Spectral Analysis of the Subelliptic Oblique Derivative Problem

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Purpose of Talk

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This talk is devoted to a functional analytic approach to the subelliptic oblique derivative problem for the Laplacian with a complex parameter.

Main Results

We prove generation theorems of analytic semigroups for the subelliptic oblique derivative problem for the Laplacian in the L^p topology and in the topology of uniform convergence.

These rather surprising results (elliptic estimates for a degenerate problem) work, since we are considering the homogeneous boundary condition.

My Works

(1) K. Taira: **Analytic semigroups for the subelliptic oblique derivative problem, Journal of Mathematical Society of Japan, Vol. 69, No. 3 (2017), 1281-1330**

DOI: 10.2969/jmsj/06931281

(2) K. Taira: **Spectral analysis of the subelliptic oblique derivative problem, Arkiv for Matematik, Vol. 55, No. 1 (2017), 243-270**

DOI: 10.4310/ARKIV.2017.v55.n1.a13

Scheme of Talk (1)

(1) Historical Background (Poincare)

(2) Motivation of Talk

(3) Formulation of the Problem

(4) Main Results

Scheme of Talk (2)

(5) Special Reduction to the Boundary

(Beals-Fefferman-Grossman)

(6) Microlocal Analysis

(Hartogs-Malgrange-Guan)

(7) Generation of Analytic Semigroups

(Agmon)

(8) Asymptotic Eigenvalue Distributions

(Boutet de Monvel)

Historical Background

Due to Poincare, it is known that the oblique derivative problem arises naturally when determining the gravitational field of the moon, the earth and the other celestial bodies.

Henri Poincare

Henri Poincare (1854-1912)
French Mathematician

H. Poincare:
Lecons de mechanique celeste, Tome III, Gauthier-Villars, Paris, 1910

Motivation of Talk

Motivation of Talk (1)

In physical geodesy, investigations of the Earth's gravity field based on surface gravity data are usually associated with a simultaneous determination of the figure of the Earth.

Motivation of Talk (2)

The precise 3D positioning by the Global Navigation Satellite Systems (GNSS) has brought new possibilities in gravity field modelling. Terrestrial gravimetric measurements located by precise satellite positioning yield oblique derivative boundary conditions in the form of surface gravity disturbances.

References (1)

(1) K.R. Koch and A.J. Pope: Uniqueness and existence for the geodetic boundary-value problem using the known surface of the Earth. Bulletin of Geodecy 46 (1972), 467-476.

(2) A. Bjerhammar and L. Svensson:

On the geodetic boundary-value problem for a fixed boundary surface - satellite approach,

Bulletin of Geodesy 57 (1983), 382-393.

References (2)

(3) P. Holota: Coerciveness of the linear gravimetric boundary-value problem and a geometrical interpretation.

Journal of Geodesy 71 (1997), 640-651.

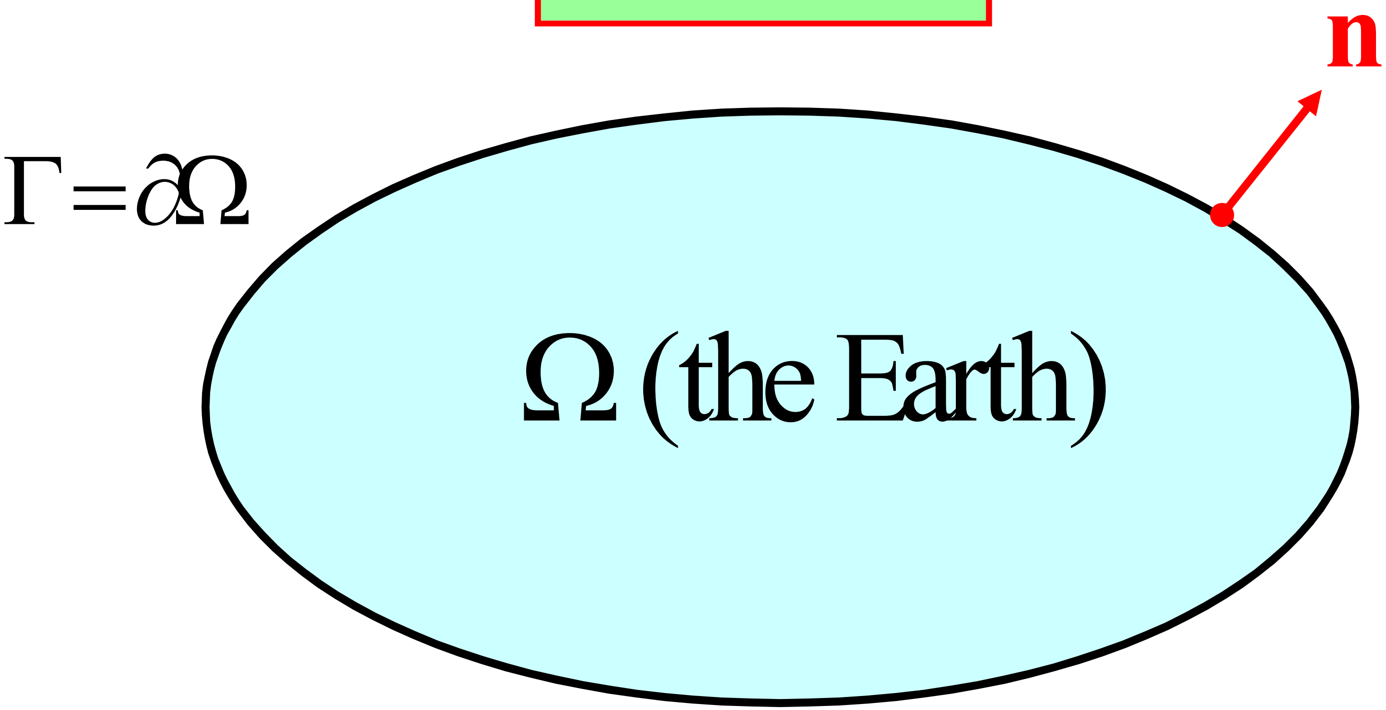
(4) R. Cunderlik, K. Mikula and M. Mojzes: Numerical solution of the linearized fixed gravimetric boundary-value problem, Journal of Geodesy 82 (2008), 15-29.

Motivation of Talk (3)

Now the shape of the Earth can be obtained by geometric satellite triangulation and satellite altimetry over the oceans. In this way, the (linearized) fixed gravimetric boundary value problem in physical geodesy is an oblique derivative problem for the Laplace equation in the Earth's exterior.

Bounded Domain

$\mathbf{R}^n, \quad n \geq 3$



Laplace Operator

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$

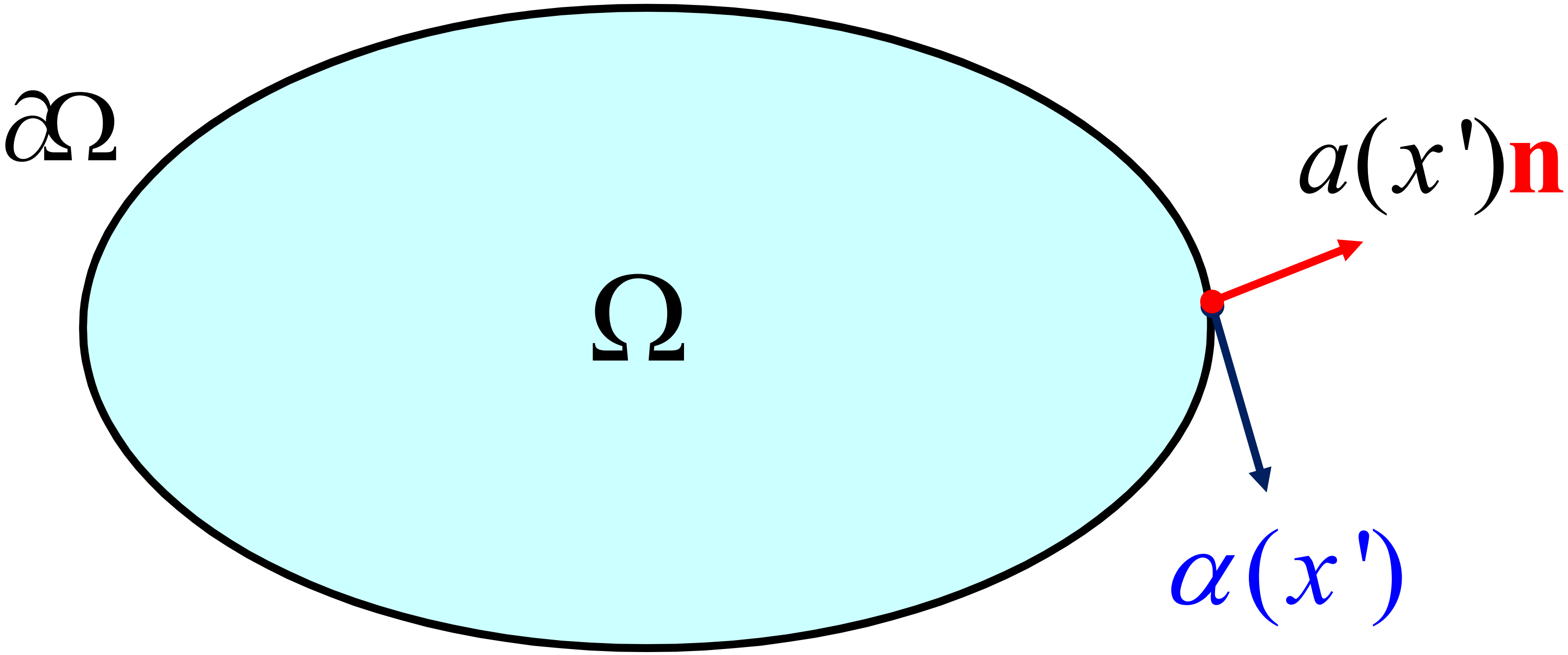
Oblique Derivative Condition

$$\frac{\partial u}{\partial \nu}(x') = a(x')\frac{\partial u}{\partial \mathbf{n}} + \alpha(x') \cdot u = 0 \quad \text{on } \Gamma = \partial\Omega$$

$\mathbf{n} = (n_1, n_2, \ldots, n_n)$: **unit outward normal**

$\alpha(x')$: **tangent vector field**

Oblique Derivative Condition



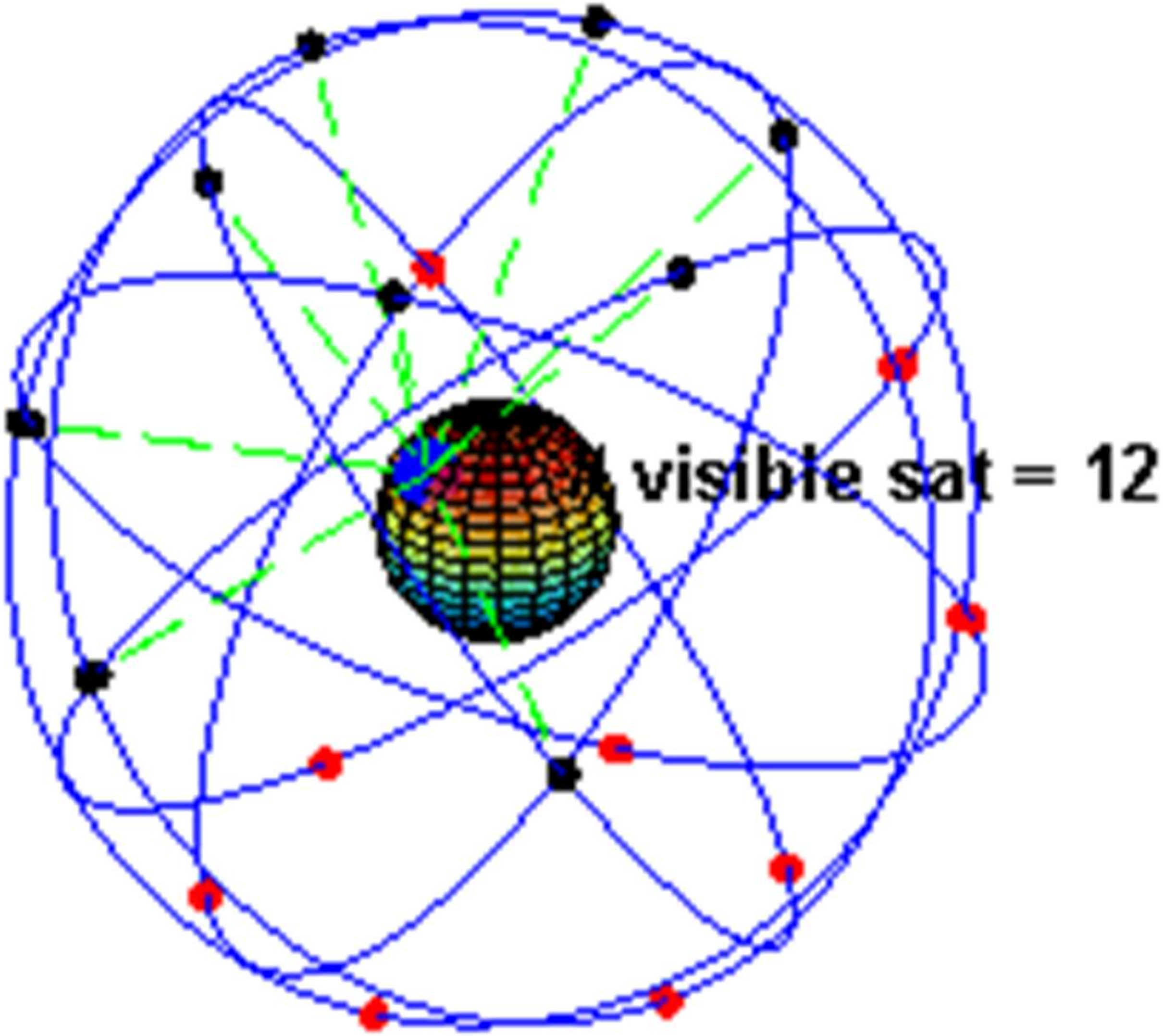
$$\nu(x') = a(x')\mathbf{n} + \alpha(x')$$

Motivation of Talk (4)

Nowadays we only pick up the data sent from the satellites not on the horizontal.

We neglect the data sent from the satellites on the horizontal.

(Elliptic case)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Conclusion (Subelliptic case)

(1) We can make use of the data sent from the satellites even on the horizontal.

(2) We can economize the number of satellites around the Earth.

Example: The viewpoint of the first approximation to the weather forecast data.

Formulation of the Problem

In this talk we will deal with an interior oblique derivative problem in a bounded domain.

The analysis of harmonic functions in an exterior domain can be reduced to that of harmonic functions in a bounded domain by using the Kelvin transform.

Brief History (1)

(1) L^2 Theory

Ju. V. Egorov and V. A. Kondratev: The oblique derivative problem, Math. USSR Sb. 7 (1969), 139-169

(2) Holder Space Theory

B. Winzell: A boundary value problem with an oblique derivative, Comm. Partial Differential Equations, 6 (1981), 305-328

Brief History (2)

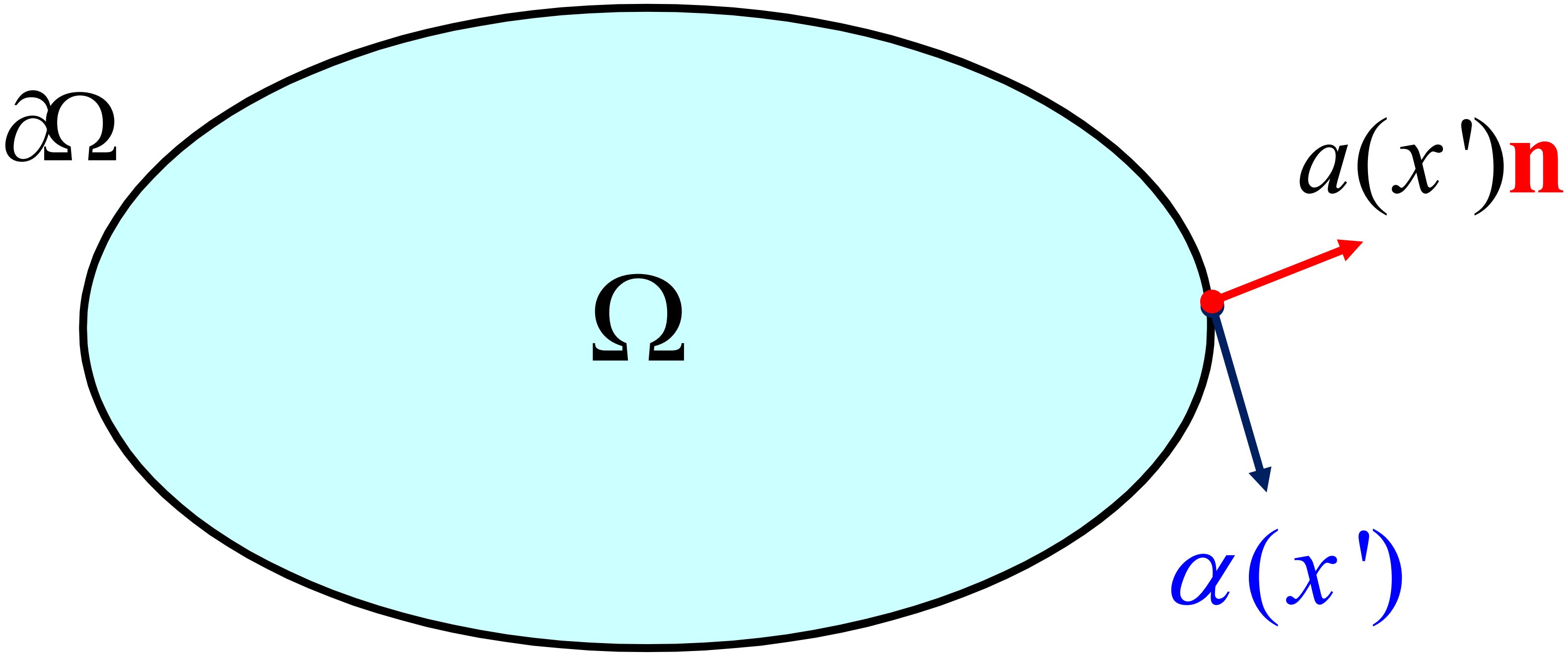
(3) L^p Theory

H. Smith: **The subelliptic oblique derivative problem, Comm. Partial Differential Equations, 15 (1990), 97-137**

P. Guan and E. Sawyer: **Regularity estimates for the oblique derivative problem, Ann. of Math. (2), 137 (1993), 1-70**

Subelliptic Case

Oblique Derivative Condition



$$\nu(x') = a(x')\mathbf{n} + \alpha(x')$$

Fundamental Condition (H)

(1) **The vector field** $\alpha(x')$ **is non-zero on the** **set** $\Gamma_0 = \{x' \in \Gamma : a(x') = 0\}$ **of tangency.**

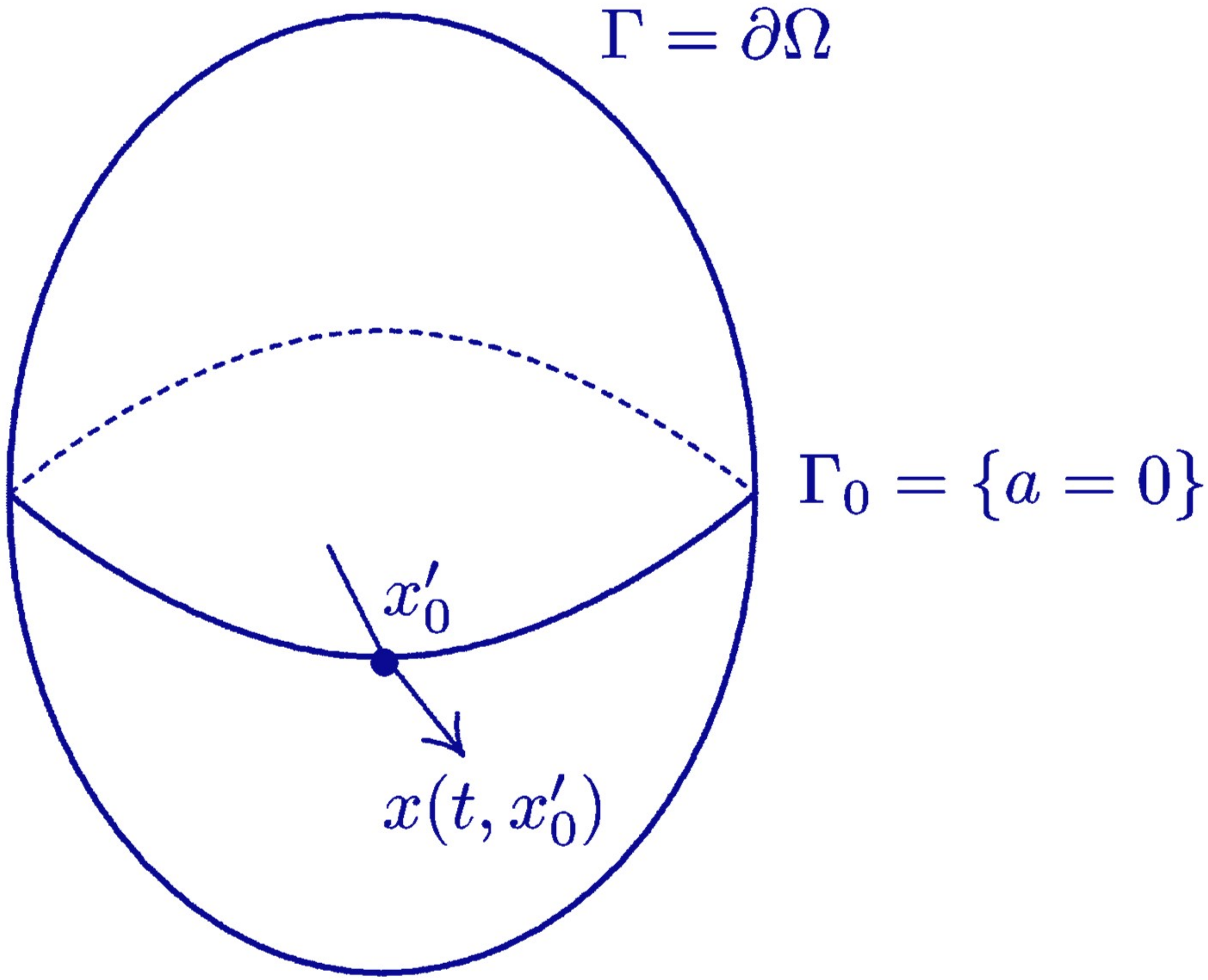
(2) **Along the integral curve** $x(t, x'_0)$ **of** $\alpha(x')$ **passing through** $x'_0 \in \Gamma_0$ **at** $t = 0$, **the function**

$t \mapsto a(x(t, x'_0))$

has zeros of even order $\leq 2k$.

$$\nu(x') = a(x')\mathbf{n} + \alpha(x')$$

Fundamental Condition (H)



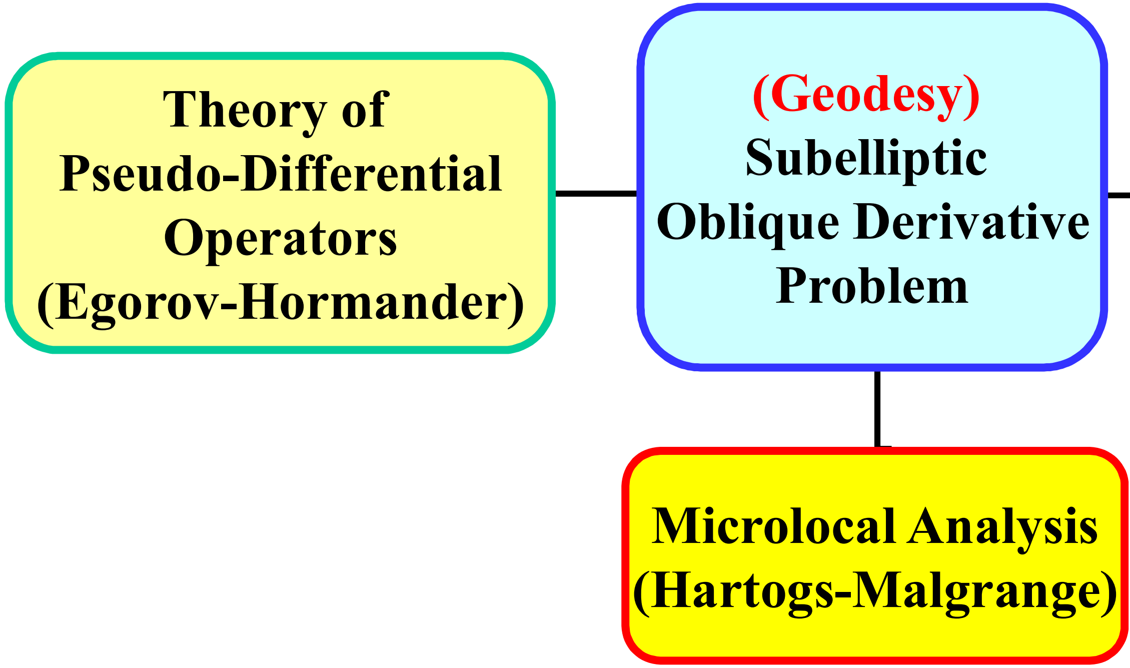
Typical Examples

$$a\left(x(t, x'_0)\right) = t^{2k} \quad : \quad \text{YES}$$

$$a\left(x(t, x'_0)\right) = e^{-1/t^2} \quad : \quad \text{NO}$$

Bird's-Eye View

Functional analytic approach to the subelliptic oblique derivative problem



Mathematical Background

Field	Subject	Mathematicians
Pseudo-Differential Operators	Subellipticity	Egorov-Hormander Smith
Microlocal Analysis	Hartogs' Theorem	Malgrange Guan
Real Analysis	Boundedness of Maximal Operators	Hardy-Littlewood Muckenhoupt Sawyer

References (1)

(1) Ju. V. Egorov: **Subelliptic operators.**

Uspekhi Mat. Nauk 30:2 (182) (1975), 57-114, 30:3 (183) (1975), 57--104 (in Russian);

English translation: Russian Math. Surveys 30:2 (1975), 59-118, 30:3 (1975), 55--105.

(2) L. Hormander: **Subelliptic operators.**

In: Seminar on singularities of solutions of linear partial differential equations, Ann. of Math. Stud., No. 91, 127-208, 1979.

References (2)

P. Guan and E. Sawyer: Regularity estimates for the oblique derivative problem, Ann. of Math. (2), 137 (1993), 1-70

References (3)

(1) B. Muckenhoupt: Hardy's inequality with weights, Studia Math. 44 (1972), 31-38

(2) B. Muckenhoupt : Weighted norm inequalities for the Hardy maximal functions, Trans. A. M. S. 165 (1972), 207-226

(3) E. Sawyer: Weighted inequalities for the one-sided Hardy-Littlewood maximal functions, Trans. A. M. S. 297 (1986), 53-61

Main Results

Regularity Theorem

My Work (1981)

$$u \in L^2(\Omega) \qquad \boxed{p = 2}$$

$$\begin{cases} \Delta u = f \in H^s(\Omega) \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma} = \varphi \in H^{s+1/2}(\Gamma) \end{cases}$$

$$\Rightarrow$$

$$u \in H^{s+2-\delta}(\Omega)$$

Loss of δ-derivatives

$$0 \le \delta = \frac{2k}{2k+1} < 1$$

$k = 0$ (Elliptic case) $\Leftrightarrow \delta = 0$

$k = \infty \Leftrightarrow \delta = 1$

Sharp Regularity Theorem (Smith, 1990)

$u \in L^p(\Omega),\ 1 < p < \infty,$

$$\begin{cases} \Delta u = f \in W^{s,p}(\Omega) \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma} = 0 \quad \text{on } \Gamma \end{cases}$$

$\Rightarrow$

$u \in W^{s+2,p}(\Omega).$

Elliptic gain of 2-derivatives from f

Remarkable Fact

The degeneracy occurs only for the boundary data.

Generation Theorem
of
Analytic Semigroups

Homogeneous Case

We define a **densely defined, closed** **operator**

$\mathfrak{A}_p : L^p(\Omega) \to L^p(\Omega) \quad (1 < p < \infty)$

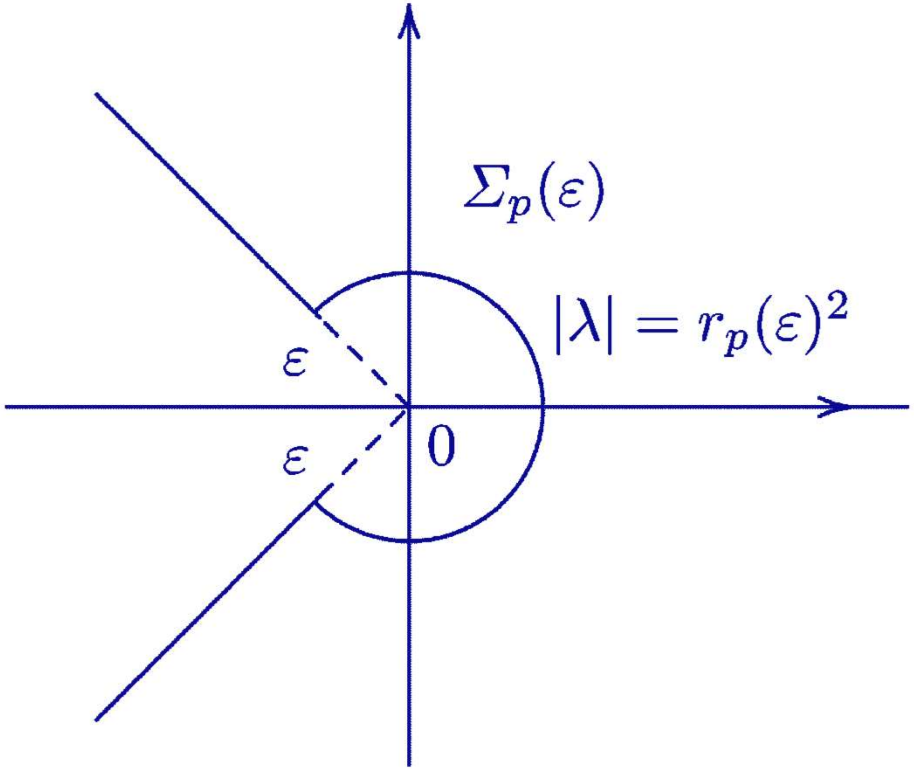
as follows:

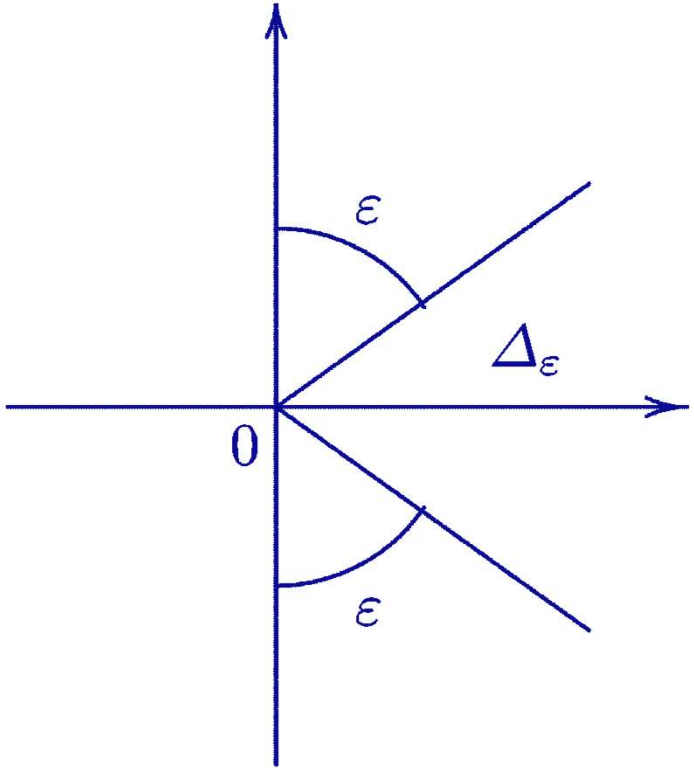
(a) $D(\mathfrak{A}_p) = \left\{ u \in W^{2,p}(\Omega) : \boxed{\frac{\partial u}{\partial \nu} = 0} \right\}$

(b) $\mathfrak{A}_p u = \Delta u, \ \forall u \in D(\mathfrak{A}_p)$

Then $\mathfrak{A}_p$ **generates an** **analytic semigroup**

$e^{z\mathfrak{A}_p}$ **on** $L^p(\Omega)$





$$\lambda = r^2 e^{i\theta}$$
$$-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$$

$$e^{z\mathfrak{A}_p} \quad (z \in \Delta_\varepsilon)$$

Minimal Growth of the Resolvent

$$\left\| \left(\mathfrak{A}_p - \lambda I \right)^{-1} f \right\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|} \| f \|_{L^p(\Omega)}$$

$$\forall f \in L^p(\Omega)$$

$$1 < p < \infty$$

Spectral Properties of the Subelliptic Oblique Derivative Problem

We define a **densely defined, closed** operator

$\mathfrak{A}_2 : L^2(\Omega) \to L^2(\Omega)$ $\boxed{p = 2}$

as follows:

(a) $D(\mathfrak{A}_2) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \right\}$

(b) $\mathfrak{A}_2 u = \Delta u, \ \forall u \in D(\mathfrak{A}_2)$

Then the operator $\mathfrak{A}_2$ enjoys the following four spectral properties:

Asymptotic Eigenvalue Distribution (1)

(1) The spectrum of $\mathfrak{A}_2$ is **discrete** and the eigenvalues λ_j of $\mathfrak{A}_2$ have **finite multiplicities**.

(2) All rays different from the **negative axis** are rays of minimal growth of the resolvent $\left(\mathfrak{A}_2 - \lambda I\right)^{-1}$.

(3) The negative axis is a **direction of condensation** of eigenvalues of $\mathfrak{A}_2$.

Asymptotic Eigenvalue Distribution (2)

(4) Let

$$N(t) := \sum_{\mathrm{Re}\lambda_j \geq -t} 1$$

the counting function

where each λ_j is repeated according to its multiplicity.

Then the **asymptotic eigenvalue distribution** formula

$$N(t) = \frac{|\Omega|}{2^n \pi^{n/2} \Gamma(n/2+1)} \cdot t^{n/2} + o(t^{n/2}) \text{ as } t \to +\infty$$

holds true. Here $|\Omega|$ denotes the volume of Ω

Conclusion

(1) We can make use of the data sent from the satellites even on the horizontal.

(2) We can economize the number of satellites around the Earth.

The viewpoint of the first approximation to the weather forecast data.

My theorem should be tested by
Numerical Analysis of
the linear fixed altimetry-gravimetry
boundary value problem

References

Zuzana Fašková, Róbert ˇCunderlík and Karol Mikula: Finite element method for solving geodetic boundary value problems Journal of Geodesy, Vol. 84, No. 2 (2010), 135-144.

https://doi.org/10.1007/s00190-009-0349-7

Special Reduction to the Boundary

The First Idea of Approach

We make use the Dirichlet-Neumann operator in the exterior domain in reducing the oblique derivative problem to the study of a pseudo-differential operator on the boundary.

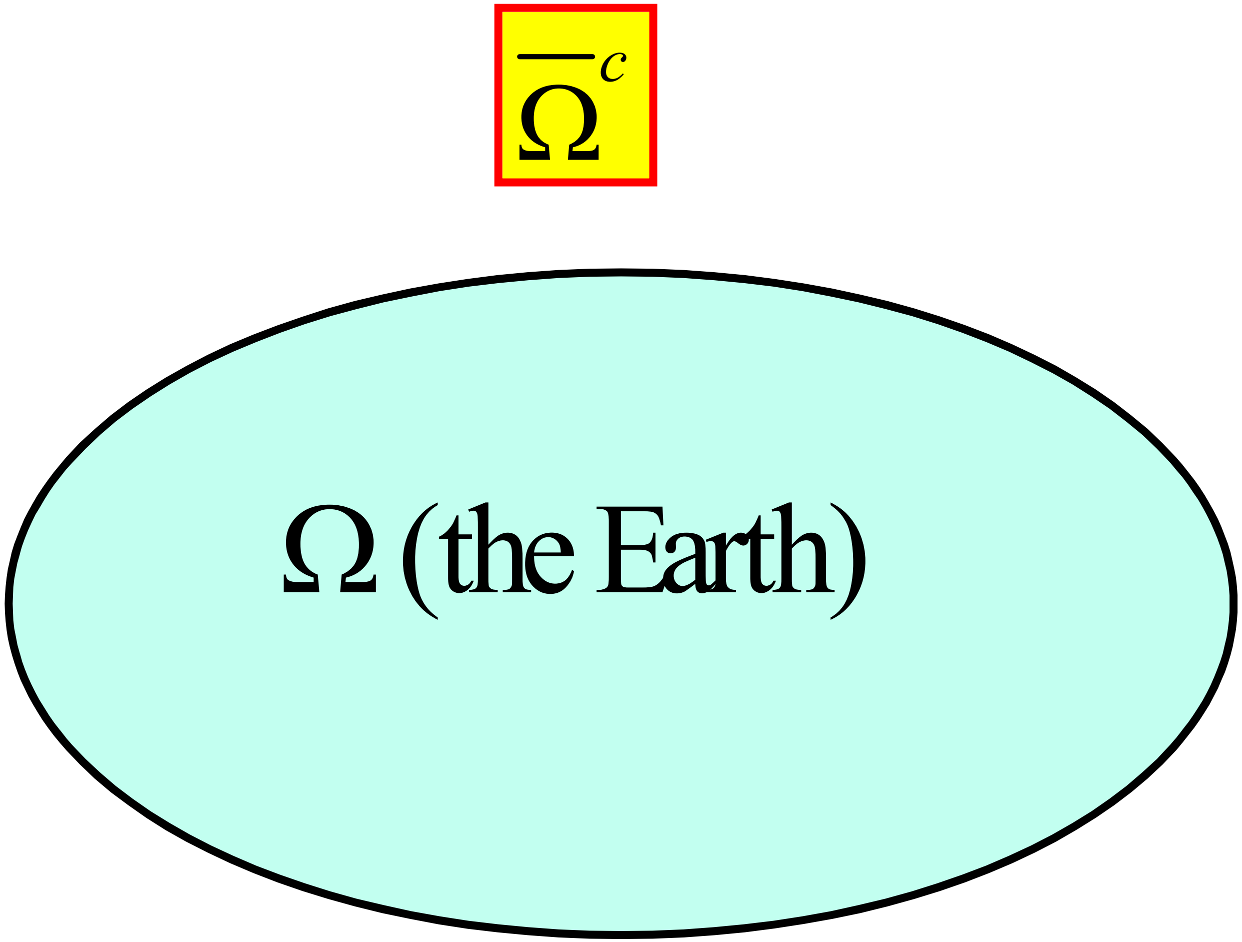
References

M. Beals, C. Fefferman and R. Grossman:
Strictly pseudoconvex domains in C^n.
Bull. A. M. S. 8 (1983), 125-322

Exterior

Dirichlet Problem

Exterior Domain



Model Case

Model Case (1)

$\Omega = \mathbf{R}^n_+$ (upper-half space)

$\Gamma = \mathbf{R}^{n-1}$

$\overline{\Omega}^c = \mathbf{R}^n_-$ (lower-half space)

$x = (x', x_n)$

$x' = (x'', t)$ (tangential variables)

Model Case (2)

$$\alpha(x') = \alpha(x'', t) = \frac{\partial}{\partial t} \quad \textbf{(tangent vector field)}$$

$$\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial x_n} \quad \textbf{(outward normal derivative)}$$

$$a\left(x'', t\right) = a(t)$$

Model Case (3)

$$\gamma_1 = \alpha(x') + a(x'', t)\frac{\partial}{\partial \mathbf{n}}$$

$$= \frac{\partial}{\partial t} - a(t)\frac{\partial}{\partial x_n}$$

Zero Extension Operator

$$f_0(x) := \begin{cases} f(x) & \text{for all } x \in \mathbf{R}^n_+, \\ 0 & \text{for all } x \in \mathbf{R}^n_- \end{cases}$$

Newtonian Potential (1)

$$\left(N * f_0\right)(x) = \frac{1}{(n-2)\omega_n} \int_{\mathbf{R}_+^n} \frac{f(y)}{|x-y|^{n-2}}\, dy$$

Transmission Property of the Newtonian Potential

$$\left(r^{+}N\right)f := \left(N * f_{0}\right)\Big|_{\mathbf{R}_{+}^{n}}$$

$$r^{+}N : W^{s,p}(\mathbf{R}_{+}^{n}) \to W^{s+2,p}(\mathbf{R}_{+}^{n})$$

Newtonian Potential (2)

$$W^{s+2,p}(\mathbf{R}^n_+) \xrightarrow{r^+\Delta=\Delta} W^{s,p}(\mathbf{R}^n_+)$$

$$W^{s+2,p}(\mathbf{R}^n_+) \xleftarrow[r^+N]{} W^{s,p}(\mathbf{R}^n_+)$$

$$\left(r^+\Delta\right)\left(r^+N\right) = I$$

Poisson Kernel

$$\Delta(P\varphi) = 0 \text{ in } \mathbf{R}^n_+,$$

$$\gamma_0(P\varphi) = P\varphi\big|_{\mathbf{R}^{n-1}} = \varphi \text{ on } \mathbf{R}^{n-1}$$

$$P : B^{s-1/p,p}(\mathbf{R}^{n-1}) \to W^{s,p}(\mathbf{R}^n_+)$$

Poisson Kernel in the Half-Space

$$P\varphi\left(x', x_n\right) = \frac{2}{\omega_n}\int_{\mathbf{R}^{n-1}} \frac{x_n\ \varphi(y')}{\left(\left|x'-y'\right|^2 + x_n^2\right)^{n/2}} dy'$$

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

Fourier Transform Version

$$P\varphi(x', x_n)$$

$$= \frac{1}{(2\pi)^{n-1}} \int_{\mathbf{R}^{n-1}} e^{ix'\xi'} e^{-x_n|\xi'|} \hat{\varphi}(\xi') d\xi'$$

Reduction to the Boundary (1)

$$\Delta u = f \quad \text{in } \mathbf{R}^n_+,$$
$$\gamma_1 u = 0 \quad \text{on } \mathbf{R}^{n-1}$$

$$\Rightarrow$$

$$\Delta \left(r^+ N f - u \right) = f - f = 0 \quad \text{in } \mathbf{R}^n_+$$

Reduction to the Boundary (2)

$$r^{+}Nf - u = P\psi$$

$$\psi = \left(r^{+}Nf - u\right)\Big|_{\mathbf{R}^{n-1}}$$

Reduction to the Boundary (3)

$$\Delta u = f$$
$$\gamma_1 u = 0$$

$$\Leftrightarrow$$

$$\gamma_1 (P\psi) = \gamma_1 \left(r^+ N f \right) \text{ on } \mathbf{R}^{n-1}$$

Fredholm Integral Equation

$$T\psi := \gamma_1(P\psi) = a(x')\frac{\partial}{\partial \mathbf{n}}\left(P\psi\right)\Big|_{\mathbf{R}^{n-1}} + \alpha(x')\cdot\psi$$

$$= a(x')\Pi\,\psi + \alpha(x')\cdot\psi$$

$$\Rightarrow$$

$$\boxed{T\psi = a(x')\Pi\,\psi + \alpha(x')\cdot\psi = \gamma_1\left(r^+Nf\right) \text{ on } \mathbf{R}^{n-1}}$$

Dirichlet-Neumann Operator

$$\Pi\,\psi = \frac{\partial}{\partial \mathbf{n}}\left(P\psi\right)\Big|_{\mathbf{R}^{n-1}} \qquad \forall\,\psi \in C^{\infty}(\mathbf{R}^{n-1})$$

$$\Pi \in L^{1}_{\mathrm{cl}}(\mathbf{R}^{n-1})$$

Dirichlet-Neumann Operator in the Half-Space

$$\Pi\varphi(x') = -\frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\mathbf{R}^{n-1}} \frac{\varphi(y')}{|x' - y'|^n} dy'$$

Fourier Transform Version in the Half-Space

$$\Pi\varphi(x') = \frac{1}{(2\pi)^{n-1}} \int_{\mathbf{R}^{n-1}} e^{ix'\xi'} |\xi'| \hat{\varphi}(\xi')\, d\xi'$$

$$\Pi = \sqrt{-\Delta_{x'}}$$

Poisson Kernel for the Exterior Domain (Kelvin)

$$\Delta\left(P^{\text{ext}}\varphi\right)=0 \ \text{ in } \mathbf{R}_{-}^{n}$$

$$\gamma_0\left(P^{\text{ext}}\varphi\right)=P^{\text{ext}}\varphi\Big|_{\mathbf{R}^{n-1}}=\varphi \ \text{ on } \mathbf{R}^{n-1}$$

Poisson Kernel for the Exterior Domain

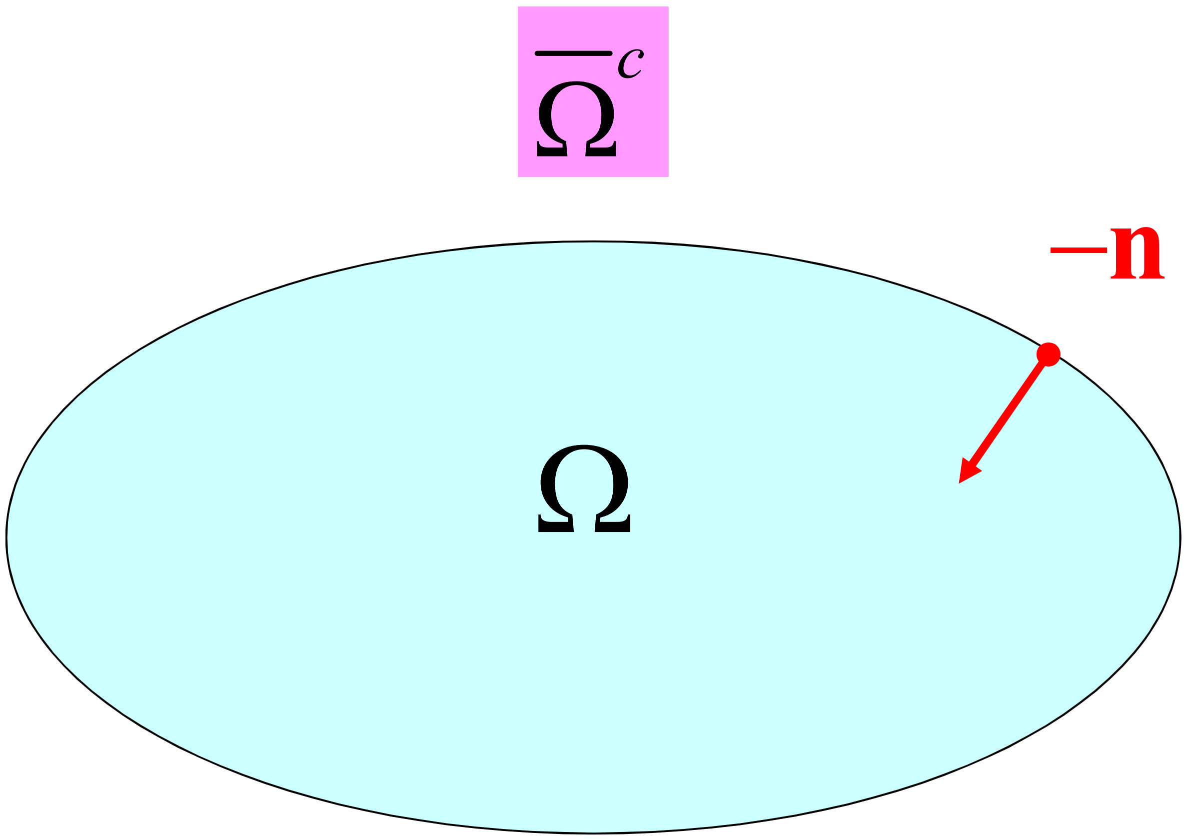
$$\Delta\left(N * f_0\right) = f_0 = 0 \quad \text{in } \mathbf{R}^n_-$$

$$\left(N * f_0\right)\Big|_{\mathbf{R}^{n-1}} \in B^{s+2-1/p,p}\left(\mathbf{R}^{n-1}\right)$$

$$\Downarrow$$

$$N * f_0 = P^{\text{ext}}\left(N * f_0\Big|_{\mathbf{R}^{n-1}}\right) \quad \text{in } \mathbf{R}^n_-$$

Outward Normal for the Exterior Domain



Reduction to the Boundary

$$N * f_0 = P^{\text{ext}}\left(N * f_0\big|_{\mathbf{R}^{n-1}}\right) \quad \text{in } \mathbf{R}^n_-$$

$$\Downarrow$$

$$\gamma_1\left(N * f_0\right) = \gamma_1\left(P^{\text{ext}}\left(N * f_0\big|_{\mathbf{R}^{n-1}}\right)\right)$$

$$= \frac{\partial}{\partial \nu}\left(P^{\text{ext}}\left(N * f_0\big|_{\mathbf{R}^{n-1}}\right)\right)$$

$$= a(x')\frac{\partial}{\partial \mathbf{n}}\left(P^{\text{ext}}\left(N * f_0\big|_{\Gamma}\right)\right)\Big|_{\mathbf{R}^{n-1}} + \alpha(x')\cdot\left(N * f_0\big|_{\mathbf{R}^{n-1}}\right)$$

$$= -a(x')\frac{\partial}{\partial(-\mathbf{n})}\left(P^{\text{ext}}\left(N * f_0\big|_{\Gamma}\right)\right)\Big|_{\mathbf{R}^{n-1}} + \alpha(x')\cdot\left(N * f_0\big|_{\mathbf{R}^{n-1}}\right)$$

Dirichlet-Neumann Operator for the Exterior Domain

$$\Pi^{\text{ext}}\psi = \frac{\partial}{\partial(-\mathbf{n})}\left(P^{\text{ext}}\psi\right)\Big|_{\mathbf{R}^{n-1}}, \qquad \forall\, \psi \in C^{\infty}(\mathbf{R}^{n-1})$$

$-\mathbf{n}$: unit **outward** normal to $\overline{\Omega}^{c} = \mathbf{R}^{n}_{-}$

Model Case (1)

$\Omega = \mathbf{R}^n_+$ (upper-half space)

$\Gamma = \mathbf{R}^{n-1}$

$\overline{\Omega}^c = \mathbf{R}^n_-$ (lower-half space)

$x = (x', x_n)$

$x' = (x'', t)$ (tangential variables)

Model Case (2)

$$\alpha(x') = \alpha(x'', t) = \frac{\partial}{\partial t}$$ **(tangent vector field)**

$$\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial x_n}$$ **(outward normal derivative)**

$$a\left(x'', t\right) = a(t)$$

$$\Rightarrow$$

$$\boxed{\gamma_1 = \alpha(x') + a(x'', t)\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial t} - a(t)\frac{\partial}{\partial x_n}}$$

Model Case (3)

$$T = \gamma_1 P = \frac{\partial}{\partial t} + a(t)\sqrt{-\Delta_{x''} - \frac{\partial^2}{\partial t^2}}$$

$$T' = \gamma_1 P^{ext}$$

$$= -\frac{\partial}{\partial t} + a(t)\sqrt{-\Delta_{x''} - \frac{\partial^2}{\partial t^2}}$$

Fredholm Integral Equation

$$T\mu = T'\nu$$

$$\nu = N * f_0 \big|_{\mathbf{R}^{n-1}},$$

$$\mu = \left(u - N * f_0\right)\big|_{\mathbf{R}^{n-1}}$$

Microlocal Analysis

Reduction of the Pseudo-Differential Operators (Microlocal Analysis)

$$T = \gamma_1 P = \frac{\partial}{\partial t} + a(t)\sqrt{-\Delta_{x''} - \frac{\partial^2}{\partial t^2}}$$

$$L = \frac{\partial}{\partial t} + a_1(t)\sqrt{-\Delta_{x''}}$$

$$a_1(t) := \frac{a(t)}{\sqrt{1 + a(t)^2}} \approx a(t)$$

The Second Idea of Approach

We make use of Hartog's theorem in applying a precise version of Malgrange's preparation theorem in micro local analysis.

Microlocal Analysis (1)

$$a(t)\sqrt{|\xi''|^2+\tau^2}+\sqrt{-1}\,\tau$$

$$=\boxed{\frac{a(t)\sqrt{|\xi''|^2+\tau^2}+\sqrt{-1}\,\tau}{a_1(t)\,|\xi''|+\sqrt{-1}\,\tau}}$$

$$\times\left(a_1(t)\,|\xi''|+\sqrt{-1}\,\tau\right)$$

Characteristic Set

$$t_1(x'', t, \xi'', \tau) = a(t)\sqrt{|\xi''|^2 + \tau^2} + \sqrt{-1}\,\tau$$

$$t'_1(x'', t, \xi'', \tau) = a(t)\sqrt{|\xi''|^2 + \tau^2} - \sqrt{-1}\,\tau$$

$$\Sigma = \{a(t) = 0,\ \xi'' \neq 0,\ \tau = 0\}$$

Microlocal Analysis (2)

$$|\tau| \le |\xi''|, \quad s := \tau / |\xi''|$$

$$\Rightarrow$$

$$\frac{a(t)\sqrt{|\xi''|^2 + \tau^2} + \sqrt{-1}\,\tau}{a_1(t)\,|\xi''| + \sqrt{-1}\,\tau}$$

$$= \frac{\sqrt{-1}\,s + a(t)\sqrt{1 + s^2}}{\sqrt{-1}\,s + \dfrac{a(t)}{\sqrt{1 + a(t)^2}}}$$

Hartogs' Theorem

$$h(s,a) := \frac{\sqrt{-1}\,s + a\sqrt{1+s^2}}{\sqrt{-1}\,s + \dfrac{a}{\sqrt{1+a^2}}}$$

is holomorphic in the unit polydisk

$$\{(s,a) : |s| \le 1, |a| \le 1\}$$

$(s,a) = (0,0)$ **removable singularity**

Friedrich Hartogs

Friedrich Hartogs (1874-1943)
German Mathematician

Micolocal Analysis (3)

$$a(t)\sqrt{|\xi''|^2+\tau^2}+\sqrt{-1}\,\tau$$

$=$ **A Bounded Symbol (Hartogs)**

$$\times\ a_1(t)\,|\xi''|+\sqrt{-1}\,\tau$$

$$a(t)\sqrt{|\xi''|^2+\tau^2}-\sqrt{-1}\,\tau$$

$=$ **A Bounded Symbol (Hartogs)**

$$\times\ a_1(t)\,|\xi''|-\sqrt{-1}\,\tau$$

Model Case (1)

$$T = \gamma_1 P = \frac{\partial}{\partial t} + a(t)\sqrt{-\Delta_{x''} - \frac{\partial^2}{\partial t^2}}$$

$$T' = \gamma_1 P^{ext} = -\frac{\partial}{\partial t} + a(t)\sqrt{-\Delta_{x''} - \frac{\partial^2}{\partial t^2}}$$

Model Case (2)

$$L_1 = \frac{\partial}{\partial t} + a_1(t)\sqrt{-\Delta_{x''}}, \quad x' = (x'', t)$$

$$L_1' = -\frac{\partial}{\partial t} + a_1(t)\sqrt{-\Delta_{x''}}, \quad x' = (x'', t)$$

The separation of variables form

Special Reduction of the Fredholm Integral Equation

$$L_1 \mu = L_1' \nu$$

$$\nu = N * f_0 \Big|_{\mathbf{R}^{n-1}} ,$$

$$\mu = \left(u - N * f_0\right)\Big|_{\mathbf{R}^{n-1}}$$

Real Analysis

Solution of the Fredholm Pseudo-Differential Equation (1)

$$L_1 \mu = L_1 ' \nu$$

$$\mu(x'',t) = -\nu(x'',t)$$
$$+2\frac{1}{(2\pi)^{n-2}}\int_{\mathbf{R}^{n-2}} e^{ix''\xi''}\left(\int_{-1}^{t} a_1(t')\left|\xi''\right| e^{-\left|\xi''\right|\int_{t'}^{t} a_1(\theta)d\theta}\tilde{\nu}(\xi'',t')dt'\right)d\xi''$$

Pseudo-Differential Operator with Parameter (Treves)

$$K\left(a_1\sqrt{-\Delta_{x''}}\right)v(x'',t) =$$

$$\frac{1}{(2\pi)^{n-2}}\int_{\mathbf{R}^{n-2}} e^{ix''\xi''}\left(\int_{-1}^{t} a_1(t')|\xi''|e^{-|\xi''|\int_{t'}^{t}a_1(\theta)d\theta}\tilde{v}(\xi'',t')dt'\right)d\xi''$$

Pseudo - differential operator in x''

parametrized by t.

$$x' = (x'', t)$$

References

F. Treves: A new method of the subelliptic estimates. Comm. Pure Appl. Math. 24 (1971), 71-115.

Solution of the Fredholm Pseudo-Differential Equation (2)

$$L_1 \mu = L_1' \nu$$

$$\Updownarrow$$

$$\mu(x'',t) = -\nu(x'',t) + K\left(a_1\sqrt{-\Delta_{x''}}\right)\nu(x'',t)$$

Regularity Theorem

$$\mu(x'',t) = -\nu(x'',t) + 2K\left(a_1\sqrt{-\Delta_{x''}}\right)\nu(x'',t)$$

$\mu(x'',t)$ **gains 0-derivatives from** $\nu(x'',t)$

⇕

$$K\left(a_1\sqrt{-\Delta_{x''}}\right) : B^{s,p}(\mathbf{R}^{n-2}\times I) \to B^{s,p}(\mathbf{R}^{n-2}\times I)$$

bounded for $s > -1/2$

Boundedness Theorem (Guan-Sawyer)

$$K\left(a_1\sqrt{-\Delta_{x''}}\right): B^{s,p}(\mathbf{R}^{n-2}_{x''}\times I_t)\to B^{s,p}(\mathbf{R}^{n-2}_{x''}\times I_t)$$

bounded for $s>-1/2$

⇕

$$\left(\frac{1}{\int_s^t a_1(\theta)\,d\theta}\int_s^t a_1(\theta)^{p'}\,d\theta\right)^{p-1}$$

$$\leq C\,\frac{1}{u-t}\int_t^u a_1(\theta)\,d\theta$$

A_p − **condition**

References

(1) B. Muckenhoupt: Hardy's inequality with weights, Studia Math. 44 (1972), 31-38

(2) B. Muckenhoupt : Weighted norm inequalities for the Hardy maximal functions, Trans. A. M. S. 165 (1972), 207-226

(3) E. Sawyer: Weighted inequalities for the one-sided Hardy-Littlewood maximal functions, Trans. A. M. S. 297 (1986), 53-61

Weighted Spaces (Muckenhoupt)

$$L^2(I,w)=\left\{\tilde{h}:\int_{-1}^{1}\left|\tilde{h}(s)\right|^2 w(s)ds<\infty\right\}$$

$$I=(-1,1)$$

$$s = A(t) = \int_0^t a_1(\theta)d\theta$$

$$w(s) = \frac{1}{a_1\left(A^{-1}(s)\right)}$$

Example

$$a(\theta) = \theta^{2k}$$

$$s = \frac{t^{2k+1}}{2k+1}$$

$$w(s) = \frac{1}{(2k+1)^{2k/(2k+1)}} \frac{1}{s^{2k/(2k+1)}}$$

One-sided Hardy-Littlewood Maximal Operator

$$M^- : L^p((0,1),w) \to L^p((0,1),w)$$

bounded

$$M^-\tilde{h}(s) = \sup_{0<\delta<s<1} \frac{1}{\delta}\int_{s-\delta}^{s} \tilde{h}(\theta)\,d\theta$$

$$\tilde{h}(s) = h\left(A^{-1}(s)\right)$$

One-Sided Version of A_p Condition

$$\left(\frac{1}{\delta}\int_t^{t+\delta} w(s)\,ds\right)\left(\frac{1}{\delta}\int_{t-\delta}^{t} w(s)^{1-p'}\,ds\right)^{p-1} \le C$$

$$0 < \delta < t < 1$$

$$s = A(t) = \int_0^t a_1(\theta)\,d\theta$$

$$w(s) = \frac{1}{a_1\left(A^{-1}(s)\right)}$$

Conclusion (Guan-Sawyer)

$$K\left(a_1\sqrt{-\Delta_{x''}}\right): B^{s,p}(\mathbf{R}^{n-2}_{x''}\times I_t)\to B^{s,p}(\mathbf{R}^{n-2}_{x''}\times I_t)$$

bounded for $s>-1/2$

⇕

$$\left(\frac{1}{\int_s^t a_1(\theta)\,d\theta}\int_s^t a_1(\theta)^{p'}\,d\theta\right)^{p-1}$$

$$\le C\,\frac{1}{u-t}\int_t^u a_1(\theta)\,d\theta$$

A_p **− condition**

Examples

$$\left(\frac{1}{\int_s^t a_1(\theta)\,d\theta} \int_s^t a_1(\theta)^{p'}\,d\theta \right)^{p-1}$$

$$\leq C \frac{1}{u-t} \int_t^u a_1(\theta)\,d\theta, \qquad -1 < s < t < u < 1$$

$$a(\theta) = \theta^{2k} : \quad \text{YES}$$

$$a(\theta) = e^{-1/\theta^2} : \quad \text{NO}$$

Agmon's Method

Shmuel Agmon

Shmuel Agmon (1922-)
Israel Mathematician

The Third Idea of Approach

We make use of Agmon's technique of treating a spectral parameter as a second-order elliptic differential operator of an extra variable on the unit circle and relating the old problem to a new one with the additional variable.

References

(1) S. Agmon: Lectures on elliptic boundary value problems. Van Nostrand, Princeton, New Jersey, 1965.

(2) K. Taira: Un theoreme d'existence et d'unicite des solutions pour des problemes aux limites non-elliptiques.

J. Functional Analysis, 43 (1981), 166-192.

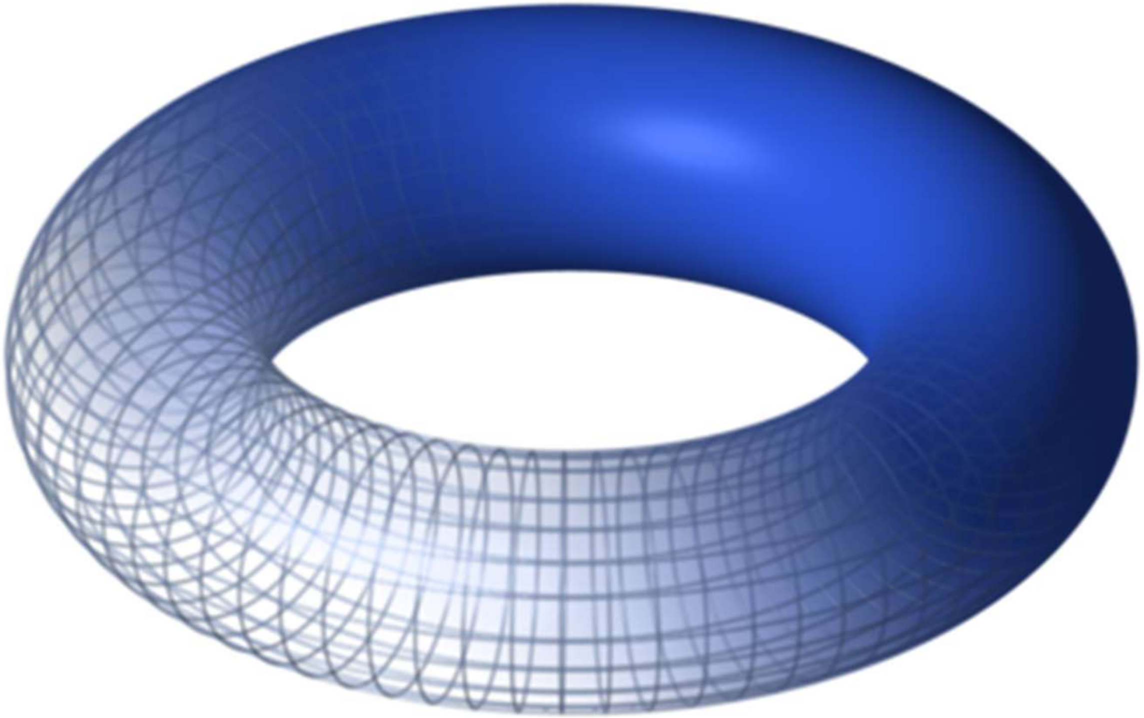
Differential Operator with a Complex Parameter

$$\Delta u - \lambda I = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \lambda I$$

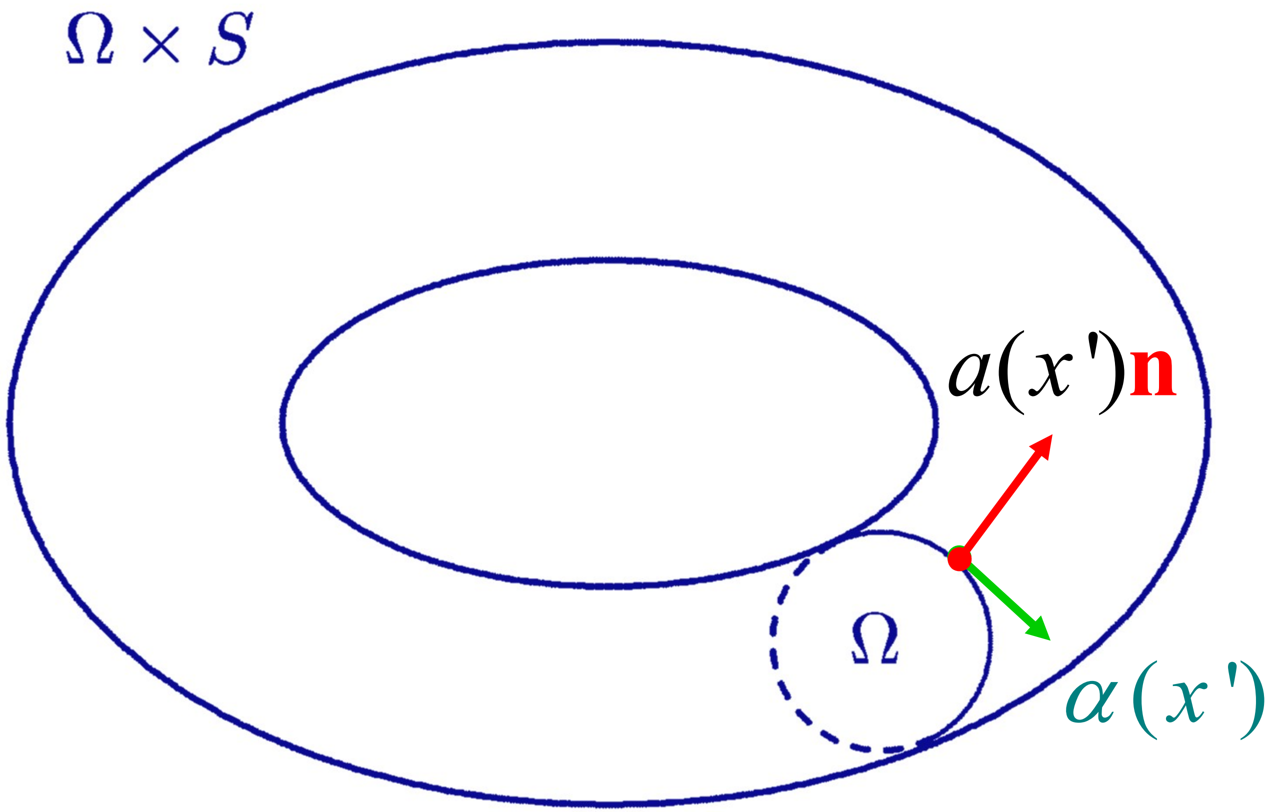
$$\lambda = r^2 e^{i\theta}$$

$$-\pi < \theta < \pi$$

Product Domain



Product Domain



Augmented Strongly Uniform Elliptic Differential Operator

$$\widetilde{\Lambda}(\theta) = \Delta u + e^{i\theta} \frac{\partial^2 u}{\partial y^2}$$

$$= \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + e^{i\theta} \frac{\partial^2 u}{\partial y^2}$$

$$-\pi < \theta < \pi$$

Augmented Oblique Derivative Condition

$$\frac{\partial \tilde{u}}{\partial \nu} := a(x')\frac{\partial \tilde{u}}{\partial \mathbf{n}} + \alpha(x') \cdot \tilde{u} = 0 \quad \text{on } \Gamma \times S$$

$$\nu(x') = a(x')\mathbf{n} + \alpha(x')$$

Indices of the Operators

Closed Realization

We define a **densely definded, closed** operator

$$\widetilde{\mathfrak{A}}_p(\theta) : L^p(\Omega \times S) \to L^p(\Omega \times S) \quad (1 < p < \infty)$$

as follows:

(a) $D(\widetilde{\mathfrak{A}}_p(\theta)) = \left\{ \tilde{u} \in W^{2,p}(\Omega \times S) : \dfrac{\partial \tilde{u}}{\partial \nu} = 0 \right\}$

(b) $\widetilde{\mathfrak{A}}_p(\theta)\tilde{u} = \widetilde{\Lambda}(\theta)\tilde{u}, \ \forall \tilde{u} \in D(\widetilde{\mathfrak{A}}_p(\theta))$

Then $\widetilde{\mathfrak{A}}_p(\theta)$ is a **Fredholm operator**.

Fundamental Relationship for Indices (1981)

$$\operatorname{ind} \widetilde{\mathfrak{A}}_2(\theta) < \infty \qquad \boxed{p = 2}$$

$$\Rightarrow$$

$\exists K = I \cup J$: **finite set** of $\mathbf{Z}$:

$$\boxed{\begin{aligned} \dim N\left(\Delta - \ell^2 e^{i\theta} I\right) = 0 \quad \forall \ell \notin K, \\ \operatorname{codim} R\left(\Delta - \ell^2 e^{i\theta} I\right) = 0 \quad \forall \ell \notin K \end{aligned}}$$

Fundamental Relationship for Kernels (1)

$$N\left(\widetilde{\mathfrak{A}}_2(\theta)\right) = \bigoplus_{\ell \in \mathbf{Z}} N\left(\Delta - \ell^2 e^{i\theta} I\right) \otimes e^{i\ell y}$$

$$\lambda = \ell^2 e^{i\theta}, \quad \ell \in \mathbf{Z}, \; -\pi < \theta < \pi$$

$$p = 2$$

Fundamental Relationship for Kernels (2)

$$N\left(\widetilde{\mathfrak{A}}_2(\theta)\right) = \bigoplus_{\ell \in I} N\left(\Delta - \ell^2 e^{i\theta} I\right) \otimes e^{i\ell y}$$

$$\dim N\left(\widetilde{\mathfrak{A}}_2(\theta)\right) < \infty$$

$$\Leftrightarrow$$

$\exists I$ **finite set** of $\mathbf{Z}$:

$$\dim N\left(\Delta - \ell^2 e^{i\theta} I\right) = 0, \quad \forall \ell \notin I$$

Fundamental Relationship
Cokernels (1)

$$N\left(\widetilde{\mathfrak{A}}_2(\theta)^*\right) = \bigoplus_{\ell \in \mathbf{Z}} N\left(\left(\Delta - \ell^2 e^{i\theta} I\right)^*\right) \otimes e^{i\ell y}$$

$$\lambda = \ell^2 e^{i\theta}, \quad \ell \in \mathbf{Z}, \; -\pi < \theta < \pi$$

$$p = 2$$

Fundamental Relationship
Cokernels (2)

$$N\left(\widetilde{\mathfrak{A}}_2(\theta)^*\right) = \bigoplus_{\ell \in J} N\left(\left(\Delta - \ell^2 e^{i\theta} I\right)^*\right) \otimes e^{i\ell y}$$

$$\operatorname{codim} R\left(\widetilde{\mathfrak{A}}_2(\theta)\right) < \infty$$

$$\Leftrightarrow$$

$\exists J$: **finite set** of $\mathbf{Z}$:

$$\boxed{\operatorname{codim} R\left(\Delta - \ell^2 e^{i\theta} I\right) = 0, \quad \forall \ell \notin J}$$

Banach's closed range theorem

Fundamental Relationship for Indices

$$\operatorname{ind} \widetilde{\mathfrak{A}}_2(\theta)$$

$$= \dim N\left(\widetilde{\mathfrak{A}}_2(\theta)\right) - \operatorname{codim} R\left(\widetilde{\mathfrak{A}}_2(\theta)\right) < \infty$$

$$\Longrightarrow$$

$\exists K = I \cup J$: **finite set** of $\mathbf{Z}$:

$$\begin{cases} \dim N\left(\Delta - \ell^2 e^{i\theta} I\right) = 0 & \forall \ell \notin K, \\ \operatorname{codim} R\left(\Delta - \ell^2 e^{i\theta} I\right) = 0 & \forall \ell \notin K \end{cases}$$

Index Formula (1)

$\forall \ell \notin K = I \cup J$: **finite set of Z** :

$$\mathrm{ind}\left(\mathfrak{A}_2 - \ell^2 e^{i\theta} I\right) = 0$$

$$p = 2$$

Rellich-Kondrachov Theorem

The injection

$$W^{2,p}(\Omega) \rightarrow L^p(\Omega)$$

is compact.

Compact perturbation

Index Formula (2)

$$\operatorname{ind}\left(\mathcal{U}_p - \lambda I\right) = \operatorname{ind}\left(\mathcal{U}_2 - \lambda I\right)$$

$$= 0, \quad \forall \lambda \in \mathbf{C}$$

$$1 < p < \infty$$

Closed Realization

We define a **densely defined, closed** operator

$\mathfrak{A}_p : L^p(\Omega) \to L^p(\Omega) \quad (1 < p < \infty)$

as follows:

(a) $D(\mathfrak{A}_p) = \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \right\}$

(b) $\mathfrak{A}_p u = \Delta u, \ \forall u \in D(\mathfrak{A}_p)$

Then $\mathfrak{A}_p - \lambda I$ is a **Fredholm operator**

with **index zero** for $\forall \lambda \in \mathbf{C}$.

A priori Estimates for the Subelliptic Oblique Derivative Problem

A priori Estimate (1)

$$\left\| \tilde{u} \right\|_{W^{2,p}(\Omega\times S)} \leq \widetilde{C}(\theta)\left(\left\| \widetilde{\Lambda}(\theta)\tilde{u} \right\|_{L^p(\Omega\times S)} + \left\| \tilde{u} \right\|_{L^p(\Omega\times S)} \right)$$

$$\forall \tilde{u} \in D(\widetilde{\mathfrak{A}}_p(\theta))$$

$$\widetilde{\Lambda}(\theta) = \Delta u + e^{i\theta} \frac{\partial^2 u}{\partial y^2}$$

Domains of Definition

$$D(\mathfrak{A}_p) = \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \right\}$$

$$D(\widetilde{\mathfrak{A}}_p(\theta)) = \left\{ \tilde{u} \in W^{2,p}(\Omega \times S) : \frac{\partial \tilde{u}}{\partial \nu} = 0 \right\}$$

Localization Function

$$(1)\ \ \zeta(y) \in C^{\infty}(S)$$

$$(2)\ \operatorname{supp}\zeta \subset \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$$

$$(3)\ \zeta(y) = 1,\ \ \forall y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

Product Functions

$$\boxed{\tilde{v}_\eta(x, y) = u(x) \otimes \zeta(y) e^{i\eta y}}$$

$$u \in D(\mathfrak{A}_p),\ \eta \geq 0$$

$$\Rightarrow$$

$$\tilde{v}_\eta(x, y) \in H^{2,p}(\Omega \times S)$$

$$\frac{\partial}{\partial \nu}\left(\tilde{v}_\eta(x, y)\right) = \frac{\partial u}{\partial \nu}(x) \otimes \zeta(y) e^{i\eta y} = 0$$

A priori Estimates (2)

$$\tilde{v}_\eta(x,y) = u(x) \otimes \zeta(y) e^{i\eta y} \in D(\tilde{\mathfrak{A}}_p(\theta))$$

$$\Downarrow$$

$$\left\| \tilde{v}_\eta \right\|_{W^{2,p}(\Omega \times S)} \leq \tilde{C}(\theta) \left(\left\| \tilde{\Lambda}(\theta) \tilde{v}_\eta \right\|_{L^p(\Omega \times S)} + \left\| \tilde{v}_\eta \right\|_{L^p(\Omega \times S)} \right)$$

A priori Estimates (3)

$$\left\|\tilde{v}_\eta\right\|_{W^{2,p}(\Omega\times S)} \leq \widetilde{C}(\theta)\left(\left\|\widetilde{\Lambda}(\theta)\tilde{v}_\eta\right\|_{L^p(\Omega\times S)} + \left\|\tilde{v}_\eta\right\|_{L^p(\Omega\times S)}\right)$$

$$\left|u\right|_{2,p} + \sqrt{|\lambda|}\left|u\right|_{1,p} + |\lambda|\left\|u\right\|_p \leq \widetilde{C}'(\theta)\left\|(\Delta-\lambda)u\right\|_p$$

$$\lambda = \eta^2 e^{i\theta}$$

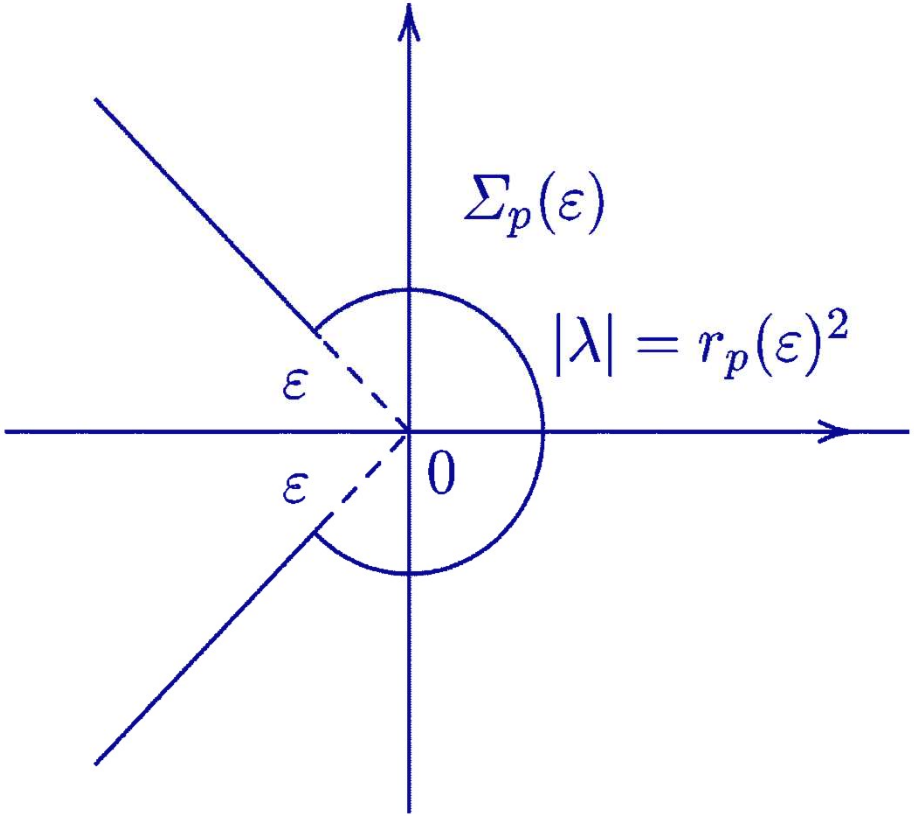
Resolvent Estimates

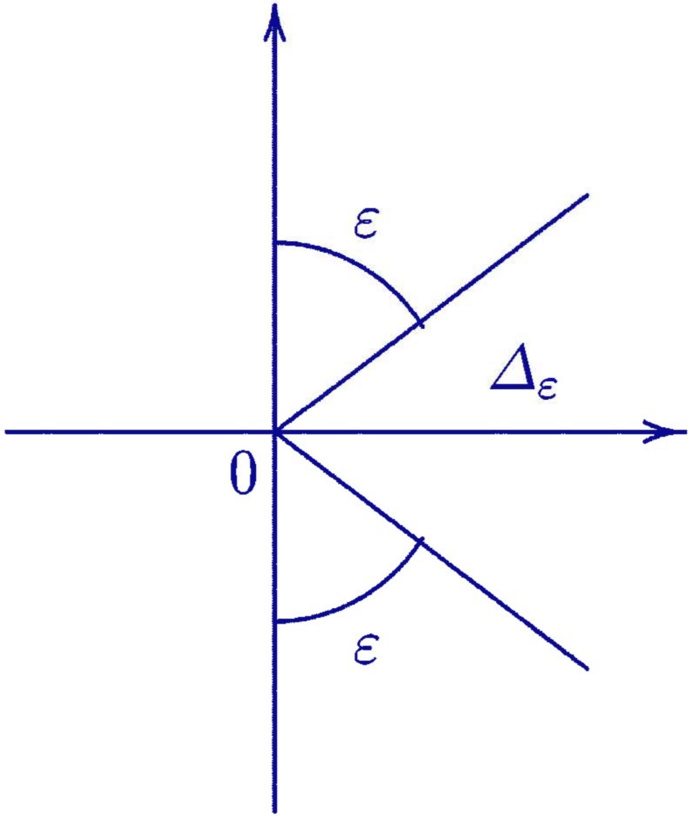
$$\left\| \left(\mathfrak{A}_p - \lambda I \right)^{-1} f \right\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|} \| f \|_{L^p(\Omega)}$$

$$\forall f \in L^p(\Omega)$$

⇑

$$\operatorname{ind} \mathfrak{A}_p = \operatorname{ind} \mathfrak{A}_2 = 0$$





$$\lambda = r^2 e^{i\theta}$$
$$-\pi + \varepsilon \le \theta \le \pi - \varepsilon$$

$$e^{z\mathfrak{A}_p} \quad (z \in \Delta_\varepsilon)$$

Main Idea
via
the Resolvent

Essential Points

(1) The **subelliptic** case:

We cannot use **Green's formula** in order to characterize the **adjoint operator** $\mathfrak{A}_2^*$.

(2) We shift our attention to the **resolvent** $\mathfrak{A}_2^{*-1}$, instead of $\mathfrak{A}_2^*$.

Boutet de Monvel Calculus

Louis Boutet de Monvel

◆**Louis Boutet de Monvel (1941-2014)**
French Mathematician

Boutet de Monvel Calculus (General form)

$$\mathfrak{A} = \begin{pmatrix} P+G & K \\ T & Q \end{pmatrix}$$

Representation Formula of the Resolvent

Representation of the Solution

$$(1-\Delta)u = f \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma$$

$\Leftrightarrow$

$$u = (I - \mathfrak{A}_2)^{-1} f$$

$$= r^+ G_2 f$$

$$- P_2 \left(S \left(\alpha(x') - a(x') \Pi_2^{ext} \right) \left(r^+ G_2 f |_\Gamma \right) \right)$$

Representation of the Resolvent

$$u = r^{+}G_2 f - P_2\left(S\left(\alpha(x') - a(x')\Pi_2^{\mathrm{ext}}\right)\left(r^{+}G_2 f\big|_{\Gamma}\right)\right)$$

$$= r^{+}G_2 f - P_2\left(R\left(r^{+}G_2 f\big|_{\Gamma}\right)\right)$$

S : **Inverse** of $T_2 = \dfrac{\partial P_2}{\partial \nu}$

$$R = S\left(\alpha(x') - a(x')\Pi_2^{\mathrm{ext}}\right)$$

Boutet de Monvel Calculus (1)

$$\mathfrak{A} = \begin{pmatrix} r^{+}G_{2} & -P_{2} \\ \gamma_{0}\left(r^{+}G_{2}\right) & R \end{pmatrix}$$

Boutet de Monvel Calculus (2)

$$\mathfrak{A}: \begin{array}{c} W^{s,p}(\Omega) \\ \oplus \\ B^{s+2-1/p,p}(\Gamma) \end{array} \to \begin{array}{c} W^{s+2,p}(\Omega) \\ \oplus \\ B^{s+2-1/p,p}(\Gamma) \end{array}$$

Bird's-Eye View

$$\begin{array}{ccc} & \gamma_{\mathbf{0}}\left(r^{+}G_{2}\right)=\gamma_{0}G_{2} & \\ W^{s,p}(\Omega) & \longrightarrow & B^{s+2-1/p,p}(\Gamma) \\ r^{+}G_{2}\downarrow & & \downarrow R \\ W^{s+2,p}(\Omega) & \longleftarrow & B^{s+2-1/p,p}(\Gamma) \\ & -P_{2} & \end{array}$$

Regularity of the Resolvent

$$\left(I - \mathfrak{A}_2 \right)^{-1} : W^{s,p}(\Omega) \to W^{s+2,p}(\Omega)$$

$$\left(I - \mathfrak{A}_2 \right)^{-k} : L^{p}(\Omega) \to W^{2k,p}(\Omega), \quad \forall k \in \mathbf{N}$$

Boutet de Monvel Calculus (3)

$$\mathfrak{A}^* = \begin{pmatrix} \left(r^+ G_2\right)^* & \left(\gamma_0\left(r^+ G_2\right)\right)^* \\ -P_2^* & R^* \end{pmatrix}$$

Boutet de Monvel Calculus (4)

$$\mathfrak{A}^* : \begin{array}{c} W^{s,p}(\Omega) \\ \oplus \\ B^{s+1-1/p,p}(\Gamma) \end{array} \rightarrow \begin{array}{c} W^{s+2,p}(\Omega) \\ \oplus \\ B^{s+1-1/p,p}(\Gamma) \end{array}$$

Bird's-Eye View

$$\begin{array}{ccc} W^{s,p}(\Omega) & \xrightarrow{-P_2^*} & B^{s+1-1/p,p}(\Gamma) \\ (r^+G_2)^* \downarrow & & \downarrow R^* \\ W^{s+2,p}(\Omega) & \xleftarrow[(\gamma_0 G_2)^*]{} & B^{s+1-1/p,p}(\Gamma) \end{array}$$

Regularity of the Adjoint of the Resolvent

$$\left(I-\mathfrak{A}_2^*\right)^{-1}: W^{s,p}(\Omega) \rightarrow W^{s+2,p}(\Omega)$$

$$\Downarrow$$

$$\left(I-\mathfrak{A}_2^*\right)^{-k}: L^p(\Omega) \rightarrow W^{2k,p}(\Omega), \quad \forall k \in \mathbf{N}$$

END