Rotational beta expansion

Jonathan Villaroza Caalim

February 2017

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Jonathan Villaroza Caalim
Doctoral Program in Mathematics

Submitted to the Graduate School of
Pure and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Science

at the
University of Tsukuba

Abstract

Let $m \in \mathbb{N}$. Let $\eta_i \in \mathbb{R}^m$ $(i = 1, 2, \ldots, m)$ be m linearly independent vectors over $\mathbb{R}$ and $\xi \in \mathbb{R}^m$. Let $\mathcal{L}$ be the lattice in $\mathbb{R}^m$ with the fundamental domain

$$\mathcal{X} = \left\{ \xi + \sum_{i=1}^{m} x_i \eta_i \;\middle|\; x_i \in [0, 1) \right\}$$

generated by the vectors η_i and ξ. Let $1 < \beta \in \mathbb{R}$ and M be an isometry in the orthogonal group $O(m)$ of dimension m. We define the rotational beta transformation map $T : \mathcal{X} \to \mathcal{X}$ with parameters β, M and $\mathcal{X}$ as the map given by

$$T(z) = \beta M z - d(z)$$

where $d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta M z \in \mathcal{X} + d$. Rotational beta transformations generalize the notions of positive and negative beta transformations in one dimension. In this study, we give necessary and sufficient conditions for the symbolic dynamical system associated to a rotational beta transformation to be sofic. Moreover, we give lower bounds for β such that the rotational beta transformation admits a unique absolutely continuous invariant measure under certain assumptions. We also give lower bounds such that these unique measures are equivalent to the Lebesgue measure.

Acknowledgement

This thesis would not be possible without the invaluable support of several individuals and institutions who have extended in many forms their assistance, encouragement and guidance throughout the duration of my doctoral studies. It is with great pleasure that I express my sincerest gratitude in this acknowledgement.

Foremost of all, I would like to extend my utmost gratitude to my Ph.D. adviser, Prof. Shigeki Akiyama, for patiently and diligently guiding me through my doctoral studies journey. I am truly indebted for Prof. Akiyama's adept supervision, one that gives me freedom to explore the areas of studies that appeal to me and at the same time, provides me direction when I am veering to far from my objectives. My graduate school experience in Japan is beyond exceptional because the kindness he has shown me.

I also thank the Hitachi Scholarship Foundation, which is now under the Hitachi Global Foundation, for generously financing my doctoral studies at the University of Tsukuba. I greatly appreciate the efforts of the Foundation for its scholars to have an holistic experience of living in Japan. The Foundation has provided numerous opportunities to know about the culture of Japan through summer trips, football matches, etc. For these and many more, I thank the Hitachi Global Foundation.

My heartfelt thanks also go to the Department of Mathematics of the Graduate School of Pure and Applied Sciences of the University of Tsukuba for giving me a quality education and hosting me while I do research for three and a half years. I also thank the Institute of Mathematics of my home university, the University of the Philippines, for granting me a leave of absence to pursue doctoral studies in Japan.

Finally, I would like to thank my family and friends for their unwavering love and support. I dedicate this dissertation to my family.

Contents

Chapter 1

Introduction

Let $m \in \mathbb{N}$. Let $\eta_i \in \mathbb{R}^m$ $(i = 1, 2, \ldots, m)$ be linearly independent over $\mathbb{R}$. Let $\xi \in \mathbb{R}^m$. We define $\mathcal{X}$ as the region

$$\mathcal{X} = \left\{ \xi + \sum_{i=1}^{m} x_i \eta_i \;\middle|\; x_i \in [0, 1) \right\}.$$

Let $\mathcal{L}$ be the lattice in $\mathbb{R}^m$ generated by the vertices of $\mathcal{X}$. Then $\mathcal{L}$ induces a disjoint partition of R^m. Indeed,

$$\mathbb{R}^m = \bigcup_{d \in \mathcal{L}} (\mathcal{X} + d).$$

Let $O(m)$ denote the orthogonal group in dimension m.

Definition 1.0.1. Let $1 < \beta \in \mathbb{R}$, $M \in O(m)$ and $\mathcal{X}$ be as given above. We define the map $d : \mathcal{X} \to \mathcal{L}$ such that for all $z \in \mathcal{X}$, $d(z)$ is the unique element in $\mathcal{L}$ where $\beta M z - d(z) \in \mathcal{X}$.

Definition 1.0.2. The rotational beta transformation with parameters β, M and $\mathcal{X}$ is the map $T : \mathcal{X} \to \mathcal{X}$ given by $T(z) = \beta M z - d(z)$.

A rotational beta transformation rotates the fundamental domain associated to it about the origin and expands the rotated region by the similarity ratio. Then the resulting expanded image is pulled back to the fundamental domain (see Figure 1.1).

For $z \in \mathcal{X}$ and $i \in \mathbb{N}$, put $d_i = d_i(z) = d(T^{i-1}(z))$. We have

$$\begin{aligned} z &= \frac{M^{-1} d_1}{\beta} + \frac{M^{-1} T(z)}{\beta} \\ &= \frac{M^{-1} d_1}{\beta} + \frac{M^{-2} d_2}{\beta^2} + \frac{M^{-2} T^2(z)}{\beta^2} \\ &= \sum_{i=1}^{\infty} \frac{M^{-i} d_i}{\beta^i}. \end{aligned}$$

In light of the above observation, we have the following definition.

Definition 1.0.3. Let T be a rotational beta transformation with parameters β, M and $\mathcal{X}$. For $z \in \mathcal{X}$, we call $D(z) := d_1(z) d_2(z) \ldots$ the *rotational beta expansion* of z with respect to T.

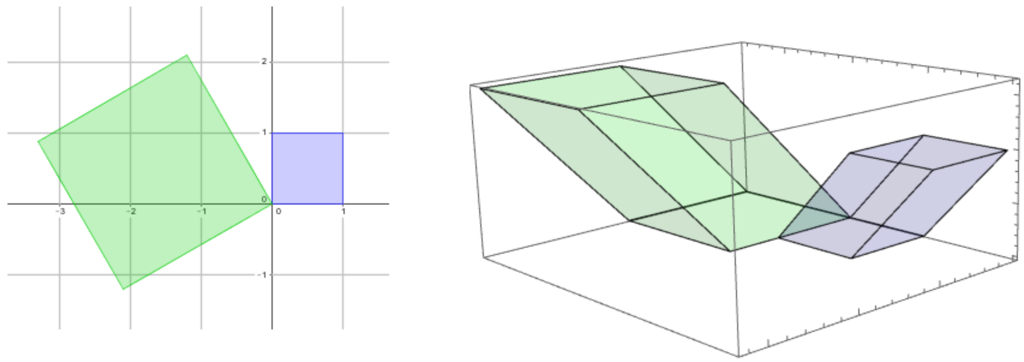


Figure 1.1: Left: $\beta = 1+\sqrt{2}$, rotation by $2\pi/3$, $\mathcal{X} = [0,1)^2$; Right: $\beta = \frac{1+\sqrt{5}}{2}$, rotation by $\pi/2$ about the z-axis, $\mathcal{X} = \{a_1(1,0,0) + a_2(0,1,0) + a_3(0,1,1) \; | a_i \in [0,1)\}$

Rotational beta expansions extend the notion of beta expansions [37, 33, 18] and negative beta expansions [17, 19] to higher dimensions. In [37], Renyi introduced the notion of beta transformations whose formulation appears below.

Definition 1.0.4. Let $1 < \beta \in \mathbb{R}$. The beta transformation is the map $T : [0,1) \to [0,1)$ given by $T(x) = \beta x - \lfloor \beta x \rfloor$ for all $x \in [0,1)$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Renyi showed that this transformation admits a unique invariant measure μ which is equivalent to the Lebesgue measure (for the pertinent definitions, see Chapter 3). Moreover, he showed that for any measurable set E in $[0,1)$, we have that

$$\mu(E) = \int_E h(x)dx,$$

where $h(x)$ is a measurable function satisfying

$$\frac{\beta-1}{\beta} \leq h(x) \leq \frac{\beta}{\beta-1}.$$

He computed the explicit form of the function $h(x)$ when β is the golden mean $(1+\sqrt{5})/2$. In [33], Parry gave the explicit form of $h(x)$ for arbitrary $\beta > 1$. He showed that

$$h(x) = \sum_{x < T^n(1^-)} \frac{1}{\beta^n},$$

where for all $n \in \mathbb{N}$, $T^n(1^-)$ denotes the limiting image under T^n of a point in $[0,1)$ very close to 1. The key idea is to determine the unique solution to the Perron-Frobenius equation. In the succeeding papers [34, 35], Parry considered generalized beta transformations and similarly determined explicitly the density $h(x)$ of the invariant measure. Other ergodic properties of beta transformations are discussed in [13, 43].

In other studies, the symbolic dynamics (see Chapter 2 for the definition) associated with beta transformations have been investigated. In particular, Bertrand showed in [9] that the dynamics has the so-called soficness property if and only if the expansion of 1 with respect to the beta transformation is eventually periodic. In other words, the soficness of the dynamical system is related to the orbit $\{T^n(1^-) \mid n \in \mathbb{N}\}$.

Analogous to the classical beta transformations, Ito and Sadahiro introduced an expansion where the base is a negative real number. In [17], they gave the definition of negative beta transformations.

Definition 1.0.5. A negative beta transformation is defined to be a map $T_{-\beta} : [-\beta/(\beta+1), -\beta/(\beta+1)+1) \to [-\beta/(\beta+1), -\beta/(\beta+1)+1)$ given by

$$T_{-\beta}(x) = -\beta x - \lfloor -\beta x + \beta/(\beta+1) \rfloor,$$

where $1 < \beta \in \mathbb{R}$.

Ito and Sadahiro proved that the dynamical system associated with the negative beta transformation is sofic if and only if the expansion of $-\beta/(\beta+1)$ with respect to $T_{-\beta}$ is eventually periodic. Meanwhile, negative beta transformations admit a unique absolutely continuous invariant measure by [30]. Ito and Sadahiro showed that the density $h_{-\beta}(x)$ of the unique invariant measure has the form

$$h_{-\beta}(x) = \sum_{x > T_{-\beta}{}^n(-\beta/(\beta+1))} \frac{1}{(-\beta)^n}.$$

They gave an example of a negative beta transformation where the map admits a unique invariant measure, but the measure is not equivalent to the Lebesgue measure as the support of the unique invariant measure contains gaps. Liao and Steiner studied these gaps in [31]. Other general beta transformations are considered in [12, 25].

Rotational beta transformations follow similar frameworks as that of the positive and negative beta transformations. The rotational beta transformations are particularly important as they give a natural algorithm of representing numbers in higher dimensions where the digits are in $\mathbb{Z}^m$. In literature, we can find various studies on non-integer representations in higher dimensions. For example, canonical number systems [15, 22, 41, 5] give representations of points in higher dimensional space using positive integer bases where the algorithm has negligible redundancy, i.e. the set of points with unique expansion has full Lebesgue measure. There are also studies that deal with expansions of complex numbers using a complex base, as in [28, 39]. In $m = 2$ case, the rotational beta expansions can be interpreted as representation of complex numbers where the digits are in $\mathbb{Z}^2$.

In our study, we find dynamical systems associated with rotational beta transformations which are sofic. These systems give rise to self-similar substitution tilings. The construction of self-similar tilings has been studied in papers like [26, 47, 29, 27].

In this paper, we discuss the symbolic dynamical systems and invariant measures associated to rotational beta transformations. In Chapter 2, we give our results on the soficness of the associated symbolic dynamical system. In Section 1 of this chapter, we give the preliminaries of the subject. In particular, we define the topological dynamics $(\mathcal{X}, T)$ and the notion of soficness. In Section 2, we detailed our result on soficness in the general setting but ultimately focusing on the two-dimensional case. In Section 3, we extend our results in $\mathbb{R}^2$ to obtain results for three-dimensional rotational beta transformations. We give example of sofic systems in Section 4 and explain in Section 5 the process of obtaining substitution tilings from sofic systems. In the last section of this chapter, we give a result on the periodicity of the rotational beta expansions in dimension 2.

In Chapter 3, we tackle the subject of invariant measures. The first section reviews basic concepts on invariant measures. The next two sections, Section 2 and Section 3, discuss the main results in this chapter. The results are categorized based on the techniques employed in the proofs. Section 2 looks at coverings of the fundamental domain $\mathcal{X}$ while Section 3 keeps track of the so-called holes inside $\mathcal{X}$. In the last section, we give some examples.

In the final chapter, Chapter 4, we sum up our results and discuss prospective studies.

Chapter 2

Symbolic Dynamical System

2.1 Preliminaries

Let $\mathcal{A} := \{d(z) \mid z \in \mathcal{X}\}$ and $\mathcal{A}^{\mathbb{Z}}$ (resp. $\mathcal{A}^*$) be the set of all bi-infinite (resp. finite) words over $\mathcal{A}$. Let $\partial(\mathcal{X})$ denote the boundary of $\mathcal{X}$ which is composed of hyperplanes of co-dimension 1. Note that the map T is not defined on $\partial(\mathcal{X})$. We extend the definition of T to $\partial(\mathcal{X})$ by introducing appropriate infinitesimal perturbations on $\partial(\mathcal{X})$ and taking the image of these under T. The set $\bigcup_{n=-\infty}^{\infty} T^n(\partial(\mathcal{X}))$ corresponds to the discontinuities of the forward and backward iterates of T. Observe that $\bigcup_{n=-\infty}^{\infty} T^n(\partial(\mathcal{X}))$ is a null set. Let

$$Ess(\mathcal{X}) := \mathcal{X} \setminus \bigcup_{n=-\infty}^{\infty} T^n(\partial(\mathcal{X})).$$

The set $Ess(\mathcal{X})$ is called the essential part of $\mathcal{X}$.

We have the following definitions.

Definition 2.1.1. Let $w \in \mathcal{A}^*$. If w appears in the expansion $D(z)$ of some $z \in \mathcal{X} \setminus \bigcup_{n=-\infty}^{\infty} T^n(\partial(\mathcal{X}))$, we say that w is admissible.

Definition 2.1.2. The symbolic dynamical system $(\mathcal{X}_T, s)$ associated with T is the topological dynamics given by

$$\mathcal{X}_T := \left\{ w = (w_j) \in \mathcal{A}^{\mathbb{Z}} \middle| w_j w_{j+1} ... w_k \text{ is admissible } \right\}.$$

and the shift operator $s : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $s((w_j)) = (w_{j+1})$.

Note that $\mathcal{X}_T$ is compact by the product topology of $\mathcal{A}^{\mathbb{Z}}$ and is invariant under s. For ease of notation, we write $(\mathcal{X}, T)$ instead of $(\mathcal{X}_T, s)$.

Definition 2.1.3. Let $(\mathcal{X}, T)$ be the symbolic dynamical system associated with a rotational beta transformation T. We say that $(\mathcal{X}, T)$ is *sofic* if there is a finite directed graph G labeled by $\mathcal{A}$ such that for each $w \in \mathcal{X}_T$, there exists a bi-infinite path in G labeled w and moreover, every labeling of a bi-infinite path in G is in $\mathcal{X}_T$.

Definition 2.1.4. Let $z \in Ess(\mathcal{X})$. The set of *predecessors* of z is defined to be

$$Pred(z) = \bigcup_{n=1}^{\infty} \left\{ d(x)d(T(x))\dots d(T^{n-1}(x)) \in \mathcal{A}^* \mid x \in T^{-n}(z) \right\}.$$

For $z \in Ess(\mathcal{X})$, the set $Pred(z)$ records all the codings of the trajectories leading to the point z. We introduce an equivalence relation on $z \in Ess(\mathcal{X})$ as follows. Let $z_i \in Ess(\mathcal{X})$, $i = 1, 2$. We say $z_1 \sim z_2$ if $Pred(z_1) = Pred(z_2)$. Then the set of equivalence classes is finite if and only if the symbolic dynamical system is sofic (cf. [32, Theorem 3.2.10]).

2.2 Results in $\mathbb{R}^2$

For a moment, we focus on rotational beta transformations in dimension 2. Let $1 < \beta \in \mathbb{R}$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| = 1$. Fix $\xi, \eta_1, \eta_2 \in \mathbb{C}$ such that η_1 and η_2 are linearly independent over $\mathbb{R}$. Let $\mathcal{X} = \{\xi + x\eta_1 + y\eta_2 \mid x \in [0, 1), y \in [0, 1)\}$. We consider the rotational beta transformation with the parameters β, ζ and $\mathcal{X}$. Then we have the following result.

Theorem 2.2.1. *Let T be a 2-dimensional rotational beta transformation on $\mathcal{X}$. The symbolic dynamical system $(\mathcal{X}, T)$ is sofic if and only if $\bigcup_{n=0}^{\infty} T^n(\partial(\mathcal{X}))$ is a finite union of segments.*

Proof We first show the forward implication. Suppose that $(\mathcal{X}, T)$ is sofic. We consider $\mathcal{X} \setminus \bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ and proceed by induction on k. By the definition of T, we know that each endpoint of a discontinuity segment falls on *another discontinuity segment of a different slope.* In the exceptional case where a discontinuity segment of $T^n(\partial(\mathcal{X}))$ falls into $\partial(\mathcal{X})$, we discard the segment as the dynamics is defined on $Ess(\mathcal{X})$ anyway. Thus, we easily see that $\mathcal{X} \setminus \bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ is partitioned into finitely many open polygons. A discontinuity segment in $T^{k+1}(\partial(\mathcal{X}))$ may cut an open polygon in $\mathcal{X} \setminus \bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ into two separate regions. If we take two points x and y from each of these two separate regions, we see that $Pred(x) \neq Pred(y)$ as one contains at least one more coding than the other. By way of contradiction, suppose that $\bigcup_{n=0}^{\infty} T^n(\partial(\mathcal{X}))$ contains infinitely many line segments. Applying T on $\bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$, at least one open polygon of $\mathcal{X} \setminus \bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ is subdivided by a line segment of $T^{k+1}(\partial(\mathcal{X}))$. Indeed, if this is not the case, we have that $T^{k+1}(\partial(\mathcal{X}))$ is contained in $\bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$. In fact, $T^l(\partial(\mathcal{X})) \subset \bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ for $l \in \mathbb{N} \geq k+1$. However, there are only finitely many segments in $\bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ whose endpoints lie on other segments of different slopes. In other words, the sequence $\{T^n(\partial(\mathcal{X}))\}_{n \in \mathbb{N} \cup \{0\}}$ is eventually periodic. Contradiction. Therefore, as the index k increases, the number of polygon also increases, i.e., the number of inequivalent predecessor sets is unbounded. Thus, $(\mathcal{X}, T)$ cannot be sofic.

We now prove the reverse implication. Suppose that $\bigcup_{n=0}^{\infty} T^n(\mathcal{X})$ contains finitely many line segments. As in above, the discontinuity line segments induce a partition of $\mathcal{X}$ into finitely many polygons (whose sides may be open or closed). Let the polygons in the partition be $P_1, \dots, P_r$ for some $r \in \mathbb{N}$. By hypothesis, we know that $\bigcup_{n=0}^{\infty} T^n(\partial(\mathcal{X}))$ is invariant under T. Let $j \in \{1, \dots, r\}$. Then, there exists an index set $\mathcal{I} \subseteq \{1, \dots, r\}$ such that

$$T(P_j) = \bigcup_{k \in \mathcal{I}} P_k.$$

For $d \in \mathcal{A}$, we define the cylinder set $[d]$ as

$$[d] := \{z \in \mathcal{X} \mid d_1(z) = d\}.$$

Suppose $P_j \cap [d]$ is not empty. From $\beta\zeta[d] = \beta\zeta\mathcal{X} \cap (\mathcal{X} + d)$, it follows that the boundary of $T([d])$ lies in $\partial(\mathcal{X}) \cup T(\partial(\mathcal{X}))$. Clearly, $T(P_j \cap [d]) \subseteq T(P_j) = \bigcup_{k\in\mathcal{I}} P_k$. Moreover,

$$\begin{aligned} \beta\zeta(P_j \cap [d]) &= \beta\zeta P_j \cap \beta\zeta[d] \\ &= \beta\zeta P_j \cap (\mathcal{X} + d). \end{aligned}$$

Therefore, we can find a subset $\mathcal{I}^*$ of $\mathcal{I}$ such that

$$T(P_j \cap [d]) = \bigcup_{k\in\mathcal{I}^*} P_k.$$

For $j \in \{1, \cdots, r\}$ and $d \in \mathcal{A}$, we call the equations above the set equations of T. of We now define a labeled directed graph G from the set equations of T. The vertex set $V(G)$ coincides with the set $\{P_1, \dots, P_r\}$ of partition polygons. For $j, k \in \{1, \dots, r\}$ and $d \in \mathcal{A}$, there is an edge labeled d from P_j to P_k if P_k is contained in $T(P_j \cap [d])$. It is clear that G is a sofic graph describing $(\mathcal{X}, T)$. □

Remark 2.2.2. The sofic symbolic dynamical system obtained in the latter part of the above proof is irreducible if $(\mathcal{X}, T)$ admits the ACIM equivalent to the Lebesgue measure. By construction, the resulting labeled graph is the minimum left resolving presentation of the irreducible system. Therefore, it is easy to determine whether the symbolic dynamical system is a shift of finite type or not by checking synchronizing words through backward reading of the labels of the paths in the graph (see [32, Theorem 3.4.17]).

An irrational rotation will result to an infinite number of discontinuity line segments. Hence, we will consider an isometry M such that M^n is the identity for some positive integer n. Let ζ be a q-th root of unity with $q > 2$. Let $\xi, \eta_1, \eta_2 \in \mathbb{Q}(\zeta, \beta)$ where η_1 and η_2 are linearly independent over $\mathbb{R}$. We consider the lattice $\mathcal{L} := \eta_1\mathbb{Z} + \eta_2\mathbb{Z}$ and $\mathcal{X} := \{\xi + x\eta_1 + y\eta_2 \mid x, y \in [0, 1)\}$. Let $\kappa : \mathcal{X} \to [0, 1)^2$ be the bijective map given by

$$\kappa(\xi + \eta_1 x + \eta_2 y) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We want to find a transformation U on $[0, 1)^2$ such that the diagram below commutes.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "T"] \arrow[d, "\kappa"'] & \mathcal{X} \arrow[d, "\kappa"] \\ {[0,1)^2} \arrow[r, "U"] & {[0,1)^2} \end{tikzcd}$$

Note that $[\mathbb{Q}(\zeta, \beta) : \mathbb{Q}(\zeta + \zeta^{-1}, \beta)] = 2$ because $\zeta^2 = (\zeta + \zeta^{-1})\zeta - 1$. Consequently, every element of

$\mathbb{Q}(\zeta, \beta)$ is uniquely expressed as a linear combination of η_1 and η_2 over $\mathbb{Q}(\zeta+\zeta^{-1}, \beta)$. In particular, there exist $a_{ij}, b_i \in \mathbb{Q}(\zeta+\zeta^{-1}, \beta)$ for $i, j \in \{1, 2\}$ such that

$$\zeta \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

and

$$(\beta\zeta - 1)\xi = b_1\eta_1 + b_2\eta_2.$$

We give the explicit form of an analog map U in the following lemma.

Lemma 2.2.3. *Let $U : [0,1)^2 \to [0,1)^2$ be the transformation defined as*

$$U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \beta(a_{11}x + a_{12}y) + b_1 - \lfloor \beta(a_{11}x + a_{12}y) + b_1 \rfloor \\ \beta(a_{21}x + a_{22}y) + b_2 - \lfloor \beta(a_{21}x + a_{22}y) + b_2 \rfloor \end{bmatrix}.$$

Then $U \circ \kappa = \kappa \circ T$.

Proof This lemma follows from a straightforward computation. □

Transferring our setting from T to U, we obtain sufficient conditions for the symbolic dynamical system $(\mathcal{X}, T)$ to be sofic. To state this result, we need the following definition.

Definition 2.2.4. A real number $\beta > 1$ is a *Pisot* number if it is an algebraic integer whose conjugates distinct from itself have modulus strictly less than 1. The number β is called a *Salem* number if it is an algebraic integer whose conjugates distinct from itself have modulus less than or equal to 1 and among the non-trivial conjugates, there exists at least one whose modulus is exactly 1.

We prove the succeeding theorem on soficness.

Theorem 2.2.5. *Let ζ be a q-th root of unity ($q > 2$) and $1 < \beta \in \mathbb{R}$ be a Pisot number. Let $\xi, \eta_1, \eta_2 \in \mathbb{Q}(\zeta, \beta)$ such that eta_1 and η_2 are linearly independent over $\mathbb{R}$. If $\zeta + \zeta^{-1} \in \mathbb{Q}(\beta)$, then the system $(\mathcal{X}, T)$ is sofic.*

Proof Let U be the analog map of T given in the previous lemma. We show that the set $\bigcup_{n=0}^{\infty} U^n(\partial([0,1)^2))$ is a finite union of line segments. Consider a discontinuity line segment in $\bigcup_{n=0}^{\infty} U^n(\partial([0,1)^2))$ whose defining equation is given by

$$f(X,Y) = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + C,$$

where $(A, B) \in \mathbb{R}^2 \setminus \{(0,0)\}$. We determine the image of f under U. Define the evidently finite set Δ as

$$\Delta := \left\{ \begin{bmatrix} \lfloor \beta(a_{11}x + a_{12}y) + b_1 \rfloor - b_1 \\ \lfloor \beta(a_{21}x + a_{22}y) + b_2 \rfloor - b_2 \end{bmatrix} \middle| \, 0 \le x, y < 1 \right\}.$$

Put

$$M := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Let (X', Y') be given by

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \beta M \begin{bmatrix} X' \\ Y' \end{bmatrix} - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \Delta.$$

Then, $U(f)$ is given by $f(X', Y') = 0$, i.e.,

$$\frac{1}{\beta} \begin{bmatrix} A & B \end{bmatrix} M^{-1} \begin{bmatrix} X + c_1 \\ Y + c_2 \end{bmatrix} + C = 0.$$

Equivalently,

$$\begin{bmatrix} A & B \end{bmatrix} M^{-1} \begin{bmatrix} X + c_1 \\ Y + c_2 \end{bmatrix} + \beta C = 0.$$

Hence, if

$$U\left(\begin{bmatrix} A^{(n)} & B^{(n)} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + C^{(n)} \right) = \begin{bmatrix} A^{(n+1)} & B^{(n+1)} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + C^{(n+1)},$$

then

$$\begin{bmatrix} A^{(n+1)} & B^{(n+1)} \end{bmatrix} = \begin{bmatrix} A^{(n)} & B^{(n)} \end{bmatrix} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{2.2.1}$$

and

$$C^{(n+1)} = \beta C^{(n)} + \begin{bmatrix} A^{(n)} & B^{(n)} \end{bmatrix} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{2.2.2}$$

with $\left(A^{(0)}, B^{(0)}, C^{(0)}\right) = (A, B, C)$.

Hence, we can define a sequence of coefficients as follows:

$$\left[A^{(n)}, B^{(n)}, C^{(n)}\right] \to \left[A^{(n+1)}, B^{(n+1)}, C^{(n+1)}\right]. \tag{2.2.3}$$

Note that the choice for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in Δ is not unique (thus, (2.2.3) is not one-to-one), but we have a trivial restriction on $C^{(n)}$. Note that the four values in the set

$$\left\{ A^{(n)}s + B^{(n)}t + C^{(n)} \;\middle|\; (s,t) \in \{(0,0),(1,0),(0,1),(1,1)\} \right\}$$

should be not simultaneously positive or negative. This is to ensure that the resulting lines intersect the closure of $\mathcal{X}$. All the same, we note that the tuples $[A^{(n)}, B^{(n)}, C^{(n)}]$ are mere candidates and among them are some tuples defining discontinuity segments. We need to drop the irrelevant tuples when constructing the sofic graph of the symbolic dynamical system associated with U. In practice, such algorithm that discards unnecessary line segments needs to keep track of the endpoints of discontinuity segments, therefore making the computation too involved.

As multiplication by ζ acts as q-fold rotation on $\mathbb{C}$, we have

$$M^q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.2.4}$$

So, the sequence $[A^{(n)}, B^{(n)}]_{n \in \mathbb{N} \cup \{0\}}$ is clearly periodic of period q. To prove the theorem, we thus need to show that there are only finitely many $C^{(n)}$'s.

The set $Int(U)$ of *intercepts* of U is defined to be the collection of all $C^{(n)}$'s arising from $\partial([0,1)^2)$. Note that $Int(U) \subset \mathbb{Q}(\beta)$. We show that the cardinality of $Int(U)$ is finite.

Recall that β is a Pisot number. Let the conjugates of β be $\beta_1 = \beta, \beta_2, \ldots, \beta_d$. Then, for $k = 2, \ldots, d$, we have $|\beta_k| < 1$. For each $k = 1, \ldots, d$, define the conjugate map $\sigma_k : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_k)$ where $\sigma_k(\beta) = \beta_k$. We show that $\sigma_k\left(C^{(n)}\right)$ is bounded for $k = 1, \ldots, d$.

Note that the four sides of the boundary of $\mathcal{X}$ are given by the coefficients

$$[A^{(0)}, B^{(0)}, C^{(0)}] \in \{[1,0,0], [1,0,-1], [0,1,0], [0,1,-1]\}.$$

We define

$$\Gamma := \left\{ \begin{bmatrix} A & B \end{bmatrix} M^{-n} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \,\middle|\, (A,B) \in \{(0,1),(1,0)\}, n \in \mathbb{N}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \Delta \right\}.$$

Clearly, the cardinality of Γ is finite. From (2.2.1),(2.2.2) and (2.2.4), we have $C^{(n+1)} = \beta C^{(n)} + m$ where $m \in \Gamma$. Note that $\Gamma \subset \mathbb{Q}(\beta)$. Taking a common denominator, there is a fixed $N \in \mathbb{N}$ such that $C^{(n)} \in \frac{1}{N}\mathbb{Z}[\beta]$. Let

$$\omega_k := \max\{1, \max_{m \in \Gamma}\{|\sigma_k(m)|\}\}.$$

For $k = 2, \ldots, d$, we have

$$\left|\sigma_k\left(C^{(n)}\right)\right| \leq |(\beta_k)^n| + \omega_k \sum_{j=0}^{n-1} |\beta_k|^j \leq \frac{\omega_k}{1 - |\beta_k|}.$$

For $k = 1$, since the line $A^{(n)}X + B^{(n)}Y + C^{(n)} = 0$ passes through $[0,1]^2$, it follows that

$$\left|\sigma_1\left(C^{(n)}\right)\right| = \left|C^{(n)}\right| \leq \max_{l=0,1,\ldots q-1}\left(\left|A^{(l)}\right| + \left|B^{(l)}\right|\right).$$

by the periodicity of $A^{(n)}$ and $B^{(n)}$. □

From the above theorem, we obtain the proceeding two corollaries.

Corollary 2.2.6. *If ζ is a 3rd, 4th or 6th root of unity and $\xi, \eta_1, \eta_2 \in \mathbb{Q}(\zeta, \beta)$ such that eta$_1$ and η_2 are linearly independent over $\mathbb{R}$. then the system $(\mathcal{X}, T)$ is sofic for any Pisot number β.*

Proof If $\zeta = \cos 2\pi/q + \sqrt{-1}\sin 2\pi/q$, then $\zeta + \zeta^{-1} = 2\cos(2\pi/q)$. For the given values of q, $\cos(2\pi/q)$ is rational. □

Corollary 2.2.7. *There exists a sofic rotational beta expansion system for any q-th root of unity.*

Proof Fix a q-th root of unity ζ. By the Minkowski Theorem for Convex Bodies, the existence of a Pisot number β such that $\mathbb{Q}(\cos(2\pi/q)) = \mathbb{Q}(\beta)$ is guaranteed (see [36]). $\square$

Example 2.2.8. Let $\beta \approx 4899$ be the root of the polynomial $x^4 - 4899x^3 - 229x^2 + 21x + 1$. It can be checked that β is Pisot and $\cos(2\pi/15) \in \mathbb{Q}(\beta)$.

In the next result, we show that the assumption $\zeta + \zeta^{-1} \in \mathbb{Q}(\beta)$ in Theorem 2.2.5 is necessary.

Theorem 2.2.9. *Let ζ be the q-th root of unity, where $q \in \mathbb{N}$ and $q \neq 1, 2, 3, 4, 6$. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta$. Then there exists a constant $G \in \mathbb{R}$ such that if $\beta > G$ and $\mathbb{Q}(\zeta + \zeta^{-1}) \cap \mathbb{Q}(\beta) = \mathbb{Q}$, the symbolic dynamical system $(\mathcal{X}, T)$ is not sofic.*

Proof From the provided parameters, we make the form of the analog transformation U explicit. Using the relation $\zeta^2 = (\zeta + \zeta^{-1})\zeta - 1$, we compute

$$\zeta \begin{bmatrix} 1 \\ \zeta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \zeta + \zeta^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \end{bmatrix}.$$

Moreover, we have $b_1 = b_2 = 0$ because $\xi = 0$. Thus,

$$U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\beta y - \lfloor -\beta y \rfloor \\ \beta(x + (\zeta + \zeta^{-1})y) - \lfloor \beta(x + (\zeta + \zeta^{-1})y) \rfloor \end{bmatrix}.$$

Let $1 < \beta \in \mathbb{R}$ such that $\mathbb{Q}(\beta) \cap \mathbb{Q}(\zeta + \zeta^{-1}) = \mathbb{Q}$. As $\mathbb{Q}(\zeta + \zeta^{-1})$ is a Galois extension over $\mathbb{Q}$, we see that $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\zeta + \zeta^{-1})$ are linearly independent. Fix a non-trivial conjugate $\gamma \in \mathbb{C}$ of $\zeta + \zeta^{-1}$. We find a Galois conjugate map σ on $\mathbb{Q}(\beta, \zeta + \zeta^{-1})$ fixing $\mathbb{Q}(\beta)$ such that $\sigma(\zeta + \zeta^{-1}) = \gamma$.

For $n \in \mathbb{N}$, define

$$\begin{bmatrix} {c_{11}}^{(n)} & {c_{12}}^{(n)} \\ {c_{21}}^{(n)} & {c_{22}}^{(n)} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}^n = \begin{bmatrix} \zeta + \zeta^{-1} & 1 \\ -1 & 0 \end{bmatrix}^n.$$

From (2.2.1) and (2.2.2), we have

$$\begin{aligned} C^{(n+1)} &= \beta C^{(n)} + \begin{bmatrix} A^{(n+1)} & B^{(n+1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \beta C^{(n)} + \begin{bmatrix} A^{(0)} & B^{(0)} \end{bmatrix} \begin{bmatrix} {c_{11}}^{(n+1)} & {c_{12}}^{(n+1)} \\ {c_{21}}^{(n+1)} & {c_{22}}^{(n+1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \beta C^{(n)} + \left[A^{(0)}{c_{11}}^{(n+1)} + B^{(0)}{c_{21}}^{(n+1)}\right] c_1 + \left(A^{(0)}{c_{12}}^{(n+1)} + B^{(0)}{c_{22}}^{(n+1)}\right) c_2, \end{aligned}$$

for some $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \Delta$.

Let $[A^{(0)}, B^{(0)}, C^{(0)}] = [1, 0, -1]$. We have

$$C^{(n+1)} = \beta C^{(n)} + {c_{11}}^{(n+1)} c_1 + {c_{12}}^{(n+1)} c_2$$

and consequently,

$$\sigma\left(C^{(n+1)}\right) = \beta\sigma\left(C^{(n)}\right) + \sigma\left({c_{11}}^{(n+1)} c_1 + {c_{12}}^{(n+1)} c_2\right).$$

Getting the absolute value of both sides, we obtain

$$\begin{aligned} \left|\sigma\left(C^{(n+1)}\right)\right| &\geq \beta\left|\sigma\left(C^{(n)}\right)\right| - \left|\sigma\left({c_{11}}^{(n+1)}c_1 + {c_{12}}^{(n+1)}c_2\right)\right| \\ &= \beta\left|\sigma\left(C^{(n)}\right)\right| - \left|\sigma\left({c_{11}}^{(n+1)}\right)c_1 + \sigma\left({c_{12}}^{(n+1)}\right)c_2\right|. \end{aligned}$$

Let

$$G_1 := \max_{n\in\mathbb{N}}\max_{\Delta}\left\{\left|\sigma\left({c_{11}}^{(n)}\right)c_1 + \sigma\left({c_{12}}^{(n)}\right)c_2\right|\right\}.$$

Then,

$$\left|\sigma\left(C^{(n+1)}\right)\right| \geq \beta\left|\sigma\left(C^{(n)}\right)\right| - G_1.$$

Now,

$$G_1 \leq \max_{n\in\mathbb{N}}\max_{\Delta}\left\{\left|\sigma\left({c_{11}}^{(n)}\right)\right||c_1| + \left|\sigma\left({c_{12}}^{(n)}\right)\right||c_2|\right\}.$$

By the periodicity of the rotation matrix, we can define the constant

$$G_2 := \max_{n\in\mathbb{N}}\left\{\left|\sigma\left({c_{1j}}^{(n)}\right)\right| \mid j = 1, 2\right\}.$$

It follows that

$$G_1 \leq G_2 \max_{\Delta}\left\{|c_1| + |c_2|\right\}.$$

By the definition of Δ, we have

$$|c_1| \leq |\lfloor -\beta \rfloor| \leq \lfloor \beta \rfloor + 1$$

and

$$|c_2| \leq \left\lfloor \beta(1 + \zeta + \zeta^{-1}) \right\rfloor.$$

Define the constant

$$G_3 = G_3(\beta) := \lfloor \beta \rfloor + 1 + \left\lfloor \beta(1 + \zeta + \zeta^{-1}) \right\rfloor.$$

Finally, let $G_4 := G_2 G_3$. Clearly,

$$\left|\sigma\left(C^{(n+1)}\right)\right| \geq \beta\left|\sigma\left(C^{(n)}\right)\right| - G_4.$$

If

$$\left|\sigma\left(C^{(n)}\right)\right| > \frac{G_4}{\beta - 1},$$

then

$$\beta\left|\sigma\left(C^{(n)}\right)\right| - G_4 > \left|\sigma\left(C^{(n)}\right)\right|.$$

Consequently,

$$\left|\sigma\left(C^{(n+1)}\right)\right| > \left|\sigma\left(C^{(n)}\right)\right|.$$

Hence, $\{\sigma(C^{(n)})\}_{n\in\mathbb{N}}$ diverges. In other words, $\bigcup_{n=0}^{\infty} U^n(\partial([0,1)^2))$ cannot be a finite collection of line segments.

Consider the case $n = 1$. Taking $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \lfloor -\beta \rfloor \\ \lfloor \beta(1+\zeta+\zeta^{-1}) \rfloor \end{bmatrix}$, we have

$$\begin{bmatrix} A^{(1)} & B^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \zeta+\zeta^{-1} & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \zeta+\zeta^{-1} & 1 \end{bmatrix},$$

and

$$\begin{aligned} C^{(1)} &= \beta C^{(0)} + \begin{bmatrix} \zeta+\zeta^{-1} & 1 \end{bmatrix} \begin{bmatrix} \lfloor -\beta \rfloor \\ \lfloor \beta(1+\zeta+\zeta^{-1}) \rfloor \end{bmatrix} \\ &= -\beta + (\zeta+\zeta^{-1})\lfloor -\beta \rfloor + \lfloor \beta(1+\zeta+\zeta^{-1}) \rfloor. \end{aligned}$$

It follows that

$$\sigma(C^{(1)}) = -\beta + \gamma\lfloor -\beta \rfloor + \lfloor \beta(1+\zeta+\zeta^{-1}) \rfloor.$$

We want

$$\left|\sigma\left(C^{(1)}\right)\right| > \frac{G_4}{\beta - 1},$$

i.e.,

$$\beta > \frac{G_4}{\left|\sigma\left(C^{(1)}\right)\right|} + 1.$$

Note that the orders of G_4 and $\left|\sigma\left(C^{(1)}\right)\right|$ are both equal to β, hence the above inequality is achieved when β is large enough. Therefore, for some constant $0 < G \in \mathbb{R}^+$, if $\beta > G$ such that $\mathbb{Q}(\beta) \cap \mathbb{Q}(\zeta+\zeta^{-1}) = \mathbb{Q}$, we have that $(\mathcal{X}, T)$ is not sofic. $\square$

Remark 2.2.10. Note that if β is algebraic over $\mathbb{Q}$ and the degree of β and that of $\zeta+\zeta^{-1}$ are relatively prime, then $\mathbb{Q}(\beta) \cap \mathbb{Q}(\zeta+\zeta^{-1}) = \mathbb{Q}$. Hence, when G in the above theorem is already computed, we can take $\beta \in \mathbb{N}$ larger than G and the result holds since the degree of $\zeta+\zeta^{-1}$ exceeds 1.

We apply the previous theorem to the cases where $\zeta = \exp(2\pi\sqrt{-1}/5)$ and $\zeta = \exp(2\pi\sqrt{-1}/7)$.

Corollary 2.2.11. *Let $\mathcal{X} = \{x + y\zeta \mid x, y \in [0,1)\}$ where $\zeta = \exp(2\pi\sqrt{-1}/5)$. If $\beta > 2.90332$ such that $\mathbb{Q}(\sqrt{5}) \cap \mathbb{Q}(\beta) = \mathbb{Q}$, then $(\mathcal{X}, T)$ is not a sofic system.*

Proof Put $\omega = (1+\sqrt{5})/2$. Clearly, $\sqrt{5} \notin \mathbb{Q}(\beta)$ implies that $\mathbb{Q}(\omega)$ and $\mathbb{Q}(\beta)$ are linearly disjoint and there exists a conjugate map $\sigma \in \mathrm{Gal}(\mathbb{Q}(\beta,\omega)/\mathbb{Q}(\beta))$ with $\sigma(\beta) = \beta$ and $\sigma(\omega) = -1/\omega$. For $n \in \mathbb{N} \cup \{0\}$,

$$\begin{bmatrix} {c_{11}}^{(n)} & {c_{12}}^{(n)} \\ {c_{21}}^{(n)} & {c_{22}}^{(n)} \end{bmatrix} = \begin{bmatrix} 1/\omega & 1 \\ -1 & 0 \end{bmatrix}^n.$$

We can show that $G_2 = \omega$ from the following direct computation:

$$\left(\sigma\left({c_{11}}^{(n)}\right), \sigma\left({c_{12}}^{(n)}\right)\right) = \begin{cases} (1,0) & n \equiv 0 \pmod 5 \\ (-\omega, 1) & n \equiv 1 \pmod 5 \\ (\omega, -\omega) & n \equiv 2 \pmod 5 \\ (-1, \omega) & n \equiv 3 \pmod 5 \\ (0,-1) & n \equiv 4 \pmod 5. \end{cases}$$

Moreover, we can take $G_3 = \lfloor \beta\omega \rfloor + \lceil \beta \rceil$. Therefore, $G_4 = \omega(\lfloor \beta\omega \rfloor + \lceil \beta \rceil)$. Now, we can verify that $[A^{(1)}, B^{(1)}, C^{(1)}] = [\omega - 1, 1, (\omega - 1)\lfloor -\beta \rfloor + \lfloor \beta\omega \rfloor - \beta]$ corresponds to a discontinuity segment. Suppose

$$\beta > \frac{1}{4}\left(13 + 3\sqrt{5} - \sqrt{70 - 2\sqrt{5}}\right) \approx 2.90332.$$

It follows that

$$\begin{aligned} \left|\sigma\left((\omega - 1)\lfloor -\beta \rfloor + \lfloor \beta\omega \rfloor - \beta\right)\right| &= \sigma\left((\omega - 1)\lfloor -\beta \rfloor + \lfloor \beta\omega \rfloor - \beta\right) \\ &= -\omega\lfloor -\beta \rfloor + \lfloor \beta\omega \rfloor - \beta, \end{aligned}$$

and

$$-\omega\lfloor -\beta \rfloor + \lfloor \beta\omega \rfloor - \beta > \frac{\omega\left(\lfloor \beta\omega \rfloor + \lceil \beta \rceil\right)}{\beta - 1}.$$

Then the sequence $\left\{\sigma\left(C^{(n)}\right) \mid n \in \mathbb{N}\right\}$ is unbounded. □

Corollary 2.2.12. *Let* $\zeta = \exp(2\pi\sqrt{-1}/7)$. *Let* $\xi = 0$, $\eta_1 = 1$ *and* $\eta_2 = \zeta$. *If* $\beta > 3.115295$ *such that* $\mathbb{Q}(\cos(2\pi/7)) \cap \mathbb{Q}(\beta) = \mathbb{Q}$, *the symbolic dynamical system* $(\mathcal{X}, T)$ *is not sofic.*

Proof Let $\rho = \zeta + \zeta^{-1}$. The minimal polynomial of ρ is $x^3 + x^2 - 2x - 1$. This polynomial has two negative roots, namely $\alpha_1 \approx -0.445042$ and $\alpha_2 \approx -1.80194$. For the computations below, we use α_1. Let $\sigma \in Gal(\mathbb{Q}(\beta, \rho), \mathbb{Q}(\beta))$ such that $\sigma(\rho) = \alpha_1$. For $n \in \mathbb{N} \cup \{0\}$,

$$\begin{bmatrix} {c_{11}}^{(n)} & {c_{12}}^{(n)} \\ {c_{21}}^{(n)} & {c_{22}}^{(n)} \end{bmatrix} = \begin{bmatrix} \rho & 1 \\ -1 & 0 \end{bmatrix}^n.$$

By the cyclicity of the rotation matrix, we have

$$\left({c_{11}}^{(n)}, {c_{12}}^{(n)}\right) = \begin{cases} (1,0) & n \equiv 0 \pmod 7 \\ (\rho, 1) & n \equiv 1 \pmod 7 \\ (\rho^2 - 1, \rho) & n \equiv 2 \pmod 7 \\ (1 - \rho^2, \rho^2 - 1) & n \equiv 3 \pmod 7 \\ (-\rho, 1 - \rho^2) & n \equiv 4 \pmod 7. \\ (-1, -\rho) & n \equiv 5 \pmod 7 \\ (0, -1) & n \equiv 6 \pmod 7 \end{cases}$$

We have

$$G_2 = \max\{|\sigma(0)| = 0, |\sigma(1)| = 1, |\sigma(\rho)|, |\sigma(\rho^2 - 1)|\} = 1,$$

and

$$G_3 = \lfloor \beta \rfloor + 1 + \lfloor \beta + \beta\rho \rfloor.$$

Hence, $G_4 = G_3$. Solving $\beta > \frac{G_4}{|\sigma(C^{(1)})|} + 1$, we get that $\beta > 3.115295$. $\square$

Remark 2.2.13. In the above proof, if we use the conjugate α_2 instead of α_1, we get a slightly smaller bound for β.

2.3 Results in $\mathbb{R}^3$

The situation in $\mathbb{R}^3$ is similar. The forward images of $\partial(\mathcal{X})$ under the iterates of T partition $\mathcal{X}$ into disjoint polyhedra. Hence, we have the following analogous result.

Theorem 2.3.1. *Let T be a 3-dimensional rotational beta transformation. The symbolic dynamical system $(\mathcal{X}, T)$ is sofic if and only if $\bigcup_{n=0}^{\infty} T^n(\partial(\mathcal{X}))$ is a finite union of polygons.*

Proof We will proceed as in Theorem 2.2.1. Suppose that $(\mathcal{X}, T)$ is sofic. We consider the set $\bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ of discontinuities of T. Note that by the action of T on $\mathcal{X}$, an element of $\bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ is a 2-dimensional polygon in $\mathbb{R}^3$. Moreover, each of the boundary line segment of this discontinuity element must lie on another non-parallel discontinuity element of $\bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$, except when the discontinuity element lies entirely on $\partial(\mathcal{X})$. As before, we discard these exceptional cases. It then follows that $\bigcup_{n=0}^{k} T^n(\partial(\mathcal{X}))$ partitions $\mathcal{X}$ into disjoint polyhedra. Two distinct partition cells give different predecessor sets. Hence, arguing as in Theorem 2.2.1, we prove that $\bigcup_{n=0}^{\infty} T^n(\partial(\mathcal{X}))$ is a finite union of polygons.

Suppose that $\bigcup_{n=0}^{\infty} T^n(\partial(\mathcal{X}))$ is a finite union of polygons. Then the fundamental domain $\mathcal{X}$ is partitioned into finitely many polyhedra. As in the two dimensional case, we can construct a sofic graph describing the dynamics of T by taking the vertex set to be the set of these polyhedral partition cells. $\square$

As in the 2-dimensional case, we give sufficient conditions for the system to be sofic.

Theorem 2.3.2. *Let β be a Pisot number and M be the rotation matrix by an angle $\theta = 2\pi/q$ $(2 < q \in \mathbb{N})$ about a unit vector $u \in \mathbb{Q}(\cos\theta, \sin\theta, \beta)^3$. Let $\mathcal{X} = [0,1)^3$. If $\cos\theta, \sin\theta \in \mathbb{Q}(\beta)$, the system $(\mathcal{X}, T)$ is sofic.*

Proof Let $u = (u_x, u_y, u_z)$ be the axis of the rotation. Let $w = 1 - \cos\theta$. We have

$$M = \begin{bmatrix} \cos\theta + {u_x}^2 w & u_x u_y w - u_z \sin\theta & u_x u_z w + u_y \sin\theta \\ u_y u_x w + u_z \sin\theta & \cos\theta + {u_y}^2 w & u_y u_z w - u_x \sin\theta \\ u_z u_x w - u_y \sin\theta & u_z u_y w + u_x \sin\theta & \cos\theta + {u_z}^2 w \end{bmatrix}.$$

For brevity, we write $M = [m_{ij}]$, $1 < i, j < 3$. Then the transformation T is given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \beta(m_{11}x + m_{12}y + m_{13}z) - \lfloor \beta(m_{11}x + m_{12}y + m_{13}z) \rfloor \\ \beta(m_{21}x + m_{22}y + m_{23}z) - \lfloor \beta(m_{21}x + m_{22}y + m_{23}z) \rfloor \\ \beta(m_{21}x + m_{22}y + m_{23}z) - \lfloor \beta(m_{21}x + m_{22}y + m_{23}z) \rfloor \end{bmatrix}.$$

Let $f(X,Y,Z) = AX + BY + CZ + D$ be a discontinuous polyhedral face in $\bigcup_{n=0}^{\infty} T^n(\partial([0,1)^3))$. As in the two dimensional case, define

$$\Delta := \left\{ \begin{bmatrix} \lfloor \beta(m_{11}x + m_{12}y + m_{13}z) \rfloor \\ \lfloor \beta(m_{21}x + m_{22}y + m_{23}z) \rfloor \\ \lfloor \beta(m_{31}x + m_{32}y + m_{33}z) \rfloor \end{bmatrix} \middle| \; 0 \le x, y, z < 1 \right\}.$$

We see that the image of f under T is composed of line segments of the form

$$\begin{bmatrix} A & B & C \end{bmatrix} M^{-1} \begin{bmatrix} X + c_1 \\ Y + c_2 \\ Z + c_3 \end{bmatrix} + \beta D = 0,$$

for some $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \Delta$. In other words, if

$$T\left(\begin{bmatrix} A^{(n)} & B^{(n)} & C^{(n)} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + D^{(n)} \right) = \begin{bmatrix} A^{(n+1)} & B^{(n+1)} & C^{(n+1)} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + D^{(n+1)},$$

then

$$\begin{bmatrix} A^{(n+1)} & B^{(n+1)} & C^{(n+1)} \end{bmatrix} = \begin{bmatrix} A^{(n)} & B^{(n)} & C^{(n)} \end{bmatrix} M^{-1}$$

and

$$D^{(n+1)} = \beta D^{(n)} + \begin{bmatrix} A^{(n)} & B^{(n)} & C^{(n)} \end{bmatrix} M^{-1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

with $\left(A^{(0)}, B^{(0)}, C^{(0)}\right) = (A, B, C)$.

Again, we can define a sequence of coefficients as follows:

$$\left[A^{(n)}, B^{(n)}, C^{(n)}, D^{(n)}\right] \to \left[A^{(n+1)}, B^{(n+1)}, C^{(n+1)}, D^{(n+1)}\right],$$

where the sequence $\{[A^{(n)}, B^{(n)}, C^{(n)}]\}_{n \in \mathbb{N}}$ is clearly periodic by the cyclicity of M. Moreover the set $\{D^{(n)} \mid n \in \mathbb{N} \cup \{0\}\}$ is contained in $\mathbb{Q}(\beta)$. We can proceed as in the two dimensional case. $\square$

Corollary 2.3.3. *There exists a sofic symbolic dynamical system associated to a three-dimensional rotational beta transformation for $\theta = 2\pi/q$, $(2 < q \in \mathbb{N})$.*

Proof First we write $\mathbb{Q}(\cos\theta, \sin\theta) = \mathbb{Q}(\alpha)$ for some algebraic number $\alpha \in \mathbb{R}$. Then we apply Minkowski's Theorem to find a Pisot number β such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. $\square$

2.4 Examples

In this section, we give some examples of sofic symbolic dynamical systems associated to rotational beta transformations.

Example 2.4.1. Let $1 < \beta \in \mathbb{R}$. Let $\zeta = \sqrt{-1}$, $\xi = -1 - \beta\sqrt{-1}$, $\eta_1 = 1$ and $\eta_2 = \beta\sqrt{-1}$. Consider the rotational beta transformation in $\mathbb{R}^2$ with the above parameters. We have

$$T\left(x + y\sqrt{-1}\right) = -\beta y - \lfloor -\beta y + 1 \rfloor + \beta x\sqrt{-1}.$$

We note that the form of T^2 is related to the 1-dimensional negative beta transformation (see [17]) $g : [-1, 0) \to [-1, 0)$ given by

$$g(x) = -\beta^2 x - \left\lfloor -\beta^2 x + 1 \right\rfloor.$$

Indeed,

$$\begin{aligned} T^2\left(x + y\sqrt{-1}\right) &= -\beta^2 x - \left\lfloor -\beta^2 x + 1 \right\rfloor + \left(-\beta^2 y - \beta \lfloor -\beta y + 1 \rfloor\right)\sqrt{-1} \\ &= g(x) + \beta g(y/\beta)\sqrt{-1}. \end{aligned}$$

From the shape of the map g, we know that if β^2 is a Pisot number, then the system $(\mathcal{X}, T)$ is sofic (cf. Theorem 3.3 in [20]). Therefore, taking β^2 a Pisot number such that β is not gives an example of a sofic system beyond Theorem 2.2.5. For instance, all non-square integers greater than 1 satisfy the above condition. Meanwhile, if we take β to be the Salem number such that $\beta^4 - \beta^3 - \beta^2 - \beta + 1 = 0$, the system is also sofic.

Example 2.4.2. Let T be the rotational beta transformation with parameters $\zeta = \exp(2\pi\sqrt{-1}/3)$, $\beta = 1 + \sqrt{2}$, $\eta_1 = 1$, $\eta_2 = \zeta^2$ and $(\beta\zeta - 1)\xi = 3 - \beta$. The digit set is given by

$$\begin{aligned} \mathcal{A} = \{ & a = -1 - \zeta^2, b = -\zeta^2, c = 1 - \zeta^2, d = 2 - \zeta^2, e = -2 - 2\zeta^2, f = -1 - 2\zeta^2, \\ & g = -2\zeta^2, h = 1 - 2\zeta^2, i = -2 - 3\zeta^2, j = -1 - 3\zeta^2, k = -3\zeta^2 \}. \end{aligned}$$

In Figure 2.1, we see that $\mathcal{X}$ is partitioned by the discontinuity segments into the polygons P_n ($n = 1, \ldots, 12$). The action of T on the polygons is described in Table 2.1 and the sofic graph is depicted in Figure 2.2.

Example 2.4.3. Let T be the rotational beta transformation with parameters $\beta = \frac{1+\sqrt{5}}{2}$, $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/5)$. We consider the analog map U. The digit set $\mathcal{A}$ is

$$\left\{ a = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, c = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, d = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, e = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, f = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}.$$

In Figure 2.3, we see the partition of $[0, 1)^2$ into polygonal cells. The action of U to these cells is given in Table 2.2.

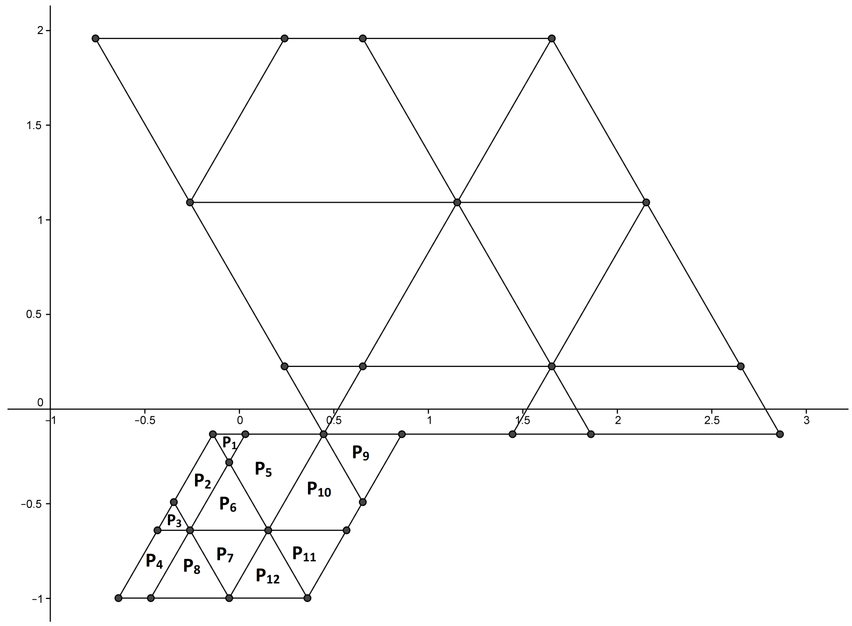


Figure 2.1: $\beta\zeta\mathcal{X}$ and the partition of $\mathcal{X}$

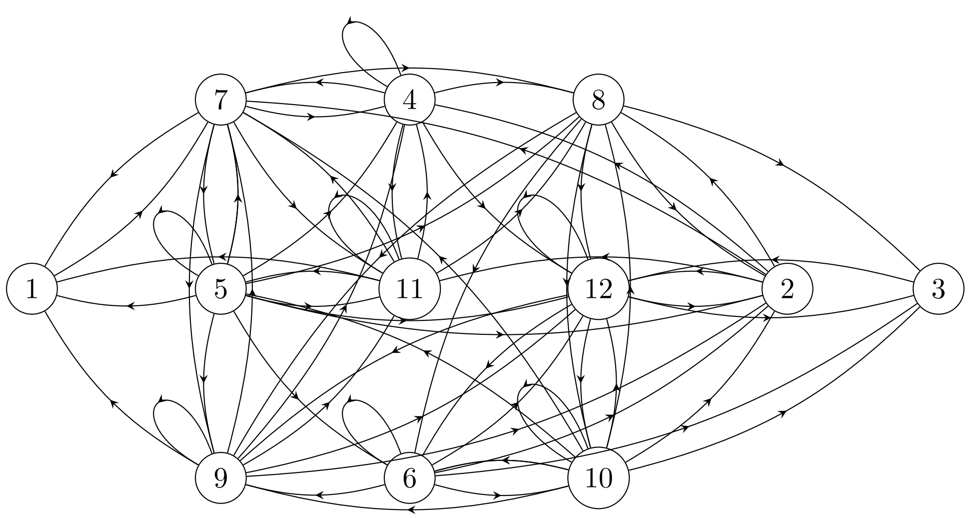


Figure 2.2: Sofic graph of the symbolic dynamical system in Example 2.4.2

P_n	$T(P_n)$	digit
1	7	b
2	11, 12	b
	4, 7, 8	c
3	12	c
4	11	c
	4, 7, 8, 12	d
5	9	a
	1, 2, 5, 6	b
	7, 11, 12	f
	4, 7, 8	g
6	9, 10	b
	2, 3, 6	c
	12	g
7	1, 5	c
	11	g
	4, 7, 8	h

P_n	$T(P_n)$	digit
8	9, 10	c
	2, 3, 6	d
	12	h
9	9	e
	1, 2	f
	7, 11, 12	i
	4	j
10	5, 6, 9, 10	f
	2, 3, 6	g
	7, 8, 12	j
11	1, 5	g
	11	j
	4, 7, 8	k
12	9, 10	g
	2, 3, 6	h
	12	k

Table 2.1: The action of T defined in Example 2.4.2 on the partition

Example 2.4.4. Let $\beta = 1 + 2\cos(2\pi/7) \approx 2.24698$ be the Pisot number satisfying $\beta^3 - 2\beta^2 - \beta + 1 = 0$. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/7)$. The symbolic dynamical system associated with the rotational beta transformation with the above parameters is sofic. The partition of $\mathcal{X}$ into more than 3000 cells by 224 discontinuity segments is depicted in Figure 2.4.

2.5 Application to tiling

Substitution tilings naturally arise from sofic dynamical systems. In this section, we explain the construction of substitution tilings from the sofic graph of a sofic dynamical system and give some examples.

2.5.1 Preliminaries

Let $Type = \{t_1, \ldots, t_n, \ldots\}$ be a set of *types* or *attributes*. A tile P in $\mathbb{R}^m$ is a pair (A, t) where A is a compact subset of $\mathbb{R}^m$ which is the closure of its interior and $t \in Type$. We call A the support of P and write $\mathrm{supp}(P) = A$. Meanwhile, we call t the type of P. We assume that a tile is connected in $\mathbb{R}^m$. Two tiles are distinct if they have different types or their supports are not equal. A finite set of tiles in $\mathbb{R}^m$ forms a patch $\mathcal{P}$ if whenever $P_i, P_j \in \mathcal{P}$ are distinct, then $int(\mathrm{supp}(P_i)) \cap int(\mathrm{supp}(P_j)) = \emptyset$ and the union of the supports of the tiles is a connected subset of $\mathbb{R}^m$. We now define a tiling in $\mathbb{R}^m$.

Definition 2.5.1. A tiling $\mathcal{T}$ in $\mathbb{R}^m$ is a set of tiles in $\mathbb{R}^m$ such that

$$\mathbb{R}^m = \bigcup_{P \in \mathcal{T}} \mathrm{supp}(P)$$

and for $P_i, P_j \in \mathcal{T}$ such that $P_i \neq P_j$, we have $int(\mathrm{supp}(P_i)) \cap int(\mathrm{supp}(P_j)) = \emptyset$.

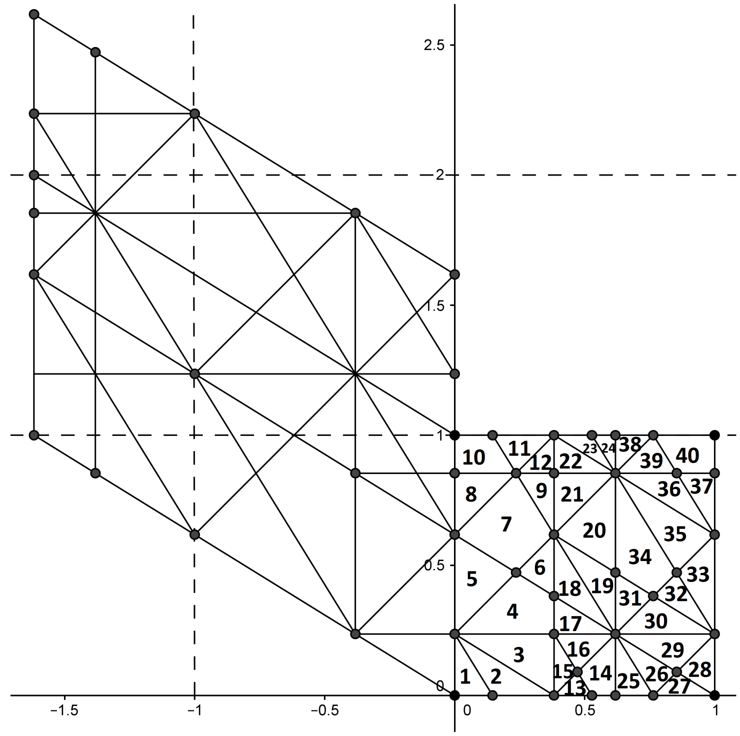


Figure 2.3: Sofic dynamical system with five-fold rotation in Example 2.4.3

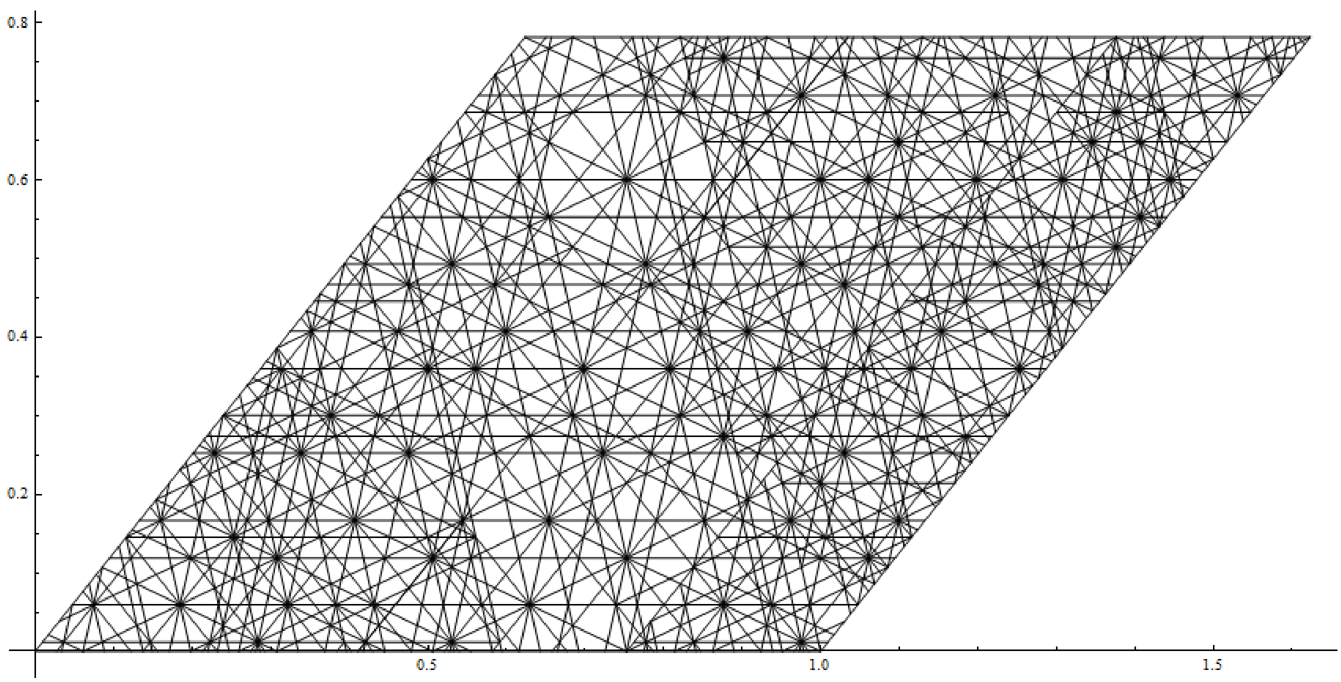


Figure 2.4: Sofic dynamical system with five-fold rotation in Example 2.4.4

P_n	$U(P_n)$	digit
1	$28, 29$	b
2	$30, 32, 33$	b
3	$31, 34, 35$	b
4	$9, 12, 19, 20, 21, 22$	b
5	$6, 7, 18$	b
6	11	b
	2	d
7	$37, 40$	a
	$8, 10$	b
	$26, 27, 28, 29$	c
	1	d
8	$36, 38, 39$	a
	25	c
9	$30, 31$	c
10	$23, 24$	a
	$13, 14, 15, 16$	c
11	$17, 18$	c
12	19	c
13	37	b
14	40	b
	$26, 27$	d
15	36	b
16	$38, 39$	b
	25	d
17	$23, 24$	b
	$13, 14$	d
18	$3, 15, 16$	d
19	$4, 17$	d
20	$33, 35$	c

P_n	$U(P_n)$	digit
20	5	d
21	$32, 34$	c
22	20	c
23	21	c
24	22	c
25	$28, 29$	d
26	$30, 32$	d
27	33	d
28	35	d
29	$31, 34$	d
30	$19, 20$	d
31	$6, 18$	d
32	$9, 21$	d
33	$11, 12, 22$	d
	2	f
34	$36, 37$	c
	$7, 8$	d
35	$39, 40$	c
	10	d
	$27, 28$	e
	1	f
36	38	c
	$25, 26, 29$	e
37	$30, 31$	e
38	23	c
	$13, 15$	e
39	24	c
	$14, 16$	e
40	$17, 18, 19$	e

Table 2.2: The action of U defined in Example 2.4.3 on the partition cells

Two tiles $T_i = (A_i, t_i)$ and $T_j = (A_j, t_j)$ are said to be equivalent up to translations if $t_i = t_j$ and $A_j = A_i + d$ for some vector $d \in \mathbb{R}^m$. This notion of equivalence readily extends to patches. Generally, we will be interested in tilings composed of finitely many inequivalent tiles up to translations. We say that a tiling $\mathcal{T}$ has a *finite local complexity* (FLC) if for any $R \in \mathbb{R} > 0$, the number of patches contained in some ball of radius R is finite up to translations. Meanwhile, we say that $\mathcal{T}$ is *repetitive* if for every compact set K in $\mathbb{R}^m$, the set $\{d \in \mathbb{R}^m \mid \mathcal{T} \cap K = (d + \mathcal{T}) \cap K\}$ is relatively dense.

In what follows, we define a special class of tilings called substitution tilings. Let $\mathbb{P} = \{P_1, P_2, \ldots, P_r\}$ be a finite set of tiles in $\mathbb{R}^m$. We will call the P_i's prototiles. Define $\mathbb{P}^*$ to be the set of all patches $\mathcal{P}$ in $\mathbb{R}^m$ such that if $P \in \mathcal{P}$, then the tile P is translationally equivalent to some element in $\mathbb{P}$. Let $\phi : \mathbb{R}^m \to \mathbb{R}^m$ be an expansive linear map, i.e., ϕ is a linear map whose eigenvalues have modulus greater than 1. A map σ from $\mathbb{P}$ to $\mathbb{P}^*$ is called a tile substitution with respect to an expansive linear map ϕ if for all $i \in \{1, \ldots, r\}$, there are finite sets D_{ij} of vectors in $\mathbb{R}^m$ such that

$$\sigma(P_i) = \{P_j + d \mid d \in D_{ij}, j = 1, \ldots, r\}$$

and

$$\phi(\mathrm{supp}(P_i)) = \bigcup_{j=1}^{r} \bigcup_{d \in D_{ij}} \mathrm{supp}(P_j) + d.$$

We can naturally extend the definition of σ to a patch $\mathcal{P}$ in $\mathbb{P}^*$ as follows:

$$\sigma(\mathcal{P}) = \{\sigma(P) \mid P \in \mathcal{P}\}.$$

Moreover, $\sigma^k(\mathcal{P}) = \sigma^{k-1}(\mathcal{P})$ for any integer $k \geq 2$. Given a tile substitution σ, we define an associated matrix $M \in M_r(\mathbb{Z}_{\geq 0})$ with non-negative integer entries given by $M_{ij} = |D_{ij}|$. We call this matrix the substitution matrix of σ. If M is primitive, i.e., there exists $k \in \mathbb{N}$ such that $M^k > 0$, then we say that σ is primitive. A tiling $\mathcal{T}$ is a substitution tiling if it is a fixed point of some tile substitution σ, i.e., $\sigma(\mathcal{T}) = \mathcal{T}$. If the associated ϕ of a substitution tiling is a similarity, we say that the tiling is self-similar. If the substitution tiling is primitive, repetitive and has FLC, we say that the tiling is self-affine.

2.5.2 Construction of the substitution tiling

Let $T : \mathcal{X} \to \mathcal{X}$ be a two-dimensional rotational beta transformation with parameters $\beta \in \mathbb{R} > 1$ and $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$. Suppose that the symbolic dynamical system $(\mathcal{X}, T)$ associated with T is sofic. Then $\mathcal{X}$ is partitioned into polygonal cells $P_1, \ldots, P_r$. Let $\mathcal{I} = \{1, \ldots, r\}$. For $i, j \in \mathcal{I}$, define

$$D_{ij} := \{d \in \mathcal{A} \mid P_i + d \subseteq \beta\zeta P_j\}.$$

Then,

$$\beta\zeta P_j = \bigcup_{i \in I} \bigcup_{d \in D_{ij}} (P_i + d).$$

We call the previous line the set equations for P_j. The set equations for P_i $(i = 1, \ldots, r)$ give rise to substitution tilings. Let the prototile set be $\mathbb{P} = \{P_1, \ldots, P_r\}$, the set of partition cells. Let the expansive linear map be $\phi : \mathbb{R}^m \to \mathbb{R}^m$ given by $\phi(x) = \beta x$. Define the substitution rule σ by the set equations for the prototiles.

Suppose that $(\mathcal{X}, T)$ is sofic. If, in addition, the rotational beta transformation T has the property that for all $z \in \mathcal{X}$, the set $\bigcup_{n=0}^{\infty} T^{-n}(z)$ is dense in $\mathcal{X}$, then the adjacency matrix of its sofic graph is primitive. Hence, a substitution tiling arising from T is also primitive.

2.5.3 Examples

Example 2.5.2. Let $\beta = (1 + \sqrt{5})/2$. Let $\zeta = \sqrt{-1}$. Take $\mathcal{X} = \{x + y\sqrt{-1} \mid x, y \in [0, 1)\}$. The associated symbolic dynamical system is sofic. The fundamental domain $\mathcal{X}$ is partitioned into the cells $P_1 = \{x + y\sqrt{-1} \mid 0 < x < 1/\beta^2, 0 < y < 1/\beta\}$, $P_2 = \{x + y\sqrt{-1} \mid 0 < x < 1/\beta^2, 1/\beta < y < 1\}$, $P_3 = \{x + y\sqrt{-1} \mid 1/\beta^2 < x < 1, 0 < y < 1/\beta\}$, and $P_4 = \{x + y\sqrt{-1} \mid 1/\beta^2 < x < 1, 1/\beta < y < 1\}$. The substitution rule σ is given by the following set equations:

$$\begin{aligned}
\beta\zeta P_1 &= \{P_1 \cup P_3\} + (-1,0) \\
\beta\zeta P_2 &= P_3 + (-2,0) \\
\beta\zeta P_3 &= \{(P_2 \cup P_4) + (-1,0)\} \cup \{(P_1 \cup P_3) + (-1,1)\} \\
\beta\zeta P_4 &= \{P_4 + (-2,0)\} \cup \{P_3 + (-2,1)\}.
\end{aligned}$$

The substitution matrix is primitive. We consider the tiling in Figure 2.5 given by the limit of P_3 under σ. The tiling has local finite complexity. Indeed, we introduce pseudo-vertices on the prototiles P_1, P_3 and P_4. On each side of these prototiles, mark two points which are $1/\beta^3$ distance away from the endpoints as pseudo-vertices. Then the tiling becomes edge-to-edge. It follows that when the size of the patch is bounded, there are just finitely many ways the tiles can glue themselves together because of the edge-to-edge condition on the tiles. Thus, for any given $R > 0$, there exist only finitely many inequivalent patches up to translations contained in a ball of radius R. In other words, the tiling has FLC. Notice in Figure 2.5 that a copy of $\sigma^k(P_3)$ rotated by $\pi/2$ clockwise appears in $\sigma^{k+1}(P_3)$.

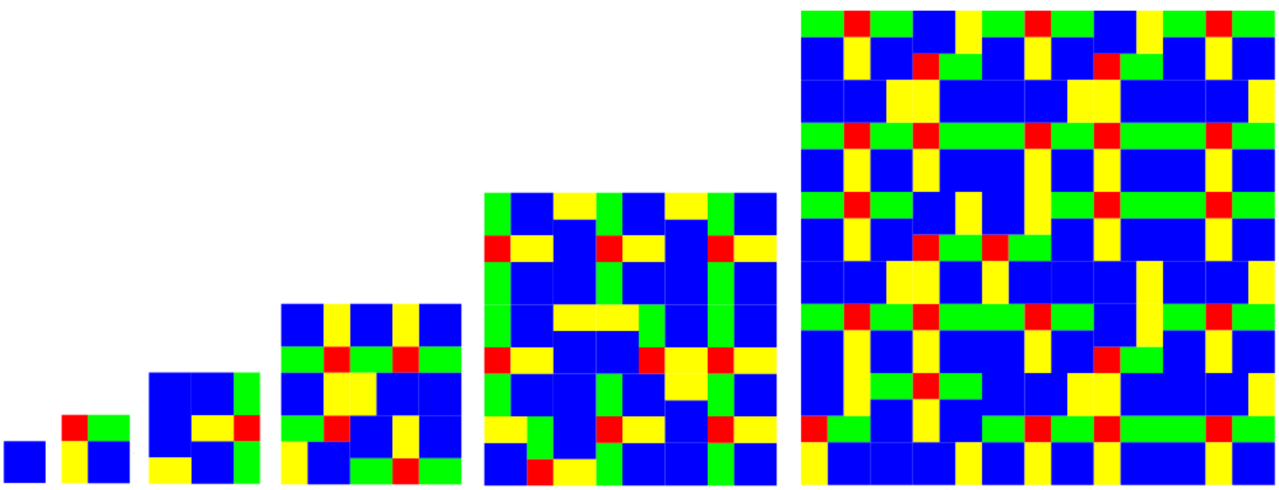


Figure 2.5: Tiling from 4-fold rotational beta transformation. The patches $\sigma^k(P_3)$ $(k = 0,1,2,3,4,5)$ are shown here where the translates of the prototiles are given this color assignment: P_1 - green, P_2 - red, P_3 - blue, P_4 - yellow

Example 2.5.3. This example is based on Example 2.4.3 where $\beta = (1+\sqrt{5})/2$ and $\zeta = \exp(2\pi\sqrt{-1}/5)$. The partition of $\mathcal{X}$ in colored rendition is given in Figure 2.6. The tile substitution σ is given by Table 2.2. Any substitution tiling arising from σ is primitive since σ is primitive. A particular substitution tiling is depicted in Figure 2.7.

2.6 Periodic expansions

A very well-known fact about decimal expansions is that the expansion of a number is eventually periodic if and only if the number is rational. For beta expansion, we have the result given by Schmidt in [42]

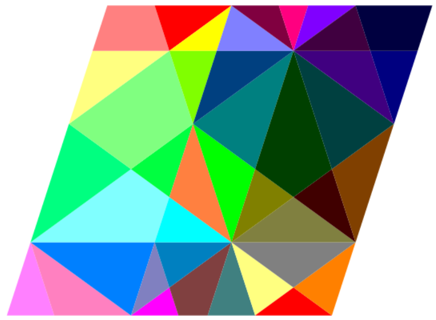


Figure 2.6: Partition of $\mathcal{X}$. The partition cell P_7 (green trapezoid) is given by the vertices $(1/\beta^3, 0) + 2(a, b)/\beta^3$, $(a, b)/\beta$, $(1/\beta^3, 0) + (a, b)(1/\beta + 1/\beta^3)$ and $(1/\beta^2, 0) + (a, b)/\beta$ where $(a, b) = (\cos(2\pi/5), \sin(2\pi/5))$.

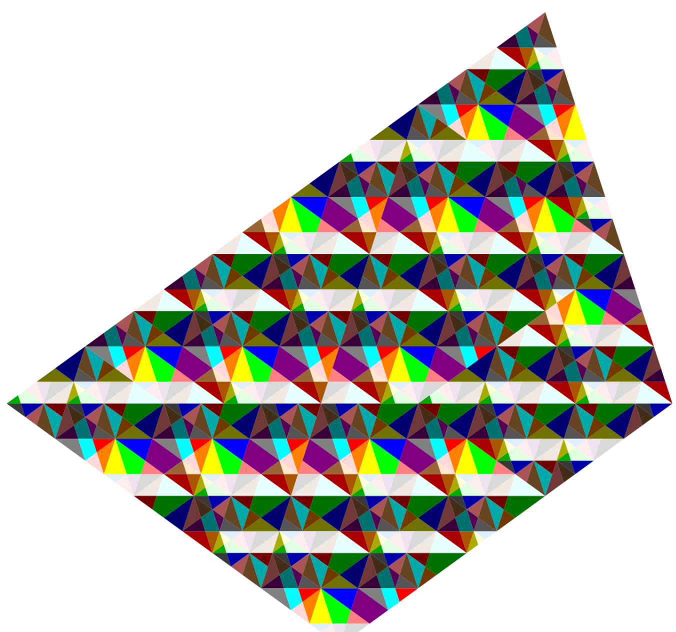


Figure 2.7: Tiling from a 5-fold rotational beta transformation. This is the image of P_7 under σ^6.

that if β is Pisot, then the set $Per(\beta)$ of numbers in $[0,1)$ with eventually periodic expansions coincides with $\mathbb{Q}(\beta)\cap[0,1)$. On the other hand, if all numbers in $\mathbb{Q}$ have an eventually periodic expansion, then β is Pisot or Salem. It is an open problem whether $Per(\beta)=\mathbb{Q}(\beta)\cap[0,1)$ when β is a Salem number.

In this section, we describe the set of eventually periodic expansions for some classes of rotational beta transformations. Let $1<\beta\in\mathbb{R}$ and $\zeta\in\mathbb{C}\setminus\mathbb{R}$ such that ζ has modulus 1. Let $\eta_1,\eta_2\in\mathbb{C}$ such that η_1 and η_2 are linearly independent over $\mathbb{R}$.

Definition 2.6.1. Let T be a rotational beta transformation with parameters β,ζ and $\mathcal{X}$. We define $Per(\beta,\zeta,\mathcal{X})$ to be the set of points $z\in\mathcal{X}$ having the property that $\{T^n(z)\mid n\in\mathbb{N}\cup\{0\}\}$ is finite.

Remark 2.6.2. Note that when $\{T^n(z)\mid n\in\mathbb{N}\cup\{0\}\}$ is finite, then it is also an eventually periodic sequence. Moreover, $\{d_n(z)\mid n\in\mathbb{N}\cup\{0\}\}$ is likewise eventually periodic.

Let $z\in Per(\beta,\zeta,\mathcal{X})$. For some $m\in\mathbb{N}\cup\{0\}$ and $l\in\mathbb{N}$, we can write z as

$$\begin{aligned} z &= \frac{d_1}{\beta\zeta}+\cdots+\frac{d_m}{(\beta\zeta)^m}+\left(\frac{d_{m+1}}{(\beta\zeta)^{m+1}}+\cdots+\frac{d_{m+l}}{(\beta\zeta)^{m+l}}\right)+\left(\frac{d_{m+1}}{(\beta\zeta)^{m+l+1}}+\cdots+\frac{d_{m+l}}{(\beta\zeta)^{m+2l}}\right)+\cdots \\ &= \sum_{i=1}^{m}\frac{d_i}{(\beta\zeta)^i}+\sum_{i=1}^{l}\frac{d_{m+i}}{(\beta\zeta)^{m+i}}\sum_{k=1}^{\infty}\frac{1}{(\beta\zeta)^{kl}}. \end{aligned}$$

Thus, we obtain the following statement.

Proposition 2.6.3. The set $Per(\beta,\zeta,\mathcal{X})$ is a subset of $\mathcal{X}\cap\mathbb{Q}(\beta\zeta,\eta_1,\eta_2)$.

Let $\eta\in\mathbb{C}\setminus\mathbb{R}$ be an algebraic integer over $\mathbb{Q}$. Set $\eta_1=1$ and $\eta_2=\eta$ so that the lattice $\mathcal{L}$ is $\mathbb{Z}+\mathbb{Z}\eta$ with the fundamental domain $\mathcal{X}=\{x+y\eta\mid x,y\in[0,1)\}$. Let ζ be an algebraic integer over $\mathbb{Q}$. We prove the following result.

Proposition 2.6.4. Suppose that $[\mathbb{Q}(\eta):\mathbb{Q}]=2$ and $\zeta\in\mathbb{Q}(\eta)$. If β is Pisot, then $\mathcal{X}\cap\mathbb{Q}(\beta,\eta)\subseteq Per(\beta,\zeta,\mathcal{X})$.

Proof The assumption $[\mathbb{Q}(\eta):\mathbb{Q}]=2$ implies that the integer ring $\mathcal{L}=\mathbb{Z}[\eta]$ is a discrete set in $\mathbb{C}$ under the standard topology. Moreover, the assumption $\zeta\in\mathbb{Q}(\eta)$ implies that $\mathbb{Q}(\beta,\eta)=\mathbb{Q}(\eta)(\beta\zeta)$. Let d be the degree $[\mathbb{Q}(\eta)(\beta\zeta):\mathbb{Q}(\eta)]=[\mathbb{Q}(\eta,\beta):\mathbb{Q}(\eta)]$. Since β and ζ are algebraic integers over $\mathbb{Q}$, $\beta\zeta$ is also an algebraic integer over $\mathbb{Q}$. In addition, $\beta\zeta$ is an algebraic integer over $\mathbb{Q}(\eta)$. So there are coefficients $(a_i)_{i<d}\in\mathbb{Z}[\eta]^d$ such that

$$(\beta\zeta)^d=\sum_{i=0}^{d-1}a_i(\beta\zeta)^i.$$

Fix an element x of $\mathcal{X}\cap\mathbb{Q}(\beta,\eta)=\mathcal{X}\cap\mathbb{Q}(\eta)(\beta\zeta)$. We can write

$$x=\frac{1}{v}\sum_{i=0}^{d-1}b_i(\beta\zeta)^i,$$

where $v\in\mathbb{Z}$ and $b_i\in\mathbb{Z}[\eta]$ for each $i<d$. Taking v with the minimum absolute value, the $(b_i)_i$'s are unique up to multiplication with units. By induction, the following holds.

Claim 2.6.5. *For all $n \in \mathbb{N}$, there exists a unique $(r_{n,1}, \ldots, r_{n,d-1})_i \in \mathbb{Z}[\eta]^d$ such that*

$$T^n(x) = \frac{1}{v}\sum_{i=0}^{d-1} r_{n,i}(\beta\zeta)^i.$$

Let $\beta^{(1)} = \beta, \beta^{(2)}, \ldots . \beta^{(d)}$ be the conjugates of β over $\mathbb{Q}(\eta)$. For each conjugate $\beta^{(j)}$, we consider the conjugate map on $\mathbb{Q}(\eta, \beta)$ fixing $\mathbb{Q}(\eta)$ that sends β to $\beta^{(j)}$. Since $\zeta \in \mathbb{Q}(\eta)$, we have $(\beta\zeta)^{(j)} = \beta^{(j)}\zeta$.

Claim 2.6.6. *There exists $C > 0$ such that for all $n \in \mathbb{N}$ and $j \leq d$, we have $|T^n(x)^{(j)}| < C$.*

Proof of Claim If $x \in \mathcal{X}$, then $|d(x)| < |\beta\zeta x| + |\eta| + 1 < (|\eta| + 1)(|\beta| + 1) = D$. So $|d_n(x)|$ is uniformly bounded by D. We can show by induction that

$$T^n(x) = (\beta\zeta)^n x - \sum_{i=1}^{n-1}(\beta\zeta)^{n-i}d_i(x).$$

Fix $j > 1$. Since β is Pisot, then $|\beta^{(j)}| < 1$. So,

$$\begin{aligned}
|T^n(x)^{(j)}| &= |(\beta^{(j)}\zeta)^n x - \sum_{i=1}^{n-1}(\beta^{(j)}\zeta)^{n-i}d_i(x)| \\
&\leq |\beta^{(j)}|^n|x| + \sum_{i=1}^{n-1}|\beta^{(j)}|^{n-i}|d_i(x)| \\
&\leq D\sum_{i=1}^{n}|\beta^{(j)}|^i = D\frac{|\beta^{(j)}|}{1 - |\beta^{(j)}|}
\end{aligned}$$

In case $j = 1$, since $T^n(x) \in \mathcal{X}$, we obtain $|T^n(x)| \leq |\eta| + 1$. Then

$$C = \max\left\{\left\{D\frac{|\beta^{(j)}|}{1 - |\beta^{(j)}|} : d \geq j > 1\right\} \cup \{|\eta| + 1\}\right\}$$

is required.

By the first claim,

$$\begin{pmatrix} T^n(x)^{(1)} \\ T^n(x)^{(2)} \\ \vdots \\ T^n(x)^{(d)} \end{pmatrix} = \frac{1}{v}\begin{pmatrix} 1 & \beta^{(1)}\zeta & (\beta^{(1)}\zeta)^2 & \cdots & (\beta^{(1)}\zeta)^{d-1} \\ 1 & \beta^{(2)}\zeta & (\beta^{(2)}\zeta)^2 & \cdots & (\beta^{(2)}\zeta)^{d-1} \\ \vdots & & & \ddots & \\ 1 & \beta^{(d)}\zeta & (\beta^{(d)}\zeta)^2 & \cdots & (\beta^{(d)}\zeta)^{d-1} \end{pmatrix}\begin{pmatrix} r_{n,0} \\ r_{n,1} \\ \vdots \\ r_{n,d-1} \end{pmatrix}.$$

By the second claim, the vectors $(T_n(x)^{(1)}, \ldots, T_n(x)^{(d)})$ on the left-hand side of the equation are bounded. Since the $\beta^{(j)}$'s are distinct, we can show by using Vandermonde's determinant that the matrix on the right-hand side is non-singular. So, the vectors $(r_{n,1}, \ldots, r_{n,d-1})$ on the right-hand side are also bounded. Since $\mathbb{Z}[\eta]^d$ is discrete, there are only finitely many such vectors. This means that $\{(r_{n,i})_i \mid n \in \mathbb{N}\}$ is finite and so is $\{T_n(x) \mid n \in \mathbb{N} \cup \{0\}\}$. $\square$

Chapter 3

Invariant Measure

In this chapter, we explore some results on the invariant measures of rotational beta transformations.

3.1 Preliminaries

Let T be a rotational beta transformation on $\mathcal{X}$. Let $\mathcal{B}$ be a σ-algebra on $\mathcal{X}$. Suppose that T preserves a probability measure μ on $\mathcal{X}$, i.e., for any measurable set $A \in \mathcal{B}$, we have that $\mu(T^{-1}(A)) = \mu(A)$, where $T^{-1}(A) := \{z \in \mathcal{X} \mid T(z) \in A\}$. We say that μ is absolutely continuous to the Lebesgue measure λ (or equivalently, μ is an absolutely continuous invariant measure, ACIM, with respect to the Lebesgue measure λ) if $\mu(A) = 0$ whenever $\lambda(A) = 0$, $A \in \mathcal{B}$. If, in addition, $\lambda(A) = 0$ whenever $\mu(A) = 0$, $A \in \mathcal{B}$, then we say that μ is equivalent to the Lebesgue measure λ. The map T is said to be ergodic with respect to μ (or, alternatively, μ is an ergodic measure of T) if for every $A \in \mathcal{B}$ such that $T^{-1}(A) = A$, then either $\mu(A) = 0$ or $\mu(A) = 1$.

A general theory developed in [23, 24, 16, 40, 45, 10, 46] tells us that rotational beta transformations have at least one and at most a finite number of ergodic ACIMs. Moreover, any two ergodic ACIMs have disjoint interiors. In the theorems presented here, we investigate the uniqueness of the invariant measure. In the case the invariant measure is unique, we give sufficient conditions for the measure to be equivalent to the Lebesgue measure.

3.2 Technique I: Coverings

Before we present our results, we prepare some definitions.

Definition 3.2.1. A *strip* in $\mathbb{R}^m$ is the closed region between two parallel hyperplanes of co-dimension one. Its *width* is the distance between the two hyperplanes that define it. The *width* $w(J)$ of a nonempty set $J \subset \mathbb{R}^m$ is defined to be the minimum among the widths of the strips which contain J.

Definition 3.2.2. Let $X \subseteq \mathbb{R}^m$ and let P be a point set in X. We define the *covering radius* $r(P)$ of P relative to X to be

$$r(P) = \sup_{x \in X} \inf_{y \in P} \|x - y\|.$$

We say that P is an $r(P)$-covering of X.

If $w(J)$ or $r(P)$ does not assume a value in $\mathbb{R}$, we assign it the value ∞.

Recall the notation $Ess(\mathcal{X}) = \mathcal{X} \setminus \bigcup_{n=-\infty}^{\infty} T^n(\partial(\mathcal{X}))$. For $z \in \mathcal{X}$ and $n \in \mathbb{N}$, define

$$T^{-n}(z) := \{x \in \mathcal{X} \mid T^n(x) = z\}.$$

Let A be a subset of $\mathbb{R}^m$. For $0 < t \in \mathbb{R}$, define $B_{-t}(A)$ to be the set of points in A which have distance at least t from $\partial(A)$.

We prove the following result.

Theorem 3.2.3. *Suppose that for any $\varepsilon > 0$, there exist $z \in Ess(\mathcal{X})$ and $n \in \mathbb{N}$ such that $\bigcup_{j=0}^{n} T^{-j}(z)$ is an ε-covering of $B_{-\varepsilon}(\mathcal{X})$. Then $(\mathcal{X}, T)$ has a unique absolutely continuous invariant measure.*

Proof Note that the rotational beta transformations fall under the category of piecewise expanding maps considered in [40]. We deduce from the proof of Theorem 5.2 in [40] that the support of each ACIM contains an open ball where the associated Radon-Nikodym density has a positive lower bound. Two such balls arising from distinct ergodic ACIMs must have an empty intersection.

Suppose $(\mathcal{X}, T)$ has μ_1 and μ_2 as distinct ergodic ACIMs. Let h_j be the density corresponding to μ_j for $j = 1, 2$. Note that h_j is the fixed point of the Perron-Frobenius operator with a constant positive Jacobian. As such, for almost all z such that $h_j(z) > 0$, we have $h_j(T(z)) > 0$. For $j \in \{1, 2\}$, take open balls $B(x_j, s)$ satisfying $\operatorname{essinf}_{B(x_j,s)} h_j > 0$. By the hypothesis, we can take $z \in Ess(\mathcal{X})$ and $n_j \in \mathbb{N}$ such that

$$T^{-n_j}(z) \cap B(x_j, s) \neq \emptyset.$$

Let $u_j \in B(x_j, s)$ such that $T^{n_j}(u_j) = z$. Letting $\delta_j > 0$ such that the ball $B(u_j, \delta_j)$ is contained in $B(x_j, s)$, we see that

$$T^{n_j}(B(u_j, \delta_j)) = B(T^{n_j}(u_j), \delta'_j)$$

for some $\delta'_j > 0$. But

$$B(T^{n_j}(u_j), \delta'_j) = B(z, \delta'_j).$$

We conclude that $\operatorname{essinf}_{B(z,\delta'_j)} h_j > 0$, which is a contradiction. Hence, system has a unique ergodic component. $\square$

Theorem 3.2.4. *If the set $\bigcup_{n=1}^{\infty} T^{-n}(z)$ is dense in $\mathcal{X}$ for all $z \in \mathcal{X}$, then $(\mathcal{X}, T)$ has a unique absolutely continuous invariant measure and moreover, this invariant measure is equivalent to the Lebesgue measure.*

Proof The first assertion of the theorem follows readily from Theorem 3.2.3. Hence, we only need to show the equivalence of the unique ACIM to the Lebesgue measure. Let μ be the unique ACIM and h be its density. Denote by λ the Lebesgue measure. For $\varepsilon > 0$, let

$$N_\varepsilon := \{x \in \mathcal{X} \mid \operatorname{essinf}_{B(x,\varepsilon)} h = 0\}$$

and define

$$N := \bigcap_{\varepsilon > 0} N_\varepsilon.$$

From Proposition 5.1 in [40], we know that $\mu(N)=0$. We claim that

$$N \subset \bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X})).$$

Let $z \in \mathcal{X}$ such that $z \notin \bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X}))$. We take a ball $B(x,s) \subset \mathcal{X}$ inside the support of μ satisfying $\operatorname{essinf}_{B(x,s)} h > 0$. By hypothesis, we can find some positive integer n such that $B(x,s)$ intersects $T^{-n}(z)$. In other words, there exists $w \in \mathcal{X}$ such that $T^n(w) = z$ and the ball $B(w,\varepsilon)$ is contained in $B(x,s)$ for small $\varepsilon > 0$. Let $\varepsilon' > 0$ such that

$$\begin{aligned} T^n(B(w,\varepsilon)) &= B(T^n(w),\varepsilon') \\ &= B(z,\varepsilon'). \end{aligned}$$

So $\operatorname{essinf}_{B(z,\varepsilon')} h > 0$, implying that $z \notin N$. This proves that $N \subset \bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X}))$. Consequently, the Lebesgue measure of N is 0. Let $S \subset \mathcal{X}$ be a measurable set such that $\lambda(S) > 0$. Note that $\lambda(S \setminus N) = \lambda(S) - \lambda(N) = \lambda(S) > 0$. We take a Lebesgue density point $z \in S \setminus N$ such that

$$\lim_{t \to 0} \frac{\lambda(B(z,t) \cap (S \setminus N))}{\lambda(B(z,t))} = 1.$$

Because $z \notin N$, there exist constants $c > 0$ and $\varepsilon_0 > 0$ where $\operatorname{essinf}_{B(z,\varepsilon_0)} h > c$. For $0 < \varepsilon \leq \varepsilon_0$, we therefore have

$$\mu(S) = \mu(S \setminus N) = \int_{S\setminus N} h d\lambda > \int_{B(z,\varepsilon)\cap(S\setminus N)} c d\lambda > 0.$$

This shows that the Lebesgue measure λ is absolutely continuous to μ. □

We apply the two preceding theorems on rotational beta transformations in $\mathbb{R}^2$. Let $1 < \beta \in \mathbb{R}$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| = 1$. Fix $\xi, \eta_1, \eta_2 \in \mathbb{C}$ such that η_1 and η_2 are linearly independent over $\mathbb{R}$. We have $\mathcal{X} = \{\xi + x\eta_1 + y\eta_2 \mid x \in [0,1), y \in [0,1)\}$ and $\mathcal{L} = \eta_1\mathbb{Z} + \eta_2\mathbb{Z}$.

Remark 3.2.5. We mention the following obvious relation which we use repeatedly throughout this study. For $z \in \mathcal{X}$ and $n \in \mathbb{N}$,

$$T^{-(n+1)}(z) = \frac{1}{\beta\zeta}[(T^{-n}(z) + \mathcal{L}) \cap \beta\zeta\mathcal{X}].$$

The width $w(\mathcal{X})$ is the minimum height of the parallelogram formed by $\mathcal{X}$. We have

$$w(\mathcal{X}) = \min\{|\eta_1|, |\eta_2|\} \sin\theta(\mathcal{X}),$$

where $\theta(\mathcal{X}) \in (0,\pi)$ is the angle between η_1 and η_2. For $n \in \{1,2\}$, we define

$$B_n(\theta(\mathcal{X})) = \max\left\{\nu_n(\theta(\mathcal{X})), \frac{2r(\mathcal{L})}{w(\mathcal{X})}\right\},$$

where

$$\nu_1(\theta(\mathcal{X})) := \begin{cases} 2 & \text{if } \frac{1}{2} < \tan\frac{\theta(\mathcal{X})}{2} < 2 \\ \frac{1+|\cos\theta(\mathcal{X})|}{2(\sin\theta(\mathcal{X})+|\cos\theta(\mathcal{X})|-1)} & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \nu_2(\theta(\mathcal{X})) &:= 1+\frac{\sqrt{2}}{\sin\theta(\mathcal{X})\sqrt{1+|\cos\theta(\mathcal{X})|}} \\ &= 1+\frac{1}{\sin\theta(\mathcal{X})\max\left\{\sin\frac{\theta(\mathcal{X})}{2},\cos\frac{\theta(\mathcal{X})}{2}\right\}}. \end{aligned}$$

Notice that the definitions of the constants $B_1(\theta(\mathcal{X}))$ and $B_2(\theta(\mathcal{X}))$ are independent of the choice of the translation ξ. The constants are determined only by η_1 and η_2. Furthermore, we can readily check from the graphs (see Figure 3.1) of $\nu_1(\theta(\mathcal{X}))$ and $\nu_2(\theta(\mathcal{X}))$ that $\nu_1(\theta(\mathcal{X})) \le \nu_2(\theta(\mathcal{X}))$.

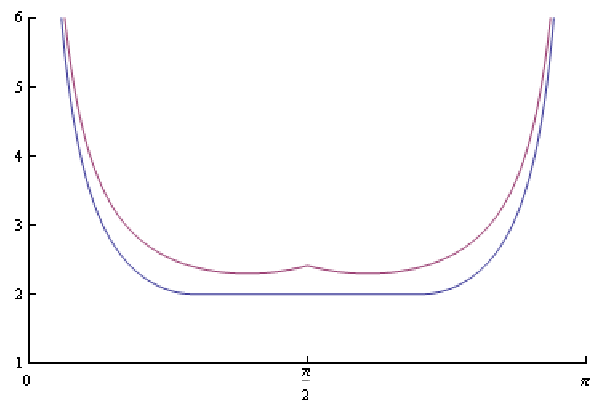


Figure 3.1: $\nu_1(\theta(\mathcal{X}))$ and $\nu_2(\theta(\mathcal{X}))$

Theorem 3.2.6. *If $\beta > B_2(\theta(\mathcal{X}))$, then $(\mathcal{X}, T)$ has a unique absolutely continuous invariant measure. Moreover, this measure is equivalent to the Lebesgue measure.*

Proof We study the backward images under T of points in $\mathcal{X}$. In particular, we look at $T^{-n}(z)$ for $n \in \mathbb{N}$ and $z \in \mathcal{X}$. For brevity, put $r = r(\mathcal{L}), w = w(\mathcal{X}), \theta = \theta(\mathcal{X})$, $\nu_2 = \nu_2(\theta(\mathcal{X}))$ and $B_2 = B_2(\theta(\mathcal{X}))$.

Let $\beta > B_2$. We claim that $\bigcup_{n=1}^{\infty} T^{-n}(z)$ is dense in $\mathcal{X}$ for all $z \in \mathcal{X}$. From Remark 3.2.5,

$$T^{-1}(z) = \frac{1}{\beta\zeta}\left((z+\mathcal{L}) \cap \beta\zeta\mathcal{X}\right).$$

Since $r(z+\mathcal{L}) = r$, we know $z+\mathcal{L}$ is an r-covering of $\mathbb{C}$. Let us consider the region $B_{-r}(\beta\zeta\mathcal{X})$. Since $\beta > B_2 \ge 2r/w$, $B_{-r}(\beta\zeta\mathcal{X})$ is not empty. We note that $B_{-r}(\beta\zeta\mathcal{X})$ does not intersect any ball $B(x,r)$ centered at $x \in \mathbb{C} \setminus \beta\zeta\mathcal{X}$ of radius r. Because of this, the set $(z+\mathcal{L}) \cap \beta\zeta\mathcal{X}$ is not empty. For each $x \in B_{-r}(\beta\zeta\mathcal{X})$, we can find $d \in \mathcal{L}$ where $z+d \in \beta\zeta\mathcal{X}$ and the ball $B(z+d,r)$ contains x. In other words, $z+\mathcal{L}\cap\beta\zeta\mathcal{X}$ is an r-covering of $B_{-r}(\beta\zeta\mathcal{X})$. Consequently, $T^{-1}(z)$ is an r/β-covering of $B_{-r/\beta}(\mathcal{X})$. Considering translates, it follows naturally that $T^{-1}(z)+\mathcal{L}$ is an r/β-covering of $B_{-r/\beta}(\mathcal{X})+\mathcal{L}$ (see Figure

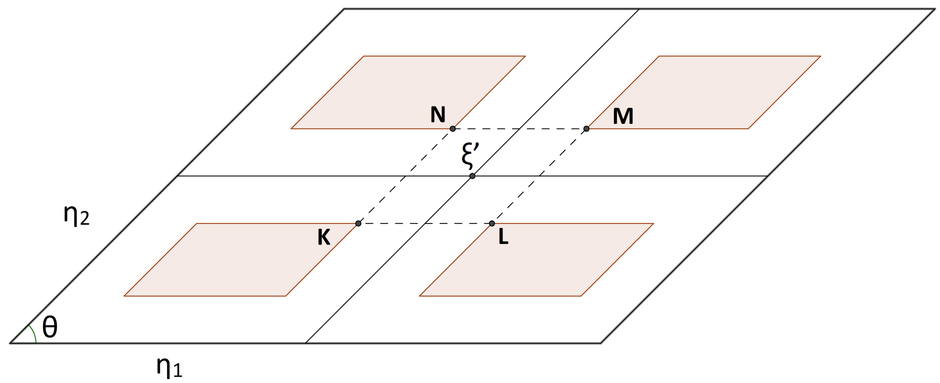


Figure 3.2: $B_{-\frac{r}{\beta}}(\mathcal{X}) + \mathcal{L}$

3.2) . We extend the radius r/β by a factor of ν_2 to cover the complex space in its entirety. Indeed, owing to the inequality

$$\nu_2 > 1 + 1/\sin\theta,$$

it suffices to consider a rhombus KLMN in Figure 3.2 determined by some adjacent translates of $B_{-r/\beta}(\mathcal{X})$ and confirm that it is covered. By the symmetry of ν_2 about $\pi/2$, it is enough to look at the cases where $\theta \in (0, \pi/2]$. Take the Voronoï diagram induced by the four vertices K, L, M and N of the rhombus KLMN. From this diagram, we can readily see that the factor needed is given by the circumradii of the triangles ΔKLN and ΔLMN, which are the acute triangles determined by the smaller diagonal of the rhombus. This gives the constant ν_2. Refer to Figure 3.3 below to compare the Voronoï diagrams arising from the two cases $\theta \in (0, \pi/2]$ and $\theta \in [\pi/2, \pi)$.

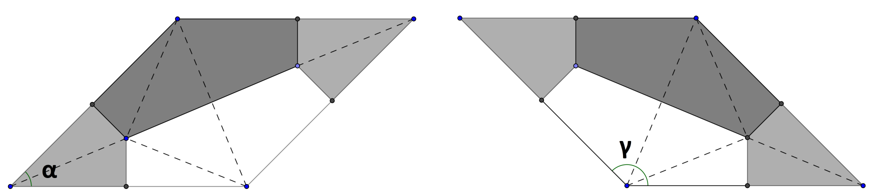


Figure 3.3: Voronoï diagrams where $\alpha \in (0, \pi/2]$ and $\gamma \in (\pi/2, \pi)$

We claim that $T^{-n}(z)$ is an r_n-covering of $B_{-r_n}(\mathcal{X})$ for all $n \in \mathbb{N}$, where

$$r_n = r{\nu_2}^{n-1}/\beta^n.$$

We have exhibited previously that $T^{-1}(z)$ is an r_1-covering of $B_{r_1}(\mathcal{X})$. Suppose now that the claim holds for all $k \le n$ for some $n \in \mathbb{N}$. We note that

$$T^{-(n+1)}(z) = \frac{1}{\beta\zeta}\left(\left(T^{-n}(z) + \mathcal{L}\right) \cap \beta\zeta\mathcal{X}\right).$$

Because $\beta > \nu_2$, we have $r_n < r$. Therefore, $\beta w > 2r_n$. This implies that $B_{-r_n}(\beta\zeta\mathcal{X})$ is not empty. By hypothesis, $T^{-n}(z)+\mathcal{L}$ is an r_n-covering of $B_{-r_n}(\mathcal{X})+\mathcal{L}$. We enlarge the radius r_n by a factor of ν_2 to cover $\mathbb{C}$ entirely. By doing so, we consequently obtain a covering of $\beta\zeta\mathcal{X}$. For all $c\in\mathbb{C}\setminus\beta\zeta\mathcal{X}$, we have

$$B(c,r_n\nu_2)\cap B_{-r_n\nu_2}(\beta\zeta\mathcal{X})=\emptyset.$$

This means that

$$(T^{-n}(z)+\mathcal{L})\cap\beta\zeta\mathcal{X}$$

is an $r_n\nu_2$-covering of $B_{-r_n\nu_2}(\beta\zeta\mathcal{X})$. For this reason, we know that $T^{-(n+1)}(z)$ is an r_{n+1}-covering of $B_{-r_{n+1}}(\mathcal{X})$. This finishes the induction which completes the proof of the claim. Finally, noting that $\lim_{n\to\infty}r_n=0$, we see that $\bigcup_{n=1}^{\infty}T^{-n}(z)$ is dense in $\mathcal{X}$. $\square$

In proving the previous theorem, we look at coverings of $\mathcal{X}$ arising from the sets $T^{-n}(z)$ for $n\in\mathbb{N}$ and $z\in\mathcal{X}$. We can make these coverings finer by considering an appropriate $z\in\mathcal{X}$. Hence, we have the following result.

Theorem 3.2.7. *If $\beta>B_1(\theta(\mathcal{X}))$, then $(\mathcal{X},T)$ has a unique absolutely continuous invariant probability measure.*

Proof Let $z=\xi$. Put $r=r(\mathcal{L})$, $w=w(\mathcal{X})$, $\theta=\theta(\mathcal{X})$, $\nu_1=\nu_1(\theta(\mathcal{X}))$ and $B_1=B_1(\theta(\mathcal{X}))$. Suppose $\beta>B_1$. We proceed as in the proof of the preceding theorem. Notice that we can find a point $\xi'\in\xi+\mathcal{L}$ (see Figure 3.2) inside the parallelogram $KLMN$. Along the perimeter of the translates of $\mathcal{X}$, we can cover some rectangular strips by balls $B(x,2r/\beta)$ centered at $x\in T^{-1}(\xi)+\mathcal{L}$ of radius $2r/\beta$ as illustrated in Figure 3.4. Taking this into account, we need to cover an area composed of four kite-shaped regions around ξ' (see Figure 3.5).

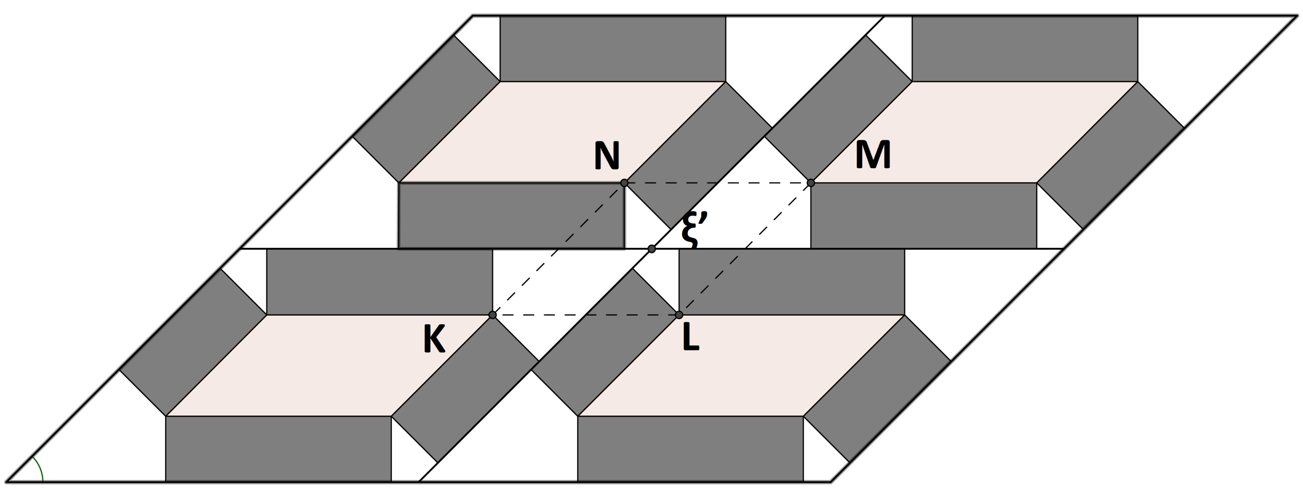


Figure 3.4: The rectangular strips in dark gray

Suppose $1/2<\tan(\theta/2)<2$. The ball $B(\xi',2r/\beta)$ contains the bases of the two perpendiculars emanating from M to the lines ℓ_1 and ℓ_2, where ℓ_j is the line parallel to η_j and passing through ξ' for $j=1,2$. This implies that the balls $B(\xi',2r/\beta)$ and $B(M,r/\beta)$ cover the kite containing M. By the

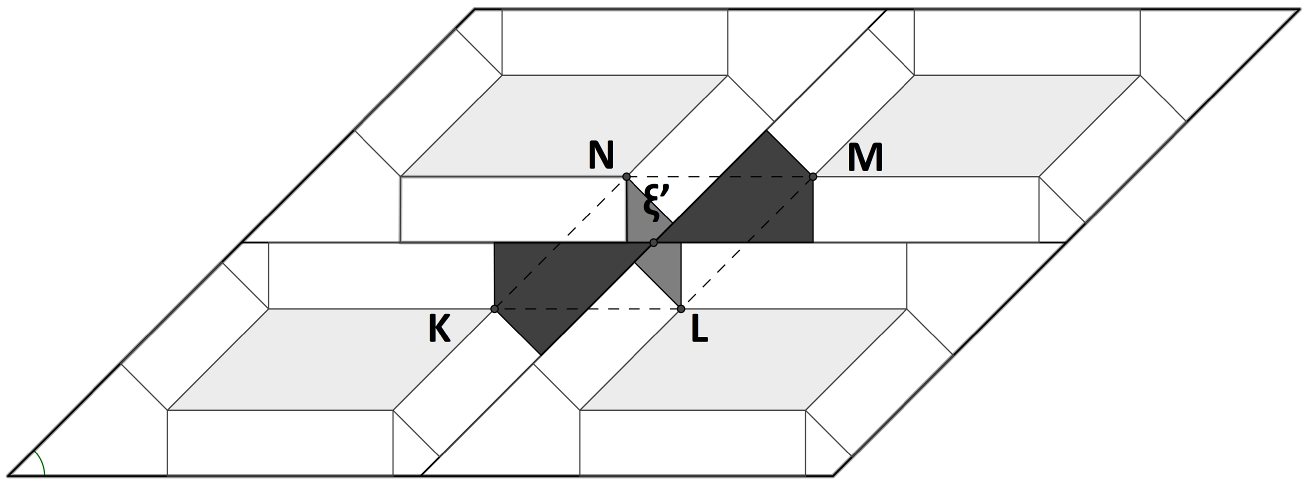


Figure 3.5: The kites

same argument, we can exhibit that remaining kites are similarly covered. Therefore, $(\xi \cup T^{-1}(\xi)) + \mathcal{L}$ is a $2r/\beta$-covering of $\mathbb{C}$.

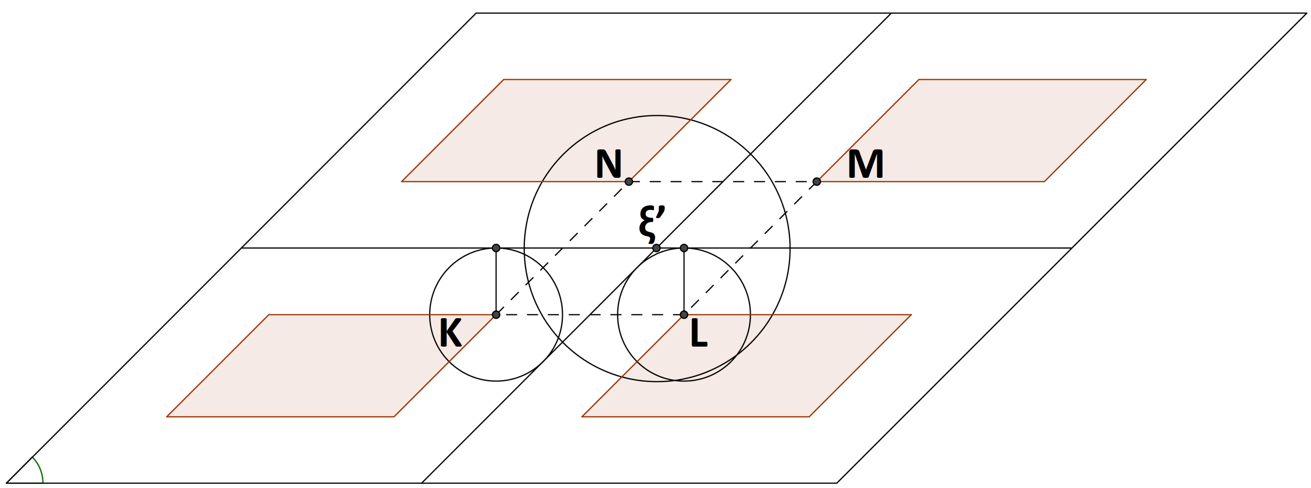


Figure 3.6: The balls $B(\xi', 2r/\beta)$, $B(K, r/\beta)$ and $B(L, r/\beta)$

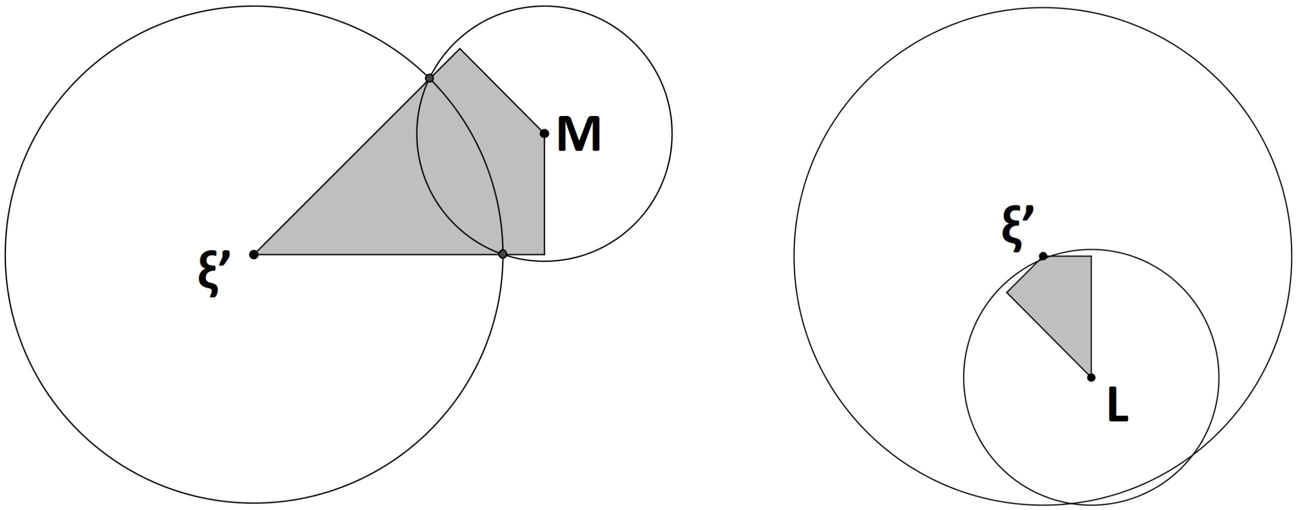
Suppose $\tan(\theta/2) \notin (1/2, 2)$. In this case, such as in Figure 3.6, there exist sections yet to be covered after doubling the radius r/β. Refer to Figure 3.7. We find $\rho > 1$ such that the balls $B(\xi', (\rho+1)r/\beta)$ and $B(M, \rho r/\beta)$ intersect on the boundary of the kite. A straightforward calculation gives the minimal value of ρ to be

$$\frac{1+\cos\theta}{2(-1+\sin\theta+\cos\theta)}.$$

Therefore, we see that $\mathbb{C}$ is be covered by balls centered at the elements of $(\xi \cup T^{-1}(\xi)) + \mathcal{L}$ of radius $\rho r/\beta$.

As in the previous theorem, the set $\bigcup_{j=0}^{n} T^{-j}(\xi)$ is an $r{\nu_1}^{n-1}/\beta^n$-covering of $B_{-r{\nu_1}^{n-1}/\beta^n}(\mathcal{X})$ by

Figure 3.7: Covering the kites



induction.

The choice of $z = \xi$, a point lying on the boundary of $\mathcal{X}$, is not suitable for use in the remainder of the proof. Hence, we introduce a small perturbation to this point to obtain a point in the interior of $\mathcal{X}$ which is very close to ξ. Indeed, there exists $\varepsilon_0 > 0$ such that in the above induction, the base step holds for all $z \in B(\xi, \varepsilon_0)$. Let $z \in Ess(\mathcal{X}) = \mathcal{X} \setminus \bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X}))$ such that for some $2 \leq n \in \mathbb{N}$,

$$z \in B(\xi, \varepsilon_0 {\nu_1}^{n-2}/\beta^{n-1}).$$

Then $\bigcup_{j=0}^{k} T^{-j}(z)$ is an $r{\nu_1}^{k-1}/\beta^k$-covering of $B_{-r{\nu_1}^{k-1}/\beta^k}(\mathcal{X})$ for $k = 1, \ldots, n$. We can choose such z because the set $\bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X}))$ is a null set. To sum up, we can find $z \in Ess(\mathcal{X})$ and $n \in \mathbb{N}$ such that $\bigcup_{j=0}^{n} T^{-j}(z)$ is an ε-covering of $B_{-\varepsilon}(\mathcal{X})$ for $\varepsilon > 0$. $\square$

Modifying the technique of using coverings employed in the previous theorems, we get some results on a particular class of rotational beta transformations (which was introduced by K. Scheicher and P. Surer via personal communication). Let $\beta \in \mathbb{R} > 1$ and let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ such that $|\zeta| = \beta$. Let $\mathcal{X}$ be the set $\{x - \overline{\zeta} y \mid x, y \in [0, 1)\}$, where $\overline{\zeta}$ is the complex conjugate of ζ. We define the transformation $T_\zeta : \mathcal{X} \mapsto \mathcal{X}$ as $z \mapsto \zeta z - d$, where $d \in \mathbb{Z}$ is the unique element of the lattice $\mathcal{L} = \mathbb{Z} - \overline{\zeta}\mathbb{Z}$ such that $\zeta z \in \mathcal{X} + d$. Note that

$$\zeta \mathcal{X} \subset \mathcal{X} + \mathbb{Z}.$$

We first give some auxiliary definitions. For $z \in \mathcal{X}$, define

$$R(z) := \sup_{x \in \mathcal{X} + \mathbb{Z}} \inf_{n \in \mathbb{Z}} \{\|x - (z + n)\|\}.$$

We also define

$$R(\mathcal{X}) := \sup_{z \in \mathcal{X}} R(z).$$

We can easily check that

$$R(\mathcal{X}) = \sqrt{1 + 4\beta^2 \sin^2\theta}/2,$$

where $\theta = \theta(\mathcal{X})$. For $\theta \in (0, \pi)$, we define

$$\begin{aligned} C_1 = C_1(\theta) : &= \max\{2, \sqrt{2 + 2\sqrt{1 + 4\csc^2\theta}}/2\} \\ &= \begin{cases} 2 & \text{if } \sin\theta \geq \sqrt{3}/6 \\ \sqrt{2 + 2\sqrt{1 + 4\csc^2\theta}}/2 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$C_2 = C_2(\theta) := \sqrt{2 + \sqrt{4 + \csc^2\theta}}.$$

Remark 3.2.8. Note that $C_2 \geq \sqrt{2 + \sqrt{5}} \approx 2.05817$ with equality when $\theta = \pi/2$. Moreover, $C_2 > C_1 \geq 2$ for all $\theta \in (0, \pi)$. Refer to Figure 3.8 for the comparison of the graphs of C_1 and C_2.

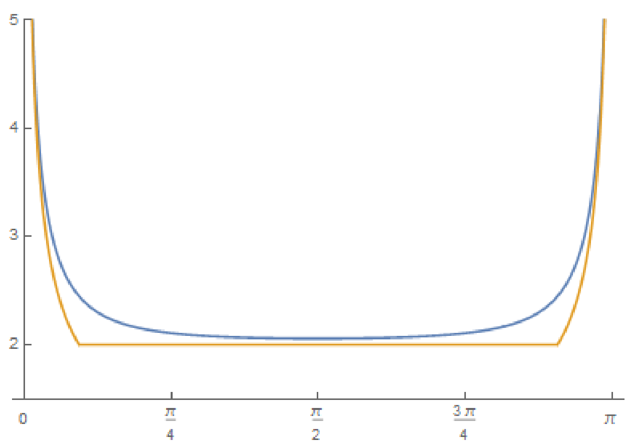


Figure 3.8: $C_1(\theta(\mathcal{X})) > C_2(\theta(\mathcal{X}))$

Then we have the following result.

Theorem 3.2.9. *If $\beta > C_1(\theta(\mathcal{X}))$, then T_ζ admits a unique ACIM. If, moreover, $\beta > C_2(\theta(\mathcal{X}))$, then the unique ACIM is equivalent to the 2-dimensional Lebesgue measure.*

Proof First we show that if $\beta > C_2$, then T_ζ admits a unique ACIM and this measure is equivalent to the 2-dimensional Lebesgue measure. Let $\beta > C_2$. It suffices to show that the set $\bigcup_{n=0}^{\infty} {T_\zeta}^{-n}(z)$ is dense in $\mathcal{X}$ for all $z \in \mathcal{X}$. For brevity, let $r = R(\mathcal{X})$ and set $r_n = 2^{n-1}r/\beta^n$ for $n \in \mathbb{N}$. Define

$$\mathbb{B}(\zeta\mathcal{X}, r/\sin\theta) := [\zeta\mathcal{X} + r/\sin\theta] \cap [\zeta\mathcal{X} - r/\sin\theta].$$

Observe that when $\beta^2 > 2r/\sin\theta$, the set $\mathbb{B}(\zeta\mathcal{X}, r/\sin\theta)$ is nonempty. Note that the constant C_2 is derived from the inequality $\beta^2 > 2r/\sin\theta$. Therefore, assuming $\beta > C_2$, we have $\mathbb{B}(\zeta\mathcal{X}, r/\sin\theta) \neq \emptyset$. Let

$z \in \mathcal{X}$. For all $n \in \mathbb{N}$,

$$\begin{aligned}\zeta T_\zeta^{-n}(z) &= (T_\zeta^{-(n-1)}(z) + \mathcal{L}) \cap \zeta\mathcal{X} \\ &= (T_\zeta^{-(n-1)}(z) + \mathbb{Z}) \cap \zeta\mathcal{X}.\end{aligned}$$

It is clear that $z+\mathbb{Z}$ is an r-covering of $\mathcal{X}+\mathbb{Z}$. We claim that $(z+\mathbb{Z})\cap\zeta\mathcal{X}$ is not empty. Indeed, for any $z' \in (\mathcal{X}+\mathbb{Z}) \setminus \zeta\mathcal{X}$, the ball centered at z' of radius r does not intersect $\mathbb{B}(\zeta\mathcal{X}, r/\sin\theta)$. For this reason, we conclude that $(z+\mathbb{Z})\cap\zeta\mathcal{X}$ is an r-covering of $\mathbb{B}(\zeta\mathcal{X}, r/\sin\theta)$. Consequently, $T_\zeta^{-1}(z)$ is an r_1-covering of the similarly defined set

$$\mathbb{B}(\mathcal{X}, -\overline{\zeta}r_1/(\beta\sin\theta)) := [\mathcal{X} + r_1(-\overline{\zeta})/(\beta\sin\theta)] \cap [\mathcal{X} - r_1(-\overline{\zeta})/(\beta\sin\theta)].$$

Now, $T_\zeta^{-1}(z) + \mathbb{Z}$ covers $\mathbb{B}(\mathcal{X}, -\overline{\zeta}r_1/(\beta\sin\theta)) + \mathbb{Z}$ by balls of radius r_1. Extending the radius to $2r_1$, we see that $T_\zeta^{-1}(z)+\mathbb{Z}$ covers $\mathcal{X}+\mathbb{Z}$ fully. Therefore, $(T_\zeta^{-1}(z)+\mathbb{Z})\cap\zeta\mathcal{X}$ is an $2r_1$-covering of $\mathbb{B}(\zeta\mathcal{X}, 2r_1/\sin\theta)$. Consequently, $T_\zeta^{-2}(z)$ is an r_2-covering of $\mathbb{B}(\mathcal{X}, -\overline{\zeta}r_2/(2\beta\sin\theta))$. By induction, we have for all $2 \le n \in \mathbb{N}$ that $T_\zeta^{-n}(z)$ is an r_n-covering of $\mathbb{B}(\mathcal{X}, -\overline{\zeta}r_n/(2\beta\sin\theta))$. Finally, $\lim_{n\to\infty} r_n = 0$ because $C_2 > 2$. This proves the latter part of the theorem.

For the remaining statement of the theorem, we let $z \in \mathcal{X}$ be the intersection point of the two diagonals of $\mathcal{X}$. Then

$$R(z) = \sqrt{1+\beta^2\sin^2\theta}/2.$$

Let $r = R(z)$. As in the discussion above, we want $\beta^2 > 2r/\sin\theta$ so that the set $\mathbb{B}(\zeta\mathcal{X}, r/\sin\theta)$ is nonempty. Solving $\beta^2 > 2r/\sin\theta$, we obtain the constant C_1. We now assume $\beta > C_1$ and proceed as in the proof above to yield the desired conclusion. □

3.3 Technique II: Holes

So far, we have discussed results on invariant measures for 2-dimensional rotational beta transformations. From here on, we provide results in the general m-dimensional setting. We first give a definition.

Definition 3.3.1. We say that a rotational beta transformation T has property (S) if for all $z \in \mathcal{X}$, there exists $n \in \mathbb{N}\cup\{0\}$ such that

$$2r(T^{-n}(z)+\mathcal{L}) \le \beta w(\mathcal{X}).$$

Remark 3.3.2. There are many rotational beta transformations that satisfy the property (S) or the stronger condition that there exists $n \in \mathbb{N}$ such that for any $z \in \mathcal{X}$, $2r(T^{-n}z+\mathcal{L}) \le \beta\omega(\mathcal{X})$. If $n = 0$, the above inequality simplifies to

$$2r(\mathcal{L}) = 2r(z+\mathcal{L}) \le \beta\omega(\mathcal{X}).$$

On the other hand, there are rotational beta transformations which do not possess the property (S). For example, non-surjective rotational beta transformations do not have the property (S).

Proposition 3.3.3. Suppose T satisfies the property (S). Then the set $\mathcal{X}\setminus T^{-n}(z)$ does not contain any open ball of radius $\omega(\mathcal{X})/2$ for all $z \in \mathcal{X}$ and $n \in \mathbb{N}$.

Proof Suppose the contrary. Let

$$V := \{y + d \ \mid y \in T^{-(n-1)}(z), d \in \mathcal{L}\}.$$

We have

$$\beta M T^{-n}(z) = \beta M \mathcal{X} \cap V.$$

If $\beta M \mathcal{X} \setminus V$ contains a ball of radius $\beta\omega(\mathcal{X})/2$, necessarily its center is at least $\beta w(\mathcal{X})/2$ away from any point in V, which is a contradiction. □

In the following, we introduce the notion of *holes* inside the fundamental domain.

Definition 3.3.4. Let $n \in \mathbb{N}$. A *hole of n-th level* is a ball $B(x, r)$ such that

$$B(x, r) \subset \mathcal{X} \setminus \bigcup_{i=1}^{n} T^{-i}(z).$$

The boundedness of $\mathcal{X}$ implies that for fixed $n \in \mathbb{N}$, the set

$$\{(x, r) \in \mathcal{X} \times \mathbb{R}_{\geq 0} \mid B(x, r) \text{ is an } n\text{-th level hole}\}$$

is compact.

We have a result which covers the family of transformations T_ζ.

Theorem 3.3.5. *Assume that the property (S) holds and there is an $\eta \in \mathcal{L}$ such that*

$$\beta M \mathcal{X} \subset \bigcup_{j \in \mathbb{Z}} (\mathcal{X} + j\eta).$$

If $\beta > 2$, then $(\mathcal{X}, T)$ has a unique ACIM. Moreover, this measure is equivalent to the m-dimensional Lebesgue measure.

Proof Let r_n be the maximum among the radii of the holes of n-th level. Let $z \in \mathcal{X}$ and $n \in \mathbb{N}$ such that $\mathcal{X} \setminus T^{-n}(z)$ does not contain a ball of radius $w(\mathcal{X})/2$. Note that

$$\beta M T^{-(n+1)}(z) = \beta M \mathcal{X} \cap \bigcup_{d \in \mathcal{L}} (T^{-n}(z) + d).$$

By hypothesis,

$$\beta M \mathcal{X} \subset \bigcup_{j \in \mathbb{Z}} (\mathcal{X} + j\eta).$$

We consider a ball

$$B(x, r) \subset \bigcup_{j \in \mathbb{Z}} (\mathcal{X} \setminus T^{-n}(z)) + j\eta,$$

such that the radius r is maximal. Then $r_{n+1} = r/\beta$. Suppose that $B(x, r)$ intersects $(\mathcal{X}+i\eta) \cap (\mathcal{X}+(i+1)\eta)$ and $(\mathcal{X}+(i+1)\eta) \cap (\mathcal{X}+(i+2)\eta)$ for some $i \in \mathbb{Z}$. Then $B(x, r)$ contains a segment of length greater than

$w(\mathcal{X})$. Consequently, the radius of the hole should be greater than $w(\mathcal{X})/2$, which is a contradiction. So, $B(x,r)$ intersects $\bigcup_{j\in\mathbb{Z}} \partial(\mathcal{X}) + j\eta$.

Next, we claim that under the assumption $r > 2r_n$, the hole $B(x,r) \subset \beta M\mathcal{X}$ must contain an element

$$y \in \bigcup_{i\in\mathbb{Z}} (T^{-n}(z) + i\eta).$$

Suppose the contrary. As a hemisphere of $B(x,r)$ belongs to one side of the hyperplane, we can find a ball in $\mathcal{X}$ of radius at least $r/2$ not containing any element of $T^{-n}(z)$ (see Figure 3.9). But $r/2 > r_n$. So,

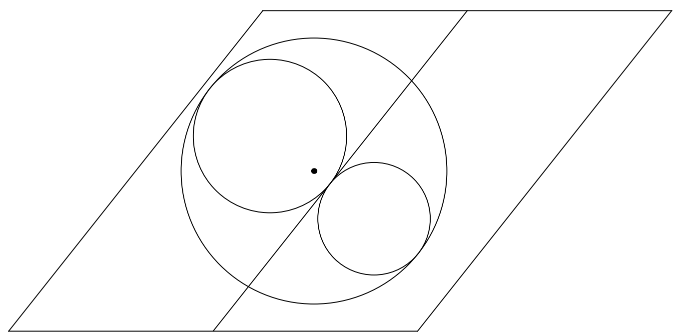


Figure 3.9: The largest hole $B(x,r) \subset \beta M\mathcal{X}$ containing a translate of a point in $T^{-n}(z)$

we have a contradiction. In other words, $r \leq 2r_n$. Finally, we assert that $\lim_{n\to\infty} r_n = 0$. Indeed, we have the inequality $r_{n+1} \leq 2r_n/\beta$ derived from $r_{n+1} = r/\beta$. This ends the proof. $\square$

We return to the family of two-dimensional rotational beta transformations T_ζ we have previously defined. Applying the above theorem, we see that if the expansion ratio β is bigger than some constant dependent on $\theta(\mathcal{X})$, we have a result on the invariant measure of $(\mathcal{X}, T_\zeta)$ as stated in the corollary below. First we define for $\theta = \theta(\mathcal{X}) \in (0,\pi)$,

$$C(\theta) := \sqrt{\frac{1+\sqrt{1+4\sin^4\theta}}{2\sin^2\theta}}.$$

Corollary 3.3.6. *If $\beta > \max\{2, C(\theta)\}$, then $(\mathcal{X}, T_\zeta)$ has a unique ACIM. This ACIM is equivalent to the 2-dimensional Lebesgue measure.*

Proof From the assumption, we have $\beta^2 > 2$. This means that for all $z \in \mathcal{X}$, there are at least two integer translates of z in the region $\zeta\mathcal{X}$ which are aligned with distance 1. Rotating and shrinking back to $\mathcal{X}$, we similarly have at least two points in ${T_\zeta}^{-1}(z)$ aligned with distance $1/\beta$. These two points are lying in a line parallel to $-\overline{\zeta}$. Denote by L the line segment adjoining the midpoints of the longer sides of $\mathcal{X}$. We take $x \in {T_\zeta}^{-1}(z)$ such that it is closest to L. From L to x, we consider the line segment parallel to $-\overline{\zeta}$. This segment is of length at most $1/(2\beta)$. Inside $\zeta\mathcal{X}$, we take a circle whose radius is $\beta w(\mathcal{X})/2 = (\beta\sin\theta)/2$ (see Figure 3.10). We consider the diameter parallel to the longer side of $\zeta\mathcal{X}$. Parallel to this diameter, we look at the chord of the circle extending to x of length $2h$. Observing the right triangle of hypotenuse equal to the radius of the circle, a side of length h and the other side of length at most $\sin\theta/(2\beta)$, we see

$$h \geq \sqrt{(\beta \sin\theta/2)^2 - (\sin\theta/(2\beta))^2}.$$

If $2h > 1$, then the chord contains a translate $x + d$ with $d \in \mathbb{Z}$. Hence, an open ball of radius $w(\mathcal{X})/2$ in $\mathcal{X}$ must contain a point of $T_\zeta{}^{-2}(z)$. Thus if $\beta > \max\{2, C(\theta)\}$ then $\mathcal{X} \setminus T_\zeta{}^{-2}(z)$ cannot contain an open ball of radius $w(\mathcal{X})/2$. The theorem now follows from Theorem 3.3.5. $\square$

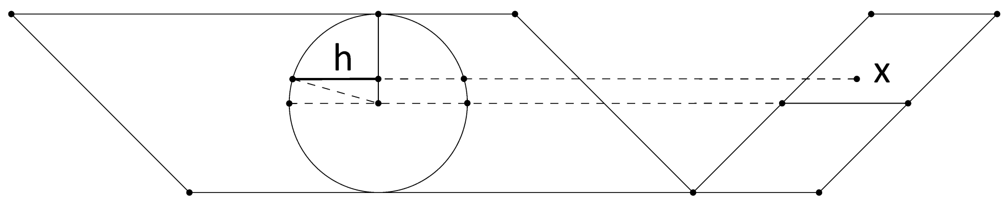


Figure 3.10: T_ζ when $\zeta = \beta\exp(3\pi\sqrt{-1}/4)$

Remark 3.3.7. If $\theta \in [\sin^{-1}(2/\sqrt{15}), \pi - \sin^{-1}(2/\sqrt{15})]$, then $C(\theta) < 2$. Hence, the corollary holds for all $\beta > 2$.

Remark 3.3.8. We compare the results given in Theorem 3.2.9 and Corollary 3.3.6. Referring to Figure 3.11, we can see that $C_2(\theta) < C(\theta)$ for $\theta \in (0, \alpha) \cup (180° - \alpha, 180°)$ where $\alpha \approx 27.28°$.

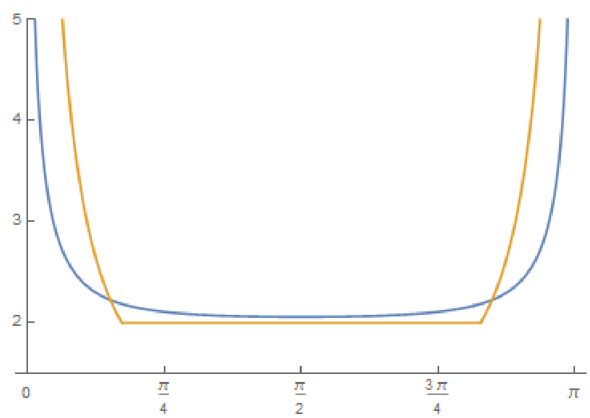


Figure 3.11: $C(\theta(\mathcal{X}))$ and $C_2(\theta(\mathcal{X}))$

The next result relies on the so called Tarski's Plank Problem [44]. The Plank Problem asks whether or not the following statement is true: Given a convex set X covered without gaps by k planks (or rectangular strips) of width w_i ($i = 1, \ldots, k$), then

$$\sum_{i=1}^{k} w_i \geq w(X).$$

Bang settled this problem in [8] using an ingenious technique.

Lemma 3.3.9. *Let $m \in \mathbb{N}$. Consider an m-dimensional sphere intersected by k hyperplanes $H_1, \ldots, H_k$ of co-dimension 1 for some $k \in \mathbb{N}$. Then there exists an open ball W inside the sphere whose radius is*

$1/(k+1)$ and $W \cap H_i = \emptyset$. The maximum radius of such balls attains its minimal when the hyperplanes are parallel and consecutive ones are of distance $2/(k+1)$.

Proof Let B be the unit sphere. Take a concentric ball W of radius

$$k/(k+1) = 1 - 1/(k+1).$$

Fix the hyperplanes $H_1, \ldots, H_k$ intersecting B. For $i \in \{1, \ldots, k\}$ and small $\varepsilon > 0$, define the strip $P_i(\varepsilon)$ whose points are of distance at most $1/(k+1) - \varepsilon$ from H_i. Applying the result of Bang, we have

$$W \setminus \bigcup_{i=1}^{k} P_i(\varepsilon) \neq \emptyset$$

because

$$k(2/(k+1) - 2\varepsilon) < 2k/(k+1).$$

Hence, there is an open ball whose center is a point in $W \setminus \bigcup_{i=1}^{k} P_i(\varepsilon)$ and of radius $1/(k+1) - \varepsilon$ inside a cell generated by B and the hyperplanes. Therefore, the intersection

$$\bigcap_{\varepsilon > 0} W \setminus \bigcup_{i=1}^{k} \mathrm{Int}(P_i(\varepsilon)),$$

is nonempty. This implies that there exists an open ball of radius $1/(k+1)$ entirely contained in a cell generated by B and H_i $(i = 1, \ldots, k)$ as desired.

Let u_i be a vector orthogonal to H_i of length $1/(k+1) - \varepsilon$ for $i = 1, \ldots, k$. In [8], it is proved that

$$\bigcap (W \pm u_1 \pm u_2 \cdots \pm u_k) \neq \emptyset,$$

where the intersection is taken over all combinations of the $+$ and $-$ signs. Notice that the intersection remains empty even if $\varepsilon = 0$. If not, then necessarily the hyperplanes are parallel. Suppose that the hyperplanes H_i $(1 \leq i \leq t < k)$ are parallel and the remaining ones, i.e. H_j $(j > t)$, are not parallel to H_1. Then $\bigcap (W \pm u_1 \pm u_2 \cdots \pm u_t)$ has the shape of a lens

$$L := (W - tu_1) \cap (W + tu_1).$$

The width $w(L)$ of L is $2(k-t)/(k+1)$ which is the distance of the two hyperplanes tangent to the topmost and lowermost peaks of the lens L (see Figure 3.12).

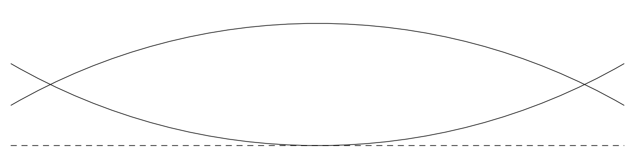


Figure 3.12: The lens L and its minimum strip

We can deduce from [8] that

$$\bigcap (L \pm u_{t+1} \cdots \pm u_k) \neq \emptyset.$$

This intersection contains a translation of κL for some $\kappa > 0$ because for $j > t$, the minimal strip perpendicular to u_j which contain L has width greater than $w(L)$. In addition, the strips $P_i(0)$ ($i = 1, \ldots, k$) do not entirely cover W. In other words, if the hyperplanes are not all parallel, we find a ball of radius strictly larger than $1/(k+1)$ contained in B not touching the hyperplanes. □

The succeeding theorem makes use the above lemma.

Theorem 3.3.10. *Suppose that the rotational beta transformation T has the property (S). If $\beta \geq m+1$, then $(\mathcal{X}, T)$ has a unique ACIM. Moreover, this measure is equivalent to the m-dimensional Lebesgue measure.*

Proof The proof follows essentially that of Theorem 3.3.5. Let $z \in \mathcal{X}$. We show that $\bigcup_{n=1}^{\infty} T^{-n}(z)$ is dense in $\mathcal{X}$. Let r_n denote the maximum among the radii of the holes of n-th level. As before, consider a ball $B(x, r) \subset \beta M \mathcal{X}$ of maximal radius r not intersecting $\bigcup_{d \in \mathcal{L}} (T^{-n}(z) + d)$. By Proposition 3.3.3, this ball cannot intersect two parallel hyperplanes containing some faces of $\partial(\mathcal{X}) + d$ where $d \in \mathcal{L}$. So at most m hyperplanes H_i ($i = 1, \ldots, m$) containing a face of $\partial(\mathcal{X}) + d$ slice $B(x, r)$. By Lemma 3.3.9, there is a ball W of radius $r/(m+1)$ inside $B(x, r)$ not intersecting the hyperplanes H_i. The ball W is clearly in the interior of $\mathcal{X} \setminus T^{-n}(z) + d$ for some $d \in \mathcal{L}$. It now follows that $r_{n+1} = r/\beta \leq (m+1)r_n/\beta$. Thus if $\beta > m+1$, we get the desired conclusion.

We are left to show the theorem for $\beta = m+1$. When $m = 1$, the rotational beta transformation T on $[0, 1)$ takes one of the two forms below:

$$T(x) = 2x + a - \lfloor 2x + a \rfloor \text{ or } T(x) = -2x + a - \lfloor -2x + a \rfloor.$$

In this case, T has the 1-dimensional Lebesgue measure as its unique ACIM. For $m \geq 2$, note that any two hyperplanes of the m hyperplanes are not parallel. By Lemma 3.3.9, any ball $B(x, r) \subset \beta M \mathcal{X}$ contains a maximal hole of radius strictly greater than $r/(m+1)$, proving the theorem for $\beta = m+1$. □

Remark 3.3.11. For $m = 1$, we always have $2r(\mathcal{L}) = \omega(\mathcal{X})$. So, all 1-dimensional rotational beta transformations satisfy property (S). Thus, the assumptions in Theorem 3.3.10 reduce to $\beta \geq 2$. We assert that the inequality $\beta \geq 2$ in Theorem 3.3.10 is sharp when $m = 1$. Indeed, the symmetric beta transformation [6] $U : [-1/2, 1/2) \to [-1/2, 1/2)$ given by

$$U(x) = \beta x - \left\lfloor \beta x + \frac{1}{2} \right\rfloor$$

has no ACIM equivalent to the 1-dimensional Lebesgue measure when $\beta < 2$.

The idea of keeping track of n-th level holes of the fundamental domain $\mathcal{X}$ can be applied to rotational beta transformations in $\mathbb{R}^2$ to improve the lower bounds B_1 and B_2 given in Theorems 3.2.6 and 3.2.7. Let us review the parameters defining a rotational beta transformation T in $\mathbb{R}^2$. Let $\eta_1, \eta_2, \xi \in \mathbb{R}^2$ such that η_1 and η_2 are linearly independent over $\mathbb{R}$. We take the lattice $\mathcal{L} = \eta_1\mathbb{Z} + \eta_2\mathbb{Z}$ in $\mathbb{R}^2$ and its fundamental domain $\mathcal{X} = \{\xi + x_1\eta_1 + x_2\eta_2 \mid x_1, x_2 \in [0,1)\}$. From the orthogonal group $O(2)$ in dimension 2, we fix an isometry M. As before, $\theta(\mathcal{X}) \in (0,\pi)$ denotes the angle between η_1 and η_2. Define for $\theta = \theta(\mathcal{X}) \in (0, \pi/2]$,

$$B_3 = B_3(\theta) := \begin{cases} 2 & \text{if } \frac{1}{2} < \tan(\theta/2) \\ 1 + \frac{2}{1+\sin(\theta/2)} & \text{if } \sin\theta < \sqrt{5} - 2 \\ \frac{3}{2} + \frac{\cot^2(\theta/2)}{16} + \tan^2(\theta/2) & \text{otherwise} \end{cases}$$

and

$$B_4 = B_4(\theta) := \begin{cases} 1 + \frac{1}{\sin(\theta)\cos(\theta/2)} & \text{if } \frac{\pi}{3} < \theta \\ 1 + \frac{2}{1+\sin(\theta/2)} & \text{otherwise.} \end{cases}$$

For $\theta \in [\pi/2, \pi)$, define $B_i(\theta) := B_i(\pi - \theta)$, $(i = 3, 4)$. We can check that $B_3 \le B_4 < 3$ (see Figure 3.13).

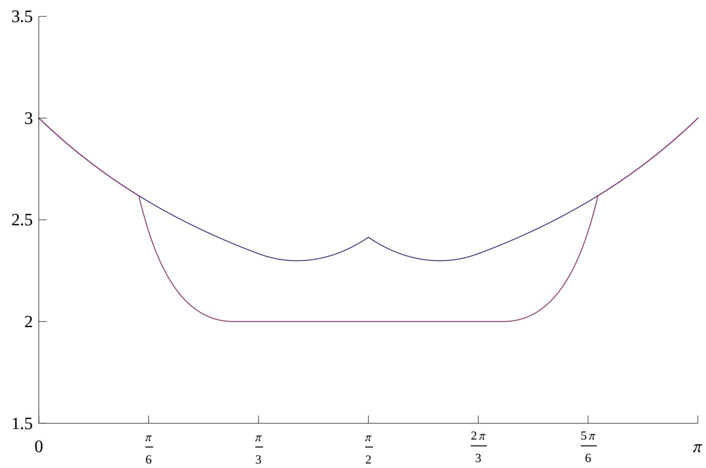


Figure 3.13: Comparison of B_3 and B_4

Theorem 3.3.12. *Suppose that T is a rotational beta transformation satisfying (S). If $\beta > B_3$, then $(\mathcal{X}, T)$ has a unique ACIM. If, in addition, $\beta > B_4$, then the unique ACIM is equivalent to the 2-dimensional Lebesgue measure.*

Proof We only show the case where $\xi = 0$ as the other cases can be derived by a similar proof. Furthermore, we only need to consider the case where $0 < \theta \le \pi/2$ because the graphs of $B_3(\theta)$ and $B_4(\theta)$ are symmetric with respect the line $\theta = \pi/2$. Let us derive the constant $\beta > B_4$. We pattern the proof from those of Theorems 3.3.5 and 3.3.10. We determine the infimum of radius r such that for every ball $B(x,r) \subset \beta M\mathcal{X}$, there exists a $d \in \mathcal{L}$ where $B(x,r) \cap (\mathcal{X} + d)$ contains a ball of radius r_n. Then r_{n+1} can be chosen to be r/β. Equivalently, we determine the supremum of r such that one can find a ball $B(x,r) \subset \beta M\mathcal{X}$ that for all $d \in \mathcal{L}$, $B(x,r) \cap (\mathcal{X} + d)$ does not contain a ball of radius r_n. The supremum is clearly realized by the configuration where for at least one $d \in \mathcal{L}$, a ball of radius r_n is inscribed in $B(x,r) \cap (\mathcal{X} + d)$ and it is touching at least one side of $\mathcal{X} + d$ and the circumference of $B(x,r)$. If only one side of $\mathcal{X} + d$ is passing through $B(x,r)$, then the supremum would be $2r_n$ as in Theorem 3.3.5 or Figure 3.9. Since $B_2 > 4$ for any θ, we will see that this case does not give the supremum. By Proposition 3.3.3, the ball $B(x,r)$ does not intersect two parallel hyperplanes (lines).

Let us consider the case that $B(x, r)$ intersects two non-parallel lines. For example, in Figure 3.14, the ball $B(0, r)$ is touching two maximal balls of radius r_n inside the cells. However, this configuration does not give the supremum, because shifting a little the ball $B(0, r)$ along the bisector of the acute angle, we easily see that every cell no longer contains a ball of radius r_n. Since a planar circle is determined

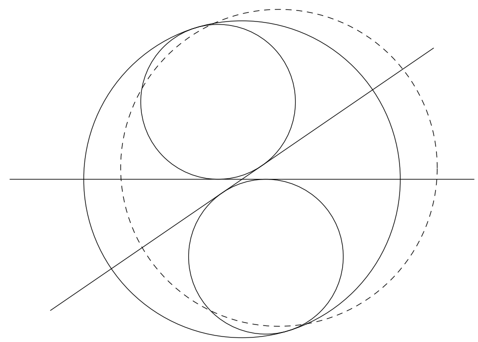
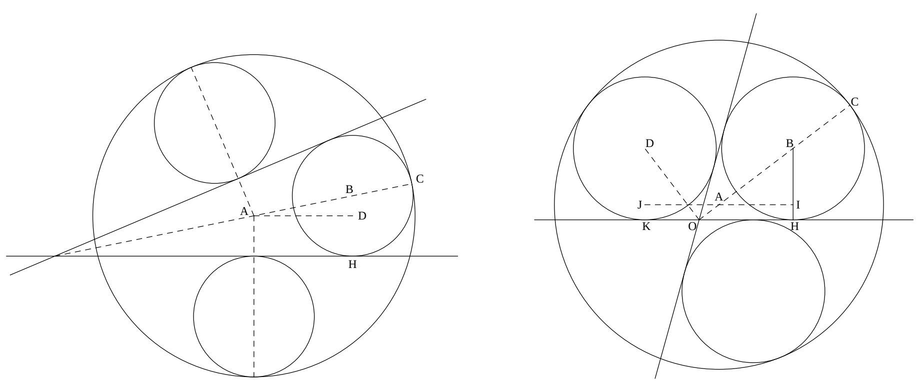


Figure 3.14: A configuration which does not correspond to the supremum case

by three intersections, by similar small perturbation argument, in the supremum configuration $\partial(B(x, r))$ must touch $B(y_i, r_n)$ $(i = 1, 2, 3)$ where $B(y_i, r_n)$ are the balls inscribed in three distinct $\mathcal{X} + d_i$ with $d_i \in \mathcal{L}$. Each $B(y_i, r_n)$ is a maximal ball within a cell in $\partial(\mathcal{X}) + d_i$. There are exactly two possible supremum configurations as depicted in Figure 3.15(a) and 3.15(b).



(a) Small angle case

(b) Large angle case

Figure 3.15: Two possible supremum cases

Figure 3.15(a) is for $\theta(\mathcal{X}) < \pi/3$. Denote by A and B the centers of $B(x, r)$ and one of the $B(y_i, r_n)$'s, respectively. Let the line segment BH be the perpendicular from B to the x-axis and AD be parallel to the x-axis which intersects BH. Since $AC = r$, $BC = BH = r_n$, putting $\ell = DH$, we have

$$r = \ell + 2r_n, \qquad \frac{r_n - \ell}{r_n + \ell} = \sin\left(\frac{\theta}{2}\right)$$

and we obtain

$$\frac{r}{r_n} = 1 + \frac{2}{1 + \sin(\theta/2)}.$$

For $\theta(\mathcal{X}) \geq \pi/3$, the configuration in Figure 3.15(a) is no longer possible as the three inscribed circled are forced to have different radii. Hence, we configuration depicted in Figure 3.15(b). In Figure 3.15(b), A is the center of the ball $B(x, r)$ while B and D are the centers of two of the inscribed circles of radius r_n. The point O is the origin and the horizontal line is taken to be the x-axis. The segments BH and DK are the perpendiculars from B and D to the x-axis. Let JI be the segment passing through A and parallel to x-axis connecting the two perpendiculars. Then we have

$$\begin{aligned} KH &= KO + OH = \frac{r_n}{\cot(\theta/2)} + \frac{r_n}{\tan(\theta/2)} \\ r &= AB + r_n \\ AB\cos(\theta/2) &= AI = AD\cos(\theta/2) = AJ = KH/2. \end{aligned}$$

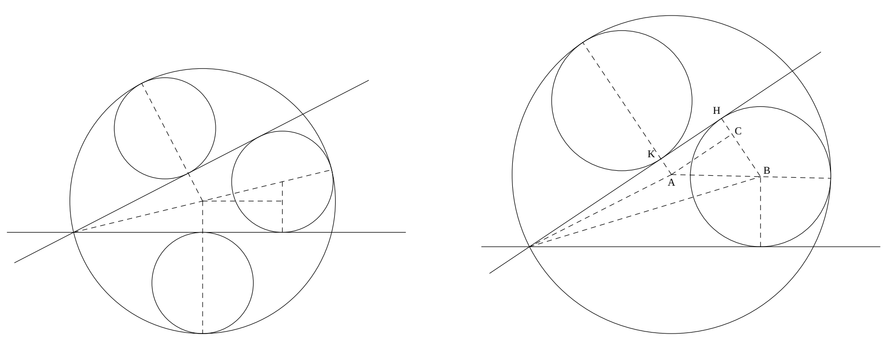
From these, we get

$$\begin{aligned} \frac{r}{r_n} &= 1 + \frac{\cot(\theta/2) + \tan(\theta/2)}{2\cos(\theta/2)} \\ &= 1 + \frac{1}{\sin(\theta)\cos(\theta/2)}. \end{aligned}$$

Next we show that if $\beta > B_3$, then T satisfies the hypothesis of Theorem 3.2.3. It suffices to consider the point $z = 0$ as we have seen previously. So we introduce the restriction that $B(x, r) \cap \mathcal{L} = \emptyset$. That is, $B(x, r)$ does not contain the origin in its interior. In Figure 3.15(a), the origin is outside of $B(x, r)$ and so we get the same value for r/r_n. As we increase $\theta(\mathcal{X})$, the large circle passes through the origin (Figure 3.16(a)) at some angle. This happens when additionally

$$\frac{r_n}{\sin(\theta/2)} = r + \ell + r_n$$

holds, i.e., when $\sin(\theta/2) = \sqrt{5} - 2$. If $\sin(\theta/2) > \sqrt{5} - 2$, the supremum configuration turn into the one in Figure 3.16(b).



(a) First critical configuration (b) Another configuration

Figure 3.16: Supremum configurations

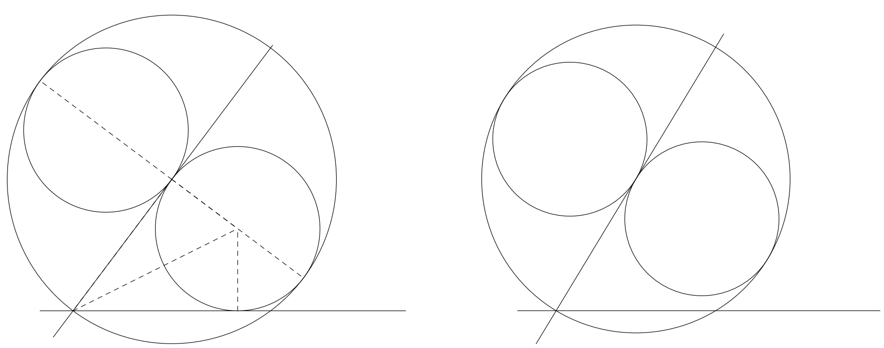
This is the configuration that $\partial(B(x, r))$ passes through the origin and touches $B(y_i, r_n)$ $(i = 1, 2)$ where $B(y_i, r_n)$ are the balls inscribed in two distinct $\mathcal{X} + d_i$ with $d_i \in \mathcal{L}$. Each $B(y_i, r_n)$ is a maximal ball in a cell in $\partial(\mathcal{X}) + d_i$. Let A denote the center of $B(x, r)$, B the center of one of the $B(y_i, r_n)$'s and O be the origin. The segments AK and BH are perpendiculars from A and B, respectively, to the line of slope $\tan\theta$. The segment AC is parallel to the line and intersects BH. Putting $AK = CH = \ell$, we have

$$\begin{aligned} r &= \ell + 2r_n \\ OK^2 &= r^2 - \ell^2 \\ AB &= r_n + \ell \\ BC &= r_n - \ell \\ AC &= KH = \sqrt{4r_n\ell} \\ OH &= OK + KH = \frac{r_n}{\tan(\theta/2)}. \end{aligned}$$

Thus, we have

$$\frac{r}{r_n} = \frac{3}{2} + \frac{1}{16}\cot^2\left(\frac{\theta}{2}\right) + \tan^2\left(\frac{\theta}{2}\right).$$

This configuration works provided $OK \leq OH$. The next critical configuration is depicted in Figure 3.17(a).



(a) Second critical configuration (b) Larger angle

Figure 3.17: Critical configuration and beyond

It is plain to confirm from Figure 3.17(a) that the case of Figure 3.16(b) appears only when $\tan(\theta/2) \leq 1/2$. For the case $1/2 < \tan(\theta/2)$, the maximal ball inscribed in the cell bounded by the x-axis and the line of slope $\tan(\theta)$ does not touch the x-axis as in Figure 3.17(b). Therefore the configuration is reduced to the case of Theorem 3.3.5, i.e., $B(x, r)$ is cut essentially by a single line, and we get $r/r_n = 2$. All the statements of Theorem 3.3.12 are proved.

Remark 3.3.13. We remark several observations regarding the previous theorem. First, as already pointed out, the constants B_3 and B_4 are uniformly bounded by 3. For this reason, Theorem 3.3.12 gives an improvement of Theorems 3.2.7 and 3.2.6. Moreover, the theorem applies when the isometry is either a rotation or reflection. When $\theta(\mathcal{X})$ or $\pi - \theta(\mathcal{X})$ is small, the new bounds in Theorem 3.3.10 are better.

In the following proposition, we show that some rotational beta transformations have at least two distinct ACIMs.

Proposition 3.3.14. Let $\mathcal{X} = [-1/2, 1/2)^2$ and $1 < \beta \in \mathbb{R}$. Let $T : \mathcal{X} \to \mathcal{X}$ be the rotational beta transformation given by

$$T(z) = \beta z - d(z).$$

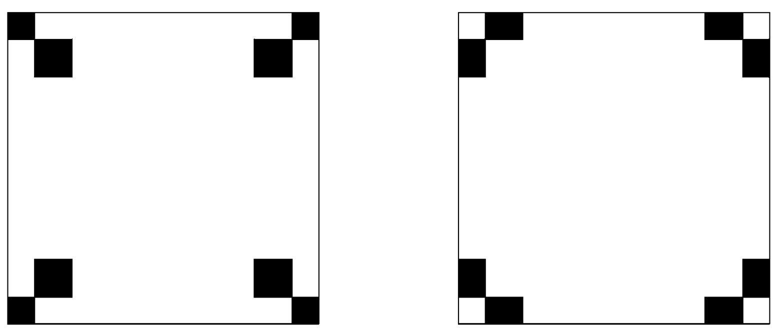
If $\beta \leq \sqrt{2}$, then $(\mathcal{X}, T)$ has more than one ACIM.

Proof Let $U : [-1/2, 1/2) \to [-1/2, 1/2)$ be the map given in Remark 3.3.11 Then T is the Cartesian product of U with itself. Let

$$Z = [-1/2, \beta/2 - 1] \cup [1 - \beta/2, 1/2].$$

When $\beta < 2$, one can check that the support of the ACIM of U is inside Z. Consequently, the support of an ACIM of T is contained in $Z \times Z$. This evidently implies that an ACIM of T, in this case, is not equivalent to the Lebesgue measure. Let $\beta \leq \sqrt{2}$. Let $Y_1 \subset \mathcal{X}$ be the region composed of eight squares of size $(\beta - 1)/(\beta(\beta + 1))$ and $(\beta - 1)/2(\beta + 1)$ and $Y_2 \subset \mathcal{X}$ be the region composed of eight $(\beta - 1)/(\beta(\beta + 1)) \times (\beta - 1)/2(\beta + 1)$ rectangles as in Figure 3.18. Under T, a small square in Y_1 is sent

Figure 3.18: Y_1 (Left) and Y_2 (Right)



to the diagonally opposite big square. Meanwhile, a big square in Y_1 is sent to the four corner squares. Thus, Y_1 is closed under T. We conclude that T has an ACIM supported in Y_1. Similarly, we can consider the restriction of T on Y_2. We obtain a similar result that T has an ACIM supported in Y_2. Since Y_1 and Y_2 are disjoint, then the ACIMs are distinct. $\square$

We can construct a class of isomorphic transformations to the given map above where $\theta(\mathcal{X})$ is arbitrary.

Corollary 3.3.15. *Let the fundamental domain $\mathcal{X}$ be the rhombus of side length 1 whose diagonals meet at the origin. Let $\beta \in \mathbb{R}$ such that $1 < \beta \leq \sqrt{2}$. Let T be the rotational beta transformation such that $T(z) = \beta z - d(z)$. Then $(\mathcal{X}, T)$ has at least two ACIMs.*

Remark 3.3.16. We know from [30] that U has a unique ACIM for $\beta > 1$. So, it is interesting that the Cartesian product $U \times U$ has more than one ACIM. Moreover, the two different ACIMs given above give the same projections on the first coordinate. This can be explained by the fact that U is not totally ergodic, i.e., U is ergodic but U^2 is not. Meanwhile, the unique ACIM invariant under U is equivalent to the Lebesgue measure when $\beta \geq 2$ (see Remark 3.3.11). Thus, T admits a unique ACIM and this measure is equivalent to the 2-dimensional Lebesgue measure when $\beta \geq 2$. Taking these examples into account, any improvements ${B_3}^*$ and ${B_4}^*$ of B_3 and B_4 of Theorem 3.3.12, respectively, should satisfy ${B_3}^* > \sqrt{2}$ and ${B_4}^* \geq 2$.

We give another rotational beta transformation with more than one ACIM. Note that in this transformation, the rotation component is not trivial.

Proposition 3.3.17. Let T be the rotational beta transformation with parameters $\beta = 1.039$, $\zeta = \sqrt{-1}$, $\eta_1 = 2.92$, $\eta_2 = \exp(\pi\sqrt{-1}/3)$ and $\xi = 0$. Then T admits at least two ACIMs.

Proof We look at the forward orbits of random points in $\mathcal{X}$ under T. The orbits are depicted in Figures 3.19 and 3.20. As in the preceeding proposition, we determine disjoint ergodic components of $\mathcal{X}$ corresponding to the two figures where the restriction of T is well-defined.

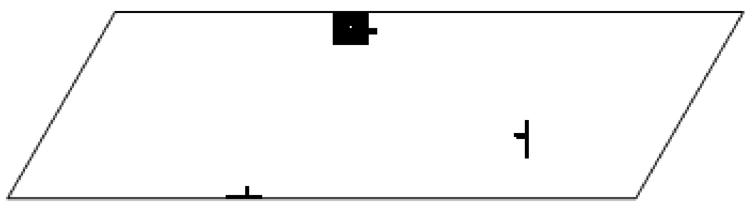


Figure 3.19: First Component

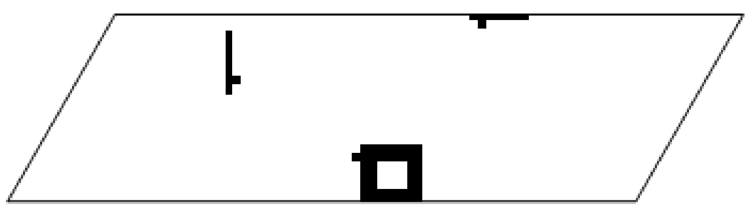


Figure 3.20: Second Component

In Figure 3.19, the biggest polygon is composed of two shapes E and F given in Figure 3.21. The rectangle E has dimensions of ratio $1 : \beta$. Applying iterates of T, we can determine the vertices of E as

$$x + \frac{\sqrt{-3}}{2},\ x + y\sqrt{-1},\ x + \frac{1}{\beta}\left(\frac{\sqrt{3}}{2} - y\right) + y\sqrt{-1},\ x + \frac{1}{\beta}\left(\frac{\sqrt{3}}{2} - y\right) + \frac{\sqrt{-3}}{2},$$

where $x = \eta_1 - \frac{\sqrt{3}}{2}\beta - \frac{1}{2}$ and $y = \beta x - \frac{\sqrt{3}}{2}$. Meanwhile, the vertices of F in counter-clockwise ordering are

$$x + \frac{1}{\beta}\left(\frac{\sqrt{3}}{2} - y\right) + y\sqrt{-1},\ \gamma + y\sqrt{-1},\ \gamma + v\sqrt{-1},\ u + v\sqrt{-1},$$

$$u + v'\sqrt{-1},\ \gamma + v'\sqrt{-1},\ \gamma + \frac{\sqrt{-3}}{2},\ x + \frac{1}{\beta}\left(\frac{\sqrt{3}}{2} - y\right) + \frac{\sqrt{-3}}{2},$$

where $\gamma = -\beta y + \eta_1 - \frac{1}{2}$, $u + v\sqrt{-1} = T^3(\gamma + \sqrt{-3}/2)$ and $u + v'\sqrt{-1} = T^2(x - 1/2)$. The two other shapes in Figure 3.19 are $T(F)$ and $T^2(F)$ which are both similar to F of ratio β and β^2, respectively. An easy calculation yields (see Figure 3.22)

$$E \cup F = T(E) \cup T^3(F).$$

Therefore, the restriction of T on

$$Y_1 := E \cup F \cup T(F) \cup T^2(F)$$

is well-defined. Hence, T admits an ACIM supported in Y_1.

Similarly, we can find polygons corresponding to Figure 3.20 where the restriction of T is well-defined. In Figure 3.20, we look at the biggest polygon as a composition of a rectangle, which we will call J, and an octagon, which we will call K. The four vertices of the rectangle J are

$$p, q, q + \beta(q - p)\sqrt{-1}, p + \beta(q - p)\sqrt{-1},$$

where $p = \sqrt{3}/\beta$ and $q = \eta_1 - 1$. On the other hand, the vertices of K include

$$p, p + \beta(q - p)\sqrt{-1}, r, r + \beta(q - p)\sqrt{-1},$$

where $r = q - \beta^2(q - p)$. The other vertices are given by

$$g_1 = \beta\sqrt{-1}s + \eta_1 - \eta_2, g_2 = \beta\sqrt{-1}t + \eta_1 - \eta_2,$$

where $s = T(q + \eta_2)$ and $t = T(\beta\sqrt{-1}r + \eta_1 - \eta_2)$ and the remaining vertices are $r + Im(g_1)\sqrt{-1}$ and $r + Im(g_2)\sqrt{-1}$, where Im denotes the imaginary part of the given complex number. We can confirm that the two other octagons in Figure 3.20 are given by $T(K)$ and $T^2(K)$. Moreover, $J \cup K = T(J) \cup T^3(K)$. Setting $Y_2 = J \cup K \cup T(K) \cup T^2(K)$, we have that the restriction of T on Y_2 is well-define. Thus, Y_2 gives rise to an ACIM of T. Since Y_1 and Y_2 are obviously disjoint, the ACIMs arising from Y_1 and Y_2 must be distinct.

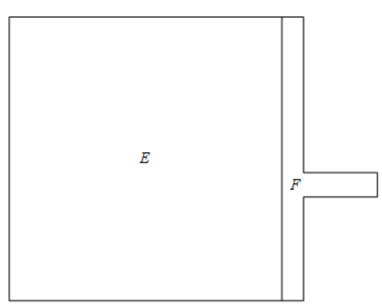


Figure 3.21: E and F

□

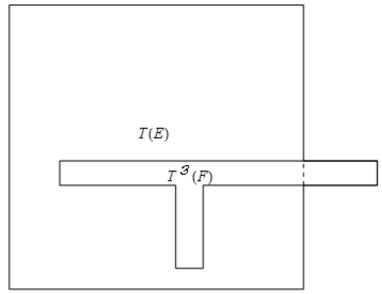


Figure 3.22: Partition of E and F

Remark 3.3.18. In the above result, we get the same conclusion when

$$\frac{\sqrt{3}}{2}\beta + 1 + \frac{\sqrt{3}}{\beta} - \frac{\sqrt{3}}{2\beta^3} \le \eta_1 \le \frac{1}{2} + \frac{\sqrt{3}}{\beta} + \frac{\sqrt{3}}{2\beta^3}.$$

The region defined by these inequalities is shown in Figure 3.23.

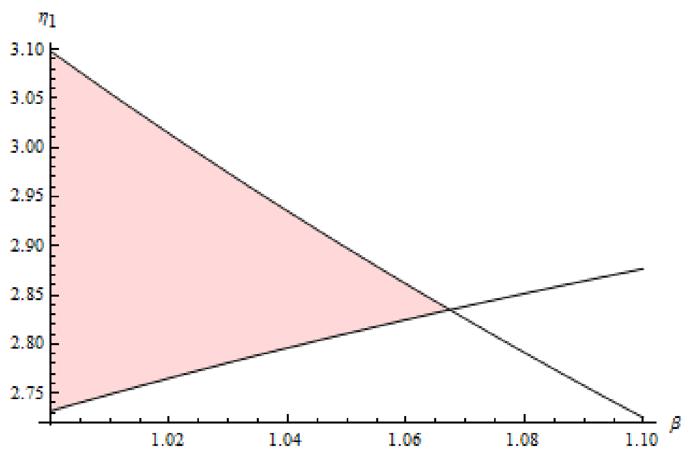


Figure 3.23: Non-ergodic parameters

3.4 Examples

In this section, we give some examples relating invariant measures. The examples in Chapter 2 Section 2.4 are also discussed.

Example 3.4.1. In Example 2.5.2, the map T has a unique ACIM and this is equivalent to the Lebesgue measure. Indeed, we can determine explicitly the density of the invariant measure, call this μ, from the primitive adjacency matrix of the sofic graph. Let P_i $(i = 1, 2, 3, 4)$ be as they are defined in Example 2.5.2. Let λ be the Lebesgue measure. For any rectangle $B \subseteq \mathcal{X}$, we have

$$\mu(B) = \frac{1}{5-\beta}\int_B h(z)d\lambda,$$

where $h(z) = \begin{cases} \beta^3 = 2+\sqrt{5} & z \in P_1 \cup P_4 \\ \beta^2 = (3+\sqrt{5})/2 & z \in P_2 \\ \beta^4 = (3+\sqrt{5})^2/4 & z \in P_3. \end{cases}$

Example 3.4.2. We look back at Example 2.4.1. Liao-Steiner [31] proved that the unique ACIM of f is equivalent to the 1-dimensional Lebesgue measure if and only if $\beta^2 \geq (1+\sqrt{5})/2$. Consequently, the ACIM of T is equivalent to the 2-dimensional Lebesgue measure if and only if $\beta \geq \sqrt{(1+\sqrt{5})/2}$.

Example 3.4.3. In Example 2.4.2, we have $B_2 = 7/3$ because $r(\mathcal{L}) = 1/\sqrt{3}$ and $w(\mathcal{X}) = \sqrt{3}/2$. Therefore $\beta > B_2 = 7/3$. By Theorem 3.2.7, the map T admits a unique ACIM. This ACIM is equivalent to the Lebesgue measure.

Example 3.4.4. In Example 2.4.3, we can check that the adjacency matrix of the sofic graph is primitive. We can derive the from the set equations of the partition cells the unique ACIM whose density is positive and constant at each cell. Moreover, the ACIM is unique. Because $\beta < 2$, this example is beyond Theorem 3.2.7.

Example 3.4.5. In Example 2.4.4, we can likewise show that adjacency matrix of the sofic graph is primitive. Therefore, T admits a unique ACIM and this measure is Lebesgue.

Chapter 4

Summary and Outlook

In this dissertation, we have introduced a class of piecewise expanding maps in $\mathbb{R}^m$. These maps called rotational beta transformations generalize the notion of positive and negative beta transformations defined on the one-dimensional real space. A rotational beta transformation, as in the classical case, induces a numeration system on the domain of definition of the transformation. In general, the radix of the associated numeration, which we called the rotational beta expansion, is βM where $1 < \beta \in \mathbb{R}$ and M is an isometry in the m-th dimensional orthogonal group. In Chapter 2, we defined a topological dynamical system $(\mathcal{X}, T)$ associated with a rotational beta transformation T. This system is called the symbolic dynamical system. We discussed the so-called soficness property of the symbolic dynamical systems. In Theorem 2.2.1, we give a necessary and sufficient condition for $(\mathcal{X}, T)$ to be sofic. Moreover, we gave conditions on the parameters of a two-dimensional rotational beta transformation ensuring that $(\mathcal{X}, T)$ is sofic in Theorem 2.2.5. In this result, the expansive constant β is, in particular, a Pisot number. Further investigation is needed to conclude something when β is not Pisot (e.g., when β is a Salem number or in general, just an algebraic number). In a similar light, we can research on other classes of non-sofic dynamical systems which are distinct from those given in Theorem 2.2.9. Meanwhile, in Proposition 2.6.4, we gave a partial result on periodic expansions association to rotational beta transformations. In future research, we endeavor to discuss periodicity of expansion on a larger class of rotational beta transformations. Another problem we need to look at regarding rotational beta expansions is on the admissibility of words built from the digit set arising from the transformation. We wish to give a result on admissibility which is analogous to the admissibility criteria given by Parry in [33] for the classical beta expansion in terms of the itinerary of the discontinuity point 1 under the map.

In Chapter 3, we look at the rotational beta transformation in the context of a measure-preserving dynamical system. In particular, the common theme in the results discussed in Chapter 3 is the provision of lower bounds on the expansive constant β ensuring that transformation has a unique ACIM and that this ACIM is equivalent to the Lebesgue measure. The results are obtained using two different strategies - one using the notion of covering radius and the other the notion of holes. In $\mathbb{R}^2$, we have Theorems 3.2.6, 3.2.7 and 3.3.12 for a general class of rotational beta transformation. Specializing on a particular family of rotational beta transformation, we obtained Theorem 3.2.9 and Corollary 3.3.6. For higher dimensional spaces, we have Theorems 3.3.5 and 3.3.10. In the above results, there are two important properties that we assumed, namely $\beta w(\mathcal{X}) > 2r(\mathcal{L})$ and the property (S). It is interesting to derive results when the rotational beta transformation satisfies neither of these two conditions. Also, it is expected that the bounds given here can be lowered. Future studies can focus on developing a strategy to obtain better bounds. In Propositions 3.3.14 and 3.3.17 and Corollary 3.3.15, we gave examples of rotational beta transformations with at least two ACIMs. We want to describe other classes of rotational beta transformations admitting more than one ACIM. Finally, if the ACIM of the rotational beta transformation, it is a good subject of future research to determine the explicit form of the density of the invariant measure.

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