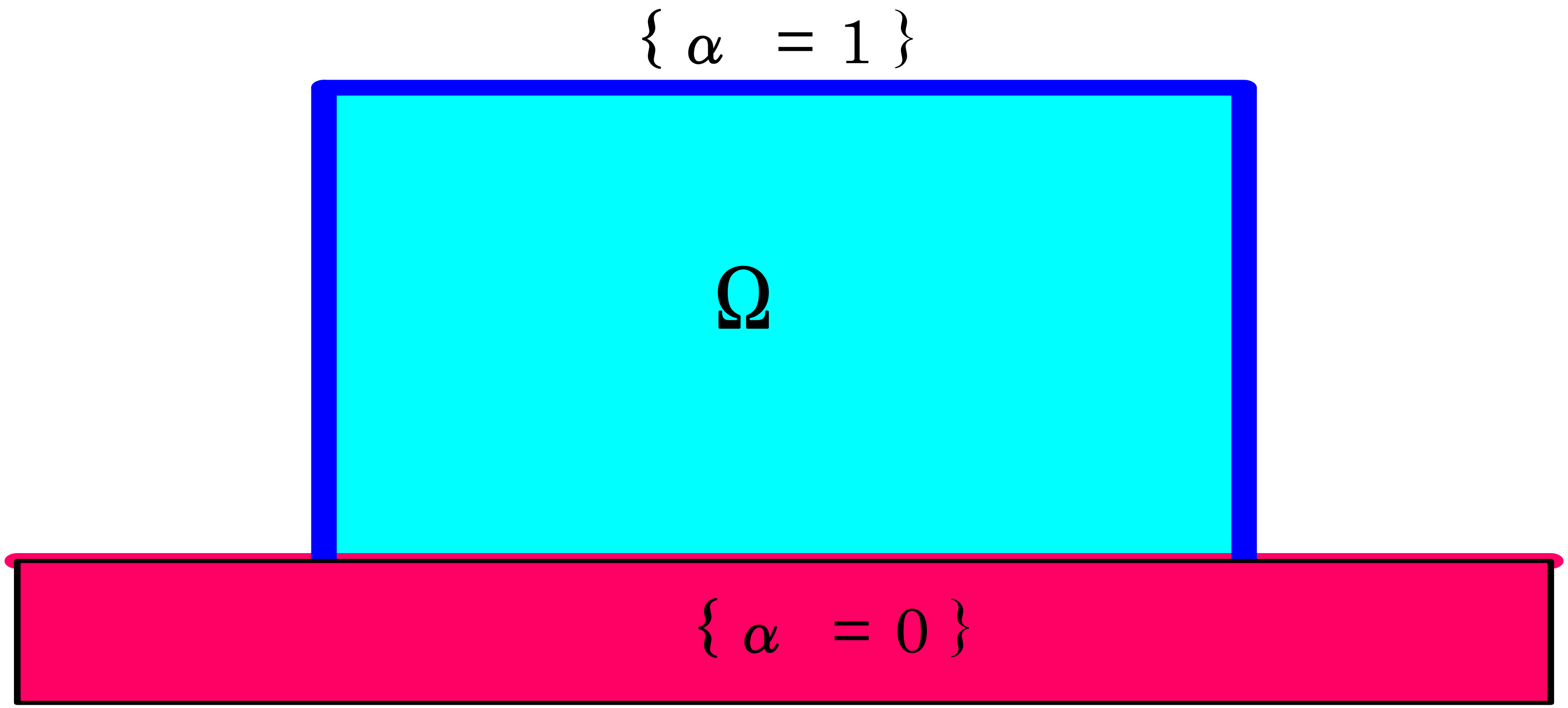
Boundary Value Problems of Nonlinear Elastostatics

Kazuaki TAIRA
Institute of Mathematics
University of Tsukuba
Tsukuba 305-8571
Japan

Domain with Corner Singularities



Abstract

- This talk is devoted to a **semigroup approach** to an initial-boundary value problem of **nonlinear elastodynamics** in the case where the boundary condition is a **regularization** of the **genuine mixed** displacement-traction boundary condition.

My Works

- **Taira: On boundary value problems of nonlinear elastostatics, Osaka Journal of Mathematics, 33 (1996), 555-585.**

  http://projecteuclid.org/euclid.ojm/1200786927

- **Taira: Introduction to boundary value problems of nonlinear elastostatics, Tsukuba Journal of Mathematics, 32 (2008), 67-138**

- **Taira : A mixed problem of linear elastodynamics, Journal of Evolution Equations, 13 (2013), 481-507**

  DOI: 10.1007/s00028-013-0187-1

References (1)

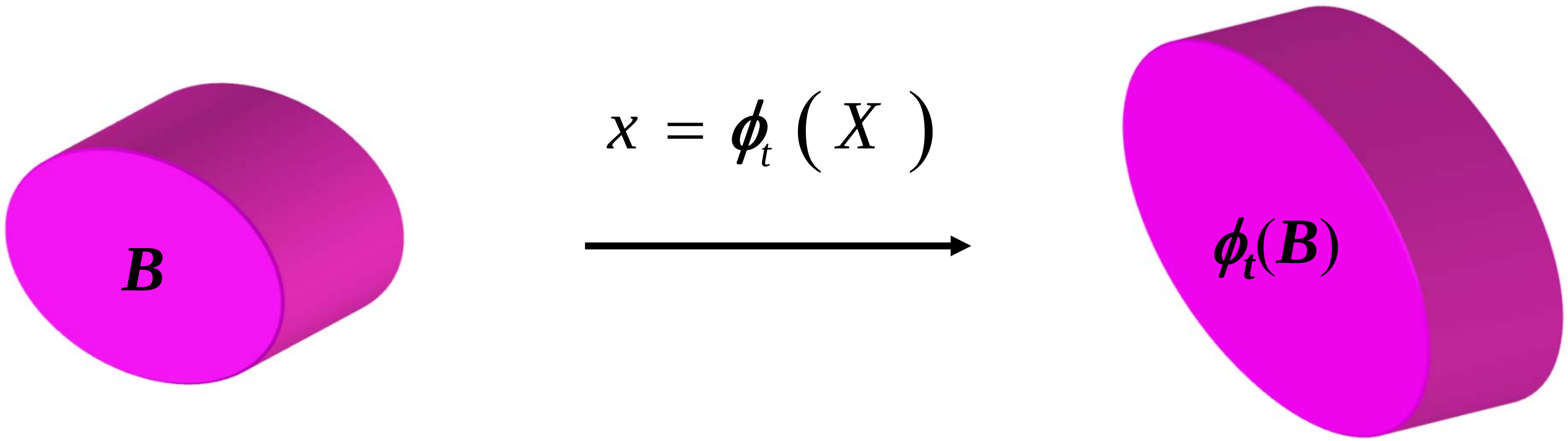
- **Ciarlet**: **Mathematical elasticity, North-Holland, 1988**
- **Duvaut et Lions**: **Les inéqations en mécanique et en physique, Dunod, 1972**
- **Marsden and Hughes**: **Mathematical foundations of elasticity, Prentice-Hall, 1983**
- **Valent**: **Boundary value problems of finite elasticity, Springer-Verlag, 1988**

References (2)

- **Goldstein: Semigroups of linear operators and applications, Oxford University Press, 1985**
- **Weiss: Abstract vibrating systems, J. Math. Mech. 17 (1967), 241—255.**

Formulation of the Problem

Nonlinear Elasticity



Reference configuration

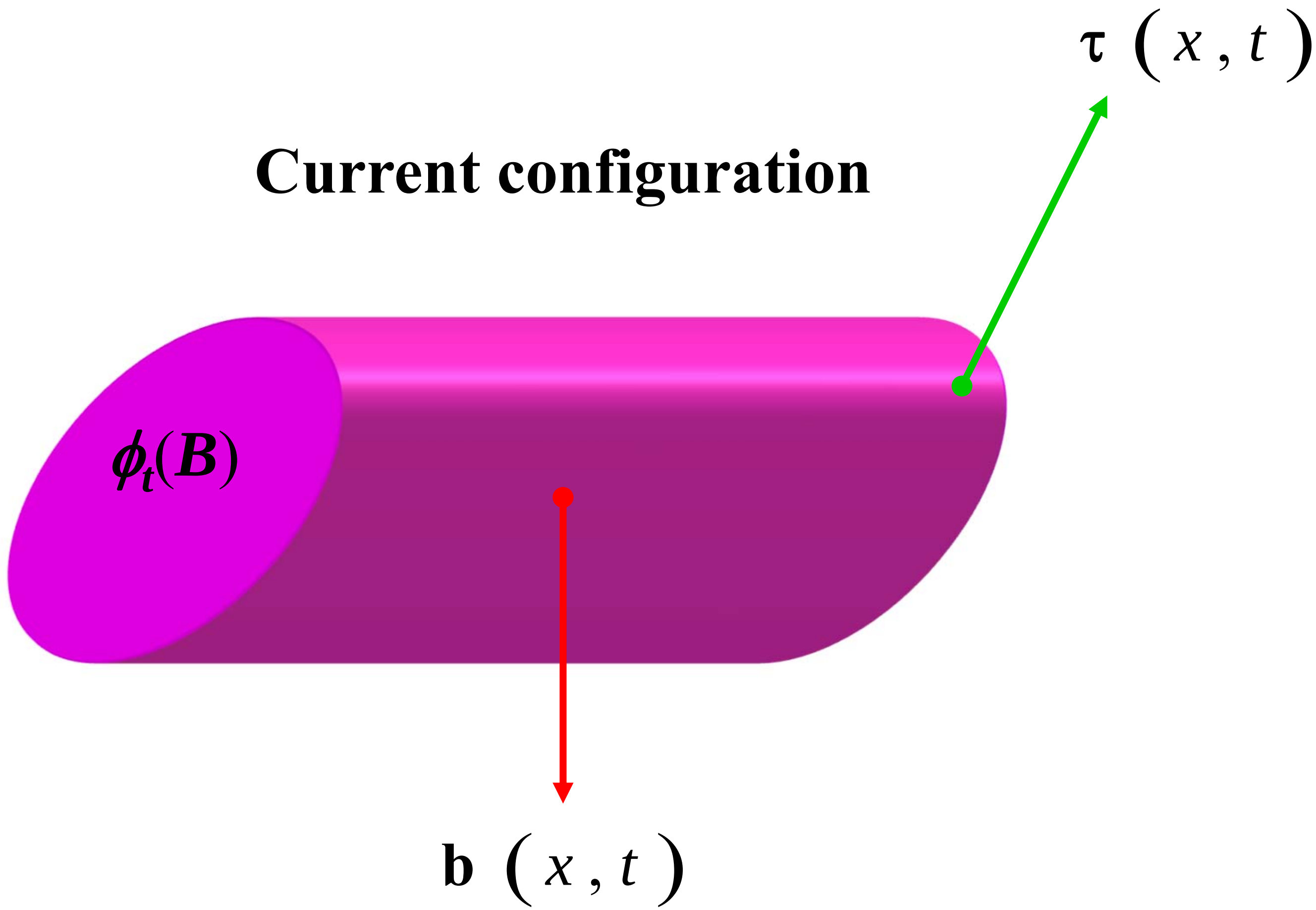
Body after time *t*

Notation

Notation	Meaning
Ω	Domain in R^3
$\boldsymbol{B} = \overline{\Omega}$	Reference configuration
$\boldsymbol{\phi}_t$	Deformation of $\boldsymbol{B}$
$X = (X_1, X_2, X_3)$	Material point
$x = (x_1, x_2, x_3)$	Spatial point

Eulerian Description (1)

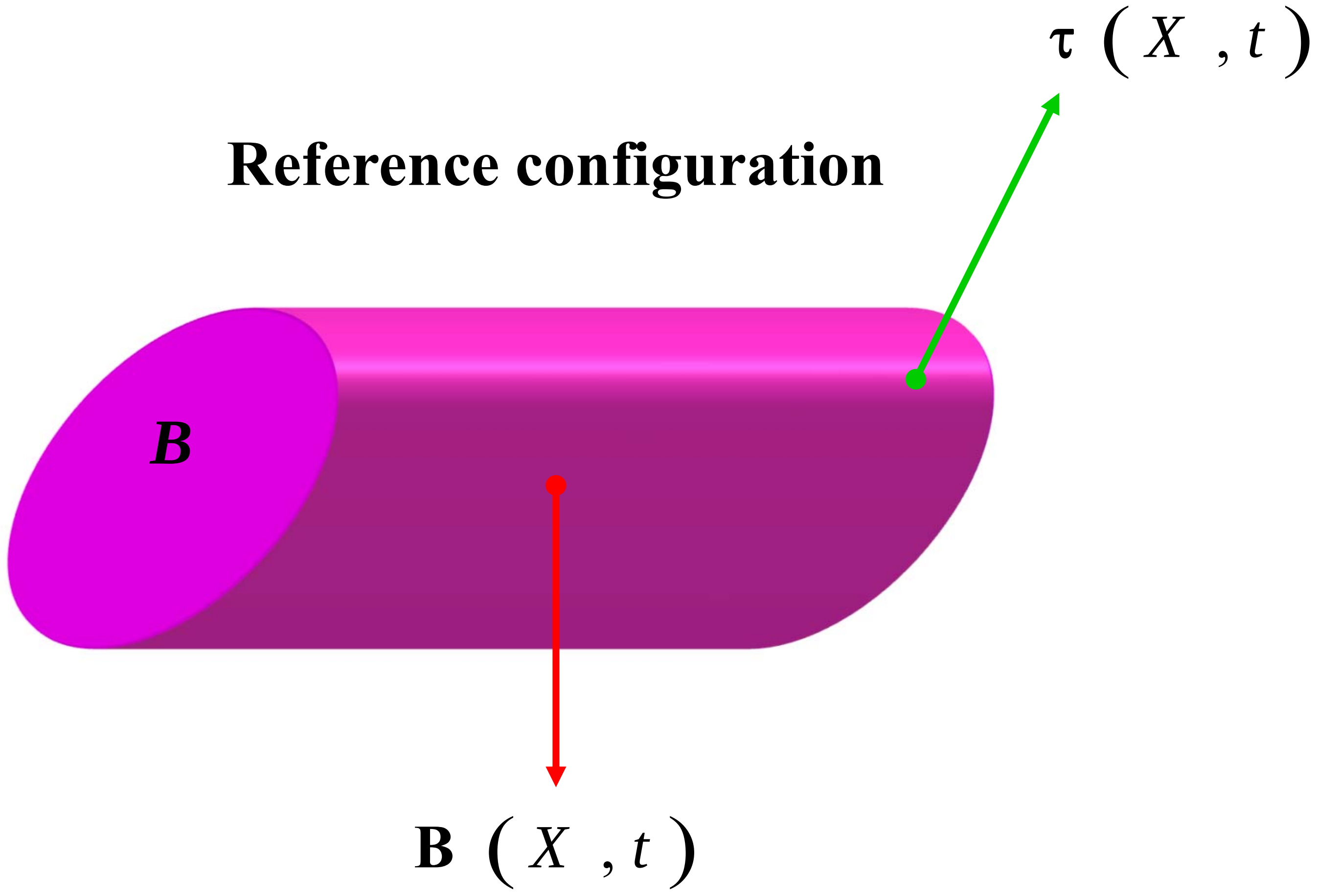
Current configuration



Leonhard Euler

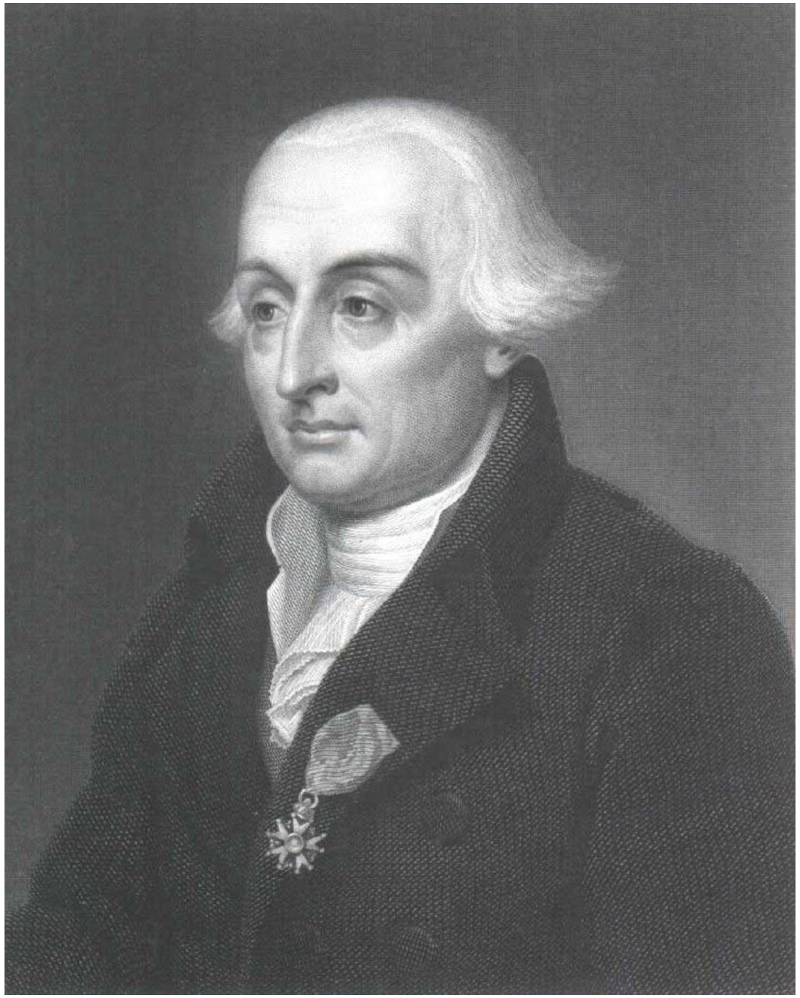
Leonhard Euler (1707-1783)
Swiss Mathematician, Physicist, Astronomer and Engineer

Lagrangian Description



Joseph-Louis Lagrange

Joseph-Louis Lagrange (1736-1813)
Italian Mathematician and Astronomer



Continuum Mechanics (1)

Form	Conservation of mass	Balance of linear momentum
Eulerian form	$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$	$\rho \dot{\mathbf{v}} = \operatorname{div} \sigma + \rho \mathbf{b}$
Lagrangian form	$\rho_0(X) = \rho(\phi_t(X), t) J(X, t)$	$\rho_0 \dfrac{\partial \mathbf{V}}{\partial t} = \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B}$

Divergence of a Tensor Filed

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$\Rightarrow$$

$$\mathrm{div}\,\mathbf{T} = \begin{pmatrix} \partial_1 T_{11} + \partial_2 T_{12} + \partial_3 T_{13} \\ \partial_1 T_{21} + \partial_2 T_{22} + \partial_3 T_{23} \\ \partial_1 T_{31} + \partial_2 T_{32} + \partial_3 T_{33} \end{pmatrix}$$

The divergence theorem for a tensor filed

$$\int_{\Omega} \mathrm{div}\,\mathbf{T}\, dx = \int_{\partial\Omega} \mathbf{T} \bullet \mathbf{n}\, d\sigma$$

Notation (1)

Notation	Meaning
$\rho_0(X),\ \rho(x,t)$	Density
$\mathbf{V}(X,t),\ \mathbf{v}(x,t)$	Velocity
$\mathbf{B}(X,t),\ \mathbf{b}(x,t)$	Body force
$\boldsymbol{\sigma}(x,t)$	Cauchy stress tensor
$\mathbf{P}(X,t)$	First Piola-Kirchhoff stress tensor

Louis Augustin Cauchy

Louis Augustin Cauchy (1789-1857)
French Mathematician and Physicist

Notation (2)

Notation	Meaning
$\boldsymbol{\phi}_t(X)$	Deformation
$\mathbf{F}(X,t) = D\boldsymbol{\phi}_t(X)$	Deformation gradient
$J(X,t) = \det \mathbf{F}(X,t)$	Jacobian
$\dot{} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \bullet \nabla$	Material derivative

Deformation Gradient

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} : \overline{\Omega} \to \mathbf{R}^3$$

$$\Rightarrow$$

$$\nabla \boldsymbol{\varphi} = \begin{pmatrix} \partial_1 \varphi_1 & \partial_2 \varphi_1 & \partial_3 \varphi_1 \\ \partial_1 \varphi_2 & \partial_2 \varphi_2 & \partial_3 \varphi_2 \\ \partial_1 \varphi_3 & \partial_2 \varphi_3 & \partial_3 \varphi_3 \end{pmatrix}$$

Displacement gradient

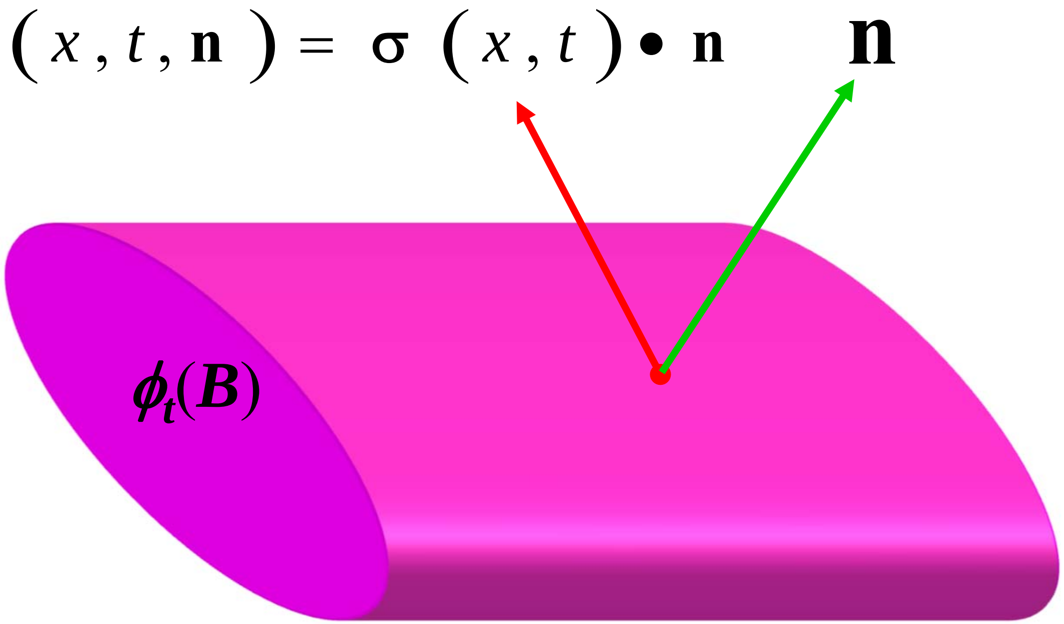
$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \varphi_1 - x_1 \\ \varphi_2 - x_2 \\ \varphi_3 - x_3 \end{pmatrix} : \overline{\Omega} \to \mathbf{R}^3$$

$$\Rightarrow$$

$$\nabla \boldsymbol{u} = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix} = \nabla \boldsymbol{\varphi} - \mathbf{I}$$

Eulerian Description (2)

$$\mathbf{t}(x, t, \mathbf{n}) = \boldsymbol{\sigma}(x, t) \bullet \mathbf{n}$$



Current configuration

Continuum Mechanics (2)

Form	Cauchy's Theorem
Eulerian form	$\mathbf{t}(x, t, \mathbf{n}) = \boldsymbol{\sigma}(x, t) \bullet \mathbf{n}$
Lagrangian form	$\mathbf{T}(X, t, \mathbf{N}) = \mathbf{P}(X, t) \bullet \mathbf{N}$

Notation (3)

Notation	Meaning
$\mathbf{t}(x, t, \mathbf{n})$	Cauchy stress vector
$\boldsymbol{\sigma}(x, t)$	Cauchy stress tensor
$\mathbf{T}(X, t, \mathbf{N})$	First Piola-Kirchhoff stress vector
$\mathbf{P}(X, t)$	First Piola-Kirchhoff stress tensor
$\mathbf{n}$, $\mathbf{N}$	Outward unit normal vector

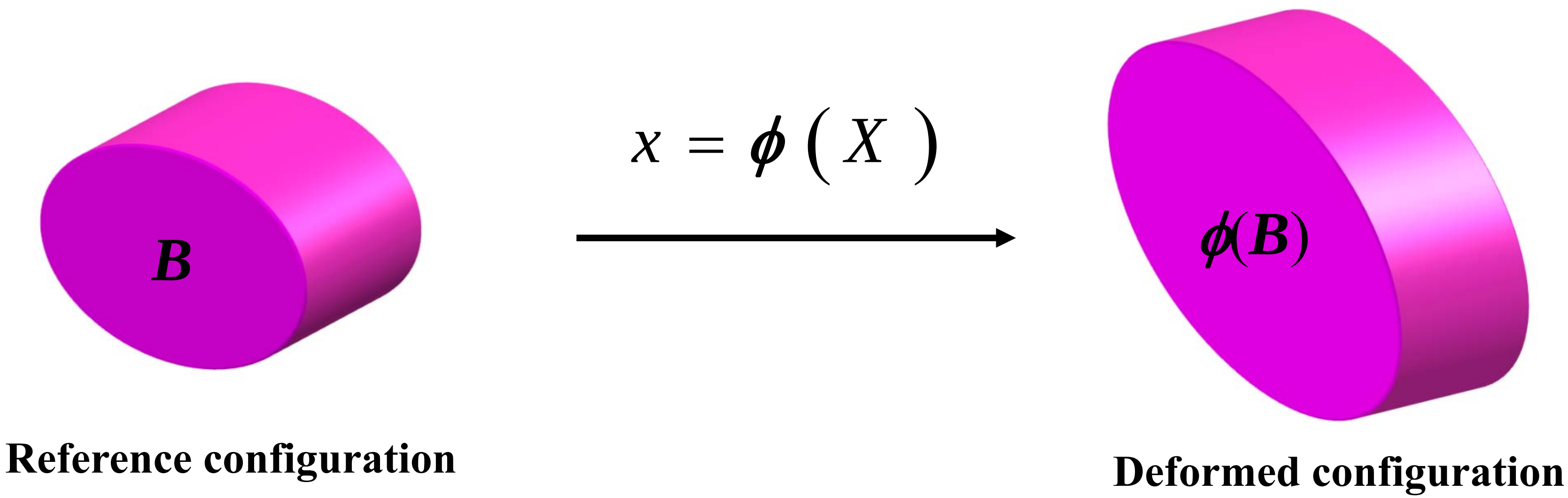
Continuum Mechanics (3)

Form	Balance of angular momentum	Conservation of energy
Eulerian form	$\sigma = {}^t\sigma$	$\rho \dot{e} + \mathrm{div}\, \mathbf{q} = \mathrm{tr}(\sigma \mathbf{d}) + \rho r$
Lagrangian form	$\mathbf{S} = {}^t\mathbf{S}$	$\rho_0 \frac{\partial E}{\partial t} + \mathrm{Div}\, \mathbf{Q} = \mathrm{tr}(\mathbf{S}\mathbf{D}) + \rho_0 R$

Notation (4)

Notation	Meaning
$\mathbf{S}(X,t)$	Second Piola-Kirchhoff stress tensor
$E(X,t)$, $e(x,t)$	Internal energy function
$\mathbf{D}(X,t)$, $\mathbf{d}(x,t)$	Material rate of deformation tensor
$R(X,t)$, $r(x,t)$	Heat supply
$\mathbf{Q}(X,t)$, $\mathbf{q}(x,t)$	Heat flux vector

Nonlinear Elastostatics



Notation

Notation	Meaning
Ω	Bounded domain in R^3 with smooth boundary
$\boldsymbol{B} = \overline{\Omega}$	Reference configuration
$\boldsymbol{\phi}$	Deformation of ***B***
$X = (X_1, X_2, X_3)$	Material point
$x = (x_1, x_2, x_3)$	Spatial point

Elasticity

A material is said to be **elastic**

$\Leftrightarrow$

$\exists A$ function $\widehat{\mathbf{P}}(X, \mathbf{F})$ of points $X \in B$ and 3×3 matrices $\mathbf{F} = (F_{ij})$ with $\det \mathbf{F} > 0$ such that

$$\boxed{\mathbf{P}(X) = \widehat{\mathbf{P}}\big(X, D\boldsymbol{\phi}(X)\big)}$$

Constitutive Function

$\widehat{\mathbf{P}}(X, \mathbf{F})$ is called a **constitutive function**

Hyper-Elasticity

A material is said to be **hyperelastic**

$\Leftrightarrow$

$\exists A$ function $W(X, \mathbf{F})$ of points $X \in B$

and 3×3 matrices $\mathbf{F} = (F_{ij})$ with $\det \mathbf{F} > 0$

such that

$$\widehat{\mathbf{P}}(X, \mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}}(X, \mathbf{F})$$

$$\boxed{\widehat{P}_{ij}(X, \mathbf{F}) = \frac{\partial W}{\partial F_{ij}}(X, \mathbf{F})}$$

Stored Energy Function

$W(X, \mathbf{F})$ is called a **stored energy function**

First Elasticity Tensor

The **four-index tensor**

defined by the formula

$$\mathbf{A} = \frac{\partial \widehat{\mathbf{P}}}{\partial \mathbf{F}} = \frac{\partial^2 W}{\partial \mathbf{F}\, \partial \mathbf{F}},$$

$$\boxed{\mathrm{A}_{ij\ell m}(X, \mathbf{F}) = \frac{\partial \widehat{P}_{ij}}{\partial F_{\ell m}}(X, \mathbf{F}) = \frac{\partial^2 W}{\partial F_{ij}\, \partial F_{\ell m}}(X, \mathbf{F})}$$

is called the **first elasticity tensor**.

Typical Examples

Hencky-Nadai Elasto-Plastic Material

The **stored energy function** has the form

$$W(X,\mathbf{F}) = \frac{3}{4}\int_0^{\Gamma(\mathbf{F})} g(\xi)\,d\xi + \frac{K}{2}\left(\sum_{k=1}^{3} F_{kk} - 3\right)^2$$

$$g(\xi) \in C^{\infty}([0,\infty),\mathbf{R})$$

$$\Gamma(\mathbf{F}) = \frac{4}{3}\sum_{i,j=1}^{3}\left(\frac{1}{2}\left(F_{ij}+F_{ji}\right) - \frac{1}{3}\left(\sum_{k=1}^{3} F_{kk}F_{kk}\right)\delta_{ij}\right)^2$$

The Modulus of Compression

K is called the **modulus of compression**

Saint Venant-Kirchhoff Isotropic Material

The **stored energy function** has the form

$$W(X,\mathbf{F})$$

$$= \frac{\lambda(X)}{8}\left(\sum_{k=1}^{3} C_{kk}(\mathbf{F}) - 3\right)^2 + \frac{\mu(X)}{4}\sum_{i,j=1}^{3}\left(C_{ij}(\mathbf{F}) - \delta_{ij}\right)^2$$

$$C_{ij}(\mathbf{F}) = \sum_{k=1}^{3} F_{ki}F_{kj}$$

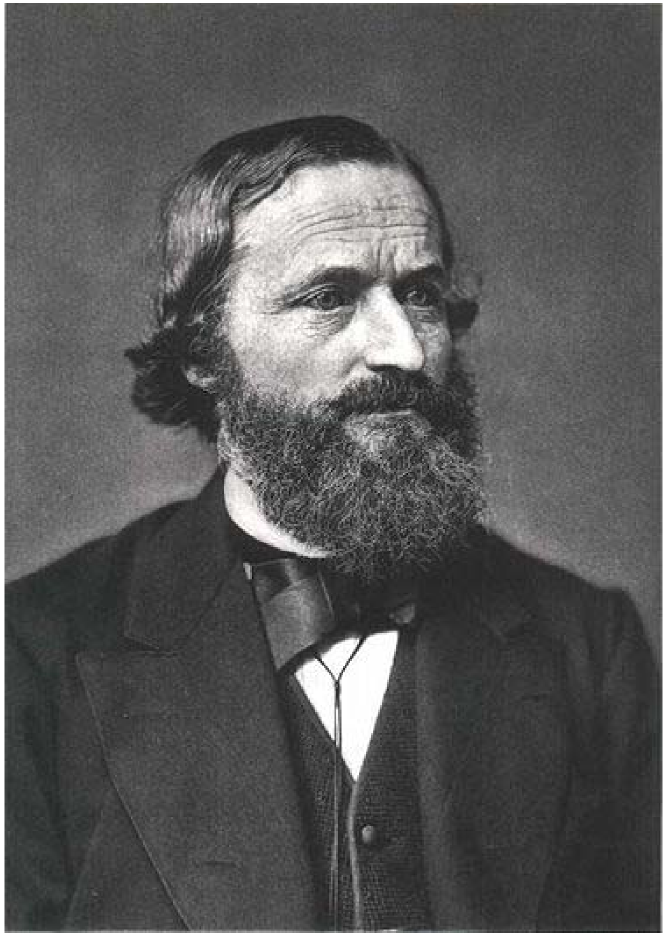
Saint Venant

Adhémar Jean Claude Barré de Saint Venant (1797-1886)

French Mechanician and Mathematician

Gustav Robert Kirchhoff

Gustav Robert Kirchhoff (1824-1887)
Prussian/German Physicist and Chemist



Lamé functions

$\lambda(X)$, $\mu(X)$ are called **Lame functions**

The Right Cauchy-Green tensor

$$C_{ij}(\mathbf{F}) = \sum_{k=1}^{3} F_{ki} F_{kj}$$

Green-Saint Venant strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

$$\mathbf{E} = (E_{ij})$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} \right)$$

Saint Venant-Kirchhoff Isotropic Material

$$W\left(X, \mathbf{F}\right) = \frac{\lambda\left(X\right)}{2}\left(\operatorname{tr}\mathbf{E}\right)^2 + \mu\left(X\right)\operatorname{tr}\left(\mathbf{E}^2\right)$$

Formulation of the Results

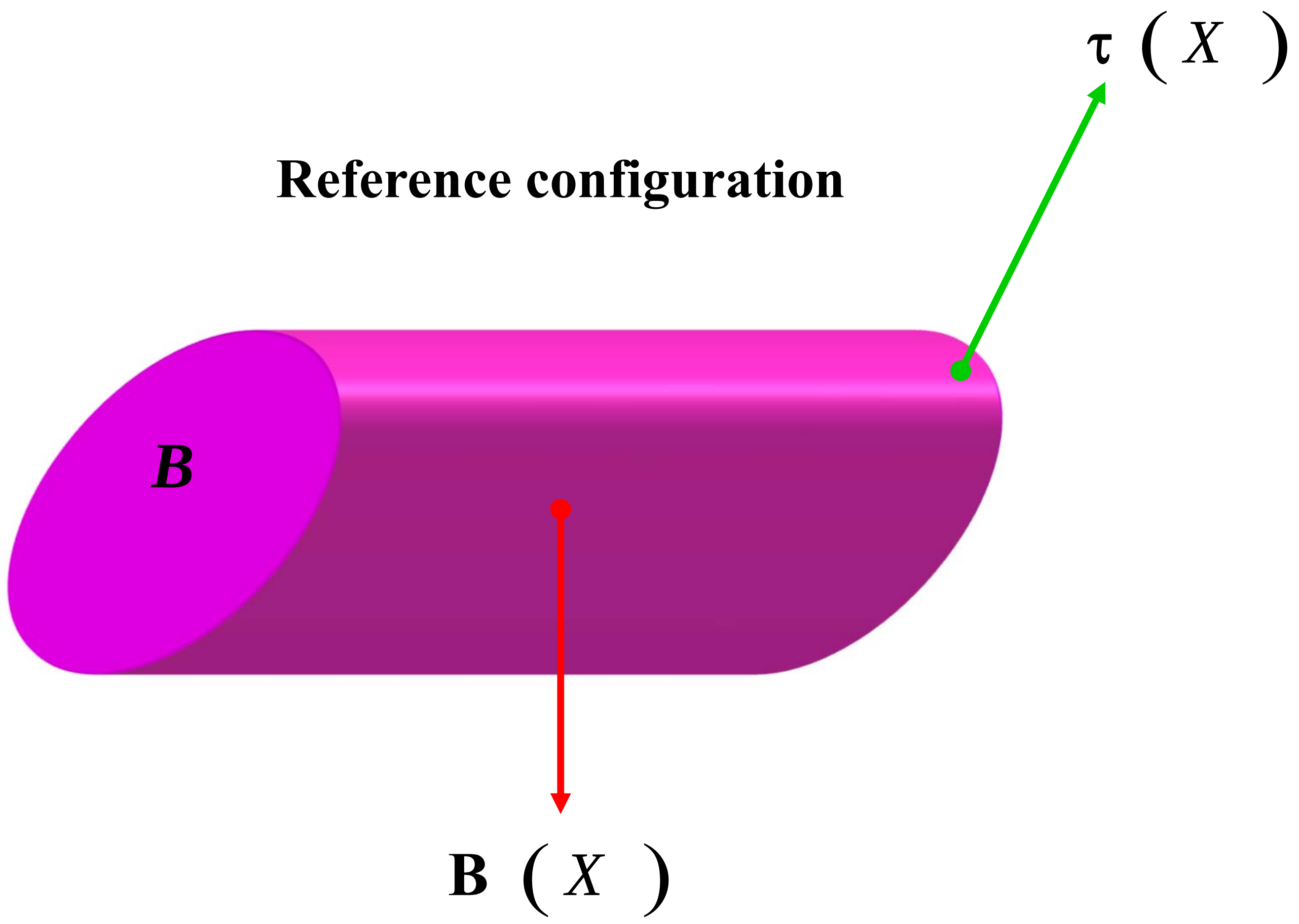
Nonlinear Boundary Value Problem
(Lagrange)

This talk is devoted to an L^p approach to the following equilibrium equations for the unknown deformation $\boldsymbol{\phi}(X)$

$$\operatorname{Div} \widehat{\mathbf{P}}\left(X, D\boldsymbol{\phi}(X)\right) + \mathbf{B}(X) = \mathbf{0} \quad \text{in} \quad \Omega,$$

$$\alpha(\mathrm{X})\widehat{\mathbf{P}}\left(X, D\boldsymbol{\phi}(X)\right)\cdot\mathbf{N}(X) + \left(1 - \alpha(X)\right)\boldsymbol{\phi}(X) = \boldsymbol{\tau}(X) \quad \text{on} \quad \partial\Omega.$$

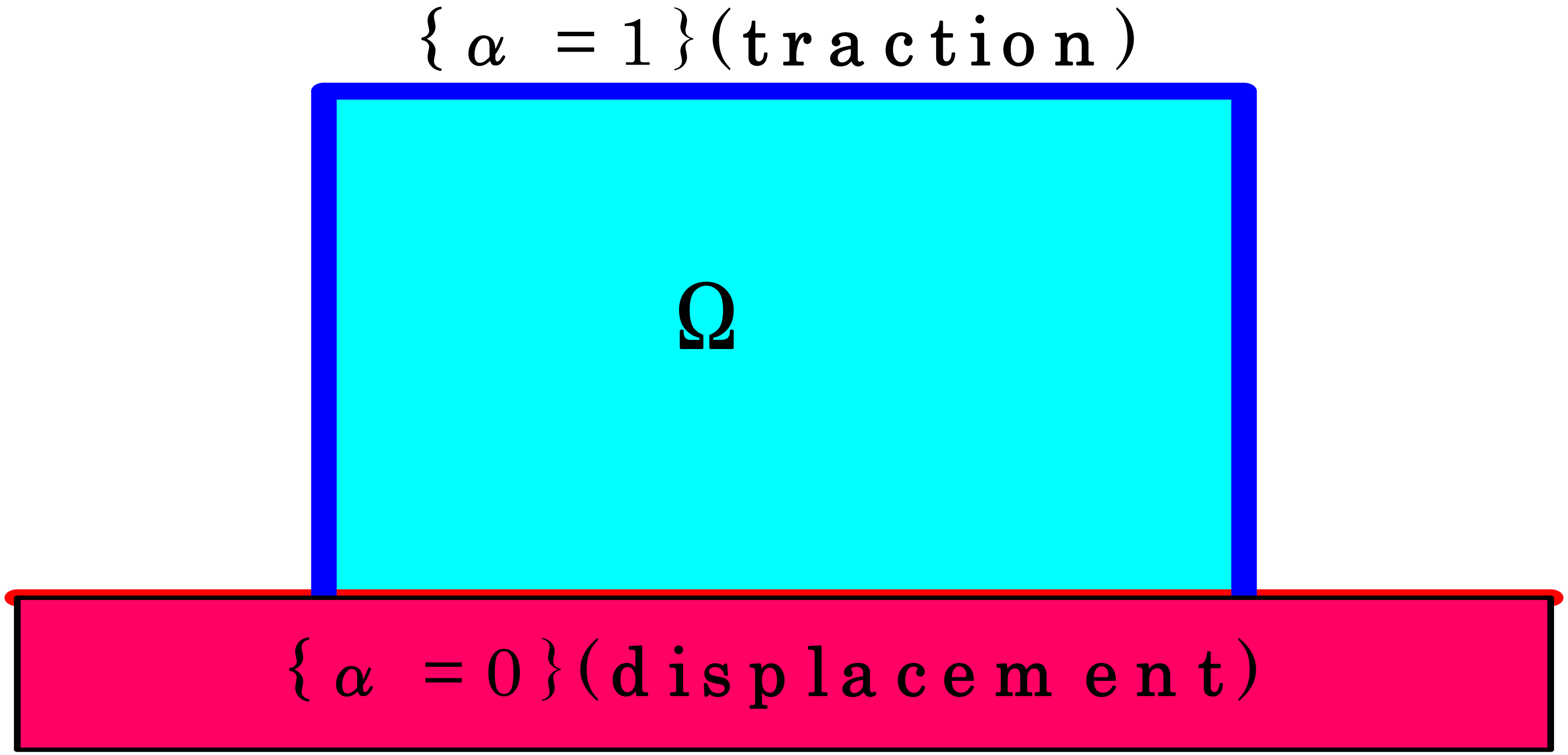
Lagrangian Description



Notation

Notation	Meaning
$\widehat{\mathbf{P}}(X, \mathbf{F})$	**Constitutive function**
$\mathbf{B}(X)$	**Body force**
$\tau(X)$	**Surface force**
$\mathbf{N}(X)$	**Outward unit normal**
$\alpha(X)$	$\alpha(X) \in C^{\infty}(\partial\Omega)$
	$0 \leq \alpha(X) \leq 1 \text{ on } \partial\Omega$

Boundary Condition



Example of Regularization (Real axis)

$$\alpha(x_1) = \begin{cases} e^{-\frac{1}{x_1}} & \text{for } x_1 > 0, \\ 0 & \text{for } x_1 \leq 0 \end{cases}$$

Example of Regularization (Unit Circle)

$$a(x_1, x_2) = a(\cos\theta, \sin\theta) = \begin{cases} \exp[2/\pi + 1/(\theta + \pi/2)]\left(1 - \exp[2/\pi - 1/(\theta + \pi)]\right), \\ \theta \in \left(-\pi, -\pi/2\right) \\ 0, \theta \in \left[-\pi/2, 0\right] \\ \exp[2/\pi - 1/\theta]\left(1 - \exp[2/\pi + 1/(\theta - \pi/2)]\right), \\ \theta \in \left(0, \pi/2\right), \\ 1, \theta \in \left[\pi/2, \pi\right] \end{cases}$$

Main Results

1. **If the linearized problem has unique solutions, then so does the nonlinear one, nearby. This is done by using the linear L^p theory and the inverse mapping theorem.**
2. **Our results can be applied to the Saint Venant-Kirchhoff elastic material and the Hencky-Nadai elasto-plastic material.**

Crucial Point

- The crucial point is how to find a function space associated with the degenerate boundary condition in which the linearized problem has unique solutions.

Function Spaces (1)

For all real s, we define **Sobolev spaces**

$$\mathbf{H}^{s,p}\left(\Omega,\mathbf{R}^3\right)=\mathbf{H}^{s,p}\left(\mathbf{R}^3,\mathbf{R}^3\right)\Big|_{\Omega}$$

For all s > 1/p, we define **Besov spaces**

$$\mathbf{B}^{s-1/p,p}\left(\partial\Omega,\mathbf{R}^3\right)=\mathbf{H}^{s,p}\left(\Omega,\mathbf{R}^3\right)\Big|_{\partial\Omega}.$$

Function Spaces (2)

$$s > 1 + 1/p$$

$$\mathbf{B}_{(\alpha)}^{s-1-1/p,p}\left(\partial\Omega, \mathbf{R}^3\right) = \alpha(X)\mathbf{B}^{s-1-1/p,p}\left(\partial\Omega, \mathbf{R}^3\right) + \left(1-\alpha(X)\right)\mathbf{B}^{s-1/p,p}\left(\partial\Omega, \mathbf{R}^3\right)$$

$$(A) \quad 0 \le \alpha(X) \le 1 \text{ and } \alpha(X) \not\equiv 1 \text{ on } \partial\Omega.$$

Definition of a norm

$$\left|\varphi\right|_{\mathbf{B}_{(\alpha)}^{s-1-1/p,p}\left(\partial\Omega,\mathbf{R}^3\right)}$$
$$= \inf\left\{\left|\varphi_1\right|_{s-1-1/p,p} + \left|\varphi_2\right|_{s-1/p,p} : \varphi = \alpha(X)\varphi_1 + \left(1-\alpha(X)\right)\varphi_2\right\}$$

$$(A) \quad 0 \le \alpha\left(X\right) \le 1 \text{ and } \alpha\left(X\right) \not\equiv 1 \text{ on } \partial\Omega.$$

Function Spaces (3)

$$\mathbf{B}_{(\alpha)}^{s-1-1/p,p}\left(\partial\Omega,\mathbf{R}^3\right)=\mathbf{B}^{s-1/p,p}\left(\partial\Omega,\mathbf{R}^3\right)$$

$$\text{if} \quad \alpha(X)\equiv 0 \quad \text{on} \quad \partial\Omega \quad \textbf{(Dirichlet)}$$

$$\mathbf{B}_{(\alpha)}^{s-1-1/p,p}\left(\partial\Omega,\mathbf{R}^3\right)=\mathbf{B}^{s-1-1/p,p}\left(\partial\Omega,\mathbf{R}^3\right)$$

$$\text{if} \quad \alpha(X)\equiv 1 \quad \text{on} \quad \partial\Omega \quad \textbf{(Neumann)}$$

Function Spaces (4)

$$X = \mathbf{H}^{s,p}\left(\Omega, \mathbf{R}^3\right),$$

$$Y = \mathbf{H}^{s-2,p}\left(\Omega, \mathbf{R}^3\right) \times \mathbf{B}_{(\alpha)}^{s-1-1/p,p}\left(\partial\Omega, \mathbf{R}^3\right).$$

Nonlinear Map

$$F : X \rightarrow Y$$

$$F(\boldsymbol{\phi}) = \left(-\mathrm{Div}\,\hat{\mathbf{P}}(D\boldsymbol{\phi}),\ \alpha\hat{\mathbf{P}}(D\boldsymbol{\phi})\bullet\mathbf{N} + (1-\alpha)\,\boldsymbol{\phi}\big|_{\partial\Omega} \right)$$

Fundamental Assumptions (1)

$$(A) \quad 0 \leq \alpha(X) \leq 1 \text{ and } \alpha(X) \not\equiv 1 \quad \text{on} \quad \partial\Omega.$$

$$(P) \quad \overset{o}{\mathbf{P}}\left(D\overset{o}{\boldsymbol{\phi}}\right) = 0 \text{ when } \overset{o}{\boldsymbol{\phi}} = \mathbf{I}_{\Omega}.$$

Remarks

1. Condition (P) implies that the reference configuration is a natural state.
2. Condition (A) implies that our boundary condition is not equal to the pure traction boundary condition

Fundamental Assumptions (2)

$$(H) \qquad \overset{\circ}{A} = \frac{\partial^2 W}{\partial \mathbf{F}\, \partial \mathbf{F}} \quad \text{at } \overset{\circ}{\boldsymbol{\phi}} = \mathbf{I}_\Omega$$

enjoys the property of **symmetry**

$$\boxed{\overset{\circ}{A}_{ij\ell m} = \overset{\circ}{A}_{\ell m ij} = \overset{\circ}{A}_{ji\ell m}}$$

and is **uniformly pointwise stable**

$$\boxed{\frac{1}{2}\mathbf{e} \bullet \overset{\circ}{A} \bullet \mathbf{e} \geq \exists\, \eta\, \|\mathbf{e}\|}$$

Main Theorem (1)

Main Theorem states that if the linearized problem is uniformly pointwise stable, then, for slight perturbations of the load or boundary conditions from their values at the natural state, then the nonlinear problem has a unique solution.

Main Theorem (2)

$1 < p < \infty, \quad s > 3/p + 1.$

$\exists$ A neighborhood U of $\overset{\circ}{\boldsymbol{\phi}} = \mathbf{I}_\Omega$

and a neighborhood V of the point

$$\left(-\mathrm{Div}\, \widehat{\mathbf{P}}\left(D\overset{\circ}{\boldsymbol{\phi}} \right),\ \alpha \widehat{\mathbf{P}}\left(D\overset{\circ}{\boldsymbol{\phi}} \right) \bullet \mathbf{N} + (1-\alpha)\overset{\circ}{\boldsymbol{\phi}}\Big|_{\partial\Omega} \right)$$

such that the **nonlinear map**

$$\boxed{F : U \to V}$$

is **bijective.**

Typical Examples

Hencky-Nadai Elasto-Plastic Material

$$(A) \quad 0 \le \alpha(X) \le 1 \text{ and } \alpha(X) \not\equiv 1 \text{ on } \partial\Omega .$$

$$(G) \quad g(0) > 0 \text{ and } K > 0$$

Condition (H) is satisfied and so Main Theorem applies.

Saint Venant-Kirchhoff Isotropic Material

(A) $0 \leq \alpha(X) \leq 1$ and $\alpha(X) \not\equiv 1$ on $\partial\Omega$.

(M) $\exists c_1 > 0, \ \exists c_2 > 0$ such that

$$\mu(X) \geq c_1 \quad \text{on } \Omega,$$

$$\lambda(X) + \frac{2}{3}\mu(X) \geq c_2 \quad \text{on } \Omega$$

Condition (H) is satisfied and so Main Theorem applies.

Linearized Problems
(Frechet Derivatives)

Mixed Problem of Linear Elastodynamics

We study the following mixed problem of **linear elastodynamics**:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}\left(\mathbf{a}\bullet\nabla\mathbf{u}\right) = \mathbf{f} \quad \text{in } \Omega \times (0,\infty),$$

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\alpha(x')(\mathbf{a}\bullet\nabla\mathbf{u}\bullet\mathbf{n}) + \left(1-\alpha(x')\right)\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,\infty).$$

Example (I)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}\left(\lambda\ \mathrm{tr}(\mathbf{e}(\mathbf{u}))\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u})\right) = \mathbf{f} \quad \text{in } \Omega \times (0,\infty),$$

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\frac{\partial \mathbf{u}}{\partial t}\bigg|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\alpha(x')(\boldsymbol{\tau}(\mathbf{u})\bullet\mathbf{n}) + \left(1-\alpha(x')\right)\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,\infty).$$

Example (I)

$$\mathbf{e}(\mathbf{u}) = (e_{ij}) = \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right), \quad \text{strain tensor}$$

$$\boldsymbol{\tau}(\mathbf{u}) = (\tau_{ij}) = \left(\lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \right), \quad \text{stress tensor}$$

Example (Ia)

$$(a_{ij\ell m}) = \left(\lambda\delta_{ij}\delta_{\ell m} + \mu(\delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell})\right)$$

λ, μ : **Lame moduli**

$$(H)\quad \mu > 0, \quad k = \frac{3\lambda + 2\mu}{3} > 0.$$

k : **modulus of compression**

Example (Ib)

$$(\mathrm{a}_{ij\ell m}) = \left(\left(K - \frac{2}{3} g(0) \right) \delta_{ij} \delta_{\ell m} + g(0) (\delta_{i\ell} \delta_{jm} + \delta_{im} \delta_{j\ell}) \right)$$

K is the **modulus of compression**

$$g(\xi) \in C^{\infty}([0,\infty), \mathbf{R})$$

$$(G) \quad K > 0, \quad g(0) > 0.$$

Example (II)

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{in } \Omega \times (0, \infty),$$

$$u\big|_{t=0} = u_0 \quad \text{in } \Omega,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1 \quad \text{in } \Omega,$$

$$\alpha(x')\frac{\partial u}{\partial \mathbf{n}} + \big(1 - \alpha(x')\big)u = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

Example (II)

$$\left(a_{ij\ell m}\right) = \left(\delta_{i\ell}\delta_{jm}\right).$$

Linear Boundary Value Problem

We are reduced to the study of a problem of **linear elastostatics** for the unknown vector function **v**:

$$\mathbf{A}\mathbf{v} = \mathrm{div}\left(\mathbf{a}\bullet\nabla\mathbf{v}\right) = \mathbf{f} \quad \text{in } \Omega,$$

$$\mathbf{B}_\alpha \mathbf{v} = \alpha(x')(\mathbf{a}\bullet\nabla\mathbf{v}\bullet\mathbf{n}) + \left(1-\alpha(x')\right)\mathbf{v} = \varphi \quad \text{on } \partial\Omega$$

$\mathbf{a} = (\mathrm{a}_{ij\ell m})$: smooth elasticity tensor.

n : the outward unit normal

Fundamental Existence and Uniqueness Theorem

$(A)\quad 0 \leq \alpha(X) \leq 1 \text{ and } \alpha(X) \not\equiv 1 \text{ on } \partial\Omega .$

$(H)\quad \begin{array}{|l|} \hline \mathrm{a}_{ij\ell m} = \mathrm{a}_{\ell m ij} = \mathrm{a}_{ji\ell m} \\ \frac{1}{2}\mathbf{e}\bullet\mathbf{a}\bullet\mathbf{e} \geq \exists\, \eta \left\|\mathbf{e}\right\|^2 \\ \hline \end{array}$

$$(\mathbf{A},\mathbf{B}_\alpha): \mathbf{H}^{s,p}\left(\Omega,\mathbf{R}^3\right) \to \mathbf{H}^{s-2,p}\left(\Omega,\mathbf{R}^3\right) \times \mathbf{B}_{(\alpha)}^{s-1-1/p,p}\left(\partial\Omega,\mathbf{R}^3\right)$$

is an algebraic and topological **isomorphism**

Regularity Theorem

Regularity Theorem

$$(H) \quad \boxed{\begin{array}{l} a_{ij\ell m} = a_{\ell m ij} = a_{ji\ell m} \\ \frac{1}{2}\mathbf{e} \bullet \mathbf{a} \bullet \mathbf{e} \geq \exists\, \eta \left\|\mathbf{e}\right\|^2 \end{array}}$$

$$\Downarrow$$

$$1 < p < \infty, \quad s > 1/p + 1$$

$$\boxed{\begin{array}{l} \mathbf{u} \in \mathbf{L}^p\left(\Omega, \mathbf{R}^3\right), \ \mathbf{A}\mathbf{u} \in \mathbf{H}^{s-2,p}\left(\Omega, \mathbf{R}^3\right), \\ \mathbf{B}_\alpha \mathbf{u} \in \mathbf{B}_{(\alpha)}^{s-1-1/p,p}\left(\partial\Omega, \mathbf{R}^3\right) \Rightarrow \mathbf{u} \in \mathbf{H}^{s,p}\left(\Omega, \mathbf{R}^3\right) \end{array}}.$$

Construction of a Parametrix

Symbol of a Pseudo-Differential Operator

$$\mathbf{t}(x',\xi')$$
$$= \alpha(x')\mathbf{p}_1(x',\xi')$$
$$+ \left[(1-\alpha(x'))\mathbf{I} + \alpha(x')\mathbf{p}_0(x',\xi')\right]$$
$$+ \cdots$$

Here:

$$\mathbf{p}_1(x',\xi') \geq \exists c_0 \left|\xi'\right|\mathbf{I} \text{ on } T^*\left(\partial\Omega\right)$$

Elementary Lemma

$$\boxed{\begin{array}{l} f(x) \in C^2(\mathbf{R}), \\ f(x) \geq 0 \text{ on } \mathbf{R}, \\ \sup_{x \in \mathbf{R}} |f''(x)| \leq \exists c \end{array}}$$

$$\Rightarrow$$

$$|f'(x)| \leq \sqrt{2c}\,\big(f(x)\big)^{1/2} \text{ on } \mathbf{R}.$$

Criteria for Parametrices (1)

$$A = p(x, D) \in L^m_{1,0}(\Omega).$$

Assume that:

$$\left| D_\xi^\alpha D_x^\beta p(x,\xi) \right| \le \exists C_{K,\alpha,\beta} \left| p(x,\xi) \right| \left(1 + |\xi|\right)^{-|\alpha| + (1/2)|\beta|},$$

$$\left| p(x,\xi)^{-1} \right| \le \exists C_K, \ \forall x \in K \subset \Omega, \ \forall \ |\xi| \ge C_K.$$

Criteria for Parametrices (2)

$$\exists B \in L^0_{1,1/2}(\Omega) \quad \text{such that}$$

$$AB \equiv I \mod L^{-\infty}(\Omega),$$

$$BA \equiv I \mod L^{-\infty}(\Omega).$$

Uniqueness Theorem

Uniqueness Theorem

$$\mathbf{v} \in \mathbf{H}^{s,p}\left(\Omega, \mathbf{R}^3\right)$$
$$\mathbf{A}\mathbf{v} = \mathrm{div}\left(\mathbf{a} \bullet \nabla \mathbf{v}\right) = \mathbf{0} \quad \text{in} \quad \Omega,$$
$$\mathbf{B}_\alpha \mathbf{v} = \alpha(x')(\mathbf{a} \bullet \nabla \mathbf{v} \bullet \mathbf{n}) + \left(1 - \alpha(x')\right)\mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial\Omega,$$

$(A)\quad 0 \le \alpha\left(X\right) \le 1 \text{ and } \alpha\left(X\right) \not\equiv 1 \text{ on } \partial\Omega.$

$(H)\quad \begin{array}{l} a_{ij\ell m} = a_{\ell m ij} = a_{ji\ell m} \\ \frac{1}{2}\mathbf{e} \bullet \mathbf{a} \bullet \mathbf{e} \ge \exists\, \eta \left\|\mathbf{e}\right\|^2 \end{array}$

⇓

$$\mathbf{v} = \mathbf{0} \text{ in } \Omega$$

Korn's Inequality

A non-empty set : $\gamma \subset \partial \Omega$

$\mathbf{u} \in \mathbf{H}^{1,2}(\Omega, \mathbf{R}^3)$

$\mathbf{u} = \mathbf{0}$ on γ

$$\int_\Omega \|\mathbf{e}\|^2 dx \geq \exists c(\gamma)\left(\int_\Omega \|\mathbf{u}\|^2 dx + \int_{\partial\Omega} \|\nabla \mathbf{u}\|^2 dx\right)$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$: the **strain tensor**

Existence Theorem

Existence Theorem

$$\mathbf{f} \in \mathbf{H}^{s-2,p}\left(\Omega, \mathbf{R}^3\right), \quad \varphi \in \mathbf{B}^{s-1-1/p,p}_{(\alpha)}\left(\partial\Omega, \mathbf{R}^3\right),$$

(A) $\quad 0 \le \alpha(X) \le 1$ and $\alpha(X) \not\equiv 1$ on $\partial\Omega$.

(H) $\quad \begin{array}{l} a_{ij\ell m} = a_{\ell m ij} = a_{ji\ell m} \\ \frac{1}{2}\mathbf{e} \bullet \mathbf{a} \bullet \mathbf{e} \ge \exists\, \eta \, \|\mathbf{e}\|^2 \end{array}$

$$\exists\, \mathbf{v} \in \mathbf{H}^{s,p}\left(\Omega, \mathbf{R}^3\right)$$

$$\begin{array}{l} \mathbf{A}\mathbf{v} = \mathrm{div}\left(\mathbf{a} \bullet \nabla \mathbf{v}\right) = \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{B}_{\alpha}\mathbf{v} = \alpha(x')(\mathbf{a} \bullet \nabla \mathbf{v} \bullet \mathbf{n}) + \left(1 - \alpha(x')\right)\mathbf{v} = \varphi \quad \text{on } \partial\Omega \end{array}$$

The Operator A

$$\mathbf{A} : L^2(\Omega, \mathbf{R}^3) \to L^2(\Omega, \mathbf{R}^3)$$

$$\text{(a)} \quad D(\mathbf{A}) = \{\mathbf{u} \in H^2(\Omega, \mathbf{R}^3) : \mathbf{B}_\alpha \mathbf{u} = \mathbf{0}\}.$$

$$\text{(b)} \quad \mathbf{A}\mathbf{u} = \operatorname{div}(\mathbf{a} \bullet \nabla \mathbf{u}), \ \forall \mathbf{u} \in D(\mathbf{A}).$$

$-A$: **positive definite, self-adjoint**

Function Space (1)

$\mathbf{B} = \sqrt{-\mathbf{A}}$: the **square root** of $-\mathbf{A}$

$H_{\mathbf{A}}$ = the **domain** $D(\mathbf{B})$ with the inner product $(\mathbf{u}, \mathbf{v})_{\mathrm{A}} = (\mathbf{B}\mathbf{u}, \mathbf{B}\mathbf{v})$.

$(A) \quad 0 \leq \alpha(X) \leq 1$ and $\alpha(X) \not\equiv 1$ on $\partial\Omega$.

Function Space (2)

$\mathsf{H}_{\mathbf{A}}$ = the **completion** of $D(\mathbf{A})$ with respect to the inner product

$$(\mathbf{u},\mathbf{v})_{A} = -(\mathbf{A}\mathbf{u},\mathbf{v})$$

$$= \int_{\Omega} \nabla \mathbf{u} \bullet \mathbf{a} \bullet \overline{\nabla \mathbf{v}}\, dx + \int_{\{\alpha \neq 0\}} \frac{1-\alpha}{\alpha} \mathbf{u} \bullet \overline{\mathbf{v}}\, d\sigma$$

Function Spaces (4)

$$\mathbf{H}_A = H_0^1\left(\Omega, \mathbf{R}^3\right)$$

if $\alpha(X) \equiv 0$ on $\partial\Omega$ (**Dirichlet**)

$$\mathbf{H}_A = H^1\left(\Omega, \mathbf{R}^3\right)$$

if $\alpha(X) \equiv 1$ on $\partial\Omega$ (**Neumann**)

Embedding Properties

$$D(\mathbf{A}) \subset \mathbb{H}_{\mathbf{A}} \subset H^1\left(\Omega, \mathbf{R}^3\right)$$

Existence and Uniqueness Theorem

$$\mathbf{u}_0 \in D(\mathbf{A}),\ \mathbf{u}_1 \in \mathrm{H}_{\mathbf{A}},\ \mathbf{f} \in C^1\left([0,\infty), L^2(\Omega,\mathbf{R}^3)\right),$$

⇓

$$\begin{cases} \dfrac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}\left(\mathbf{a}\bullet\nabla\mathbf{u}\right) = \mathbf{f} \quad \text{in } \Omega \times (0,\infty), \\ \mathbf{u}\big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial t}\Big|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega, \\ \alpha(x')(\mathbf{a}\bullet\nabla\mathbf{u}\bullet\mathbf{n}) + \left(1-\alpha(x')\right)\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

has a **unique solution**

$$\mathbf{u} \in C^2\left([0,\infty), L^2(\Omega,\mathbf{R}^3)\right) \cap C^1\left([0,\infty), \mathrm{H}_{\mathbf{A}}\right) \cap C\left([0,\infty), D(\mathbf{A})\right)$$

Energy Estimate

$$
\begin{aligned}
&\|\mathbf{u}(t)\|_{H^2}^2 + \|\mathbf{u}'(t)\|_{H_A}^2 + \|\mathbf{u}''(t)\|_{L^2}^2 \\
&\le \exists C \left(\|\mathbf{u}_0\|_{H^2}^2 + \|\mathbf{u}_1\|_{H_A}^2 + \|\mathbf{f}(0)\|_{L^2}^2 + \int_0^t \|\mathbf{f}'(s)\|_{L^2}^2 \, ds \right).
\end{aligned}
$$

Regularity Theorem

$$\forall \mathbf{u}_0 \in D(\mathbf{A}),\ \forall \mathbf{u}_1 \in \mathsf{H}_{\mathbf{A}},\ \forall \mathbf{f} \in C^1\left([0,\infty), L^2(\Omega,\mathbf{R}^3)\right)$$

$$\Rightarrow$$

$$\boxed{\exists \mathbf{u} \in C^2\left([0,\infty), L^2(\Omega,\mathbf{R}^3)\right) \cap C^1\left([0,\infty), \mathsf{H}_{\mathbf{A}}\right) \cap C\left([0,\infty), D(\mathbf{A})\right)}$$

such that

$$\begin{aligned}
&\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}\left(\mathbf{a} \bullet \nabla \mathbf{u}\right) = \mathbf{f} \quad \text{in } \Omega \times (0,\infty),\\
&\mathbf{u}\big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,\\
&\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,\\
&\alpha(x')(\mathbf{a} \bullet \nabla \mathbf{u} \bullet \mathbf{n}) + \left(1-\alpha(x')\right)\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,\infty)
\end{aligned}$$

Semigroup Approach

Semigroup Approach (1)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}\left(\mathbf{a} \bullet \nabla \mathbf{u}\right) = \mathbf{f} \quad \text{in } \Omega \times (0,\infty),$$

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\alpha(x')(\mathbf{a} \bullet \nabla \mathbf{u} \bullet \mathbf{n}) + \left(1 - \alpha(x')\right)\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,\infty).$$

$$\mathbf{u}''(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{f}(t),$$

$$\mathbf{u}(0) = \mathbf{u}_0, \ \mathbf{u}'(0) = \mathbf{u}_1$$

Semigroup Approach (2)

$$\mathbf{u}''(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{f}(t),$$
$$\mathbf{u}(0) = \mathbf{u}_0, \ \mathbf{u}'(0) = \mathbf{u}_1$$

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{u}(t) \\ \mathbf{u}'(t) \end{pmatrix}, \quad \mathbf{U}(0) = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \end{pmatrix},$$
$$\mathbf{F}(t) = \begin{pmatrix} 0 \\ \mathbf{f}(t) \end{pmatrix}, \quad \mathfrak{A} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{A} & 0 \end{pmatrix}$$

Semigroup Approach (3)

$$\mathbf{U}'(t) = \mathfrak{A}\mathbf{U}(t) + \mathbf{F}(t),$$
$$\mathbf{U}(0) = \mathbf{U}_0.$$

$$X = \mathsf{H}_{\mathbf{A}} \times L^2(\Omega, \mathbf{R}^3),$$
$$D(\mathfrak{A}) = D(\mathbf{A}) \times \mathsf{H}_{\mathbf{A}}.$$

Semigroup Approach (4)

$$e^{t\mathfrak{A}} = \cos(t\mathbf{B})\begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix} + \mathbf{B}^{-1}\sin(t\mathbf{B})\begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{A} & 0 \end{pmatrix}$$

$$\mathbf{B} = \sqrt{-\mathbf{A}}, \quad \mathbf{B}^2 = -\mathbf{A}.$$

Representation Formula

$$\mathbf{u}(t) = \cos(t\mathbf{B})\mathbf{u}_0 + \mathbf{B}^{-1}\sin(t\mathbf{B})\mathbf{u}_1 + \int_0^t \mathbf{B}^{-1}\sin((t-s)\mathbf{B})\mathbf{f}(s)ds.$$

$$\mathbf{B} = \sqrt{-\mathbf{A}}, \quad \mathbf{B}^2 = -\mathbf{A}.$$

Open Problems

Open Problems

1. The first problem is to generalize Main Theorem to the case where the domain has **corner singularities**.
2. The second problem is to study the case where the function $\alpha(X)$ is the **characteristic function** of a subset of the boundary.

References (4)

- **Ito: On a mixed problem of linear elastodynamics with time-dependent discontinuous boundary condition, Osaka J. Math. 27 (1990), 667—707.**

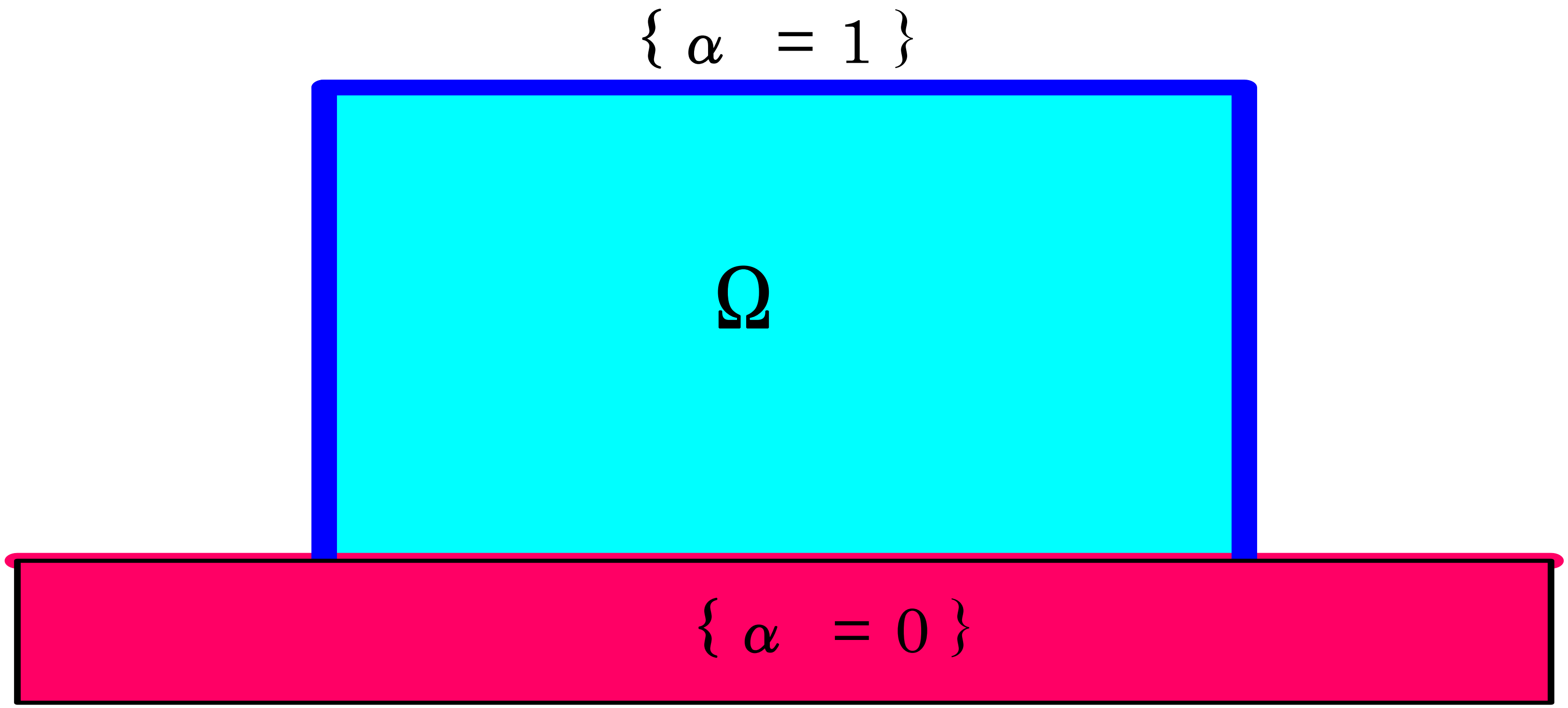
Nonlinear Boundary Value Problem (Lagrange)

This talk is devoted to an L^p approach to the following equilibrium equations for the unknown deformation $\boldsymbol{\phi}(X)$

$$\mathrm{Div}\,\widehat{\mathbf{P}}\big(X, D\boldsymbol{\phi}(X)\big) + \mathbf{B}(X) = \mathbf{0} \quad \text{in} \quad \Omega,$$

$$\alpha(\mathrm{X})\widehat{\mathbf{P}}\big(X, D\boldsymbol{\phi}(X)\big)\bullet\mathbf{N}(X) + \big(1-\alpha(X)\big)\boldsymbol{\phi}(X) = \boldsymbol{\tau}(X) \quad \text{on} \quad \partial\Omega.$$

Domain with Corner Singularities

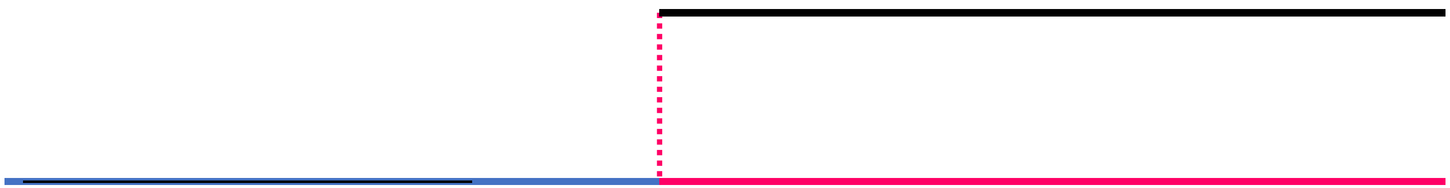


Boundary Condition (1)

$$\alpha(x) = \begin{cases} 1 & \text{on } \partial_N \Omega, \\ 0 & \text{on } \partial_D \Omega \end{cases}$$

Boundary Condition (2)

$$\alpha(x')(\mathbf{a}\bullet\nabla\mathbf{u}\bullet\mathbf{n})+\left(1-\alpha(x')\right)\mathbf{u}=\mathbf{0}\quad \text{on } \partial\Omega\times(0,\infty)$$

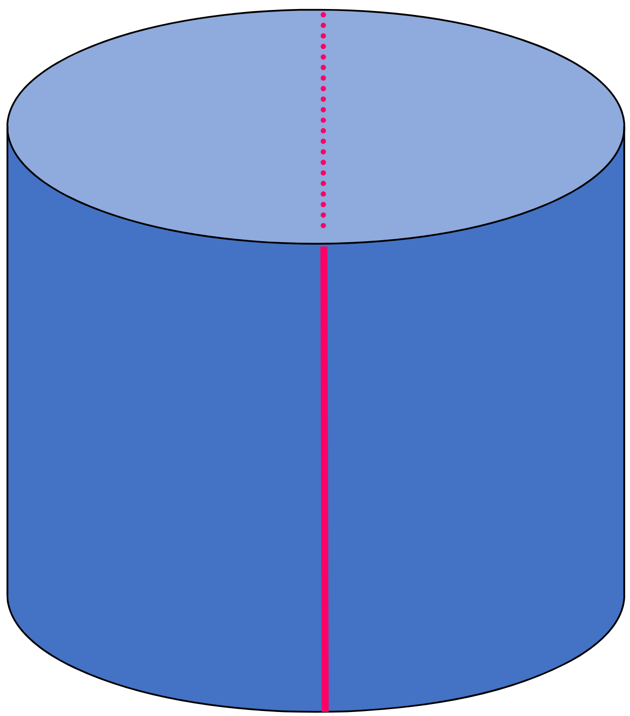


$$\partial_{D}\Omega=\{\alpha=0\}$$

$$\partial_{N}\Omega=\{\alpha=1\}$$

Geometry of the Second Problem

$$\Omega \times (0, T)$$



$$\partial_D \Omega \times (0, T)$$

$$\partial_N \Omega \times (0, T)$$

Fundamental Function Space

$$\mathbf{V}_D(0) = \left\{ \mathbf{u} \in H^1(\Omega, \mathbf{R}^3) : \mathbf{u} = \mathbf{0} \ \text{ on } \partial_{\mathrm{D}}\Omega \right\}$$

Definition of a Weak Solution (1)

$$\mathbf{u}_0 \in \mathbf{V}_D(0),\ \mathbf{u}_1 \in L^2(\Omega, \mathbf{R}^3),\ \mathbf{f} \in L^2(\Omega \times (0,T), \mathbf{R}^3)$$

A function

$$\mathbf{u} \in H^1(\Omega \times (0,T), \mathbf{R}^3)$$

is called a **weak solution** if it satisfies the conditions

$$\mathbf{u}(0,\bullet) = \mathbf{u}_0,$$

$$\mathbf{u} = \mathbf{0} \text{ on } \partial_D \Omega \times (0,T)$$

Definition of a Weak Solution(2)

$$-\int_0^T \left(\frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \eta}{\partial t} \right)_{L^2} dt + \int_0^T \left(\mathbf{a} \bullet \nabla \mathbf{u}, \nabla \eta \right)_{L^2} dt$$

$$= \left(\mathbf{u}_1, \eta(0,\bullet) \right)_{L^2} + \int_0^T \left(\mathbf{f}, \eta \right)_{L^2} dt$$

$$\forall \eta \in H^1(\Omega \times (0,T), \mathbf{R}^3), \; \eta = 0 \text{ on } \partial_D \Omega \times (0,T), \; \eta(T,\bullet) = 0.$$

Existence and Uniqueness Theorem

$$\begin{cases} \dfrac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div}\,(\mathbf{a} \bullet \nabla \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \\ \mathbf{u}\big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial t}\bigg|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{on } \partial_D \Omega \times (0,T), \\ \mathbf{a} \bullet \nabla \mathbf{u} \bullet \mathbf{n} = \mathbf{0} \quad \text{on } \partial_N \Omega \times (0,T) \end{cases}$$

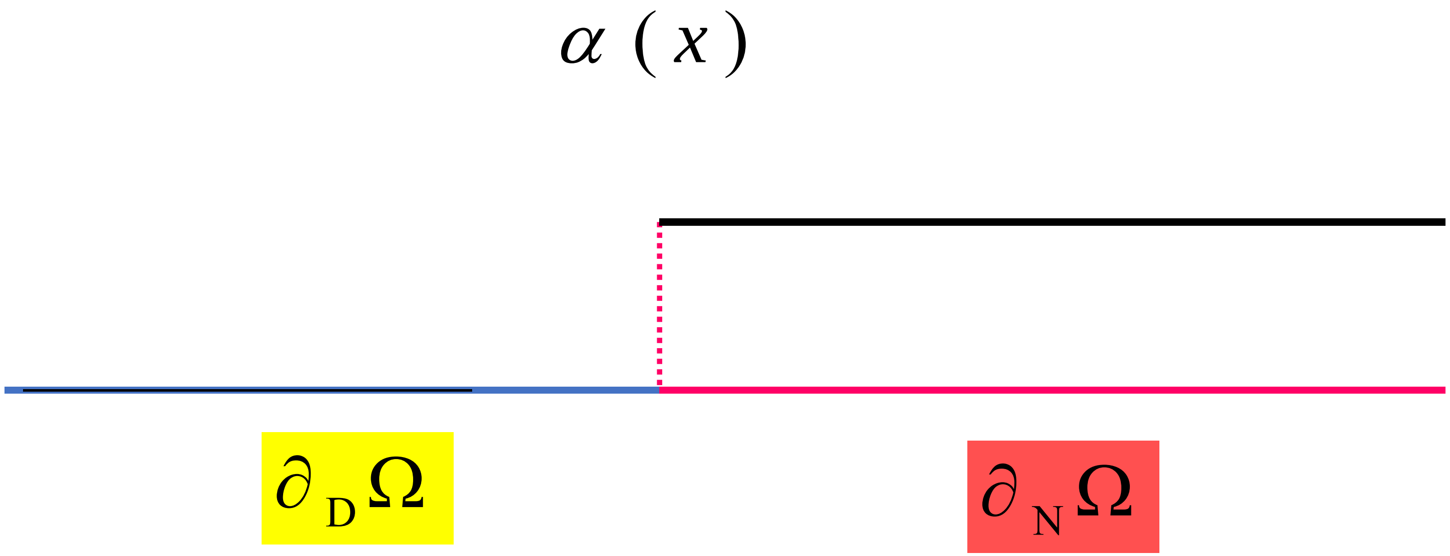
has a **unique weak solution**

$$\mathbf{u} \in H^1(\Omega \times (0,T), \mathbf{R}^3)$$

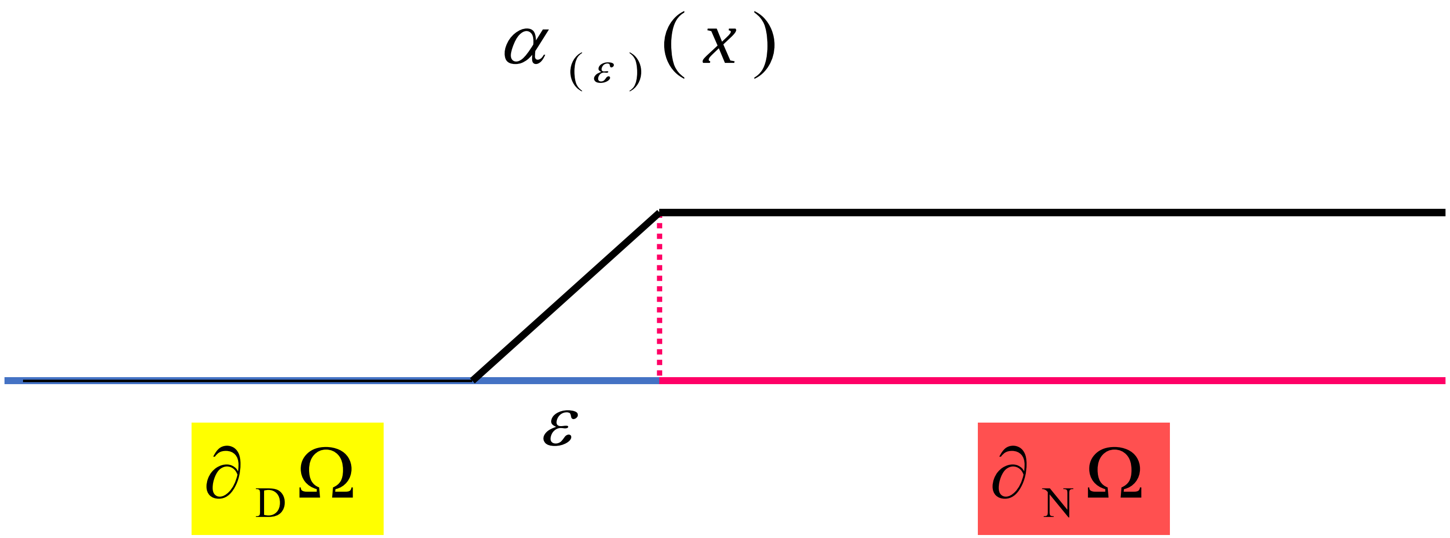
Energy Estimate for Weak Solutions

$$\|\mathbf{u}(t)\|_{H^1}^2 + \|\mathbf{u}'(t)\|_{L^2}^2$$
$$\leq \exists C(T)\left(\|\mathbf{u}_0\|_{H^1}^2 + \|\mathbf{u}_1\|_{L^2}^2 + \int_0^t \|\mathbf{f}(s)\|_{L^2}^2 \, ds\right).$$

The Original Boundary Condition



Approximations to the Boundary Condition



Regularized Mixed Problems

We study the following **regularized** mixed problem:

$$\begin{cases} \dfrac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - \operatorname{div}\left(\mathbf{a} \bullet \nabla \mathbf{u}_\varepsilon\right) = \mathbf{f}^{(\varepsilon)} \quad \text{in } \Omega \times (0, \infty), \\ \mathbf{u}_\varepsilon \big|_{t=0} = \mathbf{u}_0^{(\varepsilon)} \quad \text{in } \Omega, \\ \left.\dfrac{\partial \mathbf{u}_\varepsilon}{\partial t}\right|_{t=0} = \mathbf{u}_1^{(\varepsilon)} \quad \text{in } \Omega, \\ \alpha_{(\varepsilon)}(\mathbf{a} \bullet \nabla \mathbf{u}_\varepsilon \bullet \mathbf{n}) + \left(1 - \alpha_{(\varepsilon)}\right)\mathbf{u}_\varepsilon = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

Energy Estimate for Regularized Solutions

$$\forall \varepsilon > 0$$

$$\|\mathbf{u}_\varepsilon(t)\|_{H^1}^2 + \|\mathbf{u}'_\varepsilon(t)\|_{L^2}^2$$
$$\leq \exists C(T)\left(\|\mathbf{u}_0^{(\varepsilon)}\|_{H^1}^2 + \|\mathbf{u}_1^{(\varepsilon)}\|_{L^2}^2 + \int_0^t \|\mathbf{f}^{(\varepsilon)}(s)\|_{L^2}^2\, ds\right).$$

Energy Estimate for Weak Solutions

$\varepsilon \downarrow 0$

$$\|\mathbf{u}(t)\|_{H^1}^2 + \|\mathbf{u}'(t)\|_{L^2}^2$$
$$\leq \exists C(T)\left(\|\mathbf{u}_0\|_{H^1}^2 + \|\mathbf{u}_1\|_{L^2}^2 + \int_0^t \|\mathbf{f}(s)\|_{L^2}^2\, ds\right).$$

END