Suppl. Rend. Circ. Mat. Palermo
Vol. 52 (1998), 157–173

FELLER SEMIGROUPS WITH BOUNDARY CONDITIONS

Kazuaki TAIRA

Abstract. This expository paper is devoted to the problem of construction of Feller semigroups with Ventcel' (Wentzell) boundary conditions for *elliptic* Waldenfels operators. Intuitively our result may be stated as follows: We can construct a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space until it "dies" at which time it reaches the set where the absorption phenomenon occurs.

Introduction and Results

Let D be a bounded, *convex* domain of Euclidean space $\mathbf{R}^N$ with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional compact smooth manifold with boundary.

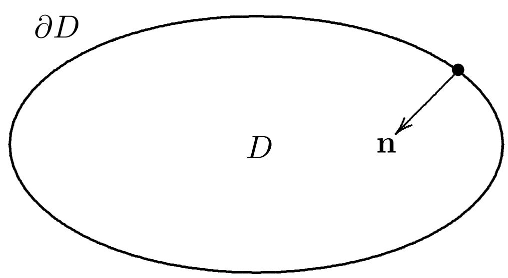


Figure 1

Let $C(\overline{D})$ be the space of real-valued, continuous functions on $\overline{D}$. We equip the space $C(\overline{D})$ with the topology of uniform convergence on the whole $\overline{D}$; hence it is a Banach space with the maximum norm

$$\|f\| = \max_{x \in \overline{D}} |f(x)|.$$

A strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on the Banach space $C(\overline{D})$ is called a *Feller semigroup* on $\overline{D}$ if it is non-negative and contractive on $C(\overline{D})$:

$$f \in C(\overline{D}),\ 0 \leq f \leq 1 \quad \text{on } \overline{D} \implies 0 \leq T_t f \leq 1 \quad \text{on } \overline{D}.$$

1991 *Mathematics Subject Classification.* Primary 47D07, 35J25; Secondary 47D05, 60J35, 60J60.

Key words and phrases. Feller semigroup, Markov process, elliptic boundary value problem.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

It is known (cf. [Ta1]) that if T_t is a Feller semigroup on $\overline{D}$, then there exists a unique Markov transition function p_t on $\overline{D}$ such that

$$T_t f(x) = \int_{\overline{D}} p_t(x, dy) f(y), \quad f \in C(\overline{D}).$$

It can be shown that the function p_t is the transition function of some strong *Markov process*; hence the value $p_t(x, E)$ expresses the transition probability that a Markovian particle starting at position x will be found in the set E at time t.

Furthermore, it is known (cf. [BCP], [SU], [Ta1], [We]) that the infinitesimal generator of a Feller semigroup $\{T_t\}_{t \geq 0}$ is described analytically by a Waldenfels operator W and a Ventcel' boundary condition L, which we formulate precisely.

Let W be a second-order *elliptic* integro-differential operator with real coefficients such that

$$\begin{aligned}
(0.1) \qquad Wu(x) &= Pu(x) + S_r u(x) \\
&:= \left(\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) \right) \\
&\quad + \int_D s(x,y) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right] dy,
\end{aligned}$$

where:

(1) $a^{ij} \in C^\infty(\mathbf{R}^N)$, $a^{ij} = a^{ji}$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x) \xi_i \xi_j \geq a_0 |\xi|^2, \quad x \in \mathbf{R}^N, \ \xi \in \mathbf{R}^N.$$

(2) $b^i \in C^\infty(\mathbf{R}^N)$.

(3) $c \in C^\infty(\mathbf{R}^N)$ and $c \leq 0$ in D.

(4) The integral kernel $s(x,y)$ is the distribution kernel of a properly supported, pseudo-differential operator $S \in L^{2-\kappa}_{1,0}(\mathbf{R}^N)$, $\kappa > 0$, which has the *transmission property* with respect to ∂D, and $s(x,y) \geq 0$ off the diagonal $\{(x,x) : x \in \mathbf{R}^N\}$ in $\mathbf{R}^N \times \mathbf{R}^N$. The measure dy is the Lebesgue measure on $\mathbf{R}^N$.

The operator W is called a second-order *Waldenfels operator* (cf. [BCP]). The differential operator P is called a diffusion operator which describes analytically a strong Markov process with continuous paths (diffusion process) in the interior D. The operator S_r is called a second-order Lévy operator which is supposed to correspond to the jump phenomenon in the interior D; a Markovian particle moves by jumps to a random point, chosen with kernel $s(x,y)$, in the interior D. Therefore, the Waldenfels operator W is supposed to correspond to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space D (see Figure 2 below).

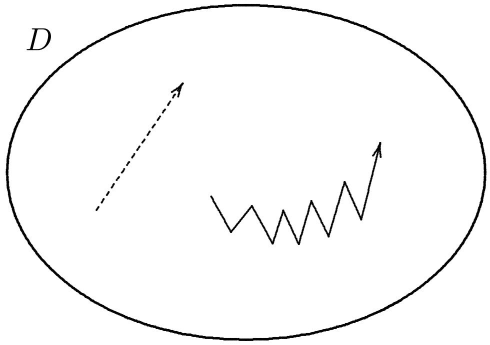


Figure 2

Let L be a second-order boundary condition such that we have, in terms of local coordinates $(x_1, x_2, \ldots, x_{N-1})$,

(0.2)

$$
\begin{aligned}
Lu(x') &= Qu(x') + \mu(x')\frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x')Wu(x') + \Gamma u(x') \\
&:= \left(\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x')\frac{\partial u}{\partial x_i}(x') + \gamma(x')u(x')\right) \\
&\quad + \mu(x')\frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x')Wu(x') \\
&\quad + \left(\int_{\partial D} r(x', y')\left[u(y') - u(x') - \sum_{j=1}^{N-1}(y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right] dy'\right. \\
&\quad \left. + \int_D t(x', y)\left[u(y) - u(x') - \sum_{j=1}^{N-1}(y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right] dy\right),
\end{aligned}
$$

where:

(1) The operator Q is a second-order degenerate elliptic differential operator on ∂D with non-positive principal symbol. In other words the α^{ij} are the components of a smooth symmetric contravariant tensor of type $\binom{2}{0}$ on ∂D satisfying the condition

$$
\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j \geq 0, \quad x' \in \partial D,\ \xi' = \sum_{j=1}^{N-1} \xi_j dx_j \in T^*_{x'}(\partial D).
$$

Here $T^*_{x'}(\partial D)$ is the cotangent space of ∂D at x'.

(2) $Q1 = \gamma \in C^\infty(\partial D)$ and $\gamma \leq 0$ on ∂D.

(3) $\mu \in C^\infty(\partial D)$ and $\mu \geq 0$ on ∂D.

(4) $\delta \in C^\infty(\partial D)$ and $\delta \geq 0$ on ∂D.

(5) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D.

(6) The integral kernel $r(x',y')$ is the distribution kernel of a pseudo-differential operator $R \in L^{2-\kappa_1}_{1,0}(\partial D)$, $\kappa_1 > 0$, and $r(x',y') \geq 0$ off the diagonal $\Delta_{\partial D} = \{(x',x') : x' \in \partial D\}$ in $\partial D \times \partial D$. The density dy' is a strictly positive density on ∂D.

(7) The integral kernel $t(x,y)$ is the distribution kernel of a properly supported, pseudo-differential operator $T \in L^{2-\kappa_2}_{1,0}(\mathbf{R}^N)$, $\kappa_2 > 0$, which has the transmission property with respect to the boundary ∂D, and $t(x,y) \geq 0$ off the diagonal $\{(x,x) : x \in \mathbf{R}^N\}$ in $\mathbf{R}^N \times \mathbf{R}^N$.

The boundary condition L is called a second-order *Ventcel' boundary condition*. The six terms of L

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i}(x'),$$

$$\gamma(x')u(x'), \quad \mu(x')\frac{\partial u}{\partial \mathbf{n}}(x'), \quad \delta(x')Wu(x'),$$

$$\int_{\partial D} r(x',y') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy',$$

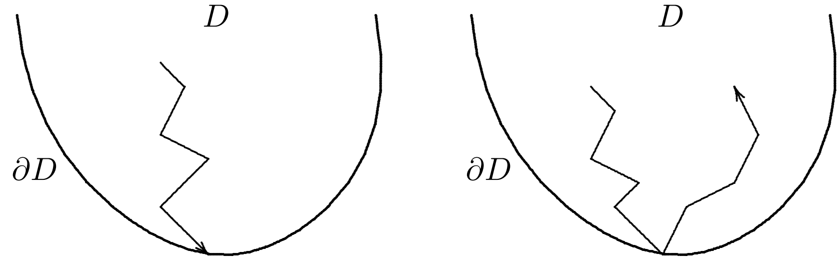
$$\int_D t(x',y) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy$$

are supposed to correspond to the diffusion along the boundary, the absorption phenomenon, the reflection phenomenon, the viscosity phenomenon and the jump phenomenon on the boundary and the inward jump phenomenon from the boundary, respectively (see Figures 3 through 5 below).

This paper is devoted to the functional analytic approach to the problem of construction of Feller semigroups with Ventcel' boundary conditions. More precisely, we consider the following problem:

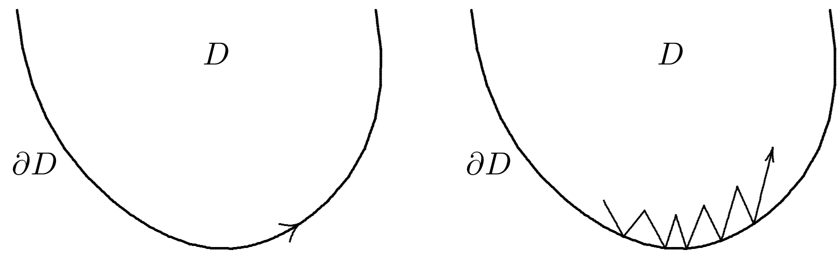
Problem. *Conversely, given analytic data (W, L), can we construct a Feller semigroup $\{T_t\}_{t\geq 0}$ whose infinitesimal generator is characterized by (W, L) ?*

We shall only restrict ourselves to some aspects which have been discussed in our papers [Ta1] through [Ta5]. Our approach is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of partial differential equations. It focuses on the relationship between two interrelated subjects in analysis; Feller semigroups and elliptic boundary value problems, providing powerful methods for future research.



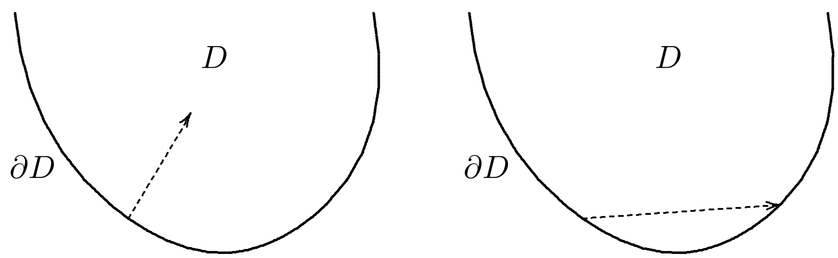
absorption reflection

Figure 3



diffusion along the boundary viscosity

Figure 4



jump into the interior jump on the boundary

Figure 5

We say that the boundary condition L is *transversal* on the boundary ∂D if it satisfies the condition

$$\int_D t(x', y)dy = +\infty \quad \text{if } \mu(x') = \delta(x') = 0. \tag{0.3}$$

The intuitive meaning of condition (0.3) is that a Markovian particle jumps away "instantaneously" from the points $x' \in \partial D$ where neither reflection nor viscosity phenomenon occurs (which is similar to the reflection phenomenon). The situation may be represented schematically by Figure 6 below. Probabilistically, condition

(0.3) means that every Markov process on the boundary ∂D is the "trace" on ∂D of trajectories of some Markov process on the closure $\overline{D} = D \cup \partial D$.

The next theorem asserts that there exists a Feller semigroup on $\overline{D}$ corresponding to such a diffusion phenomenon that one of the reflection phenomenon, the viscosity phenomenon and the inward jump phenomenon from the boundary occurs at each point of the boundary ∂D (cf. [Ta3, Theorem 1]):

Theorem 1. *We define a linear operator $\mathfrak{A}$ from the space $C(\overline{D})$ into itself as follows:*

(a) The domain of definition $D(\mathfrak{A})$ of $\mathfrak{A}$ is the set

$$D(\mathfrak{A}) = \left\{u \in C(\overline{D}) : Wu \in C(\overline{D}), Lu = 0\right\}. \tag{0.4}$$

(b) $\mathfrak{A}u = Wu$, $u \in D(\mathfrak{A})$.
Here Wu and Lu are taken in the sense of distributions.

Assume that the boundary condition L is transversal *on the boundary ∂D. Then the operator $\mathfrak{A}$ generates a Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D}$.*

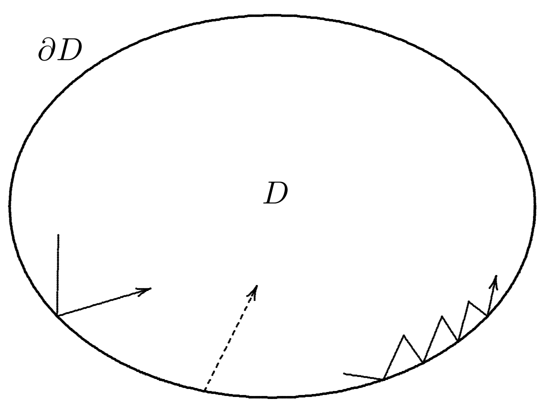


Figure 6

We remark that Theorem 1 was proved before by Taira [Ta1, Theorem 10.1.3] under some additional conditions, and also by Cancelier [Ca, Théorème 3.2]). On the other hand Takanobu and Watanabe [TW] proved a probabilistic version of Theorem 1 in the case where the domain D is the half space $\mathbf{R}^N_+$ (see [TW, Corollary]).

Next we generalize Theorem 1 to the *non-transversal* case. To do so, we assume that:

(H) There exists a second-order Ventcel' boundary condition L_ν such that

$$Lu = m\, L_\nu u + \gamma\, u,$$

where

(3′) $m \in C^\infty(\partial D)$ and $m \geq 0$ on ∂D, and the boundary condition L_ν is given in local coordinates $(x_1, x_2, \dots, x_{N-1})$ by the formula

$$\begin{aligned}
&L_\nu u(x') \\
&= \overline{Q}u(x') + \overline{\mu}(x')\frac{\partial u}{\partial \mathbf{n}}(x') - \overline{\delta}(x')Wu(x') + \overline{\Gamma}u(x') \\
&:= \left(\sum_{i,j=1}^{N-1} \overline{\alpha}^{ij}(x')\frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \overline{\beta}^{i}(x')\frac{\partial u}{\partial x_i}(x')\right) \\
&\quad + \overline{\mu}(x')\frac{\partial u}{\partial \mathbf{n}}(x') - \overline{\delta}(x')\, Wu(x') \\
&\quad + \left(\int_{\partial D} \overline{r}(x', y')\left[u(y') - u(x') - \sum_{j=1}^{N-1}(y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right] dy'\right. \\
&\quad + \left.\int_D \overline{t}(x', y)\left[u(y) - u(x') - \sum_{j=1}^{N-1}(y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right] dy\right),
\end{aligned}$$

and satisfies the *transversality* condition

$$\int_D \overline{t}(x', y)\, dy = +\infty \quad \text{if } \overline{\mu}(x') = \overline{\delta}(x') = 0. \tag{0.3'}$$

We let

$$M = \{x' \in \partial D : \mu(x') = \delta(x') = 0, \int_D t(x', y)\, dy < \infty\}.$$

Then, by condition (0.3′) it follows that

$$M = \{x' \in \partial D : m(x') = 0\},$$

since we have $\mu(x') = m(x')\,\overline{\mu}(x')$, $\delta(x') = m(x')\,\overline{\delta}(x')$, and $t(x', y) = m(x')\,\overline{t}(x', y)$. Hence we find that the boundary condition L is *not* transversal on ∂D.

Furthermore, we assume that:

(A) $m(x') - \gamma(x') > 0$ on ∂D.

The intuitive meaning of conditions (H) and (A) is that a Markovian particle does not stay on ∂D for any period of time until it "dies" at the time when it reaches the set M where the particle is definitely absorbed. We remark that condition (0.3) is a special case of conditions (H) and (A) if we take $m \equiv 1$ and $\gamma \equiv 0$ on ∂D.

Now we introduce a subspace of $C(\overline{D})$ which is associated with the boundary condition L.

By condition (A), we find that the boundary condition

$$Lu = m\, L_\nu u + \gamma\, u = 0 \quad \text{on } \partial D$$

includes the condition

$$u = 0 \quad \text{on } M.$$

With this fact in mind, we let

$$C_0(\overline{D} \setminus M) = \{u \in C(\overline{D}) : u = 0 \text{ on } M\}.$$

The space $C_0(\overline{D} \setminus M)$ is a closed subspace of $C(\overline{D})$; hence it is a Banach space.

A strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on the space $C_0(\overline{D} \setminus M)$ is called a *Feller semigroup* on $\overline{D} \setminus M$ if it is non-negative and contractive on $C_0(\overline{D} \setminus M)$:

$$f \in C_0(\overline{D} \setminus M),\ 0 \leq f \leq 1 \text{ on } \overline{D} \setminus M \implies 0 \leq T_t f \leq 1 \text{ on } \overline{D} \setminus M.$$

We define a linear operator $\mathfrak{W}$ from $C_0(\overline{D} \setminus M)$ into itself as follows:

(a) The domain of definition $D(\mathfrak{W})$ of $\mathfrak{W}$ is the set

$$D(\mathfrak{W}) = \left\{u \in C_0(\overline{D} \setminus M) : Wu \in C_0(\overline{D} \setminus M), Lu = 0\right\}. \tag{0.5}$$

(b) $\mathfrak{W}u = Wu$, $u \in D(\mathfrak{W})$.

The next theorem is a generalization of Theorem 1 to the non-transversal case (cf. [Ta5, Theorem 2]):

Theorem 2. *Assume that conditions (A) and (H) are satisfied. Then the operator* $\mathfrak{W}$ *defined by formula (0.5) generates a Feller semigroup* $\{T_t\}_{t \geq 0}$ *on* $\overline{D} \setminus M$.

If T_t is a Feller semigroup on $\overline{D} \setminus M$, then there exists a unique Markov transition function p_t on $\overline{D} \setminus M$ such that

$$T_t f(x) = \int_{\overline{D} \setminus M} p_t(x, dy) f(y), \quad f \in C_0(\overline{D} \setminus M),$$

and further that p_t is the transition function of some strong Markov process. On the other hand, the intuitive meaning of conditions (A) and (H) is that the absorption phenomenon occurs at each point of the set $M = \{x' \in \partial D : \mu(x') = \delta(x') = 0\}$. Therefore, Theorem 2 asserts that there exists a Feller semigroup on $\overline{D} \setminus M$ corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space $\overline{D} \setminus M$ until it "dies" at which time it reaches the set M. The situation may be represented schematically by Figure 7 below.

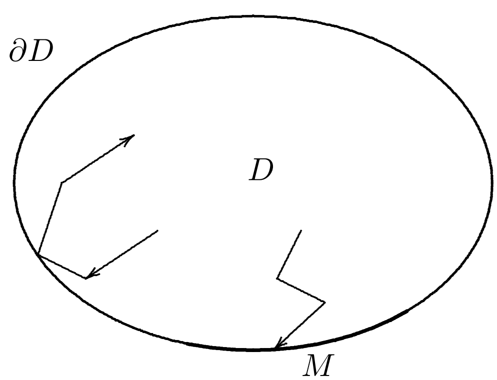


Figure 7

We remark that Taira [Ta2] has proved Theorem 2 under the condition that $L_\nu = \partial/\partial \mathbf{n}$ and $\delta \equiv 0$ on ∂D, by using the L^p theory of pseudo-differential operators (see [Ta2, Theorem 4]).

1. Theory of Feller Semigroups

We give a brief description of basic definitions and results about a class of semigroups (Feller semigroups) associated with Markov processes which forms a functional analytic background for the proof of Theorems 1 and 2. The results discussed here are adapted from [Ta1, Chapter 9].

1.1 Markov Transition Functions and Feller Semigroups. Let (K, ρ) be a locally compact, separable metric space and $\mathcal{B}$ the σ-algebra of all Borel sets in K.

A function $p_t(x, E)$, defined for all $t \geq 0$, $x \in K$ and $E \in \mathcal{B}$, is called a (temporally homogeneous) *Markov transition function* on K if it satisfies the following four conditions:

(a) $p_t(x, \cdot)$ is a non-negative measure on $\mathcal{B}$ and $p_t(x, K) \leq 1$ for each $t \geq 0$ and each $x \in K$.

(b) $p_t(\cdot, E)$ is a Borel measurable function for each $t \geq 0$ and each $E \in \mathcal{B}$.

(c) $p_0(x, \{x\}) = 1$ for each $x \in K$.

(d) (The Chapman-Kolmogorov equation) For any t, $s \geq 0$, $x \in K$ and any $E \in \mathcal{B}$, we have

$$p_{t+s}(x, E) = \int_K p_t(x, dy) p_s(y, E). \tag{1.1}$$

Here is an intuitive way of thinking about the above definition of a Markov transition function. The value $p_t(x, E)$ expresses the transition probability that a physical particle starting at position x will be found in the set E at time t. Equation (1.1) expresses the idea that a transition from the position x to the set E in time $t + s$ is composed of a transition from x to some position y in time t, followed by a transition from y to the set E in the remaining time s; the latter transition has probability $p_s(y, E)$ which depends only on y (see Figure 8 below). Thus a particle "starts afresh"; this property is called the *Markov property*.

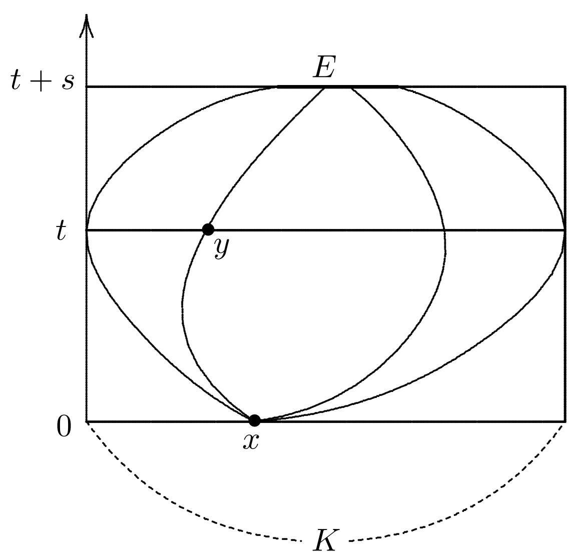


Figure 8

We add a point ∂ to K as the point at infinity if K is not compact, and as an isolated point if K is compact; so the space $K_\partial = K \cup \{\partial\}$ is compact.

Let $C(K)$ be the space of real-valued, bounded continuous functions on K. The space $C(K)$ is a Banach space with the supremum norm

$$\|f\| = \sup_{x \in K} |f(x)|.$$

We say that a function $f \in C(K)$ converges to zero as $x \to \partial$ if, for each $\varepsilon > 0$, there exists a compact subset E of K such that

$$|f(x)| < \varepsilon, \quad x \in K \setminus E,$$

and we then write $\lim_{x \to \partial} f(x) = 0$. We let

$$C_0(K) = \left\{ f \in C(K) : \lim_{x \to \partial} f(x) = 0 \right\}.$$

The space $C_0(K)$ is a closed subspace of $C(K)$; hence it is a Banach space. Note that $C_0(K)$ may be identified with $C(K)$ if K is compact.

If we introduce a useful convention

> Any real-valued function f on K is extended to the space $K_\partial = K \cup \{\partial\}$ by setting $f(\partial) = 0$,

then the space $C_0(K)$ may be identified with the subspace of $C(K_\partial)$ which consists of all functions f satisfying $f(\partial) = 0$:

$$C_0(K) = \{ f \in C(K_\partial) : f(\partial) = 0 \}.$$

Moreover, we can extend a Markov transition function p_t on K to a Markov transition function p'_t on K_∂ as follows:

$$\begin{cases} p'_t(x, E) = p_t(x, E), \quad x \in K, \ E \in \mathcal{B}; \\ p'_t(x, \{\partial\}) = 1 - p_t(x, K), \quad x \in K; \\ p'_t(\partial, K) = 0, \ p'_t(\partial, \{\partial\}) = 1. \end{cases}$$

Intuitively, this means that a Markovian particle moves in the space K until it "dies" at which time it reaches the point ∂; hence the point ∂ is called the *terminal point*.

Now we introduce some conditions on the measures $p_t(x, \cdot)$ related to continuity in $x \in K$, for fixed $t \geq 0$.

A Markov transition function p_t is called a *Feller function* if the function

$$T_t f(x) = \int_K p_t(x, dy) f(y)$$

is a continuous function of $x \in K$ whenever f is in $C(K)$, that is, if we have

$$f \in C(K) \implies T_t f \in C(K).$$

In other words, the Feller property is equivalent to saying that the measures $p_t(x, \cdot)$ depend continuously on $x \in K$ in the usual weak topology, for every fixed $t \geq 0$.

We say that p_t is a C_0-*function* if the space $C_0(K)$ is an invariant subspace of $C(K)$ for the operators T_t:

$$f \in C_0(K) \implies T_t f \in C_0(K).$$

The Feller or C_0-property deals with continuity of a Markov transition function $p_t(x, E)$ in x, and does not, by itself, have no concern with continuity in t. We give a necessary and sufficient condition on $p_t(x, E)$ in order that its associated operators $\{T_t\}_{t \geq 0}$ be strongly continuous in t on the space $C_0(K)$:

$$\lim_{s \downarrow 0} \|T_{t+s} f - T_t f\| = 0, \quad f \in C_0(K).$$

A Markov transition function p_t on K is said to be *uniformly stochastically continuous* on K if the following condition is satisfied: For each $\varepsilon > 0$ and each compact $E \subset K$, we have

$$\lim_{t \downarrow 0} \sup_{x \in E} [1 - p_t(x, U_\varepsilon(x))] = 0,$$

where $U_\varepsilon(x) = \{y \in K : \rho(x, y) < \varepsilon\}$ is an ε-neighborhood of x.

Then we have the following (see [Ta1, Theorem 9.2.3]):

Theorem 1.1. *Let p_t be a C_0-transition function on K. Then the associated operators $\{T_t\}_{t \geq 0}$, defined by*

$$T_t f(x) = \int_K p_t(x, dy) f(y), \quad f \in C_0(K), \tag{1.2}$$

is strongly continuous in t on $C_0(K)$ if and only if p_t is uniformly stochastically continuous on K and satisfies the following condition:

(L) For each $s > 0$ and each compact $E \subset K$, we have

$$\lim_{x \to \partial} \sup_{0 \leq t \leq s} p_t(x, E) = 0.$$

A family $\{T_t\}_{t \geq 0}$ of bounded linear operators acting on $C_0(K)$ is called a *Feller semigroup* on K if it satisfies the following three conditions:

(i) $T_{t+s} = T_t \cdot T_s$, t, $s \geq 0$; $T_0 = I$.

(ii) The family $\{T_t\}$ is strongly continuous in t for $t \geq 0$:

$$\lim_{s \downarrow 0} \|T_{t+s} f - T_t f\| = 0, \quad f \in C_0(K).$$

(iii) The family $\{T_t\}$ is non-negative and contractive on $C_0(K)$:

$$f \in C_0(K), 0 \leq f \leq 1 \quad \text{on } K \implies 0 \leq T_t f \leq 1 \quad \text{on } K.$$

The next theorem gives a characterization of Feller semigroups in terms of Markov transition functions (see [Ta1, Theorem 9.2.6]):

Theorem 1.2. *If p_t is a uniformly stochastically continuous C_0-transition function on K and satisfies condition (L), then its associated operators $\{T_t\}_{t\geq 0}$ form a Feller semigroup on K.*

Conversely, if $\{T_t\}_{t\geq 0}$ is a Feller semigroup on K, then there exists a uniformly stochastically continuous C_0-transition p_t on K, satisfying condition (L), such that formula (1.2) holds.

1.2 Generation Theorems of Feller Semigroups. If $\{T_t\}_{t\geq 0}$ is a Feller semigroup on K, we define its *infinitesimal generator* A by the formula

$$Au = \lim_{t\downarrow 0} \frac{T_t u - u}{t}, \tag{1.3}$$

provided that the limit (1.3) exists in the space $C_0(K)$. More precisely, the generator A is a linear operator from the space $C_0(K)$ into itself defined as follows.

(1) The domain $D(A)$ of A is the set

$$D(A) = \{u \in C_0(K) : \text{the limit (1.3) exists}\}.$$

(2) $Au = \lim_{t\downarrow 0} \frac{T_t u - u}{t}$, $u \in D(A)$.

The next theorem is a version of the Hille-Yosida theorem adapted to the present context (see [Ta1, Theorem 9.3.1 and Corollary 9.3.2]):

Theorem 1.3. *(i) Let $\{T_t\}_{t\geq 0}$ be a Feller semigroup on K and A its infinitesimal generator. Then we have the following:*

(a) The domain $D(A)$ is everywhere dense in the space $C_0(K)$.

(b) For each $\alpha > 0$, the equation $(\alpha I - A)u = f$ has a unique solution u in $D(A)$ for any $f \in C_0(K)$. Hence, for each $\alpha > 0$, the Green operator $(\alpha I - A)^{-1} : C_0(K) \longrightarrow C_0(K)$ can be defined by the formula

$$u = (\alpha I - A)^{-1} f, \quad f \in C_0(K).$$

(c) For each $\alpha > 0$, the operator $(\alpha I - A)^{-1}$ is non-negative on the space $C_0(K)$:

$$f \in C_0(K),\ f \geq 0 \quad \text{on } K \implies (\alpha I - A)^{-1} f \geq 0 \quad \text{on } K.$$

(d) For each $\alpha > 0$, the operator $(\alpha I - A)^{-1}$ is bounded on the space $C_0(K)$ with norm

$$\|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha}.$$

(ii) Conversely, if A is a linear operator from the space $C_0(K)$ into itself satisfying condition (a) and if there is a constant $\alpha_0 \geq 0$ such that, for all $\alpha > \alpha_0$, conditions (b) through (d) are satisfied, then the operator A is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t\geq 0}$ on K.

2. Proof of Theorem 1

We reduce the problem of construction of Feller semigroups to the problem of *unique solvability* for the boundary value problem

$$\begin{cases} (\alpha - W)u = f & \text{in } D, \\ (\lambda - L)u = \varphi & \text{on } \partial D, \end{cases}$$

and then prove existence theorems for Feller semigroups. Here α is a positive number and λ is a non-negative number.

The idea of our approach is stated as follows (cf. [BCP], [SU], [Ta1]).

(1) First, we consider the following *Dirichlet problem*:

$$\begin{cases} (\alpha - W)v = f & \text{in } D, \\ v|_{\partial D} = 0 & \text{on } \partial D. \end{cases}$$

The existence and uniqueness theorem for this problem is well established in the framework of Hölder spaces. We let

$$v = G_\alpha^0 f.$$

The operator G_α^0 is the Green operator for the Dirichlet problem. Then it follows that a function u is a solution of the problem

$$(*) \qquad \begin{cases} (\alpha - W)u = f & \text{in } D, \\ Lu = 0 & \text{on } \partial D \end{cases}$$

if and only if the function

$$w = u - v$$

is a solution of the problem

$$\begin{cases} (\alpha - W)w = 0 & \text{in } D, \\ Lw = -Lv = -LG_\alpha^0 f & \text{on } \partial D. \end{cases}$$

However, we know that every solution w of the equation

$$(\alpha - W)w = 0 \quad \text{in } D$$

can be expressed by means of a single layer potential in the following form:

$$w = H_\alpha \psi.$$

The operator H_α is the harmonic operator for the Dirichlet problem. Thus, by using the Green and harmonic operators we can reduce the study of problem $(*)$ to that of the equation

$$LH_\alpha \psi = -LG_\alpha^0 f \quad \text{on } \partial D.$$

This is a generalization of the classical Fredholm integral equation.

(2) Next we recall the notion of transmission property due to Boutet de Monvel [Bo], which is a condition about symbols in the normal direction at the boundary.

If $m \in \mathbf{R}$, we let

$$L^m_{1,0}(\overline{\mathbf{R}^N_+}) = \text{the space of pseudo-differential operators in } L^m_{1,0}(\mathbf{R}^N_+) \text{ which can be extended to a pseudo-differential operator in } L^m_{1,0}(\mathbf{R}^N).$$

A pseudo-differential operator $A \in L^m_{1,0}(\overline{\mathbf{R}^N_+})$ is said to have the *transmission property* with respect to the boundary $\mathbf{R}^{N-1}$ if the restriction of $A(u^0)$ to $\mathbf{R}^N_+$ has a smooth extension to $\mathbf{R}^N$ for every $u \in C^\infty_0(\overline{\mathbf{R}^N_+})$, where u^0 is the extension of u to $\mathbf{R}^N$ by 0 outside $\overline{\mathbf{R}^N_+}$.

We remark that the notion of transmission property may be transferred to manifolds with boundary. Indeed, if Ω is a relatively compact open subset of an N-dimensional, paracompact smooth manifold M without boundary, then the notion of transmission property can be extended to the class $L^m_{1,0}(M)$, upon using local coordinate systems flattening out the boundary $\partial\Omega$.

It is known (cf. [Ho], [RS]) that if $T \in L^{2-\kappa_2}_{1,0}(\mathbf{R}^N)$ has the transmission property with respect to the boundary ∂D, then the operator

$$\varphi(x') \longmapsto \int_D t(x', y) H_\alpha \varphi(y)\, dy$$

is a classical, pseudo-differential operator of order $2 - \kappa_2$ on the boundary ∂D. Therefore, we find that the operator LH_α is the sum of a degenerate elliptic differential operator of second order and a classical pseudo-differential operator of order $2 - \min(\kappa_1, \kappa_2)$.

(3) The next unique solvability theorem for pseudo-differential operators will play an essential role in the construction of Feller semigroups (see [Ta3, Theorem 2.1]):

Theorem 2.1. *Let T be a classical pseudo-differential operator of second order on an n-dimensional compact smooth manifold M without boundary such that*

$$T = P + S,$$

where:

(a) The operator P is a second-order degenerate elliptic differential operator *on M with non-positive principal symbol, and $P1 \le 0$ on M.*

(b) The operator S is a classical pseudo-differential operator of order $2 - \kappa$, $\kappa > 0$, on M and its distribution kernel $s(x, y)$ is non-negative *off the diagonal $\Delta_M = \{(x, x) : x \in M\}$ in $M \times M$.*

(c) $T1 = P1 + S1 \le 0$ on M.

Then, for each integer $k \ge 1$, there exists a constant $\lambda = \lambda(k) > 0$ such that for any $f \in C^{k+\theta}(M)$ we can find a function $\varphi \in C^{k+\theta}(M)$ satisfying the equation

$$(T - \lambda I)\varphi = f \quad \text{on } M,$$

and the estimate

$$\|\varphi\|_{C^{k+\theta}(M)} \leq C_{k+\theta}(\lambda)\|f\|_{C^{k+\theta}(M)}.$$

Here $C_{k+\theta}(\lambda) > 0$ *is a constant independent of* f.

By applying Theorem 2.1 to our situation, we can show that if the boundary condition L is transversal on the boundary ∂D, then the operator LH_α is *bijective* in the framework of Hölder spaces. The crucial point in the proof is that we consider the term $\delta(Wu|_{\partial D})$ of viscosity in the boundary condition

$$Lu = L_0 u - \delta(Wu|_{\partial D})$$

as a term of "perturbation" of the boundary condition $L_0 u$.

Therefore, we find that a unique solution u of problem $(*)$ can be expressed as follows:

$$u = G_\alpha^0 f - H_\alpha \left(LH_\alpha^{-1} \left(LG_\alpha^0 f\right)\right). \tag{2.1}$$

This formula allows us to verify that the operator $\mathfrak{A}$, defined by formula (0.4), satisfies conditions (a) through (d) in Theorem 1.3. Intuitively, formula (2.1) tells us that if the boundary condition L is transversal on the boundary ∂D, then we can "piece together" a Markov process on the boundary ∂D with W-diffusion in the interior D to construct a Markov process on the closure $\overline{D} = D \cup \partial D$. The situation may be represented schematically by Figure 9 below.

It seems that our method of construction of Feller semigroups is, in spirit, not far removed from the probabilistic method of construction of diffusion processes by means of Poisson point processes of Brownian excursions used by Watanabe [Wa].

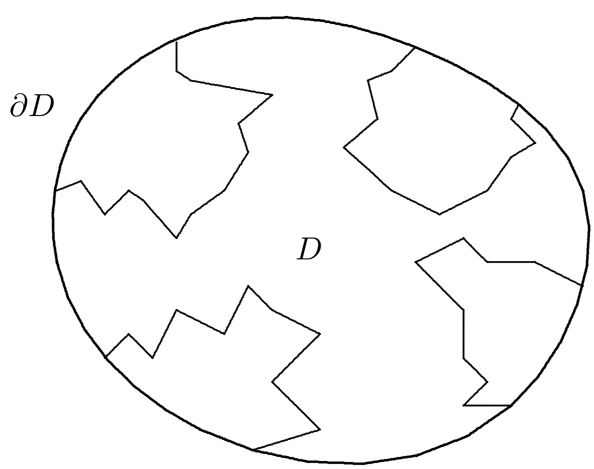


Figure 9

3. Proof of Theorem 2

We explain the idea of the proof of Theorem 2.

(1) First, we remark that if condition (H) is satisfied, then the boundary condition L can be written in the following form:

$$Lu = m\, L_\nu u + \gamma\, u \quad \text{on } \partial D,$$

where the boundary condition L_ν is *transversal* on ∂D. Hence, applying Theorem 1 to the boundary condition L_ν we can solve uniquely the following boundary value problem:

$$\begin{cases} (\alpha - W)v = f & \text{in } D, \\ L_\nu v = 0 & \text{on } \partial D. \end{cases}$$

We let

$$v = G_\alpha^\nu f.$$

The operator G_α^ν is the Green operator for the boundary condition L_ν. Then it follows that a function u is a solution of the problem

$$(**) \qquad \begin{cases} (\alpha - W)u = f & \text{in } D, \\ Lu = m\, L_\nu u + \gamma\, u = 0 & \text{on } \partial D \end{cases}$$

if and only if the function

$$w = u - v$$

is a solution of the problem

$$\begin{cases} (\alpha - W)w = 0 & \text{in } D, \\ Lw = -Lv = -\gamma\, v & \text{on } \partial D. \end{cases}$$

Thus, as in the proof of Theorem 1 we can reduce the study of problem $(**)$ to that of the equation

$$LH_\alpha \psi = -LG_\alpha^\nu f = -\gamma\, G_\alpha^\nu f \quad \text{on } \partial D.$$

(2) By applying Theorem 2.1 as in the proof of Theorem 1, we can show that if condition (A) is satisfied, then the operator LH_α is *bijective* in the framework of Hölder spaces.

Therefore, we find that a unique solution u of problem $(**)$ can be expressed as follows:

$$u = G_\alpha^\nu f - H_\alpha \left(LH_\alpha^{-1} \left(LG_\alpha^\nu f \right) \right).$$

This formula allows us to verify all the conditions of the generation theorem of Feller semigroups (Theorem 1.3), especially the *density* of the domain $D(\mathfrak{W})$ in the space $C_0(\overline{D} \setminus M)$.

It is worth while pointing out that if we use instead of G_α^ν the Green operator G_α^0 for the Dirichlet problem as in the proof of Theorem 1, our proof would break down.

4. Open Problems

In this final section we give two open problems concerning the problem of construction of Feller semigroups with boundary conditions.

(I) The first problem is to generalize Theorem 1 to the *genuine* non-transversal case.

(II) The second problem is to generalize Theorems 1 and 2 to the *non-elliptic* case, that is, the case where the Waldenfels operator W is degenerate.

For example, Taira [Ta4] treated the case where W is a second-order, degenerate elliptic *differential operator* such that we have, in terms of local coordinates $(x_1, \dots, x_{N-1}, x_N)$,

$$W = \frac{\partial^2}{\partial x_N^2} + x_N^{2k} \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_{N-1}^2} \right).$$

Here k is a positive integer. Then Theorem 1 remains valid for this Waldenfels operator W (see [Ta4, Main Theorem]).

Acknowledgement. This work was supported in part by a grant for Dispatch of Researchers for International Research Conferences from the Ministry of Education, Science and Culture of Japan.

References

[BCP] Bony J.-M., Courrège P., Priouret P., *Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum*, Ann. Inst. Fourier (Grenoble) **18** (1968), 369–521.

[Bo] Boutet de Monvel L., *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[Ca] Cancelier C., *Problèmes aux limites pseudo-différentiels donnant lieu au principe du maximum*, Comm. P.D.E. **11** (1986), 1677–1726.

[Ho] Hörmander L., *Pseudodifferential operators and non-elliptic boundary problems*, Ann. of Math. **83** (1966), 129–209.

[RS] Rempel S., Schulze B.-W., *Index theory of elliptic boundary problems*, Akademie-Verlag, Berlin, 1982.

[SU] Sato K., Ueno T., *Multi-dimensional diffusion and the Markov process on the boundary*, J. Math. Kyoto Univ. **14** (1965), 529–605.

[Ta1] Taira K., *Diffusion processes and partial differential equations*, Academic Press, San Diego New York London Tokyo, 1988.

[Ta2] Taira K., *Boundary value problems and Markov processes*, Lecture Notes in Math. No. 1499, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1991.

[Ta3] Taira K., *On the existence of Feller semigroups with boundary conditions*, Memoirs Amer. Math. Soc. No. 475 (1992).

[Ta4] Taira K., *On the existence of Feller semigroups with boundary conditions II*, J. Functional Analysis **129** (1995), 108–131.

[Ta5] Taira K., *On the existence of Feller semigroups with boundary conditions III*, Hiroshima Math. J. **27** (1997), 77–103.

[TW] Takanobu S., Watanabe S., *On the existence and uniqueness of diffusion processes with Wentzell's boundary conditions*, J. Math. Kyoto Univ. **28** (1988), 71–80.

[Wa] Watanabe S., *Construction of diffusion processes with Wentzell's boundary conditions by means of Poisson point processes of Brownian excursions*, Probability Theory, Banach Center Publications, Vol. 5, PWN-Polish Scientific Publishers, Warsaw, 1979, pp. 255–271.

[We] Wentzell (Ventcel') A. D., *On boundary conditions for multidimensional diffusion processes* (in Russian), Teoriya Veroyat. i ee Primen. **4** (1959), 172–185; English translation in Theory Prob. and its Appl. **4** (1959), 164–177.

Department of Mathematics, Hiroshima University, Higashi-Hiroshima 739, Japan