Duke Mathematical Journal manuscript No.
(will be inserted by the editor)

Semigroups and boundary value problems

Kazuaki Taira

October 29, 1981

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1 Introduction

Let D be a bounded domain in Euclidean space $\mathbf{R}^N$ with smooth boundary ∂D and let $C(\overline{D})$ be the space of real-valued, continuous functions on the closure $\overline{D} = D \cup \partial D$.

A strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on the Banach space $C(\overline{D})$ is called a *Feller semigroup* on $\overline{D}$ if it satisfies the condition

$$f \in C(\overline{D}),\ 0 \leq f \leq 1 \text{ on } \overline{D} \implies 0 \leq T_t f \leq 1 \text{ on } \overline{D}.$$

It is known (cf. [2], [5], [26]) that there corresponds to a Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D}$ a *strong Markov process* $\mathscr{X}$ on $\overline{D}$ whose transition function $p(t,x,dy)$ satisfies the formula

$$T_t f(x) = \int_{\overline{D}} f(y)\, p(t,x,dy) \quad \text{for all } f \in C(\overline{D}), \tag{1.1}$$

and further that, under certain continuity hypotheses concerning the transition function $p(t,x,dy)$ such as

$$\lim_{t \downarrow 0} \int_{|y-x|>\varepsilon} p(t,x,dy) = 0 \quad \text{for all } \varepsilon > 0 \text{ and } x \in \overline{D}, \tag{1.2}$$

Kazuaki Taira
The Institute for Advanced Study, Princeton, New Jersey 08540, U.S.A.
Institute of Mathematics, University of Tsukuba, Tsukuba 305–8571, Japan
E-mail: taira@math.tsukuba.ac.jp

the infinitesimal generator $\mathfrak{A}$ of $\{T_t\}_{t\geq 0}$ is described analytically as follows:

(i) Let x be a (fixed) point of the *interior* D of the domain. For every C^2-function $u \in D(\mathfrak{A})$ of $\mathfrak{A}$, by expanding $u(y)-u(x)$, we obtain from formulas (1.1) and (1.2) that

$$\begin{aligned}
&\mathfrak{A}u(x) \qquad (1.3)\\
&= \lim_{t\downarrow 0} \frac{T_t u(x)-u(x)}{t}\\
&= \lim_{t\downarrow 0} \frac{1}{t}\left(\int_{\overline{D}} P(t,x,dy)\,u(y)-u(x)\right)\\
&= \lim_{t\downarrow 0} \left[\frac{1}{t}\int_{\overline{D}} P(t,x,dy)\,(u(y)-u(x)) + \frac{1}{t}\left(\int_{\overline{D}} P(t,x,dy)-1\right)u(x)\right]\\
&= \lim_{t\downarrow 0} \left\{ \frac{1}{t}\left(\int_{|y-|\leq\varepsilon} P(t,x,dy)-1\right)u(x)\right.\\
&\quad + \sum_{i=1}^{N} \frac{1}{t}\int_{|y-|\leq\varepsilon} (y_i-x_i)\,P(t,x,dy)\,\frac{\partial u}{\partial x_i}(x)\\
&\quad + \sum_{i,j=1}^{N} \frac{1}{t}\int_{|y-|\leq\varepsilon} (y_i-x_i)\,(y_j-x_j)\,P(t,x,dy)\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x)\\
&\quad \left. + \text{ remainder terms}\right\}\\
&= c(x)u(x) + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i}(x) + \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x).
\end{aligned}$$

Here the limits

$$\begin{aligned}
c(x) &:= \lim_{t\downarrow 0} \frac{1}{t}\left(\int_{|y-|\leq\varepsilon} P(t,x,dy)-1\right),\\
b^i(x) &:= \lim_{t\downarrow 0} \frac{1}{t}\int_{|y-|\leq\varepsilon} (y_i-x_i)\,P(t,x,dy),\\
a^{ij}(x) &:= \lim_{t\downarrow 0} \frac{1}{t}\int_{|y-|\leq\varepsilon} (y_i-x_i)\,(y_j-x_j)\,P(t,x,dy)
\end{aligned}$$

exist independently of sufficiently small $\varepsilon > 0$ and satisfy the conditions

1° $c(x) \leq 0$.
2° $a^{ij}(x) = a^{ji}(x)$ and

$$\sum_{i,j=1}^{N} a^{ij}(x)\,\xi_i\,\xi_j \geq 0 \quad \text{for all } \xi = (\xi_1,\xi_2,\ldots,\xi_N) \in \mathbf{R}^N.$$

If we let

$$Au(x) := \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x), \tag{1.4}$$

then we have, by formula (1.3),

$$\mathfrak{A}u(x) = Au(x) \quad \text{for every } u \in D(\mathfrak{A}) \cap C^2(D). \tag{1.5}$$

(ii) Similarly, for a fixed point x' of the boundary ∂D of the domain, by choosing a system $x = (x_1, x_2, \ldots, x_{N-1}, x_N)$ of local coordinates as $x \in D$ if $x_N > 0$ and $x \in \partial D$ if $x_N = 0$, we then have the formula

$$\begin{aligned} &Lu(x') \\ &= \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i}(x') + \gamma(x') u(x') \\ &\quad + \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x') A u(x') \\ &= 0 \quad \text{for every } u \in D(\mathfrak{A}) \cap C^2(\overline{D}). \end{aligned} \tag{1.6}$$

Here:

1° $\alpha^{ij}(x') = \alpha^{ji}(x')$ and

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \xi_i \xi_j \geq 0 \quad \text{for all } \xi' = (\xi_1, \xi_2, \ldots, \xi_{N-1}) \in \mathbf{R}^{N-1}.$$

2° $\gamma(x') \leq 0$.
3° $\mu(x') \geq 0$.
4° $\delta(x') \geq 0$.
5° $\mathbf{n}$ is the unit inward normal to the boundary ∂D at x'.

The condition L is called a *Ventcel's boundary condition*.

Probabilistically, the above result may be interpreted as follows. A Markovian particle in the diffusion process (strong Markov process with continuous paths) $\mathscr{X}$ on $\overline{D}$ is governed by the operator A in the interior D of the domain, and it obeys the condition L on the boundary ∂D of the domain. Note that the terms

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial}{\partial x_i},$$

$$\gamma(x') u, \quad \mu(x') \frac{\partial u}{\partial \mathbf{n}}, \quad \delta(x') A u$$

of L are supposed to correspond to the diffusion along the boundary, absorption, reflection and viscosity phenomena, respectively.

Analytically, via the celebrated Hille–Yosida theorem in the theory of semigroups, it may be interpreted as follows. A Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D}$ is described by a degenerate elliptic differential operator A of second-order and a Ventcel's boundary condition L if the paths of its corresponding strong Markov process $\mathscr{X}$ are continuous. Hence we are reduced to the study of non-elliptic boundary value problems for (A, L) in the theory of partial differential equations.

We are interested in the following:

Problem 1 Conversely, given analytic data (A, L), can we construct a Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D}$?

In the case $N = 1$, this problem is solved completely both from probabilistic and analytic viewpoints by Feller [7], [8], Dynkin [4], Itô–McKean Jr. [14] and Ray [18]. So we shall consider the multi-dimensional case $N \geq 2$.

In [19], Sato and Ueno studied the case when the operator A is *elliptic* on $\overline{D}$ and proved that there exists a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ if the boundary value problem

$$\begin{cases} (\alpha - A)u = 0 & \text{in } D, \\ (\lambda - L)u = \varphi & \text{on } \partial D \end{cases} \tag{1.7}$$

is solvable for sufficiently many functions φ in $C(\partial D)$. Here α and λ are non-negative parameters.

One main purpose of this paper is to generalize their results to the *non-elliptic* case (Theorem 2.1 and Corollary 2.1). Intuitively, our non-ellipticity hypothesis concerning the operator A is stated as follows (see hypothesis (H)):

A Markovian particle governed by the operator A (A-diffusion) diffuses everywhere in D and exits $\overline{D} = D \cup \partial D$ *through any point of* ∂D *in finite time*. (1.8)

The probabilistic meaning of the condition that the boundary value problem (1.7) is solvable for sufficiently many functions φ in $C(\partial D)$ is that there exists a strong Markov process $\mathscr{Y}$ (with discontinuous paths) on ∂D. So, by hypothesis (1.8) we can "piece out" the Markov process $\mathscr{Y}$ with A-diffusion in the interior D to construct a strong Markov process $\mathscr{X}$ on the closure $\overline{D} = D \cup \partial D$ and hence a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$. This should seem to be very close to a probabilistic method of construction of diffusion processes by Watanabe [25].

On the other hand, in [2], Bony, Courrège and Priouret proved that, under the ellipticity condition on the operator A, if either the matrix $\left(\alpha^{ij}(x')\right)$ is positive definite on ∂D or if $\left(\alpha^{ij}(x')\right) \equiv 0$ and $\mu(x') > 0$ on ∂D, then there exists a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ whose infinitesimal generator $\mathfrak{A}$ satisfies conditions (1.5) and (1.6). Intuitively, their results imply that if either a Markovian particle diffuses everywhere along the boundary or if it reflects always at the boundary, then there exists a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ corresponding to such a diffusion phenomenon.

In [21], the author generalized their results to the case where the matrix $\left(\alpha^{ij}(x')\right)$ is non-negative definite on ∂D and $\mu(x') \geq 0$ on ∂D, under some hypothesis concerning the boundary condition L. However, the intuitive meaning of this hypothesis is not so clear from the probabilistic viewpoint.

The other purpose of this paper is to prove that, under the ellipticity condition on the operator A, if (Hypothesis (A))

A Markovian particle goes through the set $M = \{x' \in \partial D : \mu(x') = 0\}$, where no reflection phenomenon occurs, in finite time, (1.9)

then there exists a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ corresponding to such a diffusion phenomenon (Theorem 2.2), which is an improvement on the result of [21].

We sum up the contents of this paper briefly. In Section 2, we state general existence theorems for Feller semigroups $\{T_t\}_{t\geq 0}$ on $\overline{D}$ as Theorem 2.1 and Corollary 2.1 and further, as a simple application of these results to the *elliptic* case, we state Theorem 2.2. In Section 7, we explain the reason why we confine ourselves to the elliptic case. The proofs of Theorem 2.1 and Corollary 2.1 are based on versions of the Hille–Yosida theorem and they are carried out in Section 4 just as in the elliptic case studied by Sato–Ueno [19] and by Bony–Courrège–Priouret [2], if we use results of Stroock–Varadhan [20] on the Dirichlet problem for non-elliptic operators satisfying such hypothesis as (1.8) instead of classical results in the elliptic case. Theorem 2.2 is proved in Section 5 by showing that, under the ellipticity condition on the operator A, if such hypothesis as (1.9) is satisfied, then the boundary value problem

$$\begin{cases} (\alpha - A)u = 0 & \text{in } D, \\ Lu = \varphi & \text{on } \partial D \end{cases} \tag{$*$}$$

has a unique solution u in $C^\infty(\overline{D})$ for any $\varphi \in C^\infty(\partial D)$. Here α is a positive spectral parameter. As in [21], the proof of this unique and existence theorem for problem $(*)$ is based on the maximum principle and versions of the *a priori* estimates used by Oleĭnik–Radkevič [17] and by Hörmander [13] in studying the hypoellipticity of pseudo-differential operators with non-negative principal symbols. We make use of these estimates, on one hand, to prove the regularity theorem for problem $(*)$ and, on the other hand, to show that problem $(*)$ has index zero, by using a method essentially due to Agmon–Nirenberg [1]. By the regularity theorem and the maximum principle, we have the uniqueness theorem and hence the existence theorem for problem $(*)$, since problem $(*)$ has index *zero*. The fundamental *a priori* estimates are proved separately in Section 6 because of the length of their proof. In Section 3, we summarize basic results such as versions of the Hille–Yosida theorem in the theory of semigroups, the uniqueness and existence theorem for the Dirichlet problem and the maximum principle for non-elliptic operators from the probabilistic viewpoint, and an interpretation of boundary conditions in terms of distributions.

A summary of this paper is given in [23].

The author would like to express his hearty thanks to Charles Rockland who kindly read through the original version of this paper and suggested many revisions and corrections. He is also indebted to Junjiro Noguchi for formulating hypothesis (A) in terms of differential geometry.

2 Statement of results

We start by stating general existence theorems for Feller semigroups $\{T_t\}_{t\geq 0}$ on $\overline{D}$ in terms of boundary value problems for (A,L) in the case where the operator A is *non-elliptic* on $\overline{D}$. In the elliptic case, similar results are obtained by Sato–Ueno [19] and also by Bony–Courrège–Priouret [2].

For the differential operator A given by formula (1.4), assume that there exists an open subset G of $\mathbf{R}^N$, containing $\overline{D}$, such that the coefficients of A satisfy the following conditions:

(1) $a^{ij} \in C^\infty(G)$, $a^{ij}(x) = a^{ji}(x)$ for all $x \in G$ and $1 \leq i, j \leq N$, and

$$\sum_{i,j=1}^{N} a^{ij}(x)\,\xi_i\,\xi_j \geq 0 \quad \text{for all } x \in G \text{ and } \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbf{R}^N. \tag{2.1}$$

(2) $b^i \in C^\infty(G)$ for $1 \leq i \leq N$.
(3) $c \in C^\infty(G)$ and $c(x) \leq 0$ in D.

The fundamental hypothesis concerning the operator A is the following:

The Lie algebra $\mathscr{L}(X_1, X_2, \dots, X_N)$ over $\mathbf{R}$ generated by the vector fields (H)

$$X_i = \sum_{j=1}^{N} a^{ij}(x)\frac{\partial}{\partial x_j}, \quad 1 \leq i \leq N,$$

over $\mathbf{R}$ *has rank N at every point of D and the boundary* ∂D *is non-characteristic with respect to the operator A, that is*,

$$\sum_{i,j=1}^{N} a^{ij}(x') n_i n_j > 0 \quad \text{on } \partial D.$$

Here $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit inward normal to the boundary ∂D at x'.

The intuitive meaning of hypothesis (H) is that a Markovian particle starting at any point of D can diffuse *everywhere* in D and exit the closure $\overline{D} = D \cup \partial D$ through *any* point of ∂D in finite time (cf. Remark 3.1). From the viewpoint of the theory of partial d1fferential equations, the hypothesis that rank $\mathscr{L}(X_1 1, X_2, \ldots, X_N) = N$ in D is a sufficient condition for the operator A to be *hypoelliptic* in D (see [17]), while the hypothesis that

$$\sum_{i,j=1}^{N} a^{ij}(x') n_i n_j > 0 \quad \text{on } \partial D$$

is a sufficient condition for the operator A to be *partially hypoelliptic* with respect to ∂D ([11]).

Assume that the coefficients of the Ventcel's boundary condition L given by formula (1.6) satisfy the following conditions:

(1) The $\alpha^{ij}(x')$ are the components of a C^∞ symmetric contravariant tensor of type $\binom{2}{0}$ on the boundary ∂D and

$$\sum_{i,j=1}^{N} \alpha^{ij}(x') \xi_i \xi_j \geq 0 \quad \text{for all } x' \in \partial D \text{ and } \xi' \in T^*_{x'}(\partial D). \tag{2.2}$$

Here $T^*_{x'}(\partial D)$ is the cotangent space of ∂D at x'.

(2) $\beta^i \in C^\infty(\partial D)$ for $1 \leq i \leq N-1$.
(3) $\gamma \in C^\infty(\partial D)$ and $\gamma(x') \leq 0$ on ∂D.
(4) $\mu \in C^\infty(\partial D)$ and $\mu(x') \geq 0$ on ∂D.
(5) $\delta \in C^\infty(\partial D)$ and $\delta(x') \geq 0$ on ∂D.

In constructing a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$, we shall make use of a class $\{S_t^\alpha\}_{t\geq 0}$ ($\alpha \geq 0$) of Feller semigroups on the boundary ∂D (cf. Remark 4.1). For this purpose, we introduce the following:

Definition 2.1 A Ventcel's boundary condition L is said to be *transversal* on the boundary ∂D if it satisfies the condition

$$\mu(x') + \delta(x') > 0 \quad \text{on } \partial D. \tag{2.3}$$

Intuitively, the transversality condition (2.3) implies that either reflection or viscosity phenomenon occurs on the boundary ∂D.

By virtue of the transversality condition (2.3), we find that a Markovian particle starting at any point of ∂D does not stay in the boundary ∂D all the time and enters the interior D some time or other. Probabilistically, this means that a Markov process on ∂D is the "trace" on the boundary of trajectories of a Markov process on $\overline{D}$ (see [24]).

First, we state the following:

Theorem 2.1 *Let the differential operator A satisfy conditions* (2.1) *and let the boundary condition L satisfy conditions* (2.2). *Assume that hypothesis (H) is satisfied and that L is* transversal *on ∂D, and further that the following two conditions are satisfied:*

(I) (the existence) For some constants $\alpha \geq 0$ and $\lambda \geq 0$, the boundary value problem

$$\begin{cases} (\alpha - A)u = 0 & \text{in } D, \\ (\lambda - L)u = \varphi & \text{on } \partial D \end{cases} \tag{2.4}$$

has a solution $u \in C^\infty(\overline{D})$ for any $\varphi \in C^\infty(\partial D)$.

(II) (the uniqueness) For some constant $\alpha > 0$, we have the assertion

$$\begin{aligned} & u \in C(\overline{D}), \ (\alpha - A)u = 0 \text{ in } D, \ Lu = 0 \text{ on } \partial D \\ & \implies u = 0 \text{ in } D. \end{aligned}$$

Then there exists a Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D}$ whose infinitesimal generator $\mathfrak{A}$ is characterized as follows:

(a) The domain $D(\mathfrak{A})$ of $\mathfrak{A}$ is the space

$$D(\mathfrak{A}) = \left\{ u \in C(\overline{D}) : Au \in C(\overline{D}), \ Lu = 0 \text{ on } \partial D \right\}. \tag{2.5}$$

(b) $\mathfrak{A}u = Au$ for every $u \in D(\mathfrak{A})$.

Remark 2.1 In Theorem 2.1, Au is taken in the sense of distributions and the boundary condition Lu can be defined as a distribution on ∂D for $u \in C(\overline{D})$ such that $Au \in C(\overline{D})$, since the boundary ∂D is non-characteristic with respect to the operator A (cf. Subsection 3.5).

In general, there is a close relationship between the uniqueness and regularity properties of solutions of boundary value problems. Indeed, we shall obtain the following:

Corollary 2.1 *Let A and L be as in Theorem 2.1. Assume that condition (I) and the following condition (replacing condition (II)) are satisfied:*

(III) (the regularity) For some constant $\alpha > 0$, we have the assertion

$$\begin{aligned} & u \in C(\overline{D}), \ (\alpha - A)u = 0 \text{ in } D, \ Lu \in C^\infty(\partial D) \\ & \implies u \in C^\infty(\overline{D}). \end{aligned}$$

Then there exists a Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D}$ whose infinitesimal generator $\mathfrak{A}$ satisfies condition (2.5) *and coincides with the minimal closed extension in $C(\overline{D})$ of the restriction of A to the space $\{u \in C^2(\overline{D}) : Lu = 0 \text{ on } \partial D\}$.*

As a simple application of Corollary 2.1, we consider the case where the differential operator A is *elliptic* on $\overline{D}$, that is, there exists a constant $c_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\, \xi_i \, \xi_j \geq c_0 \, |\xi|^2 \quad \text{for all } x \in \overline{D} \text{ and } \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbf{R}^N,$$

since $\overline{D}$ is compact.

To state a hypothesis concerning the boundary condition L, we introduce some notation and definitions.

For the coefficients $\alpha^{ij}(x')$ of L, we let

$$\Phi = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j},$$

which lies in the space $\Gamma(\partial D, T(\partial D) \otimes_S T(\partial D))$ of C^∞ symmetric contravariant tensor fields of type $\binom{2}{0}$ on ∂D. Here $\otimes_S$ stands for the symmetric tensor product. Denote by $\Gamma(\partial D, T^*(\partial D))$ and $\Gamma(\partial D, T(\partial D))$ the space of C^∞ covariant vector fields and contravariant vector fields on ∂D, respectively. Then, by making use of Φ, we can define a mapping

$$\Psi : \Gamma(\partial D, T^*(\partial D)) \longrightarrow \Gamma(\partial D, T(\partial D))$$

by the formula

$$\Psi(\zeta') = \Phi\left(\zeta', \cdot\right) \quad \text{for every } \zeta' \in \Gamma(\partial D, T^*(\partial D)).$$

In terms of a local coordinate $x' = (x_1, x_2, \ldots, x_{N-1})$ on ∂D, we have the formula

$$\zeta' = \sum_{i=1}^{N-1} \zeta_i \, dx_i \longmapsto \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \, \zeta_i \frac{\partial}{\partial x_j}.$$

We let

$$\begin{aligned} \mathscr{Y} &= \text{the image of } \Psi \\ &= \left\{ \Psi\left(\zeta'\right) : \zeta' \in \Gamma(\partial D, T^*(\partial D)) \right\}. \end{aligned}$$

The fundamental hypothesis concerning the boundary condition L is the following:

The Lie algebra $\mathscr{L}(\mathscr{Y})$ over $\mathbf{R}$ generated by $\mathscr{Y}$ has rank $N-1$ at every point of the set $M = \{x' \in \partial D : \mu(x') = 0\}$. (A)

The intuitive meaning of hypothesis (A) is that a Markovian particle starting at any point of the set M, where no reflection phenomenon occurs, can *exit* M in finite time (cf. Remark 3.1).

Now we can state the main result, which is an improvement on [22, Théorème 1]:

Theorem 2.2 *Assume that the differential operator A satisfies conditions* (2.1) *and the boundary condition L satisfies conditions* (2.2)*, respectively. If A is* elliptic *on $\overline{D}$ and if L is* transversal *on ∂D and hypothesis (A) is satisfied, then we have the conclusion of Corollary 2.1.*

3 Theory of Feller semigroups

In this section, we summarize basic results such as versions of the Hille–Yosida theorem in the theory of semigroups, the uniqueness and existence theorem for the Dirichlet problem and the maximum principle for non-elliptic operators from the probabilistic viewpoint, and an interpretation of boundary conditions in terms of distributions.

3.1 Definition of a Feller semigroup

First, we give the precise definition of Feller semigroups (cf. [5]):.

Definition 3.1 Let K be a *compact* metric space and let $C(K)$ be the space of real-valued, continuous functions on K with norm

$$\|f\| = \max_{x \in K} |f(x)|.$$

A family $\{T_t\}_{t \geq 0}$ of bounded linear operators on $C(K)$ is called a *Feller semigroup* on K if it satisfies the following three conditions:

(i) $T_t \cdot T_s = T_{t+s}$ for all t, $s \geq 0$ and $T_0 =$ the identity.
(ii) $\{T_t\}$ is strongly continuous in t on the interval $[0, \infty)$, that is,

$$\lim_{t \downarrow 0} \|T_{t+s} f - T_s f\| = 0 \quad \text{for every } f \in C(K) \quad (0 < s < \infty).$$

(iii) $\{T_t\}$ is non-negative and contractive on $C(K)$, that is,

$$f \in C(K),\ 0 \leq f \leq 1 \text{ on } K \implies 0 \leq T_t f \leq 1 \text{ on } K.$$

3.2 Generation theorems for Feller semigroups

We state versions of the Hille–Yosida theorem which will play a fundamental role in the construction of Feller semigroups in Section 4 ([2], [10], [19], [27]):

Theorem 3.1 (Hille–Yosida) *(i) Let $\{T_t\}_{t \geq 0}$ be a Feller semigroup on $\overline{D}$. Its infinitesimal generator $|mathfrakA : C(\overline{D}) \to C(\overline{D})$ is defined by the formula*

$$\mathfrak{A} u = \lim_{t \downarrow 0} \frac{T_t f - f}{t} \quad \text{in } C(\overline{D}). \tag{3.1}$$

Here the domain $D(\mathfrak{A})$ of $\mathfrak{A}$ consists of all $f \in C(\overline{D})$ for which the limit in formula (3.1) *exists.*

Then the generator $\mathfrak{A}$ satisfies the following conditions:

(a) The domain $D(\mathfrak{A})$ is dense in $C(\overline{D})$.
(b) For each $\alpha > 0$, the equation $(\alpha - \mathfrak{A})u = f$ has a unique solution $u \in C(\overline{D})$ for any $f \in C(\overline{D})$. Hence, for each $\alpha > 0$, the Green operator $(\alpha - \mathfrak{A})^{-1} : C(\overline{D} \to C(\overline{D})$ can be defined by the formula

$$u = (\alpha - \mathfrak{A})^{-1} f \quad \text{for every } f \in C(\overline{D}).$$

(c) The operator $(\alpha - \mathfrak{A})^{-1}$ for $\alpha > 0$ is non-negative on $C(\overline{D})$, that is,

$$f \in C(\overline{D}),\ f \geq 0 \text{ on } \overline{D} \implies (\alpha - \mathfrak{A})^{-1} f \geq 0 \text{ on } \overline{D}.$$

(d) The operator $(\alpha - \mathfrak{A})^{-1}$ is bounded on $C(\overline{D})$ with norm

$$\left\|(\alpha - \mathfrak{A})^{-1}\right| \leq \frac{1}{\alpha} \quad \text{for } \alpha > 0.$$

(ii) Conversely, if $\mathfrak{A}$ is a linear operator on $C(\overline{D})$ satisfying condition (a) and if there exists a constant $\alpha_0 \geq 0$ such that conditions (b) through (d) hold true for all $\alpha > \alpha_0$, then $\mathfrak{A}$ is the infinitesimal generator of a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$.

Theorem 3.2 (Hille–Yosida–Ray) *(i) Let B be a linear operator on the space $C(\partial D)$ satisfying the following conditions:*

(a) The domain $D(B)$ is dense in $C(\partial D)$.
(b) If $f \in D(B)$ takes a positive maximum on ∂D, then there exists a point $x' \in \partial D$ such that $f(x') = \max_{x\in\partial D} f(x)$ and $Bf(x') \leq 0$.

Then the operator B is closable in $C(\partial D)$. Denote by $\overline{B}$ its minimal closed extension in $C(\partial D)$.

(ii) Let B be a linear operator as in part (i). Assume that the following condition is satisfied:

(c) For some $\alpha_0 \geq 0$, the range $R(\alpha_0 - B)$ of $\alpha_0 - B$ is dense in $C(\partial D)$.

Then the minimal closed extension $\overline{B}$ of B is the infinitesimal generator of a Feller semigroup $\{S_t\}_{t\geq 0}$ on the boundary ∂D.

3.3 Probabilistic meaning of hypotheses (H) and (A)

As stated in Section 2, we shall construct a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ by making use of a class $\{S_t^\alpha\}_{t\geq 0}$ of Feller semigroups on ∂D, where $\alpha \geq 0$. In other words, we shall reduce the problem of construction of Feller semigroups on the closure $\overline{D}$ to the same problem for Feller semigroups on the boundary ∂D.

The following theorem allows us to realize this plan:

Theorem 3.3 *Let the differential operator A satisfy* (2.1) *and hypothesis (H). For each $\alpha \geq 0$, the Dirichlet problem*

$$\begin{cases} (\alpha - A)u = f & \text{in } D, \\ u = \psi & \text{on } \partial D \end{cases} \tag{D}$$

has a unique solution $u \in C(\overline{D})$ for any $f \in C(\overline{D})$ and any $\psi \in C(\partial D)$.

Remark 3.1 We give a probabilistic interpretation of hypothesis (H). In [20], Stroock and Varadhan showed that the diffusion process

$$\xi(t) = (\xi_1(t), \xi_2(t), \dots, \xi_N(t))$$

which has

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial}{\partial x_i}$$

as differential generator, starting at a point $x = (x_1, x_2, \dots, x_N)$ of D, can be approximated by the function

$$\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_N(t))$$

defined by the formula

$$\phi_i(t) = x_i + 2\int_0^t \sum_{j=1}^N a^{ij}(\phi(s))\, \psi_j(s)\, ds \tag{3.2}$$

$$+\int_0^t \left(b^i(\phi(s)) - \sum_{j=1}^N \frac{\partial a^{ij}}{\partial x_j}(\phi(s)) \right) ds,$$

where

$$\psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_N(t)) : [0,\infty) \longrightarrow \mathbf{R}^N$$

is an arbitrary bounded measurable function, approximating an N-dimensional, standard Brownian motion

$$B(t) = (B_1(t), B_2(t), \ldots, B_N(t)).$$

On the other hand, we have the following:

Theorem 3.4 (Chow) *Let D be a domain in $\mathbf{R}^N$ and let $\{Z_i\}_{i=1}^r$ be a system of real C^∞ vector fields on D. If the Lie algebra $\mathscr{L}(Z_1, Z_2, \ldots, Z_r)$ over $\mathbf{R}$ generated by the vector fields $\{Z_i\}$ has rank N at a point x_0 of D, then there exists a neighborhood $U(x_0)$ of x_0 such that every point x of $U(x_0)$ can be joined to x_0 by a finite chain of trajectories of $\{\pm Z_i\}_{i=1}^r$.*

Now, by choosing the functions $\psi_j(t)$ in formula (3.2) so large that the diffusion terms

$$\sum_{j=1}^N a^{ij}(x)\, \psi_j(t)$$

dominates the drift terms

$$b^i(x)\,(\phi(s)) - \sum_{j=1}^N \frac{\partial a^{ij}}{\partial x_j}(x)$$

and by using Theorem 3.4 with

$$Z_i := \sum_{j=1}^N a^{ij} \frac{\partial}{\partial x_j}, \quad 1 \le i \le N,$$

we find that the probabilistic meaning of hypothesis (H) is that a Markovian particle starting at any point x of D can diffuse *everywhere* in D and exit $\overline{D} = D \cup \partial D$ through any point of ∂D in finite time (cf. [20, Remark 5.2]).

Similarly, we find that the probabilistic meaning of hypothesis (A) is that a Markovian particle starting at any point of the set $M = \{x' \in \partial D : \mu(x') = 0\}$, where no reflection phenomenon occurs, can *exit* M in finite time.

3.4 Maximum principles

We shall make use of the following *maximum principle* to verify condition (b) in Theorem 3.2 and to prove the uniqueness theorem for problem $(*)$ in Section 4 and Section 5, respectively ([16], [17]):

Theorem 3.5 *Let the differential operator A satisfy conditions* (2.1) *and let $\alpha \geq 0$. If hypothesis (H) is satisfied, then we have the assertions:*

(i) (The strong maximum principle) If $u \in C^2(D)$, $(A-\alpha)u \geq 0$ in D and if u takes a non-negative maximum in D, then u is constant in D.

(ii) *(The Hopf boundary point lemma) If $u \in C^2(D) \cap C(\overline{D})$, $(A-\alpha)u \geq 0$ in D and if u is not constant in D and takes a non-negative maximum at a point x_0' of ∂D, then we have the inequality*

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') < 0,$$

if u is differentiable at x_0'.

Remark 3.2 Some important remarks are in order:

1° Stroock–Varadhan [20] revealed the underlying probabilistic mechanism of propagation of the (nonnegative) maximum. Intuitively, their result may be stated as follows:
The maximum is propagated both in the positive and negative directions through the trajectories of the diffusion vector fields

$$X_i = \sum_{j=1}^{N} a^{ij}(x) \frac{\partial}{\partial x_j}, \quad 1 \leq i \leq N,$$

and only in the positive direction through the trajectories of the drift vector field

$$X_0 = \sum_{i=1}^{N} \left(b^i - \sum_{j=1}^{N} \frac{\partial a^{ij}}{\partial x_j} \right) \frac{\partial}{\partial x_i}$$

(cf. formula (3.2)).
Hence, part (i) of Theorem 3.5 follows from this result and Theorem 3.4.

2° In view of the fact that the boundary ∂D is non-characteristic with respect to the operator A, we can prove part (ii) of Theorem 3.5 just as in Oleĭnik [16].

3.5 Trace theorems

In order to give a precise meaning for the boundary condition Lu in terms of distributions, we need the following result, which follows easily from [11, Theorem 4.3.1 and Theorem 2.5.6].

Proposition 3.1 *Assume that the boundary ∂D is non-characteristic with respect to the differential operator A. Then, for every $u \in L^2(D)$ such that $Au \in L^2(D)$, we can define the boundary value $u|_{\partial D}$ as an element of $H^{-1/2}(\partial D)$ and the normal derivative $(\partial u/\partial \mathbf{n})|_{\partial D}$ as an element of $H^{-3/2}(\partial D)$, respectively. Furthermore, we have the inequality*

$$|u|_{\partial D}|_{H^{-1/2}(\partial D)} + \left| \frac{\partial u}{\partial \mathbf{n}} \right|_{H^{-3/2}(\partial D)} \leq C \left(\|Au\|_{L^2(D)} + \|u\|_{L^2(D)} \right),$$

with a constant $C > 0$ independent of u. Here $H^s(\partial D)$ is the Sobolev space of order s on the boundary ∂D with norm $|\cdot|_{H^s(\partial D)}$.

Since $C(\overline{D}) \subset L^2(D)$, it follows from formula (1.6) and conditions (2.2) that the boundary condition Lu can be defined as an element

$$Lu \in H^{-5/2}(\partial D)$$

for every $u \in C(\overline{D})$ such that $Au \in C(\overline{D})$.

4 Construction of Feller semigroups

In this section we shall prove existence theorems for Feller semigroups on $\overline{D}$.

4.1 Statement of main theorem

The basic result is the following theorem, from which we can easily obtain Theorem 2.1 and Corollary 2.1.

Theorem 4.1 *Let the differential operator A satisfy conditions* (2.1) *and let the boundary condition L satisfy conditions* (2.2)*. If hypothesis (H) is satisfied and if L is transversal on ∂D and condition (I) in Theorem 2.1 is satisfied, then there exists a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ whose infinitesimal generator $\mathfrak{A}$ is characterized as follows:*

(a) The domain $D(\mathfrak{A})$ satisfies the condition

$$D(\mathfrak{A}) \subset \left\{u \in C(\overline{D}) : Au \in C(\overline{D}),\, Lu = 0 \text{ on } \partial D\right\}. \tag{4.1}$$

(b) $\mathfrak{A}u = Au$ for every $u \in D(\mathfrak{A})$.

The proof of Theorem 4.1 is carried out just as in the case where the differential operator A is elliptic on $\overline{D}$, which is studied by Sato–Ueno [19], if we use Theorem 3.3, Theorem 3.5 and Proposition 3.1 instead of classical results on the Dirichlet problem (D) in the elliptic case (see [15]). So we give only a sketch of the proof.

4.2 Green and harmonic operators for the Dirichlet problem

For the proof of Theorem 4.1, we prepare some lemmas.

(I) By Theorem 3.3, it follows that the Dirichlet problem (D) is uniquely solvable for $\alpha \geq 0$. Hence we can define linear operators

$$G_\alpha^0 : C(\overline{D}) \longrightarrow C(\overline{D}) \quad \text{(Green operator)},$$
$$H_\alpha : C(\partial D) \longrightarrow C(\overline{D}) \quad \text{(harmonic operator)}$$

as follows.

$$\begin{cases} (\alpha - A)G_\alpha^0 f = f & \text{in } D, \\ G_\alpha^0 f = 0 & \text{on } \partial D. \end{cases} \tag{4.2}$$

$$\begin{cases} (\alpha - A)H_\alpha \psi = 0 & \text{in } D, \\ H_\alpha \psi = \psi & \text{on } \partial D. \end{cases} \tag{4.3}$$

Then we have the following:

Lemma 4.1 *(i) (a) The operator $G_\alpha^0 : C(\overline{D}) \to C(\overline{D})$ is non-negative and bounded for $\alpha \geq 0$. Furthermore, we have the inequality*

$$\left\|G_\alpha^0\right\| \leq \frac{1}{\alpha} \quad \textit{for all } \alpha > 0.$$

(b) For any $f \in C(\overline{D})$, we have the assertion

$$\lim_{\alpha\to\infty} \alpha G_\alpha^0 f(x) = f(x) \quad \text{for each } x \in D.$$

Furthermore, if $f|_{\partial D} = 0$, then this convergence is uniform in $x \in \overline{D}$.

(ii) The operator $H_\alpha : C(\partial D) \to C(\overline{D})$ is non-negative and bounded with norm

$$\|H_\alpha\| \leq 1 \quad \text{for all } \alpha > 0.$$

This lemma follows from the probabilistic formulas for G_α^0 and H_α due to Stroock–Varadhan [20].

Step (2): The following lemma shows that the operators G_α and H_α preserve regularity up to the boundary:

Lemma 4.2 *(i) The Green operator G_α^0 maps $C^\infty(\overline{D})$ into itself for each $\alpha \geq 0$.*
(ii) The harmonic operator H_α maps $C^\infty(\partial D)$ into $C^\infty(\overline{D})$ for each $\alpha \geq 0$.

Proof First, it follows from Oleĭnik–Radkevič [17, Theorem 2.6.2] that if the Lie algebra $\mathscr{L}(X_1, X_2, \ldots, X_N)$ has rank N at every point x of D, then the operator $\alpha - A$ is *hypoelliptic* in D, that is,

$$u \in \mathscr{D}'(D),\ (\alpha - A)u \in C^\infty(D) \implies u \in C^\infty(D).$$

Hence, by formulas (4.2) and (4.3) we have the following *interior regularity* properties:

$$f \in C^\infty(D) \implies G_\alpha^0 f \in C^\infty(D), \tag{4.4}$$

$$\psi \in C(\partial D) \implies H_\alpha \psi \in C^\infty(D), \tag{4.5}$$

Furthermore, it follows from Hörmander [11, Corollary 4.3.1] that if the boundary ∂D is non-characteristic with respect to A, then the operator $\alpha - A$ is *partially hypoelliptic* with respect to ∂D, that is,

If $(\alpha - A)u$ in $C^\infty(\overline{D})$ and the derivatives of u with respect to the boundary variables are all continuous, then $u \in C^\infty(\overline{D})$.

Hence, Part (i) of the lemma follows from assertions (4.2) and (4.4) and Part (ii) of the lemma follows from assertions (4.3) and (4.5), respectively.

The proof of Lemma 4.2 is complete.

Step (3): By Lemma 4.2, we can define linear operators

$$LG_\alpha^0 : C(\overline{D}) \longrightarrow C(\partial D),$$
$$LH_\alpha : C(\partial D) \longrightarrow C(\partial D)$$

as follows.

(a) The domain $D\left(LG_\alpha^0\right)$ is the space $C^\infty(\overline{D})$:

$$D\left(LG_\alpha^0\right) = C^\infty(\overline{D}).$$

(b) $LG_\alpha^0 f = L\left(G_\alpha^0 f\right)$ for every $f \in D\left(LG_\alpha^0\right)$.

(c) The domain $D(LH_\alpha)$ is the space $C^\infty(\partial D)$:

$$D(LH_\alpha) = C^\infty(\partial D).$$

(d) $LH_\alpha \psi = L(H_\alpha \psi)$ for every $\psi \in D(LH_\alpha)$.

Then we have the following:

Lemma 4.3 *(i) The operator LG_α^0 can be uniquely extended to a non-negative, bounded linear operator*

$$\overline{LG_\alpha^0} : C(\overline{D}) \longrightarrow C(\partial D)$$

for every $\alpha \geq 0$. The situation can be visualized in the following diagram:

$$\boxed{\begin{array}{ccc} C(\overline{D}) & \xrightarrow{\overline{LG_\alpha^0}} & C(\partial D) \\ \uparrow & & \uparrow \\ C^\infty(\overline{D}) & \xrightarrow[LG_\alpha^0]{} & C^\infty(\partial D) \end{array}}$$

(ii) The operator LH_α has the minimal closed extension

$$\overline{LH_\alpha} : C(\partial D) \longrightarrow C(\partial D)$$

for every $\alpha \geq 0$. The situation can be visualized in the following diagram:

$$\boxed{\begin{array}{ccc} C(\partial D) & \xrightarrow{\overline{LH_\alpha}} & C(\partial D) \\ \uparrow & & \uparrow \\ C^\infty(\partial D) & \xrightarrow[LH_\alpha]{} & C^\infty(\partial D) \end{array}}$$

Proof Part (i) follows from the non-negativity of G_α^0. Indeed, we have, by formulas (1.6) and (4.2),

$$LG_\alpha^0 f(x') = \delta(x') f(x') + \mu(x') \frac{\partial}{\partial \mathbf{n}} \left(LG_\alpha^0 f \right)(x') \geq 0$$
$$\text{for every non-negative function } f \in C^\infty(\overline{D}).$$

Part (ii) follows from an application of part (i) of Theorem 3.2 with

$$B := LH_\alpha,$$

by using Theorem 3.5 to verify conditions (a) and (b) of Theorem 3.2, just as in the proofs of [19, Lemma 4.2 and Corollary to Lemma 4.1].

The proof of Lemma 4.3 is complete. □

Step (4): By applying Proposition 3.1 to the operator $A - \alpha$ with $\alpha \geq 0$, we find that the boundary condition $L(G_\alpha f)$ for every $f \in\in C(\overline{D})$ can be defined as a distribution on ∂D, since $G_\alpha f$ satisfies formulas (4.2).

Similarly, the boundary condition $L(H_\alpha \psi)$ for every $\psi \in C(\partial D)$ can be defined as a distribution on ∂D, since $H_\alpha \psi$ satisfies formulas (4.3).

The following lemma shows that the boundary operators $\widetilde{LG_\alpha^0}$ and $\widetilde{LH_\alpha}$ thus defined are an extension of the operators $\overline{LG_\alpha}$ and $\overline{LH_\alpha}$, respectively:

Lemma 4.4 *Let $\alpha \geq 0$. Then we have the following assertions:*

(i) If we define a linear operator

$$\widetilde{LG_\alpha^0} : C(\overline{D}) \longrightarrow \mathscr{D}'(\partial D)$$

by the formula

$$\widetilde{LG_\alpha^0} f = L\left(G_\alpha^0 f\right) \quad \text{for every } f \in C(\overline{D}),$$

then it follows that

$$\overline{LG_\alpha^0} \subset \widetilde{LG_\alpha^0} \quad \text{on } C(\overline{D}).$$

The situation can be visualized as follows:

$$\boxed{\begin{array}{ccc} C(\overline{D}) & \xrightarrow{\widetilde{LG_\alpha^0}} & \mathscr{D}'(\partial D) \\ \Big\uparrow & & \Big\uparrow \\ C(\overline{D}) & \xrightarrow[\overline{LG_\alpha^0}]{} & C(\partial D) \end{array}}$$

(ii) Similarly, if we define a linear operator

$$\widetilde{LH_\alpha} : C(\partial D) \longrightarrow \mathscr{D}'(\partial D)$$

by the formula

$$\widetilde{LH_\alpha} \psi = L(H_\alpha \psi) \quad \text{for every } \psi \in C(\partial D),$$

then it follows that

$$\overline{LH_\alpha} \subset \widetilde{LH_\alpha} \quad \text{on } C(\partial D).$$

The situation can be visualized as follows:

$$\boxed{\begin{array}{ccc} C(\partial D) & \xrightarrow{\widetilde{LH_\alpha}} & \mathscr{D}'(\partial D) \\ \Big\uparrow & & \Big\uparrow \\ C(\partial D) & \xrightarrow[\overline{LH_\alpha}]{} & C(\partial D) \end{array}}$$

Proof Part (i) follows from the boundedness of G_α^0 and an application of Proposition 3.1 with $A := A - \alpha$, while Part (ii) follows from the boundedness of H_α and an application of Proposition 3.1 with $A := A - \alpha$. □

4.3 Proof of Theorem 4.1

Now the proof can be carried out in the following way, just as in the proof of [19, Theorem 5.2].

Step 1: If condition (I) is satisfied, then the operator $\overline{LH_\alpha}$ is the infinitesimal generator of a Feller semigroup $\{S_t^\alpha\}_{t\geq 0}$ on the boundary ∂D.

Step 2: If $\overline{LH_\alpha}$ generates a Feller semigroup $\{S_t^\alpha\}_{t\geq 0}$ on ∂D for some $\alpha \geq 0$, then the operator $\overline{LH_\beta}$ generates a Feller semigroup $\{S_t^\beta\}_{t\geq 0}$ on ∂D for *any* $\beta \geq 0$.

Step 3: If the boundary condition L is *transversal* on ∂D, then the operator $\overline{LH_\alpha}$ is em bijective for any $\alpha > 0$ and its inverse

$$\overline{LH_\alpha}^{-1} : C(\partial D) \longrightarrow C(\partial D)$$

is non-positive and bounded.

Step 4: For any $\alpha > 0$, we can define a linear operator

$$G_\alpha : C(\overline{D}) \longrightarrow C(\overline{D})$$

by the formula

$$G_\alpha f := G_\alpha^0 f - H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\overline{LG_\alpha^0 f} \right) \right) \quad \text{for every } f \in C(\overline{D}). \tag{4.6}$$

Furthermore, we can define a linear operator

$$\mathfrak{A} : C(\overline{D}) \longrightarrow C(\overline{D})$$

as follows:

(a) The domain $D(\mathfrak{A})$ is the space

$$D(\mathfrak{A}) = \left\{ u \in C(\overline{D}) : Au \in C(\overline{D}),\ u|_{\partial D} \in \widetilde{D}, Lu = 0 \right\}. \tag{4.1$'$}$$

(b) $\mathfrak{A}u = Au$ for every $u \in D(\mathfrak{A})$.

Here $\widetilde{D}$ is the common domain of the operators $\{\overline{LH_\alpha}\}_{\alpha \geq 0}$:

$$\widetilde{D} = \bigcap_{\alpha \geq 0} D\left(\overline{LH_\alpha}\right).$$

Then we have the formula

$$G_\alpha = (\alpha - \mathfrak{A})^{-1} \quad \text{for every } \alpha > 0. \tag{4.7}$$

Indeed, we assume that

$$u \in D(\mathfrak{A}),$$
$$(\alpha - \mathfrak{A})u = 0.$$

Then it follows from the uniqueness property of solutions of the Dirichlet problem (D) that u can be written uniquely in the form

$$u = H_\alpha \left(u|_{\partial D} \right), \quad u|_{\partial D} \in \widetilde{D},$$

and satisfies the condition

$$\overline{LH_\alpha}\,(u|_{\partial D}) = Lu = 0 \quad \text{on } \partial D.$$

Since the operator $\overline{LH_\alpha}$ is bijective for any $\alpha > 0$, it follows that

$$u|_{\partial D} = 0.$$

so that

$$u = 0 \quad \text{in } D.$$

This proves that the operator $\alpha - \mathfrak{A}$ is injective.

On the other hand, we find from formulas (4.2), (4.3) and Lemma 4.4 that, for any $f \in C(\overline{D})$ the function $u = G_\alpha f$, defined by formula (4.6), satisfies the conditions

$$\begin{cases} (\alpha - A)u = f & \text{in } D, \\ u|_{\partial D} = -\overline{LH_\alpha}^{-1}\left(\overline{LG_\alpha^0 f}\right) \in \widetilde{D} = \cap_{\alpha \geq 0} D\left(\overline{LH_\alpha}\right), & \\ Lu = 0 & \text{on } \partial D. \end{cases}$$

This implies that $u \in D(\mathfrak{A})$ and that $(\alpha - \mathfrak{A})u = f$.

Consequently, we have proved the desired formula (4.7).

Step 5: In light of expression (4.6) of $G_\alpha = (\alpha - \mathfrak{A})^{-1}$, it follows that the operator $\mathfrak{A}$, defined by $(4.1)'$, satisfies conditions (a) through (d) in Theorem 3.1. Hence it follows from an application of part (ii) of the same theorem that the operator $\mathfrak{A}$ is the infinitesimal generator of a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$.

The proof of Theorem 4.1 is complete. □

Remark 4.1 Note that, as is seen from expression (4.6), we constructed the Green operator $G_\alpha = (\alpha - \mathfrak{A})^{-1}$ of a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ for each $\alpha > 0$, by making use of the Green operator $-\overline{LH_\alpha}^{-1}$ of a Feller semigroup $\{S_t^\alpha\}_{t\geq 0}$ on the boundary ∂D.

4.4 Proof of Theorem 2.1

By Theorem 4.1, it suffices to show that if conditions (I) and (II) are satisfied, then we have the assertion

$$\begin{aligned} D(\mathfrak{A}) &= \left\{u \in C(\overline{D}) : Au \in C(\overline{D}),\, u|_{\partial D} \in \widetilde{D},\, Lu = 0 \text{ on } \partial D\right\} \\ &= \left\{u \in C(\overline{D}) : Au \in C(\overline{D}),\, Lu = 0 \text{ on } \partial D\right\}. \end{aligned}$$

Assume that

$$\begin{cases} u \in C(\overline{D}), \\ Au \in C(\overline{D}), \\ Lu = 0 \text{ on } \partial D. \end{cases}$$

Then, by letting

$$w := u - G_\alpha\left((\alpha - A)u\right),$$

we obtain from formulas (4.7) and $(4.1)'$ that

$$\begin{cases} (\alpha - A)w = 0 & \text{in } D, \\ Lw = 0 & \text{on } \partial D, \end{cases}$$

and hence from condition (II) with $u := w$ that

$$w = 0 \quad \text{in } D.$$

This implies that

$$u = G_\alpha\left((\alpha - A)u\right) \in D(\mathfrak{A}).$$

The proof of Theorem 2.1 is complete. □

4.5 Proof of Corollary 2.1

Step 1: First, we show that conditions (I) and (III) imply condition (II).

Assume that

$$\begin{cases} u \in C(\overline{D}), \\ (\alpha - A)u = 0 & \text{in } D, \\ Lu = 0 & \text{on } \partial D. \end{cases}$$

Then it follows from condition (Ill) that $u \in C^\infty(\overline{D})$ and hence from the uniqueness property of solutions of the Dirichlet problem (D) that u can be written in the form

$$u = H_\alpha\left(u|_{\partial D}\right), \quad u|_{\partial F} \in D\left(LH_\alpha\right)\left(= C^\infty\left(\partial D\right)\right)$$

and satisfies the condition

$$LH_\alpha\left(u|_{\partial D}\right) = Lu = 0 \quad \text{on } \partial D. \tag{4.8}$$

As stated in Step 3 in the proof of Theorem 4.1, if condition (I) is satisfied and the boundary condition L is transversal on ∂D, then the minimal closed extension $\overline{LH_\alpha}$ in $C(\partial D)$ of LH_α is bijective for any $\alpha > 0$.

Therefore, we have, by condition (4.8),

$$u|_{\partial D} = 0,$$

and so

$$u = 0 \quad \text{in } D.$$

This proves that condition (II) is satisfied.

Step 2: Next we show that if condition (III) is satisfied, then we have the regularity property

$$f \in C^\infty(\overline{D}) \implies G_\alpha f \in C^\infty(\overline{D}). \tag{4.9}$$

Let $f \in C^\infty(\overline{D})$. Then it follows from part (i) of Lemma 4.2 that

$$G_\alpha f \in C^\infty(\overline{D}).$$

Furthermore, by letting

$$w := H_\alpha\left(\overline{LH_\alpha}^{-1}(LG_\alpha^0 f)\right),$$

we obtain from formula (4.2) and part (ii) of Lemma 4.4 that

$$\begin{cases} (\alpha - A)w = 0 & \text{in } D, \\ Lw = LG_\alpha^0 f \in C^\infty(\partial D) & \text{on } \partial D, \end{cases}$$

and hence from condition (III) with $u := w$ that

$$w \in C^\infty(\overline{D}).$$

By formula (4.6) this proves that

$$G_\alpha f = G_\alpha^0 f - w \in C^\infty(\overline{D}).$$

Step 3: Finally, we show that the operator $\mathfrak{A}$, defined by formula (2.5), coincides with the minimal closed extension in $C(\overline{D})$ of the restriction of A to the space

$$\left\{ u \in C^2(\overline{D}) : Lu = 0 \text{ on } \partial D \right\}.$$

For $u \in D(\mathfrak{A})$, choose a sequence $\{f_j\}_{j=1}^\infty$ in $C^\infty(\overline{D})$ such that

$$f_j \longrightarrow (\alpha - \mathfrak{A})u \quad \text{in } C(\overline{D}) \text{ as } j \to \infty. \tag{4.10}$$

If we let

$$u_j = G_\alpha f_j,$$

then it follows from assertion (4.9) and formula (4.7) that

$$\begin{cases} u_j \in C^\infty(\overline{D}), \\ (\alpha - A)u_j = 0 & \text{in } D, \\ Lu_j = 0 & \text{on } \partial D. \end{cases}$$

In particular, we have the assertion

$$u_j \in D(\mathfrak{A}) \cap C^\infty(\overline{D}).$$

Furthermore, since the operator $G_\alpha : C(\overline{D}) \to C(\overline{D})$ is bounded, it follows from assertion (4.10) and formula (4.7) that

$$u_j = G_\alpha f_j \longrightarrow G_\alpha(\alpha - \mathfrak{A})u = u \quad \text{in } C(\overline{D}) \text{ as } j \to \infty,$$

and hence that

$$Au_j = \alpha u_j - f_j \longrightarrow \alpha u - (\alpha - \mathfrak{A})u = \mathfrak{A}u \quad \text{in } C(\overline{D}) \text{ as } j \to \infty.$$

Summing up, we have proved that

$$(u_j, Au_j) \longrightarrow (u, \mathfrak{A}u) \quad \text{in } C(\overline{D}) \oplus C(\overline{D}) \text{ as } j \to \infty.$$

Consequently, we obtain that

$$\begin{aligned} \text{The graph of } \mathfrak{A} &:= \{(u, \mathfrak{A}u) : u \in D(\} \\ &= \text{the closure in } C(\overline{D}) \oplus C(\overline{D}) \text{ of the graph} \\ &\quad \left\{ u \in C^2(\overline{D}) : Lu = 0 \text{ on } \partial D \right\}. \end{aligned}$$

The proof of Corollary 2.1 is complete. □

5 Existence, uniqueness and regularity theorem for problem $(*)$

The purpose of this section is to prove the following existence, uniqueness and regularity theorem for problem $(*)$. By virtue of Sobolev's lemma, we find that conditions (I) and (III) are satisfied. Hence, Theorem 2.2 follows from an application of Corollary 2.1.

Theorem 5.1 *Let the differential operator A satisfy conditions* (2.1) *and let the boundary condition L satisfy conditions* (2.2)*. Assume that A is elliptic on $\overline{D}$ and further that L is transversal on ∂D and that hypothesis (A) is satisfied. Then there exists a constant* $0 < \kappa \leq 1$ *such that, for each constant* $\alpha > 0$ *the boundary value problem*

$$\begin{cases} (\alpha - A)u = f & \text{in } D, \\ Lu = \varphi & \text{on } \partial D \end{cases} \tag{$*$}$$

has a unique solution $u \in H^{s-2+\kappa}(D)$ for any $f \in H^{s-2}(D)$ and any $\varphi in H^{s-5/2}(\partial D)$, where $s \geq 3$.

Furthermore, for each constant $\alpha \geq 0$ *we have the regularity property*

$$\begin{aligned} & u \in H^t(D),\ t \in \mathbf{R},\ (\alpha - A)u \in C^\infty(\overline{D}),\ Lu \in C^\infty(\partial D) \\ & \Longrightarrow\ u \in C^\infty(\overline{D}). \end{aligned} \tag{5.1}$$

Here $H^s(D)$ (resp. $H^s(\partial D)$) denotes the Sobolev space of order s on D (resp. ∂D).

Proof The proof is essentially the same as that of [21, Théorème 4.1] except that we use Lemma 5.1 and Lemma 5.2 below instead of [21, Lemme 4.6]. So we give only a sketch of the proof.

Step (1): First, by using the Green operator G_α and the harmonic operator H_α of the Dirichlet problem (D), we reduce the study of problem $(*)$ to that of the operator LH_α on the boundary ∂D.

It is well known (cf. [15]) that if the differential operator A is elliptic on $\overline{D}$, then for a ;;i, 0 the Dirichlet problem (D) has a unique solution $u \in H^s(D$ (s ;;i, 2) for any $f \in H^{s-2}(D)$ and $\varphi \in (\partial D)$. Hence we can define linear operators

$$\begin{aligned} G_\alpha^0 &: H^{s-2}(D) \longrightarrow H^t(D) \quad (s \geq 2), \\ H_\alpha &: H^t(\partial D) \longrightarrow H^t(D) \quad (t \in \mathbf{R}) \end{aligned}$$

by formulae (4.2) and (4.3), respectively.

Then we can easily obtain the following:

Proposition 5.1 *Let A and L be as in Theorem 5.1 and let* $\alpha \geq 0$*. For given $f \in H^{s-2}(D)$ and $\varphi \in H^{s-5/2}(\partial D)$ with $s \geq 3$, there exists a solution $u \in H^t(D)$ of problem $(*)$ for $t \leq s$ if and only if there exists a solution $\psi \in H^{t-1/2}(\partial D)$ of the equation:*

$$LH_\alpha \psi = \varphi - LG_\alpha^0 f \quad \text{on } \partial D.$$

Furthermore, the solutions u and ψ are related to each other by the following relation:

$$u = H_\alpha \psi + G_\alpha^0 f. \tag{5.2}$$

By formula (4.3), we can write the operator LH_α in the form

$$LH_\alpha \psi = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial \psi}{\partial x_i} + \gamma(x') \psi \qquad (5.3)$$
$$+ \mu(x') \frac{\partial}{\partial \mathbf{n}} (H_\alpha \psi) \Big|_{\partial D} - \alpha\, \delta(x') \psi.$$

Hence we find (see [12]) that LH_α is a second-order, pseudo-differential operator on the boundary ∂D and further (see [21]) that its symbol is given by the formula

$$\left[- \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \xi_i \xi_j \right] + \left[-\mu(x') \left| \xi' \right| + \sqrt{-1} \sum_{i=1}^{N-1} \beta^i(x') \xi_i \right] \qquad (5.4)$$
$$+ \text{ terms of order } \leq 0 \text{ depending on } \alpha.$$

Here $|\xi'|$ denotes the length of a covector $\xi' = (\xi_1, \xi_2, \ldots, \xi_{N-1})$ with respect to the Riemannian metric induced on the boundary ∂D by the Riemannian metric (a_{ij}) (the inverse matrix of $\left(a^{ij}\right)$ of $\mathbf{R}^N$.

By virtue of the fact that LH_α is a first-order, pseudo-differential operator on ∂D, we can associate with problem $(*)$ a closed linear operator

$$\mathscr{L}(\alpha) : H^{s-5/2+\kappa}(\partial D) \longrightarrow H^{s-5/2}(\partial D)$$

as follows.

(a) The domain $D(\mathscr{L}(\alpha))$ is the space

$$D(\mathscr{L}(\alpha)) = \left\{ \psi \in H^{s-5/2+\kappa}(\partial D) : LH_\alpha \psi \in H^{s-5/2}(\partial D) \right\}. \qquad (5.5)$$

(b) $\mathscr{L}(\alpha)\psi = LH_\alpha \psi$ for every $\psi \in D(\mathscr{L}(\alpha))$.

Here $\kappa > 0$ is a constant and will be fixed later on (see Lemma 5.1 below).

Then it is easily seen from Proposition 5.1 with $t := s - 2 + \kappa$ that the problems of existence, uniqueness and regularity of solutions of problem $(*)$ are reduced to the same problems for the operator $\mathscr{T}(\alpha)$, respectively.

Step (2): Next we show that if hypothesis (A) is satisfied, then the operator LH_α is *hypoelliptic* on ∂D and further an *a priori* estimate holds true for LH_α. This proves regularity property (5.1) for problem $(*)$.

By formula (5.3), we can decompose the pseudo-differential operator LH_α in the form

$$LH_\alpha = Q_\alpha + \mu(x') \Pi_\alpha. \qquad (5.6)$$

Here the operator

$$Q_\alpha : \psi \longmapsto \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 \psi}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial \psi}{\partial x_i} + (\gamma(x') - \alpha \delta(x')) \psi$$

is a second-order, differential operator with non-positive principal symbol

$$- \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \xi_i \xi_j \leq 0 \quad \text{on the cotangent bundle } T^*(\partial D),$$

and the operator

$$\Pi_\alpha : \psi \longmapsto \frac{\partial}{\partial \mathbf{n}} (H_\alpha \psi)\Big|_{\partial D}$$

is a classical pseudo-differential operator of first order with principal symbol

$$-|\xi'|.$$

By considering $\mu(x')\Pi_\alpha$ as a term of "perturbation" of Q_α and by using the argument in the proof of Oleĭnik–Radkevič [17, Theorem 2.6.2] and Hörmander [13, Theorem 5.9], we can prove the following:

Lemma 5.1 *Let A and L be as in Theorem 5.1 and assume that hypothesis (A) is satisfied. Then there exists a constant $0 < \kappa \leq 1$ such that, for each $s \in \mathbf{R}$, we have the regularity property*

$$\psi \in \mathscr{D}'(\partial D),\ LH_\alpha \psi \in H^s(\partial D) \implies \psi \in H^{s+\kappa}(\partial D). \tag{5.7}$$

Furthermore, for any $t < s + \kappa$ there exists a constant $C_{s,t} > 0$ such that the a priori *estimate*

$$|\psi|_{H^{s+\kappa}(\partial D)} \leq C_{s,t} \left(|(LH_\alpha)\,\psi|^2_{H^s(\partial D)} + |\psi|^2_{H^t(\partial D)} \right) \tag{5.8}$$

holds true.

Remark 5.1 The constant κ in the lemma can be chosen as follows:

$$\kappa = \begin{cases} 1 & \text{in a neighborhood of } x_0' \text{ such that } \mu(x_0') > 0, \\ 2^{1-R(x_0'} & \text{in a neighborhood of } x_0' \text{ such that } \mathscr{L}(\mathscr{Y}) \text{ has rank } N-1 \text{ at } x_0'. \end{cases}$$

Here $R(x_0') \geq 1$ is the length of $\mathscr{L}(\mathscr{Y})$ at x_0' (cf. the proof of Proposition 6.2).

Lemma 5.1 is the essential step in the proof of Theorem 5.1 and will be proved in the next Section 6 due to its length.

By virtue of Sobolev's lemma, the regularity property (5.1) for problem $(*)$ follows immediately from formula (5.2) and the regularity property (5.7).

Step (3): By the regularity property (5.1), it follows that any homogeneous solution of problem $(*)$ is smooth up to the boundary. Hence the uniqueness theorem for problem $(*)$ is an immediate consequence of the following maximum principle:

Proposition 5.2 (the maximum principle) *Let the differential operator A satisfy conditions* (2.1) *and let the boundary condition L satisfy conditions* (2.2). *If the hypothesis (H) is satisfied and if L is transversal on ∂D, then we have, for each $\alpha > 0$,*

$$\begin{aligned} &u \in C^2(\overline{D}),\ (A-\alpha)u \geq 0 \text{ in } D,\ Lu \geq 0 \text{ on } \partial D \\ &\implies u \leq 0 \text{ on } \overline{D}. \end{aligned}$$

Proof If u is constant in D, then it follows that

$$0 \leq (A-\alpha)\,u = (c(x)-\alpha)\,u \quad \text{in } D,$$

and hence that u is non-positive constant in D, since $c(x) \leq 0$ in D and $\alpha > 0$.

Thus we may assume that u is not constant in D. Assume, to the contrary, that

$$\max_{x \in \overline{D}} u(x) > 0.$$

Then it follows from an application of Theorem 3.5 that there exists a point x_0' of ∂D such that

$$\begin{cases} u(x_0') = \max_{x \in \overline{D}} u(x) > 0, \\ \frac{\partial u}{\partial \mathbf{n}}(x_0') < 0. \end{cases}$$

Furthermore, we remark that

$$\begin{cases} \frac{\partial u}{\partial x_i}(x_0') = 0 \quad \text{for } 1 \leq i \leq N-1, \\ Au(x_0') \geq \alpha u(x_0') > 0, \end{cases}$$

and that

$$\sum_{i.j=1}^{N-1} \alpha^{ij}(x_0') \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0') \leq 0.$$

since the matrix $\left(\alpha^{ij}(x)\right)$ is non-negative definite.

Hence we have, by conditions (2.2) and (2.3),

$$\begin{aligned} & Lu(x_0') \\ & = \sum_{i.j=1}^{N-1} \alpha^{ij}(x_0') \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0') + \gamma(x_0')\, u(x_0') + \mu(x_0')\, \frac{\partial u}{\partial \mathbf{n}}(x_0') - \delta(x_0') Au(x_0') \\ & \leq \mu(x_0')\, \frac{\partial u}{\partial \mathbf{n}}(x_0') - \delta(x_0') Au(x_0') \\ & < 0. \end{aligned}$$

This contradicts the assumption that

$$Lu \geq 0 \quad \text{on } \partial D.$$

The proof of Proposition 5.2 is complete.

Step (4): Finally, we prove the existence theorem for problem $(*)$. For this purpose, we make use of a method essentially due to Agmon–Nirenberg ([1], [15]). This is a technique of treating a spectral parameter α as a second-order elliptic differential operator of an extra variable y on the unit circle S, and relating the old problem to a new one with the additional variable. Our presentation of this technique is due to Fujiwara [9].

Substep (4-i): By replacing the parameter α in problem $(*)$ by the differential operator

$$-\frac{\partial^2}{\partial y^2}$$

on the unit circle $S = \mathbf{R}/\mathbf{Z}$, we consider the following boundary value problem:

$$\begin{cases} \left(-\frac{\partial^2}{\partial y^2} - A\right) \widetilde{u} = \widetilde{f} & \text{in } D \times S, \\ L\widetilde{u} = \widetilde{\varphi} & \text{on } \partial D \times S. \end{cases} \qquad (\widetilde{*})$$

Then, roughly speaking, the most important relation between problem $(*)$ and problem $(\widetilde{*})$ is stated as follows (see [22]):

$$\begin{cases} \textit{If the index of problem } (\widetilde{*}) \textit{ is finite, then the index of problem } (*) \\ \textit{is equal to zero for all } \alpha \geq 0. \end{cases} \tag{5.9}$$

We formulate this assertion more precisely. Note (see [15]) that the Dirichlet problem

$$\begin{cases} \left(-\frac{\partial^2}{\partial y^2} - A\right)\widetilde{u} = \widetilde{f} & \text{in } D \times S, \\ \widetilde{u} = \widetilde{\psi} & \text{on } \partial D \times S \end{cases} \tag{$\widetilde{D}$}$$

has a unique solution $\widetilde{u}$ in $H^s(D\times S)$ for any $\widetilde{f} \in H^{s-2}(D\times S)$ and any $\widetilde{\psi} \in H^{s-1/2}(\partial D \times S)$, where $s \geq 2$.

Therefore, we can define linear operators

$$\widetilde{G} : H^{s-2}(D\times S) \longrightarrow H^s(D\times S), \quad s \geq 2,$$
$$\widetilde{H} : H^{t-1/2}(\partial D\times S) \longrightarrow H^t(D\times S), \quad t \in \mathbf{R}$$

as follows.

$$\begin{cases} \left(-\frac{\partial^2}{\partial y^2} - A\right)\widetilde{G}\widetilde{f} = \widetilde{f} & \text{in } D \times S, \\ \widetilde{G}\widetilde{f} = 0 & \text{on } \partial D \times S. \end{cases} \tag{5.10}$$

$$\begin{cases} \left(-\frac{\partial^2}{\partial y^2} - A\right)\widetilde{H}\widetilde{\psi} = 0 & \text{in } D \times S, \\ \widetilde{H}\widetilde{\psi} = \widetilde{\psi} & \text{on } \partial D \times S. \end{cases} \tag{5.11}$$

By formula (5.11), it follows that the operator $L\widetilde{H}$ can be written in the form

$$\begin{aligned} &L\widetilde{H}\widetilde{\psi} \qquad\qquad (5.12) \\ &= \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 \widetilde{\psi}}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial \widetilde{\psi}}{\partial x_i} + \gamma(x')\widetilde{\psi} + \mu(x') \frac{\partial}{\partial \mathbf{n}} \left(\widetilde{H}\widetilde{\psi}\right)\Big|_{\partial D \times S} + \delta(x') \frac{\partial^2 \widetilde{\psi}}{\partial y^2}. \end{aligned}$$

Hence we find that $L\widetilde{H}$ is a second-order, pseudo-differential operator on the boundary $\partial D \times S$ with symbol

$$\begin{aligned} &\left[-\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j - \delta(x')\right] \qquad (5.13) \\ &+ \left[-\mu(x')\sqrt{|\xi'|^2 + \eta^2} + \sqrt{-1}\sum_{i=1}^{N-1} \beta^i(x')\xi_i + \sqrt{-1}\sum_{i,j=1}^{N-1} \frac{\partial \alpha^{ij}}{\partial x_j}\xi_i\right] \\ &+ \text{terms of order} \leq 0. \end{aligned}$$

Here η is the dual variable of y in the cotangent bundle $T^*(S)$.

Therefore, we can associate with problem $(\widetilde{*})$ a *closed* linear operator

$$\widetilde{\mathscr{T}} : H^{s-5/2+\kappa}(\partial D \times S) \longrightarrow H^{s-5/2}(\partial D \times S)$$

as follows:

(a) The domain $D\left(\widetilde{\mathscr{T}}\right)$ is the space

$$D\left(\widetilde{\mathscr{T}}\right) = \left\{ \widetilde{\psi} \in H^{s-5/2+\kappa}(\partial D \times S) : \left(L\widetilde{H}\right) \widetilde{\psi} \in H^{s-5/2}(\partial D \times S) \right\}. \tag{5.14}$$

(b) $\widetilde{\mathscr{T}}\widetilde{\psi} = \left(L\widetilde{H}\right)\widetilde{\psi}$ for every $\widetilde{\psi} \in D\left(\widetilde{\mathscr{T}}\right)$.

Then, as in problem $(*)$, it is easy to see that the study of problem $(\widetilde{*})$ is reduced to that of the operator $\widetilde{\mathscr{T}}$ on the boundary $\partial D \times S$.

Recall the following:

Definition 5.1 Let X and Y be Banach spaces and let $T : X \to Y$ be a closed linear operator with domain $D(T)$. We say that the *index* of T is finite if the dimension of the kernel $N(T)$ of T is finite and if the range $R(T)$ of T is closed in Y and its codimension is also finite. Then the index $\operatorname{ind} T$ of T is defined by the formula

$$\operatorname{ind} T = \dim N(T) - \operatorname{codim} R(T).$$

Now we can formulate assertion (5.9) precisely, by using the operators $\mathscr{T}(\alpha)$ and $\widetilde{\mathscr{T}}$ defined by formulas (5.5) and (5.14), respectively:

$$\text{If the index of } \widetilde{\mathscr{T}} \text{ is } \textit{finite}\text{, then the index of } \mathscr{T}(\alpha) \text{ is equal to } \textit{zero}. \tag{5.9$'$}$$

The proof of assertion $(5.9)'$ is essentially a repetition of that of [22, Théorème], so we may omit it.

In Step (3), we proved that if hypothesis (A) is satisfied, then the uniqueness theorem for problem $(*)$ is valid for any $\alpha > 0$, or equivalently,

$$\dim N(\mathscr{T}(\alpha)) = 0 \quad \text{for any } \alpha > 0.$$

Thus, if we show that the index of $\widetilde{\mathscr{T}}$ is finite, then it follows from assertion $(5.9)'$ that

$$\operatorname{codim} R(\mathscr{T}(\alpha)) = 0 \quad \text{for any } \alpha > 0,$$

and hence that the existence theorem for problem $(*)$ is valid for any $\alpha > 0$.

Substep (4-ii): It remains to prove that if hypothesis (A) is satisfied, then the index of $\widetilde{\mathscr{T}}$ is finite.

Step 1°: First, by formula (5.12) we can express the pseudo-differential operator $L\widetilde{H}$ in the form

$$L\widetilde{H} = \widetilde{Q} + \mu(x')\,\widetilde{\Pi}. \tag{5.15}$$

Here:

(1) $\widetilde{Q}$ is a second-order, differential operator

$$\widetilde{Q} = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2}{\partial x_i \partial x_j} + \delta(x') \frac{\partial^2}{\partial y^2} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial}{\partial x_i} + \gamma(x')$$

with non-positive principal symbol

$$-\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j \leq 0 \quad \text{on the cotangent bundle } T^*(\partial D),$$

(2) The operator

$$\widetilde{\Pi} : \widetilde{\psi} \longmapsto \frac{\partial}{\partial \mathbf{n}} \left(\widetilde{H} \widetilde{\psi} \right) \Big|_{\partial D \times S}$$

is a first-order, elliptic pseudo-differential operator with principal symbol

$$-\sqrt{|\xi'|^2 + \eta^2}.$$

As in Section 2, we let

$$\widetilde{\Phi} = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j} + \delta(x') \frac{\partial}{\partial y} \otimes_S \frac{\partial}{\partial y},$$

and define a mapping

$$\widetilde{\Psi}_0 : \Gamma(\partial D \times S, T^*(\partial D \times S)) \longrightarrow \Gamma(\partial D \times S, T(\partial D \times S))$$

by the formula

$$\widetilde{\Psi}_0(\widetilde{\zeta}) = \widetilde{\Phi}\left(\widetilde{\zeta}, \cdot\right) \quad \text{for every } \widetilde{\zeta} \in \Gamma(\partial D \times S, T^*(\partial D \times S)).$$

Let $p_1 : \partial D \times S \longrightarrow \partial D$ and $p_2 : \partial D \times S \longrightarrow S$ be the projection on ∂D and S, respectively. Then we can define a mapping

$$\widetilde{\Psi} : \Gamma(\partial D, T^*(\partial D)) \times \Gamma(S, T^*(S)) \longrightarrow \Gamma(\partial D \times S, T(\partial D \times S))$$

by the formula

$$\widetilde{\Psi}\left(\zeta', \sigma'\right) = \widetilde{\Psi}_0\left(p_1^* \zeta' + p_2^* \sigma'\right) \quad \text{for all } (\zeta', \sigma') \in \Gamma(\partial D, T^*(\partial D)) \times \Gamma(S, T^*(S)).$$

In terms of a local coordinate $(x', y) = (x_1, x_2, \ldots, x_{N-1}, y)$ on the boundary $\partial D \times S$, we have, for $\zeta' = \sum_{i=1}^{N-1} \zeta_i \, dx_i$ and $\sigma' = \sigma \, dy$,

$$\widetilde{\Psi}\left(\zeta', \sigma'\right) = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \, \zeta_i \, \frac{\partial}{\partial x_j} + \delta(x') \, \frac{\partial}{\partial y}.$$

We let

$$\begin{aligned} \widetilde{Y} &= \text{the image of } \widetilde{\Psi} \\ &= \left\{ \widetilde{\Psi}((\zeta', \sigma') : \zeta' \in \Gamma(\partial D, T^*(\partial D)), \, \sigma' \in \Gamma(S, T^*(S)) \right\}. \end{aligned}$$

By condition (2.3), we find that if hypothesis (A) is satisfied, then the following hypothesis is satisfied:

$(\widetilde{A})$ The Lie algebra $\mathscr{L}(\widetilde{Y})$ over $\mathbf{R}$ generated by $\widetilde{Y}$ has rank N at every point of the set $\widetilde{M} = \{(x', y) \in \partial D \times S : \mu(x') = 0\} = M \times S$.

Hence, by considering $\mu(x')\widetilde{\Pi}$ as a term of "perturbation" of $\widetilde{Q}$ and by using the argument just in the proof of [17, Theorem 2.6.2] and [13, Theorem 5.9], we can obtain the following result, analogous to Lemma 5.1:

Lemma 5.2 *Let A and L be as in Theorem 5.1. Assume that hypothesis (A) is satisfied, and let κ be the same constant as in Lemma 5.1. Then, for each $s \in \mathbf{R}$ we have the regularity property*

$$\begin{aligned} &\widetilde{\psi} \in \mathscr{D}'(\partial D \times S), \ \left(L\widetilde{H}\right) \widetilde{\psi} \in H^s(\partial D \times S) \\ &\Longrightarrow \ \widetilde{\psi} \in H^{s+\kappa}(\partial D \times S). \end{aligned} \tag{5.16}$$

Furthermore, for any $t < s + \kappa$ there exists a constant $\widetilde{C}_{s,t} > 0$ such that the a priori *estimate*

$$|\widetilde{\psi}|_{H^{s+\kappa}(\partial D \times S)} \leq \widetilde{C}_{s,t} \left(\left| \left(L\widetilde{H}\right) \widetilde{\psi} \right|^2_{H^s(\partial D \times S)} + |\widetilde{\psi}|^2_{H^t(\partial D \times S)} \right) \tag{5.17}$$

holds true.

As stated above, the proof of Lemma 5.2 is essentially the same as that of Lemma 5.1, so we may omit it.

Step 2°: To complete the proof, we need two lemmas. The first one is a version of Peetre's lemma (cf. [15]).

Lemma 5.3 **(Peetre)** *Let X, Y and Z be Banach spaces such that $X \subset Z$ with* compact *injection and let $T : X \to Y$ be a closed linear operator with domain $D(T)$. Then the following two conditions are equivalent:*

(a) The kernel $N(T)$ of T is finite-dimensional and the range $R(T)$ of T is closed in Y.
(b) There exists a constant $c > 0$ such that the inequality

$$|x|_X \leq c\,(|Tx|_Y + |x|_Z)$$

holds true for all $x \in D(T)$.

The second lemma characterizes the adjoint operator of ([22]):

Lemma 5.4 *Define a linear operator*

$$\widetilde{\mathscr{T}_1}^* : H^{-s+5/2}(\partial D \times S) \longrightarrow H^{-s+5/2-\kappa}(\partial D \times S)$$

as follows:

(a) The domain $D\left(\widetilde{\mathscr{T}_1}^*\right)$ is the space

$$D\left(\widetilde{\mathscr{T}_1}^*\right) = \left\{ \widetilde{\varphi} \in H^{-s+5/2}(\partial D \times S) : \left(L\widetilde{H}\right)^* \widetilde{\varphi} \in H^{-s+5/2-\kappa}(\partial D \times S) \right\}.$$

(b) $\widetilde{\mathscr{T}_1}^* \widetilde{\varphi} = \left(L\widetilde{H}\right)^* \widetilde{\varphi}$ for every $\widetilde{\varphi} \in D\left(\widetilde{\mathscr{T}_1}^*\right)$.

Here $\left(L\widetilde{H}\right)^*$ is the formal adjoint of $L\widetilde{H}$.

Then it follows that

$$\widetilde{\mathscr{T}}^* \subset \widetilde{\mathscr{T}_1}^*.$$

Step 3°: Now we are able to prove that the index of $\widetilde{\mathscr{T}}$ is finite.

By Rellich's compactness theorem, it follows that the injection $H^s(\partial D\times S)\subset H^t\partial D\times S)$ is compact for $t<s$. Hence, by using Lemma 5.3 with

$$\begin{aligned}
X &:= H^{s-5/2+\kappa}(\partial D\times S),\\
Y &:= H^{s-5/2}(\partial D\times S),\\
Z &:= H^{t}(\partial D\times S), \quad t<s-5/2+\kappa,\\
T &:= \widetilde{\mathscr{T}},
\end{aligned}$$

we obtain from estimate (5.17) with $s:=s-5/2$ that $\dim N(\widetilde{\mathscr{T}})<\infty$ and the range $R(\widetilde{\mathscr{T}})$ is closed in $H^{s-5/2}(\partial D\times S)$.

On the other hand, it follows from formula (5.13) that the symbol of the *formal adjoint* $\left(L\widetilde{H}\right)^*$ is given by the formula

$$\begin{aligned}
&\left[-\sum_{i,j=1}^{N-1}\alpha^{ij}(x')\xi_i\xi_j-\delta(x')\right]\\
&+\left[-\mu(x')\sqrt{|\xi'|^2+\eta^2}-\sqrt{-1}\sum_{i=1}^{N-1}\beta^i(x')\xi_i+\sqrt{-1}\sum_{i,j=1}^{N-1}\frac{\partial\alpha^{ij}}{\partial x_j}\xi_i\right]\\
&+\text{ terms of order}\leq 0.
\end{aligned}$$

Hence, just as in the proof of Lemma 5.2, we can obtain the following results:

(a) For each $s\in\mathbf{R}$, we have the regularity property

$$\begin{aligned}
&\widetilde{\varphi}\in\mathscr{D}'(\partial D\times S),\ \left(L\widetilde{H}\right)^*\widetilde{\varphi}\in H^{-s+5/2-\kappa}(\partial D\times S)\\
&\Longrightarrow\ \widetilde{\varphi}\in H^{-s+5/2}(\partial D\times S).
\end{aligned}\tag{5.18}$$

(b) For each $t<-s+5/2$, there exists a constant $\widetilde{C}^*_{s,t}>0$ such that the *a priori* estimate

$$\begin{aligned}
&|\widetilde{\varphi}|_{H^{-s+5/2}(\partial D\times S)}\\
&\leq\widetilde{C}^*_{s,t}\left(\left|\left(L\widetilde{H}\right)^*\widetilde{\varphi}\right|^2_{H^{-s+5/2-\kappa}(\partial D\times S)}+|\widetilde{\varphi}|^2_{H^t(\partial D\times S)}\right)
\end{aligned}\tag{5.19}$$

holds true.

Consequently, it follows from Lemma 5.4 and the regularity property (5.18) that

$$N\left(\widetilde{\mathscr{T}}^*\right)=\left\{\widetilde{\varphi}\in C^\infty(\partial D\times S):\left(L\widetilde{H}\right)^*=0\right\}$$

and hence from estimate (5.19) that

$$\operatorname{codim}R\left(\widetilde{\mathscr{T}}\right)=\dim N\left(\widetilde{\mathscr{T}}^*\right)<\infty.$$

Indeed, it suffices to note that $R\left(\widetilde{\mathscr{T}}\right)$ is closed in $H^{s-5/2}(\partial D\times S)$ and further from Rellich's compactness theorem that the injection $H^{s-5/2}(\partial D\times S)\subset H^t(\partial D\times S)$ is compact for $t<-s+5/2$.

Summing up, we have proved that the index of $\widetilde{\mathscr{T}}$ is finite.

Now the proof of Theorem 5.1 is complete. □

6 Fundamental *a priori* estimates

In this section we shall prove Lemma 5.1 which plays a fundamental role in the proof of Theorem 4.1. We remark that the proof of Lemma 5.2 is essentially the same as that of Lemma 5.1, as stated in Section 5.

The essential step in the proof is Proposition 6.2 below, which can be obtained by arguing as in the proof of Oleĭnik–Radkevič [17, Theorem 2.6.2] in a neighborhood of a point x_0' such that $\mu(x_0') = 0$ and by using Hörmander [13, Theorem 5.9] in a neighborhood of a point x_0' such that $\mu(x_0') > 0$.

6.1 Energy estimates

First, we prove the following *energy estimate*:

Proposition 6.1 *Let A and L be as in Theorem 5.1 and let U be a coordinate patch of the boundary ∂D with local coordinate $x' = (x_1, x_2, \ldots, x_{N-1})$. Then, for every compact set $K \subset U$ and $s \geq 0$ there exists a constant $C_{K,s,t} > 0$ such that the* energy estimate

$$\sum_{j=1}^{N-1}\left[\left|\sum_{i=1}^{N-1}\alpha^{ij}D_i\psi\right|^2_{H^s(\partial D)} + \left|\sum_{k,\ell=1}^{N-1}\frac{\partial\alpha^{k\ell}}{\partial x_j}D_kD_\ell\psi\right|^2_{H^{s-1}(\partial D)}\right] \leq C_{K,s}\left(|LH_\alpha\psi|^2_{L^2(\partial D)} + |\psi|^2_{H^{2s}(\partial D)}\right) \tag{6.1}$$

holds true for all $\psi \in C_0^\infty(K)$. Here

$$D_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}, \quad 1 \leq j \leq N-1.$$

The proof of Proposition 6.1 is carried out just as in the case where $\mu(x') \equiv 0$ on ∂D, that is, $LH_\alpha = Q_\alpha$, if we consider $\mu(x')\,\Pi_\alpha$ as a term of "perturbation" of Q_α and by using the following sharp Gårding inequality for $\mu(x')\,\Pi_\alpha$ ([12]):

Lemma 6.1 (the sharp Gårding inequality) *Let P be a properly supported, first-order pseudo-differential operator in $U \subset \mathbf{R}^{N-1}$. Assume that its principal symbol $p_1(x', \xi')$ is non-negative when $x' \in U$ and $\xi' \in \mathbf{R}^{N-1}$. Then, for every compact set $K \subset U$ there exists a constant $C_K > 0$ such that the inequality*

$$\operatorname{Re}\int_{\mathbf{R}^{N-1}} P\psi\cdot\overline{\psi}\,dx' \geq -C_K\int_{\mathbf{R}^{N-1}}|\psi|^2\,dx' \tag{6.2}$$

holds true for all $\psi \in C_0^\infty(K)$.

Proof (of Proposition 6.1) We remark that the case where $LH_\alpha = Q_\alpha$ is studied in great detail by Oleĭnik–Radkevič [17]. We shall denote by C a generic positive constant depending only on K.

We rewrite the differential operator Q_α in formula (5.6) as follows:

$$Q_\alpha\psi = -\sum_{i,j=1}^{N-1}D_j\left(\alpha^{ij}(x')D_i\psi\right) + \sum_{i=1}^{N-1}\left(\sum_{j=1}^{N-1}D_j\alpha^{ij}(x') + \sqrt{-1}\,\beta^i(x')\right)D_i\psi$$

$$+\big(\gamma(x') - \alpha\,\delta(x')\big)\,\psi.$$

Then we have, by integration by parts,

$$\operatorname{Re}(Q_\alpha\psi,\psi)_{L^2(\partial D)} = -\sum_{i,j=1}^{N-1}\big(\alpha^{ij}D_i\psi, D_j\psi\big)_{L^2(\partial D)} + \operatorname{Re}(h\,\psi,\psi)_{L^2(\partial D)} \tag{6.3}$$

where $h(x')$ is some function in $C^\infty(\partial D)$. Hence it follows from formula (6.3) that

$$\sum_{i,j=1}^{N-1}\big(\alpha^{ij}D_i\psi, D_j\psi\big)_{L^2(\partial D)} \le -\operatorname{Re}(Q_\alpha\psi,\psi)_{L^2(\partial D)} + C\,|\psi|^2_{L^2(\partial D)}\,. \tag{6.4}$$

On the other hand, since $\mu(x')\,\Pi_\alpha$ in formulaeq:5.6 is a first-order, pseudo-differential operator with non-positive principal symbol

$$-\mu(x')\,|\xi'| \le 0 \quad \text{on the cotangent bundle } T^*(\partial D),$$

by applying Lemma 6.1 to $-\mu(x')\,\Pi_\alpha$ we obtain that

$$0 \le -\operatorname{Re}(\mu\,\Pi_\alpha\psi,\psi)_{L^2(\partial D)} + C\,|\psi|^2_{L^2(\partial D)}\,. \tag{6.5}$$

Hence it follows from inequalities (6.4) and (6.5) that

$$\sum_{i,j=1}^{N-1}\big(\alpha^{ij}D_i\psi, D_j\psi\big)_{L^2(\partial D)} \le -\operatorname{Re}(LH_\alpha\psi,\psi)_{L^2(\partial D)} + C\,|\psi|^2_{L^2(\partial D)}\,. \tag{6.6}$$

The desired estimate (6.1) follows from estimate (6.6) just as in the proof of Oleĭnik–Radkevič [17, Theorem 2.6.1].

The proof of Proposition 6.1 is complete. □

6.2 Local *a priori* estimates

Next we prove a local version of the *a priori* estimate (5.8).

For a pseudo-differential operator P with symbol $p(x',\xi')$, we shall denote by $P^{(j)}$ and $P_{(j)}$ $(1 \le j \le N-1)$ pseudo-differential operators with symbols $\partial p(x',\xi')/\partial\xi_j$ and $D_j p(x',\xi')$, respectively.

Proposition 6.2 *Let A and L be as in Theorem 5.1. Assume that hypothesis (A) is satisfied. Then, for any point x'_0 of ∂D, we can find a neighborhood $U(x'_0)$ of x'_0 such that:*

For every compact set $K \subset U(x'_0)$, there exists a constant $0 < \kappa(K) \le 1$ such that we have, for any $s \in \mathbf{R}$ and $t < s+\kappa$,

$$\sum_{j=1}^{N-1}\left(\left|(LH_\alpha)^{(j)}\,\psi\right|^2_{H^{s+\kappa/2}(\partial D)} + \left|(LH_\alpha)_{(j)}\,\psi\right|^2_{H^{s-1+\kappa/2}(\partial D)}\right) + |\psi|^2_{H^{s+\kappa}(\partial D)} \tag{6.7}$$
$$\le C_{K,s,t}\left(|LH_\alpha\psi|^2_{H^s(\partial D)} + |\psi|^2_{H^t(\partial D)}\right), \quad \psi \in C_0^\infty(K),$$

. with a constant $C_{K,s,t} > 0$.

Step (1): Before the proof, we formulate hypothesis (A) in terms of local coordinates.

(i) Let x_0' be a point of $M = \{x' \in \partial D : \mu(x') = 0\}$ and let $U(x_0')$ be a coordinate patch of x_0' with local coordinate $x' = (x_1, x_2, \ldots, x_{N-1})$. We let

$$Y_j = \sum_{j=1}^{N-1} \alpha^{ij} \frac{\partial}{\partial x_j} \quad \text{for } 1 \leq j \leq N-1.$$

Then it is clear that hypothesis (A) at x_0' is equivalent to the following:

$(A)_{x_0'}$ The Lie algebra $\mathscr{L}(Y_1, Y_2, \ldots, Y_{N_1})$ over $\mathbf{R}$ generated by the vector fields $\{Y_i\}_{i=1}^{N-1}$ has rank $N-1$ at x_0'.

(ii) Furthermore, we can give a more useful formulation of hypothesis (A) in terms of symbols of differential operators.

For any multi-index $I = (i_1, i_2, \ldots, i_k)$ with $1 \leq i_\ell \leq N-1$ for $1 \leq \ell \leq k$, we associate a first-order differential operator

$$Y_I = \left[Y_{i_1}, \left[\cdots, \left[Y_{i_{k-1}}, Y_{i_k}\right]\right]\right],$$

and denote by $Y_I(x', \xi')$ its symbol. Then it is also clear that hypothesis $(A)_{x_0'}$ is equivalent to the following:

$(A)'_{x_0'}$ There exists an integer $R(x_0') \geq 1$ such that:

$$\sum_{|I| \leq R(x_0')} \left|Y_I(x_0', \xi')\right| > 0 \quad \text{for all } \xi' \in \mathbf{R}^{N-1} \setminus \{0\}, \tag{6.8}$$

where $|I| = k$ for $I = (i_1, i_2, \ldots, i_k)$.

Since the symbol $Y_I(x', \xi')$ is a positively homogeneous function of ξ' of first degree and continuous with respect to x' and ξ', we find from inequality (6.8) that hypothesis (A) is equivalent to the following:

$(A)'$ For every point x_0' of the set $M = \{x' \in \partial D : \mu(x') = 0\}$, we can find a neighborhood $U(x_0')$ of x_0' such that, for every compact set $K \subset U(x_0')$ there exist an integer $R(K) \geq 1$ and a constant $C_K > 0$ such that

$$1 + \sum_{|I| \leq R(x_0')} \left|Y_I(x', \xi')\right|^2 \geq C_K \left(1 + |\xi'|^2\right) \quad \text{for all } x \in K \text{ and } \xi' \in \mathbf{R}^{N-1}. \tag{6.9}$$

Step (2): The proof of Proposition 6.2 is divided into three steps.

Substep (2-1): First, we obtain the following:

Lemma 6.2 *Let U be a coordinate patch of the boundary ∂D. Then, for every compact set $K \subset U$, $s \geq 0$ and $k \geq 1$, there exists a constant $C_{K,s,t} > 0$ such that the inequality*

$$\sum_{|I|=k} |Y_I \psi|^2_{H^{s-1+2^{1-k}}(\partial D)} \leq C_{K,s,t} \left(|LH_\alpha \psi|^2_{L^2(\partial D)} + |\psi|^2_{H^{2s}(\partial D)}\right) \tag{6.10}$$

holds true for all $\psi \in C_0^\infty(K)$.

For $k = 1$, inequality (6.10) follows from inequality (6.1), and in the general case it is proved by induction on k as in the proof of Oleĭnik–Radkevič [17, Lemma 2.6.4].

Substep (2-2): Now we prove estimate (6.7) in the case where $\mu(x_0') = 0$. We shall denote by C a generic positive constant depending only on K, s and t.

As stated in Step (I), if hypothesis (A) is satisfied, then we can find a neighborhood $U(x_0')$ of x_0' such that:

For every compact set $K \subset U(x'_0)$ inequality (6.9) holds true with some integer $R(K) \geq 1$ and a constant $C_K > 0$. Thus, by applying Oleĭnik–Radkevič [17, Theorem 2.2.9] to the differential operators Y_I, $|I| \leq R(K)$, we obtain that

$$|\psi|^2_{H^{s+1}(\partial D)} \leq C \left(\sum_{|I| \leq R(K)} |Y_I \psi|^2_{H^s(\partial D)} + |\psi|^2_{H^s(\partial D)} \right),$$

and hence from inequality (6.10) with $s := s + 1 - 2^{1-R(K)}$ that

$$|\psi|^2_{H^{s+1}(\partial D)} \leq C \left(|LH_\alpha \psi|^2_{L^2(\partial D)} + |\psi|^2_{H^{2(s+1-2^{1-R(K)})}(\partial D)} + |\psi|^2_{H^s(\partial D)} \right). \tag{6.11}$$

We now choose

$$s := 2^{1-R(K)} - 1,$$

and let

$$\kappa(K) := 2^{1-R(K)}.$$

Then we note that

$$s \leq 0,$$
$$0 < \kappa(K) \leq 1,$$

since $1 \leq R(K) < \infty$.

Therefore, by the well-known *interpolation inequality* (cf. Oleĭnik–Radkevič [17, Theorem 2.1.12]), we have, for $t < \kappa(K)$,

$$|\psi|^2_{H^\kappa(\partial D)} \leq C \left(|LH_\alpha \psi|^2_{L^2(\partial D)} + |\psi|^2_{H^t(\partial D)} \right). \tag{6.12}$$

Consequently, just as in the proof of Oleĭnik–Radkevič [17, Theorem 2.6.2], we obtain from inequalities (6.1) and (6.12) the desired estimate (6.7) in the case where $\mu(x'_0) = 0$.

Substep (2-3): Finally, we prove estimate (6.7) in the case where $\mu(x'_0) > 0$ by using the following result due to Hörmander [13]:

Theorem 6.1 *Let P be a properly supported, pseudo-differential operator in $U \subset \mathbf{R}^{N-1}$ of order m and denote by $p_m(x', \xi')$ (resp. $p'_{m-1}(x', \xi')$) its principal (resp. subprincipal) symbol. Assume that the range of $p_m(x', \xi')$ belongs to a closed angle Γ with opening $< \pi$ and that the range of $-p'_{m-1}(x', \xi')$ on the characteristic set*

$$\Sigma = \left\{ (x', \xi') \in U \times \left(\mathbf{R}^{N-1} \setminus \{0\} \right) : p_m(x', \xi') = 0 \right\}$$

belongs to another closed angle Γ' with $\Gamma \cap \Gamma' = \{0\}$.

Then the following two conditions are equivalent:

(a) For every compact set $K \subset U$, $s \in \mathbf{R}$ and $t < s + m - 1$, there exists a constant $C_{K,s,t} > 0$ such that the inequality

$$|\psi|^2_{H^{s+m-1}(\mathbf{R}^{N-1})} \leq C_{K,s,t} \left(|P\psi|^2_{H^s(\mathbf{R}^{N-1})} + |\psi|^2_{H^t(\mathbf{R}^{N-1})} \right)$$

holds true for all $\psi \in C_0^\infty(K)$.

(b) At every point (x', ξ') of Σ, either $p'_{m-1}(x', \xi') \neq 0$ or else the Hamiltonian map of the Hessian of p_m is not nilpotent.

By condition (2.2) and formula (5.4), we have the following:

(a) The principal symbol

$$p_2(x', \xi' = -\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\, \xi_i\, \xi_j$$

of LH_α satisfies the condition

$$p_2(x', \xi') \leq 0 \quad \text{on } T^*(\partial D) \setminus \{0\}.$$

(b) The characteristic set Σ of LH_α is given by the formula

$$\Sigma = \left\{ (x', \xi') \in T^*(\partial D) \setminus \{0\} : \sum_{i,j=1}^{N-1} \alpha^{ij}(x')\, \xi_i\, \xi_j = 0 \right\}.$$

(c) The subprincipal symbol $p'_1(x', \xi')$ of LH_α on Σ is equal to the following:

$$p'_1(x', \xi') = -\mu(x')\,|\xi'| + \sqrt{-1} \left(\sum_{j=1}^{N-1} \beta^j(x')\, xi_j - \sum_{i,j=1}^{N-1} \frac{\partial \alpha^{ij}}{\partial x_j}\, \xi_i \right). \tag{6.13}$$

In this way, we find that all the hypotheses of Theorem 6.1 are satisfied for the operator LH_α. Furthermore, if $\mu(x'_0) > 0$, then we can find a neighborhood $U(x'_0)$ of x'_0 such that $\mu(x') > 0$ in $U(x'_0)$ and hence from condition (6.13)

$$p_1(x', \xi') \neq 0 \quad \text{in } U(x'_0) \times \left(\mathbf{R}^{N-1} \setminus \{0\} \right).$$

Therefore, by applying Theorem 6.1 to LH_α, we have, for every compact set $K \subset U(x'_0)$,

$$|\psi|^2_{H^1(\partial D)} \leq C \left(|LH_\alpha \psi|^2_{L^2(\partial D)} + |\psi|^2_{H^t(\partial D)} \right). \tag{6.14}$$

Here $t < 1$.

Consequently, in the case where $\mu(x'_0) > 0$, the desired estimate (6.7) with $\kappa(K) = 1$ follows from inequalities (6.1) and (6.14) in the same way as the desired estimate (6.7) with $\kappa(K) = 2^{1-R(K)}$ follows from inequalities (6.1) and (6.12) in the case where $\mu(x'_0) = 0$.

The proof of Proposition 6.2 is complete. □

6.3 Two-parameter family of Sobolev norms

Before the proof of Lemma 5.1, we must introduce a two-parameter family of norms on the boundary ∂D as in Hörmander [12] ([6], [17]).

(I) First, we introduce a two-parameter family of norms on the Sobolev spaces $H^s(\mathbf{R}^n)$. If $m > 0$ and $0 < \rho < 1$, we let

$$\|u\|^2_{H^{(s,m,\rho)}(\mathbf{R}^n)} = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \left(1 + |\xi|^2\right)^s \left(1 + |\rho\xi|^2\right)^{-m} |\widehat{u}(\xi)|^2 d\xi. \tag{6.15}$$

We list two results which follow at once:

(1) For all $u \in H^{s-m}(\mathbf{R}^n)$, we have the inequalities

$$\rho^m \|u\|_{H^{(s,m,\rho)}(\mathbf{R}^n)} \leq \|u\|_{H^{s-m}(\mathbf{R}^n)} \leq \|u\|_{H^{(s,m,\rho)}(\mathbf{R}^n)},$$

that is, the norm $\|u\|_{H^{(s,m,\rho)}(\mathbf{R}^n)}$ is equivalent to the norm $\|u\|_{H^{s-m}(\mathbf{R}^n)}$.

(2) If $u \in H^s(\mathbf{R}^n)$, then we have the assertion

$$\|u\|_{H^{(s,m,\rho)}(\mathbf{R}^n)} \uparrow \|u\|_{H^s(\mathbf{R}^n)} \quad \text{as } \rho \downarrow 0,$$

so that

$$\|u\|_{H^s(\mathbf{R}^n)} = \sup_{0<\rho<1} \|u\|_{H^{(s,m,\rho)}(\mathbf{R}^n)}.$$

This is an immediate consequence of the dominated convergence theorem.

The two-parameter family $\|\cdot\|_{H^{(s,1,\rho)}(\mathbf{R}^n)}$ of norms was introduced by Hörmander [11], and was used to prove regularity theorems for linear partial differential equations. See also Hörmander [12], Fediĭ [6] and Oleĭnik–Radkevič [17].

(II) Let $s \in \mathbf{R}$, $m > 0$ and $0 < \rho < 1$. In the Sobolev space $H^{s-m}(\partial D)$, we consider a norm $|\cdot|_{H^{(s,m,\rho)}(\partial D)}$ by the formula

$$|\psi|_{H^{(s,m,\rho)}(\partial D)} = \inf_{\psi'+\psi''=\psi} \left(\frac{1}{\rho^m} \left|\psi'\right|_{H^{s-m}(\partial D)} + \left|\psi''\right|_{H^s(\partial D)} \right). \tag{6.16}$$

Then the above results are also true for the spaces $H^s(\partial D)$. More precisely, we have the following results (cf. Hörmander [11], [12]):

(1) The norm $|\cdot|_{H^{(s,m,\rho)}(\partial D)}$ increases as $\rho \downarrow 0$, and we have the formula

$$|\psi|_{H^s(\partial D)} = \sup_{0<\rho<1} |\psi|_{(s,m,\rho)}$$

if $\psi \in H^s(\partial D)$.

(2) The norm $|\cdot|_{H^{(s,m,\rho)}(\partial D)}$ has locally an equivalent expression such as formula (6.15); hence it is equivalent to the norm $|\cdot|_{H^{s-m}(\partial D)}$.

(III) We need two lemmas on the norms $|\cdot|_{H^{(s,m,\rho)}(\partial D)}$, which can be proved just as in the proof of [12, Theorem 1.4.9] in the case where $m = 1$. So we may omit their proofs.

The first lemma explains a motivation of introduction of the norms $|\cdot|_{H^{(s,m,\rho)}(\partial D)}$:

Lemma 6.3 *If there exists a constant $C > 0$, independent of ρ, such that*

$$\sup_{0<\rho<1} |\psi|_{H^{(s,m,\rho)}(\partial D)} \leq C \quad \text{for } \psi \in H^{(s,m,\rho)}(\partial D),$$

then it follows that $\psi \in H^s(\partial D)$ and we have the inequality $|\psi|_{H^s(\partial D)} \leq C$.

The second lemma gives another expression for the norms $|\cdot|_{H^{(s,m,\rho)}(\partial D)}$ via Friedrichs' mollifiers. Let U be a coordinate patch of the boundary ∂D with local coordinate $x' = (x_1, x_2, \ldots, x_{N-1})$ such that

$$U = \left\{ \left|x'\right| < l \right\}$$

and let

$$K = \left\{ \left|x'\right| < r \right\}, \quad 0 < r < l,$$

be a compact subset of U. Choose a (fixed) function $\chi \in C_0^\infty(U)$ such that

$$\begin{cases} \operatorname{supp}\chi \subset \{|x'|1 - r\}, & \\ \widehat{\chi}(\xi') = O\left(|\xi'|^k\right) & \text{as } |\xi'| \to 0, \text{ for an integer } k \geq 0, \\ \widehat{\chi}(\tau\xi') = 0 & \text{for all } \tau \in \mathbf{R} \text{ only when } \xi' = 0. \end{cases} \tag{6.17}$$

For example, we may take

$$\chi(x') = \Delta^k \left(\theta(x')\right)$$

where $\theta(x')$ is a function in $C_0^\infty(\mathbf{R}^{n-1})$ such that

$$\widehat{\theta}(0) = \int_{\mathbf{R}^{n-1}} \theta(x')\, dx' \neq 0.$$

Furthermore, we let

$$\chi_\varepsilon(x') = \frac{1}{\varepsilon^{N-1}}\, \chi\left(\frac{x'}{\varepsilon}\right) \quad \text{for } 0 < \varepsilon < 1.$$

The next lemma gives another equivalent expression for the norm $|\psi|_{H^{(s,m,\rho)}(\partial D)}$ in terms of the regularizations $\psi * \chi_\varepsilon$ of ψ:

Lemma 6.4 *Let U, K and $\chi(x')$ be as above. Assume that the function $\chi(x')$ satisfies condition* (6.17) *for $k > s$. Then, for any $s_1 \in \mathbf{R}$ and $t < s + s_1 - m$, there exist constants $C_{s,s_1,t} > 0$ and $C'_{s,s_1,t} > 0$, independent of ρ, such that*

$$\begin{aligned} &|\psi|^2_{H^{(s+s_1,m,\rho)}(\partial D)} \\ &\leq C_{s,s_1,t} \left(\int_0^1 |\psi * \chi_\varepsilon|^2_{H^{s_1}(\partial D)} \left(1 + \frac{\rho^2}{\varepsilon^2}\right)^{-m} \varepsilon^{-2s}\, \frac{d\varepsilon}{\varepsilon} + |\psi|^2_{H^{(s+s_1,m,\rho)}(\partial D)} \right) \\ &\leq C'_{s,s_1,t}\, |\psi|^2_{H^{(s+s_1,m,\rho)}(\partial D)} \end{aligned} \tag{6.18}$$

for all $\psi \in H^{s+s_1-m}(\partial D)$ with support in K.

6.4 Proof of Lemma 5.1

By Proposition 6.2, we can cover the boundary ∂D by a finite number of coordinate patches $\{U_j\}_{j=1}^d$ in each of which estimate (6.7) holds true for all $\psi \in C_0^\infty(U_j)$. Let $\{\psi_j\}_{j=1}^d$ be a partition of unity subordinate to the covering $\{U_j\}_{j=1}^d$, and choose functions $\varphi_j \in C_0^\infty(U_j)$ such that $\varphi_j \equiv 1$ on $\operatorname{supp} \psi_j$. We let

$$\kappa = \min_{1 \leq j \leq d} \kappa\left(\operatorname{supp} \varphi_j\right).$$

Without loss of generality, we may assume that $\psi \in H^t(\partial D)$ for some $t < s + \kappa$. Thus, in order to prove the lemma, it suffices to show that, for each ψ_j we have the assertions

$$\psi_j \psi \in H^t(\partial D),\ LH_\alpha \psi \in H^s(\partial D) \implies \psi_j \psi \in H^{s+\kappa}(\partial D). \tag{5.7$'$}$$

$$\left|\psi_j \psi\right|^2_{H^{s+\kappa}(\partial D)} \leq C \left(|LH_\alpha \psi|^2_{H^s(\partial D)} + \left|\psi_j \psi\right|^2_{H^t(\partial D)} \right). \tag{5.8$'$}$$

We shall denote by C a generic positive constant depending only on s and t. Now we choose $0 < m < s + \kappa - t$ for m in norm (6.15) and $k = [s] + 1$ for k in condition (6.17) where $[s]$ stands for the integral part of s. Then, by applying the first inequality of (6.18) with $s_1 := \kappa$ to $\psi := \psi_j \psi$ and further inequality (6.7) with $s := 0$ and $t := t - s (< \kappa)$ to $\psi := (\psi_j \psi) * \chi_\varepsilon$, we obtain the inequality

$$\left|\psi_j \psi\right|^2_{H^{(s+\kappa,m,\rho)}(\partial D)} \tag{6.19}$$

$$\leq C\left(\int_0^1 \left|((\psi_j\psi)*\chi_\varepsilon)\right|^2_{H^\kappa(\partial D)}\left(1+\frac{\rho^2}{\varepsilon^2}\right)^{-m}\varepsilon^{-2s}\frac{d\varepsilon}{\varepsilon}+\left|\psi_j\psi\right|^2_{H^t(\partial D)}\right)$$
$$\leq C\left(\int_0^1 \left|LH_\alpha\left((\psi_j\psi)*\chi_\varepsilon\right)\right|^2_{L^2(\partial D)}\left(1+\frac{\rho^2}{\varepsilon^2}\right)^{-m}\varepsilon^{-2s}\frac{d\varepsilon}{\varepsilon}\right.$$
$$\left.+\int_0^1 \left|((\psi_j\psi)*\chi_\varepsilon)\right|^2_{H^{t-s}(\partial D)}\left(1+\frac{\rho^2}{\varepsilon^2}\right)^{-m}\varepsilon^{-2s}\frac{d\varepsilon}{\varepsilon}+\left|\psi_j\psi\right|^2_{H^t(\partial D)}\right).$$

By using the second inequality of (6.18) and inequality (6.16) to estimate the second term of the last inequality of (6.19), we have the inequality

$$\begin{aligned}&\int_0^1 \left|((\psi_j\psi)*\chi_\varepsilon)\right|^2_{H^{t-s}(\partial D)}\left(1+\frac{\rho^2}{\varepsilon^2}\right)^{-m}\varepsilon^{-2s}\frac{d\varepsilon}{\varepsilon} \\ &\leq C\left|\psi_j\psi\right|^2_{H^{(t,m,\rho)}(\partial D)} \\ &\leq C\left|\psi_j\psi\right|^2_{H^t(\partial D)}.\end{aligned} \tag{6.20}$$

Furthermore, in light of the *pseudo locality* for pseudo-differential operators, by arguing as in the proof of Oleĭnik–Radkevič [17, Theorem 2.4.2], we can estimate the first term of the last inequality of (6.19) as follows (cf. [17, inequality (2.4.46)]):

$$\begin{aligned}&\int_0^1 \left|LH_\alpha\left((\psi_j\psi)*\chi_\varepsilon\right)\right|^2_{L^2(\partial D)}\left(1+\frac{\rho^2}{\varepsilon^2}\right)^{-m}\varepsilon^{-2s}\frac{d\varepsilon}{\varepsilon} \\ &\leq C\left(\left|LH_\alpha\psi\right|^2_{H^s(\partial D)}+\left|\psi_j\psi\right|^2_{H^t(\partial D)}\right).\end{aligned} \tag{6.21}$$

Hence it follows from inequalities (6.19), (6.20) and (6.21) that

$$\left|\psi_j\psi\right|^2_{H^{(s+\kappa,m,\rho)}(\partial D)} \leq C\left(\left|LH_\alpha\psi\right|^2_{H^s(\partial D)}+\left|\psi_j\psi\right|^2_{H^t(\partial D)}\right).$$

Therefore, by virtue of Lemma 6.3, we obtain regularity property $(5.7)^{|prime}$ and estimate $(5.8)'$.

The proof of Lemma 5.1 is complete. □

7 Concluding remark

In applying Corollary 2.1 we confined ourselves to the case where the differential operator A is elliptic on $\overline{D}$. The reason is that when the operator A satisfies only hypothesis (H) we do not know whether the operator LH_α written as formula (5.3), which played a fundamental role in the proof of Theorem 5.1, is a pseudo-differential operator on the boundary or not. It is an open problem to extend Theorem 2.2 to the case where the differential operator A satisfies hypothesis (H).

Acknowledgements This research is partially supported by NSF grant MCS 77-18723 A03.

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