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ALGEBRA OF KODAIRA-SPENCER GRAVITY AND DEFORMATION OF CALABI-YAU MANIFOLD

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We study the algebraic structure of the configuration space of the Kodaira-Spencer gravity theory on a Calabi-Yau threefold. We then investigate the deformation problem of the Kodaira-Spencer gravity at the classical level using the algebraic tools obtained here.

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1. Introduction

Generally speaking, second quantization of string theory is string field theory [1,2]. In the case of topological closed B string theory [3], however, its second quantization is known to reduce to a six-dimensional field theory on a Calabi-Yau three-fold X [4], which is the Kodaira-Spencer (KS) theory of gravity [5]. The configuration space of the KS gravity theory $\mathcal{B}$, which we will define in (2.2) below, has a structure of the differential Gerstenhaber-Batalin-Vilkovisky (dGBV in short) algebra [6,7].

In this paper we study the algebraic structure of the configuration space $\mathcal{B}$ more closely. We show, in particular, that some operators such as the Hodge dual operator or the Lefshetz operators, which are originally defined on the space of the differential forms $\mathcal{A}$ and are transferred to $\mathcal{B}$ by the isomorphism between $\mathcal{A}$ and $\mathcal{B}$ as vector spaces, behave better in $\mathcal{B}$ than in $\mathcal{A}$ in some sense. We also find another dGBV algebra structure in $\mathcal{B}$. Then we consider the problem of holomorphic deformation of the KS theory using the algebraic tools developed above.

The organization of the present paper is as follows. In Section 2, the configuration space $\mathcal{B}$ and the action of the KS gravity theory, as well as the space of the differential forms $\mathcal{A}$, are introduced. In Section 3, we describe the algebraic structure of the configuration space $\mathcal{B}$; we convert the various linear operators on $\mathcal{A}$ to $\mathcal{B}$ and express these as differential operators of the sigma model variables; in particular we find that the Hodge dual operator is super-algebra homomorphism modulo a phase factor, and the Lefshetz operators are derivations on $\mathcal{B}$. We give another dGBV algebra structure on $\mathcal{B}$; two dGBV algebras are related to each other by the Hodge duality. In Section 4, we construct the solution of the KS equation, which is the classical equation of the KS action, using the algebraic tools developed above. Then we describe the deformation of the action and the supercharges associated with the condensation of the solution. In Section 5, we identify the deformation of the previous section as the holomorphic limit of the deformation of the complex structure of the Calabi-Yau three-fold X by a direct computation. We also propose a deformation of the states under the holomorphic deformation. In Section 6, we discuss rather briefly future directions of the present paper.

2. Kodaira-Spencer Theory of Gravity

2.1. *Calabi-Yau threefold*

Let X be a Calabi-Yau threefold with a fixed complex structure. We also fix a holomorphic three-form $\Omega = s(z)\,\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3$, where z^is are local holomorphic coordinates of X, and a Kähler form $\omega = \mathrm{i}\sum g_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}z^{\bar{j}}$. We do not assume that the Kähler metric $g_{i\bar{j}}$ is Ricci-flat.

There are two volume forms: the one is $\mathrm{d}\,\mathrm{vol}_\omega = \omega^3/3!$, the other is $\mathrm{d}\,\mathrm{vol}_\Omega = \mathrm{i}\,\Omega \wedge \bar{\Omega}$, the ratio of which yields a scalar function σ on X:

$$\exp(\sigma) = |s|^2(\det g)^{-1}. \tag{2.1}$$

Note that the Ricci tensor can be written as $R_{i\bar{j}} = \partial_i\partial_{\bar{j}}\sigma$.

2.2. *Configuration space of the Kodaira-Spencer theory*

Two dimensional B-twisted topological sigma model with X as a target [3] coupled to topological gravity defines a topological B-string theory [4,5].

The string field theory of the topological B-string is reduced to a six dimensional field theory on X [4], known as the Kodaira-Spencer theory of gravity [5].

To explain the field content of the KS theory, we introduce the following space:

$$\mathcal{B} = \bigoplus_{p,q=0}^{3} B^{p,q}, \quad B^{p,q} = \Gamma(X, \textstyle\bigwedge^p TX \otimes \bigwedge^q \overline{T^*X}), \tag{2.2}$$

where $TX(\overline{T^*X})$ is the (anti-)holomorphic (co)tangent bundle of X. Note that $\mathcal{B}$ is a subspace of the state space of the topological B-sigma model [3] which has X as the target with all stringy excitations supressed [4].

It is convenient to use the sigma model variables $\boldsymbol{\theta}_i = \partial/\partial z^i$, $\boldsymbol{\eta}^{\bar{j}} = \mathrm{d}z^{\bar{j}}$ so that an element α of $B^{p,q}$ is expressed as

$$\alpha = \frac{1}{p!q!}\sum \alpha^{i_1\cdots i_p}_{\bar{j}_1\cdots\bar{j}_q}(z,\bar{z})\,\boldsymbol{\theta}_{i_1}\cdots\boldsymbol{\theta}_{i_p}\boldsymbol{\eta}^{\bar{j}_1}\cdots\boldsymbol{\eta}^{\bar{j}_q}. \tag{2.3}$$

For $\alpha \in B^{p,q}$, $|\alpha| = p + q$ is called the ghost number of α.

2.3. *Space of the differential forms*

We have also the space of differential forms on X:

$$\mathcal{A} = \bigoplus_{p,q=0}^{3} A^{p,q}, \quad A^{p,q} = \Gamma(X, \textstyle\bigwedge^p T^*X \otimes \bigwedge^q \overline{T^*X}). \tag{2.4}$$

We define the Hermite metric on $\mathcal{A}$ by the Hodge dual operator $* : A^{p,q} \to A^{3-q,3-p}$ as

$$(a|b) = \int_X a \wedge *\bar{b}, \quad a, b \in \mathcal{A}. \tag{2.5}$$

The exterior differential operators are given by $\partial = \sum \mathrm{d}\, z^i \wedge \partial/\partial z^i$, $\bar{\partial} = \sum \mathrm{d}\, z^{\bar{i}} \wedge \partial/\partial z^{\bar{i}}$, and their conjugates with respect to the metric (2.5) are $\partial^\dagger = - * \bar{\partial} *$, $\bar{\partial}^\dagger = - * \partial *$.

These four operators are all nilpotent and their anti-commutators all vanish except for two pairs, both of which give the same Laplacian:

$$\Box = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial} = \partial\partial^\dagger + \partial^\dagger\partial. \tag{2.6}$$

The harmonic states are its kernel: $\mathbb{H} = \mathrm{Ker}\,\Box$, the orthogonal complement of which with respect to the metric (2.5) admits further orthogonal decompositions:

$$\mathbb{H}^\perp = \mathrm{Im}\,\bar{\partial} \oplus \mathrm{Im}\,\bar{\partial}^\dagger = \mathrm{Im}\,\partial \oplus \mathrm{Im}\,\partial^\dagger. \tag{2.7}$$

In particular the subspace $\mathrm{Ker}\,\partial \subset \mathcal{A}$ is decomposed as

$$\mathrm{Ker}\,\partial = \mathbb{H} \oplus \mathrm{Im}(\partial\bar{\partial}) \oplus \mathrm{Im}(\partial\bar{\partial}^\dagger), \tag{2.8}$$

where $\mathbb{H}$ is called the physical states, $\mathrm{Im}(\partial\bar{\partial})$ the trivial states, and $\mathrm{Im}(\partial\bar{\partial}^\dagger)$ the unphysical states in [2] in the context of closed string field theory.

The Green's operator $\mathbf{G}$ is defined to be zero on $\mathbb{H}$, and $\Box^{-1}$ on $\mathbb{H}^\perp$ [8]. Let $\Pi : \mathcal{A} \to \mathbb{H}$ be the orthogonal projection onto the harmonic forms, then

$$\mathrm{id}_{\mathcal{A}} = \Pi + \mathbf{G}\Box. \tag{2.9}$$

Form the Lefshetz operators $L = \omega\wedge : A^{p,q} \to A^{p+1,q+1}$, $\Lambda = L^\dagger = *^{-1} \circ L \circ *$, we can form a $SU(2)$ algebra which is self-adjoint with respect to the metric (2.5):

$$J_1 = \frac{1}{2}(L + \Lambda), \quad J_2 = \frac{1}{2\,\mathrm{i}}(\Lambda - L), \quad J_3 = \frac{1}{2}(3 - P - Q), \tag{2.10}$$

where $(P, Q)|_{A^{p,q}} = (p, q)$, and $[J_a, J_b] = \mathrm{i} \sum \epsilon_{abc} J_c$.

The commutation relations among the differential operators and the Lefshetz operators above are known as the Hodge-Kähler identities [9]:

$$[L, \partial] = 0, \qquad [L, \bar{\partial}] = 0, \tag{2.11}$$

$$[\Lambda, \partial^\dagger] = 0, \qquad [\Lambda, \bar{\partial}^\dagger] = 0, \tag{2.12}$$

$$[L, \partial^\dagger] = \mathrm{i}\,\bar{\partial}, \quad [L, \bar{\partial}^\dagger] = -\,\mathrm{i}\,\partial, \tag{2.13}$$

$$[\Lambda, \partial] = \mathrm{i}\,\bar{\partial}^\dagger, \quad [\Lambda, \bar{\partial}] = -\,\mathrm{i}\,\partial^\dagger. \tag{2.14}$$

Using the holomorphic three-form Ω, we can define an isomorphism $\rho : \mathcal{B} \to \mathcal{A}$ of vector spaces [10]

$$\rho(1) = s\, \mathrm{d} z^1 \wedge \mathrm{d} z^2 \wedge \mathrm{d} z^3, \tag{2.15}$$

$$\rho(\boldsymbol{\theta}_i) = \tfrac{1}{2} s \textstyle\sum_{jk} \epsilon_{ijk}\, \mathrm{d} z^j \wedge \mathrm{d} z^k, \tag{2.16}$$

$$\rho(\boldsymbol{\theta}_i \boldsymbol{\theta}_j) = -s \textstyle\sum_k \epsilon_{ijk}\, \mathrm{d} z^k, \tag{2.17}$$

$$\rho(\boldsymbol{\theta}_1 \boldsymbol{\theta}_2 \boldsymbol{\theta}_3) = -s, \tag{2.18}$$

where ϵ_{ijk} is the Levi-Civita symbol. Note that ρ maps $B^{p,q}$ to $A^{3-p,q}$, so that it inverts the Grassmann parity.

The significance of the isomorphism ρ is that we can convert each linear operator f on $\mathcal{A}$ to that of $\mathcal{B}$ by $\hat{f} := \rho^{-1} \circ f \circ \rho$. In particular, we give the special names for the operators below

$$\Delta = \hat{\partial} : B^{p,q} \to B^{p-1,q}, \tag{2.19}$$

$$S = \widehat{\partial^\dagger} : B^{p,q} \to B^{p+1,q}, \tag{2.20}$$

$$R = \widehat{\bar{\partial}^\dagger} : B^{p,q} \to B^{p,q-1}. \tag{2.21}$$

In fact, $\mathcal{B}$ itself has a Hermite metric, $\bar{\partial}_{\mathcal{B}}$, which is the BRST operator of the topological string [3], and its conjugate $\bar{\partial}^\dagger_{\mathcal{B}}$ [8,9]. Note that here we are temporarily using subscripts in order to distinguish $\bar{\partial}_{\mathcal{B}}$ from $\bar{\partial}_{\mathcal{A}}$. Then we find that

$$\bar{\partial}_{\mathcal{B}} = -\widehat{\bar{\partial}_{\mathcal{A}}}, \quad \bar{\partial}^\dagger_{\mathcal{B}} = -e^{\sigma} \circ R \circ e^{-\sigma}, \tag{2.22}$$

where σ is the scalar function defined in (2.1) [10].

2.4. *Kodaira-Spencer gravity action*

Let us define the trace map $\mathrm{Tr} : \mathcal{B} \to \mathbb{C}$ (complex numbers) by

$$\mathrm{Tr}(\alpha) = \int_X \rho(\alpha) \wedge \Omega, \quad \alpha \in \mathcal{B}. \tag{2.23}$$

It is clear that $\mathrm{Tr}(\alpha) \neq 0$ only if $\alpha \in B^{3,3}$.

Lemma 2.1. *Let $\alpha \in B^{p,q}$, $\beta \in B^{3-p,3-q}$, then $\rho(\alpha) \wedge \rho(\beta) = (-1)^{q+1} \rho(\alpha \wedge \beta) \wedge \Omega$.*

Proof. We use ordered multi-indices to write the elements

$$\alpha = \sum_{I,J}^{<} \alpha^I_{\bar{J}} \boldsymbol{\theta}_I \boldsymbol{\eta}^{\bar{J}} \in B^{p,q}, \quad \beta = \sum_{K,L}^{<} \beta^K_{\bar{L}} \boldsymbol{\theta}_K \boldsymbol{\eta}^{\bar{L}} \in B^{3-p,3-q},$$

where each multi-index has the form $I = (i_1, \ldots, i_p)$, $i_1 < i_2 < \cdots < i_p$. For $I = (i_1, \ldots, i_p)$, let $I^* = (i^*_1, \ldots, i^*_{3-p})$ be the complementary multi-index, i.e., $\{i_1, \cdots, i_p, i^*_1, \ldots, i^*_{3-p}\} = \{1, 2, 3\}$ as an unordered set. Then we have $\rho(\boldsymbol{\theta}_I) = (-1)^{|I|(|I|-1)/2} \epsilon_{I,I^*} s \boldsymbol{\eta}^{I^*}$, where $\boldsymbol{\eta}^i = \mathrm{d} z^i$, Simple computation shows that

$$\rho(\alpha) \wedge \rho(\beta) = (-1)^{pq} s \sum_{I,J}^{<} \epsilon_{I,I^*} \epsilon_{J,J^*} \alpha^I_{\bar{J}} \beta^{I^*}_{\bar{J}^*} \boldsymbol{\eta}^{\bar{1}} \boldsymbol{\eta}^{\bar{2}} \boldsymbol{\eta}^{\bar{3}} \wedge \Omega,$$

$$\alpha \wedge \beta = (-1)^{q+pq} \sum_{I,J}^{<} \epsilon_{I,I^*} \epsilon_{J,J^*} \alpha^I_{\bar{J}} \beta^{I^*}_{\bar{J}^*} \boldsymbol{\theta}_1 \boldsymbol{\theta}_2 \boldsymbol{\theta}_3 \boldsymbol{\eta}^{\bar{1}} \boldsymbol{\eta}^{\bar{2}} \boldsymbol{\eta}^{\bar{3}},$$

$$\rho(\alpha \wedge \beta) = (-1)^{1+q+pq} s \sum_{I,J}^{<} \epsilon_{I,I^*} \epsilon_{J,J^*} \alpha^I_{\bar{J}} \beta^{I^*}_{\bar{J}^*} \boldsymbol{\eta}^{\bar{1}} \boldsymbol{\eta}^{\bar{2}} \boldsymbol{\eta}^{\bar{3}}. \qquad \square$$

An immediate corollary is the useful formula

$$\mathrm{Tr}(\alpha \wedge \beta) = (-1)^{q+1} \int_X \rho(\alpha) \wedge \rho(\beta), \quad \alpha \in B^{p,q},\ \beta \in B^{3-p,3-q}, \tag{2.24}$$

from which we obtain the "integration by parts" formulas below:

$$\mathrm{Tr}(\bar{\partial}\alpha \wedge \beta) = (-1)^{|\alpha|+1} \mathrm{Tr}(\alpha \wedge \bar{\partial}\beta), \tag{2.25}$$

$$\mathrm{Tr}(\Delta\alpha \wedge \beta) = (-1)^{|\alpha|} \mathrm{Tr}(\alpha \wedge \Delta\beta), \tag{2.26}$$

$$\mathrm{Tr}(S\alpha \wedge \beta) = (-1)^{|\alpha|+1} \mathrm{Tr}(\alpha \wedge S\beta), \tag{2.27}$$

$$\mathrm{Tr}(R\alpha \wedge \beta) = (-1)^{|\alpha|} \mathrm{Tr}(\alpha \wedge R\beta). \tag{2.28}$$

Now we are in a position to write down the action of the KS gravity theory [5]. For simplicity, we here restrict ourselves to the classical theory, which means that the "string field" φ takes values in $B^{1,1}$. It is known that in closed string field theory, to obtain a gauge invariant action, we must put the subsidiary condition $\Delta\varphi = 0$ on the state $\varphi \in \mathcal{B}$, so that the configuration space must be reduced to

$$B^{1,1} \cap \mathrm{Ker}\,\Delta = \rho^{-1}(\mathbb{H}^{2,1}) \oplus \mathrm{Im}\,\Delta \cap B^{1,1}, \tag{2.29}$$

where the first direct summand of the right hand side is called massless, while the second massive [5]. According to (2.29), we decompose $\varphi \in B^{1,1} \cap \mathrm{Ker}\,\Delta$ as $\varphi = \varphi_1 \oplus \Phi$, where $\varphi_1 \in \rho^{-1}(\mathbb{H}^{2,1})$ and $\Phi \in \mathrm{Im}\,\Delta$. Then the action evaluated at φ is given by

$$S_X[\varphi_1|\Phi] = \mathrm{Tr}\left(\frac{1}{6}(\varphi_1 + \Phi) \wedge (\varphi_1 + \Phi) \wedge (\varphi_1 + \Phi) - \frac{1}{2}\bar{\partial}\Delta^{-1}\Phi \wedge \Phi\right). \tag{2.30}$$

We note that $\bar{\partial}$ above is the BRST operator of the string theory. From the Hodge decomposition (2.7) and the partial integration (2.25), we can see that the value of (2.30) does not depend on the choice of $\Delta^{-1}\Phi$. Note that from (2.9) we can write $\varphi_1 = \hat{\Pi}(\varphi)$ and $\Phi = \Delta S\,\hat{\mathbf{G}}(\varphi)$.

We note that in (2.30) the massless mode φ_1 does not has a kinetic term so that it acts as a background field [5]. Let $\mathcal{M}$ be the moduli space of the complex structures of Calabi-Yau manifolds, and $T\mathcal{M}$ be its holomorphic tangent bundle. Then the complex structure of X determines a point $[X] \in \mathcal{M}$ and φ_1 an element of $T_{[X]}\mathcal{M}$. Thus the total background of the action (2.30) can be regarded as the total space of $T\mathcal{M}$ [11]. It is also interesting to note that integration of the holomorphic anomaly equation is performed by quantization of the massless modes [5,12,13].

The equation of motion, which is known as the Kodaira-Spencer equation, construction of its solutions, and deformations of the action (2.30), we will discuss in later sections.

3. Algebraic Structure of $\mathcal{B}$

3.1. *Linear operators in sigma model variables*

We have seen in the previous section that the space of differential forms $\mathcal{A}$ have a rich set of linear operators on it, which we can transfer to those on $\mathcal{B}$ via the isomorphism $\rho : \mathcal{B} \to \mathcal{A}$ (2.15–2.18). In this subsection, we describe these linear operators on $\mathcal{B}$ as those acting on the bosonic variables $z^i, z^{\bar{j}}$ as well as the fermionic ones $\boldsymbol{\theta}_i, \boldsymbol{\eta}^{\bar{j}}$.

3.1.1. *The Hodge dual operator*

The action of the Hodge dual operator $\hat{*} : B^{p,q} \to B^{q,p}$ clearly corresponds to the exchange of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$. Form this observation, we find the following description.

First define a map κ between the sigma model variables by

$$\kappa(\boldsymbol{\theta}_i) = -\sum g_{i\bar{j}}\,\boldsymbol{\eta}^{\bar{j}}, \quad \kappa(\boldsymbol{\eta}^{\bar{j}}) = \sum g^{i\bar{j}}\boldsymbol{\theta}_i. \tag{3.1}$$

Then we extend κ to $\mathcal{B}$ as a super-algebra homomorphism. Finally the action of the transferred Hodge dual operator on a homogeneous element $\alpha \in \mathcal{B}$ is given by

$$\hat{*}\alpha = \mathrm{i}\, e(|\alpha|)\kappa(\alpha), \tag{3.2}$$

where $e(n)$ is the "phenomenological" sign factor defined by $e(n) = (-1)^{(n+1)(n+2)/2}$. We can easily see that (3.2) satisfies $\hat{*}^2\alpha = (-1)^{|\alpha|+1}\alpha$.

The operator $\hat{*}$ is well-defined for the wedge product of $\mathcal{B}$. Indeed from the formula $e(n)e(m)e(n+m) = (-1)^{nm+1}$, we have

$$\hat{*}(\alpha \wedge \beta) = \mathrm{i}\,(-1)^{|\alpha|\cdot|\beta|}\,\hat{*}(\alpha) \wedge \hat{*}(\beta). \tag{3.3}$$

3.1.2. *Four differential operators*

The exterior differentials and their Hermite conjugates viewed in $\mathcal{B}$ are given by

$$\bar{\partial} = \sum \boldsymbol{\eta}^{\bar{i}} \frac{\partial}{\partial z^{\bar{i}}}, \tag{3.4}$$

$$\Delta = s^{-1} \circ \sum \frac{\partial}{\partial z^i}\frac{\partial}{\partial \boldsymbol{\theta}_i} \circ s \tag{3.5}$$

$$S = -\sum g^{\bar{j}k}\boldsymbol{\theta}_k \left(\frac{\partial}{\partial z^{\bar{j}}} - \Gamma^{\bar{i}}_{\bar{j}\bar{l}}\,\boldsymbol{\eta}^{\bar{l}}\frac{\partial}{\partial \boldsymbol{\eta}^{\bar{i}}}\right), \tag{3.6}$$

$$R = e^{-\sigma} \circ \sum g^{\bar{i}j}\left(\frac{\partial}{\partial z^j} + \Gamma^k_{jl}\,\boldsymbol{\theta}_k \frac{\partial}{\partial \boldsymbol{\theta}_l}\right)\frac{\partial}{\partial \boldsymbol{\eta}^{\bar{i}}} \circ e^{\sigma}, \tag{3.7}$$

where $\Gamma^i_{jk} = \sum_{\bar{l}} g^{i\bar{l}}\partial_j g_{k\bar{l}}$, $\partial_j = \partial/\partial z^j$, is the Christoffel symbol, and it is unavoidable to use sometimes Einstein's summation rule.

3.1.3. *Lefshetz operators*

The Lefshetz operators, $\hat{L} : B^{p,q} \to B^{p-1,q+1}$, $\hat{\Lambda} : B^{p,q} \to B^{p+1,q-1}$, are given by the following differential operators

$$\hat{L} = \mathrm{i} \sum g_{i\bar{j}} \boldsymbol{\eta}^{\bar{j}} \frac{\partial}{\partial \boldsymbol{\theta}_i}, \tag{3.8}$$

$$\hat{\Lambda} = -\mathrm{i} \sum g^{i\bar{j}} \boldsymbol{\theta}_i \frac{\partial}{\partial \boldsymbol{\eta}^{\bar{j}}}. \tag{3.9}$$

Let us check the first formula (3.8). Pick up an element of $B^{p,q}$ and differentiate it by $\boldsymbol{\theta}_i$

$$\alpha = \sum_{I,J}^{<} \alpha^I_{\bar{J}} \boldsymbol{\theta}_I \boldsymbol{\eta}^{\bar{J}} \in B^{p,q}, \quad \frac{\partial}{\partial \boldsymbol{\theta}_i} \alpha = \sum_{I,J:i\in I}^{<} \alpha^I_{\bar{J}} \, \epsilon(I_i) \boldsymbol{\theta}_{I_i} \boldsymbol{\eta}^{\bar{J}},$$

where $I_i = I \backslash \{i\}$, and $\epsilon(I_i)$ is a sign factor. Then we have

$$\sum_{i,j} \mathrm{i}\, g_{i\bar{j}} \boldsymbol{\eta}^{\bar{j}} \frac{\partial}{\partial \boldsymbol{\theta}_i} \alpha = (-1)^{p-1} \sum_{i,j} \sum_{I,J:i\in I, j\notin J}^{<} \mathrm{i}\, g_{i\bar{j}} \, \alpha^I_{\bar{J}} \, \epsilon(I_i) \boldsymbol{\theta}_{I_i} \boldsymbol{\eta}^{\bar{j}} \boldsymbol{\eta}^{\bar{J}}. \tag{3.10}$$

On the other hand, the action of $\hat{L}$ is computed as

$$\rho(\alpha) = (-1)^{p(p-1)/2} \sum_{I,J}^{<} s \alpha^I_{\bar{J}} \, \epsilon_{I,I^*} \boldsymbol{\eta}^{I^*} \boldsymbol{\eta}^{\bar{J}},$$

$$\omega \wedge \rho(\alpha) = (-1)^{p(p-1)/2+3-p} \sum_{i,j} \sum_{I,J:i\in I, j\notin J}^{<} \mathrm{i}\, g_{i\bar{j}} \, s\alpha^I_{\bar{J}} \, \epsilon_{I,I^*} \boldsymbol{\eta}^{i} \boldsymbol{\eta}^{I^*} \boldsymbol{\eta}^{\bar{j}} \boldsymbol{\eta}^{\bar{J}},$$

$$\rho^{-1}(\omega \wedge \rho(\alpha)) = \sum_{i,j} \sum_{I,J:i\in I, j\notin J}^{<} \mathrm{i}\, g_{i\bar{j}} \, \alpha^I_{\bar{J}} \, \epsilon_{I,I^*} \epsilon'(I_i) \boldsymbol{\theta}_{I_i} \boldsymbol{\eta}^{\bar{j}} \boldsymbol{\eta}^{\bar{J}}, \tag{3.11}$$

where $\epsilon'(I_i)$ is the sign factor which satisfies $\rho(\boldsymbol{\theta}_{I_i}) = (-1)^{(p-1)(p-2)/2} s \, \epsilon'(I_i) \boldsymbol{\eta}^i \boldsymbol{\eta}^{I^*}$.

If $i \in I$ is the kth element, then the sign factors are $\epsilon(I_i) = (-1)^{k-1}$, $\epsilon'(I_i) = (-1)^{p-k} \epsilon_{I,I^*}$, which shows that (3.10) and (3.11) coincide with each other.

Then the relation $\circledast \circ \hat{\Lambda} = \hat{L} \circ \circledast$ yields the second formula (3.9). We can also verify the Hodge-Kähler identities (2.11–2.14) among the converted operators $\hat{L}, \hat{\Lambda}$, and $\bar{\partial}, \Delta, S, R$.

The configuration space $\mathcal{B}$ of the KS gravity is a subspace of the state space of the topological B sigma model with stringy excitations omitted, so that $\mathcal{B}$ has some remnants of the $N = 2$ supersymmetry algebra, among which are the left/right $U(1)$ charges $(1/2)(\pm \mathrm{i}(\hat{L} - \hat{\Lambda}) + P_{\mathcal{B}} + Q_{\mathcal{B}})$ [5], the sum of which yields the ghost number operator

$$P_{\mathcal{B}} + Q_{\mathcal{B}} = \sum \boldsymbol{\theta}_i \frac{\partial}{\partial \boldsymbol{\theta}_i} + \sum \boldsymbol{\eta}^{\bar{j}} \frac{\partial}{\partial \boldsymbol{\eta}^{\bar{j}}}. \tag{3.12}$$

As $\widehat{Q_{\mathcal{A}}} = Q_{\mathcal{B}}$, $\widehat{P_{\mathcal{A}}} = 3 - P_{\mathcal{B}}$, the generators of the $SU(2)$ (2.10) become

$$\hat{J}_1 = \frac{1}{2}(\hat{\Lambda} + \hat{L}), \quad \hat{J}_2 = \frac{1}{2\,\mathrm{i}}(\hat{\Lambda} - \hat{L}), \quad \hat{J}_3 = \frac{1}{2}(P_{\mathcal{B}} - Q_{\mathcal{B}}). \tag{3.13}$$

Notice that these operators are of order one [14], which means that they act on $\mathcal{B}$ as derivations with respect to the wedge product. Therefore it can be said that the wedge product of $\mathcal{B}$ conserves the $SU(2)$ symmetry.

3.1.4. *Hodge duality, $SU(2)$, and $B^{1,1}$*

Note that the Hodge dual operator $\hat{*}$ maps each diagonal sector $B^{p,p}$ to itself with the eigenvalues $\pm\,\mathrm{i}$. It is easy to see that $\hat{*}$ acts as $-\,\mathrm{i}$ on $B^{0,0}$, and as $+\,\mathrm{i}$ on $B^{3,3}$. As for the most important sector $B^{1,1}$, we find the following fact:

Lemma 3.1. *For $\alpha \in B^{1,1}$, $\hat{*}\alpha = +\,\mathrm{i}\,\alpha$ if and only if $\hat{L}\alpha = 0$.*

Proof. Let us denote an element of $B^{1,1}$ by $\alpha = \sum \alpha^i_{\bar{j}} \boldsymbol{\theta}_i \boldsymbol{\eta}^{\bar{j}}$. Then $\hat{L}\alpha = \mathrm{i} \sum g_{i\bar{k}} \alpha^i_{\bar{j}} \boldsymbol{\eta}^{\bar{k}} \boldsymbol{\eta}^{\bar{j}}$, so that $\hat{L}\alpha = 0 \Leftrightarrow \sum g_{i\bar{k}} \alpha^i_{\bar{j}} = \sum g_{i\bar{j}} \alpha^i_{\bar{k}}$. On the other hand, from $\hat{*}\alpha = \mathrm{i} \sum g_{i\bar{k}} g^{\bar{j}l} \alpha^i_{\bar{j}} \boldsymbol{\theta}_l \boldsymbol{\eta}^{\bar{k}}$, we see that $\hat{\alpha} = +\,\mathrm{i}\,\alpha \Leftrightarrow \sum g_{i\bar{k}} g^{\bar{j}l} \alpha^i_{\bar{j}} = \alpha^l_{\bar{k}}$. $\square$

In other words, $\hat{*}\alpha = +\,\mathrm{i}\,\alpha$ if α belongs to a trivial representation (singlet), and $\hat{*}\alpha = -\,\mathrm{i}\,\alpha$ if α belongs to an adjoint representation (triplet) of $SU(2)$.

3.2. *dGBV algebras on $\mathcal{B}$*

3.2.1. *Algebra associated with the pair $(\Delta, \bar{\partial})$*

We recall here the dGBV algebra on $\mathcal{B}$ discovered in [6]. A good reference of the subject is [7]. Let us define the odd bracket on $\mathcal{B}$ by

$$[\alpha \bullet \beta] = (-1)^{|\alpha|}\Delta(\alpha \wedge \beta) - (-1)^{|\alpha|}\Delta(\alpha) \wedge \beta - \alpha \wedge \Delta(\beta), \tag{3.14}$$

where the symbol $\bullet$ carries the ghost number minus one. It should be remarked here that if $\alpha, \beta \in \operatorname{Ker}\Delta$, then $[\alpha \bullet \beta] = (-1)^{|\alpha|}\Delta(\alpha \wedge \beta) \in \operatorname{Im}\Delta$.

This bracket satisfies the following relations:

$$[\beta \bullet \alpha] = -(-1)^{(|\alpha|+1)(|\beta|+1)}\,[\alpha \bullet \beta], \tag{3.15}$$

$$[\alpha \bullet [\beta \bullet \gamma]] = [[\alpha \bullet \beta] \bullet \gamma] + (-1)^{(|\alpha|+1)(|\beta|+1)}\,[\beta \bullet [\alpha \bullet \gamma]], \tag{3.16}$$

$$[\alpha \bullet (\beta \wedge \gamma)] = [\alpha \bullet \beta] \wedge \gamma + (-1)^{(|\alpha|+1)|\beta|}\beta \wedge [\alpha \bullet \gamma], \tag{3.17}$$

$$\Delta[\alpha \bullet \beta] = [\Delta\alpha \bullet \beta] + (-1)^{|\alpha|+1}\,[\alpha \bullet \Delta\beta], \tag{3.18}$$

$$\bar{\partial}[\alpha \bullet \beta] = [\bar{\partial}\alpha \bullet \beta] + (-1)^{|\alpha|+1}\,[\alpha \bullet \bar{\partial}\beta]. \tag{3.19}$$

The first two relations define a structure of the odd Lie algebra on $\mathcal{B}$, the third means that $[\alpha\bullet\ ]$ is a derivation with respect to the wedge product, while the last two show that both Δ and $\bar{\partial}$ are derivations with respect to the odd bracket.

A most useful presentation of the bracket (3.14) is given by the odd Poisson bracket

$$[\alpha \bullet \beta] = \sum \left(\alpha \frac{\overleftarrow{\partial}}{\partial z^i} \wedge \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\theta}_i}\beta - \alpha \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\theta}_i} \wedge \frac{\overrightarrow{\partial}}{\partial z^i}\beta \right). \tag{3.20}$$

Note that the right hand side does not depend on the holomorphic three-form Ω.

Recall that $\mathcal{B}$ is a hybrid of polyvectors and differential forms (2.2), and polyvectors close themselves under the Schouten-Neienhuis bracket [14,15]. This puts to $\mathcal{B}$ another product structure; it is essentially the same as the odd bracket (3.14). To see this, let us use multi-indices to present two homogeneous elements of $\mathcal{B}$

$$\alpha = \sum_{|I|=p,|J|=q}^{<} \alpha^I_{\bar{J}}\, \boldsymbol{\theta}_I \boldsymbol{\eta}^{\bar{J}} \in B^{p,q}, \quad \beta = \sum_{|K|=r,|L|=s}^{<} \beta^K_{\bar{L}}\, \boldsymbol{\theta}_K \boldsymbol{\eta}^{\bar{L}} \in B^{r,s}, \tag{3.21}$$

Then the odd bracket can be written as

$$[\alpha \bullet \beta] = -(-1)^{q(r+1)} \sum_{I,J,K,L}^{<} [\![\alpha^I_{\bar{J}}\boldsymbol{\theta}_I, \beta^K_{\bar{L}}\boldsymbol{\theta}_K]\!]_{\text{sn}}\, \boldsymbol{\eta}^{\bar{J}} \boldsymbol{\eta}^{\bar{L}}, \tag{3.22}$$

where $[\![\ ,\ ]\!]_{\text{sn}}$ is the Schouten-Neienhuis bracket.The formula (3.22) is known as the generalized Tian's Lemma. The original Tian's Lemma refers to the case of $p = r = 1$ [16], where the Schouten-Neienhuis bracket reduces to the Lie bracket of vector fields.

3.2.2. *Algebra associated with the pair* (R, S)

The dGVB algebra in the previous subsubsection has been constructed using the two operators: Δ of ghost number minus one and of order two [14] and $\bar{\partial}$ of ghost number one and of order one, which satisfy $\Delta^2 = 0$, $\bar{\partial}^2 = 0$ and $\Delta\bar{\partial} + \bar{\partial}\Delta = 0$.

We have another pair (R, S) which has the same properties as $(\Delta, \bar{\partial})$, which leads us to define the following odd bracket

$$[\alpha \star \beta] = (-1)^{|\alpha|} R(\alpha \wedge \beta) - (-1)^{|\alpha|} R(\alpha) \wedge \beta - \alpha \wedge R(\beta). \tag{3.23}$$

The two odd brackets are related by the Hodge dual operator as

$$[\alpha \star \beta] = \mathrm{i}\,(-1)^{|\alpha|\cdot|\beta|+1}\, \hat{*}[\hat{*}\alpha \bullet \hat{*}\beta], \tag{3.24}$$

from which we can show the dGBV algebra relations:

$$[\beta \star \alpha] = -(-1)^{(|\alpha|+1)(|\beta|+1)}\, [\alpha \star \beta], \tag{3.25}$$

$$[\alpha \star [\beta \star \gamma]] = [[\alpha \star \beta] \star \gamma] + (-1)^{(|\alpha|+1)(|\beta|+1)}\, [\beta \star [\alpha \star \gamma]], \tag{3.26}$$

$$[\alpha \star (\beta \wedge \gamma)] = [\alpha \star \beta] \wedge \gamma + (-1)^{(|\alpha|+1)|\beta|} \beta \wedge [\alpha \star \gamma], \tag{3.27}$$

$$R[\alpha \star \beta] = [R\alpha \star \beta] + (-1)^{|\alpha|+1}\, [\alpha \star R\beta], \tag{3.28}$$

$$S\,[\alpha \star \beta] = [S\,\alpha \star \beta] + (-1)^{|\alpha|+1}\, [\alpha \star S\beta]. \tag{3.29}$$

We have also obtained the following formulas

$$\begin{aligned} &\Delta[\alpha \star \beta] - [\Delta\alpha \star \beta] + (-1)^{|\alpha|}[\alpha \star \Delta\beta] \\ &= -R[\alpha \bullet \beta] + [R\alpha \bullet \beta] - (-1)^{|\alpha|}[\alpha \bullet R\beta], \end{aligned} \tag{3.30}$$

$$\begin{aligned} &S\,[\alpha \bullet \beta] - [S\,\alpha \bullet \beta] + (-1)^{|\alpha|}[\alpha \bullet S\beta] \\ &= -\bar{\partial}[\alpha \star \beta] + [\bar{\partial}\alpha \star \beta] - (-1)^{|\alpha|}[\alpha \star \bar{\partial}\beta] \\ &= (-1)^{|\alpha|}\left(\hat{\Box}(\alpha \wedge \beta) - \hat{\Box}\alpha \wedge \beta - \alpha \wedge \hat{\Box}\beta\right), \end{aligned} \tag{3.31}$$

where we recall that the Laplacian is given by $\hat{\Box} = -\bar{\partial}R - R\bar{\partial} = \Delta S + S\,\Delta$.

The odd Poisson bracket form is given by

$$[\alpha \star \beta] = \sum g^{\bar{i}j} \left(\alpha \left(\frac{\overleftarrow{\partial}}{\partial z^j} + \frac{\overleftarrow{\partial}}{\partial \theta_l} \theta_k \Gamma^k_{jl} \right) \wedge \frac{\overrightarrow{\partial}}{\partial \eta^{\bar{i}}} \beta - \alpha \frac{\overleftarrow{\partial}}{\partial \eta^{\bar{i}}} \wedge \left(\frac{\overrightarrow{\partial}}{\partial z^j} + \Gamma^k_{jl} \theta_k \frac{\overrightarrow{\partial}}{\partial \theta_l} \right) \beta \right), \tag{3.32}$$

the right hand side of which is independent of the scalar field σ.

The dGBV algebras associated with the space of differential forms $\mathcal{A}$, which are expected to be a mirror dual to those of $\mathcal{B}$ when the sigma model instanton effects are incorporated, have been studied in [17,18,19].

4. KS Action and its Deformation

The algebraic tools developed in the previous sector enables a systematic treatment of the classical KS gravity action

$$S_X[\varphi_1|\Phi] = \mathrm{Tr}\left(\frac{1}{6}\varphi \wedge \varphi \wedge \varphi - \frac{1}{2}\bar{\partial}\Delta^{-1}\Phi \wedge \Phi\right), \tag{4.1}$$

where we recall that $\varphi = \varphi_1 + \Phi$ is an element of $B^{1,1} \cap \mathrm{Ker}\,\Delta$, the massless part $\varphi_1 = \hat{\Pi}(\varphi) \in \rho^{-1}(\mathbb{H}^{2,1})$ is a background field, and the massive part $\Phi \in \mathrm{Im}\,\Delta \cap B^{1,1}$ carries dynamical degrees of freedom.

We obtain the equation of motion of φ from variation of the action (4.1)

$$\bar{\partial}\varphi + \frac{1}{2}[\varphi \bullet \varphi] = 0. \tag{4.2}$$

In components, $\varphi = \sum \varphi^i_{\bar{j}} \boldsymbol{\theta}_i \boldsymbol{\eta}^{\bar{j}}$, it reads

$$\partial_{\bar{j}}\varphi^i_{\bar{k}} - \partial_{\bar{k}}\varphi^i_{\bar{j}} + \sum_l \left(\varphi^l_{\bar{j}}\partial_l\varphi^i_{\bar{k}} - \varphi^l_{\bar{k}}\partial_l\varphi^i_{\bar{j}}\right) = 0. \tag{4.3}$$

This is the famous Kodaira-Spencer equation [8], which describes deformation of complex structures on X.

The action has the gauge symmetry; the infinitesimal form of it is

$$\delta\Phi = \bar{\partial}\xi + [\varphi \bullet \xi], \tag{4.4}$$

where $\xi \in B^{1,0} \cap \mathrm{Ker}\,\Delta$ is a gauge parameter.

4.1. *Solution to KS equation*

We construct a solution to the KS equation (4.2) following [16,20].

Let $\varphi_1 \in \rho^{-1}(\mathbb{H}^{2,1})$ a massless mode. Then we claim that there is a series of massive modes $\varphi_n \in \rho^{-1}(\mathrm{Im}(\partial\bar{\partial}^\dagger)) \cap B^{1,1}$, $n \geq 2$, such that $\varphi = \sum_{n\geq 1}\varphi_n$ solves the KS equation (4.2). In fact, we can solve order by order the equation for φ_n:

$$\bar{\partial}\rho(\varphi_n) = \psi_n := \frac{1}{2}\rho\left(\sum_{i+j=n}[\varphi_i \bullet \varphi_j]\right). \tag{4.5}$$

For $n = 1$, (4.5) is trivially satisfied by φ_1.

For $n = 2$, (4.5) becomes

$$\bar{\partial}\rho(\varphi_2) = \psi_2 = \rho([\varphi_1 \bullet \varphi_1]). \tag{4.6}$$

From the definition of the odd bracket (3.14) and the fact $\varphi_1 \in \operatorname{Ker}\Delta$, we see that $\psi_2 \in \operatorname{Im}\partial$. The formula (3.19) shows that $\psi_2 \in \operatorname{Ker}\bar{\partial}$ because $\varphi_1 \in \operatorname{Ker}\bar{\partial}$. Then $\partial\bar{\partial}$-Lemma implies that $\psi_2 \in \operatorname{Im}(\partial\bar{\partial}) \subset \operatorname{Im}\bar{\partial}$, from which we see that (4.6) has solutions. In particular the gauge condition $\rho(\varphi_2) \in \operatorname{Im}\bar{\partial}^\dagger$ picks up the unique one: $\rho(\varphi_2) = \bar{\partial}^\dagger \mathbf{G}\psi_2 \in \operatorname{Im}(\partial\bar{\partial}^\dagger)$.

Suppose that we have solutions $\varphi_k = \bar{\partial}^\dagger \mathbf{G}\psi_n \in \rho^{-1}\operatorname{Im}(\partial\bar{\partial}^\dagger)$ for $2 \leq k \leq n$. Then we see that $\psi_{n+1} \in \operatorname{Im}\partial$ because each term $[\varphi_i \bullet \varphi_j]$ of ψ_{n+1} belongs to $\operatorname{Im}\Delta$ by the inductive assumption. We can also show that $\psi_{n+1} \in \operatorname{Ker}\bar{\partial}$ since

$$\bar{\partial}\,\frac{1}{2}\sum_{i+j=n+1}[\varphi_i \bullet \varphi_j] = \sum_{i+j=n+1}[\bar{\partial}\varphi_i \bullet \varphi_j] = -\frac{1}{2}\sum_{i+j=n+1}\sum_{k+l=i}[[\varphi_k \bullet \varphi_l] \bullet \varphi_j], \tag{4.7}$$

the right hand side of which vanishes due to the Jacobi identity of the odd bracket (3.16). Therefore again from the $\partial\bar{\partial}$-Lemma, we see that $\psi_{n+1} \in \operatorname{Im}(\partial\bar{\partial}) \subset \operatorname{Im}(\bar{\partial})$, and we obtain the $n+1$st solution $\rho(\varphi_{n+1}) = \bar{\partial}^\dagger \mathbf{G}\psi_{n+1}$. Thus we have obtained a solution $\varphi = \sum_{n\geq 1}\varphi_n$ of the KS equation (4.2). Note that φ satisfies $\Delta\varphi = 0$, $R\varphi = 0$ by construction. For the proof of the convergence of the infinite sum $\sum_{n\geq 1}\varphi_n$ for sufficiently small φ_1s, see [21].

Let us define the massive propagator [11] by

$$\boldsymbol{P} = \rho^{-1} \circ \bar{\partial}^\dagger \partial \mathbf{G} \circ \rho : B^{p,q} \to B^{p-1,q-1}. \tag{4.8}$$

Then we get the recursion formula for $n \geq 2$:

$$\varphi_n = \frac{1}{2}\sum_{i=1}^{n-1}\boldsymbol{P}(\varphi_i \wedge \varphi_{n-i}). \tag{4.9}$$

Proposition 4.1. *The solution $\varphi = \sum_{n\geq 1}\varphi_n$ of* (4.2) *constructed above satisfies $\bar{*}\varphi = +\,\mathrm{i}\,\varphi$.*

Proof. From the lemma 3.1, it suffices to show that $\hat{L}\varphi = 0$. First note that $\hat{L}\varphi_1 = 0$ because $\hat{L}$ maps $\rho^{-1}(\mathbb{H}^{2,1})$ to $\rho^{-1}(\mathbb{H}^{3,2}) = \{0\}$. Second, we observe that $\hat{L}$ commutes with $\boldsymbol{P}$ as $[L, \mathbf{G}] = 0$ and $[L, \bar{\partial}^\dagger\partial] = 0$, so that $\hat{L}\varphi_2 = \frac{1}{2}\boldsymbol{P}\hat{L}(\varphi_1 \wedge \varphi_1) = \frac{1}{2}\boldsymbol{P}(\hat{L}\varphi_1 \wedge \varphi_1 + \varphi_1 \wedge \hat{L}\varphi_1) = 0$, where we have used the fact that $\hat{L}$ acts as a derivation with respect to the wedge product. Finally, from the recursion relation (4.9), we see $\hat{L}\varphi_n = 0$ for each n. □

4.2. *Condensation of string and deformation of supercharges*

Let $\Phi[\varphi_1] = \sum_{n\geq 2}\varphi_n$, where $\varphi_1 \in \rho^{-1}(\mathbb{H}^{2,1})$ and φ_n, $n \geq 2$, is the solution of (4.5) constructed above. If we let Φ have vacuum expectation value $\Phi[\varphi_1]$ and expand the KS action (4.1) with respect to the fluctuation Φ' around it, we get [5]

$$S_X[\varphi_1|\Phi[\varphi_1] + \Phi'] = S_X[\varphi_1|\Phi[\varphi_1]] + \tilde{S}[\Phi'], \tag{4.10}$$

$$\tilde{S}[\Phi'] = \operatorname{Tr}\left(\frac{1}{6}\Phi' \wedge \Phi' \wedge \Phi' + \frac{1}{2}\Delta^{-1}(\bar{\partial}\Phi' + [\varphi \bullet \Phi']) \wedge \Phi'\right), \tag{4.11}$$

where $\varphi = \varphi_1 + \Phi[\varphi_1]$ is the solution of the KS equation considered above.

The first term of (4.10) is known as the prepotential of topological string and has a following expansion [22]:

$$S_X[\varphi_1|\Phi[\varphi_1]] = \frac{1}{6}\operatorname{Tr}(\varphi_1 \wedge \varphi_1 \wedge \varphi_1) + \frac{1}{8}\operatorname{Tr}(\varphi_1 \wedge \varphi_1 \wedge \boldsymbol{P}(\varphi_1 \wedge \varphi_1)) \\ + \frac{1}{8}\operatorname{Tr}(\varphi_1 \wedge \varphi_1 \wedge \boldsymbol{P}(\varphi_1 \wedge \boldsymbol{P}(\varphi_1 \wedge \varphi_1))) + \cdots . \tag{4.12}$$

We note that in the kinetic term of $\tilde{S}[\Phi']$ (4.11) appearance of the new BRST operator $\bar{\partial} + [\varphi\bullet\ ]$, which is nilpotent if φ solves the KS equation, as is always the case for string field theories [23]. The same deformation of the BRST operator $\bar{\partial}$ has also been considered in the first quantization approach to topological B sigma model [24].

In the reference [25], the authors have found that in addition to the BRST operator $\bar{\partial}$, S also deforms to $S + [\varphi\star\ ]$ in our notation. Here we remark the identities $[\varphi\bullet\ ] = [\Delta, \varphi\wedge]$, $[\varphi\star\ ] = [R, \varphi\wedge]$. They have also given the equation for this to be nilpotent:

$$\left(D^i\varphi^j_{\bar{k}} + \varphi^i_{\bar{l}}D^{\bar{l}}\varphi^j_{\bar{k}}\right)\boldsymbol{\theta}_i\boldsymbol{\theta}_j\boldsymbol{\eta}^{\bar{k}} = 0, \tag{4.13}$$

where $D^i = \sum g^{i\bar{j}}D_{\bar{j}}$ is the covariant derivative of the Levi-Civita connection. However we can easily show that the left hand side of (4.13) can be written as

$$-\left(S\varphi + \frac{1}{2}[\varphi \star \varphi]\right) = -\,\mathrm{i}\,\hat{*}\left(\bar{\partial}\varphi + \frac{1}{2}[\varphi \bullet \varphi]\right), \tag{4.14}$$

which shows that (4.13) does not put any constraint on φ other than the KS equation. The remaining operators of order two Δ and R are left unchanged under the deformation [25]. We note that from the relation between two odd brackets (3.24), it is easily seen that

$$S + [\varphi\star\ ] = \hat{*} \circ (\bar{\partial} + [\varphi\bullet\ ]) \circ \hat{*}. \tag{4.15}$$

The commutation relations between the Lefshetz operators and $[\varphi\bullet\ ]$, $[\varphi\star\ ]$ are given by

$$[\hat{L}, [\varphi\bullet\ ]] = [\hat{L}\varphi\bullet\ ] = 0, \quad [\hat{L}, [\varphi\star\ ]] = -\,\mathrm{i}[\varphi\bullet\ ], \tag{4.16}$$

$$[\hat{\Lambda}, [\varphi\star\ ]] = [\hat{\Lambda}\varphi\star\ ] = 0, \quad [\hat{\Lambda}, [\varphi\bullet\ ]] = \mathrm{i}[\varphi\star\ ], \tag{4.17}$$

which shows that the Hodge-Kähler identities (2.11–2.14) are preserved under the deformation $\bar{\partial} \to \bar{\partial} + [\varphi\bullet\ ]$, $S \to S + [\varphi\star\ ]$.

We want to find a "Calabi-Yau manifold" $\tilde{X}$ which realizes the identity $\tilde{S}[\Phi'] = S_{\tilde{X}}[0|\Phi']$; it is clear that $\tilde{X}$ has a close relation to the deformation of the complex structure of X induced by the solution of the KS equation φ. Therefore we describe the deformation of complex structure below.

5. Deformation of Complex Structure

5.1. *Classical deformation*

Let $\varphi_1 \in \rho^{-1}(\mathbb{H}^{2,1})$ be a massless mode of the KS action, so small in magnitude that the solution of the KS equation constructed above $\varphi = \sum_{n\geq 1}\varphi_n$ converges, and X_{φ_1} the Calabi-Yau manifold with the complex structure defined by φ, that is, a local function f on X_{φ_1} is holomorphic if $(\partial_{\bar{i}} + \sum \varphi^j_{\bar{i}}\partial_j)f = 0$ for all $\bar{i}$.

Local frames of $(1,0)$ and $(0,1)$ vector fields and 1-forms on X_{φ_1} are given by

$$\boldsymbol{e}_i = \frac{\partial}{\partial z^i} + \sum \bar{\varphi}_i^{\bar{j}} \frac{\partial}{\partial z^{\bar{j}}}, \quad \boldsymbol{e}_{\bar{j}} = \frac{\partial}{\partial z^{\bar{j}}} + \sum \varphi_{\bar{j}}^i \frac{\partial}{\partial z^i}, \tag{5.1}$$

$$\boldsymbol{f}^i = \mathrm{d}\, z^i - \sum \varphi_{\bar{j}}^i \,\mathrm{d}\, z^{\bar{j}}, \quad \boldsymbol{f}^{\bar{j}} = \mathrm{d}\, z^{\bar{j}} - \sum \bar{\varphi}_i^{\bar{j}} \,\mathrm{d}\, z^i, \tag{5.2}$$

where $\bar{\varphi}_i^{\bar{j}}$ is the complex conjugate of $\varphi_{\bar{i}}^j$. Their pairing is $\langle \boldsymbol{f}^i, \boldsymbol{e}_j \rangle = \delta^i_j - \sum \varphi^i_{\bar{k}} \bar{\varphi}^{\bar{k}}_j = (I - \varphi\bar{\varphi})^i_j$. If we define the matrix $N = (I - \varphi\bar{\varphi})^{-1}$, then we have the identities $\sum N^i_j \varphi^j_{\bar{k}} = \sum \varphi^i_{\bar{l}} \bar{N}^{\bar{l}}_{\bar{k}}$, and $\sum g_{m\bar{l}} N^m_k = \sum g_{k\bar{j}} \bar{N}^{\bar{j}}_{\bar{l}}$, where $\bar{N}$ is the complex conjugate of N, and $\partial/\partial z^i = \sum N^j_i (\boldsymbol{e}_j - \bar{\varphi}^{\bar{k}}_j \boldsymbol{e}_{\bar{k}})$, $\mathrm{d}\, z^l = \sum N^l_i (\boldsymbol{f}^i + \varphi^i_{\bar{k}} \boldsymbol{f}^{\bar{k}})$.

We claim that a holomorphic three-form on X_{φ_1} is given by

$$\Omega_{\varphi_1} = s\boldsymbol{f}^1\boldsymbol{f}^2\boldsymbol{f}^3 = \rho\left(1 - \varphi + \frac{1}{2!}\varphi\wedge\varphi - \frac{1}{3!}\varphi\wedge\varphi\wedge\varphi\right). \tag{5.3}$$

As Ω_{φ_1} is a $(3,0)$-form on X_{φ_1}, we have only to show that $\mathrm{d}\,\Omega_{\varphi_1} = 0$, which is equivalent to

$$(-\bar{\partial} + \Delta)\left(1 - \varphi + \frac{1}{2!}\varphi\wedge\varphi - \frac{1}{3!}\varphi\wedge\varphi\wedge\varphi\right) = 0. \tag{5.4}$$

It is easy to check (5.4) by the KS equation and the dGBV algebra relations. We also note the volume form that it defines is $\mathrm{i}\,\Omega_{\varphi_1} \wedge \overline{\Omega}_{\varphi_1} = \det(I - \varphi\bar{\varphi})\,\mathrm{i}\,\Omega \wedge \overline{\Omega}$.

The Kähler form ω is still of the type $(1,1)$ under the new complex structure, and given by $\omega = \mathrm{i}\sum g_{k\bar{j}} \bar{N}^{\bar{j}}_{\bar{l}} \boldsymbol{f}^k \boldsymbol{f}^{\bar{l}}$.

The configuration space $\mathcal{B}_{\varphi_1}$ for the KS gravity defined on X_{φ_1} can be written as

$$\mathcal{B}_{\varphi_1} = \bigoplus_{p,q} B^{p,q}_{\varphi_1}, \quad B^{p,q}_{\varphi_1} = \left\{\beta = \frac{1}{p!q!}\sum \beta^{i_1,\ldots,i_p}_{\bar{j}_1,\ldots,\bar{j}_q}(z,\bar{z})\,\boldsymbol{e}_{i_1}\cdots\boldsymbol{e}_{i_p}\boldsymbol{f}^{\bar{j}_1}\cdots\boldsymbol{f}^{\bar{j}_q}\right\}. \tag{5.5}$$

Note that $\{\mathcal{B}_{\varphi_1}\}_{\varphi_1 \in \rho^{-1}(\mathbb{H}^{2,1})}$ defines a vector bundle, or a bundle of super-algebras on a neighborhood of $[X]$ in the moduli space $\mathcal{M}$. It would be nice to know interesting connections on this bundle [26].

It is also convenient to make a choice of local holomorphic coordinates w^α of X_{φ_1} [27,28]. If we put the relations between local frames of $T^*X_{\varphi_1}$ and TX_{φ_1} by

$$\boldsymbol{f}^i = \sum_\alpha A^i_\alpha \,\mathrm{d}\, w^\alpha, \tag{5.6}$$

$$\boldsymbol{e}_i = \sum_\alpha B^\alpha_i \frac{\partial}{\partial w^\alpha}, \tag{5.7}$$

then we have the matrix relation $AB = N^{-1}$ and

$$\mathrm{d}\, w^\alpha = (A^{-1})^\alpha_i \,\mathrm{d}\, z^i - (A^{-1}\varphi)^\alpha_{\bar{j}} \,\mathrm{d}\, z^{\bar{j}}, \tag{5.8}$$

$$\mathrm{d}\, z^i = (B^{-1})^i_\alpha \,\mathrm{d}\, w^\alpha + (\varphi\bar{B}^{-1})^i_{\bar{\beta}} \,\mathrm{d}\, w^{\bar{\beta}}. \tag{5.9}$$

Among the consistency conditions derived from (5.8) are

$$\partial_i (A^{-1})^\alpha_j - \partial_j (A^{-1})^\alpha_i = 0, \tag{5.10}$$

$$\sum_\alpha (A^{-1})^\alpha_l e_{\bar{j}} A^i_\alpha = \partial_l \varphi^i_{\bar{j}}, \tag{5.11}$$

where $e_{\bar{j}} = \partial_{\bar{j}} + \sum \varphi^s_{\bar{j}} \partial_s$ is the differential operator corresponding to the vector field $\boldsymbol{e}_{\bar{j}}$. Similarly, from (5.9), we obtain the consistency condition

$$\sum B^\alpha_i e_{\bar{j}} (B^{-1})^l_\alpha = \sum N^l_k M^k_{i\bar{j}}, \quad M^k_{i\bar{j}} := e_i \varphi^k_{\bar{j}} + \sum \varphi^k_{\bar{m}} e_{\bar{j}} \bar{\varphi}^{\bar{m}}_i. \tag{5.12}$$

The Kähler two-form ω is rewritten as $\omega = \mathrm{i}\, \tilde{g}_{\alpha\bar{\beta}}\, \mathrm{d}\, w^\alpha \wedge \mathrm{d}\, w^{\bar{\beta}}$, where

$$\tilde{g}_{\alpha\bar{\beta}} = \sum_{j,k} A^k_\alpha (\bar{B}^{-1})^{\bar{j}}_{\bar{\beta}} g_{k\bar{j}}, \quad \tilde{g}^{\alpha\bar{\beta}} = \sum_{l,m} \bar{B}^{\bar{\beta}}_{\bar{l}} (A^{-1})^\alpha_m g^{\bar{l}m}, \tag{5.13}$$

from which we can compute the Christoffel symbols $\tilde{\Gamma}^\alpha_{\beta\gamma}$ for the Kähler metric $\tilde{g}_{\alpha\bar{\beta}}$.

5.2. *Differential operators in new complex structure*

In this subsection, we compute the four differential operators introduced in 3.1.2 for the KS gravity theory on the deformed Calabi-Yau manifold X_{φ_1}, which we denote by $\bar{\partial}_{\varphi_1}$, Δ_{φ_1}, S_{φ_1}, and R_{φ_1}, and show that these are just reduced to $\bar{\partial} + [\varphi \bullet\ ]$, Δ, $S + [\varphi \star\ ]$, and R, respectively in the holomorphic limit where we make an analytic continuation of the moduli parameters so that $\bar{\varphi}_1$ is set to zero while φ_1 is kept fixed [5].

Let us explain the "analytic continuation" above more detail. Let $\overline{\mathcal{M}}$ be the same manifold as $\mathcal{M}$ with the opposite complex structure. Then we extend the moduli space to be $\mathcal{M} \times \overline{\mathcal{M}}$ [5,29,30] and the original moduli space is diagonally embedded as $\mathcal{M}_{\mathrm{diag}}$ in Fig. 1. Let (t^a) be local holomorphic coordinates of $\mathcal{M}$, and the point $[X] \in \mathcal{M}$ correspond to $t = t_0$. By the classical deformation described in the previous subsection, $[X] \in \mathcal{M} \times \overline{\mathcal{M}}$, with coordinates $(t_0, \bar{t}_0)$ in Fig. 1, moves along the diagonal line to the point corresponding to X_{φ_1}; then setting $\bar{\varphi}_1 = 0$ we arrive at a point on the horizontal line $\bar{t} = \bar{t}_0$. We call the point $X^{\mathrm{hol}}_{\varphi_1}$ and the deformation from X to $X^{\mathrm{hol}}_{\varphi_1}$ the holomorphic deformation.

It is easy to give the differential operators on X_{φ_1} in terms of the local holomorphic coordinates w^α according to the formulas (3.4–3.7). To perform the holomorphic limit above, however, it is necessary to change coordinates from $(w^\alpha, w^{\bar{\beta}})$ to $(z^i, z^{\bar{j}})$.

The four differential operators acting on $\mathcal{B}_{\varphi_1}$ of the form (5.5) are given by

$$\bar{\partial}_{\varphi_1} = \bar{N}^{\bar{k}}_{\bar{r}} \boldsymbol{f}^{\bar{r}} \left(\frac{\partial}{\partial z^{\bar{k}}} + \varphi^l_{\bar{k}} \frac{\partial}{\partial z^l} - N^p_s M^s_{i\bar{k}} \boldsymbol{e}_p \frac{\partial}{\partial \boldsymbol{e}_i} \right), \tag{5.14}$$

$$\Delta_{\varphi_1} = (s \det N^{-1})^{-1} \circ \left(\frac{\partial}{\partial z^i} + \bar{\varphi}^{\bar{j}}_i \frac{\partial}{\partial z^{\bar{j}}} \right) \frac{\partial}{\partial \boldsymbol{e}_i} \circ (s \det N^{-1}) - \boldsymbol{f}^{\bar{m}} \partial_{\bar{m}} \bar{\varphi}^{\bar{j}}_i \frac{\partial}{\partial \boldsymbol{e}_i} \frac{\partial}{\partial \boldsymbol{f}^{\bar{j}}}, \tag{5.15}$$

$$\begin{aligned} S_{\varphi_1} = {} & g^{\bar{u}l} \bar{N}^{\bar{k}}_{\bar{u}} \boldsymbol{e}_l \left(- \frac{\partial}{\partial z^{\bar{k}}} - \varphi^i_{\bar{k}} \frac{\partial}{\partial z^i} + N^n_r M^r_{i\bar{k}} \boldsymbol{e}_n \frac{\partial}{\partial \boldsymbol{e}_i} + g_{\bar{m}n} g^{\bar{j}q} \bar{N}^{\bar{m}}_{\bar{s}} M^n_{q\bar{k}} \boldsymbol{f}^{\bar{s}} \frac{\partial}{\partial \boldsymbol{f}^{\bar{j}}} \right) \\ & + g^{\bar{u}l} \bar{N}^{\bar{k}}_{\bar{u}} \boldsymbol{e}_l \left(\Gamma^{\bar{j}}_{\bar{k}\bar{s}} + g^{m\bar{j}} g_{\bar{s}q} \Gamma^q_{mn} \varphi^n_{\bar{k}} \right) \boldsymbol{f}^{\bar{s}} \frac{\partial}{\partial \boldsymbol{f}^{\bar{j}}}, \end{aligned} \tag{5.16}$$

$$\begin{aligned} R_{\varphi_1} = {} & e^{-\tilde{\sigma}} \circ g^{k\bar{j}} \left(\frac{\partial}{\partial z^k} + \bar{\varphi}^{\bar{l}}_k \frac{\partial}{\partial z^{\bar{l}}} \right) \frac{\partial}{\partial \boldsymbol{f}^{\bar{j}}} \circ e^{\tilde{\sigma}} + g^{\bar{l}k} \partial_{\bar{l}} \bar{\varphi}^{\bar{j}}_k \frac{\partial}{\partial \boldsymbol{f}^{\bar{j}}} - g^{\bar{j}k} \partial_{\bar{n}} \bar{\varphi}^{\bar{l}}_k \boldsymbol{f}^{\bar{n}} \frac{\partial}{\partial \boldsymbol{f}^{\bar{j}}} \frac{\partial}{\partial \boldsymbol{f}^{\bar{l}}} \\ & + N^m_p \left(g^{\bar{j}k} e_i (N^{-1})^p_k + g^{\bar{l}p} \bar{M}^{\bar{j}}_{\bar{l}i} + g^{\bar{j}s} (N^{-1})^k_s (\Gamma^p_{ik} + g^{\bar{l}p} g_{k\bar{q}} \Gamma^{\bar{q}}_{\bar{n}\bar{l}} \bar{\varphi}^{\bar{n}}_i) \right) \boldsymbol{e}_m \frac{\partial}{\partial \boldsymbol{e}_i} \frac{\partial}{\partial \boldsymbol{f}^{\bar{j}}}, \end{aligned} \tag{5.17}$$

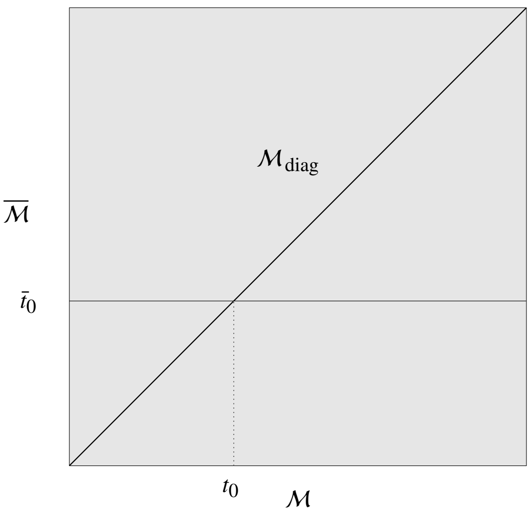

Fig. 1. Analytic continuation of the moduli space

where $e^{\tilde\sigma} = \det N e^{\sigma}$ and use of the Einstein summation convention is unavoidable.

In the holomorphic limit $\bar\varphi \to 0$, $\boldsymbol{e}_i \to \boldsymbol{\theta}_i$, $\boldsymbol{f}^{\bar j} \to \boldsymbol{\eta}^{\bar j}$, $N \to I$, $M^k_{i\bar j} \to \partial_i\varphi^k_{\bar j}$.
Let us see the limit of each operators. The cases of $\bar\partial$, Δ and R are easy:

$$\bar\partial_{\varphi_1} \to \boldsymbol{\eta}^{\bar k}\left(\frac{\partial}{\partial z^{\bar k}} + \varphi^l_{\bar k}\frac{\partial}{\partial z^l} - \partial_i\varphi^p_{\bar k}\boldsymbol{\theta}_p\frac{\partial}{\partial\boldsymbol{\theta}_i}\right) = \bar\partial + [\varphi\bullet\ ], \tag{5.18}$$

where we have used the formula (3.20) and

$$\Delta_{\varphi_1} \to s^{-1}\circ\frac{\partial}{\partial z^i}\frac{\partial}{\partial\boldsymbol{\theta}_i}\circ s = \Delta, \tag{5.19}$$

$$R_{\varphi_1} \to e^{-\sigma}\circ g^{k\bar j}\frac{\partial}{\partial z^k}\frac{\partial}{\partial\boldsymbol{\eta}^{\bar j}}\circ e^{\sigma} + g^{k\bar j}\Gamma^m_{ik}\boldsymbol{\theta}_m\frac{\partial}{\partial\boldsymbol{\theta}_i}\frac{\partial}{\partial\boldsymbol{\eta}^{\bar j}} = R. \tag{5.20}$$

For the case of S, we have

$$\begin{aligned} S_{\varphi_1} \to & -g^{\bar k l}\boldsymbol{\theta}_l\left(\frac{\partial}{\partial z^{\bar k}} - \Gamma^{\bar j}_{\bar s\bar k}\boldsymbol{\eta}^{\bar s}\frac{\partial}{\partial\boldsymbol{\eta}^{\bar j}}\right) \\ & - g^{\bar k l}\boldsymbol{\theta}_l\left(-\varphi^t_{\bar k}\frac{\partial}{\partial z^t} + \partial_i\varphi^n_{\bar k}\boldsymbol{\theta}_n\frac{\partial}{\partial\boldsymbol{\theta}_i} + g_{\bar k n}g^{\bar j q}D_q\varphi^n_{\bar s}\boldsymbol{\eta}^{\bar s}\frac{\partial}{\partial\boldsymbol{\eta}^{\bar j}}\right). \end{aligned} \tag{5.21}$$

On the other hand we know from the formula (3.32)

$$[\varphi\star\ ] = -g^{\bar a b}\varphi^i_{\bar a}\boldsymbol{\theta}_i\left(\frac{\partial}{\partial z^b} + \Gamma^c_{bd}\boldsymbol{\theta}_c\frac{\partial}{\partial\boldsymbol{\theta}_d}\right) + g^{\bar a b}D_b\varphi^c_{\bar j}\boldsymbol{\theta}_c\boldsymbol{\eta}^{\bar j}\frac{\partial}{\partial\boldsymbol{\eta}^{\bar a}}, \tag{5.22}$$

and the difference between the second term of the right hand side of (5.21) and (5.22) $D_i(g^{\bar k l}\varphi^n_{\bar k})\boldsymbol{\theta}_l\boldsymbol{\theta}_n\partial/\partial\boldsymbol{\theta}_i$ vanishes owing to Prop. 4.1. This shows that the holomorphic limit of S_{φ_1} is $S + [\varphi\star\ ]$.

In conclusion, we have identified the the deformation of the differential operators observed in (4.11) with the one caused by the holomorphic limit of the deformation of the complex structure, so that we can write $\tilde{S}[\Phi'] = S_{X^{\rm hol}_{\varphi_1}}[0|\Phi']$.

At first sight, it may seem that for the KS action on $X^{\rm hol}_{\varphi_1}$ we must use the deformed trace map

$$\mathrm{Tr}^{\rm hol}_{\varphi_1}(\alpha) = \int_X \rho^{\rm hol}_{\varphi_1}(\alpha) \wedge \Omega_{\varphi_1}, \tag{5.23}$$

where $\rho^{\rm hol}_{\varphi_1}(\boldsymbol{\theta}_I) = (-1)^{|I|(|I|-1)/2}\epsilon_{I,I^*}s\boldsymbol{f}^{I^*}$. However $\mathrm{Tr}^{\rm hol}_{\varphi_1}(\alpha) = \mathrm{Tr}(\alpha)$ for any $\alpha \in \mathcal{B}$.

5.3. *Deformation of states*

Let us denote the deformed operators by $(\bar\partial)^{\rm hol}_{\varphi_1} = \bar\partial + [\varphi\bullet\ ]$, $S^{\rm hol}_{\varphi_1} = S + [\varphi\star\ ]$. Then it is easy to see that

$$(\bar\partial)^{\rm hol}_{\varphi_1} \circ \Delta + \Delta \circ (\bar\partial)^{\rm hol}_{\varphi_1} = 0, \quad S^{\rm hol}_{\varphi_1} \circ R + R \circ S^{\rm hol}_{\varphi_1} = 0. \tag{5.24}$$

We can also show

$$(\bar\partial)^{\rm hol}_{\varphi_1} \circ S^{\rm hol}_{\varphi_1} + S^{\rm hol}_{\varphi_1} \circ (\bar\partial)^{\rm hol}_{\varphi_1} = 0. \tag{5.25}$$

To see this, we calculate the action of the left hand side of (5.25) on $\alpha \in \mathcal{B}$;

$$\begin{aligned}
(\bar\partial)^{\rm hol}_{\varphi_1} S^{\rm hol}_{\varphi_1}(\alpha) &= \bar\partial(S\alpha + [\varphi\star\alpha]) + [\varphi\bullet(S\alpha + [\varphi\star\alpha])], \\
S^{\rm hol}_{\varphi_1}(\bar\partial)^{\rm hol}_{\varphi_1}(\alpha) &= S(\bar\partial\alpha + [\varphi\bullet\alpha]) + [\varphi\star(\bar\partial\alpha + [\varphi\bullet\alpha])].
\end{aligned}$$

The sum of the two above becomes

$$\left\{(\bar\partial)^{\rm hol}_{\varphi_1}, S^{\rm hol}_{\varphi_1}\right\}(\alpha) = [S\varphi\bullet\alpha] + [\bar\partial\varphi\star\alpha] + [\varphi\star[\varphi\bullet\alpha]] + [\varphi\bullet[\varphi\star\alpha]], \tag{5.26}$$

where we have used the formula (3.31).

Using the KS equation of motion $\bar\partial\varphi = -(1/2)\Delta(\varphi\wedge\varphi)$ and its Hodge dual form $S\varphi = -(1/2)R(\varphi\wedge\varphi)$, we can rewrite the first two terms of the right hand side of (5.26) as

$$\begin{aligned}
&-\frac{1}{2}[R(\varphi\wedge\varphi)\bullet\alpha] - \frac{1}{2}[\Delta(\varphi\wedge\varphi)\star\alpha] \\
&= \frac{1}{2}\Delta(R(\varphi\wedge\varphi)\wedge\alpha) + \frac{1}{2}R(\Delta(\varphi\wedge\varphi)\wedge\alpha) + \frac{1}{2}R(\varphi\wedge\varphi)\wedge\Delta\alpha + \frac{1}{2}\Delta(\varphi\wedge\varphi)\wedge R\alpha,
\end{aligned}$$

while the last two terms of the right hand side of (5.26) are

$$\begin{aligned}
[\varphi\star[\varphi\bullet\alpha]] &= R(\varphi\wedge\Delta(\varphi\wedge\alpha) - \varphi\wedge\varphi\wedge\Delta\alpha) - \varphi\wedge R\Delta(\varphi\wedge\alpha) + \varphi\wedge R(\varphi\wedge\Delta\alpha), \\
[\varphi\bullet[\varphi\star\alpha]] &= \Delta(\varphi\wedge R(\varphi\wedge\alpha) - \varphi\wedge\varphi\wedge R\alpha) - \varphi\wedge\Delta R(\varphi\wedge\alpha) + \varphi\wedge\Delta(\varphi\wedge R\alpha).
\end{aligned}$$

At this point, we have

$$
\begin{aligned}
&\left\{(\bar{\partial})^{\mathrm{hol}}_{\varphi_1}, S^{\mathrm{hol}}_{\varphi_1}\right\}(\alpha)\\
=&\Delta\left(\frac{1}{2}R(\varphi\wedge\varphi)\wedge\alpha+\varphi\wedge R(\varphi\wedge\alpha)-\varphi\wedge\varphi\wedge R\alpha\right)\\
&+R\left(\frac{1}{2}\Delta(\varphi\wedge\varphi)\wedge\alpha+\varphi\wedge\Delta(\varphi\wedge\alpha)-\varphi\wedge\varphi\wedge\Delta\alpha\right)\\
&+\frac{1}{2}R(\varphi\wedge\varphi)\wedge\Delta\alpha+\varphi\wedge R(\varphi\wedge\Delta\alpha)\\
&+\frac{1}{2}\Delta(\varphi\wedge\varphi)\wedge R\alpha+\varphi\wedge\Delta(\varphi\wedge R\alpha).
\end{aligned}
\tag{5.27}
$$

By the seven-term relation [14] which holds for operators of order two such as Δ or R, and is merely the expansion of the dGBV relation (3.17) or (3.27):

$$
\begin{aligned}
\Delta(\alpha\wedge\beta\wedge\gamma)=&\ \Delta(\alpha\wedge\beta)\wedge\gamma+(-1)^{|\alpha|}\alpha\wedge\Delta(\beta\wedge\gamma)+(-1)^{|\beta|\cdot(|\alpha|+1)}\beta\wedge\Delta(\alpha\wedge\gamma)\\
&-\Delta(\alpha)\wedge\beta\wedge\gamma-(-1)^{|\alpha|}\alpha\wedge\Delta(\beta)\wedge\gamma-(-1)^{|\alpha|+|\beta|}\alpha\wedge\beta\wedge\Delta(\gamma),
\end{aligned}
\tag{5.28}
$$

we have $R(\varphi\wedge\varphi\wedge\alpha)=R(\varphi\wedge\varphi)\wedge\alpha+2\varphi\wedge R(\varphi\wedge\alpha)-\varphi\wedge\varphi\wedge R(\alpha)$; thus the first term of the right hand side of (5.27) becomes

$$
\frac{1}{2}\Delta R(\varphi\wedge\varphi\wedge\alpha)-\frac{1}{2}\Delta(\varphi\wedge\varphi\wedge R\alpha).
$$

Similarly, the second term of the right hand side of (5.27) becomes

$$
\frac{1}{2}R\Delta(\varphi\wedge\varphi\wedge\alpha)-\frac{1}{2}R(\varphi\wedge\varphi\wedge\Delta\alpha),
$$

so that their sum is

$$
-\frac{1}{2}\Delta(\varphi\wedge\varphi\wedge R\alpha)-\frac{1}{2}R(\varphi\wedge\varphi\wedge\Delta\alpha).
\tag{5.29}
$$

Then another use of the seven-term relation (5.28) shows that (5.29) cancels out the third and the fourth terms of (5.27).

We can define the deformed Lapacian by

$$
\hat{\Box}^{\mathrm{hol}}_{\varphi_1}=-(\bar{\partial})^{\mathrm{hol}}_{\varphi_1}\circ R-R\circ(\bar{\partial})^{\mathrm{hol}}_{\varphi_1}=S^{\mathrm{hol}}_{\varphi_1}\circ\Delta+\Delta\circ S^{\mathrm{hol}}_{\varphi_1}.
\tag{5.30}
$$

To see that the two operators above give the same $\hat{\Box}^{\mathrm{hol}}_{\varphi_1}$, we have only to show

$$
-[\varphi\bullet R\alpha]-R[\varphi\bullet\alpha]=[\varphi\star\Delta\alpha]+\Delta[\varphi\star\alpha]
\tag{5.31}
$$

for each $\alpha\in\mathcal{B}$. However this is the special case of the formula (3.30).

Let us define a linear map $f_{\varphi_1}:\mathcal{B}\to\mathcal{B}$ labeled by an element $\varphi_1\in\rho^{-1}(\mathbb{H}^{2,1})$ as follows. First for $\alpha\in\mathcal{B}$, set $\alpha_0=\alpha$, and define α_n, $n\geq 1$, recursively by

$$
\alpha_n=\sum_{k=1}^{n}\boldsymbol{P}(\varphi_k\wedge\alpha_{n-k}),
\tag{5.32}
$$

where we recall that $\boldsymbol{P}$ is the massive propagator defined in (4.8), and φ_n is the solution of (4.5) and given by the recursion relation (4.9). Then the map f_{φ_1} is simply given by

$$f_{\varphi_1}(\alpha) = \sum_{n=0}^{\infty} \alpha_n. \tag{5.33}$$

Presumably, the infinite sum in (5.33) will converge for φ_1 so small enough that the infinite sum $\varphi = \sum_{n\geq 1} \varphi_n$ converges.

In fact, f_{φ_1} has nice properties as a map on $\operatorname{Ker}\Delta$:

Proposition 5.1. *f_{φ_1} maps $\operatorname{Ker}\Delta \cap \operatorname{Ker}\bar\partial$ to $\operatorname{Ker}\Delta \cap \operatorname{Ker}(\bar\partial)^{\mathrm{hol}}_{\varphi_1}$.*

Proof. We will show by mathematical induction

$$\bar\partial \alpha_n + \sum_{k=1}^{n} [\varphi_k \bullet \alpha_{n-k}] = 0. \tag{5.34}$$

By the induction hypothesis and the Jacobi identity (3.16), we have

$$\begin{aligned}
\bar\partial \sum_{k=1}^{n} [\varphi_k \bullet \alpha_{n-k}] &= \sum_{k=1}^{n} \Big([\bar\partial \varphi_k \bullet \alpha_{n-k}] - [\varphi_k \bullet \bar\partial \alpha_{n-k}] \Big) \\
&= -\frac{1}{2} \sum_{a+b+c=n} [[\varphi_a \bullet \varphi_b] \bullet \alpha_c] + \sum_{a+b+c=n} [\varphi_a \bullet [\varphi_b \bullet \alpha_c]] \\
&= 0.
\end{aligned}$$

It is also easy to see that $[\varphi_k \bullet \alpha_{n-k}] \in \operatorname{Im}\Delta$ for each k in (5.34). Then the $\Delta\bar\partial$-Lemma implies that

$$\sum_{k=1}^{n} [\varphi_k \bullet \alpha_{n-k}] \in \operatorname{Im}(\bar\partial \Delta) \subset \operatorname{Im}(\bar\partial). \tag{5.35}$$

Therefore

$$\bar\partial \bar\partial^{\dagger} \mathbf{G} \rho \sum_{k=1}^{n} [\varphi_k \bullet \alpha_{n-k}] = \rho \sum_{k=1}^{n} [\varphi_k \bullet \alpha_{n-k}], \tag{5.36}$$

which is equivalent to (5.34). □

Proposition 5.2. *f_{φ_1} maps $\operatorname{Ker}\Delta \cap \operatorname{Im}\bar\partial$ to $\operatorname{Ker}\Delta \cap \operatorname{Im}(\bar\partial)^{\mathrm{hol}}_{\varphi_1}$.*

Proof. In this case, $\alpha_0 \in \operatorname{Ker}\Delta \cap \operatorname{Im}\bar\partial = \operatorname{Im}(\bar\partial\Delta)$. Thus we can set $\alpha_0 = \bar\partial \beta_0$, $\beta_0 \in \operatorname{Im}(\Delta)$.

We should find a series $\beta_1, \beta_2, \beta_3, \ldots$, which satisfies the equation

$$f_{\varphi_1}(\alpha) = \sum_{n=0}^{\infty} \alpha_n = \bar\partial \sum_{n=0}^{\infty} \beta_n + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [\varphi_m \bullet \beta_n] = 0,$$

the nth order terms of which are

$$\alpha_n = \bar\partial \beta_n + \sum_{k=1}^{n} [\varphi_k \bullet \beta_{n-k}]. \tag{5.37}$$

However, it is easily verified that β_ns defined by the recursion relation

$$\beta_n = \boldsymbol{P} \sum_{k=1}^{n} (\varphi_k \wedge \beta_{n-k}) \tag{5.38}$$

solve the equation (5.37). □

To sum up, the diagram below is commutative:

$$\begin{array}{ccc} \mathrm{Ker}\,\Delta & \xrightarrow{f_{\varphi_1}} & \mathrm{Ker}\,\Delta \\ {\scriptstyle\bar\partial}\big\downarrow & & \big\downarrow{\scriptstyle(\bar\partial)^{\mathrm{hol}}_{\varphi_1}} \\ \mathrm{Ker}\,\Delta & \xrightarrow{f_{\varphi_1}} & \mathrm{Ker}\,\Delta. \end{array} \tag{5.39}$$

We can also show that $f_{\varphi_1} \circ \hat{*} = \hat{*} \circ f_{\varphi_1}$ from $\hat{*} \circ \boldsymbol{P} = -\boldsymbol{P} \circ \hat{*}$, and that f_{φ_1} maps $\rho^{-1}(\mathbb{H}) = \mathrm{Ker}\,\hat{\Box}$ to $\mathrm{Ker}\,\Delta \cap \mathrm{Ker}(\bar\partial)^{\mathrm{hol}}_{\varphi_1} \cap \mathrm{Ker}\,R \cap \mathrm{Ker}\,S^{\mathrm{hol}}_{\varphi_1}$.

6. Outlook

In this paper we have analyzed the equation of motion and deformations of the classical KS gravity theory using the algebraic structure of the configuration space $\mathcal{B}$. Quantization of the KS gravity can be performed by the Batalin-Vilkovisky formalism [31,1,32,2,5,11], where we relax the condition on the ghost number of the field; in the action (2.30), the field φ has a contribution from each sector $B^{p,q} \cap \mathrm{Ker}\,\Delta$, except for $p \neq 3$, where Δ^{-1} cannot be defined [5]. Then the massless part of the field φ necessarily induces a extended deformation of the Calabi-Yau manifold [3,33,6]. It should be clear that the algebraic tools developed in this paper are useful in the quantization problem of the KS gravity.

It would also be interesting to study the open-closed topological B-model [34] from the point of view of the second quantization [4,35].

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References

[1] C.B. Thorn, String Field Theory, *Phys. Rep.* **175** (1989) 1–101.
[2] B. Zwiebach, Closed String Field Theory: Quantum Action and the B-V Master Equation, *Nucl. Phys.* **B390** (1993) 33–152, hep-th/9205075.
[3] E. Witten, Mirror Manifolds and Topological Field Theory, in *Essays on Mirror Manifolds*, ed. S.-T. Yau, (International Press, Hong Kong, 1992), pp. 120–159, hep-th/9112056.
[4] E. Witten, Chern-Simons Gauge Theory as a String Theory, in *The Floer Memorial Volume*, Prog. Math. vol. 133, eds. H. Hofer et al., (Birkhäuser, Boston, 1995), pp. 637–678, hep-th/9207094.
[5] M. Bershadsky, S. Ceccoti, H. Ooguri and C. Vafa, Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes, *Commun. Math. Phys.* **165** (1994) 311–427, hep-th/9309140.

[6] S. Barannikov and M. Kontsevich, Frobenius Manifolds and Formality of Lie Algebras of Polyvector Fields, *Int. Math. Res. Notices* (1998), no.4, 201–215, alg-geom/9710032.
[7] Y. Manin, *Frobenius Manifolds, Quantum Cohomology and Moduli Spaces*, (Amer. Math. Soc., Providence, 1999); Three Constructions of Frobenius Manifolds: a Comparative Study, math/9801006 [math.QA].
[8] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Classics in Mathematics, (Springer-Verlag, Berlin, 2005).
[9] S. Kobayashi, *Complex Geometry* 2, Foundations of Modern Mathematics vol. 30 (in Japanese), (Iwanami, Tokyo, 1998).
[10] D. Huybrechts, *Complex Geometry: An Introduction*, Universitext, (Springer-Verlag, Berlin, 2005)
[11] M. Bershadsky and V. Sadov, Theory of Kähler Gravity, *Int. J. Mod. Phys*. **A11** (1996) 4689-4730, hep-th/9410011.
[12] S. Yamaguchi and S.-T. Yau, Topological String Partition Functions as Polynomials, *J. High Energy Phys.* **0407** (2004) 047 (20 pages), hep-th/0406078.
[13] M. Alim, E. Scheidegger, S.-T. Yau and Jie Zhou, Special Polynomial Rings, Quasi Modular Forms and Duality of Topological Strings, *Adv. Theor. Math. Phys.* **18** (2014), no.2, 401–467, arXiv:1306.0002 [hep-th].
[14] C. Roger, Gerstenhaber and Batalin-Vilkovisky Algebras; Algebraic, Geometric and Physical Aspects, *Arch. Math. (Brno)* **45** (2009) 301–324.
[15] K. Fukaya, *Symplectic Geometry*, Developments in Modern Mathematics vol. 21 (in Japanese), (Iwanami, Tokyo, 1999).
[16] G. Tian, Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and Its Petersson-Weil Metric, in *Mathematical Aspects of String Theory*, ed. S.-T. Yau, (World Scientific, Singapore, 1987), pp. 629–646.
[17] S. A. Merkulov, Formality of Canonical Symplectic Complexes and Frobenius Manifolds, *Int. Math. Res. Notices* (1998), no. 14, 727–733, math/9805072 [math.SG].
[18] H.-D. Cao and Jian Zhou, Frobenius Manifold Structure on Dolbeault Cohomology and Mirror Symmetry, *Commun. Anal. Geom*. **8** (2000) , no.4, 795-808, math/9805094 [math.DG].
[19] J.-S. Park, Topological Open p-Branes, in *Symplectic Geometry and Mirror Symmetry*, eds. K. Fukaya et al., (World Scientific, New Jersey, 2001), pp. 311–384, hep-th/0012141.
[20] A. N. Todorov, The Weil-Petersson Geometry of the Moduli Space of $SU(n \geq 3)$ (Calabi-Yau) Manifolds I, *Commun. Math. Phys*. **126** (1989) 325–346.
[21] Y. Itagaki, Convergence of the Normalized Solution of the Maurer-Cartan Equation in the Barannikov-Kontsevich Construction, *J. Math. Soc. Japan* **54** (2002) 309–328.
[22] A. Losev and S. Shadrin, From Zwiebach Invariants to Getzler Relation, *Commun. Math. Phys*. **271** (2007) 649–679, math/0506039 [math.QA].
[23] A. Sen, Equations of Motion in Non-Polynomial Closed String Field Theory and Conformal Invariance of Two Dimensional Field Theory, *Phys. Lett.* **B241** (1990) 350–356.
[24] J. M. F. Labastida and M. Mariño, Type B Topological Matter, Kodaira-Spencer Theory, and Mirror Symmetry, *Phys. Lett.* **B333** (1994) 386–395, hep-th/9405151.
[25] N. Marcus and S. Yankielowicz, The Topological B-Model as a Twisted Spinning Particles, *Nucl. Phys.* **B432** (1994) 225–245, hep-th/9408116.
[26] K. Ranganathan, H. Sonoda and B. Zwiebach, Connections on the State Space over Conformal Field Theories, *Nucl. Phys*. **B414** (1994) 405–460, hep-th/9304053.
[27] C. Becchi, S. Giusto and C. Imbimbo, The Holomorphic Anomaly of Topological Strings, *Fortsch. Phys*. **47** (1999) 195–200, hep-th/9801100.
[28] G. Bandelloni and S. Lazzarini, Kodaira-Spencer Deformation of Complex Structures and Lagrangian Field Theory, *J. Math. Phys*. **39** (1998) 3619–3642, hep-th/9802086.
[29] E. Witten, Quantum Background Independence in String Theory, in *Salamfestschrift*, eds. A.

Ali et al., (World Scientific, Singapore, 1994), pp. 257–275, hep-th/9306122.
[30] M. Kontsevich, Mirror Symmetry in Dimensions 3, *Sém. N. Bourbaki* **37** (1995) 275–293.
[31] M. Bochicchio, Gauge Fixing of the Field Theory of the Bosonic String, *Phys. Lett*. **B193** (1987) 31–36.
[32] H. Hata, BRS Invariance and Unitarity in Closed String Field Theory, *Nucl. Phys*. **B329** (1990) 698–722; Construction of the Quantum Action for Path Integral Quantization of String Field Theory, *Nucl. Phys*. **B339** (1990) 663–686.
[33] Z. Ran, Thickening Calabi-Yau Moduli Spaces, in *Mirror Symmetry II*, AMS/IP Stud. Adv. Math. vol. 1, eds. B. Greene and S.-T. Yau, (International Press of Boston, Cambridge, 1996), pp. 393–400, alg-geom/9312003.
[34] C. Hofman, On the Open-Closed B-Model, *J. High Energy Phys.* **0311** (2003) 069, 47 pp, hep-th/0204154.
[35] B. Zwiebach, Oriented Open-Closed String Theory Revisited, *Ann. Phys*. **267** (1998) 193–248, hep-th/9705241.