Some Results on Wavelet Expansions

Toshio Suzuki

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Toshio Suzuki
Doctral Program in Mathematics

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1 Wavelet Analysis

1.1 Introduction

Wavelet Analysis is one of the time-frequency analysis and it is applied to the various fields like physics, chemistry, industry and so on. It is said that the first wavelet is Haar basis by Alfréd Haar in 1909. Wavelets are introduced in the beginning of the 1980s. In 1975, Jean Morlet, who is an oil explorer, found the idea of wavelet analysis. After that, many researchers contributed to the development of continuous wavelet transform, discrete wavelet transform. They first used the word *ondelette* meaning of small wave in French and after translation to English, the word wavelet had born.

Wavelet analysis is the method like Fourier analysis. We use trigonometric functions (sine and cosine function) to analyze signal (function) in Fourier analysis. This method can analyze various signals in each frequency since we can represent the signal by the superposition of trigonometric functions. Wavelet analysis also represent signals with the superposition of wavelets. Trigonometric functions have Infinity length supports. On the other hand, a wavelet has compact support or decay at infinity. Recently, wavelet analysis is one of the useful methods of analyze time and frequency space at the same time.

In this thesis, we are going to consider three topics of the wavelet analysis. Especially, three topics and results are closely related to Sobolev space $W^{1,1}(\mathbf{R})$. The functions of this space is similar to the absolute continuous functions space.

1.2 Preliminary

Let Ω be an open set of $\mathbf{R}$. In this thesis, $C^\alpha(\Omega)$ $(0 < \alpha < 1)$ denotes the space of the Hölder continuous functions. When $\alpha = 1$, $C^\alpha(\Omega)$ is the space $Lip(\Omega)$ of Lipschitz continuous functions. A function of bounded variation on Ω is a real valued function whose total variation is finite. The space of bounded variations functions on Ω is denoted by $BV(\Omega)$ with the norm $\|f\|_{BV} := \|f\|_{L^1} + V(f,\Omega)$, where V is the total variation.

For $1 \le p < \infty$, $L^p(\Omega)$ denotes

$$L^p(\Omega) = \left\{ f : f \text{ is a Lebesgue measurable function and } \int_\Omega |f(x)|^p dx < \infty \right\}$$

and

$$L^\infty(\Omega) = \{ f : f \text{ is a Lebesgue measurable function and ess. } \sup_{x\in\Omega} |f(x)| < \infty \}.$$

The norm of the L^p spaces are defined by $\|f\|_{L^p} = (\int_\Omega |f(x)|^p dx)^{1/p}$ and $\|f\|_{L^\infty} = \text{ess. } \sup_{x\in\Omega} |f(x)|$. In particular, $L^2(\Omega)$ is a Hilbert space and its inner product is defined by $(f,g)_{L^2} = \int_\Omega f(x)\overline{g(x)}dx$.

For $f \in L^2(\mathbf{R})$, the Fourier transform and the inverse Fourier transform are given by

$$\begin{aligned} \mathcal{F}f(\xi) \left(= \hat{f}(\xi)\right) &= \int_{\mathbf{R}} f(x) e^{-ix\xi} dx, \\ \mathcal{F}^{-1}f(\xi) \left(= \check{f}(\xi)\right) &= \frac{1}{2\pi} \int_{\mathbf{R}} f(x) e^{ix\xi} dx. \end{aligned}$$

For $1 \le p < \infty$, $m \in \mathbf{N}$, we also introduce the Sobolev space $W^{m,p}(\Omega)$ as

$$W^{m,p}(\mathbf{R}) = \left\{ f \in L^p(\Omega) \middle| \text{ the weak derivative } D^\alpha f \ (|\alpha| \le m) \text{ exists and } D^\alpha f \in L^p(\mathbf{R}) \right\}.$$

By the Sobolev embedding theorem, the following set inclusions hold:

$$Lip(\mathbf{R}) \subset W^{1,1}_{loc}(\mathbf{R}), \quad W^{1,1}(\mathbf{R}) \subset C^0(\mathbf{R}) \cap L^\infty(\mathbf{R}). \tag{1}$$

1.3 Discrete Wavelet Transform

Definition 1.1. *The collection $\{V_j\}_{j\in\mathbf{Z}}$ of closed subspaces of $L^2(\mathbf{R})$ is called Multiresolution analysis (MRA) if the following conditions are satisfied :*

(i) $V_j \subset V_{j+1}$ *for all* $j \in \mathbf{Z}$.

(ii) $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$.

(iii) $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$.

(iv) $f \in V_j \Leftrightarrow f(2t) \in V_{j+1}$.

(v) *There exists* $\varphi \in V_0$ *s.t.* $\{\varphi(t-k) \mid k \in \mathbf{Z}\}$ *consists orthonormal basis for* V_0. *This function* φ *is called the scaling function.*

Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA. Since each V_j is a Hilbert space and by the MRA condition (i), there exists W_j such that

$$V_{j+1} = V_j \oplus W_j \text{ for all } j \in \mathbf{Z}.$$

By condition (ii), we get the orthogonal decomposition

$$L^2(\mathbf{R}) = \bigoplus_{j\in\mathbf{Z}} W_j.$$

So, there exists a function $\psi \in W_0$ such that $\{\psi(t-k) \mid k \in \mathbf{Z}\}$ forms an orthonormal basis for W_0 and the scalings and translations of $\psi \in W_0$ form an orthonormal basis for $L^2(\mathbf{R})$.

Definition 1.2. *A function* $\psi \in L^2(\mathbf{R})$ *is called an orthonormal wavelet if the set* $\{2^{j/2}\psi(2^j \cdot -k) \mid j,k \in \mathbf{Z}\}$ *is an orthonormal basis for* $L^2(\mathbf{R})$.

Since $\varphi(\cdot/2) \in V_{-1} \subset V_0$, there exists a sequence $\{\alpha_k\}_{k\in\mathbf{Z}}$ satisfying the two-scale equation

$$\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbf{Z}} \alpha_k \varphi(x-k).$$

By the Fourier transform, we have

$$\hat{\varphi}(2\xi) = m_0(\xi)\varphi(\xi)$$

where $m_0(\xi) = \sum_{k\in\mathbf{Z}} \alpha_k e^{-ik\xi} \in L^2(-\pi,\pi)$ is called the low-pass filter associated with the scaling function $\varphi \in V_0$. The low-pass filter has important relations with wavelets. Indeed, the MRA and the low-pass filter are enable us to give us the wavelet.

Theorem 1.3. *Let* φ *be a scaling function for an MRA* $\{V_j\}_{j\in\mathbf{Z}}$ *and* m_0 *be a low-pass filter associated with the MRA. Suppose that* ν *is a measurable function satisfying* $|\nu(\xi)| = 1$ *and we define*

$$\hat{\psi}(\xi) = e^{i\xi/2}\nu(\xi)\overline{m_0\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right) \quad a.e. \ \mathbf{R}.$$

Then, the function ψ *is an orthonormal wavelet for the MRA.*

2 Unconditional Convergence of Wavelet Expansion

2.1 Introduction

In this section, we discuss the unconditional convergence of wavelet expansions. As for the Fourier expansion $f(t)=\sum_{j\in\mathbf{Z}} c_j e_j(t)$ on Ω, the following results on the convergences of Fourier series are well-known (see [26] etc.):

$(i)_F$ If $f\in C^\alpha(\Omega)$ for $\alpha>1/2$, the Fourier series converges uniformly and absolutely, i.e., $\sum_{j\in\mathbf{Z}}|c_j|<\infty$.

$(ii)_F$ If $f\in W^{1,1}(\Omega)\cap C^\alpha(\Omega)$ for $\alpha>0$, the Fourier series converges uniformly and absolutely, i.e., $\sum_{j\in\mathbf{Z}}|c_j|<\infty$. In fact, $W^{1,1}(\Omega)$ can be relaxed to $BV(\Omega)$.

$(iii)_F$ For the function $f(t)=\sum_{n=1}^{\infty}\frac{\sin nt}{n\log(1+n)}\in W^{1,1}(\Omega)$ with $\Omega=(-\pi,\pi)$, its Fourier series does not converge absolutely.

For a Scauder basis $\{e_j(t)\}$, the order of the basis is important in the sense of stable convergence. Let X be a Banach space. The series $\sum_{j\in\mathbf{Z}} c_j e_j(t)$ converges unconditionally to $f(t)$ in X if and only if $\sum_{j\in\mathbf{Z}} c_j e_{\sigma(j)}(t)$ converges to $f(t)$ in X for any permutation $\sigma:\mathbf{Z}\to\mathbf{Z}$. This is equivalent to the condition that for any $\varepsilon_j=\pm1$, $\sum_{j\in\mathbf{Z}}\varepsilon_j c_j e_j(t)$ converges to $f(t)$ in X. In the Hilbert space, the unconditional convergence holds with an orthonormal basis $\{e_j(t)\}$ thanks to the Parseval's identity $\|\sum_{j\in\mathbf{Z}} c_j e_j(t)\|_X^2=\sum_{j\in\mathbf{Z}}|c_j|^2$. We can see that for a Banach space X, the absolute convergence is stronger than the unconditional convergence since

$$\left\|\sum_{j\in\mathbf{Z}}\varepsilon_j c_j e_j(t)\right\|_X\le\sum_{j\in\mathbf{Z}}\|c_j e_j(t)\|_X=\sum_{j\in\mathbf{Z}}|c_j|. \tag{2}$$

For a Banach space $X=L^\infty(\mathbf{R})$, since $\text{ess.sup}|e_j(t)|=1$, we find that the Fourier series converges to $f(t)$ unconditionally in $L^\infty(\mathbf{R})$ in the case of $(i)_F$ or $(ii)_F$. Here we remark that the Banach space $X=L^\infty(\Omega)$ for the convergence and the Banach space $\tilde{X}=C^\alpha(\Omega)$ or $W^{1,1}(\Omega)$ for the limit f are different ($\tilde{X}\subset X$) in the Fourier series.

Now we consider the wavelet expansion $f(t)=\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{j,k}\psi_{j,k}(t)$, where $\psi_{j,k}(t)=2^{j/2}\psi(2^j t-k)$. The Fourier basis consists of only the analytic function space $\mathcal{A}$. Conversely, there are various wavelet bases ψ. To classify wavelets ψ, we denote Y the space (set) which restricts the regularity or the decay at infinity. The following results about wavelet expansions are known:

$(iv)_w$ If $\psi\in Y=\{y\in C^1(\mathbf{R});|y(t)|+|y'(t)|\le g(|t|)\}$ with a decreasing $g\in L^1[0,\infty)$ such that $|g(0)|<\infty$ and $\|tg(\cdot)\|_{L^1[0,\infty)}<\infty$, $\{\psi_{j,k}(t)\}$ is an unconditional basis in $X=\tilde{X}=L^p(\mathbf{R})$ with $1<p<\infty$ (see [13]).

$(v)_w$ If $\psi\in Y=\{y\in\mathcal{A}(\mathbf{R});\mathcal{F}[y]$ is characteristic functions of a finite sum of bounded closed intervals (unimodular wavelets)$\}$, $\{\psi_{j,k}(t)\}$ is an unconditional basis in $X=\tilde{X}=L^p(\mathbf{R})$ with $1<p<\infty$ (see [3], [12]).

Let us choose the Banach space $X=\tilde{X}=W^{1,1}(\mathbf{R})$, and also $\psi\in Y=W^{1,1}(\mathbf{R})$. By the Sobolev embedding theorem $W^{1,1}(\mathbf{R})\subset L^2(\mathbf{R})$, the coefficients $c_{j,k}:=(f,\psi_{j,k})_{L^2}$ are well-defined. Thus, we see the following basic observation:

Proposition 2.1. *Assume that $\psi\in W^{1,1}(\mathbf{R})$. Then, the wavelet expansion $\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{j,k}\psi_{j,k}(t)$ converges to $f(t)$ unconditionally in $W^{1,1}(\mathbf{R})$ if the coefficients satisfy $\{2^{|j|/2}c_{j,k}\}_{(j,k)\in\mathbf{Z}^2}\in\ell^1$.*

Proof. Some calculations give us

$$\begin{aligned}
\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}}\|c_{j,k}\psi_{j,k}\|_{W^{1,1}} &\leq \sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} 2^{j/2}|c_{j,k}|\|\psi(2^j\cdot -k)\|_{W^{1,1}} \\
&= \sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} 2^{j/2}|c_{j,k}|\int_{\mathbf{R}}\Big\{|\psi(2^jt-k)|+2^j|\psi'(2^jt-k)|\Big\}dt \\
&\leq \Big(\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} 2^{-j/2}|c_{j,k}|\Big)\|\psi\|_{L^1}+\Big(\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} 2^{j/2}|c_{j,k}|\Big)\|\psi'\|_{L^1} \\
&\leq \Big(\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} 2^{|j|/2}|c_{j,k}|\Big)\|\psi\|_{W^{1,1}}.
\end{aligned}$$

□

Thus, if the scalar series $\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} 2^{|j|/2}|c_{j,k}|$ converges, the wavelet expansion $\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{j,k}\psi_{j,k}$ for $\psi\in W^{1,1}(\mathbf{R})$ converges absolutely and also unconditionally in $W^{1,1}(\mathbf{R})$. However, it is not clear whether the condition $\{2^{|j|/2}c_{j,k}\}_{(j,k)\in\mathbf{Z}^2}\in\ell^1$ really restricts the limit $f\in W^{1,1}(\mathbf{R})$.

From Sobolev embedding theorem(1), we see that the Sobolev space $W^{1,1}(\mathbf{R})$ can be regarded as a function space between $Lip(\mathbf{R})$ (with a sufficient decay at infinity) and $C^0(\mathbf{R})\cap L^\infty(\mathbf{R})$. The gap between the regularities $\psi\in Y$ and f should make the convergence worse. Indeed, some results on the convergences of the Fourier series come from the gap of functions f and the basis $e_j\in\mathcal{A}$. Therefore, we choose the suitable function space as $X=\Big\{C^0(\mathbf{R})\cap L^\infty(\mathbf{R})\Big\}\backslash W^{1,1}(\mathbf{R})$ and topology space as $\tilde{X}=L^\infty(\mathbf{R})$ to take a little bit weak topology than $W^{1,1}(\mathbf{R})$. Thus, we prove the following result of the wavelet expansion which corresponding to $(iii)_F$ in case of the Fourier expansion:

Theorem 2.2. *There exists $f_0\in\{C^0(\mathbf{R})\cap L^\infty(\mathbf{R})\}\backslash W^{1,1}(\mathbf{R})$ satisfying the following:*

- *f_0 has the wavelet expansion $f_0(t)=\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{j,k}\psi_{j,k}(t)$ in $L^2(\mathbf{R})$ for some $\psi\in Lip(\mathbf{R})$ and $\{c_{j,k}\}_{(j,k)\in\mathbf{Z}^2}\in\ell^2$ such that $\{2^{|j|/2}c_{j,k}\}_{(j,k)\in\mathbf{Z}^2}\notin\ell^1$.*
- *$\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{j,k}\psi_{j,k}(t)$ converges to $f_0(t)$ uniformly and non-unconditionally in $L^\infty(\mathbf{R})$.*

To prove Theorem 2.2, we make a concrete $f_0\in\{C^0(\mathbf{R})\cap L^\infty(\mathbf{R})\}\backslash W^{1,1}(\mathbf{R})$. Since the Strönberg wavelet has Lipschitz continuity and exponential decay at infinity, we are able to construct f_0 simply. The Strönberg wavelet is given by

$$\psi^{St}(t)=\sum_{k\in\mathbf{Z}} b_kN_2(2t-k),$$

where the coefficients b_k are defined by

$$b_k=\begin{cases} -4(\sqrt{3}-2)^k & \text{if} \quad k\geq 1, \\ -\dfrac{5}{2}+\dfrac{\sqrt{3}}{2} & \text{if} \quad k=0, \\ -(2-\sqrt{3})^{-\frac{k}{2}}\left(\cos\dfrac{k\pi}{2}+\sqrt{2}\sin\dfrac{k\pi}{2}\right) & \text{if} \quad k\leq -1, \end{cases}$$

and N_2 is the B-spline of order 2 given by

$$N_2(t)=\begin{cases} t & \text{for} \quad 0\leq t\leq 1, \\ 2-t & \text{for} \quad 1\leq t\leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

(see [8]). For the Franklin wavelet $\psi^{Fr} \in Lip(\mathbf{R})$, the following fact is known:

$(vi)_w$ $\{\psi^{Fr}_{j,k}(t)\}$ is an unconditional basis in $X = \tilde{X} = L^p(\mathbf{R})$ with $1 < p < \infty$ (see Theorem 6.23 in §5 of [13]).

This holds for the spline wavelets of the same order. Therefore, we have the following also for $\psi^{St} \in Lip(\mathbf{R})$:

$(vi)'_w$ $\{\psi^{St}_{j,k}(t)\}$ is an unconditional basis in $X = \tilde{X} = L^p(\mathbf{R})$ with $1 < p < \infty$ (see Theorem 6.14 in §5 of [13]).

These two facts $(vi)_w$ and $(vi)'_w$ are obtained by introducing a small modification on the C^1 assumption of $(iv)_w$. We remark that $\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} |c_{j,k}|\psi^{St}_{j,k}(t) \in X = L^p(\mathbf{R})$ with $1 < p < \infty$ and f_0 is not a counter example for $(vi)'_w$. The function space $X = L^\infty(\mathbf{R})$ in Theorem 2.2 is locally stronger than $X = L^p(\mathbf{R})$ with $1 < p < \infty$ in $(vi)'_w$. This causes the non-unconditionality even for the continuous function f_0.

2.2 Proof of the Theorem 2.2

We consider the following function:

$$f_0(t) = \sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{j,k}\psi^{St}_{j,k}(t) \quad \text{with} \quad c_{j,k} = \begin{cases} \dfrac{(-1)^j}{(j+1)2^{\frac{j}{2}}} & \text{for } j \geq 0 \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Our purpose is to prove that $f_0 \notin W^{1,1}(\mathbf{R})$ and that $\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} c_{j,k}\psi^{St}_{j,k}(t)$ converges to $f_0(t)$ uniformly and non-unconditionally in $L^\infty(\mathbf{R})$. Let us put $t_n = 2^{-n}$ $(n \geq 1)$. We can write the function f_0 as

$$f_0(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}}\psi^{St}_{j,0}(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1}\psi^{St}(2^j t) = \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t - k).$$

Furthermore, using the fact that $\mathrm{supp} N_2 \subset [0,2]$, we shall compute $f_0(t)$ for $t \in [2^{-n}, 2^{-n+1}] = [t_n, t_{n-1}]$ as follows:

- *Case* $n = 1$) For $t \in [2^{-1}, 2^0] = [t_1, t_0]$, we have

$$\begin{aligned} f_0(t) &= \frac{(-1)^0}{0+1}\Big\{b_0 N_2(2^{0+1}t - 0) + b_1 N_2(2^{0+1}t - 1)\Big\} \\ &\quad + \frac{(-1)^1}{1+1}\Big\{b_1 N_2(2^{1+1}t - 1) + b_2 N_2(2^{1+1}t - 2) + b_3 N_2(2^{1+1}t - 3)\Big\} \\ &\quad + \frac{(-1)^2}{2+1}\Big\{b_3 N_2(2^{2+1}t - 3) + b_4 N_2(2^{2+1}t - 4) + b_5 N_2(2^{2+1}t - 5) \\ &\qquad\qquad + b_6 N_2(2^{2+1}t - 6) + b_7 N_2(2^{2+1}t - 7)\Big\} \\ &\quad + \cdots \\ &= \sum_{j=0}^{\infty}\sum_{k=2^j-1}^{2^{j+1}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t - k). \end{aligned}$$

Here we used the fact that $2^{0+1}t \in [1,2]$, $2^{1+1}t \in [2,4]$, $2^{2+1}t \in [4,8]$, and $2^{j+1}t \in [2^j, 2^{j+1}]$.

• *Case* $n=2$) For $t\in[2^{-2},2^{-1}]=[t_2,t_1]$, we have

$$\begin{aligned}
f_0(t) &= \frac{(-1)^0}{0+1}\Big\{b_{-1}N_2(2^{0+1}t+1)+b_0N_2(2^{0+1}t-0)\Big\}\\
&\quad+\frac{(-1)^1}{1+1}\Big\{b_0N_2(2^{1+1}t-0)+b_1N_2(2^{1+1}t-1)\Big\}\\
&\quad+\frac{(-1)^2}{2+1}\Big\{b_1N_2(2^{2+1}t-1)+b_2N_2(2^{2+1}t-2)+b_3N_2(2^{2+1}t-3)\Big\}\\
&\quad+\cdots\\
&= \frac{(-1)^0}{0+1}\Big\{b_{-1}N_2(2^{0+1}t+1)+b_0N_2(2^{0+1}t-0)\Big\}\\
&\quad+\sum_{j=1}^{\infty}\sum_{k=2^{j-1}-1}^{2^j-1}\frac{(-1)^j}{j+1}b_kN_2(2^{j+1}t-k).
\end{aligned}$$

Here we used the fact that $2^{0+1}t\in[2^{-1},1]$, $2^{1+1}t\in[1,2]$, $2^{2+1}t\in[2,4]$, and $2^{j+1}t\in[2^{j-1},2^j]$.

• *Case* $n=3$) For $t\in[2^{-3},2^{-2}]=[t_3,t_2]$, we have

$$\begin{aligned}
f_0(t) &= \frac{(-1)^0}{0+1}\Big\{b_{-1}N_2(2^{0+1}t+1)+b_0N_2(2^{0+1}t-0)\Big\}\\
&\quad+\frac{(-1)^1}{1+1}\Big\{b_{-1}N_2(2^{1+1}t+1)+b_0N_2(2^{1+1}t-0)\Big\}\\
&\quad+\frac{(-1)^2}{2+1}\Big\{b_0N_2(2^{2+1}t-0)+b_1N_2(2^{2+1}t-1)\Big\}\\
&\quad+\cdots\\
&= \sum_{j=0}^{1}\frac{(-1)^j}{j+1}\Big\{b_{-1}N_2(2^{j+1}t+1)+b_0N_2(2^{j+1}t-0)\Big\}\\
&\quad+\sum_{j=2}^{\infty}\sum_{k=2^{j-2}-1}^{2^{j-1}-1}\frac{(-1)^j}{j+1}b_kN_2(2^{j+1}t-k).
\end{aligned}$$

Here we used the fact that $2^{0+1}t\in[2^{-2},2^{-1}]$, $2^{1+1}t\in[2^{-1},1]$, $2^{2+1}t\in[1,2]$, and $2^{j+1}t\in[2^{j-2},2^{j-1}]$.

Thus, if $n\geq 2$, for $t\in[2^{-n},2^{-n+1}]=[t_n,t_{n-1}]$, we recursively have

$$\begin{aligned}
f_0(t) &= \sum_{j=0}^{n-2}\frac{(-1)^j}{j+1}\Big\{b_{-1}N_2(2^{j+1}t+1)+b_0N_2(2^{j+1}t-0)\Big\}\\
&\quad+\sum_{j=n-1}^{\infty}\sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1}\frac{(-1)^j}{j+1}b_kN_2(2^{j+1}t-k).
\end{aligned}$$

Since $2^{j+1}t\in[2^{j-n+1},2^{j-n+2}]\subset[0,1]$ for $0\leq j\leq n-2$, we see that

$$N_2(2^{j+1}t+1)=2-(2^{j+1}t+1)=1-2^{j+1}t,\quad N_2(2^{j+1}t-0)=2^{j+1}t$$

and get

$$\begin{aligned}
f_0(t) &= \sum_{j=0}^{n-2}\frac{(-1)^j}{j+1}\Big\{(1-2^{j+1}t)b_{-1}+2^{j+1}tb_0\Big\}\\
&\quad+\sum_{j=n-1}^{\infty}\sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1}\frac{(-1)^j}{j+1}b_kN_2(2^{j+1}t-k).
\end{aligned}\tag{3}$$

Remark 2.3. *In the same way, we also get for* $t \in [-2^{-n+1}, -2^{-n}] = [-t_{n-1}, -t_n]$ *that*

$$\begin{aligned} f_0(t) &= \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \Big\{ (1-2^{j+1}t)b_{-1} - 2^{j+1}tb_{-2} \Big\} \\ &\quad + \sum_{j=n-1}^{\infty} \sum_{k=-2^{j-n+2}-1}^{-2^{j-n+1}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t-k). \end{aligned} \tag{4}$$

- **Step 1(Unbounded Variation)**

It is sufficient to show that $f_0 \notin BV(\mathbf{R})$ instead of $f_0 \notin W^{1,1}(\mathbf{R})$ since $W^{1,1}(\mathbf{R}) \subset BV(\mathbf{R})$. Especially when $t = t_n$, noting that $N_2(2^{j+1}t_n - k) = \delta_{1,2^{j+1}t_n-k}$, that is, the summation with respect to k runs over only $k = 2^{j+1}t_n - 1 = 2^{j-n+1} - 1$, (3) can be changed into

$$f_0(t_n) = \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \Big\{ (1-2^{j-n+1})b_{-1} + 2^{j-n+1} b_0 \Big\} + \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+1}-1}.$$

Hence it follows that for $n \geq 3$

$$\begin{aligned} f_0(t_{n-1}) - f_0(t_n) &= \sum_{j=0}^{n-3} \frac{(-1)^j}{j+1} \left\{ \left(1-2^{j-n+2}\right) b_{-1} + 2^{j-n+2} b_0 \right\} + \sum_{j=n-2}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+2}-1} \\ &\quad - \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \left\{ \left(1-2^{j-n+1}\right) b_{-1} + 2^{j-n+1} b_0 \right\} - \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+1}-1} \\ &= \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \left\{ \left(1-2^{j-n+2}\right) b_{-1} + 2^{j-n+2} b_0 \right\} - \frac{(-1)^{n-2}}{n-1} b_0 \\ &\quad + \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+2}-1} + \frac{(-1)^{n-2}}{n-1} b_0 \\ &\quad - \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \left\{ \left(1-2^{j-n+1}\right) b_{-1} + 2^{j-n+1} b_0 \right\} - \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+1}-1} \\ &= \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} 2^{j-n+1} (b_0 - b_{-1}) + \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} (b_{2^{j-n+2}-1} - b_{2^{j-n+1}-1}). \end{aligned}$$

Our next task is to find a suitable wavelet whose coefficients b_k satisfy $\sum_{n=1}^{\infty} |f_0(t_{n-1}) - f_0(t_n)| = \infty$.

In particular, for convenience, we shall choose the Strömberg wavelet $\psi^{St} \in Lip(\mathbf{R})$ given by

$$\psi^{St}(t) = \sum_{k \in \mathbf{Z}} b_k N_2(2t-k),$$

where

$$b_k = \begin{cases} -4(\sqrt{3}-2)^k & \text{if} \quad k \geq 1, \\ -\dfrac{5}{2} + \dfrac{\sqrt{3}}{2} & \text{if} \quad k = 0, \\ -(2-\sqrt{3})^{-\frac{k}{2}} \left(\cos \dfrac{k\pi}{2} + \sqrt{2} \sin \dfrac{k\pi}{2} \right) & \text{if} \quad k \leq -1, \end{cases}$$

(see [8]). Noting that

$$\sum_{j=0}^{n-2} \frac{(-2)^j}{j+1} = 2^{-1} \int_0^2 \frac{1-(-y)^{n-1}}{1+y} dy = 2^{-1} \log 3 - 2^{n-1} \int_0^1 \frac{(-z)^{n-1}}{1+2z} dz,$$

we can rewrite

$$\begin{aligned} f_0(t_{n-1}) - f_0(t_n) &= (b_0 - b_{-1}) 2^{1-n} \sum_{j=0}^{n-2} \frac{(-2)^j}{j+1} + \frac{(-1)^{n-1}}{n}(b_1 - b_0) \\ &\quad + \sum_{j=n}^{\infty} \frac{(-1)^j}{j+1}(b_{2^{j-n+2}-1} - b_{2^{j-n+1}-1}) \\ &= (b_{-1} - b_0)\Big(\int_0^1 \frac{(-z)^{n-1}}{1+2z} dz - 2^{-n}\log 3 \Big) + \frac{(-1)^{n-1}}{n}(b_1 - b_0) \\ &\quad + \sum_{j=n}^{\infty} \frac{(-1)^j}{j+1}(b_{2^{j-n+2}-1} - b_{2^{j-n+1}-1}). \end{aligned}$$

Especially, for $n = 2m+1$ ($m \geq 1$), we obtain

$$\begin{aligned} t_{2m} - t_{2m+1} &= 2^{-2m-1}, \\ f_0(t_{2m}) - f_0(t_{2m+1}) &= (b_{-1} - b_0) \int_0^1 \frac{z^{2m}}{1+2z} dz + \frac{b_1 - b_0}{2m+1} \\ &\quad -(b_{-1} - b_0) 2^{-2m-1} \log 3 + \sum_{j=2m+1}^{\infty} \frac{(-1)^j}{j+1} \{ b_{2^{j-2m+1}-1} - b_{2^{j-2m}-1} \} \\ &=: I + II - III + IV. \end{aligned}$$

Using $b_{-1} - b_0 = \frac{1}{2}(3+\sqrt{3})$ and $b_1 - b_0 = \frac{3}{2}(7 - 3\sqrt{3})$, we get

$$\begin{aligned} I + II &\geq (b_{-1} - b_0) \int_0^1 \frac{z^{2m}}{1+2} dz + \frac{b_1 - b_0}{2m+1} > \frac{13(2-\sqrt{3})}{6(m+1)}, \\ |IV| &\leq \frac{1}{2m+2} \sum_{j=2m+1}^{\infty} |b_{2^{j-2m+1}-1} - b_{2^{j-2m}-1}| \\ &= \frac{2}{(m+1)(2-\sqrt{3})} \sum_{j=2m+1}^{\infty} \Big\{ (2-\sqrt{3})^{2^{j-2m}} - (2-\sqrt{3})^{2^{j-2m+1}} \Big\} \\ &\leq \frac{2}{(m+1)(2-\sqrt{3})} (2-\sqrt{3})^{2^{(2m+1)-2m}} = \frac{2(2-\sqrt{3})}{m+1}. \end{aligned}$$

Since $|III| < \frac{2-\sqrt{3}}{7(m+1)}$ for $m \geq 2$, there exists $c > 0$ such that for $m \geq 2$

$$\begin{aligned} |f_0(t_{2m}) - f_0(t_{2m+1})| &\geq \Big| I + II + IV \Big| - |III| \geq \Big| I + II - |IV| \Big| - |III| \\ &\geq \frac{c}{m+1}. \end{aligned} \tag{5}$$

Since

$$\sum_{n=1}^{\infty} |f_0(t_{n-1}) - f_0(t_n)| \geq \sum_{m=1}^{\infty} |f_0(t_{2m}) - f_0(t_{2m+1})| = \infty,$$

we find that $f_0 \notin BV(\mathbf{R})$. Therefore we can conclude that $f_0 \notin W^{1,1}(\mathbf{R})$.

Remark 2.4. *Thanks to the Strömberg wavelet $\psi^{St} \in Lip(\mathbf{R})$, we can know that $|I + II + IV| \neq 0$. The information on the exact values of b_k is required to find (5). Therefore, it would be difficult to get (5) for general piecewise linear spline wavelets $\psi \in Lip(\mathbf{R})$ or even for the Franklin wavelet $\psi^{Fr} \in Lip(\mathbf{R})$ whose values of b_k are very complicated (see [8]).*

- **Step 2 (Continuity and Uniform Convergence)**
  As for the continuity (at $t = 0$), with

$$f_0(0) = \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}0 - k) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} b_{-1},$$

by (3) we get

$$\begin{aligned} |f_0(t) - f_0(0)| &\leq \Big| \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} \Big| |b_{-1}| + \Big| \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} 2^{j+1} \Big| |b_0 - b_{-1}| t \\ &\quad + \Big| \sum_{j=n-1}^{\infty} \sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t - k) \Big| \\ &=: I' + II' + III'. \end{aligned}$$

We can estimate I', II' and III' as follows:

$$\begin{aligned} I' &= \Big| \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} \Big| (\sqrt{3} - 1) = \Big| \int_0^1 \frac{(-z)^{n-1}}{1+z} dz \Big| (\sqrt{3} - 1) \\ &\leq \int_0^1 \frac{z^{n-1}}{1+0} dz (\sqrt{3} - 1) \leq \frac{C}{n}, \\ II' &\leq \Big| \sum_{j=0}^{n-2} \frac{(-2)^j}{j+1} \Big| \cdot 2 |b_0 - b_{-1}| t_{n-1} \\ &= \Big| 2^{-1} \log 3 - 2^{n-1} \int_0^1 \frac{(-z)^{n-1}}{1+2z} dz \Big| \cdot 2 \Big| -\frac{1}{2}(3 + \sqrt{3}) \Big| 2^{-n+1} \\ &\leq \Big(2^{-1} \log 3 + 2^{n-1} \int_0^1 \frac{z^{n-1}}{1 + 2 \cdot 0} dz \Big) \cdot (3 + \sqrt{3}) 2^{-n+1} \leq \frac{C}{n}, \\ III' &\leq \sum_{j=n-1}^{\infty} \sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1} \frac{|b_k|}{j+1} \\ &\leq \frac{1}{n} \sum_{j=n-1}^{\infty} \Big(\sup_{2^{j-n+1}-1 \leq k \leq 2^{j-n+2}-1} |b_k| \Big) \sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1} 1 \\ &\leq \frac{1}{n} \sum_{j=n-1}^{\infty} 4(2 - \sqrt{3})^{2^{j-n+1}-1} (2^{j-n+1} + 1) \leq \frac{C}{n}. \end{aligned}$$

Thus we find that $|f_0(t) - f_0(0)| \to 0$ as $n \to \infty$ for $t \in [2^{-n}, 2^{-n+1}] = [t_n, t_{n-1}]$. This implies the right continuity of f_0. Similarly, the left continuity of f_0 follows from (4) instead of (3), and we get $f_0 \in C^0(\mathbf{R})$.

Remark 2.5. *We remark that f_0 is not only continuous but also uniformly continuous. More precisely, f_0 satisfies $|f_0(t) - f_0(s)| \leq C / \log |t-s|^{-1}$ for $0 < |t-s| < 1/2$, that is, f_0 has log-Hölder continuous.*

In general, f_0 is continuous if $f_J \to f_0$ uniformly, but the converse does not hold. It is known that $f_J \to f_0$ uniformly if $\{f_J(t)\}_J$ is uniformly equicontinuous and $f_J(t) \to f_0(t)$ pointwise. Let us take the sequence of partial sums

$$f_J(t) = \sum_{j=0}^{J} \frac{(-1)^j}{(j+1) 2^{\frac{j}{2}}} \psi_{j,0}^{St}(t).$$

Similarly, we also find that $|f_J(t) - f_J(0)| \to 0$ *as* $n \to \infty$ *for* $t \in [t_n, t_{n-1}]$. *We remark that the corresponding* I'_J, II'_J *and* III'_J *tend to* 0 *independently of* J. *This means that* $\{f_J(t)\}_J$ *is uniformly equicontinuous. We see that* $f_J(t_0) \to f_0(t_0)$ *for a fixed* $t_0 \neq 0$, *since* $|b_k|$ *is rapidly decreasing at* $\pm\infty$ *and*

$$f_J(t_0) = \sum_{j=0}^{J} \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \psi^{St}_{j,0}(t_0) = \sum_{j=0}^{J} \sum_{k\in\mathbf{Z}} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t_0 - k) \sim \sum_{j=0}^{J} \frac{(-1)^j}{j+1} b_{[2^{j+1}t_0]},$$

where $[\alpha]$ *is the largest integer not greater than* α. *Meanwhile, we immediately see that* $f_J(0) \to f_0(0)$ *as an alternating series. Thus,* $f_J(t) \to f_0(t)$ *pointwise and we can conclude that* $f_J \to f_0$ *uniformly.*

- **Step 3(Non-unconditional Convergence)**

The non-unconditional convergence implies that there exists a sequence $\beta_j \in \{1, -1\}$ such that the series $\sum_{j\in\mathbf{Z}} \beta_j c_j e_j(t)$ does not converge. In order to know the non-unconditional convergence of $f_0(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \psi^{St}_{j,0}(t)$, with $\beta_j = \overline{c_j}/|c_j|$ especially for $\sum_{j\in\mathbf{Z}} \beta_j c_j e_j(t)$ we shall consider the divergence of

$$\tilde{f}_0(t) := \sum_{j=0}^{\infty} \Big| \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \Big| \psi^{St}_{j,0}(t).$$

We remark that we can not deal with $\sum_{j=0}^{\infty} \Big| \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \Big| |\psi^{St}_{j,0}(t)|$ instead of $\tilde{f}_0(t)$. Let us define the interval $I_J = (2^{-J-3}, 2^{-J-3} + 2^{-J-4})$ for $J \geq 1$. Taking $L^\infty(\mathbf{R})$-norm, we have

$$\begin{aligned}
\|\tilde{f}_0\|_{L^\infty(\mathbf{R})} &\geq \|\tilde{f}_0\|_{L^\infty(I_J)} = \Big\| \sum_{j=0}^{\infty} \frac{\psi^{St}_{j,0}(t)}{(j+1)2^{\frac{j}{2}}} \Big\|_{L^\infty(I_J)} \\
&\geq \Big\| \sum_{j=0}^{J-1} \frac{\psi^{St}_{j,0}(t)}{(j+1)2^{\frac{j}{2}}} \Big\|_{L^\infty(I_J)} - \Big\| \sum_{j=J}^{\infty} \frac{\psi^{St}_{j,0}(t)}{(j+1)2^{\frac{j}{2}}} \Big\|_{L^\infty(I_J)} \\
&=: L_J - M_J.
\end{aligned}$$

We note that $0 < 2^{j+1}t - k < 2$ if $N_2(2^{j+1}t - k) \neq 0$, i.e., $2^{j+1}t - 2 < k < 2^{j+1}t$. Therefore we may consider $[2^{j+1}t] - 1 \leq k \leq [2^{j+1}t]$. As for the 1st term, we get

$$\begin{aligned}
L_J &\geq \operatorname*{ess\,sup}_{t\in I_J} \sum_{j=0}^{J-1} \frac{\psi^{St}_{j,0}(t)}{(j+1)2^{\frac{j}{2}}} = \operatorname*{ess\,sup}_{t\in I_J} \sum_{j=0}^{J-1} \frac{1}{j+1} \sum_{k\in\mathbf{Z}} b_k N_2(2^{j+1}t - k) \\
&= \operatorname*{ess\,sup}_{t\in I_J} \sum_{j=0}^{J-1} \frac{1}{j+1} \Big\{ b_{-1} N_2(2^{j+1}t + 1) + b_0 N_2(2^{j+1}t - 0) \Big\} \\
&= \operatorname*{ess\,sup}_{t\in I_J} \sum_{j=0}^{J-1} \frac{1}{j+1} \Big\{ b_{-1}(1 - 2^{j+1}t) + b_0 2^{j+1}t \Big\} \\
&= \operatorname*{ess\,sup}_{t\in I_J} \sum_{j=0}^{J-1} \frac{1}{j+1} \Big\{ b_{-1} - 2^{j+1}t(b_{-1} - b_0) \Big\}.
\end{aligned}$$

Here we used that $0 < 2^{j+1}t \leq \frac{3}{8}$ for $0 \leq j \leq J$ and $t \in I_J$. Since $b_{-1} = \sqrt{4 - 2\sqrt{3}} = \sqrt{3} - 1$ and $b_{-1} - b_0 = \frac{\sqrt{3}+3}{2}$, we see that

$$L_J \geq \sum_{j=0}^{J-1} \frac{1}{j+1} \Big\{ b_{-1} - 2^{j+1} \cdot (2^{-J-3} + 2^{-J-4}) \cdot (b_{-1} - b_0) \Big\}$$

$$
\begin{aligned}
&\geq \sum_{j=0}^{J-1} \frac{1}{j+1}\Big\{b_{-1} - 2^{(J-1)+1}\cdot(2^{-J-3}+2^{-J-4})\cdot(b_{-1}-b_0)\Big\} \\
&\geq \sum_{j=0}^{J-1} \frac{c}{j+1} \quad (c>0).
\end{aligned}
$$

As for the 2nd term, noting that $|b_k|$ is decreasing for $k \geq 0$, we get

$$
\begin{aligned}
M_J &\leq \operatorname*{ess.\,sup}_{t\in I_J} \sum_{j=J}^{\infty} \frac{|\psi_{j,0}^{St}(t)|}{(j+1)2^{\frac{j}{2}}} = \operatorname*{ess.\,sup}_{t\in I_J} \sum_{j=J}^{\infty} \frac{1}{j+1}\Big|\sum_{k\in\mathbf{Z}} b_k N_2(2^{j+1}t-k)\Big| \\
&= \operatorname*{ess.\,sup}_{t\in I_J} \sum_{j=J}^{\infty} \frac{1}{j+1}\Big|b_{[2^{j+1}t]-1} N_2(2^{j+1}t-[2^{j+1}t]+1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + b_{[2^{j+1}t]} N_2(2^{j+1}t-[2^{j+1}t])\Big| \\
&\leq \operatorname*{ess.\,sup}_{t\in I_J} \sum_{j=J}^{\infty} \frac{|b_{[2^{j+1}t]-1}|+|b_{[2^{j+1}t]}|}{j+1} \\
&\leq \frac{C}{J} + \operatorname*{ess.\,sup}_{t\in I_J} \sum_{j=J+2}^{\infty} \frac{2|b_{[2^{j+1}t]-1}|}{j+1} \leq \frac{C}{J} + \sum_{j=J+2}^{\infty} \frac{2|b_{2^{j-J-2}-1}|}{j+1}.
\end{aligned}
$$

Here we used that $[2^{j+1}t]-1 \geq [2^{j-J-2}]-1 = 2^{j-J-2}-1 (\geq 0)$ for $j \geq J+2$ and $t \in I_J$. Moreover, we easily see that

$$
M_J \leq \frac{C}{J} + c_1 + \sum_{h=2}^{\infty} \frac{8(2-\sqrt{3})^{2^h-1}}{J+h+2} \leq C' + \sum_{h=2}^{\infty} \frac{8(2-\sqrt{3})^h}{1+1+2} \leq C'',
$$

where C' is independent of J. Thus, it follows that

$$
\|\tilde{f}_0\|_{L^\infty(\mathbf{R})} \geq L_J - M_J \geq \sum_{j=0}^{J-1} \frac{c}{j+1} - C''.
$$

This holds for all $J \geq 0$, that is, $\|\tilde{f}_0\|_{L^\infty(\mathbf{R})} = \infty$ and this completes the proof of Theorem 2.2.

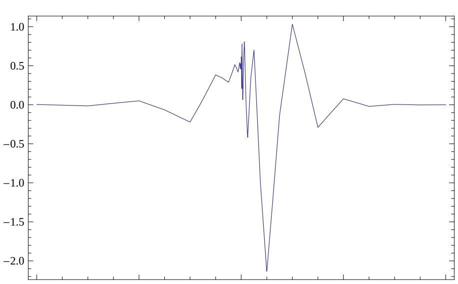

Figure 1: the graph of f_0

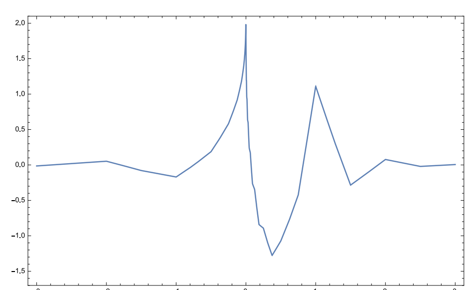

Figure 2: the graph of $\tilde{f}_0$

Remark 2.6. *If we take only $B^0(\mathbf{R})$-norm (*sup*-norm) instead of $L^\infty(\mathbf{R})$-norm (*ess. sup*-norm), by substituting $t=0$, we immediately find that*

$$
\|\tilde{f}_0\|_{B^0(\mathbf{R})} \geq \Big|\sum_{j=0}^{\infty}\Big|\frac{(-1)^j}{(j+1)2^{\frac{j}{2}}}\Big|\psi_{j,0}^{St}(0)\Big| = \Big|\sum_{j=0}^{\infty} \frac{\sum_{k\in\mathbf{Z}} b_k N_2(-k)}{j+1}\Big| = \sum_{j=0}^{\infty} \frac{b_{-1}}{j+1} = \infty.
$$

In the above estimate of $\|\tilde{f}_0\|_{L^\infty(\mathbf{R})}$ *, we used the sequence of the interval* $\{I_J\}$ *since the essential supremum excludes the measure zero set* $\{t=0\}$*. We also remark that* $\sum_{j=0}^{\infty}\left|\frac{(-1)^j}{(j+1)2^{\frac{j}{2}}}\right|\psi_{j,0}^{St}(t)$ *does not converge to* $\tilde{f}_0(t)$ *uniformly (while* $\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)2^{\frac{j}{2}}}\psi_{j,0}^{St}(t)$ *converges to* $f_0(t)$ *uniformly).*

3 Takagi Function and its Wavelet Expansion

3.1 Takagi Function

The Takagi function is well-known as a nowhere differentiable continuous function. Let $S(x) = \min_{k\in\mathbf{N}}|x-k|$ be the sawtooth function. Then the Takagi function is defined by

$$T(x) = \sum_{j=0}^{\infty} 2^{-j} S(2^j x). \tag{6}$$

Using B-spline $N_2(x)$, restricted the support only on the interval $[0,1]$ and multiply the height by 2, we can get another representation of the Takagi function as

$$T(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} 2^{-j-1} N_2(2^{j+1}x - 2k).$$

From this representation, we see that the Takagi function can be constructed from dilations and translations of a single function as in the case of wavelet expansions. In this section, we are going to consider the generalized Takagi function having the form of

$$F(t,x) = \sum_{j=0}^{\infty} c_j t^j G\Big(\Psi^j(x)\Big), \tag{7}$$

where Ψ^j is a j-fold iteration by Ψ. It is known that $c_j = (j+1)^{-1}$ is the critical case of differentiability in x (see [17]). Yamaguti and Hata [14] showed that (7) with $c_j = 1$ gives $F(t,x) = \sum_{j=0}^{J} t^j G(\Psi^j(x)) + t^{J+1}F(t,\Psi^{J+1}(x))$, and that by taking the limit as $J\to\infty$ the function $F(t,x)$ is characterized by the solution of the functional equation

$$F(t,x) = tF\Big(t,\Psi(x)\Big) + G(x), \quad (t,x) = [0,1)\times\mathbf{R},$$

with an initial function G such that supp $G\subset[0,1]$. As for $c_j=(j+1)^{-1}$, we prove the following:

Proposition 3.1. *The function* $F(t,x)$ *of (7) with* $c_j=(j+1)^{-1}$ *satisfies the functional equation*

$$F(t,x) = tF\Big(t,\Psi(x)\Big) - t\int_0^1 sF\Big(ts,\Psi(x)\Big)ds + G(x), \quad (t,x) = [0,1)\times\mathbf{R},$$

with an initial function G *such that* supp $G\subset[0,1]$.

Proof. Integration by parts gives

$$\int_0^1 1\cdot\int_0^s \zeta F\Big(t\zeta,\Psi^2(x)\Big)d\zeta ds = \int_0^1 sF\Big(ts,\Psi^2(x)\Big)ds - \int_0^1 s^2F\Big(ts,\Psi^2(x)\Big)ds.$$

Hence, we have

$$\begin{aligned}
& F(t,x) \\
= \;& t\Big\{tF\Big(t,\Psi^2(x)\Big) - t\int_0^1 sF\Big(ts,\Psi^2(x)\Big)ds + G\Big(\Psi(x)\Big)\Big\} \\
& -t\int_0^1 s\Big\{tsF\Big(ts,\Psi^2(x)\Big) - ts\int_0^1 \tau F\Big(ts\tau,\Psi^2(x)\Big)d\tau + G\Big(\Psi(x)\Big)\Big\}ds + G(x) \\
= \;& t^2F\Big(t,\Psi^2(x)\Big) + \sum_{j=0}^{1}\frac{t^j}{j+1}G\Big(\Psi^j(x)\Big) - t^2\int_0^1 sF\Big(ts,\Psi^2(x)\Big)ds \\
& -t^2\int_0^1 s^2F\Big(ts,\Psi^2(x)\Big)ds + t^2\int_0^1 1\cdot\int_0^s \zeta F\Big(t\zeta,\Psi^2(x)\Big)d\zeta ds \\
= \;& t^2F\Big(t,\Psi^2(x)\Big) + \sum_{j=0}^{1}\frac{t^j}{j+1}G\Big(\Psi^j(x)\Big) - 2t^2\int_0^1 s^2F\Big(ts,\Psi^2(x)\Big)ds.
\end{aligned}$$

Recursively, we also get

$$\begin{aligned}
& F(t,x) \\
= \;& t^2\Big\{tF\Big(t,\Psi^3(x)\Big) - t\int_0^1 sF\Big(ts,\Psi^3(x)\Big)ds + G\Big(\Psi^2(x)\Big)\Big\} + \sum_{j=0}^{1}\frac{t^j}{j+1}G\Big(\Psi^j(x)\Big) \\
& -2t^2\int_0^1 s^2\Big\{tsF\Big(ts,\Psi^3(x)\Big) - ts\int_0^1 \tau F\Big(ts\tau,\Psi^3(x)\Big)d\tau + G\Big(\Psi^2(x)\Big)\Big\}ds \\
= \;& t^3X\Big(t,\Psi^3(x)\Big) + \sum_{j=0}^{2}\frac{t^j}{j+1}G\Big(\Psi^j(x)\Big) - t^3\int_0^1 sF\Big(ts,\Psi^3(x)\Big)ds \\
& -2t^3\int_0^1 s^3F\Big(ts,\Psi^3(x)\Big)ds + 2t^3\int_0^1 s\cdot\int_0^s \zeta F\Big(t\zeta,\Psi^3(x)\Big)d\zeta ds \\
= \;& t^3F\Big(t,\Psi^3(x)\Big) + \sum_{j=0}^{2}\frac{t^j}{j+1}G\Big(\Psi^j(x)\Big) - 3t^3\int_0^1 s^3F\Big(ts,\Psi^3(x)\Big)ds \\
= \;& t^JF\Big(t,\Psi^J(x)\Big) + \sum_{j=0}^{J-1}\frac{t^j}{j+1}G\Big(\Psi^j(x)\Big) - Jt^J\int_0^1 s^JF\Big(ts,\Psi^J(x)\Big)ds.
\end{aligned}$$

If we take $J\to\infty$, since $0<t<1$ it follows that

$$F(t,x) = \sum_{j=0}^{\infty}\frac{t^j}{j+1}G\Big(\Psi^j(x)\Big).$$

□

As an application of (7) with $c_j = 1$, Yamaguti and Hata introduced in [14] the Takagi function by choosing $G(x) = \Psi(x) = N_2(2x)$ and $t = 2^{-1}$. We remark that the famous tent map is defined by $x_{n+1} = N_2(2x_n)$ and the iteration by $\Psi(x) = N_2(2x)$ yields a chaotic dynamical system in the sense of Devaney. In this paper, for better match with the wavelet analysis, we shall propose another chaotic dynamical system

$$B_2(x) = \begin{cases} 2x & \text{if } \; 0 < x \le 2^{-1}, \\ 2x-1 & \text{if } \; 2^{-1} < x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

which is called the Bernoulli shift map, and more generally

$$B_p(x) = \begin{cases} px & \text{if } \ 0 < x \le 1 \cdot p^{-1}, \\ px - 1 & \text{if } \ 1 \cdot p^{-1} < x \le 2 \cdot p^{-1}, \\ px - 2 & \text{if } \ 2 \cdot p^{-1} < x \le 3 \cdot p^{-1}, \\ \vdots & \\ px - (p-1) & \text{if } \ (p-1) \cdot p^{-1} < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2. *One can also take compactly supported (non-orthogonal) Riesz bases as* $G(x) = N_2(2x)$*. Then, we remark that* $G(\Psi^j(x))$ *with* $\Psi(x) = B_2(x)$ *coincides with the one with* $\Psi(x) = N_2(2x)$ *(the Takagi function case), i.e.,*

$$G\Big(B_2^j(x)\Big) = G^{j+1}(x). \tag{8}$$

The left-hand side enables us to detect the orbit from x *easier than the right-hand side. Besides* $G(x) = N_2(2x)$ *and* $\Psi(x) = B_2(x)$*,* G *can be generalized for the form* $G(B_p^j(x))$*.*

Now we shall take the initial (piecewise linear) function

$$G(x) = \sum_{k=0}^{p-2} g(k+1) N_2(px - k), \tag{9}$$

where $g(k) \in \mathbf{R}$ for $1 \le k \le p-1$. Since supp $G \subset [0,1]$, we also suppose that $g(0) = g(p) = 0$. For given $x \in [0,1]$, define the numbers $0 \le \xi_j \le p-1$ by the base-p numeral system

$$x = 0.\xi_1 \xi_2 \cdots = 0 + \xi_1 p^{-1} + \xi_2 p^{-2} + \cdots.$$

and define

$$D_J^{(p)} = \sum_{j=1}^{J} c_{j-1} \Big(g(\xi_j + 1) - g(\xi_j) \Big).$$

We call p-adic rational x of the form $x = Kp^{-j}$ with $K \in \mathbf{Z}$ and $j \in \mathbf{N}$, i.e., finite fraction. When x is a non p-adic rational, there exist infinite number of digits including "ξ_J" such that $\xi_J \neq 0$. Therefore, for the non p-adic rational x, we can take a subsequence $\{J_m^-\}$ such that $\xi_{J_m^-} \neq 0$ and put

$$r_m^- := c_{J_m^- - 1} \Big(2g(\xi_{J_m^-}) - g(\xi_{J_m^-} - 1) - g(\xi_{J_m^-} + 1) \Big) \xi_{J_m^-},$$

here we remark that $g(\xi_{J_m^-} - 1)$ is well-defined since $\xi_{J_m^-} \neq 0$ and $\xi_{J_m^-} - 1 \ge 0$.

We also note that there exist infinite number of digits including "ξ_J" such that $\xi_J \neq p-1$. Otherwise, after the last "ξ_J", we have $\xi_J \ (p-1) \ (p-1) \cdots$ which results in $(\xi_J + 1) \ 00 \cdots$. This contradicts that ξ_j is the last. Therefore, we can also take a subsequence $\{J_m^+\}$ such that $\xi_{J_m^+} \neq p-1$ for all x, and put

$$r_m^+ := c_{J_m^+ - 1} \Big(2g(\xi_{J_m^+} + 1) - g(\xi_{J_m^+}) - g(\xi_{J_m^+} + 2) \Big) (\xi_{J_m^+} + 1),$$

here we remark that $g(\xi_{J_m^+} + 2)$ is well-defined since $\xi_{J_m^+} \neq p-1$ and $\xi_{J_m^+} + 2 \le p$. We remark that $\lim_{m \to \infty} r_m^{\pm} = 0$ when $\lim_{j \to \infty} c_j = 0$.

Our purpose is to find a sufficient condition for the non-differentiability of the generalized Takagi function. For the Takagi function $T(x)$, Allaart and Kawamura [1] and Krüppel [18] paid attention very carefully to the left-hand side derivative, and independently discovered the necessary and sufficient condition for the improper infinite derivative $T'(x) = +\infty$.

Proposition 3.3 (Allaart and Kawamura, Krüppel). *Let $x \in (0,1)$ be non-dyadic, and write $x = \sum_{n=1}^{\infty} 2^{-a_n}$ where $\{a_n\}$ is a strickly increasing sequence of positive integers. Then, $T'(x) = \infty$ if and only if*

$$a_{n+1} - 2a_n + 2n - \log_2(a_{n+1} - a_n) \to -\infty,$$

In the Takagi function case, the parameter $t = 2^{-1}$ has been fixed in (7). For the general case $p \geq 2$, the different choice of $t = p^{-1}$ is crucial to deal with $B_p(x)$. Indeed, taking into a consideration the case $c_j = (j+1)^{-1}$ as in Proposition 3.1, we can prove the following theorem. This is a generalization of Proposition 3.3.

Theorem 3.4. *Let $p \geq 2$ and $g(k) \in \mathbf{R}$ for $1 \leq k \leq p-1$ ($g(0) = g(p) = 0$). Suppose that $t = p^{-1}$, $\Psi(x) = B_p(x)$, $G(x) = \sum_{k=0}^{p-2} g_{k+1} N_2(px - k)$, and put*

$$\mathbf{T}(x) := F(p^{-1}, x) = \sum_{j=0}^{\infty} c_j p^{-j} G\Big(B_p^j(x)\Big).$$

Then, $\mathbf{T}(x)$ is not differentiable at $x \in [0,1]$, if one of the following holds:
(i) $\left\{D_{J_m^+}^{(p)} + r_m^+\right\}_{m \in \mathbf{N}}$ *does not converge;*
(ii) $\left\{D_{J_m^-}^{(p)} + r_m^-\right\}_{m \in \mathbf{N}}$ *does not converge if x is a non p-adic rational;*
(iii) $\lim_{m\to\infty}\left(D_{J_m^+}^{(p)} + r_m^+\right) \neq \lim_{m\to\infty}\left(D_{J_m^-}^{(p)} + r_m^-\right)$ *if x is a non p-adic rational.*

In the Takagi function $T(x)$, we see that $p = 2$, $c_j = 1$, $G(x) = g(1)N_2(2x)$ with $g(1) = 1$ ($g(0) = g(2) = 0$) and

$$D_J^{(2)} = \sum_{j=1}^{J}(1 - 2\xi_j) = \sum_{j=1}^{J}(-1)^{\xi_j}.$$

On the other hand, we are forced to take $\xi_{J_m^+} = 0$ and $\xi_{J_m^-} = 1$ for $p = 2$, and then $r_m^{\pm} = 2$ for all $m \in \mathbf{N}$. Therefore, for p-adic rationals, we get the non-differentiability of $T(x)$ immediately by (i). As for non p-adic rationals, it is possible that (i) or (ii) does not work. For instance, for the non p-adic rational $x = 0.101010\cdots$, we see that $J_m^+ = 2m$ and $J_m^- = 2m-1$ and have

$$D_{J_m^+}^{(2)} = \sum_{j=1}^{2m}(1 - 2\xi_j) = (1-1) + \cdots + (1-1) = 0 \quad \text{for all } m \in \mathbf{N},$$

and

$$D_{J_m^-}^{(2)} = \sum_{j=1}^{2m-1}(1 - 2\xi_j) = 1 + (-1+1) + \cdots + (-1+1) = 1 \quad \text{for all } m \in \mathbf{N},$$

and $r_m^{\pm} = 2$ for all $m \in \mathbf{N}$. Thanks to (iii), we can know that $T(x)$ is not differentiable at $x = 0.101010\cdots$. For the case $p = 2$, we need not consider $r_m^{\pm}$ which plays an important role in the case $p \neq 2$ unless $\lim_{j\to\infty} c_j = 0$ (see Example 3.7).

Example 3.5. ($p = 3$)*: Let us consider*

$$G(x) = g(1)N_2(3x) + g(2)N_2(3x-1)$$

with $g(1) = 1$, $g(2) = -1$ ($g(0) = g(3) = 0$), and $\mathbf{T}(x) = \sum_{j=0}^{\infty} c_j 3^{-j} G\Big(B_3^j(x)\Big)$. The Cantor set C is the uncountable set having measure zero of real numbers whose ternary expansion in base 3 does not contain the digit 1. Then we have for $x \in C$

$$D_{J_m^{\pm}}^{(3)} = \sum_{j=1}^{J_m^{\pm}} c_{j-1}\Big(g(\xi_j + 1) - g(\xi_j)\Big) = \sum_{j=1}^{J_m^{\pm}} c_{j-1},$$

and $r_m^{\pm} = \pm 3c_{J_m^{\pm}-1}$ *since* $\xi_{J_m^+} = 0$ *and* $\xi_{J_m^-} = 2$*. Applying Theorem 2.2 with not only* $c_j = 1$ *but also* $c_j = (j+1)^{-1}$*, we see that* $\mathbf{T}(x)$ *is not differentiable at* $x \in C$*.*

Remark 3.6. *We remark that* $\{c_j\} = \{(j+1)^{-1}\}$ *in Example 3.5 (*$p=3$*) belongs to* ℓ^2*. Kôno [17] considered* $\{c_j\}$ *also for the Takagi function* $T(x)$ *and showed that* $T(x)$ *is absolutely continuous and hence differentiable almost everywhere if* $\{c_j\} \in \ell^2$*. So, the measure-zero set* C *in Example 3.5 is excluded for the differentiability.*

Example 3.7. **(**$p = 4$**)***: Let us consider "table-top" function*

$$G(x) = g(1)N_2(4x) + g(2)N_2(4x-1) + g(3)N_2(4x-2)$$

with $g(1) = g(2) = g(3) = 1$ *(*$g(0) = g(4) = 0$*), and* $\mathbf{T}(x) = \sum_{j=0}^{\infty} 4^{-j} G\Big(B_4^j(x)\Big)$*. In fact, this* $\mathbf{T}(x)$ *coincides with the Takagi function* $T(x)$ *in the sense that*

$$T(x) = \lim_{m\to\infty} \sum_{j=0}^{2m-1} 2^{-j} N_2\Big(2B_2^j(x)\Big) = \sum_{j=0}^{\infty} \Big(2^{-2j-1} N_2\Big(2B_2^{2j+1}(x)\Big) + 2^{-2j} N_2\Big(2B_2^{2j}(x)\Big)\Big).$$

We remark that $x = 0.1212\cdots$ *is a undesirable point for* $D_{J_m}^{(4)}$*. Indeed, we find that* $J_m^{\pm} = m$ *and have*

$$D_{J_m^{\pm}}^{(4)} = \sum_{j=1}^{m} \Big(g(\xi_j + 1) - g(\xi_j)\Big) = 0 \quad \text{for all } m \in \mathbf{N},$$

and $r_m^+ = 3(\xi_m - 1)$ *and* $r_m^- = (2 - \xi_m)$*. Applying Theorem 3.4 with* $c_j = 1$*, we see that* $\mathbf{T}(x)$ *is not differentiable at* $x = 0.1212\cdots$*, due to the oscillation of* $r_m^{\pm}$*.*

The Baire category theorem also proves abstractly the existence of nowhere differentiable continuous functions. Indeed, the Baire category theorem says that a non-empty complete metric space $X = C^0[0,1]$ is not the countable union of nowhere dense closed sets V_j $(j \in \mathbf{N})$ defined by

$$X_j = \Big\{ f \in X;\ \min_{x\in[0,1]} \sup_{h\neq 0} \Big| \frac{f(x+h) - f(x)}{h} \Big| \leq j \Big\}.$$

Thus, $f \in X - \cup_{j=1}^{\infty} X_j \neq \phi$ means that $f \notin X_j$ for all $j \in \mathbf{N}$ and hence

$$\lim_{h\to 0} \Big| \frac{f(x+h) - f(x)}{h} \Big| = +\infty.$$

If we also consider $X = L^2(\mathbf{R})$ and the multiresolution space $X_j = V_j$ where piecewise linear continuous (Lipschitz continuous) functions on the intervals $[k, k+1]$ for all $k \in \mathbf{Z}$, is given by the Riesz basis $\{N_2(2^j x - k); k \in \mathbf{Z}\}$, then we can know the existence of functions which are nowhere piecewise linear continuous. This suggests that nowhere Lipschitz continuous functions could be expanded with MRA wavelets.

Now we devote ourselves to the absolutely continuous $T(x)$ with $c_j = (j+1)^{-1}$ (see [17]). The absolute continuity allows us to differentiate $T(x)$ under the integral excluding the measure zero set. Then we can prove the following:

Theorem 3.8. *For* $c_j = 1$*,* $(j+1)^{-1}$*, the function*

$$T(x) = \sum_{j=0}^{\infty} (j+1)^{-1} 2^{-j} N_2\Big(2B_2^j(x)\Big)$$

can be expanded as

$$T(x) = \sum_{J \in \mathbf{Z}} \sum_{K \in \mathbf{Z}} d_{J,K} \psi^H_{J,K}(x),$$

where ψ^H *is the Haar wavelet and*

$$d_{J,K} = \begin{cases} 2^{-3/2J} \sum_{j=0}^{J-1} c_j \left(2 \left[\dfrac{K \bmod 2^{J-j}}{2^{J-j-1}} \right] - 1 \right) & \text{if } 0 \le j \le J-1, K \ge 0, \\ 2^{J/2-1} \sum_{j=0}^{\infty} c_j 2^{-j} & \text{if } J \le -1, K = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

In paticular, for $c_j = (j+1)^{-1}$*, we have* $\|T\|^2_{L^2(\mathbf{R})} = \frac{1}{3}\mathrm{Li}_2\left(\frac{1}{4}\right) + (\log 2)^2$ *and* $\|T'\|^2_{L^2(\mathbf{R})} = \frac{2}{3}\pi^2$ *where* $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ *is called polylogarithm.*

We can know from the coefficients $d_{J,K}$ that the translation parameter K indicate singularity points in of $T(x)$ and the terms with J in the coefficients $d_{J,K}$ defines the value of $T(x)$. We think that this behavior of coefficients explain the characteristics of the Takagi function $T(x)$.

Remark 3.9. *By using the Sobolev embedding theorem, we can find that* $T \in C^{1/2}(\mathbf{R})$ *since* $\|T\|_{W^{1,2}(\mathbf{R})} \Big(= \|T\|_{L^2(\mathbf{R})} + \|T'\|_{L^2(\mathbf{R})} \Big) < \infty$*. But, it is known that even the Takagi function with* $c_j = 1$ *is Hölder continuous of any order* $\alpha < 1$ *(see [2], [23]).*

3.2 Proof of Theorem 3.4

Let us consider

$$\mathbf{T}(x) = \sum_{j=0}^{\infty} c_j p^{-j} G\Big(\Psi^j(x)\Big).$$

As stated in (8), the form $G(\Psi^j(x))$ with $\Psi(x) = B_p(x)$ immediately gives

$$\Psi^j(x) = \begin{cases} p^j x & \text{if } 0 < x \le 1 \cdot p^{-j}, \\ p^j x - 1 & \text{if } 1 \cdot p^{-j} < x \le 2 \cdot p^{-j}, \\ p^j x - 2 & \text{if } 2 \cdot p^{-j} < x \le 3 \cdot p^{-j}, \\ \vdots & \\ p^j x - (p^j - 1) & \text{if } (p^j - 1) \cdot p^{-j} < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

At first, we shall suppose that $x = x_J$ is a p-adic rational and put

$$x_J = 0.\xi_1 \xi_2 \cdots \xi_J = \sum_{i=1}^{J} \xi_i p^{-i} \quad (\xi_J \ne 0).$$

• For the level $j = 0$, obviously we get

$$c_0 p^{-0} G(\Psi^0(x_J)) = c_0 G(x_J) = c_0 G\Big(\sum_{i=1}^{J} \xi_i p^{-i}\Big).$$

• For the level $j = 1$, we rewrite $\Psi(x)$ as

$$\Psi(x) = \begin{cases} px & \text{if } 0 < (x - 0 \cdot p^{-1}) \le p^{-1}, \\ p(x - q \cdot p^{-1}) & \text{if } 0 < (x - q \cdot p^{-1}) \le p^{-1} \text{ for } 1 \le q \le p-1, \\ 0 & \text{otherwise.} \end{cases}$$

For all cases $0 \le \xi_1 (= q) \le p-1$, we find that

$$\begin{aligned} c_1 p^{-1} G(\Psi^1(x_J)) &= c_1 p^{-1} G\Big(p\Big(\sum_{i=1}^{J} \xi_i p^{-i} - \xi_1 \cdot p^{-1}\Big)\Big) \\ &= \begin{cases} c_1 p^{-1} G(0) & \text{if } \ J = j, \\ c_1 p^{-1} G\Big(\sum_{i=2}^{J} \xi_i p^{1-i}\Big) & \text{if } \ J \ge j+1. \end{cases} \end{aligned}$$

• For the level $j = 2$, we also rewrite $\Psi^2(x)$ as

$$\Psi^2(x) = \begin{cases} p^2 x & \text{if } \ 0 < (x - 0 \cdot p^{-1} - 0 \cdot p^{-2}) \le p^{-2}, \\ p^2(x - q \cdot p^{-1} - r \cdot p^{-2}) & \text{if } \ 0 < (x - q \cdot p^{-1} - r \cdot p^{-2}) \le p^{-2}, \\ & \text{for } \ 0 \le q, r \le p-1 \ \text{ and } \ q + r \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

For all cases $0 \le \xi_1 (= q), \xi_2 (= r) \le p-1$, we find that

$$\begin{aligned} c_2 p^{-2} G(\Psi^2(x_J)) &= c_2 p^{-2} G\Big(p\Big(\sum_{i=1}^{J} \xi_i p^{-i} - \xi_1 \cdot p^{-1} - \xi_2 \cdot r^{-1}\Big)\Big) \\ &= \begin{cases} c_2 p^{-2} G(0) & \text{if } \ 1 \le J \le j, \\ c_2 p^{-2} G\Big(\sum_{i=3}^{J} \xi_i p^{2-i}\Big) & \text{if } \ J \ge j+1. \end{cases} \end{aligned}$$

• For the level $j = J-1$, similarly we get

$$c_{J-1} p^{-(J-1)} G(\Psi^{J-1}(x_J)) = \begin{cases} c_{J-1} p^{-(J-1)} G(0) & \text{if } \ 1 \le J \le j, \\ c_{J-1} p^{-(J-1)} G\Big(\sum_{i=J}^{J} \xi_i p^{(J-1)-i}\Big) & \text{if } \ J \ge j+1. \end{cases}$$

• For the level $j = J$, similarly we get

$$c_J p^{-J} G(\Psi^J(x_J)) = \begin{cases} c_J p^{-J} G(0) & \text{if } \ 1 \le J \le j, \\ c_J p^{-J} G\Big(\sum_{i=J+1}^{J} \xi_i p^{J-i}\Big) & \text{if } \ J \ge j+1. \end{cases}$$

We remark that $\sum_{i=J+1}^{J}$ does not make a sense. So, we see that

$$c_J p^{-J} G(\Psi^J(x_J)) = c_J p^{-J} G(0).$$

• For the level $j \ge J+1$, we also see that

$$c_j p^{-j} G(\Psi^j(x_J)) = c_j p^{-j} G(0).$$

Thus, noting that $G(0) = 0$, we have

$$\mathbf{T}(x_J) = \sum_{j=0}^{J-1} c_j p^{-j} G\Big(\sum_{i=j+1}^{J} \xi_i p^{j-i}\Big) = \sum_{j=1}^{J} c_{j-1} p^{-j+1} G\Big(\sum_{i=j}^{J} \xi_i p^{j-i-1}\Big).$$

Since G, given by (9), is a piecewise linear function, we rewrite G as

$$G(x) = \begin{cases} g(1)px & \text{if } \ 0 < x \leq p^{-1}, \\ \Big(g(q+1) - g(q)\Big)px + \Big(g(q) - g(q+1)\Big)q + g(q) & \\ & \text{if } \ q \cdot p^{-1} < x \leq (q+1) \cdot p^{-1} \text{for } \ 1 \leq q \leq p-1, \\ \quad 0 & \text{otherwise.} \end{cases}$$

Hence, for all the cases $0 \leq \xi_j (= q) \leq p - 1$ we find that

$$G\Big(\sum_{i=j}^{J} \xi_i p^{j-i-1}\Big) = \Big(g(\xi_j + 1) - g(\xi_j)\Big)p\Big(\sum_{i=j}^{J} \xi_i p^{j-i-1}\Big) + \Big(g(\xi_j) - g(\xi_j + 1)\Big)\xi_j + g(\xi_j).$$

Therefore, exchanging the order of integration, we have

$$\begin{aligned} \mathbf{T}(x_J) &= \sum_{j=1}^{J} c_{j-1}\Big(g(\xi_j + 1) - g(\xi_j)\Big)\Big(\sum_{i=j}^{J} \xi_i p^{-i+1}\Big) \\ &\quad + \sum_{j=1}^{J} c_{j-1} p^{-j+1}\Big\{\Big(g(\xi_j) - g(\xi_j + 1)\Big)\xi_j + g(\xi_j)\Big\} \\ &= \sum_{i=1}^{J} \xi_i p^{-i+1} \sum_{j=1}^{i} c_{j-1}\Big(g(\xi_j + 1) - g(\xi_j)\Big) \\ &\quad + \sum_{i=1}^{J} c_{i-1} p^{-i+1}\Big\{\Big(g(\xi_i) - g(\xi_i + 1)\Big)\xi_i + g(\xi_i)\Big\} \\ &= \sum_{i=1}^{J} p^{-i+1}\Big\{\xi_i D_i^{(p)} + c_{i-1}\Big\{\Big(g(\xi_i) - g(\xi_i + 1)\Big)\xi_i + g(\xi_i)\Big\}\Big\}. \end{aligned}$$

Taking $J \to +\infty$, we have

$$\mathbf{T}(x_\infty) = \sum_{i=1}^{\infty} p^{-i+1}\Big\{\xi_i D_i^{(p)} + c_{i-1}\Big\{\Big(g(\xi_i) - g(\xi_i + 1)\Big)\xi_i + g(\xi_i)\Big\}\Big\}$$

for the non p-adic rational x_∞. We remark that this representation has a meaning also for p-adic rational by regarding x_∞ as $x_\infty = 0.\xi_1\xi_2 \cdots \xi_J 00 \cdots$.

As for the right-hand side derivative, we see that $\xi_{J_m^+} \neq p - 1$ for all $m \in \mathbf{N}$. So, let us put

$$x_m := x_\infty + p^{-J_m^+} = 0.\xi_1\xi_2 \cdots \xi_{J_m^+ - 1}\ (\xi_{J_m^+} + 1)\ \xi_{J_m^+ + 1} \cdots.$$

Then, it follows that

$$\begin{aligned} \frac{\mathbf{T}(x_m) - \mathbf{T}(x_\infty)}{x_m - x_\infty} &= pD_{J_m^+}^{(p)} + pc_{J_m^+ - 1}\Big\{\Big(g(\xi_{J_m^+} + 1) - g(\xi_{J_m^+} + 2)\Big)(\xi_{J_m^+} + 1) \\ &\qquad + g(\xi_{J_m^+} + 1) - \Big(g(\xi_{J_m^+}) - g(\xi_{J_m^+} + 1)\Big)\xi_{J_m^+} - g(\xi_{J_m^+})\Big\} \\ &= p\Big(D_{J_m^+}^{(p)} + r_m^+\Big). \end{aligned}$$

As for the left-hand side derivative, we see that $\xi_{J_m^-} \neq 0$ for all $m \in \mathbf{N}$. So, let us put

$$x_m := x_\infty - p^{-J_m^-} = 0.\xi_1\xi_2 \cdots \xi_{J_m^- - 1}\ (\xi_{J_m^-} - 1)\ \xi_{J_m^- + 1} \cdots.$$

Then, it follows that

$$\begin{aligned}\frac{\mathbf{T}(x_\infty)-\mathbf{T}(x_m)}{x_\infty - x_m} &= pD^{(p)}_{J_m^-} + pc_{J_m^- -1}\Big\{\Big(g(\xi_{J_m^-}) - g(\xi_{J_m^-}+1)\Big)\xi_{J_m^-} + g(\xi_{J_m^-}) \\ &\quad -\Big(g(\xi_{J_m^-}-1) - g(\xi_{J_m^-})\Big)(\xi_{J_m^-}-1) - g(\xi_{J_m^-}-1)\Big\} \\ &= p\Big(D^{(p)}_{J_m^-} + r_m^-\Big).\end{aligned}$$

Thus, we can conclude that $\mathbf{T}(x_\infty)$ is not differentiable at $x_\infty \in [0,1]$, if one of (i), (ii) and (iii) holds. This completes the proof of Theorem 2.2.

3.3 Proof of Theorem 3.8

First, let us compute the Fourier expansion of $T(x)$ with $c_j = (j+1)^{-1}$ on [0,1]. Another representation of $T(x)$ using sawtooth function $S(x)$ is $\tilde{T}(x) = \sum_{j=0}^{\infty}(j+1)^{-1}2^{-j+1}S(2^j x)$ on $[0,1]$. The Fourier expansion of $S(x)$ is given by

$$S(x) = \sum_{n\in\mathbf{Z}} \beta_n e^{2\pi i n x}.$$

We can get the Fourier coefficients β_n as

$$\int_0^1 S(2x)e^{-2\pi i n x}dx = \begin{cases} -\frac{1}{\pi^2 n^2} & \text{if } n \text{ is odd },\\ 0 & \text{if } n \text{ is even } (\neq 0),\\ \frac{1}{4} & \text{if } n = 0.\end{cases}$$

Since $S(2^j x) = \sum_{n\in\mathbf{Z}} \beta_n e^{2\pi i (2^j n)x}$, the Fourier coefficients β_n^j of $S(2^j x)$ is

$$\beta_n^j = \begin{cases} \beta_{2k+1} & \text{if } n = 2^j(2k+1) \text{ with some } k \in \mathbf{Z}\ ,\\ \frac{1}{4} & \text{if } n = 0,\\ 0 & \text{otherwise.}\end{cases}$$

Noting that $\tilde{T}(x)$ converges uniformly, we see that,

$$\begin{aligned}\tilde{T}(x) &= \sum_{j=0\infty}(j+1)^{-1}2^{-j+1}S(2^j x) \\ &= \sum_{j=0}^{\infty}(j+1)^{-1}2^{-j+1}\sum_{n\in\mathbf{Z}}\beta_n^j e^{2\pi i n x} \\ &= \sum_{n\in\mathbf{Z}}\left(\sum_{j=0}^{\infty}(j+1)^{-1}2^{-j+1}\beta_n^j\right)e^{2\pi i n x}.\end{aligned}$$

So, using the Parseval theorem, we can compute the L^2-norm of $T(x)$ as

$$\begin{aligned}\|T\|_{L^2(\mathbf{R})} &= \|\tilde{T}\|_{L^2[0,1]} \\ &= \sum_{n\in\mathbf{Z}}\left(\sum_{j=0}^{\infty}(j+1)^{-1}2^{-j+1}\beta_n^j\right)^2 \\ &= \sum_{m=0}^{\infty}\sum_{k\in\mathbf{Z}}\left(\frac{1}{2^{m-1}(m+1)(2k+1)\pi^2}\right)^2 + \left(\sum_{j=0}^{\infty}(j+1)^{-1}2^{-j+1}\frac{1}{4}\right)^2 \\ &= \frac{1}{3}\mathrm{Li}_2\left(\frac{1}{4}\right) + (\log 2)^2.\end{aligned}$$

The wavelet coefficients of $T(x)$ with the Haar wavelet(10) is given by the inner product: i.e.,

$$d_{J,K} = \int_{\mathbf{R}} T(t)\overline{\psi^H_{J,K}(t)}dt = \sum_{j=0}^{\infty} c_j 2^{-j} 2^{J/2} \int_0^1 N_2\left(2B_2^j(t)\right) \overline{\psi^H(2^J t - K)}dt.$$

• For $K \leq -1$, or $0 \leq J$ and $2^J \leq K$, the supports of $N_2\left(2B_2^j(t)\right)$ and $\psi^H(2^J t - K)$ are disjoint. So, $d_{J,K} = 0$.
• For $0 \leq J \leq j$ and $0 \leq K \leq 2^J - 1$, $N_2\left(2B_2^j(x)\right)$ has 2C number of hats in supp $\psi^H_{J,K}$. Then, change of variables gives

$$\int_0^1 N_2\left(2B_2^j(t)\right) \overline{\psi^H(2^J t - K)}dt = C\int_0^{1/2^j} N_2\left(2B_2^j(t)\right)dt - C\int_0^{1/2^j} N_2\left(2B_2^j(t)\right)dt = 0$$

and $d_{J,K} = 0$.
• For $J \leq -1$ and $K = 0$, we find that

$$\begin{aligned} d_{J,K} &= \sum_{j=0}^{\infty} c_j 2^{-j} 2^{J/2} \int_0^1 N_2\left(2B_2^j(t)\right) \overline{\psi^H(2^J t - K)}dt \\ &= \sum_{j=0}^{\infty} c_j 2^{-j} 2^{J/2} \int_0^1 N_2\left(2B_2^j(t)\right)dt \\ &= 2^{J/2-1} \sum_{j=0}^{\infty} c_j 2^{-j} \end{aligned}$$

• For $0 \leq j \leq J-1$ and $2^{J-j}\ell \leq K \leq 2^{J-j-1}(2\ell+1) - 1$ ($\ell = 0, 1, \ldots, 2^j - 1$), by the definition of the Haar wavelet,

$$\begin{aligned} \int_0^1 N_2\left(2B_2^j(t)\right)\overline{\psi(2^J t - K)}dt &= \int_{2^{-J}K}^{2^{-J}(K+\frac{1}{2})} 2^{j+1}t dt - \int_{2^{-J}(K+\frac{1}{2})}^{2^{-J}(K+1)} 2^{j+1}t dt \\ &= -2^{j-2J-1}. \end{aligned}$$

• For $0 \leq j \leq J-1$ and $2^{J-j-1}(2\ell+1) \leq K \leq 2^{J-j}(\ell+1) - 1(\ell = 0, 1, \ldots, 2^j - 1)$, in the same way as above,

$$\begin{aligned} \int_0^1 N_2\left(2B_2^j(t)\right)\overline{\psi(2^J t - K)}dt &= \int_{2^{-J}K}^{2^{-J}(K+\frac{1}{2})} (2 - 2^{j+1}t)dt - \int_{2^{-J}(K+\frac{1}{2})}^{2^{-J}(K+1)} (2 - 2^{j+1}t)dt \\ &= 2^{j-2J-1}. \end{aligned}$$

We can rewrite the last two cases as

$$\int_0^1 N_2\left(2B_2^j(t)\right)\overline{\psi(2^J t - K)}dt = \left(2\left[\frac{K \bmod 2^{J-j}}{2^{J-j-1}}\right] - 1\right)2^{j-2J-1}$$

and we obtain (10).

Finally, we can also compute L^2-norm of the derivative $T'(x)$. Noting the fact that $\partial_x N_2(2x) = \psi^H(x)$, by the Parseval theorem we have

$$\|T'\|^2_{L^2(\mathbf{R})} = \left\| \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} (j+1)^{-1} 2^{-j} \partial_x \{N_2(2^{j+1}x - 2k)\} \right\|^2_{L^2(\mathbf{R})}$$

$$= \left\| \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} (j+1)^{-1} 2^{-j/2+1} \psi_{j,k}^H(x) \right\|_{L^2(\mathbf{R})}^2$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} (j+1)^{-2} 2^{-j+2} = \frac{2}{3}\pi^2.$$

This completes the proof of Theorem 3.4.

4 Feature Extraction of Distortion Sounds

4.1 Introduction

Music dictation is a very popular way to play music without using a score. However, the skill of music dictation depends on one's experience and sense. Recently, music using electric guitar employs guitar effectors like distortion, chorus, modulation and etc. We will focus on distortion which is one of the most famous guitar effectors. Our motivation is to improve music dictation regardless of one's skills and experience.

In this section, we assume the amplitude of the signal $f(t)$ is normalized, i.e., $\max_t |f(t)| = 1$ and has a compact support. Distortion effector is a nonlinear transform [20], which is constructed by the following two steps: amplification and clipping. Amplification is a transformation from the original signal $f(t)$ to $\check{f}(t) = Cf(t)$ for some constant $C > 1$. Clipping means cutting the signal off, which transforms from $\check{f}(t)$ to $\tilde{f}(t) = \max\{-1, \min\{1, \check{f}(t)\}\}$ (see Fig. 3). Since the distortion effector process has clipping, which is a nonlinear transformation, it is difficult to analyze distortion sounds by Fourier method.

From the preceding study, distortion is a relative measurement and there are several methods used for distortion. One of the values of the describing distortions level is *Total Harmonic Distortion* (D_{THD}) [5] [7] which is defined by,

$$D_{\mathrm{THD}} = \frac{\sqrt{H_2^2 + H_3^2 + \cdots + H_N^2}}{\sqrt{H_1^2 + H_2^2 + H_3^2 + \cdots + H_N^2}}$$

or

$$D_{\mathrm{THD}} = \frac{\sqrt{H_2^2 + H_3^2 + \cdots + H_N^2}}{H_1}$$

where H_N denotes the N-th harmonic response and H_1 does fundamental response and it is denoted in percent(%) or decibel(dB). Nevertheless, D_{THD} does not consider the original sound harmonic and we have to recognize the original sound if we want to calculate D_{THD}. Therefore, we defined the feature quantities of distortion sounds.

4.2 Proposed Method

Here, we consider the new feature of distortion sounds with wavelets. As mentioned, Fourier method is not suitable to analyze distortion sound because of clipping. We define three ways of extracting the feature of distortion sounds. First, we define the feature quantity based on the differential, that is, we would focus on amplification as

$$E_1(a) = \max_b \frac{1}{a} \left| \int_{\mathbf{R}} f(t) \psi^H \left(\frac{t-b}{a} \right) dt \right| ,$$

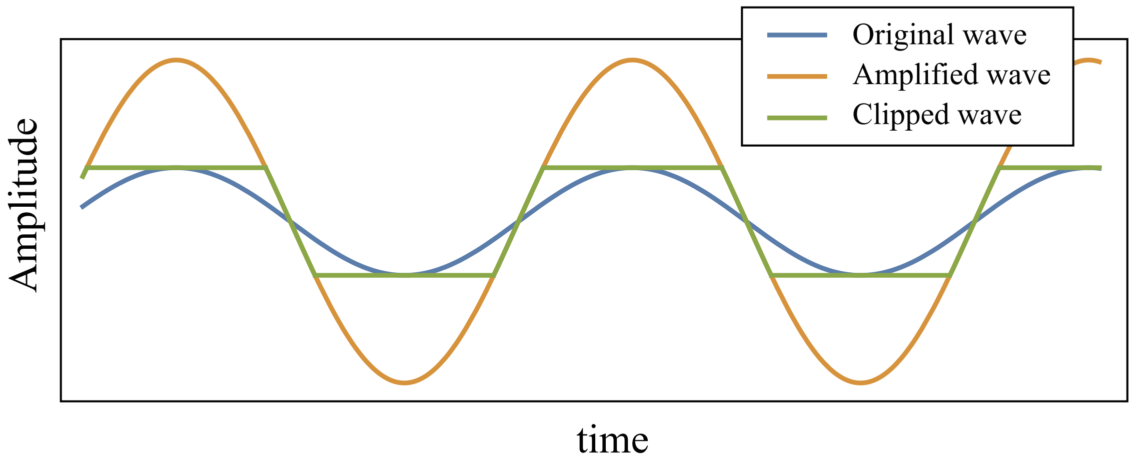

Figure 3: Schematic wave shape of $f(t)$, $\check{f}(t)$, and $\tilde{f}(t)$

where we choose $\psi(t)$ as Haar wavelet (10) and we identify the sounds are more distorted as this value is larger. The reason this E_1 is based on the differentiation is that it measures the gradient of the signal $f(t)$. The correlation with Haar wavelet can treat the gradient of a signal in the integral translation.

Because Haar wavelet has strong localization, we can extract the feature of distortion sound. Previous method would not be useful to analyze distortion sound since distortion filter is a nonlinear transform. The proposed method can be applied even though we do not know the original sound. For each pitch sound, we can choose the wavelet dilation $a \in \mathbf{R}$ to compare the feature of distortion sound. This method is independent of the state of signal (i.e., stable or unstable of sounds).

The second feature is based on the area of the signal.

$$E_2 \quad = \quad \int_{\mathbf{R}} |f(t)|^2 \, dt,$$

i.e., E_2 is the square of L^2-norm of the signal f. We identify the sounds are more distorted as these value is larger. Hence the area of the graph of the distortion sound get larger, we can judge the distortion level. However, this method depends on the state of the sounds. Therefore, for unsteady signal, we have to withdraw the one wave from the signal f and use the method. Moreover, because instruments that have a lot of harmonic tones make various waveforms for each time, different sounds cannot be compared in this way even if they are from same instrumental or same level of distortion.

The third feature focuses on the clipped part of the waveform. We define

$$E_3(a) = \int_{\mathbf{R}} \left| \int_{\mathbf{R}} |f(t)|^{1/4} \psi^H \left(\frac{t-b}{a} \right) dt \right| db$$

where ψ denotes the Haar wavelet, and we identify the sounds are more distorted as this value is smaller. The more clipped part the signal has, the less the value of E_3 is since $\int_{\mathbf{R}} \psi(t) dt = 0$. The inner integral is similar to E_1, differential way. Thus, we can think E_3 is the norm of the homogeneous Sobolev space $\dot{W}^{1,1}$. The 1/4 power of the signal makes the difference of clipped parts and others clear. This method also depends the state of sounds so we have to apply it for one wave.

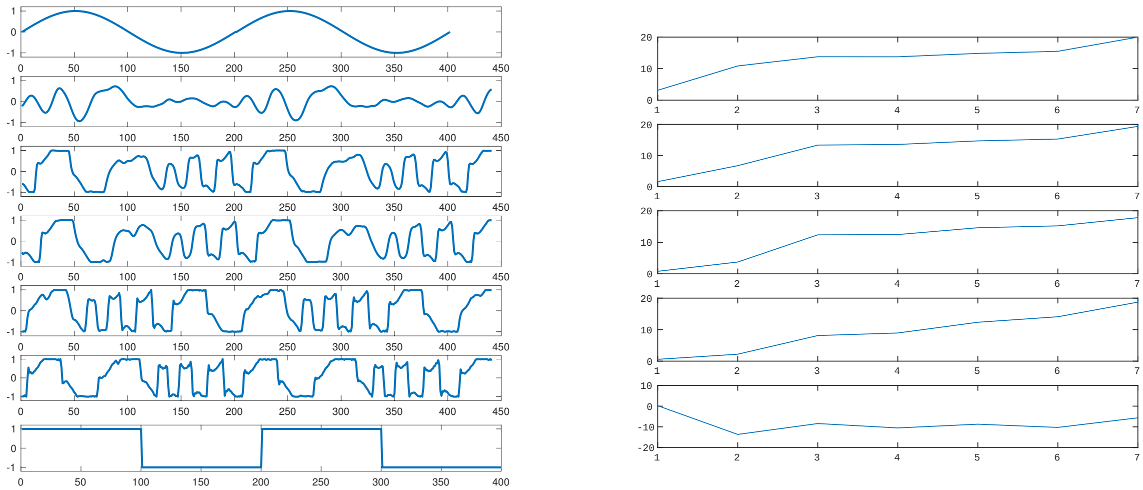

Figure 4: 7 levels of distortion sounds (left) and features on the proposed method (right)

4.3 Experiments on the Proposed Method

In the rest of this section, we focus on E_1 and analyze distortion sounds with it. In the left of Figure 4, there are 7 waves of sounds from the pure sound (top) to the most distorted sound, square wave (bottom). The right of Figure 4 is the features for 7 sounds with appropriate Haar wavelet scale. The scales of the wavelets are smaller from the top to the 4th graph and the bottom graph is the features on total harmonic distortion (THD). The vertical axes of the graphs on the right side of Figure 4 represent the features using the proposed method while the horizontal axes represent the seven sounds. On a horizontal axis, each number corresponds to a graph on the left side of Figure 4 with 1 corresponding to the topmost graph and 7 corresponding to the bottommost graph.

Figure 4 explains that proposed method extracts the feature of the distortion sounds since the upper four graphs grow. Conversely, the THD doesn't extract the distortion features. Therefore, we can conclude that the THD is inappropriate to analyze distortion sounds and we could define the appropriate feature to the distortion sounds.

4.4 Subjective Experiments

To compare our proposed method and human ears, we conducted subjective experiments for 14 people (4 people are experienced in playing the guitar and 10 are not). We prepared different 6 levels,18 sounds from clean sound to high level distortion sound. The experimental procedure is the following:

- Let subjects listen to some sounds to recognize distortion sounds.
- Let subjects listen to two sounds, and subject would answer which is higher level of distortion sounds.

We summarize the comparison of the subjective experiment result and our method in Fig. 4.4. The left side of figure is the result from people who had experienced the guitar, the right side is the one from people who had not experienced the guitar. The horizontal line shows that our proposed method and the vertical number denotes the order of distortion sound that human ears distinguish its level. The graphs in the Fig. 4.4 are growing and we can conclude our proposed method can

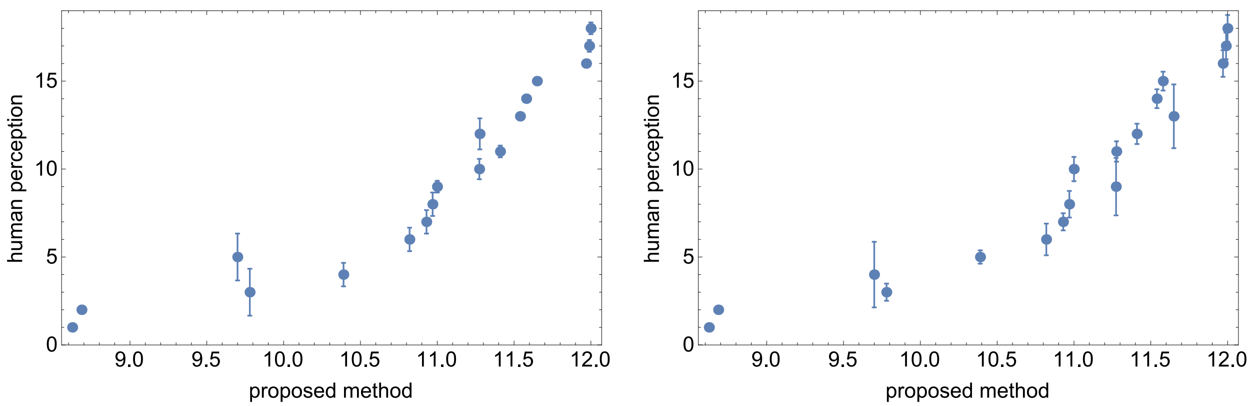

Figure 5: the experiment results with guitar experienced people (left) and guitar inexperienced people (right)

extract the distortion sound features that human auditory perception can listen to. Since this method can be utilized if there are no information about original sound, it is valuable to compare the distortion sounds.

As guitar experienced people are less, the error of the left graph in Fig. 5 is larger than the left ones. However, there is no big difference between the both graphs. This is the advantage of our proposal method since the its measure of distortion sound is independent of one's skills and experience.

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