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SUBLINEAR HIGSON CORONA OF EUCLIDEAN CONE

By

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Abstract. Let X be a proper metric space. The sublinear Higson compactification $h_L X$ is a variant of the Higson compactification. Its boundary $h_L X \setminus X$ is denoted $\nu_L X$, and is called the sublinear Higson corona of X. The sublinear Higson corona is a functor from the category of coarse spaces to that of compact Hausdorff spaces. Let P be a compact metric space and X be an unbounded proper metric space. We show that the sublinear Higson corona of a product space $P \times X$ equipped with a cone metric is homeomorphic to a product $P \times \nu_L X$. Especially, the sublinear Higson corona of the n-dimensional Euclidean space is homeomorphic to the product of an $(n-1)$-dimensional sphere and the sublinear Higson corona of natural numbers.

1. Introduction

Compactifications of product spaces have been studied long time. Glicksberg [4] showed that the Stone-Čech compactification $\beta(X \times Y)$ of a product of two spaces X and Y (at least one of X and Y is infinite) is homeomorphic to $\beta X \times \beta Y$ if and only if $X \times Y$ is pseudo-compact (see also [11]). Tomoyasu [10] studied the Higson Compactifications of product spaces. Kawamura and Tomoyasu [6] studied approximations of Stone-Čech compactifications by Higson compactifications.

In this paper, we study the sublinear Higson compactification of product spaces. Let M be a non-compact complete Riemannian manifold. From considerations of index theory, Higson introduced a certain compactification of M defined as the maximal ideal space of the commutative C^*-algebra generated by

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the smooth functions on M whose gradient vanishes at infinity. Let X be a proper metric space. Roe gave an analogue of Higson's construction for X to define the Higson compactification hX. The boundary $\nu X = hX \backslash X$ is called the Higson corona of X.

The sublinear Higson corona is a variant of the Higson corona. A relation between the sublinear Higson corona and the Gromov-Lawson conjecture was studied in [2, Lemma 3.3]. (In [2] the sublinear Higson corona of an open manifold M was denoted by $\nu_{1/x}(M)$.)

We introduce the sublinear Higson corona $\nu_L X$ for a proper metric space X and show that it is a faithful functor from the category of coarse spaces to that of compact Hausdorff spaces. Let P be a compact metric space and $P \times_{\mathrm{cone}} X$ is a product space of P and X equipped with a cone metric defined in Section 4. We show that the sublinear Higson corona $\nu_L(P \times_{\mathrm{cone}} X)$ is homeomorphic to a product $P \times \nu_L X$ (Theorem 4.5). For example, $\nu_L \mathbf{R}^n$ is homeomorphic to $S^{n-1} \times \nu_L \mathbf{N}$. There is an application. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map of a positive determinant. We show that the induced map $\nu_L T : \nu_L \mathbf{R}^n \to \nu_L \mathbf{R}^n$ is homotopic to the identity.

The organization of this paper is as follows: In Section 2, we briefly review the coarse category. In Section 3, we define the sublinear Higson corona and study functorial properties. In Section 4, we define the Euclidean cone and study a decomposition of the sublinear Higson corona of it. In Section 5, we give an application.

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2. Coarse Category

A metric space X is *proper* if every closed bounded set in X is compact. For a positive number C, a proper metric space X is a *C-quasi-geodesic space* if for any $x, x' \in X$, there exists a map $f : [0, d(x, x')] \to X$ such that $f(0) = x$, $f(d(x, x')) = x'$ and $(1/C)|a - b| - C \le d(f(a), f(b)) \le C|a - b| + C$ for all $a, b \in [0, d(x, x')]$. We choose a base point $e \in X$ and define $|x| := d(e, x)$ for $x \in X$. In this paper, we say that a proper metric space X is a *coarse space* if X is a C_X-quasi-geodesic space for some constant C_X, and is equipped with the base point $e \in X$.

Definition 2.1. Let X and Y be coarse spaces and let $f : X \to Y$ be a map (not necessarily continuous). The map $f : X \to Y$ is a *quasi-isometry* if there exists a constant $A > 0$ such that

$$\frac{1}{A}d(x,x') - A \leq d(f(x), f(x')) \leq Ad(x,x') + A$$

for all points x, x' in X.

Let $f, g : X \to Y$ be maps. Then f is *close* to g if there exists a constant C such that $d(f(x), g(x)) \leq C$ for all $x \in X$. Moreover, f is *sublinearly close* to g if for any $\varepsilon > 0$, there exists a constant C_ε such that $d(f(x), g(x)) \leq \varepsilon|x| + C_\varepsilon$ for all $x \in X$.

Coarse spaces X and Y are *quasi-isometric* if there exist quasi-isometries $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to the identities of X and Y, respectively.

3. Sublinear Higson Corona

All arguments in this section are based on those in [9, Section 2.3]. Let X be a coarse space and $\varphi : X \to \mathbf{C}$ be a bounded continuous function. We say that φ is a *sublinear Higson function* on X if there exists a constant C_φ such that for all $R > 0$ and all $x, x' \in X \backslash B(R)$ satisfying $d(x,x') < R/2$, we have

$$|\varphi(x) - \varphi(x')| < \frac{C_\varphi d(x,x')}{R}. \tag{3.1}$$

Here $B(R)$ denotes the open ball of radius R centered at the base point e. Let $C_b(X)$ denotes the C^*-algebra of all bounded continuous functions on X. For $\varphi \in C_b(X)$, the norm of φ is the supremum, that is, $\|\varphi\| = \sup_{x \in X}|\varphi(x)|$. Let $C_{h_L}(X)$ denote the space of all sublinear Higson functions. Then it is easy to see that $C_{h_L}(X)$ is closed under multiplications and $*$-operations, so $C_{h_L}(X)$ is a sub $*$-algebra of $C_b(X)$ and its closure, denoted by $\overline{C_{h_L}}(X)$, is a unital C^*-algebra. By the Gelfand-Naimark theorem, $\overline{C_{h_L}}(X)$ is the algebra of continuous functions on a compactification of X.

Definition 3.1. The compactification $h_L X$ of X characterized by $C(h_L X) = \overline{C_{h_L}}(X)$ is called the *sublinear Higson compactification*. The boundary $h_L X \backslash X$ is denoted by $\nu_L X$, and is called the *sublinear Higson corona* of X.

Let $C_0(X)$ denote the algebra of continuous functions on X which vanish at infinity. Then we have $C(\nu_L X) \cong \overline{C_{h_L}}(X)/C_0(X)$.

Proposition 3.2. *The sublinear Higson corona of an unbounded coarse space is never second countable and its cardinal number is greater than or equal to* $2^{2^{\aleph_0}}$.

Proof. For second countability, it is enough to show that $\overline{C_{h_L}}(X)$ is not separable. We can choose a sequence $\{x_n\}$ such that $|x_n| > 2|x_{n-1}|$ for all $n \geq 0$. We define a continuous map $\varphi_n : X \to \mathbf{C}$ as follows:

$$\varphi_n(x) = \begin{cases} 1 - \frac{4d(x,x_n)}{|x_n|} & \text{if } d(x,x_n) \leq \frac{|x_n|}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

For a map $P : \mathbf{N} \to \{0,1\}$, we define a continuous map $\psi_P : X \to \mathbf{C}$ by $\psi_P(x) = \sum_{n \in \mathbf{N}} P(n)\varphi_n(x)$. Thus we obtain a family of uncountably many sublinear Higson functions such that $\|\psi_P - \psi_{P'}\| = 1$ for any pair $(\psi_P, \psi_{P'})$ of distinct $P, P' \in \{0,1\}^{\mathbf{N}}$. This shows that $\overline{C_{h_L}}(X)$ is not separable.

For $\psi \in C_b(\{x_n\})$, an extension $\hat{\psi} \in \overline{C_{h_L}}(X)$ of ψ is given by $\hat{\psi}(x) = \sum_{n \in \mathbf{N}} \psi(x_n)\varphi_n(x)$. Therefore $\overline{C_{h_L}}(X) \to C_b(\{x_n\})$ is surjective, which means the inclusion $\{x_n\} \hookrightarrow X$ extends to an embedding $\beta\{x_n\} \to h_L X$. Here $\beta\{x_n\}$ denotes the Stone-Čech compactification of $\{x_n\}$. Since $\beta\{x_n\}$ is homeomorphic to $\beta\mathbf{N}$, the cardinal number of $h_L X$ is greater than or equal to that of $\beta\mathbf{N}$, that is, $2^{2^{\aleph_0}}$. Arguments similar to those given here can be seen in the proof of [7, Theorem 3]. □

Let X and Y are coarse spaces and let $f : X \to Y$ be a continuous quasi-isometry. Then it is easy to see that f extends to the continuous map $h_L f : h_L X \to h_L Y$. The restriction $h_L f$ to $\nu_L X$ is denoted by $\nu_L f$. Assuming that X and Y are of bounded geometry (Definition 3.3), we can construct a continuous map $\nu_L f : \nu_L X \to \nu_L Y$ even if f is not continuous. To see this, we reformulate the construction of the sublinear Higson corona. Let X be a coarse space. Let $B(X)$ denote the C^*-algebra of all bounded functions on X, and let $B_0(X)$ denote the closed ideal of all bounded functions which vanish at infinity. Now let $B_{h_L}(X)$ denote the sub $*$-algebra of all bounded functions $\varphi : X \to \mathbf{C}$ which satisfy the following condition: There exists a constant C_φ such that for all $R > 0$ and all $x, x' \in X \backslash B(R)$ satisfying $d(x,x') < R/2$, we have

$$|\varphi(x) - \varphi(x')| < \frac{C_\varphi d(x,x') + C_\varphi}{R}.$$

Let $\overline{B_{h_L}}(X)$ be the closure of $B_{h_L}(X)$. Assuming the following condition on X, we can construct the sublinear Higson corona $\nu_L X$ using $B_h(X)$ instead of $C_h(X)$.

Definition 3.3. A coarse space X is of *bounded geometry* if there exists a uniformly bounded cover $\mathscr{U} = \{U_\alpha\}$ of X with a positive Lebesgue number and of finite degree. That is, there exist constants L, d and N such that the Lebesgue number of $\mathscr{U}$ is L, the diameter of all $U_\alpha \in \mathscr{U}$ is bounded by d and no more than N member of $\mathscr{U}$ have non-empty intersection.

Lemma 3.4. *Let X be a coarse space of bounded geometry. Then*
(a) $C_0(X) = \overline{C_{h_L}}(X) \cap B_0(X)$.
(b) $\overline{B_{h_L}}(X) = \overline{C_{h_L}}(X) + B_0(X)$.

Proof. The part (a) is obvious. The proof of part (b) is based on the proof of [9, Lemma 2.40]. Let $\mathscr{U} = \{U_\alpha\}$ be a cover of X described in Definition 3.3 and L, d, N be corresponding constants. Then we can construct a partition of unity $\{\pi_\alpha\}$ subordinate to $\mathscr{U}$ such that all of whose constituent functions are D-Lipschitz for a constant $D = D(L, d, N)$. (See the proof of [9, Theorem 9.9].)

Choose a point $x_\alpha \in U_\alpha$ for each α. Let $f \in B_{h_L}(X)$ and C_f be the constant which appears in the definition of $B_{h_L}(X)$. Define

$$g(x) := \sum_\alpha \pi_\alpha(x) f(x_\alpha).$$

The function g is continuous and bounded. For all $x \in X$, we have

$$f(x) - g(x) = \sum_\alpha \pi_\alpha(x)(f(x) - f(x_\alpha))$$

and $d(x, x_\alpha) < d$ whenever $\pi_\alpha(x) \neq 0$. Thus we have $f - g \in B_0(X)$. Next we show that g satisfies (3.1). By assumption, X is a C_X-quasi-geodesic for some $C_X > 0$. Let $R > 2d$ and $x, x' \in X \backslash B(R)$ such that $d(x, x') \leq C_X$. Set $I_+ := \{\alpha : \pi_\alpha(x) - \pi_\alpha(x') > 0\}$ and $I_- := \{\alpha : \pi_\alpha(x) - \pi_\alpha(x') < 0\}$. Set $t := \sum_{\alpha \in I_+}(\pi_\alpha(x) - \pi_\alpha(x')) = -\sum_{\alpha \in I_-}(\pi_\alpha(x) - \pi_\alpha(x'))$. Since each π_α is D-Lipschitz, we have $t \leq 2NDd(x, x')$. Set $f_{\max} := \max\{f(x_\alpha) : \alpha \in I_+\}$ and $f_{\min} := \min\{f(x_\alpha) : \alpha \in I_-\}$. For any $\alpha, \alpha' \in I_+ \cup I_-$, we have $x_\alpha, x_{\alpha'} \in X \backslash B(R - d)$ and $d(x_\alpha, x_{\alpha'}) < C_X + 2d$. It follows that $f_{\max} - f_{\min} \leq C_f(C_X + 2d + C_f)/(R - d) < 2C_f(C_X + 2d + C_f)/R$. Without loss of generality, we can assume that

$g(x) \geq g(x')$. Then we have

$$\begin{aligned}
|g(x) - g(x')| &= g(x) - g(x') \\
&= \sum_{\alpha \in I_+} (\pi_\alpha(x) - \pi_\alpha(x')) f(x_\alpha) + \sum_{\alpha \in I_-} (\pi_\alpha(x) - \pi_\alpha(x')) f(x_\alpha) \\
&\leq \sum_{\alpha \in I_+} (\pi_\alpha(x) - \pi_\alpha(x')) f_{\max} + \sum_{\alpha \in I_-} (\pi_\alpha(x) - \pi_\alpha(x')) f_{\min} \\
&= t(f_{\max} - f_{\min}) \\
&\leq \frac{4NDC_f(C_X + 2d + C_f)}{R} d(x, x').
\end{aligned}$$

This shows that $g \in \overline{C_{h_L}}(X)$ and completes the proof of Lemma 3.4. □

It follows from the second isomorphism theorem that

$$C(\nu_L X) = \frac{\overline{C_{h_L}}(X)}{C_0(X)} = \frac{\overline{C_{h_L}}(X)}{\overline{C_{h_L}}(X) \cap B_0(X)} = \frac{B_0(X) + \overline{C_{h_L}}(X)}{B_0(X)} = \frac{\overline{B_{h_L}}(X)}{B_0(X)}.$$

PROPOSITION 3.5. *Let X and Y be coarse spaces of bounded geometry. Then, a quasi-isometry $f : X \to Y$ extends to a continuous map $\nu_L f : \nu_L X \to \nu_L Y$. Moreover, two maps $f, g : X \to Y$ are sublinearly close if and only if $\nu_L f = \nu_L g$.*

PROOF. A quasi-isometry $f : X \to Y$ induces homomorphisms $f^* : \overline{B_{h_L}}(Y) \to \overline{B_{h_L}}(X)$ and $f^* : B_0(Y) \to B_0(X)$. If f is sublinearly close to g then $f^* - g^*$ maps $\overline{B_{h_L}}(Y)$ to $B_0(X)$. Thus $\nu f = \nu g$.

We suppose that f is not sublinearly close to g. There exist a constant C and a sequence $\{x_n\}$ such that $d(f(x_n), g(x_n)) \geq C|x_n|$. We can construct a sublinear Higson function φ on Y such that $\varphi(f(x_n)) = 1$ and $\varphi(g(x_n)) = 0$ for all $n \geq 0$, therefore $\nu_L f \neq \nu_L g$. □

COROLLARY 3.6. *Quasi-isometric spaces of bounded geometry have homeomorphic sublinear Higson coronae.*

Proposition 3.5 says that the sublinear Higson corona is a faithful functor from the category of coarse spaces of bounded geometry with sublinearly close classes of quasi-isometries to that of compact Hausdorff spaces.

Remark 3.7. Dranishnikov and Smith [3] introduced the *sublinear coarse structure* and showed that the sublinear Higson corona of a proper metric space is homeomorphic to its Higson corona with respect to the sublinear coarse structure ([3, Theorem 2.11]).

4. Euclidean Cone

Let X be a geodesic space with the base point e_X and P be a compact length space with the base point e_P. We define a cone metric on $P \times X$ as usual (See [8, Section 3.6.]). A *path* γ is a continuous map from an interval $I = [0, 1]$ to $P \times X$. We define the *length* of γ, denoted by $l(\gamma)$, to be

$$\sup\left\{\sum_{j=0}^{n-1} d(x_j, x_{j+1}) + \max\{1, |x_j|, |x_{j+1}|\} d(p_j, p_{j+1})\right\},$$

where the supremum is taken over all finite sequences $(p_j, x_j)_{j=0}^{n}$ of points on the path γ with (p_0, x_0) and (p_n, x_n) being the two endpoints. We make $P \times X$ into a length space by defining the distance d_{cone} of $z, z' \in P \times X$ to be

$$d_{\mathrm{cone}}(z, z') = \inf l(\gamma)$$

where the infimum is taken over all path γ joining $z = (p, x)$ and $z' = (p', x')$. The *Euclidean cone* of P and X, denoted by $P \times_{\mathrm{cone}} X$, is the metric space $P \times X$ equipped with this metric d_{cone} and the base point (e_P, e_X).

Example 4.1. The Euclidean cone $S^{n-1} \times_{\mathrm{cone}} \mathbf{R}_{\geq 0}$ is quasi-isometric to $\mathbf{R}^n$.

Let $C(P, \overline{C_{h_L}}(X))$ denotes the C^*-algebra of continuous $\overline{C_{h_L}}(X)$-valued functions on P. The norm of $F \in C(P, \overline{C_{h_L}}(X))$ is the supremum, that is,

$$\|F\| = \sup_{p \in P} \|F(p)\| = \sup_{p \in P}\left(\sup_{x \in X} |F(p)(x)|\right).$$

For $\varphi \in C_{h_L}(P \times_{\mathrm{cone}} X)$ and $p \in P$, we denotes by φ_p a map $x \mapsto \varphi(p, x)$.

We define a homomorphism $\Lambda : C_{h_L}(P \times_{\mathrm{cone}} X) \to C(P, \overline{C_{h_L}}(X))$ as follows:

$$\Lambda : C_{h_L}(P \times_{\mathrm{cone}} X) \ni \varphi \mapsto (p \mapsto \varphi_p) \in C(P, \overline{C_{h_L}}(X)).$$

Proposition 4.2. *Λ is well-defined and extends to a monomorphism of C^*-algebras:*

$$\Lambda : \overline{C_{h_L}}(P \times_{\mathrm{cone}} X) \to C(P, \overline{C_{h_L}}(X)).$$

PROOF. Let $\varphi \in C_{h_L}(P \times_{\mathrm{cone}} X)$. Fix a point $p \in P$. For any $R > 0$ and any $x, x' \in X \backslash B(R)$ with $d(x, x') < R/2$, we have

$$|\varphi_p(x) - \varphi_p(x')| < \frac{C_\varphi d_{\mathrm{cone}}((p,x),(p,x'))}{R} \leq \frac{C_\varphi d(x,x')}{R}.$$

Thus φ_p belongs to $\overline{C_{h_L}}(X)$ for all $p \in P$. Next we show that the map $\Lambda(\varphi) : P \to \overline{C_{h_L}}(X)$ is continuous. Let $p, p' \in P$. Since P is a length space, for any $\varepsilon > 0$, there exist points $p_0, \dots, p_n$ such that $p_0 = p$, $p_n = p'$, $d(p_{i+1}, p_i) < 1/2$ for $i = 0, \dots, n-1$ and $\sum_{i=0}^{n-1} d(p_{i+1}, p_i) < d(p, p') + \varepsilon$. Let $x \in X$. Since (p_{i+1}, x) and (p_i, x) belong to $(P \times_{\mathrm{cone}} X) \backslash B(|x|)$, and $d_{\mathrm{cone}}((p_{i+1}, x), (p_i, x)) \leq |x| d(p_{i+1}, p_i) < |x|/2$, it follows from (3.1) that

$$|\varphi(p_{i+1}, x) - \varphi(p_i, x)| \leq \frac{C_\varphi}{|x|}(d_{\mathrm{cone}}((p_{i+1}, x), (p_i, x)) \leq C_\varphi d(p_{i+1}, p_i).$$

Thus we have

$$\begin{aligned} |\Lambda(\varphi)(p)(x) - \Lambda(\varphi)(p')(x)| &= |\varphi(p,x) - \varphi(p',x)| \\ &\leq \sum_{i=0}^{n-1} |\varphi(p_{i+1},x) - \varphi(p_i,x)| \\ &\leq \sum_{i=0}^{n-1} C_\varphi d(p_{i+1}, p_i) \\ &\leq C_\varphi(d(p,p') + \varepsilon). \end{aligned}$$

We can choose ε as an arbitrary small number, so $\|\Lambda(\varphi)(p) - \Lambda(\varphi)(p')\| \leq C_\varphi d(p, p')$. This shows that $\Lambda(\varphi)$ is continuous and therefore $\Lambda(\varphi)$ belongs to $C(P, \overline{C_{h_L}}(X))$. By the definition, we have

$$\|\varphi\| = \sup_{(p,x) \in P \times_{\mathrm{cone}} X} |\varphi(p,x)| = \sup_{p \in P}\left(\sup_{x \in X} \varphi_p(x)\right) = \|\Lambda(\varphi)\|.$$

This shows that Λ is an isometry, so Λ extends to the isometry $\Lambda : \overline{C_{h_L}}(P \times_{\mathrm{cone}} X) \to C(P, \overline{C_{h_L}}(X))$. □

PROPOSITION 4.3. *The map* $\Omega : C(P) \otimes \overline{C_{h_L}}(X) \to \overline{C_{h_L}}(P \times_{\mathrm{cone}} X) : \varphi \otimes \psi \mapsto \varphi \cdot \psi$ *is well-defined.*

PROOF. The map $\Omega : C(P) \otimes \overline{C_{h_L}}(X) \to C_b(P \times_{\mathrm{cone}} X)$ is well-defined. We show that the image of Ω lies on $\overline{C_{h_L}}(P \times_{\mathrm{cone}} X)$. Let $\varphi \otimes \psi \in C(P) \otimes C_{h_L}(X)$.

We suppose that φ is a Lipschitz map with a Lipschitz constant C_φ. Let $R > 0$ and $(p,x),(p',x') \in P \times_{\mathrm{cone}} X$ such that $|x|,|x'| > R$. We assume that $d_{\mathrm{cone}}((p,x),(p',x')) < R/2$. By the definition of d_{cone}, for any $0 < \varepsilon < R/6$, there exists a sequence $\{(p_i,x_i)\}_{i=0}^n$ satisfying $(p_0,x_0) = (p,x)$, $(p_n,x_n) = (p',x')$ such that

$$\sum_{i=0}^{n-1} d(x_i,x_{i+1}) + \max\{1,|x_i|,|x_{i+1}|\}d(p_i,p_{i+1}) \le d_{\mathrm{cone}}((p,x),(p',x')) + \varepsilon.$$

Then $d_{\mathrm{cone}}((p_i,x_i),(e_P,e_X)) > R/3$ for $i = 0,\dots,n$. It follows that

$$|x_i| > \frac{R}{3(1+\operatorname{diam} P)}.$$

Here $\operatorname{diam} P$ denotes the diameter of P. Thus we have

$$\begin{aligned}
&|\varphi(p_{i+1})\psi(x_{i+1}) - \varphi(p_i)\psi(x_i)| \\
&\quad \le |\varphi(p_{i+1})\psi(x_{i+1}) - \varphi(p_i)\psi(x_{i+1})| + |\varphi(p_i)\psi(x_{i+1}) - \varphi(p_i)\psi(x_i)| \\
&\quad \le C_\varphi\|\psi\| d(p_{i+1},p_i) + \frac{3(1+\operatorname{diam} P)C_\psi\|\varphi\|}{R} d(x_{i+1},x_i) \\
&\quad \le \frac{3(1+\operatorname{diam} P)(C_\varphi\|\psi\| + C_\psi\|\varphi\|)}{R}(d(x_{i+1},x_i) + \max\{|x_{i+1}|,|x_i|\}d(p_{i+1},p_i)).
\end{aligned}$$

Then we have

$$|\varphi(p')\psi(x') - \varphi(p)\psi(x)| \le \frac{C_\varphi\|\psi\| + 3C_\psi\|\varphi\|}{R}(d_{\mathrm{cone}}((p',x'),(p,x)) + \varepsilon).$$

It follows that $\Omega(\varphi \otimes \psi)$ belongs to $C_{h_L}(P \times_{\mathrm{cone}} X)$. Since the set of Lipschitz maps is dense in $C(P)$, we have the desired consequence. □

To show that Ω is an isomorphism, we need the following well-known fact.

Lemma 4.4. *Let P be a compact metric space and A be a commutative C^*-algebra. Then $C(P) \otimes A \cong C(P,A)$.*

Proof. We can construct a family $\{\mathcal{U}^n\}_{n\in\mathbf{N}}$ of finite covers of P such that, the diameter of each member U_i^n of $\mathcal{U}^n$ is less than $1/n$. We choose points $p_i^n \in U_i^n$ for each $n \in \mathbf{N}$. Let $\{h_i^n\}_i$ be a partition of unity subordinate to $\mathcal{U}^n$. We define $\Psi : C(P) \otimes A \to C(P,A)$ by $\Psi(\varphi \otimes a)(p) = \varphi(p)a$ for $\varphi \in C(P)$, $a \in A$ and $p \in P$. Clearly Ψ is injective. We show that Ψ is surjective. Let $\psi \in C(P,A)$. Set

$\psi_n = \sum_i h_i^n \otimes \psi(p_i^n) \in C(P) \otimes A$. Since ψ is uniformly continuous, $\|\psi - \Psi(\psi_n)\|$ tends to 0 as n goes to infinity. Thus Ψ is surjective. $\square$

THEOREM 4.5. *The sublinear Higson compactification of the Euclidean cone $P \times_{\text{cone}} X$ is homeomorphic to the product $P \times h_L X$. Especially $\nu_L(P \times_{\text{cone}} X) = P \times \nu_L X$.*

PROOF. By Proposition 4.2, 4.3 and Lemma 4.4, $\overline{C_{h_L}}(P \times_{\text{cone}} X) \cong C(P) \otimes \overline{C_{h_L}}(X)$. $\square$

EXAMPLE 4.6. $\nu_L(\mathbf{R}^n) = S^{n-1} \times \nu_L \mathbf{N}$.

5. Application

DEFINITION 5.1. Let $f, g : X \to Y$ be quasi-isometries. f is *cone-homotopic* to g if there exists a quasi-isometry $H : [0,1] \times_{\text{cone}} X \to Y$ such that $f = H_0$ and $g = H_1$. We call H a *cone homotopy* between f and g.

THEOREM 5.2. *If f is cone-homotopic to g, then the induced map $\nu_L f$ is homotopic to $\nu_L g$.*

PROOF. Since $\nu_L([0,1] \times_{\text{cone}} X) = [0,1] \times \nu_L X$, we have that H induces a continuous map $\nu_L H : [0,1] \times \nu_L X \to \nu_L Y$ such that $\nu_L H(0,x) = \nu_L f(x)$ and $\nu_L H(1,x) = \nu_L g(x)$ for all $x \in X$. $\square$

EXAMPLE 5.3. Let T be an n by n integer matrix with a positive determinant. Then the linear map $T : \mathbf{Z}^n \to \mathbf{Z}^n$ is a quasi-isometry. Since T is not sublinearly close to the identity I_n, the induced map $\nu_L T : \nu_L \mathbf{Z}^n \to \nu_L \mathbf{Z}^n$ is different from the identity $\mathrm{id}_{\nu_L \mathbf{Z}^n}$. However $\nu_L T$ is homotopic to $\mathrm{id}_{\nu_L \mathbf{Z}^n}$.

PROOF. Since $\nu_L \mathbf{Z}^n$ is homeomorphic to $\nu_L \mathbf{R}^n$, it is enough to show that the map $T : \mathbf{R}^n \to \mathbf{R}^n$ is cone-homotopic to the identity. Since T has a positive determinant, we can choose a continuous path $\Theta : [0,1] \to GL_+(n, \mathbf{R}) = \{A \in GL(n, \mathbf{R}) : \det A > 0\}$ such that $\Theta(0) = T$ and $\Theta(1) = I_n$. A map $H(x,t) = \Theta(t)x$ is a cone homotopy between T and the identity I_n. $\square$

REMARK 5.4. Due to the result of Keesling [7, Section 3], the induced map νT on the Higson corona $\nu \mathbf{Z}^n$ is not homotopic to the identity $\mathrm{id}_{\nu \mathbf{Z}^n}$.

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