2(x)})$ and $w(c_x) \neq w(c_{\sigma_3(x)})$.

Our next theme is to find a new recipe effective to the failed groups. Furthermore, since we believe the method can also contribute to constructing non-binary codes, we want to propose a new recipe to construct Gray maps for non-binary codes.

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