On Diophantine exponents for non-Archimedean fields

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Abstract

In this thesis, we investigate Diophantine exponents w_n and w_n^* for non-Archimedean fields.

First, we study a relation between digits of p-adic numbers and Mahler's classification. We show that an irrational p-adic number which has a low complexity digits is an S-, T-, or U_1-number in the sense of Mahler's classification. Furthermore, we give an algebraic independence criterion for p-adic numbers whose digits are Sturmian.

Next, we study properties of Diophantine exponents w_n and w_n^* for Laurent series over a finite field. We prove that for an integer $n \geq 1$ and a rational number $w > 2n - 1$, there exist a strictly increasing sequence of positive integers $(k_j)_{j\geq 1}$ and a sequence of algebraic Laurent series $(\xi_j)_{j\geq 1}$ such that $\deg \xi_j = p^{k_j} + 1$ and

$$w_1(\xi_j) = w_1^*(\xi_j) = \ldots = w_n(\xi_j) = w_n^*(\xi_j) = w$$

for any $j \geq 1$. For an integer $n \geq 2$ and a real number $0 \leq \delta \leq 1$, we prove that there exist uncountably many Laurent series ξ for which $w_n(\xi) - w_n^*(\xi) = \delta$. Furthermore, we show that the set of values taken by $w_2 - w_2^*$ is the closed interval $[0, 1]$. We give a characterization that a continued fraction which has a low complexity partial quotient is U_2-number.

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Chapter 1

Introduction

1.1 Overview

The central problem of Diophantine approximation is how well a given real number is approximated by rational numbers. The irrational exponent is known as a function to measure quantities of rational approximation to a real number. Let ξ be a real number. We denote by the *irrationality exponent* $\mu(\xi)$ the supremum of the real numbers w which satisfy

$$0 < \left| \xi - \frac{p}{q} \right| \leq \frac{1}{|q|^w}$$

for infinitely many rational numbers p/q. It is easily seen that $\mu(\xi) = 1$ for all rational numbers ξ. On the other hand, Dirichlet's Theorem [29] implies that $\mu(\xi) \geq 2$ for all irrational real numbers ξ. Liouville [40] investigated the values of the irrationality exponent for algebraic real numbers. He showed that $\mu(\xi) \leq \deg \xi$ for all algebraic real numbers ξ. Using this result, he gave first explicit examples of transcendental real numbers, e.g. $\sum_{n=1}^{\infty} 1/2^{n!}$. After that, Thue [60], Siegel [56], and Dyson [32] improved Liouville's result. Finally, Roth [52] established that $\mu(\xi) = 2$ for all algebraic irrational real numbers ξ.

Mahler [41] and Koksma [37] introduced generalizations of the irrationality exponent. Let ξ be a real number and $n \geq 1$ be an integer. We denote by $w_n(\xi)$ the supremum of the real numbers w which satisfy

$$0 < |P(\xi)| \leq H(P)^{-w}$$

for infinitely many polynomials $P(X)$ with integer coefficients of degree at most n. Here, $H(P)$ is defined to be the maximum of the absolute values of the coefficients of $P(X)$. We denote by $w_n^*(\xi)$ the supremum of the real numbers w^* which satisfy

$$0 < |\xi - \alpha| \leq H(\alpha)^{-w^*-1}$$

for infinitely many algebraic numbers α of degree at most n. Here, $H(\alpha)$ is equal to $H(P)$, where $P(X)$ is the minimal polynomial of α over $\mathbb{Z}$. It is clear that $w_1(\xi) = w_1^*(\xi) = \mu(\xi) - 1$ for all real numbers ξ. Therefore, the functions w_n and w_n^* are generalizations of the irrationality exponent μ. A generalization or analogue of the irrationality exponent is called *Diophantine exponent.*

We put

$$w(\xi) := \limsup_{n\to\infty} \frac{w_n(\xi)}{n}, \quad w^*(\xi) := \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n}.$$

A real number ξ is said to be an

$$A\text{-}number \text{ if } w(\xi) = 0;$$
$$S\text{-}number \text{ if } 0 < w(\xi) < +\infty;$$
$$T\text{-}number \text{ if } w(\xi) = +\infty \text{ and } w_n(\xi) < +\infty \text{ for all } n;$$
$$U\text{-}number \text{ if } w(\xi) = +\infty \text{ and } w_n(\xi) = +\infty \text{ for some } n,$$

according to Mahler [41]. We recall known results on this classification. Two algebraically dependent real numbers are in the same class. A real number is algebraic if and only if it is an A-number. Almost all real numbers are S-numbers in the sense of Lebesgue measure. It is known that there exist uncountably many T-numbers and U-numbers. For example, the real number $\sum_{n=1}^{\infty} 1/2^{n!}$ is U-number. Replacing w_n and w with w_n^* and w^*, we define A^*-, S^*-, T^*-, and U^*-*number* as the above. It is known that the two classification of real numbers coincide. For an integer $n \geq 1$ and a U-number ξ, we say that ξ is a U_n-*number* if $w_n(\xi)$ is infinite and $w_m(\xi)$ are finite for all integers $1 \leq m < n$, similarly define U_n^*-*number*. The detail is found in [12].

Let $b \geq 2$ be an integer. We denote by $\lfloor x \rfloor$ the integer part of a real number x. Any real number ξ can be uniquely expanded to the form

$$\xi = \lfloor \xi \rfloor + \sum_{n=1}^{\infty} \frac{a_n}{b^n},$$

where $a_1, a_2, \ldots$ are in the set $\{0, 1, \ldots, b-1\}$ and $a_n \neq b-1$ for infinitely many n. This expansion is called b-*ary expansion* for ξ and the sequence $(a_n)_{n\geq 1}$ is called the *digits* of b-ary expansion for ξ.

For an infinite sequence $\mathbf{a} = (a_k)_{k\geq 0}$ and an integer $n \geq 1$, the *complexity function* of $\mathbf{a}$, denote by $p(\mathbf{a}, n)$, is defined to be the number of distinct sequences of length n which appear in $\mathbf{a}$. The Diophantine exponent of an infinite sequence $\mathbf{a}$, denote by $\mathrm{Dio}(\mathbf{a})$, is first introduced in [1] in order to measure repetition of patterns for an infinite sequence.

Adamczewski and Bugeaud [3] investigated the complexity function for the digits of b-ary expansion for algebraic irrational numbers. They proved that the digits $\mathbf{a}$ of b-ary expansion for an algebraic irrational number satisfies

$$\lim_{n\to\infty} \frac{p(\mathbf{a}, n)}{n} = +\infty.$$

They [5] extended this result as follows: Let ξ be a real number whose digits $\mathbf{a}$ of b-ary expansion satisfies

$$\limsup_{n\to\infty} \frac{p(\mathbf{a}, n)}{n} < +\infty. \tag{1.1}$$

Then the real number ξ is an S-, T-, or U_1-number. They also proved that the real number ξ satisfied (1.1) is a U_1-number if and only if $\mathrm{Dio}(\mathbf{a})$ is infinite. Analogue of these theorems for continued fraction expansions of real numbers had already been proved by Bugeaud [18, 19].

We recall some properties of the Diophantine exponents w_n and w_n^*. It follows from the Schmidt Subspace Theorem that

$$w_n(\xi) = w_n^*(\xi) = \min\{n, d-1\}$$

for all $n \geq 1$ and algebraic real numbers ξ of degree d. It is known that

$$0 \leq w_n(\xi) - w_n^*(\xi) \leq n - 1$$

for all $n \geq 1$ and real numbers ξ (see Section 3.4 in [12]). Sprindžuk [57] proved that $w_n(\xi) = w_n^*(\xi) = n$ for all $n \geq 1$ and almost all real numbers ξ. Baker [11] proved that for $n \geq 2$, there exists a real number ξ for which $w_n(\xi)$ and $w_n^*(\xi)$ are different. More precisely, he proved that the set of values taken by the function $w_n - w_n^*$ contains the set $[0, (n-1)/n]$ for $n \geq 2$. In recent years, this result has been improved. Bugeaud [18, 13] showed that the set of values taken by $w_2 - w_2^*$ is equal to the set $[0, 1]$ and the set of values taken by $w_3 - w_3^*$ contains the set $[0, 2)$. Bugeaud and Dujella [16] proved that for any $n \geq 4$, the set of values taken by $w_n - w_n^*$ contains the set $\left[0, \frac{n}{2} + \frac{n-2}{4(n-1)}\right)$.

1.2 Main results

In this thesis, we study Diophantine exponents for non-Archimedean fields, in particular, the field $\mathbb{Q}_p$ of p-adic numbers and the field $\mathbb{F}_q((T^{-1}))$ of Laurent series over a finite field $\mathbb{F}_q$. This thesis is based on the author's papers [47, 48, 49].

In Chapter 2, we recall basic notations on word combinatorics. We also give definitions and properties of some classes of infinite words, for example, automatic, morphic, Strumian words. We give a relation between the Diophantine exponent and the complexity function on words.

In Chapter 3, we study Mahler's classification for p-adic numbers. We can define an analogue of Diophantine exponents w_n and w_n^* over $\mathbb{Q}_p$ and Mahler's classification A-, S-, T-, U-numbers in a similar way to the real case. Adamczewski and Bugeaud [3] proved that an algebraic irrational p-adic number of the form $\sum_{n=0}^{\infty} a_n p^n$, where $a_n \in \{0, 1, \ldots, p-1\}$ satisfies

$$\lim_{n \to \infty} \frac{p((a_k)_{k \geq 0}, n)}{n} = +\infty.$$

We extend this result as follows:

Theorem 1.2.1 (= Theorem 3.1.2). *Let* $\mathbf{a} = (a_n)_{n \geq 0}$ *be a non-ultimately periodic sequence over* $\{0, 1, \ldots, p-1\}$. *Set* $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. *Assume that*

$$\limsup_{n \to \infty} \frac{p(\mathbf{a}, n)}{n} < +\infty. \tag{1.2}$$

Then the p*-adic number* ξ *is an* S-, T-, *or* U_1*-number.*

We also prove that an irrational p-adic number satisfied (1.2) is a U_1-number if and only if $\mathrm{Dio}(\mathbf{a})$ is infinite. As an application of these results, we give an algebraic independence criterion for p-adic numbers whose digits are Strumian.

In Chapter 4, we study properties of Diophantine exponents over $\mathbb{F}_q((T^{-1}))$. In this situation, Mahler [44] proved that an analogue of Roth Theorem does not hold, that is, there exists an algebraic Laurent series $\xi \in \mathbb{F}_q((T^{-1}))$ such that $w_1(\xi) > 1$. We investigate an approximation property of such algebraic Laurent series. In particular, we study the special class of algebraic Laurent series which is called Class IA.

Theorem 1.2.2 (= Theorem 4.1.1). *Let* $d \geq 1$ *be an integer and* $w > 2d - 1$ *be a rational number. Then there exist a strictly increasing sequence of positive integers* $(k_j)_{j \geq 1}$ *and a sequence* $(\xi_j)_{j \geq 1}$ *such that, for any* $j \geq 1$, ξ_j *is of Class IA, and degree* $p^{k_j} + 1$, *and*

$$w_1(\xi_j) = w_1^*(\xi_j) = \ldots = w_d(\xi_j) = w_d^*(\xi_j) = w.$$

As mentioned in Section 1.1, it is known that for an integer $n \geq 2$, there exist real numbers ξ for which $w_n(\xi)$ and $w_n^*(\xi)$ are different. We prove that for an integer $n \geq 2$, there exist $\xi \in \mathbb{F}_q((T^{-1}))$ for which $w_n(\xi)$ and $w_n^*(\xi)$ are different. More precisely, we have the following theorem.

Theorem 1.2.3 (= Corollary 4.1.10)**.** *Let $d \geq 2$ be an integer and δ be in the closed interval $[0, 1]$. Then there exist uncountably many $\xi \in \mathbb{F}_q((T^{-1}))$ such that $w_n(\xi) - w_n^*(\xi) = \delta$ for all $2 \leq n \leq d$. Furthermore, the set of values taken by $w_2 - w_2^*$ is the closed interval $[0, 1]$.*

As in the classical continued fraction theory of real numbers, if $\xi \in \mathbb{F}_q((T^{-1}))$, then we can write

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}},$$

where $a_0, a_1, a_2, \ldots \in \mathbb{F}_q[T]$, $\deg a_n \geq 1$ for all $n \geq 1$. For simplicity of notation, we write $\xi = [a_0, a_1, a_2, \ldots]$. The $a_0, a_1, a_2, \ldots$ are called the *partial quotients* of ξ.

Bugeaud [18] proved that for a real number ξ whose partial quotient $\mathbf{a}$ is bounded and not ultimately periodic, and satisfies

$$\limsup_{n \to \infty} \frac{p(\mathbf{a}, n)}{n} < +\infty,$$

the Diophantine exponent of $\mathbf{a}$ is infinite if and only if ξ is a U_2-number. We prove an analogue of the result for Laurent series over a finite field.

Theorem 1.2.4 (= Corollary 4.1.17)**.** *Let $\mathbf{a} = (a_n)_{n \geq 1}$ be a non-ultimately periodic sequence over $\mathbb{F}_q[T]$ with $\deg a_n \geq 1$ for $n \geq 1$. Assume that $(|a_n|)_{n \geq 1}$ is bounded and*

$$\limsup_{n \to \infty} \frac{p(\mathbf{a}, n)}{n} < +\infty.$$

Set $\xi := [0, a_1, a_2, \ldots]$. Then the Diophantine exponent of $\mathbf{a}$ is infinite if and only if ξ is a U_2-number.

Chapter 2

Preliminaries on word combinatorics

2.1 Definitions on words

A non-empty finite set is called an *alphabet*. A *letter* over an alphabet is an element of the alphabet. A *word* over an alphabet is a sequence over the alphabet. A word over an alphabet is called *finite* (resp. *infinite*) if the word is a finite (resp. an infinite) sequence. We denote a finite word $(a_0, a_1, \dots, a_n)$ and an infinite word $(a_m)_{m\geq 0} = (a_0, a_1, \dots)$ by $a_0a_1 \cdots a_n$ and $a_0a_1\cdots$, respectively. The *length* of a finite word W, denoted by $|W|$, is the number of letters of the word W. We write a word of the length zero as λ, is called the *empty word*.

Let $\mathcal{A}^*$ denote the set of all finite words over an alphabet $\mathcal{A}$. Note that the empty word is always in $\mathcal{A}^*$. Let $\mathcal{A}^+$ and $\mathcal{A}^{\mathbb{N}}$ denote the set of all non-empty finite words over an alphabet $\mathcal{A}$ and the set of all infinite words over $\mathcal{A}$. We define an operation on words, is called *concatenation* as follows: We concatenate two words V and W by juxtaposition their letters, denoted by VW. Note that concatenation on words is associative and, in general, is not commutative. For an alphabet $\mathcal{A}$, the set $\mathcal{A}^*$ with concatenation is a monoid with the identity element λ.

A word W is called a *prefix* of a word V if there exists a word U such that $V = WU$. We denote by $\lceil x\rceil$ the upper integer part, and $\{x\}$ the fractional part of a real number x. For an integer $n \geq 1$ and a finite word W, let $W^n := WW\cdots W$ (n times). For a convenience, write $W^0 := \lambda$. More generally, for a real number $w \geq 0$, let $W^w := W^{\lfloor w\rfloor}W'$, where W' is the prefix of W of length $\lceil(w - \lfloor w\rfloor)|W|\rceil$. For a finite word W, write $\overline{W} := WW\cdots$ (infinitely many). An infinite word $\mathbf{a}$ over an alphabet $\mathcal{A}$ is called *ultimately periodic* if there exist finite words $U \in \mathcal{A}^*$ and $V \in \mathcal{A}^+$ such that $\mathbf{a} = U\overline{V}$. A finite word W is called *overlap* if $W = aVaVa$, where V is a (possibly empty) finite word and a is a latter.

Define $\mathcal{A}^\infty := \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$, where $\mathcal{A}$ is an alphabet. We define a map $d : \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ by

$$d(\mathbf{a}, \mathbf{b}) = \begin{cases} 0 & \text{if } \mathbf{a} = \mathbf{b}, \\ 2^{-n} & \text{otherwise,} \end{cases}$$

where $\mathbf{a} = a_0a_1\cdots, \mathbf{b} = b_0b_1\cdots$ and $n = \min\{i \mid a_i \neq b_i\}$. It is well-known that $(\mathcal{A}^{\mathbb{N}}, d)$ is a metric space and we can extend the metric d to $\mathcal{A}^\infty$ by a letter b, not in $\mathcal{A}$ and identifying each finite word w over $\mathcal{A}$ with $w\overline{b} \in (\mathcal{A} \cup \{b\})^{\mathbb{N}}$.

2.2 Automatic, morphic, and Sturmian words

Let X be a set and $f : X \to X$ be a map. Throughout this thesis, for an integer $n \geq 0$, we denote by f^n the n-th iteration of f. Namely, we define recursively f^n by $f^0 := \mathrm{id}_X$ and $f^{n+1} := f \circ f^n$, where id_X is the identity map on X.

Let $k \geq 2$ be an integer. We denote by Σ_k the set $\{0, 1, \ldots, k-1\}$. A *k-automaton* is defined to be a sextuple

$$A = (Q, \Sigma_k, \delta, q_0, \Delta, \tau),$$

where Q is a finite set of *states*, $\delta : Q \times \Sigma_k \to Q$ is a *transition function*, $q_0 \in Q$ is an *initial state*, a finite set Δ is an *output alphabet*, and $\tau : Q \to \Delta$ is an *output function.* For $q \in Q$ and a finite word $W = w_0 w_1 \cdots w_n$ over Σ_k, we define recursively $\delta(q, W)$ by $\delta(q, W) = \delta(\delta(q, w_0 w_1 \cdots w_{n-1}), w_n)$. Let $n \geq 0$ be an integer and $W_n = w_r w_{r-1} \cdots w_0$, where $\sum_{i=0}^{r} w_i k^i$ is the k-ary expansion of n. An infinite word $\mathbf{a} = a_0 a_1 \cdots$ is said to be *k-automatic* if there exists a k-automaton $A = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that $a_n = \tau(\delta(q_0, W_n))$ for all $n \geq 0$.

Example. One of a classical example of an automatic sequence is the Thue-Morse sequence. Let $\mathbf{a} = a_0 a_1 \cdots$ be an infinite word such that

$$a_n = \begin{cases} 0 & \text{if the sum of digits of the binary expansion of } n \text{ is even,} \\ 1 & \text{if the sum of digits of the binary expansion of } n \text{ is odd.} \end{cases}$$

The infinite word $\mathbf{a} = 0110100110010110 \cdots$ is called the Thue-Morse sequence. It is known that the infinite word $\mathbf{a}$ is 2-automatic generated by a 2-automaton

$$A = (\{q_0, q_1\}, \Sigma_2, \delta, q_0, \{0, 1\}, \tau),$$

where $\delta(q_0, 0) = \delta(q_1, 1) = q_0, \delta(q_0, 1) = \delta(q_1, 0) = q_1, \tau(q_0) = 0$, and $\tau(q_1) = 1$.

Let $\mathcal{A}$ and $\mathcal{B}$ be alphabets. A map $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is a *morphism* if $\sigma(UV) = \sigma(U)\sigma(V)$ for all $U, V \in \mathcal{A}^*$. Then we have $\sigma(\lambda) = \lambda$. Therefore, we can uniquely extend a map from $\mathcal{A}$ to $\mathcal{B}$ to a morphism from $\mathcal{A}^*$ to $\mathcal{B}^*$. A morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ can be extended to a map from $\mathcal{A}^\infty$ to $\mathcal{B}^\infty$ as follows:

$$\sigma(\mathbf{a}) := \sigma(a_0)\sigma(a_1) \cdots ,$$

where $\mathbf{a} = a_0 a_1 \cdots \in \mathcal{A}^{\mathbb{N}}$.

The *width* of a morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is defined to be $\max_{a \in \mathcal{A}} |\sigma(a)|$.

A morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is said to be *k-uniform* if there exists an integer $k \geq 1$ such that $|\sigma(a)| = k$ for all $a \in \mathcal{A}$. In particular, we call a 1-uniform morphism a *coding.* A morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is *primitive* if there exists an integer $n \geq 1$ such that a occurs in $\sigma^n(b)$ for all $a, b \in \mathcal{A}$.

Let $\mathcal{A}$ be an alphabet and $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ be a morphism. A word $W \in \mathcal{A}^\infty$ is called a *fixed point of* σ if $\sigma(W) = W$. We say σ is *prolongable* on $a \in \mathcal{A}$ if $\sigma(a) = aW$, where $\sigma^n(W) \in \mathcal{A}^+$ for all $n \geq 0$. Then the sequence $(\sigma^n(a))_{n \geq 0}$ converges to the infinite word

$$\sigma^\infty(a) := aW\sigma(W)\sigma^2(W) \cdots$$

and $\sigma^\infty(a)$ is a fixed point of σ. It immediate that $\sigma^\infty(a)$ is an unique fixed point of σ start with a.

An infinite word $\mathbf{a} = a_0a_1\cdots$ is said to be *morphic* if if there exist alphabets $\mathcal{A}, \mathcal{B}$, a morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ which is prolongable on some $a \in \mathcal{A}$, and a coding $\tau : \mathcal{A}^* \to \mathcal{B}^*$ such that $\mathbf{a} = \tau(\sigma^\infty(a))$. In this case, the morphic word $\mathbf{a}$ is called *k-uniform* (resp. *primitive*) if the morphism σ is k-uniform (resp. primitive). When the morphic word $\mathbf{a}$ is a k-uniform, we call the alphabet $\mathcal{A}$ the *initial alphabet* associated with $\mathbf{a}$.

Example. The Thue-Morse sequence is a 2-uniform morphic word. In fact, let $\sigma : \{0,1\}^* \to \{0,1\}^*$ be a 2-uniform morphism and prolongable on 0 such that $\sigma(0) = 01$ and $\sigma(1) = 10$, and $\tau = \mathrm{id}_{\{0,1\}}$. Then it holds that the Thue-Morse sequence is generated by σ and τ, that is,

$$\tau(\sigma^\infty(0)) = 0110100110010110\cdots .$$

Example. A morphism $\sigma : \{0,1\}^* \to \{0,1\}^*$ which is defined by $\sigma(0) = 01$ and $\sigma(1) = 0$ is called the Fibonacci morphism. It is easily seen that the Fibonacci morphism is primitive, prolongable on 0 and not uniform. We call the infinite word generated by the Fibonacci morphism the Fibonacci infinite word;

$$\tau(\sigma^\infty(0)) = 0100101001001010010 10\cdots ,$$

where $\tau = \mathrm{id}_{\{0,1\}}$.

The *k-kernel* of an infinite word $\mathbf{a} = a_0a_1\cdots$ is the set of all sequences $(a_{k^in+j})_{n\ge 0}$, where $i \ge 0$ and $0 \le j < k^i$.

Cobham and Eilenberg characterized k-automatic sequences as follows:

Theorem 2.2.1 (Cobham [26], Eilenberg [33])**.** *Let $k \ge 2$ be an integer and $\mathbf{a}$ be an infinite word. Then the following are equivalent:*

(i) $\mathbf{a}$ *is k-automatic,*

(ii) $\mathbf{a}$ *is k-uniform morphic,*

(iii) *the k-kernel of* $\mathbf{a}$ *is finite.*

Let θ and ρ be real numbers with $0 < \theta < 1$ and θ is irrational. For $n \ge 1$, we put $s_{n,\theta,\rho} := \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor$ and $s'_{n,\theta,\rho} := \lceil (n+1)\theta + \rho \rceil - \lceil n\theta + \rho \rceil$. Note that $s_{n,\theta,\rho}, s'_{n,\theta,\rho} \in \{0,1\}$ for all $n \ge 1$. We also put $\mathbf{s}_{\theta,\rho} := s_{1,\theta,\rho}s_{2,\theta,\rho}\cdots$ and $\mathbf{s}'_{\theta,\rho} := s'_{1,\theta,\rho}s'_{2,\theta,\rho}\cdots$. An infinite word $\mathbf{a}$ is called *Sturmian* if there exist an irrational number $0 < \theta < 1$, a real number ρ, a finite set $\mathcal{A}$, and a coding $\tau : \{0,1\}^* \to \mathcal{A}^*$ with $\tau(0) \ne \tau(1)$ such that $\mathbf{a}$ is $\tau(\mathbf{s}_{\theta,\rho})$ or $\tau(\mathbf{s}'_{\theta,\rho})$. We call the irrational number θ *slope* of $\mathbf{a}$ and the real number ρ *intercept* of $\mathbf{a}$.

Example. Let ϕ be the golden ratio, that is, $\phi = (1+\sqrt{5})/2$ and τ be a coding such that $\tau(0) = 1$ and $\tau(1) = 0$. It is known that the Sturmian word $\tau(\mathbf{s}_{\{\phi\},0})$ is the Fibonacci word.

Let θ and ρ be real numbers. For an integer $n \ge 1$, we put

$$t_{n,\theta,\rho} := \begin{cases} 1 & \text{if } n = \lfloor k\theta + \rho \rfloor \text{ for some integer } k, \\ 0 & \text{otherwise,} \end{cases}$$

$$t'_{n,\theta,\rho} := \begin{cases} 1 & \text{if } n = \lceil k\theta + \rho \rceil \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

We also put $\mathbf{t}_{\theta,\rho} := t_{1,\theta,\rho}t_{2,\theta,\rho}\cdots$ and $\mathbf{t}'_{\theta,\rho} := t'_{1,\theta,\rho}t'_{2,\theta,\rho}\cdots$.

Lemma 2.2.2. *Let $\theta > 1$ be an irrational real number and ρ be a real number. Then we have $\mathbf{t}_{\theta,\rho} = \mathbf{s}'_{1/\theta,-\rho/\theta}$ and $\mathbf{t}'_{\theta,\rho} = \mathbf{s}_{1/\theta,-\rho/\theta}$.*

Proof.

$$\begin{aligned} t_{n,\theta,\rho} = 1 &\Leftrightarrow n = \lfloor k\theta + \rho \rfloor \text{ for some integer } k \\ &\Leftrightarrow n \leq k\theta + \rho < n+1 \text{ for some integer } k \\ &\Leftrightarrow \frac{n-\rho}{\theta} \leq k < \frac{n+1-\rho}{\theta} \text{ for some integer } k \\ &\Leftrightarrow \left\lceil \frac{n-\rho}{\theta} \right\rceil = k, \left\lceil \frac{n+1-\rho}{\theta} \right\rceil = k+1 \text{ for some integer } k \\ &\Leftrightarrow \left\lceil \frac{n+1-\rho}{\theta} \right\rceil - \left\lceil \frac{n-\rho}{\theta} \right\rceil = 1 \\ &\Leftrightarrow s'_{n,1/\theta,-\rho/\theta} = 1. \end{aligned}$$

Hence, we obtain $\mathbf{t}_{\theta,\rho} = \mathbf{s}'_{1/\theta,-\rho/\theta}$. Similarly, we have $\mathbf{t}'_{\theta,\rho} = \mathbf{s}_{1/\theta,-\rho/\theta}$. □

2.3 Complexity and Diophantine exponent on words

Let ρ be a real number. We say that an infinite word $\mathbf{a}$ satisfies *Condition* $(*)_\rho$ if there exist sequences of finite words $(U_n)_{n\geq1}$, $(V_n)_{n\geq1}$ and a sequence of nonnegative real numbers $(w_n)_{n\geq1}$ such that

(i) the word $U_nV_n^{w_n}$ is the prefix of $\mathbf{a}$ for all $n \geq 1$,

(ii) $|U_nV_n^{w_n}|/|U_nV_n| \geq \rho$ for all $n \geq 1$,

(iii) the sequence $(|V_n^{w_n}|)_{n\geq1}$ is strictly increasing.

The *Diophantine exponent* of an infinite word $\mathbf{a}$, denoted by $\mathrm{Dio}(\mathbf{a})$, and is defined to be the supremum of a real number ρ such that $\mathbf{a}$ satisfy Condition $(*)_\rho$. Diophantine exponent of an infinite word is first introduced in [1] in order to measure repetition of patterns for an infinite word. It is obvious that for an infinite word $\mathbf{a}$,

$$1 \leq \mathrm{Dio}(\mathbf{a}) \leq +\infty. \tag{2.1}$$

Dubickas [30] showed that (2.1) is the best possible.

Theorem 2.3.1. *Let $\mathcal{A}$ be an alphabet of cardinality at least two. For $w \in [1, +\infty]$, there exists an infinite word $\mathbf{a} \in \mathcal{A}^{\mathbb{N}}$ such that $\mathrm{Dio}(\mathbf{a}) = w$.*

Proof. See [30]. □

Lemma 2.3.2. *Let $\mathcal{A}$ be an alphabet. An ultimately periodic word $\mathbf{a} \in \mathcal{A}^{\mathbb{N}}$ satisfies $\mathrm{Dio}(\mathbf{a}) = +\infty$. If the cardinality of $\mathcal{A}$ is at least two, then there exists a non-ultimately periodic word $\mathbf{b} \in \mathcal{A}^{\mathbb{N}}$ such that $\mathrm{Dio}(\mathbf{b}) = +\infty$.*

Proof. By the definition of ultimately periodic words, there exist finite words $U \in \mathcal{A}^*$ and $V \in \mathcal{A}^+$ such that $\mathbf{a} = U\overline{V}$. For each $n \geq 1$, we put $U_n := U, V_n := V$ and $w_n := n$. It immediately seen that the triplets (U_n, V_n, w_n) satisfy the definition of Diophantine exponent (i) and (iii). It also follows that the sequence $(|U_nV_n^{w_n}|/|U_nV_n|)_{n\geq1}$ tends to infinity as $n \to \infty$. Hence, we obtain $\mathrm{Dio}(\mathbf{a}) = +\infty$.

Assume that the cardinality of $\mathcal{A}$ is at least two. Let $a, b \in \mathcal{A}$ be distinct letters. We define an infinite word $\mathbf{b} = b_0 b_1 \cdots$ by $b_n = a$ if $n = m!$ for some $m \geq 1$, $b_n = b$ otherwise. We put $U_n := b_1 \cdots b_{n!}, V_n := b, w_n := (n+1)! - n! - 1$ for any $n \geq 1$. It easily seen that the triplets (U_n, V_n, w_n) satisfy the definition of Diophantine exponent (i) and (iii), and the sequence $(|U_n V_n^{w_n}|/|U_n V_n|)_{n \geq 1}$ tends to infinity as $n \to \infty$. Hence, we obtain $\mathrm{Dio}(\mathbf{b}) = +\infty$. Since $\mathbf{b}$ contains arbitrary long consecutive b, the infinite word $\mathbf{b}$ is not ultimately periodic. □

Lemma 2.3.3 (Adamczewski and Cassaigne [7]). *Let $k \geq 2$ be an integer. Let $\mathbf{a}$ be a non-ultimately periodic and k-automatic word. Let m be a cardinality of the k-kernel of $\mathbf{a}$. Then we have*

$$\mathrm{Dio}(\mathbf{a}) < k^m.$$

Mossé's result [46] implies the lemma below.

Lemma 2.3.4. *Let $\mathbf{a}$ be a non-ultimately periodic and primitive morphic word. Then the Diophantine exponent of $\mathbf{a}$ is finite.*

Lemma 2.3.5 (Adamczewski and Bugeaud [6]). *Let $\mathbf{a}$ be a Sturmian word. Then the slope of $\mathbf{a}$ has bounded partial quotients if and only if the Diophantine exponent of $\mathbf{a}$ is finite.*

The *complexity function* of an infinite word $\mathbf{a} = a_0 a_1 \cdots$ is defined by

$$p(\mathbf{a}, n) = \mathrm{Card}\{a_i a_{i+1} \ldots a_{i+n-1} \mid i \geq 0\}, \quad \text{for} \quad n \geq 1.$$

It is known that automatic words, primitive morphic words, and Sturmian words have low complexity.

Lemma 2.3.6. *Let $k \geq 2$ be an integer and $\mathbf{a}$ be a k-automatic word. Let d be a cardinality of the internal alphabet associated with $\mathbf{a}$. Then we have*

$$p(\mathbf{a}, n) \leq k d^2 n, \text{ for } n \geq 1.$$

Proof. See [9, Theorem 10.3.1] or [26]. □

Lemma 2.3.7. *Let $\mathbf{a}$ be a primitive morphic word over an alphabet of cardinality of $b \geq 2$. Let v be the width of σ which generates the infinite word $\mathbf{a}$. Then we have*

$$p(\mathbf{a}, n) \leq 2 v^{4b-2} b^3 n, \text{ for } n \geq 1.$$

Proof. See [9, Theorem 10.4.12]. □

Lemma 2.3.8. *Let $\mathbf{a}$ be a Sturmian word. Then we have*

$$p(\mathbf{a}, n) = n + 1, \text{ for } n \geq 1.$$

Proof. See [9, Theorem 10.5.8]. □

The Diophantine exponent has relation to the complexity function as follows:

Lemma 2.3.9. *Let* $\mathbf{a}$ *be an infinite word on an alphabet* $\mathcal{A}$. *Assume that there exist integers* $\kappa \geq 2$ *such that*

$$p(\mathbf{a}, n) \leq \kappa n$$

for infinitely many n. *Then it holds that*

$$\mathrm{Dio}(\mathbf{a}) \geq 1 + \frac{1}{\kappa}.$$

Proof. The proof of this lemma is same way to the proof of Theorem 3 in [1], however we give a proof to be self-contained.

For $n \geq 1$, we denote by $A(n)$ the prefix of $\mathbf{a}$ of length n. Let $n \leq 1$ be an integer such that $p(\mathbf{a}, n) \leq \kappa n$. By Pigeonhole principle, there exists a finite word W_n of length n such that the word appears to $A((\kappa+1)n)$ at least twice. Thus, there exist finite words $B_n, D_n, E_n \in \mathcal{A}^*$ and $C_n \in \mathcal{A}^+$ such that

$$A((\kappa+1)n) = B_n W_n D_n E_n = B_n C_n W_n E_n.$$

Firstly, we consider the case of $|C_n| \geq |W_n|$. Then, there exists $F_n \in \mathcal{A}^*$ such that

$$A((\kappa+1)n) = B_n W_n F_n W_n E_n.$$

Put $U_n := B_n, V_n := W_n F_n$, and $w_n := |W_n F_n W_n|/|W_n F_n|$. Since $U_n V_n^{w_n} = B_n W_n F_n W_n$, the word $U_n V_n^{w_n}$ is a prefix of $\mathbf{a}$. It is immediate that $|V_n^{w_n}| = |W_n F_n W_n| \geq 2n$. Furthermore, we see that

$$\frac{|U_n V_n^{w_n}|}{|U_n V_n|} = 1 + \frac{n}{|B_n W_n F_n|} \geq 1 + \frac{1}{\kappa}.$$

We next consider the case of $|C_n| < |W_n|$. Since the two occurrences of W_n do overlap, there exists a rational number $d_n > 1$ such that $W_n = C_n^{d_n}$. Put $U_n := B_n, V_n := C_n$, and $w_n := d_n + 1$. Since $U_n V_n^{w_n} = B_n C_n W_n$, the word $U_n V_n^{w_n}$ is a prefix of $\mathbf{a}$. It follows that

$$\frac{|U_n V_n^{w_n}|}{|U_n V_n|} = 1 + \frac{n}{|B_n C_n|} \geq 1 + \frac{1}{\kappa},$$
$$|V_n^{w_n}| = |C_n W_n| \geq n + 1.$$

Therefore, we obtain $\mathrm{Dio}(\mathbf{a}) \geq 1 + 1/\kappa$. □

The lemma below is a slight improvement of [18, Lemma 9.1].

Lemma 2.3.10. *Let* $\mathbf{a}$ *be an infinite word on an alphabet* $\mathcal{A}$. *Assume that there exist integers* $\kappa \geq 2$ *and* $n_0 \geq 1$ *such that for all* $n \geq n_0$,

$$p(\mathbf{a}, n) \leq \kappa n.$$

Then, for each $n \geq n_0$, *there exist finite words* U_n, V_n *and a positive rational number* w_n *such that the following hold:*

(i) $U_n V_n^{w_n}$ *is a prefix of* $\mathbf{a}$,

(ii) $|U_n| \leq 2\kappa |V_n|$,

(iii) $n/2 \leq |V_n| \leq \kappa n$,

(iv) *if* U_n *is not the empty word, then the last letters of* U_n *and* V_n *are different,*

(v) $|U_nV_n^{w_n}|/|U_nV_n| \geq 1 + 1/(4\kappa+2)$,

(vi) $|U_nV_n| \leq (\kappa+1)n - 1$,

(vii) $|U_n^2V_n| \leq (2\kappa+1)n - 2$.

Proof. For $n \geq 1$, we denote by $A(n)$ the prefix of $\mathbf{a}$ of length n. By Pigeonhole principle, for each $n \geq n_0$, there exists a finite word W_n of length n such that the word appears to $A((\kappa+1)n)$ at least twice. Thus, for each $n \geq n_0$, there exist finite words $B_n, D_n, E_n \in \mathcal{A}^*$ and $C_n \in \mathcal{A}^+$ such that

$$A((\kappa+1)n) = B_nW_nD_nE_n = B_nC_nW_nE_n.$$

We can take these words in such way that if B_n is not empty, then the last letter of B_n is different from that of C_n.

Firstly, we consider the case of $|C_n| \geq |W_n|$. Then, there exists $F_n \in \mathcal{A}^*$ such that

$$A((\kappa+1)n) = B_nW_nF_nW_nE_n.$$

Put $U_n := B_n, V_n := W_nF_n$, and $w_n := |W_nF_nW_n|/|W_nF_n|$. Since $U_nV_n^{w_n} = B_nW_nF_nW_n$, the word $U_nV_n^{w_n}$ is a prefix of $\mathbf{a}$. It is obvious that $|U_n| \leq (\kappa-1)|V_n|$ and $n \leq |V_n| \leq \kappa n$. By the definition, we have (iv) and (vi). Furthermore, we see that

$$\frac{|U_nV_n^{w_n}|}{|U_nV_n|} = 1 + \frac{n}{|U_nV_n|} \geq 1 + \frac{1}{\kappa},$$
$$|U_n^2V_n| \leq |U_nV_n| + |U_n| \leq \kappa n + (\kappa-1)n = (2\kappa-1)n.$$

We next consider the case of $|C_n| < |W_n|$. Since the two occurrences of W_n do overlap, there exists a rational number $d_n > 1$ such that $W_n = C_n^{d_n}$. Put $U_n := B_n, V_n := C_n^{\lceil d_n/2 \rceil}$, and $w_n := (d_n+1)/\lceil d_n/2 \rceil$. Obviously, we have (i) and (iv). Since $\lceil d_n/2 \rceil \leq d_n$ and $d_n|C_n| \leq 2\lceil d_n/2 \rceil |C_n|$, we get $n/2 \leq |V_n| \leq n$. Using (iii) and $|U_n| \leq \kappa n - 1$, we can see (ii), (vi), and (vii). It is immediate that $w_n \geq 3/2$. Hence, we obtain

$$\frac{|U_nV_n^{w_n}|}{|U_nV_n|} = 1 + \frac{\lceil (w_n-1)|V_n| \rceil}{|U_nV_n|} \geq 1 + \frac{w_n - 1}{|U_n|/|V_n| + 1}$$
$$\geq 1 + \frac{1/2}{2\kappa+1} = 1 + \frac{1}{4\kappa+2}.$$

□

Chapter 3

Mahler's classification and p-adic numbers with low complexity digits

3.1 Introduction

Let p be a prime. We denote by $|\cdot|_p$ the absolute value of $\mathbb{Q}_p$ normalized to satisfy $|p|_p = 1/p$. We set $P := \{0, 1, \dots, p-1\}$.

Applying so-called the Schmidt Subspace Theorem, Adamczewski and Bugeaud [3] established a new transcendence criterion for p-adic numbers.

Theorem 3.1.1. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* P. *Set* $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. *Assume that there exist integers* $n_0 \geq 1$ *and* $\kappa \geq 2$ *such that for all* $n \geq n_0$,

$$p(\mathbf{a}, n) \leq \kappa n.$$

Then the p*-adic number* ξ *is transcendental.*

In this chapter, we study p-adic numbers which satisfy the assumption of Theorem 3.1.1 in more detail. We define Diophantine exponents for p-adic numbers. Let ξ be in $\mathbb{Q}_p$ and $n \geq 1$ be an integer. We denote by $w_n(\xi)$ (resp. $w_n^*(\xi)$) the supremum of the real numbers w (resp. w^*) which satisfy

$$0 < |P(\xi)| \leq H(P)^{-w-1} \quad (\text{resp. } 0 < |\xi - \alpha| \leq H(\alpha)^{-w^*-1})$$

for infinitely many polynomials $P(X) \in \mathbb{Z}[X]$ of degree at most n (resp. algebraic numbers $\alpha \in \mathbb{Q}_p$ of degree at most n). Here, $H(P)$, which is called the *height* of $P(X)$, is defined by the maximum of the usual absolute values of the coefficients of $P(X)$, and $H(\alpha)$, which is called the *height* of α, is defined by the height of the minimal polynomial of α over $\mathbb{Z}$. We put

$$w(\xi) := \limsup_{n\to\infty} \frac{w_n(\xi)}{n}, \quad w^*(\xi) := \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n}.$$

A p-adic number ξ is said to be an

A*-number* if $w(\xi) = 0$;

S*-number* if $0 < w(\xi) < +\infty$;

T*-number* if $w(\xi) = +\infty$ and $w_n(\xi) < +\infty$ for all n;

U*-number* if $w(\xi) = +\infty$ and $w_n(\xi) = +\infty$ for some n, according to Mahler [42]. We recall known results on this classification. A p-adic number is algebraic if and only if it is an A-number. Almost all p-adic numbers are S-numbers in the sense of Haar measure. It is known that there exist uncountably many T-numbers and U-numbers. For example, $\sum_{n=1}^{\infty} p^{n!}$ is U-number. Replacing w_n and w with w_n^* and w^*, we define A^*-, S^*-, T^*-, and U^**-number* as the above. It is known that the two classification of p-adic numbers coincide. Let $n \geq 1$ be an integer. For a U-number $\xi \in \mathbb{Q}_p$, we say that ξ is a U_n*-number* if $w_n(\xi)$ is infinite and $w_m(\xi)$ are finite for all $1 \leq m < n$, similarly define U_n^*-number. The detail is found in [12, Section 9.3].

We now state the main results.

Theorem 3.1.2. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* P*. Set* $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$*. Assume that there exist integers* $n_0 \geq 1$ *and* $\kappa \geq 2$ *such that for all* $n \geq n_0$*,*

$$p(\mathbf{a}, n) \leq \kappa n.$$

Then the p*-adic number* ξ *is an* S-, T-, *or* U_1*-number.*

Note that Theorem 3.1.2 is an analogue of Théorème 1.1 in [6], and there is a real continued fraction analogue of Theorem 3.1.2 in [18, Theorem 3.2].

Theorem 3.1.3. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* P*. Set* $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$*. Assume that there exist integers* $n_0 \geq 1$ *and* $\kappa \geq 2$ *such that for all* $n \geq n_0$*,*

$$p(\mathbf{a}, n) \leq \kappa n.$$

Then the Diophantine exponent of $\mathbf{a}$ *is infinite if and only if* ξ *is a* U_1*-number. Furthermore, if the Diophantine exponent of* $\mathbf{a}$ *is finite, then we have*

$$w_1(\xi) \leq 8(\kappa+1)^2(2\kappa+1)\,\mathrm{Dio}(\mathbf{a}) - 1. \tag{3.1}$$

There are various versions of Theorem 3.1.3: b-ary expansion for real numbers [6], continued fraction expansion for real numbers [18], Laurent series over a finite field, and its continued fraction expansion (see Chapter 4).

3.2 Application of the main results

In this section, we apply the main results to classes of sequences introduced in Chapter 2.

Theorem 3.2.1. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* P*. Set* $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$*. If the sequence* $\mathbf{a}$ *is automatic, primitive morphic, or Sturmian with its slope whose continued fraction expansion has bounded partial quotients, then the* p*-adic number* ξ *is an* S*- or* T*-number. Furthermore, if the sequence* $\mathbf{a}$ *is Sturmian with its slope whose continued fraction expansion has unbounded partial quotients, then the* p*-adic number* ξ *is a* U_1*-number.*

Note that Theorem 3.2.1 is an analogue of Théorèmes 3.1, 4.2, and 5.1 in [6].

Proof. Since the sequence $\mathbf{a}$ is automatic, primitive morphic, or Sturmian, ξ is an S-, T-, or U_1-number by Lemmas 2.3.6, 2.3.7, 2.3.8 and Theorem 3.1.2. It follows from Lemmas 2.3.3, 2.3.4, 2.3.5 and Theorem 3.1.3 that ξ is a U_1-number if $\mathbf{a}$ is Sturmian with its slope whose continued fraction expansion has unbounded partial quotients, and ξ is an S- or T-number otherwise. □

Lemma 3.2.2 (Mahler [42])**.** *Let ξ, η be in $\mathbb{Q}_p$. If ξ and η are algebraically dependent, then ξ and η are in the same class.*

Theorem 3.2.3. *Let $\theta > 1$ be an irrational real number whose continued fraction expansion has bounded partial quotients, $\theta' > 1$ be an irrational real number whose continued fraction expansion has unbounded partial quotients, and ρ, ρ' be real numbers. Then the p-adic numbers*

$$\sum_{n=1}^{\infty} p^{\lfloor n\theta+\rho \rfloor}, \quad \sum_{n=1}^{\infty} p^{\lfloor n\theta'+\rho' \rfloor}$$

are algebraically independent.

Note that Theorem 3.2.3 is an analogue of Corollaire 3.2 in [6].

Proof. Set $\xi := \sum_{n=1}^{\infty} p^{\lfloor n\theta+\rho \rfloor}$ and $\eta := \sum_{n=1}^{\infty} p^{\lfloor n\theta'+\rho' \rfloor}$. By the definition, the digits of ξ and η are $\mathbf{t}_{\theta,\rho}$ and $\mathbf{t}_{\theta',\rho'}$, respectively. It follows from Lemma 2.2.2 that $\mathbf{t}_{\theta,\rho}$ (resp. $\mathbf{t}_{\theta',\rho'}$) is Sturmian with its slope whose continued fraction expansion has bounded (resp. unbounded) partial quotients. Therefore, ξ is an S- or T-number and η is a U_1-number by Theorem 3.2.1. Hence, we see that ξ and η are algebraically independent by Lemma 3.2.2. □

Example. The following p-adic numbers are algebraically independent:

$$\sum_{n=1}^{\infty} p^{\lfloor ne \rfloor}, \quad \sum_{n=1}^{\infty} p^{\lfloor n\sqrt{p} \rfloor}.$$

3.3 Preliminaries

We recall several facts about the Diophantine exponents w_n and w_n^*.

Theorem 3.3.1. *Let $n \geq 1$ be an integer and ξ be in $\mathbb{Q}_p$. Then we have*

$$w_n^*(\xi) \leq w_n(\xi) \leq w_n^*(\xi) + n - 1.$$

Proof. See [45]. □

Theorem 3.3.2. *Let $n \geq 1$ be an integer and $\xi \in \mathbb{Q}_p$ be not algebraic of degree at most n. Then we have*

$$w_n(\xi) \geq n, \quad w_n^*(\xi) \geq \frac{n+1}{2}.$$

Furthermore, if $n = 2$, then $w_2^*(\xi) \geq 2$.

Proof. See [42, 45]. □

We recall Liouville inequality, that is, a non trivial lower bound of differences of two algebraic numbers.

Lemma 3.3.3. *Let $\alpha, \beta \in \mathbb{Q}_p$ be distinct algebraic numbers of degree m, n, respectively. Then we have*

$$|\alpha - \beta|_p \geq \frac{(m+1)^{-n}(n+1)^{-m}}{H(\alpha)^n H(\beta)^m}.$$

Proof. See [50, Lemma 2.5]. □

Applying Lemma 3.3.3, we give an estimate for the value of w_1.

Lemma 3.3.4. *Let ξ be in $\mathbb{Q}_p$ and $c_0, c_1, c_2, c_3, \theta, \rho, \delta$ be positive numbers. Let $(\beta_j)_{j\geq 1}$ be a sequence of positive integers with $\beta_j < \beta_{j+1} \leq c_0\beta_j^{\theta}$ for all $j \geq 1$. Assume that there exists a sequence of distinct terms $(\alpha_j)_{j\geq 1}$ with $\alpha_j \in \mathbb{Q}$ such that for all $j \geq 1$*

$$\frac{c_1}{\beta_j^{1+\rho}} \leq |\xi - \alpha_j|_p \leq \frac{c_2}{\beta_j^{1+\delta}},$$
$$H(\alpha_j) \leq c_3\beta_j.$$

Then we have

$$\delta \leq w_1(\xi) \leq (1+\rho)\frac{\theta}{\delta} - 1.$$

Note that there are several versions of Lemma 3.3.4 in [5, 6, 8, 10, 18, 20, 27, 35, 61].

Proof. Let α be a rational number with sufficiently large height. We define the integer $j_0 \geq 1$ by $\beta_{j_0} \leq c_0(4c_2c_3H(\alpha))^{\theta/\delta} < \beta_{j_0+1}$. Firstly, we consider the case of $\alpha = \alpha_{j_0}$. By the assumption, we obtain

$$|\xi - \alpha|_p \geq c_1\beta_{j_0}^{-1-\rho} \geq c_0^{-1-\rho}c_1(4c_2c_3)^{-(1+\rho)\theta/\delta}H(\alpha)^{-(1+\rho)\theta/\delta}.$$

Next, we consider the others case. Then, by the assumption, we have

$$H(\alpha) < (4c_2c_3)^{-1}(c_0^{-1}\beta_{j_0+1})^{\delta/\theta} \leq (4c_2c_3)^{-1}\beta_{j_0}^{\delta}.$$

Therefore, we obtain

$$|\alpha - \alpha_{j_0}|_p \geq (4H(\alpha)H(\alpha_{j_0}))^{-1} > c_2\beta_{j_0}^{-1-\delta} \geq |\xi - \alpha_{j_0}|_p$$

by Lemma 3.3.3. Hence, it follows that

$$\begin{aligned}|\xi - \alpha|_p = |\alpha - \alpha_{j_0}|_p &\geq (4H(\alpha)H(\alpha_{j_0}))^{-1} \\ &\geq 4^{-1-\theta/\delta}c_0^{-1}c_2^{-\theta/\delta}c_3^{-1-\theta/\delta}H(\alpha)^{-1-\theta/\delta}.\end{aligned}$$

By Theorem 3.3.1, we have $w_1(\xi) = w_1^*(\xi)$. Thus, we obtain

$$\delta \leq w_1(\xi) \leq \max\left((1+\rho)\frac{\theta}{\delta} - 1, \frac{\theta}{\delta}\right) = (1+\rho)\frac{\theta}{\delta} - 1.$$

□

We denote by $M_{\mathbb{Q}}$ the set of all prime numbers and ∞. We denote by $|\cdot|_\infty$ the usual absolute value in $\mathbb{Q}$. For $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n$ and $v \in M_{\mathbb{Q}}$, we define the *norm* and the *height* of $\boldsymbol{x}$ by $|\boldsymbol{x}|_v = \max_{1\leq i\leq n}|x_i|_v$ and $H(\boldsymbol{x}) = \prod_{w\in M_{\mathbb{Q}}}|\boldsymbol{x}|_w$.

The proof of Theorem 3.1.2 mainly depends on the following theorem which is so-called the Quantitative Subspace Theorem and consequence of Corollary 3.2 in [34].

Theorem 3.3.5. *Let $\alpha \in \mathbb{Q}_p$ be an algebraic number of degree d and $0 < \varepsilon < 1$. Define linear forms*

$$L_{1\infty}(X,Y,Z) = X, \quad L_{2\infty}(X,Y,Z) = Y, \quad L_{3\infty}(X,Y,Z) = Z,$$
$$L_{1p}(X,Y,Z) = X, \quad L_{2p}(X,Y,Z) = Y, \quad L_{3p}(X,Y,Z) = \alpha X - \alpha Y - Z.$$

Then all integer solutions $\boldsymbol{x} = (x_1, x_2, x_3)$ of

$$\prod_{v \in \{p,\infty\}} \prod_{i=1}^{3} |L_{iv}(\boldsymbol{x})|_v \leq |\boldsymbol{x}|_\infty^{-\varepsilon}$$

with

$$H(\boldsymbol{x}) \geq \max\left(\left(\sqrt{d+1}H(\alpha)\right)^{1/12d}, 27^{1/\varepsilon}\right)$$

lie in the union of at most

$$2^{16}3^{39}5^{10}\varepsilon^{-9}\log(3\varepsilon^{-1}d)\log(\varepsilon^{-1}\log 3d)$$

proper linear subspaces of $\mathbb{Q}^3$.

Consider a vector hyperplane of $\mathbb{Q}^n$

$$\mathcal{H} = \{(x_1, \ldots, x_n) \in \mathbb{Q}^n \mid y_1 x_1 + \cdots + y_n x_n = 0\},$$

where $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}, \gcd(y_1, \ldots, y_n) = 1$. The *height* of $\mathcal{H}$, denoted by $H(\mathcal{H})$, is defined to be $|\boldsymbol{y}|_\infty$.

The lemma below is easily seen.

Lemma 3.3.6. *Let $m, n \geq 1$ be integers with $m < n$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{Z}^n$ be linearly independent vectors such that $|\boldsymbol{x}_1|_\infty \leq \ldots \leq |\boldsymbol{x}_m|_\infty$. Then there exists a vector hyperplane $\mathcal{H}$ of $\mathbb{Q}^n$ such that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathcal{H}$ and*

$$H(\mathcal{H}) \leq m!|\boldsymbol{x}_m|_\infty^m.$$

Lemma 3.3.7. *Let $U \in P^*, V \in P^+$, and r, s be lengths of the words U, V, respectively. Put $(a_n)_{n\geq 0} := U\overline{V}$ and $\alpha := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. Then we have $H(\alpha) \leq p^{r+s}$.*

Proof. A straightforward computation shows that

$$\begin{aligned}\alpha &= \sum_{n=0}^{r-1} a_n p^n + \left(\sum_{m=0}^{s-1} a_{m+r} p^{m+r}\right)\left(\sum_{k=0}^{\infty} p^{ks}\right) \\ &= \frac{(p^s - 1)\sum_{n=0}^{r-1} a_n p^n - \sum_{m=0}^{s-1} a_{m+r} p^{m+r}}{p^s - 1}.\end{aligned}$$

Therefore, we have

$$H(\alpha) \leq \max\left(p^s - 1, (p^s - 1)\sum_{n=0}^{r-1} a_n p^n, \sum_{m=0}^{s-1} a_{m+r} p^{m+r}\right) \leq p^{r+s}.$$

□

In order to prove Theorem 3.1.3, we show the following lemma.

Lemma 3.3.8. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* P*. Set* $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$*. Then we have*

$$w_1(\xi) \geq \max(1, \mathrm{Dio}(\mathbf{a}) - 1). \tag{3.2}$$

Proof. Since ξ is irrational, we have $w_1(\xi) \geq 1$ by Theorem 3.3.2. Therefore, we may assume that $\mathrm{Dio}(\mathbf{a}) > 1$. Take a real number δ such that $1 < \delta < \mathrm{Dio}(\mathbf{a})$. For $n \geq 1$, there exist finite words U_n, V_n and positive rational numbers w_n such that $U_n V_n^{w_n}$ are the prefix of $\mathbf{a}$, the sequence $(|V_n^{w_n}|)_{n\geq 1}$ is strictly increasing, and $|U_n V_n^{w_n}| \geq \delta |U_n V_n|$. For $n \geq 1$, we set rational number

$$\alpha_n := \sum_{i=0}^{\infty} b_i^{(n)} p^i,$$

where $(b_i^{(n)})_{i\geq 0}$ is the infinite word $U_n \overline{V_n}$. Since ξ and α_n have the same first $|U_n V_n^{w_n}|$-th digits, we obtain

$$|\xi - \alpha_n| \leq p^{-\delta |U_n V_n|} \leq H(\alpha_n)^{-\delta}$$

by Lemma 3.3.7. Hence, we have (3.2). □

3.4 Proof of main results

Proof of Theorem 3.1.2. By Theorem 3.1.1, ξ is transcendental, that is, ξ is not an A-number. Therefore, it is sufficient to prove that if ξ is not a U_1-number, then ξ is not a U-number. For $n \geq n_0$, we take finite words U_n, V_n over P and positive rational numbers w_n satisfying Lemma 2.3.10 (i)–(vi). We define a positive integer sequence $(n_k)_{k\geq 0}$ by $n_{k+1} = 4(\kappa+1)n_k$ for $k \geq 0$. We set $r_k := |U_{n_k}|, s_k := |V_{n_k}|$, and $t_k := |U_{n_k} V_{n_k}|$ for $k \geq 0$. Then a straightforward computation shows that $2t_k \leq t_{k+1} \leq ct_k, r_k \leq 2\kappa s_k$ for $k \geq 0$, and $(s_k)_{k\geq 0}$ is strictly increasing, where $c = 8(\kappa+1)^2$. For $k \geq 0$, there exists an integer p_k such that

$$\frac{p_k}{p^{s_k} - 1} = \sum_{i=0}^{\infty} b_i^{(k)} p^i,$$

where $(b_i^{(k)})_{i\geq 0}$ is the infinite word $U_{n_k} \overline{V_{n_k}}$. Since ξ and $p_k/(p^{s_k} - 1)$ have the same first $|U_{n_k} V_{n_k}^{w_{n_k}}|$-th digits, we obtain

$$\left| \xi - \frac{p_k}{p^{s_k} - 1} \right|_p \leq p^{-wt_k},$$

where $w = 1 + 1/(4\kappa+2)$. Since the sequence $(s_k)_{k\geq 1}$ is strictly increasing, we may assume that $t_0 \geq 3$.

Let $\alpha \in \mathbb{Q}_p$ be an algebraic number of degree $d \geq 2$ with $H(\alpha) \geq \max(d+1, p^{s_0}, 27^{4\kappa+2})$. We define an integer $j \geq 1$ by $p^{s_{j-1}} \leq H(\alpha) < p^{s_j}$ and a real number χ by $|\xi - \alpha|_p = H(\alpha)^{-\chi}$. In what follows, we estimate an upper bound of χ. Therefore, we may assume that $\chi > 0$. Put $M := \max\{m \in \mathbb{Z} \mid p^{wc^{m-1}t_j} < H(\alpha)^{\chi}\}$. In what follows, we estimate an upper bound of M. Therefore, we may assume that $M \geq 1$. Then we obtain $p^{wt_{j+h}} \leq p^{wc^{M-1}t_j}$ for all $0 \leq h \leq M-1$. Therefore, we have

$$\begin{aligned} |p^{s_{j+h}}\alpha - \alpha - p_{j+h}|_p &= \left| \alpha - \frac{p_{j+h}}{p^{s_{j+h}} - 1} \right|_p \\ &\leq \max\left(\left| \xi - \frac{p_{j+h}}{p^{s_{j+h}} - 1} \right|_p, |\xi - \alpha|_p \right) \leq p^{-wt_{j+h}} \end{aligned}$$

for $0 \le h \le M-1$. We define linear forms by

$$\begin{aligned} L_{1\infty}(X,Y,Z) = X, \quad L_{2\infty}(X,Y,Z) = Y, \quad L_{3\infty}(X,Y,Z) = Z, \\ L_{1p}(X,Y,Z) = X, \quad L_{2p}(X,Y,Z) = Y, \quad L_{3p}(X,Y,Z) = \alpha X - \alpha Y - Z, \end{aligned}$$

and put $\boldsymbol{x}_h := (p^{s_{j+h}}, 1, p_{j+h})$ for $0 \le h \le M-1$. By the proof of Lemma 3.3.7, we obtain

$$\prod_{v\in\{p,\infty\}} \prod_{i=1}^{3} |L_{iv}(\boldsymbol{x}_h)|_v \le |\boldsymbol{x}_h|_\infty^{-1/(4\kappa+2)}$$

for all $0 \le h \le M-1$. We also have

$$H(\boldsymbol{x}_h) = |\boldsymbol{x}_h|_\infty \ge p^{s_{j+h}} \ge H(\alpha) \ge \max\left(\left(\sqrt{d+1}H(\alpha)\right)^{1/12d}, 27^{4\kappa+2}\right)$$

for all $0 \le h \le M-1$. Hence, by Theorem 3.3.5, for all $0 \le h \le M-1$, we obtain $\boldsymbol{x}_h$ in the union of N proper linear subspaces of $\mathbb{Q}^3$, where $N = \lfloor 2^{25}3^{39}5^{10}(2\kappa+1)^9 \log(6(2\kappa+1)d)\log(2(2\kappa+1)\log 3d)\rfloor$.

Assume that one of these linear subspaces of $\mathbb{Q}^3$ contains $L+1$ points of the set $\{\boldsymbol{x}_h \mid 0 \le h \le M-1\}$, where $L = \lceil \log_2((2\kappa+1)(4d+6+\log_p(2^{2d+1}(d+1))))\rceil$. It follows that there exist $(x,y,z) \in \mathbb{Z}^3 \setminus \{\boldsymbol{0}\}$ such that

$$xp^{s_{j+i_k}} + y + zp_{j+i_k} = 0, \quad (0 \le k \le L),$$

where $0 \le i_0 < i_1 < \ldots < i_L < M$. Since $\boldsymbol{x}_{i_0}$ and $\boldsymbol{x}_{i_1}$ are linearly independent, we chose $(x,y,z) \in \mathbb{Z}^3 \setminus \{\boldsymbol{0}\}$ such that $\max(|x|,|y|,|z|) \le 2p^{2t_{j+i_1}}$ by Lemma 3.3.6. Since $(s_k)_{k\ge 0}$ is strictly increasing, we have $z \ne 0$. A straightforward computation shows that

$$\begin{aligned} (1-p^{s_{j+i_k}})\alpha &= p^{s_{j+i_k}}\frac{x}{z} + \frac{y}{z} - (p^{s_{j+i_k}}\alpha - \alpha - p_{j+i_k}), \\ \alpha - \frac{y}{z} &= p^{s_{j+i_k}}\alpha + p^{s_{j+i_k}}\frac{x}{z} - (p^{s_{j+i_k}}\alpha - \alpha - p_{j+i_k}) \end{aligned}$$

for all $0 \le k \le L$. Therefore, we obtain

$$\begin{aligned} |\alpha|_p = |(1-p^{s_{j+i_k}})\alpha|_p &\le \max\left(p^{-s_{j+i_k}}\left|\frac{x}{z}\right|_p, \left|\frac{y}{z}\right|_p, p^{-wt_{j+i_k}}\right) \\ &\le \max(|z|, p^{-wt_{j+i_k}}) \le 2p^{2t_{j+i_1}}. \end{aligned}$$

Hence, we have

$$\left|\alpha - \frac{y}{z}\right|_p \le \max(2p^{2t_{j+i_1}-s_{j+i_L}}, |z|p^{-s_{j+i_L}}, p^{-wt_{j+i_L}}) = 2p^{2t_{j+i_1}-s_{j+i_L}}. \tag{3.3}$$

It follows from Lemma 3.3.3 that

$$\begin{aligned} \left|\alpha - \frac{y}{z}\right|_p &\ge 2^{-d}(d+1)^{-1}H(\alpha)^{-1}H\left(\frac{y}{z}\right)^{-d} \\ &\ge 2^{-2d}(d+1)^{-1}p^{-2dt_{j+i_1}-s_j}. \end{aligned} \tag{3.4}$$

By the properties of $(s_k)_{k\ge 0}$ and $(t_k)_{k\ge 0}$, we have

$$s_{j+i_L} \ge \frac{t_{j+i_L}}{2\kappa+1} = \frac{1}{2\kappa+1}\frac{t_{j+i_L}}{t_{j+i_{L-1}}}\cdots\frac{t_{j+i_2}}{t_{j+i_1}}t_{j+i_1} \ge \frac{2^L t_{j+i_1}}{4\kappa+2}. \tag{3.5}$$

Applying (3.3), (3.4), and (3.5), we obtain

$$t_{j+i_1} \leq \frac{(4\kappa+2)\log_p(2^{2d+1}(d+1))}{2^L-(4\kappa+2)(2d+3)} \leq 2,$$

which is contradiction.

Hence, we get $M \leq LN$. By the definition of M, we have

$$\begin{aligned} H(\alpha)^{\chi} &\leq p^{wc^M t_j} \leq p^{wc^{M+1} t_{j-1}} \\ &\leq p^{wc^{M+1}(2\kappa+1)s_{j-1}} \leq H(\alpha)^{wc^{M+1}(2\kappa+1)}. \end{aligned}$$

Therefore, we obtain

$$|\xi-\alpha|_p \geq H(\alpha)^{-wc^{LN+1}(2\kappa+1)},$$

which implies

$$w_d^*(\xi) \leq \max(w_1(\xi), wc^{LN+1}(2\kappa+1)).$$

This completes the proof. □

Proof of Theorem 3.1.3. We first assume that ξ is not a U_1-number, that is, $w_1(\xi)$ is finite. Then $\mathrm{Dio}(\mathbf{a})$ is finite by Lemma 3.3.8.

We next assume that $\mathrm{Dio}(\mathbf{a})$ is finite. For $n \geq n_0$, take finite words U_n, V_n and rational numbers w_n satisfying Lemma 2.3.10 (i)–(vi). For $n \geq n_0$, we set rational numbers

$$\alpha_n := \sum_{i=0}^{\infty} b_i^{(n)} p^i,$$

where $(b_i^{(n)})_{i\geq 0}$ is the infinite word $U_n\overline{V_n}$. Since ξ and α_n have the same first $|U_n V_n^{w_n}|$-th digits, we obtain

$$|\xi-\alpha_n| \leq p^{-\left(1+\frac{1}{4\kappa+2}\right)|U_n V_n|}.$$

Take a real number δ which is greater than $\mathrm{Dio}(\mathbf{a})$. Note that $\delta > 1$. By the definition of the Diophantine exponent, there exists an integer $n_1 \geq n_0$ such that for all $n \geq n_1$

$$|\xi-\alpha_n| \geq p^{-\delta|U_n V_n|}.$$

We define a positive integer sequence $(n_k)_{k\geq 1}$ by $n_{k+1} = 2(\kappa+1)n_k$ for $k \geq 1$. Set $\beta_k := p^{|U_{n_k} V_{n_k}|}$ for $k \geq 1$. It follows from Lemma 2.3.10 (iii) and (vi) that for $n \geq 1$

$$\beta_k < \beta_{k+1} \leq \beta_k^{4(\kappa+1)^2}.$$

By Lemma 3.3.7, we have $H(\alpha_{n_k}) \leq \beta_k$ for $k \geq 1$. Hence, we obtain (3.1) by Lemma 3.3.4. □

Chapter 4

Diophantine exponents for the field of Laurent series over a finite field

4.1 Introduction

In this chapter, we investigate properties of Diophantine exponents for Laurent series over a finite field. We first define Diophantine exponents w_n and w_n^* for Laurent series.

Let p be a prime and q a power of p. Let us denote by $\mathbb{F}_q$ the finite field of q elements, $\mathbb{F}_q[T]$ the ring of all polynomials over $\mathbb{F}_q$, $\mathbb{F}_q(T)$ the field of all rational functions over $\mathbb{F}_q$, and $\mathbb{F}_q((T^{-1}))$ the field of all Laurent series over $\mathbb{F}_q$. For $\xi \in \mathbb{F}_q((T^{-1})) \setminus \{0\}$, we can write

$$\xi = \sum_{n=N}^{\infty} a_n T^{-n},$$

where $N \in \mathbb{Z}$, $a_n \in \mathbb{F}_q$ for all $n \geq N$, and $a_N \neq 0$. We define an absolute value on $\mathbb{F}_q((T^{-1}))$ by $|0| := 0$ and $|\xi| := q^{-N}$. This absolute value can be uniquely extended on the algebraic closure of $\mathbb{F}_q((T^{-1}))$ and we continue to write $|\cdot|$ for the extended absolute value. We call an element of $\mathbb{F}_q((T^{-1}))$ an *algebraic Laurent series* if it is algebraic over $\mathbb{F}_q(T)$.

Throughout this chapter, we regard elements of $(\mathbb{F}_q[T])[X]$ as polynomials in X. The *height* of a polynomial $P(X) \in (\mathbb{F}_q[T])[X]$, denoted by $H(P)$, is defined to be the maximum of the absolute values of the coefficients of $P(X)$. We denote by $(\mathbb{F}_q[T])[X]_{\min}$ the set of all non-constant, irreducible and primitive polynomials in $(\mathbb{F}_q[T])[X]$ whose leading coefficients are monic polynomials in T. For $\alpha \in \overline{\mathbb{F}_q(T)}$, there exists a unique $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ such that $P(\alpha) = 0$. We call the polynomial $P(X)$ the *minimal polynomial* of α. The *height* (resp. the *degree*, the *inseparable degree*) of α, denoted by $H(\alpha)$ (resp. $\deg \alpha$, $\operatorname{insep} \alpha$), is defined to be the height of $P(X)$ (resp. the degree of $P(X)$, the inseparable degree of $P(X)$). We define Diophantine exponents for Laurent series over a finite field. Let $n \geq 1$ be an integer and ξ be in $\mathbb{F}_q((T^{-1}))$. We denote by $w_n(\xi)$ (resp. $w_n^*(\xi)$) the supremum of the real numbers w (resp. w^*) which satisfy

$$0 < |P(\xi)| \leq H(P)^{-w} \quad (\text{resp. } 0 < |\xi - \alpha| \leq H(\alpha)^{-w^*-1})$$

for infinitely many $P(X) \in (\mathbb{F}_q[T])[X]$ of degree at most n (resp. $\alpha \in \overline{\mathbb{F}_q(T)}$ of degree at most n).

We use the Vinogradov notation $A \ll B$ (resp. $A \ll_a B$) if $|A| \leq c|B|$ with some constant (resp. some constant depending at most on a) $c > 0$. We write $A \asymp B$ (resp. $A \asymp_a B$) if $A \ll B$ and $B \ll A$ (resp. $A \ll_a B$ and $B \ll_a A$) hold.

4.1.1 The values of w_n and w_n^* for algebraic Laurent series

Mahler [44] proved that an analogue of the Roth Theorem in this framework does not hold, that is, there exists an algebraic Laurent series ξ such that $w_1(\xi) > 1$. Indeed, let r be a power of p and put $\xi := \sum_{n=0}^{\infty} T^{-r^n}$. Then ξ is an algebraic Laurent series of degree r with $w_1(\xi) = r - 1$. In this subsection, we investigate a phenomenon that properties of the Diophantine exponents in characteristic zero are different from that of positive characteristic.

Schmidt [54] and Thakur [58] investigated the Diophantine exponent w_1 for algebraic Laurent series. Schmidt [54] introduced classes of algebraic Laurent series as follows: Let α be in $\mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q(T)$. We say that α is of *Class I* (resp. *Class IA*) if there exist an integer $s \geq 0$ and $R, S, T, U \in \mathbb{F}_q[T]$ with $RU - ST \neq 0$ (resp. $RU - ST \in \mathbb{F}_q^{\times}$) such that

$$\alpha = \frac{R\alpha^{p^s} + S}{T\alpha^{p^s} + U}.$$

For example, any quadratic Laurent series is of Class IA. Mathan [28] proved that values of w_1 for Laurent series of Class I are rational. However, it is not known whether or not there exists an algebraic Laurent series for which the value of w_1 is irrational. Let r be a power of p. Schmidt [54] and Thakur [58] independently proved that for any rational number $1 < w \leq r$, there exists an algebraic Laurent series $\xi \in \mathbb{F}_q((T^{-1}))$ of degree at most $r + 1$ such that $w_1(\xi) = w$. It follows from Theorems 4.4.5 and 4.4.6 that $w_1(\xi) = 1$ and $w_1(\eta) \geq 1$ hold for any quadratic Laurent series $\xi \in \mathbb{F}_q((T^{-1}))$ and irrational Laurent series $\eta \in \mathbb{F}_q((T^{-1}))$. Therefore, we derive that the set of values taken by w_1 over the set of Laurent series of Class IA is equal to the set of rational numbers greater than or equal to 1. Chen [24] refined Schmidt and Thakur's result by showing that the degree of ξ can be taken to be equal to $r + 1$.

We partially extend Chen's result to w_n and w_n^* for $n \geq 2$.

Theorem 4.1.1. *Let $d \geq 1$ be an integer and $w > 2d - 1$ be a rational number. Then there exist a strictly increasing sequence of positive integers $(k_j)_{j \geq 1}$ and a sequence $(\xi_j)_{j \geq 1}$ such that, for any $j \geq 1$, ξ_j is of Class IA, and degree $p^{k_j} + 1$, and*

$$w_1(\xi_j) = w_1^*(\xi_j) = \ldots = w_d(\xi_j) = w_d^*(\xi_j) = w.$$

Remark. By Theorem 4.4.6, we obtain $w \leq p^{k_1}$.

The key point of the proof of Theorem 4.1.1 is that we can determine the values of the Diophantine exponents by using partial quotients of continued fractions. It is known that we can determine values of w_1 by using partial quotients. In this paper, for a certain class of Laurent series, we extend the known result to w_n and w_n^* for any $n \geq 1$.

As mentioned in Chapter 1, it is known that $w_n(\xi) = w_n^*(\xi)$ for all real algebraic numbers ξ and integers $n \geq 1$. The proof of this result depends on the Schmidt Subspace Theorem which is a generalization of the Roth Theorem. However, analogues of these theorems in positive characteristic do not hold (see Section 4.1). Therefore, we address the following problem.

Problem 4.1.2. Is it true that

$$w_n(\xi) = w_n^*(\xi)$$

for an integer $n \geq 1$ and an algebraic Laurent series ξ?

Note that Theorem 4.1.1 gives a partial answer to Problem 4.1.2. If we remove the condition that ξ is algebraic, then the answer to Problem 4.1.2 is not true (see Theorems 4.1.8 and 4.1.9 below).

We state some corollaries of Theorem 4.1.1.

Corollary 4.1.3. *Let $n \geq 1$ be an integer. Then the set of values taken by w_n (resp. w_n^*) over the set of Laurent series of Class IA contains the set of all rational numbers greater than $2n-1$.*

We address the following natural problem arising from Corollary 4.1.3.

Problem 4.1.4. Let $n \geq 1$ be an integer. Determine the set of values taken by w_n (resp. w_n^*) over the set of algebraic Laurent series.

Since the degree of ξ_j tends to infinity under the conditions of Theorem 4.1.1, we deduce the following corollary.

Corollary 4.1.5. *Let $d \geq 1$ be an integer and $w > 2d-1$ be a rational number. Then there exists a set $\{\xi_j \mid j \geq 1\}$ of linearly independent Laurent series of Class IA such that, for any $j \geq 1$*

$$w_1(\xi_j) = w_1^*(\xi_j) = \ldots = w_d(\xi_j) = w_d^*(\xi_j) = w.$$

4.1.2 The values of the functions $w_n - w_n^*$

We deduce the following statement in a similar method to the proof of Theorem 4.1.1.

Theorem 4.1.6. *Let $d \geq 1$ be an integer and $w \geq 2d-1$ be a real number. Then there exist uncountably many $\xi \in \mathbb{F}_q((T^{-1}))$ such that*

$$w_1(\xi) = w_1^*(\xi) = \ldots = w_d(\xi) = w_d^*(\xi) = w.$$

Analogues of Theorem 4.1.6 for real and p-adic numbers are already shown in [15, 17].

It follows from Lemma 4.4.1, Theorems 4.4.5, and 4.4.6 that $w_n(\xi) = w_n^*(\xi) = 0$ and $w_n(\eta) = w_n^*(\eta) = 1$ for all $n \geq 1, \xi \in \mathbb{F}_q(T)$ and quadratic Laurent series $\eta \in \mathbb{F}_q((T^{-1}))$. It is immediate that for any $n \geq 1, w_n(\xi) = w_n^*(\xi) = +\infty$, where $\xi = \sum_{m=1}^{\infty} T^{-m!}$. Hence, we have the following corollary of Theorem 4.1.6.

Corollary 4.1.7. *For an integer $n \geq 1$, the set of values taken by w_n (resp. w_n^*) contains the set $\{0,1\} \cup [2n-1,+\infty]$. Furthermore, the set of values taken by w_1 (resp. w_1^*) is equal to $\{0\} \cup [1,+\infty]$.*

For each $n \geq 2$, we can construct explicitly continued fractions ξ for which $w_n(\xi)$ and $w_n^*(\xi)$ are different as follows:

Theorem 4.1.8. *Let $d \geq 2$ be an integer and $w \geq (3d+2+\sqrt{9d^2+4d+4})/2$ be a real number. Let $a, b \in \mathbb{F}_q[T]$ be distinct non-constant polynomials. We define a sequence $(a_{n,w})_{n\geq 1}$ by*

$$a_{n,w} = \begin{cases} b & \text{if } n = \lfloor w^i \rfloor \text{ for some integer } i \geq 0, \\ a & \text{otherwise.} \end{cases}$$

Set $\xi_w := [0, a_{1,w}, a_{2,w}, \ldots]$*. Then we have*

$$w_n^*(\xi_w) = w - 1, \quad w_n(\xi_w) = w$$

for all $2 \leq n \leq d$.

Theorem 4.1.9. *Let* $d \geq 2$ *be an integer,* $w \geq 121d^2$ *be a real number, and* $a, b, c \in \mathbb{F}_q[T]$ *be distinct non-constant polynomials. Let* $0 < \eta < \sqrt{w}/d$ *be a positive number, and put* $m_i := \lfloor (\lfloor w^{i+1} \rfloor - \lfloor w^i - 1 \rfloor)/\lfloor \eta w^i \rfloor \rfloor$ *for all* $i \geq 1$*. We define a sequence* $(a_{n,w,\eta})_{n \geq 1}$ *by*

$$a_{n,w,\eta} = \begin{cases} b & \text{if } n = \lfloor w^i \rfloor \text{ for some integer } i \geq 0, \\ c & \text{if } n \neq \lfloor w^i \rfloor \text{ for all integers } i \geq 0, \text{ and } n = \lfloor w^j \rfloor + m\lfloor \eta w^j \rfloor \text{ for some integer } 1 \leq m \leq m_j, j \geq 1, \\ a & \text{otherwise.} \end{cases}$$

Set $\xi_{w,\eta} := [0, a_{1,w,\eta}, a_{2,w,\eta}, \ldots]$*. Then we have*

$$w_n^*(\xi_{w,\eta}) = \frac{2w - 2 - \eta}{2 + \eta}, \quad w_n(\xi_{w,\eta}) = \frac{2w - \eta}{2 + \eta}$$

for all $2 \leq n \leq d$*. Hence, we have*

$$w_n(\xi_{w,\eta}) - w_n^*(\xi_{w,\eta}) = \frac{2}{2 + \eta}$$

for all $2 \leq n \leq d$.

The key ingredient of the proofs of Theorems 4.1.8 and 4.1.9 is that for $n \geq 3$, if $\xi \in \mathbb{F}_q((T^{-1}))$ has a dense (in a suitable sense) sequence of very good quadratic approximations, then we can determine $w_n(\xi)$ and $w_n^*(\xi)$.

In Section 4.4, we prove that for an integer $n \geq 1$ and $\xi \in \mathbb{F}_q((T^{-1}))$, if $1 \leq n < 2p$, then we have

$$0 \leq w_n(\xi) - w_n^*(\xi) \leq n - 1.$$

Therefore, the following corollary is immediate from Theorems 4.1.6, 4.1.8, and 4.1.9.

Corollary 4.1.10. *Let* $d \geq 2$ *be an integer and* δ *be in the closed interval* $[0, 1]$*. Then there exist uncountably many* $\xi \in \mathbb{F}_q((T^{-1}))$ *such that* $w_n(\xi) - w_n^*(\xi) = \delta$ *for all* $2 \leq n \leq d$*. Furthermore, the set of values taken by* $w_2 - w_2^*$ *is the closed interval* $[0, 1]$*.*

In the last part of this subsection, we mention a problem associated to Corollaries 4.1.7 and 4.1.10.

Problem 4.1.11. Let $n \geq 1$ be an integer. Determine the set of values taken by w_n (resp. $w_n^*, w_n - w_n^*$).

4.1.3 The values of w_2 and w_2^* for continued fractions with low complexity

In this subsection, we study a relation the values of w_2 and w_2^* for continued fractions and Diophantine exponent for its partial quotients.

We first recall classifications of Laurent series over a finite field. For $\xi \in \mathbb{F}_q((T^{-1}))$, we set

$$w(\xi) := \limsup_{n \to \infty} \frac{w_n(\xi)}{n}, \quad w^*(\xi) := \limsup_{n \to \infty} \frac{w_n^*(\xi)}{n}.$$

We say that a Laurent series $\xi \in \mathbb{F}_q((T^{-1}))$ is an

$$\begin{gathered}
A\text{-}number \text{ if } w(\xi) = 0; \\
S\text{-}number \text{ if } 0 < w(\xi) < +\infty; \\
T\text{-}number \text{ if } w(\xi) = +\infty \text{ and } w_n(\xi) < +\infty \text{ for all } n; \\
U\text{-}number \text{ if } w(\xi) = +\infty \text{ and } w_n(\xi) = +\infty \text{ for some } n.
\end{gathered}$$

This classification of $\mathbb{F}_q((T^{-1}))$ was first introduced by Bundschuh [22] and is called *Mahler's classification.* Replacing w_n and w with w_n^* and w^*, we define A^*-, S^*-, T^*-, and U^*-numbers as above. This classification of $\mathbb{F}_q((T^{-1}))$ was first introduced by Bugeaud [12, Section 9] and is called *Koksma's classification.* Let $n \geq 1$ be an integer, $\xi \in \mathbb{F}_q((T^{-1}))$ be a U-number, and $\zeta \in \mathbb{F}_q((T^{-1}))$ be a U^*-number. The number ξ (resp. the number ζ) is called a *U_n-number* (resp. *U_n^*-number*) if $w_n(\xi)$ is infinite and $w_m(\xi)$ is finite (resp. $w_n^*(\zeta)$ is infinite and $w_m^*(\zeta)$ is finite) for all $1 \leq m < n$.

Let ξ be in $\mathbb{F}_q((T^{-1}))$ and $[a_0, a_1, a_2, \ldots]$ denote the continued fraction expansion of ξ. We define sequences $(p_n)_{n \geq -1}$ and $(q_n)_{n \geq -1}$ by

$$\begin{cases} p_{-1} = 1, \; p_0 = a_0, \; p_n = a_n p_{n-1} + p_{n-2}, \; n \geq 1, \\ q_{-1} = 0, \; q_0 = 1, \; q_n = a_n q_{n-1} + q_{n-2}, \; n \geq 1. \end{cases}$$

We call $(p_n/q_n)_{n \geq 0}$ the *convergent sequence* of ξ.

We estimates an upper bound of $w_2(\xi)$ by using the Diophantine exponent for partial quotient of $\xi \in \mathbb{F}_q((T^{-1}))$.

Theorem 4.1.12. *Let $\kappa \geq 2$ and $A \geq q$ be integers, and $\mathbf{a} = (a_n)_{n \geq 1}$ be a sequence over $\mathbb{F}_q[T]$ with $q \leq |a_n| \leq A$ for all $n \geq 1$. Assume that there exists integers $\kappa \geq 2$ and $n_0 \geq 1$ such that*

$$p(\mathbf{a}, n) \leq \kappa n \text{ for all } n \geq n_0,$$

and the Diophantine exponent of $\mathbf{a}$ is finite. Set $\xi := [0, a_1, a_2, \ldots]$. Then we have

$$w_2(\xi) \leq 64(2\kappa + 1)^3 \operatorname{Dio}(\mathbf{a}) \left(\frac{\log A}{\log q} \right)^4. \tag{4.1}$$

In particular, if the sequence $(|q_n|^{1/n})_{n \geq 1}$ converges, then we have

$$w_2(\xi) \leq 64(2\kappa + 1)^3 \operatorname{Dio}(\mathbf{a}). \tag{4.2}$$

By Lemmas 2.3.3–2.3.8, we obtain the upper bound of w_2 of automatic, primitive morphic, and Sturmian continued fractions as follows:

Corollary 4.1.13. *Let $k \geq 2$ be an integer. Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a non-ultimately periodic and k-automatic sequence over $\mathbb{F}_q[T]$ with $\deg a_n \geq 1$ for all $n \geq 0$. Let A be an upper bound of the sequence $(|a_n|)_{n \geq 0}$, m be a cardinality of k-kernel of $\mathbf{a}$, and d be a cardinality of the initial alphabet associated with $\mathbf{a}$. Set $\xi := [0, a_0, a_1, \ldots]$. Then we have*

$$w_2(\xi) \leq 64(2kd^2 + 1)^3 k^m \left(\frac{\log A}{\log q} \right)^4.$$

Corollary 4.1.14. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately and primitive morphic sequence over* $\mathbb{F}_q[T]$ *with* $\deg a_n \geq 1$ *for all* $n \geq 0$*, which is generated by a primitive morphism* σ *over a finite set of cardinality* $b \geq 2$*. Let* v *be the width of* σ*. Set* $\xi := [0, a_0, a_1, \ldots]$*. Then we have*

$$w_2(\xi) \leq 64(4v^{4b-2}b^3 + 1)^3 \operatorname{Dio}(\mathbf{a}) \left(\frac{\log A}{\log q}\right)^4.$$

Corollary 4.1.15. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately and Sturmian sequence over* $\mathbb{F}_q[T]$ *with* $\deg a_n \geq 1$ *for all* $n \geq 0$*. Set* $\xi := [0, a_0, a_1, \ldots]$*. Then we have*

$$w_2(\xi) \leq 8000 \operatorname{Dio}(\mathbf{a}) \left(\frac{\log A}{\log q}\right)^4$$

if the slope of $\mathbf{a}$ *has bounded partial quotients, and we have* $w_2(\xi) = +\infty$ *otherwise.*

Conversely, we estimates an lower bound of $w_2(\xi)$ by using the Diophantine exponent for partial quotient of $\xi \in \mathbb{F}_q((T^{-1}))$.

Theorem 4.1.16. *Let* $\mathbf{a} = (a_n)_{n\geq 1}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q[T]$ *with* $\deg a_n \geq 1$ *for all* $n \geq 1$*. Assume that* $(|q_n|^{1/n})_{n\geq 1}$ *is bounded. Put*

$$m := \liminf_{n\to\infty} |q_n|^{1/n}, \quad M := \limsup_{n\to\infty} |q_n|^{1/n}.$$

Set $\xi := [0, a_1, a_2, \ldots]$*. Then we have*

$$w_2(\xi) \geq w_2^*(\xi) \geq \max\left(2, \frac{\log m}{\log M} \operatorname{Dio}(\mathbf{a}) - 1\right). \tag{4.3}$$

In particular, if the sequence $(|q_n|^{1/n})_{n\geq 1}$ *converges, then we have*

$$w_2(\xi) \geq w_2^*(\xi) \geq \max(2, \operatorname{Dio}(\mathbf{a}) - 1).$$

Furthermore, assume that the sequence $(|a_n|)_{n\geq 1}$ *is bounded. Then we have*

$$w_2(\xi) \geq \max\left(2, \frac{\log m}{\log M}(\operatorname{Dio}(\mathbf{a}) + 1) - 1\right). \tag{4.4}$$

In particular, if the sequence $(|q_n|^{1/n})_{n\geq 1}$ *converges, then we have*

$$w_2(\xi) \geq \max(2, \operatorname{Dio}(\mathbf{a})).$$

Theorems 4.1.12 and 4.1.16 are analogues of Theorems 2.2 and 2.3 in [18]. Theorems 4.1.12 and 4.1.16 are proved in a similar method of the proof of these analogue theorems.

We state an immediately consequence of Theorems 4.1.12 and 4.1.16.

Corollary 4.1.17. *Let* $\mathbf{a} = (a_n)_{n\geq 1}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q[T]$ *with* $\deg a_n \geq 1$ *for* $n \geq 1$*. Assume that* $(|a_n|)_{n\geq 1}$ *is bounded and*

$$\limsup_{n\to\infty} \frac{p(\mathbf{a}, n)}{n} < +\infty.$$

Set $\xi := [0, a_1, a_2, \ldots]$*. Then the Diophantine exponent of* $\mathbf{a}$ *is infinite if and only if* ξ *is a* U_2*-number.*

4.2 Liouville inequalities

Lemma 4.2.1. *Let $P(X)$ be in $(\mathbb{F}_q[T])[X]$. Assume that $P(X)$ can be factorized as*

$$P(X) = A\prod_{i=1}^{n}(X - \alpha_i),$$

where $A \in \mathbb{F}_q[T]$ and $\alpha_i \in \overline{\mathbb{F}_q(T)}$ for $1 \leq i \leq n$. Then we have

$$H(P) = |A|\prod_{i=1}^{n}\max(1, |\alpha_i|). \tag{4.5}$$

Furthermore, for $P(X), Q(X) \in (\mathbb{F}_q[T])[X]$, we have

$$H(PQ) = H(P)H(Q).$$

Proof. To be self-contained, we give a proof of this lemma which is well-known.

We expand $P(X)$ as

$$P(X) = AX^n + \sum_{k=1}^{n}\left((-1)^k A \sum_{1\leq i_1<\ldots<i_k\leq n}\alpha_{i_1}\cdots\alpha_{i_k}\right)X^k. \tag{4.6}$$

Therefore, it follows that $H(P) \leq |A|\prod_{i=1}^{n}\max(1, |\alpha_i|)$. If $|\alpha_i| \leq 1$ for all $1 \leq i \leq n$, then it follows from (4.6) that $H(P) = |A|$. Thus, we may assume that there exists $1 \leq i \leq n$ such that $|\alpha_i| > 1$. We put $k := \mathrm{Card}\{1 \leq i \leq n \mid |\alpha_i| > 1\}$. Then we have

$$\left|(-1)^k A \sum_{1\leq i_1<\ldots<i_k\leq n}\alpha_{i_1}\cdots\alpha_{i_k}\right| = |A|\prod_{i=1}^{n}\max(1, |\alpha_i|).$$

Hence, we obtain (4.5).

We write $Q(X) = B\prod_{j}^{m}(X - \beta_j)$, where $B \in \mathbb{F}_q[T]$ and $\beta_j \in \overline{\mathbb{F}_q(T)}$ for $1 \leq j \leq m$. Since

$$P(X)Q(X) = AB\left(\prod_{i=1}^{n}(X - \alpha_i)\right)\left(\prod_{j=1}^{m}(X - \beta_j)\right),$$

we have $H(PQ) = H(P)H(Q)$. □

The proposition below is an analogue of Theorem A.1 in [12].

Proposition 4.2.2. *Let $P(X), Q(X) \in (\mathbb{F}_q[T])[X]$ be non-constant polynomials of degree m, n, respectively. Let α be a root of $P(X)$ of order t and β be a root of $Q(X)$ of order u. Assume that $P(\beta) \neq 0$. Then we have*

$$|P(\beta)| \geq \max(1, |\beta|)^m H(P)^{-n/u+1}H(Q)^{-m/u}. \tag{4.7}$$

Furthermore, we have

$$|\alpha - \beta| \geq \max(1, |\alpha|)\max(1, |\beta|)H(P)^{-n/tu}H(Q)^{-m/tu}. \tag{4.8}$$

Proof. Write $P(X) = A\prod_{i=1}^{r}(X-\alpha_i)^{t_i}$ and $Q(X) = B\prod_{i=1}^{s}(X-\beta_i)^{u_i}$, where $A, B \in \mathbb{F}_q[T], \alpha = \alpha_1, \beta = \beta_1, t = t_1, u = u_1$, and α's (resp. β's) are pairwise distinct. Let $Q_1(X) = B_1\prod_{i=1}^{s_1}(X-\beta^{(i)})^g$ be the minimal polynomial of β, where $B_1 \in \mathbb{F}_q[T], \beta^{(1)} = \beta, g = \operatorname{insep}\beta$, and $\beta^{(i)}$'s are pairwise distinct. Since P and Q_1 do not have common roots, the resultant $\operatorname{Res}(P, Q_1)$ is non-zero and is in $\mathbb{F}_q[T]$. Therefore, by $H(Q_1)^{u/g} \leq H(Q), s_1 u \leq n$, and Lemma 4.2.1, we obtain

$$\begin{aligned}
1 \leq |\operatorname{Res}(P, Q_1)| &= |B_1|^m \prod_{i=1}^{s_1} |P(\beta^{(i)})|^g \\
&\leq |B_1|^m |P(\beta)|^g H(P)^{(s_1-1)g} \prod_{i=2}^{s_1} \max(1, |\beta^{(i)}|)^{mg} \\
&= |P(\beta)|^g H(P)^{(s_1-1)g} \left(\frac{H(Q_1)}{\max(1, |\beta|)^g}\right)^m \\
&\leq |P(\beta)|^g H(P)^{(n/u-1)g} H(Q)^{mg/u} \max(1, |\beta|)^{-mg},
\end{aligned}$$

which implies (4.7). It follows from Lemma 4.2.1 that

$$\begin{aligned}
|P(\beta)| &\leq |\beta-\alpha|^t |A| \max(1, |\beta|)^{m-t} \prod_{i=2}^{r} \max(1, |\alpha_i|)^{t_i} \\
&= |\beta-\alpha|^t H(P) \max(1, |\alpha|)^{-t} \max(1, |\beta|)^{m-t}.
\end{aligned}$$

Hence, we have (4.8) by (4.7). □

The lemma below is an analogue of Theorem A.3 in [12] and Lemma 2.3 in [50].

Lemma 4.2.3. *Let $P(X) \in (\mathbb{F}_q[T])[X]$ be an irreducible polynomial of degree n and inseparable degree f. Assume that $n \geq 2f$. For any distinct roots α, β of $P(X)$, we have*

$$|\alpha-\beta| \geq H(P)^{-n/f^2+1/f}. \tag{4.9}$$

Proof. We can write $P(X) = A\prod_{i=1}^{m}(X-\alpha_i)^f$, where $A \in \mathbb{F}_q[T], \alpha_1 = \alpha, \alpha_2 = \beta$, and α's are pairwise distinct. Put $Q(X) := A\prod_{i=1}^{m}(X-\alpha_i^f)$. Since $Q(X)$ is separable, the discriminant $\operatorname{Disc}(Q)$ is non-zero and is in $\mathbb{F}_q[T]$. Therefore, we obtain

$$\begin{aligned}
1 \leq |\operatorname{Disc}(Q)| &\leq |\alpha^f - \beta^f|^2 |A|^{2m-2} \prod_{\substack{1\leq i<j\leq m \\ (i,j)\neq(1,2)}} \max(1, |\alpha_i^f|)^2 \max(1, |\alpha_j^f|)^2 \\
&\leq |\alpha-\beta|^{2f} H(Q)^{2m-2}.
\end{aligned}$$

Hence, we have (4.9) by $H(P) = H(Q)$ and $n = mf$. □

The following proposition is an analogue of Corollary A.2 in [12] and Lemma 2.5 in [50], and is an extension of Theorem 1 in [44].

Proposition 4.2.4. *Let $\alpha, \beta \in \overline{\mathbb{F}_q(T)}$ be distinct algebraic numbers of degree m, n and inseparable degree f, g, respectively. Then we have*

$$|\alpha-\beta| \geq \max(1, |\alpha|)\max(1, |\beta|) H(\alpha)^{-n/fg} H(\beta)^{-m/fg}. \tag{4.10}$$

Proof. Let $P_\alpha(X)$ and $P_\beta(X)$ be the minimal polynomials of degree m and n, respectively. We first consider the case of $P_\alpha(X) \neq P_\beta(X)$. Then we have $P_\alpha(\beta) \neq 0$. (4.10) follows from Proposition 4.2.2.

Next, we consider the case of $P_\alpha(X) = P_\beta(X)$. By Lemmas 4.2.1 and 4.2.3, we obtain

$$\begin{aligned}|\alpha - \beta| &\geq H(\alpha)^{-n/f^2+1/f} \geq H(\alpha)^{1/f} H(\alpha)^{-n/fg} H(\beta)^{-m/fg} \\ &\geq \max(1, |\alpha|) \max(1, |\beta|) H(\alpha)^{-n/fg} H(\beta)^{-m/fg}.\end{aligned}$$

□

Let $\alpha \in \overline{\mathbb{F}_q(T)}$ be a quadratic number. Then we denote by α' the Galois conjugate of α which is different from α if $\operatorname{insep} \alpha = 1$, and itself if $\operatorname{insep} \alpha = 2$. The lemma below is an analogue of Lemma 3.2 in [50].

Lemma 4.2.5. *Let $\alpha \in \overline{\mathbb{F}_q(T)}$ be a quadratic number. If $\alpha \neq \alpha'$, then we have*

$$H(\alpha)^{-1} \leq |\alpha - \alpha'| \leq H(\alpha). \tag{4.11}$$

Proof. Let $P_\alpha(X) = AX^2 + BX + C$ be the minimal polynomial of α. Then we have

$$|\alpha - \alpha'| = \frac{|B^2 - 4AC|^{1/2}}{|A|} \leq \max(|B|, |AC|^{1/2}) \leq H(\alpha)$$

and

$$|\alpha - \alpha'| \geq \frac{1}{|A|} \geq H(\alpha)^{-1}.$$

□

We give a better estimate than Proposition 4.2.4 in some cases, which is an analogue of Lemma 7.1 in [18] and Lemma 4 in [20].

Proposition 4.2.6. *Let $\alpha, \beta \in \overline{\mathbb{F}_q(T)}$ be quadratic numbers. We denote by $P_\alpha(X) = A(X-\alpha)(X-\alpha'), P_\beta(X) = B(X-\beta)(X-\beta')$ the minimal polynomials of α, β, respectively. If $\alpha \neq \alpha'$ and $P_\alpha(X) \neq P_\beta(X)$, then we have*

$$|\alpha - \beta| \geq \max(1, |\alpha - \alpha'|^{-1}) H(\alpha)^{-2} H(\beta)^{-2}. \tag{4.12}$$

Proof. By Proposition 4.2.4, we may assume that $|\alpha - \alpha'| < 1$. Since $P_\alpha(X)$ and $P_\beta(X)$ does not have common roots, we have

$$\begin{aligned}1 &\leq |\operatorname{Res}(P_\alpha, P_\beta)| = |B|^2 |P_\alpha(\beta)||P_\alpha(\beta')| \\ &\leq |AB^2||\alpha - \beta||\alpha' - \beta| H(\alpha) \max(1, |\beta'|)^2 \\ &\leq |\alpha - \beta||\alpha' - \beta| H(\alpha)^2 H(\beta)^2.\end{aligned}$$

In the case of $|\alpha' - \beta| > |\alpha - \beta|$, we have $|\alpha - \alpha'| = |\alpha' - \beta|$. Hence, we get (4.12). In the other cases, using Lemma 4.2.5, we obtain

$$|\alpha - \beta|^2 \geq |\alpha - \beta||\alpha' - \beta| \geq H(\alpha)^{-2} H(\beta)^{-2} \geq |\alpha - \alpha'|^{-2} H(\alpha)^{-4} H(\beta)^{-4}.$$

Therefore, we have (4.12). □

4.3 Continued fractions

We collect fundamental properties of continued fractions for Laurent series over a finite field. The lemma below is immediate by induction on n.

Lemma 4.3.1. *Consider a continued fraction $\xi = [a_0, a_1, a_2, \ldots] \in \mathbb{F}_q((T^{-1}))$. Let $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of ξ. Then the following hold: for any $n \geq 0$,*

(i) $\dfrac{p_n}{q_n} = [a_0, \ldots, a_n]$,

(ii) $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$,

(iii) $\gcd(p_n, q_n) = 1$,

(iv) $|p_0| = |a_0|, |p_{n+1}| = \begin{cases} |a_0 a_1 \cdots a_{n+1}| & (a_0 \neq 0), \\ |a_2 a_3 \cdots a_{n+1}| & (a_0 = 0), \end{cases}$

(v) $|q_{n+1}| = |a_1 a_2 \cdots a_{n+1}|$,

(vi) $\xi = \dfrac{\xi_{n+1} p_n + p_{n-1}}{\xi_{n+1} q_n + q_{n-1}}$, *where* $\xi_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$,

(vii) $\left| \xi - \dfrac{p_n}{q_n} \right| = \dfrac{1}{|q_n||q_{n+1}|} = \dfrac{1}{|a_{n+1}||q_n|}$,

(viii) $\dfrac{q_{n+1}}{q_n} = [a_{n+1}, a_n, \ldots a_1]$.

We recall an analogue of Lagrange's theorem for Laurent series over a finite field.

Theorem 4.3.2. *Let ξ be in $\mathbb{F}_q((T^{-1}))$. Then ξ is quadratic if and only if its continued fraction expansion is ultimately periodic.*

Proof. See e.g. [23, Theorems 3 and 4]. □

Lemma 4.3.3. *Consider an ultimately periodic continued fraction*

$$\xi = [0, a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}] \in \mathbb{F}_q((T^{-1}))$$

for $r \geq 0, s \geq 1$. Let $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of ξ. Then ξ is a root of the following equation:

$$\begin{aligned}(q_{r-1}q_{r+s} - q_r q_{r+s-1})X^2 - (q_{r-1}p_{r+s} - q_r p_{r+s-1} + p_{r-1}q_{r+s} - p_r q_{r+s-1})X \\ + p_{r-1}p_{r+s} - p_r p_{r+s-1} = 0, \end{aligned} \tag{4.13}$$

and we have $H(\xi) \leq |q_r q_{r+s}|$. In particular, if $\xi = [0, \overline{a_1, \ldots, a_s}]$, then ξ is a root of the following equation:

$$q_{s-1}X^2 - (p_{s-1} - q_s)X - p_s = 0,$$

and we have $H(\xi) \leq |q_s|$.

Proof. It follows from Lemma 4.3.1 (vi) that

$$\xi = \frac{\xi_{r+1}p_r + p_{r-1}}{\xi_{r+1}q_r + q_{r-1}} = \frac{\xi_{r+s+1}p_{r+s} + p_{r+s-1}}{\xi_{r+s+1}q_{r+s} + q_{r+s-1}}.$$

By $\xi_{r+1} = \xi_{r+s+1}$, we obtain

$$\xi_{r+1} = \frac{q_{r-1}\xi - p_{r-1}}{-q_r\xi + p_r} = \frac{q_{r+s-1}\xi - p_{r+s-1}}{-q_{r+s}\xi + p_{r+s}}.$$

Therefore, we deduce that ξ satisfies the equation (4.13). We obtain $H(\xi) \le |q_r q_{r+s}|$ by Lemma 4.3.1 (iv) and (v). □

Lemma 4.3.4. *Let $M \ge q$ be an integer and $\xi = [0, a_1, a_2, \ldots], \zeta = [0, b_1, b_2, \ldots] \in \mathbb{F}_q((T^{-1}))$ be continued fractions with $|a_n|, |b_n| \le M$ for all $n \ge 1$. Assume that there exists an integer $n_0 \ge 1$ such that $a_n = b_n$ for all $1 \le n \le n_0$ and $a_{n_0+1} \ne b_{n_0+1}$. Then we have*

$$|\xi - \zeta| \ge \frac{1}{M^2|q_{n_0}|^2}.$$

Proof. See [2, Lemma 3]. □

Lemma 4.3.5. *For $n \ge 0$, consider an ultimately periodic continued fraction $\xi = [\overline{a_0, a_1, \ldots, a_n}] \in \mathbb{F}_q((T^{-1}))$ with $\deg a_0 \ge 1$. Then we have*

$$-\frac{1}{\xi'} = [\overline{a_n, a_{n-1}, \ldots, a_0}].$$

Proof. See the proof of Lemma 2 in [31]. □

The following lemma is an analogue of Lemma 6.1 in [18].

Lemma 4.3.6. *For $r, s \ge 1$, consider an ultimately periodic continued fraction $\xi = [0, a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}] \in \mathbb{F}_q((T^{-1}))$ with $a_r \ne a_{r+s}$. Let $(p_n/q_n)_{n\ge 0}$ be the convergent sequence of ξ. Then we have*

$$\frac{\min(|a_r|, |a_{r+s}|)}{|q_r|^2} \le |\xi - \xi'| \le \frac{|a_r a_{r+s}|}{|q_r|^2}. \tag{4.14}$$

Proof. Put $\tau := [\overline{a_{r+1}, \ldots, a_{r+s}}]$. By Lemma 4.3.5, we have $\tau' = -[0, \overline{a_{r+s}, \ldots, a_{r+1}}]$. Since

$$\xi = \frac{p_r\tau + p_{r-1}}{q_r\tau + q_{r-1}}, \quad \xi' = \frac{p_r\tau' + p_{r-1}}{q_r\tau' + q_{r-1}},$$

we obtain

$$|\xi - \xi'| = \frac{|\tau - \tau'|}{|q_r\tau + q_{r-1}||q_r\tau' + q_{r-1}|}$$

by Lemma 4.3.1 (ii). We see $|\tau - \tau'| = |a_{r+1}|$ and $|q_r\tau + q_{r-1}| = |q_r||a_{r+1}|$. It follows from Lemma 4.3.1 (viii) that

$$|q_r\tau' + q_{r-1}| = |q_r|\left|\tau' + \frac{q_{r-1}}{q_r}\right| = \frac{|q_r||[\overline{a_{r+s}, \ldots, a_{r+1}}] - [a_r, \ldots, a_1]|}{|[\overline{a_{r+s}, \ldots, a_{r+1}}]||[a_r, \ldots, a_1]|}$$
$$= \frac{|q_r||a_{r+s} - a_r|}{|a_r a_{r+s}|}.$$

Since $1 \le |a_{r+s} - a_r| \le \max(|a_{r+s}|, |a_r|)$, we obtain (4.14). □

The lemma below is an analogue of Lemma 6.3 in [18].

Lemma 4.3.7. *Let $b, c, d \in \mathbb{F}_q[T]$ be distinct non-constant polynomials, $n \geq 1$ be an integer, and $a_1, \dots, a_{n-1} \in \mathbb{F}_q[T]$ be non-constant polynomials. Put*

$$\xi := [0, a_1, \dots, a_{n-1}, c, \overline{b}].$$

Then ξ is quadratic and

$$H(\xi) \asymp_{b,c} |q_n|^2,$$

where $(p_k/q_k)_{k\geq 0}$ is the convergent sequence of ξ. Let $m \geq 2$ be an integer. Set

$$\zeta := [0, a_1, \dots, a_{n-1}, c, \overline{b, \dots, b, d}],$$

where the length of period part of ζ is m. Then ζ is quadratic and

$$H(\zeta) \asymp_{b,c,d} |\tilde{q}_n \tilde{q}_{n+m}|,$$

where $(\tilde{p}_k/\tilde{q}_k)_{k\geq 0}$ is the convergent sequence of ζ.

Proof. It follows from Theorem 4.3.2 that ξ and ζ are quadratic. By Lemma 4.3.3, we have $H(\xi) \ll_{b,c} |q_n|^2$ and $H(\zeta) \ll_{b,c,d} |\tilde{q}_n \tilde{q}_{n+m}|$. Let $P_\xi(X) = A(X - \xi)(X - \xi')$ be the minimal polynomial of ξ, where $A \in \mathbb{F}_q[T]$. Since $P_\xi(p_n/q_n)$ is non-zero, we obtain $|P_\xi(p_n/q_n)| \geq 1/|q_n|^2$. From Lemmas 4.3.1 (v) and 4.3.6, it follows that

$$\left|\xi - \frac{p_n}{q_n}\right|, \left|\xi' - \frac{p_n}{q_n}\right| \ll_{b,c} \frac{1}{|q_n|^2}.$$

Therefore, we obtain $|q_n|^2 \ll_{b,c} |A| \ll_{b,c} H(\xi)$. We denote by $P_\zeta(X)$ the minimal polynomial of ζ. Since P_ζ and P_ξ do not have a common root, we have

$$1 \leq |\operatorname{Res}(P_\zeta, P_\xi)| \leq H(\zeta)^2 H(\xi)^2 |\xi - \zeta||\xi' - \zeta||\xi - \zeta'||\xi' - \zeta'|.$$

Note that $q_n = \tilde{q}_n$. By Lemma 4.3.6, we obtain

$$|\xi - \zeta| \ll_{b,c,d} |\tilde{q}_{n+m}|^{-2}, \quad |\xi' - \zeta|, |\xi - \zeta'|, |\xi' - \zeta'| \ll_{b,c,d} |\tilde{q}_n|^{-2}.$$

Therefore, it follows that $1 \ll_{b,c,d} H(\zeta)^2 H(\xi)^2 |\tilde{q}_n|^{-6} |\tilde{q}_{n+m}|^{-2}$. Hence, we have the inequality $|\tilde{q}_n \tilde{q}_{n+m}| \ll_{b,c,d} H(\zeta)$. □

4.4 Properties of w_n and w_n^*

Let ξ be in $\mathbb{F}_q((T^{-1}))$ and $n \geq 1$ be an integer. We denote by $\tilde{w}_n(\xi)$ the supremum of the real numbers w which satisfy

$$0 < |P(\xi)| \leq H(P)^{-w}$$

for infinitely many $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n.

Lemma 4.4.1. *Let ξ be in $\mathbb{F}_q((T^{-1}))$. Then the sequences $(w_n(\xi))_{n\geq 1}, (\tilde{w}_n(\xi))_{n\geq 1}$ and $(w_n^*(\xi))_{n\geq 1}$ are increasing sequence and satisfy*

$$0 \leq w_n(\xi), \tilde{w}_n(\xi), w_n^*(\xi) \leq +\infty$$

for all $n \geq 1$.

Proof. It immediate that the sequences $(w_n(\xi))_{n\geq 1}, (\tilde{w}_n(\xi))_{n\geq 1}$ and $(w_n^*(\xi))_{n\geq 1}$ are increasing and $\tilde{w}_n(\xi) \leq w_n(\xi)$ for all $n \geq 1$. Therefore, it is sufficient to show that $\tilde{w}_1(\xi), w_1^*(\xi) \geq 0$. We write

$$\xi = \sum_{n=N}^{\infty} a_n T^{-n},$$

where $N \in \mathbb{Z}, a_n \in \mathbb{F}_q$ for all $n \geq N$ and $a_N \neq 0$. For $n \geq \max(1, N)$, we put

$$P_n(X) := \begin{cases} T^n X - \sum_{m=N}^{n} a_m T^{n-m} & \text{if } a_n \neq 0, \\ T^n X - \sum_{m=N}^{n} a_m T^{n-m} - 1 & \text{otherwise,} \end{cases}$$

$$\alpha_n := \begin{cases} \left(\sum_{m=N}^{n} a_m T^{n-m}\right)/T^n & \text{if } a_n \neq 0, \\ \left(\sum_{m=N}^{n} a_m T^{n-m} + 1\right)/T^n & \text{otherwise.} \end{cases}$$

Then we see that $P_n(X)$ is in $(\mathbb{F}_q[T])[X]_{\min}$ of degree one and the minimal polynomial of α_n with $q^n \leq H(P_n) = H(\alpha_n) \leq q^{n+\max(0,N)}$ for $n \geq \max(1, N)$. Therefore, we obtain

$$0 < |P_n(\xi)| \leq H(P_n)^{-0}, \quad 0 < |\xi - \alpha_n| \leq H(\alpha_n)^{-1+\max(0,N)/n}$$

for all $n \geq \max(1, N)$, which implies that $\tilde{w}_1(\xi), w_1^*(\xi) \geq 0$. □

For $\xi \in \mathbb{F}_q((T^{-1}))$ and integers $n, H \geq 1$, let $w_n(\xi, H)$ and $w_n^*(\xi, H)$ be given by

$$w_n(\xi, H) = \min\{|P(\xi)| \mid P(X) \in (\mathbb{F}_q[T])[X], H(P) \leq H, \deg_X P \leq n, P(\xi) \neq 0\},$$
$$w_n^*(\xi, H) = \min\{|\xi - \alpha| \mid \alpha \in \overline{\mathbb{F}_q(T)}, H(\alpha) \leq H, \deg \alpha \leq n, \alpha \neq \xi\}.$$

Lemma 4.4.2. *Let ξ be in $\mathbb{F}_q((T^{-1}))$ and $n \geq 1$ be an integer.*

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(\xi, H)}{\log H}, \quad w_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log H w_n^*(\xi, H)}{\log H}.$$

Proof. We put

$$w_n'(\xi) := \limsup_{H\to\infty} \frac{-\log w_n(\xi, H)}{\log H}.$$

For $w < w_n(\xi)$, there exists a sequence of distinct terms $(P_k(X))_{k\geq 1}$ with $P_k(X) \in (\mathbb{F}_q[T])[X]$ of degree at most n such that

$$0 < |P_k(\xi)| \leq H(P_k)^{-w}.$$

Since the set

$$\{P(X) \in (\mathbb{F}_q[T])[X] \mid \deg P \leq n, H(P) \leq H\}$$

is finite for each $H \geq 1$, the sequence $(H(P_k))_{k\geq 1}$ is divergent. Therefore, we have

$$0 < w_n(\xi, H(P_k)) \leq |P_k(\xi)| \leq H(P_k)^{-w},$$

which implies $w \leq w_n'(\xi)$. Hence, we obtain $w_n(\xi) \leq w_n'(\xi)$.

By Lemma 4.4.1, we may assume that $w_n'(\xi) > 0$. For $0 < w < w_n'(\xi)$, there exists a strictly increasing sequence of positive integers $(H_k)_{k\geq 1}$ such that

$$w \leq \frac{-\log w_n(\xi, H_k)}{\log H_k}.$$

We take a sequence $(P_k(X))_{k\geq 1}$ with $P_k(X)\in(\mathbb{F}_q[T])[X]$ and $|P_k(\xi)|=w_n(\xi,H_k)$ for any $k\geq 1$. Since $H(P_k)\leq H_k$ for any $k\geq 1$, we obtain

$$0<|P_k(\xi)|\leq H(P_k)^{-w}$$

for any $k\geq 1$. Therefore, we have $w_n(\xi)\leq w_n'(\xi)$, which implies $w_n(\xi)=w_n'(\xi)$.

In a similar way to the above proof, we deduce that

$$w_n^*(\xi)=\limsup_{H\to\infty}\frac{-\log Hw_n^*(\xi,H)}{\log H}.$$

□

The lemma below is a slight improvement of a result in [57, Section 3.4].

Lemma 4.4.3. *Let $n\geq 1$ be an integer and ξ be in $\mathbb{F}_q((T^{-1}))$. Then we have*

$$w_n(\xi)=\tilde{w}_n(\xi).$$

Proof. It is sufficient to show that $w_n(\xi)\leq\tilde{w}_n(\xi)$. By Lemma 4.4.1, we may assume that $w_n(\xi)>0$ and $\tilde{w}_n(\xi)$ is finite. For $0<w<w_n(\xi)$, there exist infinitely many $P(X)\in(\mathbb{F}_q[T])[X]$ of degree at most n such that

$$0<|P(\xi)|\leq H(P)^{-w}. \tag{4.15}$$

We can write $P(X)=A\prod_{i=1}^{k}P_i(X)$, where $A\in\mathbb{F}_q[T]$ and $P_i(X)\in(\mathbb{F}_q[T])[X]_{\min}$ for $1\leq i\leq k$. By the definition, for $\tilde{w}>\tilde{w}_n(\xi)$, there exists a positive number C such that for all $Q(X)\in(\mathbb{F}_q[T])[X]_{\min}$ of degree at most n,

$$|Q(\xi)|\geq CH(Q)^{-\tilde{w}}.$$

Therefore, by Lemma 4.2.1, we obtain

$$|P(\xi)|\geq\min(1,C^n)H(P)^{-\tilde{w}},$$

which implies $\min(1,C^n)H(P)^w\leq H(P)^{\tilde{w}}$. Since there exist infinitely many such polynomials $P(X)$, we have $w\leq\tilde{w}$. This completes the proof. □

Lemma 4.4.4. *Let $\xi\in\mathbb{F}_q((T^{-1}))$ be an algebraic Laurent series. Then we get* $\operatorname{insep}\xi=1$.

Proof. Let $P_\xi(X)\in(\mathbb{F}_q[T])[X]_{\min}$ be the minimal polynomial of ξ. Assume that $\operatorname{insep}\xi\neq 1$. Then we can write $P_\xi(X)=\sum_{k=0}^{m}A_kX^{kp}$ for some $m\geq 1, A_k\in\mathbb{F}_q[T]$ and $A_m\neq 0$. Therefore, we obtain

$$\sum_{k=0}^{m-1}\left(\frac{A_k}{A_m}\right)'\xi^{kp}=0.$$

Here, $'$ is formal derivative operator with respect to T. It follows from minimality of degree of the minimal polynomial that $(A_k/A_m)'=0$ for $0\leq k\leq m-1$. Thus, we can write

$$\frac{A_k}{A_m}=\sum_{n=N_k}^{\infty}a_{n,k}T^{-n},$$

where $N_k \in p\mathbb{Z}$, $a_{n,k} \in \mathbb{F}_q$ and $(a_{n,k})_{n\geq N_k}$ is ultimately periodic with $a_{n,k} = 0$ for an integer n with $n \not\equiv 0 \mod p$. For an integer $n \geq N_k/p$, we set $b_{n,k} := a_{np,k}^{q/p}$. Then the sequence $(b_{n,k})_{n\geq N_k/p}$ is also ultimately periodic. Therefore, we obtain

$$\frac{A_k}{A_m} = \sum_{n=N_k/p}^{\infty} b_{n,k}^p T^{-np} = \left(\frac{B_k}{B_{k,m}}\right)^p$$

for some $B_k, B_{k,m} \in \mathbb{F}_q[T]$ with $B_{k,m} \neq 0$. Hence, ξ satisfies

$$\xi^m + \sum_{k=0}^{m-1} \frac{B_k}{B_{k,m}} \xi^k = 0,$$

which implies a contradiction. □

Theorem 4.4.5. *Let $n \geq 1$ be an integer and $\xi \in \mathbb{F}_q((T^{-1}))$ be not algebraic of degree at most n. Then we have*

$$w_n(\xi) \geq n, \quad w_n^*(\xi) \geq \frac{n+1}{2}.$$

Furthermore, if $n = 2$, then $w_2^*(\xi) \geq 2$.

Proof. The former estimate follows from an analogue of Minkowski's theorem for Laurent series over a finite field [43] and the later estimates are Satz.1 and Satz.2 of [36]. □

Theorem 4.4.6. *Let $n \geq 1$ be an integer and $\xi \in \mathbb{F}_q((T^{-1}))$ be an algebraic Laurent series of degree d. Then we have*

$$w_n(\xi), w_n^*(\xi) \leq d - 1.$$

Proof. It follows from Proposition 4.2.2 that

$$P(\xi) = 0 \quad \text{or} \quad |P(\xi)| \geq H(\xi)^{-n} H(P)^{-d+1}$$

for all $P(X) \in (\mathbb{F}_q[T])[X]$ of degree at most n. Therefore, we have $w_n(\xi) \leq d - 1$.

By Proposition 4.2.4, we obtain

$$\xi = \alpha \quad \text{or} \quad |\xi - \alpha| \geq H(\xi)^{-n} H(\alpha)^{-d}$$

for all $\alpha \in \overline{\mathbb{F}_q(T)}$ of degree at most n. Hence, we deduce $w_n^*(\xi) \leq d - 1$. □

For real and p-adic numbers, analogues of the following theorem hold (See [12, Section 3.4 and 9.3]).

Theorem 4.4.7. *Let ξ be in $\mathbb{F}_q((T^{-1}))$. Then the following conditions are equivalent:*

(i) *ξ is an A-number,*

(ii) *ξ is an A^*-number,*

(iii) *ξ is an algebraic number.*

Proof. We first prove (i) $\Rightarrow$ (iii). Let $\xi \in \mathbb{F}_q((T^{-1}))$ be an A-number. Assume that ξ is transcendental. It follows from Theorem 4.4.5 that $w(\xi) \geq 1$, which is a contradiction. Similarly, we have (ii) $\Rightarrow$ (iii).

Next, we prove (iii) $\Rightarrow$ (i). By Theorem 4.4.6, we obtain $w(\xi) = 0$. Therefore, ξ is an A-number. Similarly, we have (iii) $\Rightarrow$ (ii). □

The following theorem is an analogue of Theorem 3.2 in [12].

Theorem 4.4.8. *Let ξ, η be in $\mathbb{F}_q((T^{-1}))$. If ξ and η are algebraically dependent, then ξ and η are in the same Mahler's class.*

Proof. By Theorem 4.4.7, we may assume that ξ and η are transcendental. For an integer $H \geq 1$, we take a polynomial $P(X) \in (\mathbb{F}_q[T])[X]$ with $H(P) \leq H, \deg_X P \leq n$, and $|P(\xi)| = w_n(\xi, H)$. There exists $F(X,Y) \in (\mathbb{F}_q[T])[X,Y]$ which is an irreducible primitive polynomial in X and Y such that $F(\xi, \eta) = 0$. We write

$$F(X,Y) = \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} X^i Y^j = \sum_{i=0}^{M} B_i(Y) X^i,$$

where $a_{ij} \in \mathbb{F}_q[T], B_i(Y) \in (\mathbb{F}_q[T])[Y]$, and $B_M(Y) \neq 0$. Since there exists $y \in \mathbb{F}_q(T)$ such that $P(X)$ and $F(X,y)$ have no common root, it follows that the resultant $R(Y) = \mathrm{Res}_X(P(X), F(X,Y))$ is non-zero polynomial and is in $(\mathbb{F}_q[T])[Y]$. Then we have $\deg_Y R(Y) \leq nN$ and there exists $c_1 > 0$ such that $H(R) \leq c_1 H^M$. By the basic property of resultants (see e.g. [38, p.199-200]), there exist polynomials $g(X,Y), h(X,Y) \in (\mathbb{F}_q[T])[X,Y]$ and $c_2 > 0$ such that $R(Y) = P(X)g(X,Y) + f(X,Y)h(X,Y)$ and all of the absolute value of coefficient of $g(X,Y)$ are less than or equal to $c_2 H^{M-1}$. Then we have $R(\eta) = P(\xi)g(\xi,\eta)$ and $|g(\xi,\eta)| \leq c_3 H^{M-1}$ for some constant $c_3 > 0$. Therefore, we obtain

$$w_n(\xi) \leq M - 1 + M w_{nN}(\eta),$$
$$w(\xi) \leq \limsup_{n\to\infty} \frac{(M-1)N + MN w_{nN}(\eta)}{nN} \leq MN w(\eta).$$

We change a role of ξ and η, which implies

$$w_n(\eta) \leq N - 1 + N w_{nM}(\xi), \quad w(\eta) \leq MN w(\xi).$$

This completes the proof. □

The lemma below is an analogue of Lemma A.8 in [12] and Lemma 2.4 in [50].

Lemma 4.4.9. *Let $P(X) \in (\mathbb{F}_q[T])[X]$ be a non-constant irreducible polynomial of degree n and inseparable degree f. Let ξ be in $\mathbb{F}_q((T^{-1}))$ and α be a root of $P(X)$ such that $|\xi - \alpha|$ is minimal. If $n \geq 2f$, then we have*

$$|\xi - \alpha| \leq |P(\xi)|^{1/f} H(P)^{n/f^2 - 2/f}. \tag{4.16}$$

Proof. We may assume that ξ and α are distinct. We first consider the case of $f = 1$. Write $P(X) = A\prod_{i=1}^{n}(X - \alpha_i)$, where $\alpha = \alpha_1$ and $|\xi - \alpha_1| \leq |\xi - \alpha_2| \leq \ldots \leq |\xi - \alpha_n|$.

Put $Q(X) := A\prod_{i=2}^{n}(X-\alpha_i)$ and $\Delta := \prod_{i=2}^{n}|\alpha-\alpha_i|$. Then we have $|\operatorname{Disc}(P)|^{1/2} = \Delta|A||\operatorname{Disc}(Q)|^{1/2}$. By the definition of discriminant, we obtain

$$\begin{aligned}|\operatorname{Disc}(Q)|^{1/2} &= |A|^{n-2}|\det(\alpha_i^j)_{2\le i\le n, 0\le j\le n-2}|\\ &\le |A|^{n-2}\prod_{i=2}^{n}\max(1,|\alpha_i|)^{n-2}\\ &= H(P)^{n-2}\max(1,|\alpha|)^{-n+2}.\end{aligned}$$

Since the polynomial P is separable, we get

$$\begin{aligned}1 \le |\operatorname{Disc}(P)|^{1/2} &\le H(P)^{n-2}\max(1,|\alpha|)^{-n+2}|A|\prod_{j=2}^{n}|\xi-\alpha_j|\\ &= H(P)^{n-2}\max(1,|\alpha|)^{-n+2}|\xi-\alpha|^{-1}|P(\xi)|.\end{aligned}$$

Therefore, we have (4.16).

We next consider the case of $f>1$. We can write $P(X)=R(X^f)$, where a separable polynomial $R(X)\in(\mathbb{F}_q[T])[X]$. Thus, in the same way, it follows that

$$|\xi^f-\alpha^f| \le |R(\xi^f)|H(R)^{n/f-2}.$$

Since $H(P)=H(R)$ and f is a power of p, we have (4.16). □

The following lemma is a well-known result (see e.g. [39, 59]).

Lemma 4.4.10. *Consider a continued fraction $\xi=[a_0,a_1,a_2,\ldots]\in\mathbb{F}_q((T^{-1}))$. Let $(p_n/q_n)_{n\ge0}$ be the convergent sequence of ξ. Then we have*

$$w_1(\xi)=w_1^*(\xi)=\limsup_{n\to\infty}\frac{\deg q_{n+1}}{\deg q_n}.$$

Lemma 4.4.11. *Let ξ be in $\mathbb{F}_q((T^{-1}))$ and $n\ge1$ be an integer. Then we have*

$$w_1(\xi)=w_1(\xi^{p^n}).$$

Proof. By Lemma 4.4.1 and Theorem 4.4.6, we may assume that ξ is not in $\mathbb{F}_q(T)$. Therefore, we can write $\xi=[a_0,a_1,\ldots]$. Then we have $\xi^{p^n}=[a_0^{p^n},a_1^{p^n},\ldots]$ by the Frobenius endmorphism. Hence, it follows from Lemmas 4.3.1 (v) and 4.4.10 that

$$w_1(\xi^{p^n})=\limsup_{k\to\infty}\frac{\sum_{i=1}^{k+1}\deg a_i^{p^n}}{\sum_{i=1}^{k}\deg a_i^{p^n}}=\limsup_{k\to\infty}\frac{p^n\deg q_{k+1}}{p^n\deg q_k}=w_1(\xi),$$

where $(p_k/q_k)_{k\ge0}$ is the convergent sequence of ξ. □

Proposition 4.4.12. *Let $n\ge1$ be an integer and ξ be in $\mathbb{F}_q((T^{-1}))$. Let $k\ge0$ be an integer such that $p^k\le n<p^{k+1}$. Then we have*

$$\frac{w_n(\xi)}{p^k}-n+\frac{2}{p^k}-1\le w_n^*(\xi)\le w_n(\xi). \tag{4.17}$$

Furthermore, if $1\le n<2p$, then we have

$$w_n(\xi)-n+1\le w_n^*(\xi)\le w_n(\xi). \tag{4.18}$$

Remark. It is known that for all n, analogues of (4.18) for real numbers and p-adic numbers hold (see [62, 45]). However, in our framework, we are not able to prove (4.18) for all n. The main difficulty is the existence of inseparable irreducible polynomials in $(\mathbb{F}_q[T])[X]$. Therefore, it seems that Proposition 4.4.12 describes the difference between approximation properties of characteristic zero and that of positive characteristic. On the other hand, when n is sufficiently small, we prove (4.18) using continued fraction theory and the Frobenius endomorphism.

Proof. By Lemma 4.4.1, we may assume that $w_n(\xi), w_n^*(\xi) > 0$. We first show that $w_n^*(\xi) \leq w_n(\xi)$. For $0 < w^* < w_n^*(\xi)$, there exist infinitely many $\alpha \in \overline{\mathbb{F}_q(T)}$ of degree at most n such that

$$0 < |\xi - \alpha| \leq H(\alpha)^{-w^*-1}.$$

Let $P_\alpha(X) = \sum_{i=0}^{d} a_i X^i$ be the minimal polynomial of α. Put

$$\begin{aligned} Q_\alpha(X) := a_d X^{d-1} + (a_d\alpha + a_{d-1})X^{d-2} + (a_d\alpha^2 + a_{d-1}\alpha + a_{d-2})X^{d-3} \\ + \cdots + (a_d\alpha^{d-1} + a_{d-1}\alpha^{d-2} + \cdots + a_1). \end{aligned}$$

Then we have $P_\alpha(X) = (X - \alpha)Q_\alpha(X)$. Since $\max(1, |\alpha|) = \max(1, |\xi|)$, we obtain $|Q_\alpha(\xi)| \leq H(P_\alpha)\max(1, |\xi|)^n$. Hence, it follows that

$$|P_\alpha(\xi)| \leq H(\alpha)^{-w^*} \max(1, |\xi|)^n,$$

which implies $w^* \leq w_n(\xi)$. Consequently, we have $w_n^*(\xi) \leq w_n(\xi)$.

Our next claim is that $w_n(\xi)/p^k - n + 2/p^k - 1 \leq w_n^*(\xi)$. For $0 < w < w_n(\xi)$, there exist infinitely many $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n such that

$$0 < |P(\xi)| \leq H(P)^{-w}$$

by Lemma 4.4.3. Let m denote the degree of $P(X)$ and f denote the inseparable degree of $P(X)$. We first consider the case of $m \geq 2f$. By Lemma 4.4.9, there exists α of root of $P(X)$ such that

$$|\xi - \alpha| \leq H(\alpha)^{-w/f + m/f^2 - 2/f} \leq H(\alpha)^{-w/p^k + n - 2/p^k}.$$

Now, assume that $m < 2f$. Then we have $m = f$ by $f | m$. Therefore, we can write $P(X) = A(X^f - \alpha^f)$, where $A \in \mathbb{F}_q[T]$ and $\alpha \in \overline{\mathbb{F}_q(T)}$. Thus, we get $|\xi - \alpha| \leq |A|^{-1/f} H(\alpha)^{-w/f}$. Since $\max(1, |\xi|) = \max(1, |\alpha|)$, we have

$$|\xi - \alpha| \leq \max(1, |\xi|)H(\alpha)^{-w/f - 1/f} \leq \max(1, |\xi|)H(\alpha)^{-w/p^k + n - 2/p^k}$$

by Lemma 4.2.1. This is our claim.

Finally, we assume $1 \leq n < 2p$ and show (4.18). Let $0 < w < w_n(\xi)$. If there exist infinitely many separable polynomials $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n such that

$$0 < |P(\xi)| \leq H(P)^{-w},$$

then we have $w - n + 1 \leq w_n^*(\xi)$ in a similar way to the above proof. Therefore, we may assume that there exist infinitely many inseparable polynomials $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ of degree at most n such that

$$0 < |P(\xi)| \leq H(P)^{-w}.$$

Then we can write such polynomials $P(X) = AX^p + B$, where $A, B \in \mathbb{F}_q[T]$. By Lemma 4.4.11 and the definition of w_n, we have $w \leq w_1(\xi)$. Therefore, we obtain $w - n + 1 \leq w_n^*(\xi)$ by $w_1(\xi) = w_1^*(\xi)$ and Lemma 4.4.1. Hence, we have (4.18). □

It follows from Proposition 4.4.12 that for an integer $n \geq 1$ and $\xi \in \mathbb{F}_q((T^{-1}))$

- $w_n(\xi)$ is finite if and only if $w_n^*(\xi)$ is finite,
- if $w(\xi)$ is finite, then $w^*(\xi)$ is finite.

Consequently, we obtain

- ξ is a U_n-number if and only if it is a U_n^*-number,
- if ξ is an S-number, then it is an S^*-number,
- if ξ is an T^*-number, then it is an T-number.

We address the following questions in the last of this section.

Problem 4.4.13. Does (4.18) hold for all $n \geq 1$ and $\xi \in \mathbb{F}_q((T^{-1}))$?

Problem 4.4.14. Does Mahler's classification coincide Koksma's classification?

Problem 4.4.15. Does an analogue of Theorem 4.4.8 for Koksma's classification hold?

Note that analogues of Problems 4.4.14 and 4.4.15 for real and p-adic numbers hold. The detail is found in [12, Section 3.4 and 9.3], [50, Chapter 6], and [53].

4.5 Applications of Liouville inequalities

The following proposition is an analogue of Lemma 7.2 in [18].

Proposition 4.5.1. *Let ξ be in $\mathbb{F}_q((T^{-1}))$. Let $c_0, c_1, c_2, c_3, \theta, \rho, \delta$ be positive numbers and $(\beta_j)_{j\geq 1}$ be a sequence of positive integers such that $\beta_j < \beta_{j+1} \leq c_0\beta_j^{\theta}$ for all $j \geq 1$. Assume that there exists a sequence of distinct terms $(\alpha_j)_{j\geq 1}$ with $\alpha_j \in \overline{\mathbb{F}_q(T)}$ is quadratic and $\alpha_j \neq \alpha_j'$ for $j \geq 1$ such that for all $j \geq 1$*

$$\frac{c_1}{\beta_j^{2+\rho}} \leq |\xi - \alpha_j| \leq \frac{c_2 \max(1, |\alpha_j - \alpha_j'|^{-1})}{\beta_j^{2+\delta}}, \quad H(\alpha_j) \leq c_3\beta_j.$$

Then we have

$$w_2^*(\xi) \leq (2+\rho)\frac{2\theta}{\delta} - 1.$$

Proof. Let $\alpha \in \overline{\mathbb{F}_q(T)}$ be an algebraic number of degree at most two with sufficiently large height. We define an integer $j_0 \geq 1$ by $\beta_{j_0} \leq c_0 c_2^{\frac{\theta}{\delta}}(c_3 H(\alpha))^{\frac{2\theta}{\delta}} < \beta_{j_0+1}$. We first consider the case of $\alpha = \alpha_{j_0}$. By the assumption, we have

$$|\xi - \alpha| \geq c_1\beta_{j_0}^{-2-\rho} \geq c_0^{-2-\rho} c_1 c_2^{-(2+\rho)\frac{\theta}{\delta}} c_3^{-(2+\rho)\frac{2\theta}{\delta}} H(\alpha)^{-(2+\rho)\frac{2\theta}{\delta}}.$$

We next consider the other cases. Then, by the assumption, we have

$$H(\alpha) < c_2^{-\frac{1}{2}} c_3^{-1}(c_0^{-1}\beta_{j_0+1})^{\frac{\delta}{2\theta}} \leq c_2^{-\frac{1}{2}} c_3^{-1}\beta_{j_0}^{\frac{\delta}{2}}.$$

Hence, it follows from Lemma 4.2.5 and Proposition 4.2.6 that

$$\begin{aligned} |\alpha - \alpha_{j_0}| &\geq \max(1, |\alpha_{j_0} - \alpha_{j_0}'|^{-1}) H(\alpha_{j_0})^{-2} H(\alpha)^{-2} \\ &> c_2 \max(1, |\alpha_{j_0} - \alpha_{j_0}'|^{-1})\beta_{j_0}^{-2-\delta} \geq |\xi - \alpha_{j_0}|. \end{aligned}$$

Therefore, we obtain

$$|\xi-\alpha| = |\alpha-\alpha_{j_0}| \geq \max(1, |\alpha_{j_0}-\alpha'_{j_0}|^{-1})H(\alpha_{j_0})^{-2}H(\alpha)^{-2}$$
$$\geq c_3^{-2}\beta_{j_0}^{-2}H(\alpha)^{-2} \geq c_0^{-2}c_2^{-\frac{2\theta}{\delta}}c_3^{-2-\frac{4\theta}{\delta}}H(\alpha)^{-2-\frac{4\theta}{\delta}}.$$

By the assumption and Lemma, we have

$$w_2^*(\xi) \leq \max\left(1+\frac{4\theta}{\delta}, (2+\rho)\frac{2\theta}{\delta}-1\right) = (2+\rho)\frac{2\theta}{\delta}-1.$$

□

Lemma 4.5.2. *Let α, β be in $\mathbb{F}_q((T^{-1}))$ and $P(X) \in (\mathbb{F}_q[T])[X]$ be a non-constant polynomial of degree d. Let $C \geq 0$ be a real number. Assume that $|\alpha-\beta| \leq C$. Then we have*

$$|P(\alpha)-P(\beta)| \leq \max(C, |\alpha|)^{d-1}|\alpha-\beta|H(P). \tag{4.19}$$

Proof. It is easily seen that

$$|P(\alpha)-P(\beta)| \leq H(P)\max_{1\leq i\leq d}|\alpha^i-\beta^i|.$$

By the assumption, we have $\max(C, |\alpha|) = \max(C, |\beta|)$. For any $1 \leq i \leq d$, we obtain

$$|\alpha^i-\beta^i| = |\alpha-\beta|\left|\sum_{j=0}^{i-1}\alpha^j\beta^{i-1-j}\right|$$
$$\leq |\alpha-\beta|\max_{0\leq j\leq i-1}|\alpha^j\beta^{i-1-j}|$$
$$\leq |\alpha-\beta|\max(C, |\alpha|)^{i-1}.$$

Hence, we have (4.19). □

Proposition 4.5.3. *Let ξ be in $\mathbb{F}_q((T^{-1})), d \geq 1$ be an integer, and θ, ρ, δ be positive numbers. Assume that there exists a sequence $(p_j/q_j)_{j\geq 1}$ with $p_j, q_j \in \mathbb{F}_q[T], q_j \neq 0, \gcd(p_j, q_j) = 1$ for any $j \geq 1$ such that $(|q_j|)_{j\geq 1}$ is an increasing and divergent sequence, and*

$$\limsup_{j\to\infty}\frac{\log|q_{j+1}|}{\log|q_j|} \leq \theta,$$
$$d+\delta \leq \liminf_{j\to\infty}\frac{-\log|\xi-p_j/q_j|}{\log|q_j|}, \quad \limsup_{j\to\infty}\frac{-\log|\xi-p_j/q_j|}{\log|q_j|} \leq d+\rho.$$

Then we have for all $1 \leq n \leq d$

$$d-1+\delta \leq w_n^*(\xi) \leq w_n(\xi) \leq \max\left(d-1+\rho, \frac{d\theta}{\delta}\right). \tag{4.20}$$

Note that Lemma 4.5.3 is a generalization of analogue of Lemma 1 in [10].

Proof. Let $0 < \iota < \delta$ be a real number. By the assumption, there exists an integer $c_0 \geq 1$ such that

$$|q_j| \leq |q_{j+1}| \leq |q_j|^{\theta+\iota}, \quad \frac{1}{|q_j|^{d+\rho+\iota}} \leq \left|\xi - \frac{p_j}{q_j}\right| \leq \frac{1}{|q_j|^{d+\delta-\iota}}$$

for all $j \geq c_0$. Since $|\xi - p_j/q_j| \leq 1$ for $j \geq c_0$, we have $|q_j| \max(1, |\xi|) = \max(|p_j|, |q_j|) = H(p_j/q_j)$ for $j \geq c_0$. Therefore, we obtain

$$0 < \left|\xi - \frac{p_j}{q_j}\right| \leq \frac{\max(1, |\xi|)^{d+\delta}}{H(p_j/q_j)^{d+\delta-\iota}}$$

for $j \geq c_0$. Since ι is arbitrary, we have

$$d - 1 + \delta \leq w_1^*(\xi) \leq w_2^*(\xi) \leq \ldots \leq w_d^*(\xi).$$

It is sufficient from Proposition 4.4.12 to show that

$$w_d(\xi) \leq \max\left(d - 1 + \rho, \frac{d\theta}{\delta}\right). \tag{4.21}$$

Put $c_1 := \max(1, |\xi|)^{d-1}$. Let $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ be a polynomial of degree at most d with $H(P) \geq c_1^{-1}|q_{c_0}|^{\frac{\delta}{\theta}}$. We first consider the case of $P(p_j/q_j) = 0$ for some $j \geq 1$. Then we can write $P(X) = a_j(q_j X - p_j)$ for some $a_j \in \mathbb{F}_q$. Therefore, we have

$$|P(\xi)| \geq |q_j|^{-d+1-\rho-\iota} \geq H(P)^{-d+1-\rho-\iota}.$$

We now turn to the case of $P(q_j/p_j) \neq 0$ for all $j \geq 1$. We define an integer $j_0 \geq c_0$ by

$$|q_{j_0}| \leq (c_1 H(P))^{\frac{\theta+\iota}{\delta-\iota}} < |q_{j_0+1}|.$$

Then we have

$$H(P) < c_1^{-1}|q_{j_0+1}|^{\frac{\delta-\iota}{\theta+\iota}} \leq c_1^{-1}|q_{j_0}|^{\delta-\iota}.$$

It follows from Lemma 4.5.2 that

$$|P(\xi) - P(p_{j_0}/q_{j_0})| \leq c_1 H(P) \left|\xi - \frac{p_{j_0}}{q_{j_0}}\right| < |q_{j_0}|^{-d}.$$

Since $|P(p_{j_0}/q_{j_0})| \geq |q_{j_0}|^{-d}$, we obtain

$$|P(\xi)| = |P(p_{j_0}/q_{j_0})| \geq |q_{j_0}|^{-d} \geq (c_1 H(P))^{-\frac{d(\theta+\iota)}{\delta-\iota}}.$$

Therefore, by Lemma 4.4.3, we have

$$w_d(\xi) \leq \max\left(d - 1 + \rho + \iota, \frac{d(\theta+\iota)}{\delta-\iota}\right).$$

Since ι is arbitrary, we obtain (4.21). $\square$

We give a key proposition for the proofs of Theorems 4.1.8 and 4.1.9 as follows:

Proposition 4.5.4. *Let $d \geq 2$ be an integer. Let ξ be in $\mathbb{F}_q((T^{-1}))$, and θ, ρ, δ be positive numbers. Assume that there exists a sequence $(\alpha_j)_{j\geq 1}$ such that $\alpha_j \in \overline{\mathbb{F}_q(T)}$ is quadratic for any $j \geq 1$ and $(H(\alpha_j))_{j\geq 1}$ is an increasing and divergent sequence, and*

$$\limsup_{j\to\infty} \frac{\log H(\alpha_{j+1})}{\log H(\alpha_j)} \leq \theta,$$

$$d+\delta \leq \liminf_{j\to\infty} \frac{-\log|\xi-\alpha_j|}{\log H(\alpha_j)}, \quad \limsup_{j\to\infty} \frac{-\log|\xi-\alpha_j|}{\log H(\alpha_j)} \leq d+\rho.$$

If $2d\theta \leq (d-2+\rho)\delta$, then we have for all $2 \leq n \leq d$,

$$d-1+\delta \leq w_n^*(\xi) \leq d-1+\rho. \tag{4.22}$$

Furthermore, assume that there exist a non-negative number ε and a positive number c such that for any $j \geq 1, 0 < |\alpha_j - \alpha_j'| \leq c$ and

$$\limsup_{j\to\infty} \frac{-\log|\alpha_j-\alpha_j'|}{\log H(\alpha_j)} \geq \varepsilon.$$

If $2d\theta \leq (d-2+\delta)\delta$, then we have for all $2 \leq n \leq d$,

$$d-1+\delta \leq w_n^*(\xi) \leq d-1+\rho, \quad \varepsilon \leq w_n(\xi) - w_n^*(\xi). \tag{4.23}$$

Finally, assume that there exists a non-negative number χ such that

$$\limsup_{i\to\infty} \frac{-\log|\alpha_i-\alpha_i'|}{\log H(\alpha_i)} \leq \chi.$$

Then we have for all $2 \leq n \leq d$,

$$d-1+\delta \leq w_n^*(\xi) \leq d-1+\rho, \quad \varepsilon \leq w_n(\xi) - w_n^*(\xi) \leq \chi. \tag{4.24}$$

Proof. Let $0 < \iota < \delta$ be a real number. Then there exists an integer $c_0 \geq 1$ such that

$$\frac{1}{H(\alpha_j)^{d+\rho+\iota}} \leq |\xi-\alpha_j| \leq \frac{1}{H(\alpha_j)^{d+\delta-\iota}}, \quad H(\alpha_j) \leq H(\alpha_{j+1}) \leq H(\alpha_j)^{\theta+\iota}$$

for all $j \geq c_0$. Since ι is arbitrary, we have $w_2^*(\xi) \geq d-1+\delta$. Let $\alpha \in \overline{\mathbb{F}_q(T)} \setminus \{\alpha_j \mid j \geq 1\}$ be an algebraic number of degree at most d with $H(\alpha) \geq H(\alpha_{c_0})^{\frac{\delta}{2\theta}}$.

Assume that $2d\theta \leq (d-2+\rho)\delta$. We define an integer $j_0 \geq c_0$ such that $H(\alpha_{j_0}) \leq H(\alpha)^{\frac{2(\theta+\iota)}{\delta-\iota}} < H(\alpha_{j_0+1})$. Since

$$H(\alpha) < H(\alpha_{j_0+1})^{\frac{\delta-\iota}{2(\theta+\iota)}} \leq H(\alpha_{j_0})^{\frac{\delta-\iota}{2}},$$

we obtain

$$|\alpha-\alpha_{j_0}| \geq H(\alpha)^{-2}H(\alpha_{j_0})^{-d} > H(\alpha_{j_0})^{-d-\delta+\iota} \geq |\xi-\alpha_{j_0}|$$

by Proposition 4.2.4. Therefore, we have

$$|\xi-\alpha| = |\alpha-\alpha_{j_0}| \geq H(\alpha)^{-2}H(\alpha_{j_0})^{-d} \geq H(\alpha)^{-2-\frac{2d(\theta+\iota)}{\delta-\iota}}, \tag{4.25}$$

which implies
$$w_d^*(\xi) \le \max\left(d-1+\rho+\iota, 1+\frac{2d(\theta+\iota)}{\delta-\iota}\right).$$
Since ι is arbitrary small, (4.22) follows.

Next, we assume that $2d\theta \le (d-2+\delta)\delta$ and there exist a non-negative number ε and a positive number c such that for any $j \ge 1, 0 < |\alpha_j - \alpha'_j| \le c$ and
$$\limsup_{j\to\infty} \frac{-\log|\alpha_j - \alpha'_j|}{\log H(\alpha_j)} \ge \varepsilon.$$
Since $\delta \le \rho$, we have (4.22). By the assumption and (4.25), the sequence $(\alpha_j)_{j\ge1}$ is the best approximation to ξ of degree at most d, that is,
$$w_d^*(\xi) = \limsup_{j\to\infty} \frac{-\log|\xi-\alpha_j|}{\log H(\alpha_j)} - 1. \tag{4.26}$$
Therefore, we have $w_2^*(\xi) = \ldots = w_d^*(\xi)$. In what follows, we show that $\varepsilon \le w_n(\xi) - w_n^*(\xi)$ for all $2 \le n \le d$. For any $j \ge 1$, we denote by $P_j(X) = A_j(X-\alpha_j)(X-\alpha'_j)$ the minimal polynomial of α_j. Since $|\xi - \alpha_j| \le 1$ and $|\alpha_j - \alpha'_j| \le c$ for $j \ge c_0$, we have
$$\max(1, |\xi|) \asymp \max(1, |\alpha_j|) \asymp \max(1, |\alpha'_j|)$$
for $j \ge c_0$. Therefore, it follows from Lemma 4.2.1 that $H(P_j) \asymp |A_j|$ for $j \ge c_0$. By Lemma 4.2.5, we have $|\xi - \alpha_j| < |\alpha_j - \alpha'_j|$ for $j \ge c_0$, which implies $|\xi - \alpha'_j| = |\alpha_j - \alpha'_j|$ for $j \ge c_0$. Hence, we have
$$|P_j(\xi)| \asymp H(P_j)|\xi - \alpha_j||\alpha_j - \alpha'_j| \tag{4.27}$$
for $j \ge c_0$. It follows from (4.26) and (4.27) that $w_d^*(\xi) + \varepsilon \le w_2(\xi)$. Hence, we have $\varepsilon \le w_n(\xi) - w_n^*(\xi)$ for all $2 \le n \le d$.

Finally, we assume that
$$\limsup_{j\to\infty} \frac{-\log|\alpha_j - \alpha'_j|}{\log H(\alpha_j)} \le \chi.$$
By (4.27), we obtain
$$\limsup_{j\to\infty} \frac{-\log|P_j(\xi)|}{\log H(P_j)} \le w_2^*(\xi) + \chi.$$
Put $c_1 := \max(1, |\xi|)^{d-1}$. Let $P(X) \in (\mathbb{F}_q[T])[X]_{\min}$ be a polynomial of degree at most d with $H(P) \ge c_1^{-\frac{1}{2}} H(\alpha_{c_0})^{\frac{\delta}{2\theta}}$ and $P(\alpha_j) \ne 0$ for all $j \ge 1$. We define an integer $j_1 \ge c_0$ such that $H(\alpha_{j_1}) \le (c_1 H(P)^2)^{\frac{\theta+\iota}{\delta-\iota}} < H(\alpha_{j_1+1})$. Since
$$H(P) < c_1^{-\frac{1}{2}} H(\alpha_{j_1+1})^{\frac{\delta-\iota}{2(\theta+\iota)}} \le c_1^{-\frac{1}{2}} H(\alpha_{j_1})^{\frac{\delta-\iota}{2}},$$
we have
$$\begin{aligned} |P(\xi) - P(\alpha_{j_1})| &\le c_1 H(P)|\xi - \alpha_{j_1}| < c_1^{\frac{1}{2}} H(\alpha_{j_1})^{-d-\frac{\delta-\iota}{2}} \\ &\le H(\alpha_{j_1})^{-d} H(P)^{-1} \le |P(\alpha_{j_1})| \end{aligned}$$

by Proposition 4.2.2 and Lemma 4.5.2. Therefore, we obtain

$$|P(\xi)| = |P(\alpha_{j_1})| \geq H(\alpha_{j_1})^{-d} H(P)^{-1} \geq c_1^{-\frac{d(\theta+\iota)}{\delta-\iota}} H(P)^{-1-\frac{2d(\theta+\iota)}{\delta-\iota}}.$$

Hence, we get

$$w_d(\xi) \leq \max\left(w_2^*(\xi) + \chi, 1 + \frac{2d(\theta+\iota)}{\delta-\iota} \right)$$

by Lemma 4.4.3. Since ι is arbitrary small, we have $w_d(\xi) \leq w_2^*(\xi) + \chi$. This completes the proof. □

4.6 Proof of the main results

4.6.1 Proof of Theorem 4.1.1

We extend Lemma 4.4.10 by using Proposition 4.5.3.

Proposition 4.6.1. *Let $d \geq 1$ be an integer and $\xi = [a_0, a_1, a_2, \ldots]$ be in $\mathbb{F}_q((T^{-1}))$. Let $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of ξ. Assume that*

$$\liminf_{n\to\infty} \frac{\deg q_{n+1}}{\deg q_n} \geq 2d - 1.$$

Then we have

$$w_1(\xi) = w_1^*(\xi) = \ldots = w_d(\xi) = w_d^*(\xi) = \limsup_{n\to\infty} \frac{\deg q_{n+1}}{\deg q_n}. \tag{4.28}$$

Proof. For $j \geq 1$, put

$$A_j := \frac{\deg q_{j+1}}{\deg q_j} = \frac{\log |q_{j+1}|}{\log |q_j|}.$$

It follows from Lemma 4.4.1, 4.4.10, and Proposition 4.4.12 that for all $n \geq 1$,

$$\limsup_{j\to\infty} A_j \leq w_n^*(\xi) \leq w_n(\xi).$$

By Lemma 4.3.1 (vii), we have

$$\frac{-\log |\xi - p_j/q_j|}{\log |q_j|} = 1 + A_j$$

for all $j \geq 1$. It follows from Lemma 4.3.1 (iii) and (v) that $q_j \neq 0$ and $\gcd(p_j, q_j) = 1$ hold for all $j \geq 1$, and the sequence $(|q_j|)_{j\geq 1}$ is increasing and divergent. Applying Proposition 4.5.3 to

$$\theta = \limsup_{j\to\infty} A_j, \quad \delta = d, \quad \rho = 1 + \limsup_{j\to\infty} A_j - d,$$

we obtain

$$w_n^*(\xi) \leq w_n(\xi) \leq \limsup_{j\to\infty} A_j$$

for all $1 \leq n \leq d$. Hence, we have (4.28). □

Schmidt [54] characterized Laurent series of Class IA by using continued fractions.

Theorem 4.6.2. *Let α be in $\mathbb{F}_q((T^{-1}))$. Then α is of Class IA if and only if the continued fraction expansion of α is of the form*

$$\alpha = [a_1, a_2, \ldots, a_t, b_1, b_2, \ldots], \tag{4.29}$$

where $t \geq 0$ and

$$b_{j+s} = \begin{cases} ab_j^{p^k} & \text{when } j \text{ is odd,} \\ a^{-1}b_j^{p^k} & \text{when } j \text{ is even} \end{cases} \tag{4.30}$$

for some $a \in \mathbb{F}_q^{\times}$, integers $s \geq 1$ and $k \geq 0$.

Proof. See [54]. □

Thakur [58] calculated ratios of degrees of denominators of partial quotients for Laurent series of Class IA.

Theorem 4.6.3. *Let $\alpha \in \mathbb{F}_q((T^{-1}))$ be as in (4.29) and (4.30), and $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of α. Put $d_i := \deg b_i$ and*

$$r_i := \frac{d_i}{p^k(\sum_{j=1}^{i-1} d_j) + \sum_{j=i}^{s} d_j}.$$

Then we have

$$\limsup_{n\to\infty} \frac{\deg q_{n+1}}{\deg q_n} = 1 + (p^k - 1)\max\{r_1, \ldots, r_s\}, \tag{4.31}$$

$$\liminf_{n\to\infty} \frac{\deg q_{n+1}}{\deg q_n} = 1 + (p^k - 1)\min\{r_1, \ldots, r_s\}. \tag{4.32}$$

Proof. We have (4.31) by Theorem 1 (1) in [58] and Lemma 4.4.10. (4.32) follows in a similar way to the proof of Theorem 1 (1) in [58]. □

Lemma 4.6.4. *Let $d \geq 1$ be an integer and $w > 2d - 1$ be a rational number. Write $w = a/b$, where $a, b \geq 1$ are integers and $a = p^m a'$, where $a' \geq 1, m \geq 0$ are integers and $\gcd(a', p) = 1$. Then there exist a strictly increasing sequence of positive integers $(k_j)_{j\geq 1}$, a sequence of integers $(n_j)_{j\geq 1}$ and a sequence of rational numbers $(u_j)_{j\geq 1}$ such that for any $j \geq 1, n_j \geq 3, u_j = a_j/b_j$, where $a_j, b_j \geq 1$ are integers, $\gcd(a_j, p) = 1, p^m | b_j$, and*

$$2d - 1 < \min\left\{w, u_j, \frac{p^{k_j}}{wu_j^{n_j-2}}\right\}, \quad \max\left\{w, u_j, \frac{p^{k_j}}{wu_j^{n_j-2}}\right\} = w. \tag{4.33}$$

Proof. The proof is by induction on j. By the assumption, we take an integer $n_1 \geq 3$ with $(w/(2d-1))^{n_1-1} > p$. Then we have $\log_p w^{n_1} - \log_p w(2d-1)^{n_1-1} > 1$. This implies that there exists an integer $k_1 \geq 1$ such that

$$w(2d-1)^{n_1-1} < p^{k_1} < w^{n_1}.$$

Then we have

$$\max\left\{2d - 1, \left(\frac{p^{k_1}}{w^2}\right)^{\frac{1}{n_1-2}}\right\} < \min\left\{w, \left(\frac{p^{k_1}}{(2d-1)w}\right)^{\frac{1}{n_1-2}}\right\}.$$

Let $r \geq 2$ be an integer such that $\gcd(r,p)=1$. By Lemma 2.5.9 in [9], the set $\{r^y/p^x \mid x,y \in \mathbb{Z}_{\geq 0}\}$ is dense in $\mathbb{R}_{>0}$. Thus, we can take a rational number $u_1 = a_1/b_1$ such that $a_1, b_1 \in \mathbb{Z}_{>0}, \gcd(a_1,p)=1, p^m | b_1$ and

$$\max\left\{2d-1, \left(\frac{p^{k_1}}{w^2}\right)^{\frac{1}{n_1-2}}\right\} < u_1 < \min\left\{w, \left(\frac{p^{k_1}}{(2d-1)w}\right)^{\frac{1}{n_1-2}}\right\}.$$

Therefore, we have (4.33) in the case of $j=1$. Assume that the cases of $j=1,\ldots,i$ hold for some $i \geq 1$. We take an integer $n_{i+1} \geq 3$ with $p^{k_i} < w(2d-1)^{n_{i+1}}$. In a similar way to the above proof, we can take an integer k_{i+1} and a rational number u_{i+1} which satisfy (4.33). Then we have $k_i < k_{i+1}$. This completes the proof. □

Proof of Theorem 4.1.1. We take a strictly increasing sequence of positive integers $(k_j)_{j\geq 1}$, a sequence of integers $(n_j)_{j\geq 1}$, and a sequence of rational numbers $(u_j)_{j\geq 1}$ as in Lemma 4.6.4. For $j \geq 1$, we put

$$d_{1,j} := \frac{b_j^{n_j-2}(a-b)}{p^m}, \quad d_{i,j} := \frac{a a_j^{i-2} b_j^{n_j-i-1}(a_j-b_j)}{p^m} \quad (2 \leq i \leq n_j-1),$$
$$d_{n_j,j} := \frac{p^{k_j} b b_j^{n_j-2} - a a_j^{n_j-2}}{p^m}.$$

Then we have $d_{i,j} \in \mathbb{Z}_{>0}$ and $\gcd(d_{n_j,j},p)=1$ for all $j \geq 1$ and $1 \leq i \leq n_j$. Now we take polynomials $A_{1,j},\ldots,A_{n_j,j} \in \mathbb{F}_q[T]$ with $\deg A_{i,j} = d_{i,j}$. Let

$$\xi_j := [A_{1,j},\ldots,A_{n_j,j}, A_{1,j}^{p^{k_j}},\ldots,A_{n_j,j}^{p^{k_j}}, A_{1,j}^{p^{2k_j}},\ldots] \in \mathbb{F}_q((T^{-1}))$$

and $(p_{n,j}/q_{n,j})_{n\geq 0}$ be the convergent sequence of ξ_j. For $1 \leq i \leq n_j$, we put

$$r_{i,j} := \frac{d_{i,j}}{p^{k_j}(\sum_{\ell=1}^{i-1} d_{\ell,j}) + \sum_{\ell=i}^{n_j} d_{\ell,j}}.$$

Then a straightforward computation shows that

$$r_{1,j} = \frac{a-b}{(p^{k_j}-1)b}, \quad r_{i,j} = \frac{a_j-b_j}{(p^{k_j}-1)b_j} \quad (2 \leq i \leq n_j-1),$$
$$r_{n_j,j} = \frac{p^{k_j} b b_j^{n_j-2} - a a_j^{n_j-2}}{(p^{k_j}-1) a a_j^{n_j-2}}.$$

By Theorem 4.6.3 and Lemma 4.6.4, we obtain

$$\limsup_{n\to\infty} \frac{\deg q_{n+1,j}}{\deg q_{n,j}} = w, \quad \liminf_{n\to\infty} \frac{\deg q_{n+1,j}}{\deg q_{n,j}} > 2d-1.$$

It follows from Proposition 4.6.1 that

$$w_1(\xi_j) = w_1^*(\xi_j) = \ldots = w_d(\xi_j) = w_d^*(\xi_j) = w.$$

In a similar way to the proof of Theorem 2 in [24], we obtain $\deg \xi_j = p^{k_j}+1$ for all $j \geq 1$. This completes the proof. □

4.6.2 Proofs of Theorems 4.1.6, 4.1.8, and 4.1.9

Proof of Theorem 4.1.6. Let $(\varepsilon_n)_{n\geq 1}$ be a sequence over the set $\{0,1\}$. We define recursively sequences $(a_n)_{n\geq 0}, (P_n)_{n\geq -1}$ and $(Q_n)_{n\geq -1}$ by

$$\begin{cases} a_0 = 0,\ a_1 = T + \varepsilon_1,\ a_n = T^{\lfloor (w-1)\deg Q_{n-1}\rfloor} + \varepsilon_n,\ n \geq 2, \\ P_{-1} = 1,\ P_0 = 0,\ P_n = a_n P_{n-1} + P_{n-2},\ n \geq 1, \\ Q_{-1} = 0,\ Q_0 = 1,\ Q_n = a_n Q_{n-1} + Q_{n-2},\ n \geq 1. \end{cases}$$

Set $\xi_w := [0, a_1, a_2, \ldots]$. Then $(P_n/Q_n)_{n\geq 0}$ is the convergent sequence of ξ_w. It follows from Lemma 4.3.1 (v) that

$$\lim_{n\to\infty} \frac{\deg Q_{n+1}}{\deg Q_n} = w.$$

Therefore, by Proposition 4.6.1, we have

$$w_1(\xi_w) = w_1^*(\xi_w) = \ldots = w_d(\xi_w) = w_d^*(\xi_w) = w.$$

□

Proof of Theorem 4.1.8. For an integer $j \geq 1$, we put

$$\xi_{w,j} := [0, a_{1,w}, \ldots, a_{\lfloor w^j\rfloor, w}, \overline{a}].$$

Then $\xi_{w,j}$ is quadratic by Theorem 4.3.2. Since ξ_w and $\xi_{w,j}$ have the same first $(\lfloor w^{j+1}\rfloor - 1)$-th partial quotients, while $\lfloor w^{j+1}\rfloor$-th partial quotients are different, we have

$$\lim_{j\to\infty} \frac{-\log|\xi_w - \xi_{w,j}|}{\log|q_{\lfloor w^{j+1}\rfloor}|} = 2 \tag{4.34}$$

by Lemmas 4.3.1 (v), (vii), and 4.3.4. It follows from Lemmas 4.3.1 (v) and 4.3.7 that

$$\lim_{j\to\infty} \frac{\log|q_{\lfloor w^{j+1}\rfloor}|}{\log|q_{\lfloor w^{j}\rfloor}|} = w, \quad \lim_{j\to\infty} \frac{\log|q_{\lfloor w^{j}\rfloor}|}{\log H(\xi_{w,j})} = 1. \tag{4.35}$$

Therefore, by (4.34), (4.35), and Lemma 4.3.6, we deduce that $(H(\xi_{w,j}))_{j\geq j_0}$ is an increasing and divergent sequence for some integer $j_0 \geq 1$, and

$$\lim_{j\to\infty} \frac{-\log|\xi_w - \xi_{w,j}|}{\log H(\xi_{w,j})} = w, \quad \lim_{j\to\infty} \frac{-\log|\xi_{w,j} - \xi'_{w,j}|}{\log H(\xi_{w,j})} = 1,$$
$$\lim_{j\to\infty} \frac{\log H(\xi_{w,j+1})}{\log H(\xi_{w,j})} = w.$$

By the definition of w, we have $2dw \leq (w-2)(w-d)$. Applying Proposition 4.5.4 to $\delta = \rho = w - d, \varepsilon = \chi = 1$ and $\theta = w$, we obtain

$$w_n^*(\xi_w) = w - 1, \quad w_n(\xi_w) = w$$

for all $2 \leq n \leq d$. □

Proof of Theorem 4.1.9. For an integer $j \geq 1$, we put

$$\xi_{w,\eta,j} := [0, a_{1,w,\eta}, \ldots, a_{\lfloor w^j \rfloor,w,\eta}, \overline{a, \ldots, a, c}],$$

where the length of period part is $\lfloor \eta w^j \rfloor$. Then $\xi_{w,\eta,j}$ is quadratic by Theorem 4.3.2. Since $\lfloor w^j \rfloor + (m_j + 1)\lfloor \eta w^j \rfloor > \lfloor w^{j+1} \rfloor$, it follows that $\xi_{w,\eta}$ and $\xi_{w,\eta,j}$ have the same first $(\lfloor w^{j+1} \rfloor - 1)$-th partial quotients, while $\lfloor w^{j+1} \rfloor$-th partial quotients are different. Therefore, we have

$$\lim_{j\to\infty} \frac{-\log|\xi_{w,\eta} - \xi_{w,\eta,j}|}{\log|q_{\lfloor w^{j+1} \rfloor}|} = 2$$

by Lemmas 4.3.1 (v), (vii), and 4.3.4. It follows from Lemmas 4.3.6 and 4.3.7 that

$$\lim_{j\to\infty} \frac{-\log|\xi_{w,\eta,j} - \xi'_{w,\eta,j}|}{\log|q_{\lfloor w^j \rfloor}|} = 2, \quad \lim_{j\to\infty} \frac{\log|q_{\lfloor w^j \rfloor} q_{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor}|}{\log H(\xi_{w,\eta,j})} = 1.$$

By Lemma 4.3.1 (v), we obtain

$$\lim_{j\to\infty} \frac{\log|q_{\lfloor w^{j+1} \rfloor}|}{\log|q_{\lfloor w^j \rfloor}|} = w, \quad \lim_{j\to\infty} \frac{\log|q_{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor}|}{\log|q_{\lfloor w^j \rfloor}|} = 1 + \eta.$$

Hence, we deduce that $(H(\xi_{w,\eta,j}))_{j\geq 1}$ is an increasing and divergent sequence, and

$$\lim_{j\to\infty} \frac{-\log|\xi_{w,\eta} - \xi_{w,\eta,j}|}{\log H(\xi_{w,\eta,j})} = \frac{2w}{2+\eta}, \quad \lim_{j\to\infty} \frac{-\log|\xi_{w,\eta,j} - \xi'_{w,\eta,j}|}{\log H(\xi_{w,\eta,j})} = \frac{2}{2+\eta},$$
$$\lim_{j\to\infty} \frac{\log H(\xi_{w,\eta,j+1})}{\log H(\xi_{w,\eta,j})} = w.$$

A direct computation shows that

$$2dw \leq \left(\frac{2w}{2+\eta} - 2\right)\left(\frac{2w}{2+\eta} - d\right).$$

Applying Proposition 4.5.4 to

$$\delta = \rho = \frac{2w}{2+\eta} - d, \quad \varepsilon = \chi = \frac{2}{2+\eta}, \quad \theta = w,$$

we have

$$w_n^*(\xi_{w,\eta}) = \frac{2w - 2 - \eta}{2+\eta}, \quad w_n(\xi_{w,\eta}) = \frac{2w - \eta}{2+\eta}$$

for all $2 \leq n \leq d$. □

4.6.3 Proofs of Theorems 4.1.12 and 4.1.16

The lemma below is an analogue of Lemma 5.6 in [4].

Lemma 4.6.5. *Consider a continued fraction* $\xi = [a_0, a_1, a_2, \ldots] \in \mathbb{F}_q((T^{-1}))$. *Let* $(p_n/q_n)_{n\geq 0}$ *be the convergent sequence of* ξ. *If the sequence* $(|q_n|^{1/n})_{n\geq 1}$ *is bounded, then* ξ *is not a* U_1*-number.*

Proof. By the assumption and Lemma 4.3.1 (v), there exists an integer $A \geq q$ such that $q^n \leq |q_n| \leq A^n$ for all $n \geq 1$. Therefore, for all $n \geq 1$, we have

$$\frac{\deg q_{n+1}}{\deg q_n} \leq \left(1 + \frac{1}{n}\right) \frac{\log A}{\log q}.$$

By Lemma 4.4.10, we obtain $w_1(\xi) \leq \log A / \log q$. □

Proof of Theorem 4.1.12. Applying Lemma 2.3.10, for $n \geq n_0$, we take finite words U_n, V_n and a rational number w_n satisfying Lemma 2.3.10 (i)–(v) and (vii). We define a positive integer sequence $(n_j)_{j\geq 0}$ by $n_{j+1} = 2(2\kappa + 1)\lceil \log A / \log q \rceil n_j$ for $j \geq 0$. Put $r_j := |U_{n_j}|, s_j := |V_{n_j}|$, and $\tilde{w}_j := w_{n_j}$ for $j \geq 0$. By Lemma 2.3.10 (iv), we have $a_{r_j} \neq a_{r_j+s_j}$ for all $j \geq 0$. Let $(p_n/q_n)_{n\geq 0}$ be the convergent sequence of ξ. By the assumption and Lemma 4.3.1 (v), we get $q^n \leq |q_n| \leq A^n$ for all $n \geq 1$. Therefore, it follows from Lemma 2.3.10 (iii) and (vi) that for $j \geq 0$

$$|q_{r_j} q_{r_j+s_j}| < |q_{r_{j+1}} q_{r_{j+1} s_{j+1}}| \leq |q_{r_j} q_{r_j+s_j}|^{4(2\kappa+1)^2 \left\lceil \frac{\log A}{\log q} \right\rceil \frac{\log A}{\log q}}. \tag{4.36}$$

Put $\alpha_j := [0, a_1, \dots, a_{r_j}, \overline{a_{r_j+1}, \dots, a_{r_j+s_j}}]$ for $j \geq 1$. By Lemma 4.3.3, we obtain

$$H(\alpha_j) \leq |q_{r_j} q_{r_j+s_j}| \tag{4.37}$$

for $j \geq 0$. Since ξ and α_j have the same first $r_j + \lceil \tilde{w}_j s_j \rceil$-th partial quotients, we have

$$\begin{aligned}|\xi - \alpha_j| &\leq \max\left(\left|\xi - \frac{p_{r_j+\lceil \tilde{w}_j s_j \rceil}}{q_{r_j+\lceil \tilde{w}_j s_j \rceil}}\right|, \left|\alpha_j - \frac{p_{r_j+\lceil \tilde{w}_j s_j \rceil}}{q_{r_j+\lceil \tilde{w}_j s_j \rceil}}\right|\right) \\ &\leq |q_{r_j+\lceil \tilde{w}_j s_j \rceil}|^{-2} q^{-1} \leq |q_{r_j+s_j}|^{-2} q^{-2(\lceil \tilde{w}_j s_j \rceil - s_j)-1}\end{aligned}$$

for $j \geq 0$ by Lemma 4.3.1 (v) and (vii). By Lemma 2.3.10 (v), we have

$$q^{2(\lceil \tilde{w}_j s_j \rceil - s_j)+1} \gg q^{\frac{r_j+s_j}{2\kappa+1}} |q_{r_j} q_{r_j+s_j}|^{\frac{\log q}{(4\kappa+2)\log A}}$$

for $j \geq 0$. It follows from Lemma 4.3.6 that

$$|q_{r_j}|^2 \leq \max(|\alpha_j - \alpha'_j|^{-1}, 1)$$

for $j \geq 0$. Hence, we obtain

$$|\xi - \alpha_j| \ll A^2 \max(|\alpha_j - \alpha'_j|^{-1}, 1) |q_{r_j} q_{r_j+s_j}|^{-2-\frac{\log q}{(4\kappa+2)\log A}} \tag{4.38}$$

for $j \geq 0$. Take a real number δ which is greater than $\mathrm{Dio}(\mathbf{a})$. Then ξ and α_j have the same at most $\lceil \delta(r_j + s_j) \rceil$-th partial quotients for sufficiently large j. By Lemmas 4.3.1 (v) and 4.3.4, we have

$$\begin{aligned}|\xi - \alpha_j| &\geq A^{-2} |q_{\lceil \delta(r_j+s_j) \rceil}|^{-2} \gg |q_{r_j} q_{r_j+s_j}|^{-4\delta \frac{(r_j+s_j)\log A}{(2r_j+s_j)\log q}} \\ &\gg |q_{r_j} q_{r_j+s_j}|^{-4\delta \frac{\log A}{\log q}}\end{aligned} \tag{4.39}$$

for sufficiently large j. Applying Proposition 4.5.1 by using (4.36), (4.37), (4.38), and (4.39), we obtain

$$w_2^*(\xi) \leq 64(2\kappa+1)^3 \,\mathrm{Dio}(\mathbf{a}) \left(\frac{\log A}{\log q}\right)^4 - 1.$$

Thus, we have (4.1) by (4.18).

Assume that the sequence $(|q_n|^{1/n})_{n\geq 1}$ converges. Let M be the limit of the sequence $(|q_n|^{1/n})_{n\geq 1}$. For any $\varepsilon > 0$, there exists an integer m such that for all $n \geq m$,

$$(M-\varepsilon)^n < |q_n| < (M+\varepsilon)^n.$$

In a similar way to the above, we see that

$$|q_{r_j}q_{r_j+s_j}| < |q_{r_{j+1}}q_{r_{j+1}s_{j+1}}| \leq |q_{r_j}q_{r_j+s_j}|^{4(2\kappa+1)^2\left\lceil\frac{\log(M+\varepsilon)}{\log(M-\varepsilon)}\right\rceil\frac{\log(M+\varepsilon)}{\log(M-\varepsilon)}},$$
$$|q_{r_j}q_{r_j+s_j}|^{-4\delta\frac{\log(M+\varepsilon)}{\log(M-\varepsilon)}} \ll |\xi-\alpha_j| \ll \max(|\alpha_j-\alpha_j'|^{-1},1)|q_{r_j}q_{r_j+s_j}|^{-2-\frac{\log(M-\varepsilon)}{(4\kappa+2)\log(M+\varepsilon)}},$$

for sufficiently large j. Applying Proposition 4.5.1, we have

$$w_2^*(\xi) \leq 64(2\kappa+1)^3\,\mathrm{Dio}(\mathbf{a})-1.$$

Thus, we have (4.2) by (4.18). □

Proof of Theorem 4.1.16. From Theorems 4.3.2, 4.4.5, and Proposition 4.4.12, we have $w_2(\xi) \geq w_2^*(\xi) \geq 2$. Therefore, we may assume that $\mathrm{Dio}(\mathbf{a}) > 1$. Take a real number δ such that $1 < \delta < \mathrm{Dio}(\mathbf{a})$. For $n \geq 1$, there exist finite words U_n, V_n and a real number w_n such that $U_nV_n^{w_n}$ is the prefix of $\mathbf{a}$, the sequence $(|V_n^{w_n}|)_{n\geq 1}$ is strictly increasing, and $|U_nV_n^{w_n}| \geq \delta|U_nV_n|$. Set $r_n := |U_n|, s_n := |V_n|$, and $\alpha_n := [0, a_1, \ldots, a_{r_n}, \overline{a_{r_n+1}, \ldots, a_{r_n+s_n}}]$. Let $\tilde{M}$ denote an upper bound of $(|q_n|^{1/n})_{n\geq 1}$. For any $\varepsilon > 0$, there exists an integer n_0 such that for all $n \geq n_0$,

$$(m-\varepsilon)^n < |q_n| < (M+\varepsilon)^n.$$

Since ξ and α_n have the same first $(r_n + \lceil w_ns_n\rceil)$-th partial quotients, we obtain

$$|\xi-\alpha_n| \leq |q_{r_n+\lceil w_ns_n\rceil}|^{-2} < (M+\varepsilon)^{-2(r_n+\lceil w_ns_n\rceil)\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}}$$

by Lemma 4.3.1 (vii).

Assume that the sequences $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ are bounded. Then, for all $n \geq 1$, we have

$$H(\alpha_n) \leq |q_{r_n}q_{r_n+s_n}| \leq \tilde{M}^{2r_n+s_n} \leq C,$$

where C is some constant and does not depend on n by Lemma 4.3.3. Therefore, the set $\{\alpha_n \mid n \geq 1\}$ is finite. Take a positive integer sequence $(n_i)_{i\geq 1}$ such that $n_i \to \infty$ as $i \to \infty$ and $\alpha_{n_1} = \alpha_{n_2} = \cdots$. Since $(s_n)_{n\geq 1}$ is bounded, we have $w_n \to \infty$ as $n \to \infty$. Hence, we obtain $\mathbf{a} = U_{n_i}\overline{V_{n_i}}$, which is a contradiction.

We next consider the case that $(r_n)_{n\geq 1}$ is unbounded. Here, if necessary, taking a subsequence of $(r_n)_{n\geq 1}$, we assume that $(r_n)_{n\geq 1}$ is increasing and $r_1 \geq n_0$. Since $H(\alpha_n) \leq (M+\varepsilon)^{2r_n+s_n}$ by Lemma 4.3.3, we have

$$|\xi-\alpha_n| \leq H(\alpha_n)^{-\frac{r_n+\lceil w_ns_n\rceil}{r_n+s_n}\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}} \leq H(\alpha_n)^{-\delta\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}}.$$

Hence, we obtain (4.3).

We consider the case that $(r_n)_{n\geq 1}$ is bounded, $(s_n)_{n\geq 1}$ is unbounded, and $\mathrm{Dio}(\mathbf{a})$ is finite. Here, if necessary, taking a subsequence of $(s_n)_{n\geq 1}$, we assume that $(s_n)_{n\geq 1}$ is increasing and $s_1 \geq n_0$. Then, for all $n \geq 1$, we have

$$H(\alpha_n) \leq \tilde{M}^{r_n}(M+\varepsilon)^{r_n+s_n} < C_1(M+\varepsilon)^{r_n+s_n},$$

where C_1 is some constant and does not depend on n. Therefore, we obtain

$$|\xi - \alpha_n| \leq (C_1 H(\alpha_n)^{-1})^{2\frac{r_n+\lceil w_n s_n \rceil}{r_n+s_n}\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}} \leq C_1^{2\,\mathrm{Dio}(\mathbf{a})} H(\alpha_n)^{-2\delta\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}}.$$

Hence, we obtain (4.3).

We consider the case that $(r_n)_{n\geq 1}$ is bounded, $(s_n)_{n\geq 1}$ is unbounded, and $\mathrm{Dio}(\mathbf{a})$ is infinite. Then, for all $n \geq 1$, we have $q^n \leq |q_n| \leq \tilde{M}^n$, which implies $H(\alpha_n) \leq \tilde{M}^{2r_n+s_n}$. Therefore, in a similar way to the above, we obtain

$$|\xi - \alpha_n| \leq H(\alpha_n)^{-\delta\frac{\log q}{\log \tilde{M}}}.$$

Hence, we have $w_2^*(\xi) = +\infty$, which implies (4.3).

Assume that the sequence $(|a_n|)_{n\geq 1}$ is bounded. We denote by A its upper bound. We consider the case that $(r_n)_{n\geq 1}$ is unbounded. Here, if necessary, taking a subsequence of $(r_n)_{n\geq 1}$, we assume that $(r_n)_{n\geq 1}$ is increasing and $r_1 \geq n_0$. Let $P_n(X)$ be the minimal polynomial of α_n. From Lemmas 4.3.1 (vii), 4.3.3, and 4.3.6, we obtain

$$\begin{aligned}|P_n(\xi)| &\leq H(\alpha_n)|\xi - \alpha_n||\xi - \alpha_n'| \leq A^2 H(\alpha_n)|q_{r_n+\lceil w_n s_n \rceil} q_{r_n}|^{-2} \\ &\leq A^2 H(\alpha_n)^{-2\frac{2r_n+\lceil w_n s_n \rceil}{2r_n+s_n}\frac{\log(m-\varepsilon)}{\log(M+\varepsilon)}+1}.\end{aligned}$$

Since

$$\frac{2r_n + \lceil w_n s_n \rceil}{2r_n + s_n} \geq \frac{r_n + \delta(r_n + s_n)}{2r_n + s_n} \geq \frac{r_n + s_n/2 + \delta(r_n + s_n/2)}{2r_n + s_n} \geq \frac{1+\delta}{2},$$

we obtain (4.4). For the remaining case, we have (4.4) in a similar way to the proof of (4.3). □

4.7 Further remarks

In this section, we give some theorems associated to the main results.

4.7.1 Relationship between automatic sequences and Diophantine exponents

Christol, Kamae, Mendes France and Rauzy [25] characterized algebraic Laurent series by using a finite automaton. More precisely, for a sequence $(a_n)_{n\geq 0}$ over $\mathbb{F}_q$, the Laurent series $\sum_{n=0}^{\infty} a_n T^{-n}$ is algebraic if and only if $(a_n)_{n\geq 0}$ is p-automatic. It is known that for $m \geq 1$ and $k \geq 2$, a sequence $(a_n)_{n\geq 0}$ is k-automatic if and only if it is k^m-automatic (see Theorem 6.6.4 in [9]). Therefore, we obtain the following corollary of Theorem 4.1.1.

Corollary 4.7.1. *Let $d, m \geq 1$ be integers and $w > 2d - 1$ be a rational number. Then there exists a p^m-automatic sequence $(a_n)_{n\geq 0}$ over $\mathbb{F}_q$ such that*

$$w_1(\xi) = w_1^*(\xi) = \ldots = w_d(\xi) = w_d^*(\xi) = w,$$

where $\xi = \sum_{n=0}^{\infty} a_n T^{-n}$.

In this subsection, we consider the problem whether or not we can extend Corollary 4.7.1 to k-automatic sequence for any integer $k \geq 2$. It is a natural problem in view of Corollary 4.7.1.

Let $k \geq 2$ be an integer. We define a set of rational numbers S_k as follows:

$$S_k = \left\{ \frac{k^a}{\ell} \middle| a, \ell \in \mathbb{Z}_{\geq 1} \right\}.$$

Bugeaud [14] proved that for an integer $k \geq 2$ and $w \in S_k$ with $w > 2$, there exists a k-automatic sequence $(a_n)_{n\geq 0}$ over $\{0, 2\}$ such that $w_1(\xi) = w - 1$, where $\xi = \sum_{n=0}^{\infty} a_n/3^n \in \mathbb{R}$. The proof of the result essentially depends on the Folding Lemma and an analogue of Lemma 4.4.10. It is known that the Folding Lemma holds for Laurent series over a finite field. For the statement and proof of the Folding Lemma, we refer the reader to [51, Proposition 2] and [55, the proof of Theorem 1]. Therefore, we have the following theorem which is similar to Bugeaud's result.

Theorem 4.7.2. *Let $k \geq 2$ be an integer and $w > 2$ be in S_k. Then there exists a k-automatic sequence $(a_n)_{n\geq 0}$ over $\mathbb{F}_q$ such that $w_1(\xi) = w - 1$, where $\xi = \sum_{n=0}^{\infty} a_n T^{-n}$.*

Using Proposition 4.5.3, we prove the following theorem.

Theorem 4.7.3. *Let $d \geq 1$ be an integer, $k \geq 2$ be an integer and $w > (2d + 1 + \sqrt{4d^2 + 1})/2$ be in S_k. Then there exists a k-automatic sequence $(a_n)_{n\geq 0}$ over $\mathbb{F}_q$ such that*

$$w_1(\xi) = w_1^*(\xi) = \ldots = w_d(\xi) = w_d^*(\xi) = w - 1,$$

where $\xi = \sum_{n=0}^{\infty} a_n T^{-n}$.

Note that $(2d - 1 + \sqrt{4d^2 + 1})/2$ is greater than $2d - 1$ for any $d \geq 1$.

Proof. Slightly modifying the proof of Theorem 1.2 in [21], we deduce that there exists a k-automatic sequence $(a_n)_{n\geq 0}$ over the set $\{0, 1\}$ which satisfies the following properties: Put

$$\{n \in \mathbb{Z}_{\geq 0} \mid a_n = 1\} =: \{n_0 < n_1 < n_2 < \ldots\},$$

the following inequality

$$\frac{2d + 1 + \sqrt{4d^2 + 1}}{2} < \frac{n_{k+1}}{n_k} \leq w$$

holds for all $k \geq 0$, and there exist infinitely many $k \geq 0$ such that $n_{k+1}/n_k = w$.

We put $q_k := T^{n_k}$ and $p_k := 1 + T^{n_k - n_{k-1}} + \cdots + T^{n_k - n_0}$ for any $k \geq 0$. Then we have for any $k \geq 0$, $\gcd(p_k, q_k) = 1$ and

$$\frac{p_k}{q_k} = \sum_{n=0}^{n_k} a_n T^{-n}.$$

We put

$$\xi := \sum_{n=0}^{\infty} a_n T^{-n}.$$

A direct computation shows that

$$\frac{-\log|\xi - p_{n_k}/q_{n_k}|}{\log|q_{n_k}|} = \frac{\log|q_{n_{k+1}}|}{\log|q_{n_k}|} = \frac{n_{k+1}}{n_k}$$

for any $k \geq 0$. It follows from the definition of w_1 that $w_1(\xi) \geq w-1$. Applying Proposition 4.5.3 to $\theta = w, \rho = w - d$ and $\delta = (1 + \sqrt{4d^2 + 1})/2$, we deduce that

$$w_1(\xi) = w_1^*(\xi) = \ldots = w_d(\xi) = w_d^*(\xi) = w - 1.$$

□

4.7.2 Analogues of Theorems 4.1.8 and 4.1.9 for real and p-adic numbers

The proof of Proposition 4.5.4 uses Lemmas 4.2.1, 4.2.5, 4.4.3, 4.5.2, and Propositions 4.2.2, 4.2.4. Analogues of Lemmas 4.2.1, 4.2.5, and Propositions 4.2.2, 4.2.4 for real and p-adic numbers are already known (see [12, Appendix A], [18, 20] and [50, Chapter 2]). Slightly modifying the proof of Lemmas 4.4.3 and 4.5.2, we obtain analogues of these lemmas for real and p-adic numbers. Therefore, we deduce analogues of Proposition 4.5.4 for real and p-adic numbers. Using the analogues of Proposition 4.5.4, we can prove analogues of Theorems 4.1.8 and 4.1.9 for real and p-adic numbers. The analogues of Theorems 4.1.8 and 4.1.9 are generalized Theorems 4.1 and 4.2 in [18], and Theorems 1 and 2 in [20].

Theorem 4.7.4. *Let $d \geq 2$ be an integer and $w \geq (3d+2+\sqrt{9d^2+4d+4})/2$ be a real number. Let $a, b \geq 1$ be distinct integers. We define a sequence $(a_{n,w})_{n\geq 1}$ by*

$$a_{n,w} = \begin{cases} b & \text{if } n = \lfloor w^i \rfloor \text{ for some integer } i \geq 0, \\ a & \text{otherwise.} \end{cases}$$

Set the continued fraction $\xi_w := [0, a_{1,w}, a_{2,w}, \ldots] \in \mathbb{R}$. Then we have

$$w_n^*(\xi_w) = w - 1, \quad w_n(\xi_w) = w$$

for all $2 \leq n \leq d$.

Theorem 4.7.5. *Let $d \geq 2$ be an integer, $w \geq 121d^2$ be a real number, and $a, b, c \geq 1$ be distinct integers. Let $0 < \eta < \sqrt{w}/d$ be a positive number, and put $m_i = \lfloor (\lfloor w^{i+1} \rfloor - \lfloor w^i - 1 \rfloor)/\lfloor \eta w^i \rfloor \rfloor$ for all $i \geq 1$. We define a sequence $(a_{n,w,\eta})_{n\geq 1}$ by*

$$a_{n,w,\eta} = \begin{cases} b & \text{if } n = \lfloor w^i \rfloor \text{ for some integer } i \geq 0, \\ c & \text{if } n \neq \lfloor w^i \rfloor \text{ for all integers } i \geq 0, \text{ and } n = \lfloor w^j \rfloor + m\lfloor \eta w^j \rfloor \text{ for some integer } 1 \leq m \leq m_j, j \geq 1, \\ a & \text{otherwise.} \end{cases}$$

Set the continued fraction $\xi_{w,\eta} := [0, a_{1,w,\eta}, a_{2,w,\eta}, \ldots] \in \mathbb{R}$. Then we have

$$w_n^*(\xi_{w,\eta}) = \frac{2w-2-\eta}{2+\eta}, \quad w_n(\xi_{w,\eta}) = \frac{2w-\eta}{2+\eta}$$

for all $2 \leq n \leq d$. Hence, we have

$$w_n(\xi_{w,\eta}) - w_n^*(\xi_{w,\eta}) = \frac{2}{2+\eta}$$

for all $2 \leq n \leq d$.

It seems that for each $d \geq 3$, the real numbers ξ defined by Theorems 4.7.4 and 4.7.5 are the first explicit continued fraction examples for which $w_d(\xi)$ and $w_d^*(\xi)$ are different.

Theorem 4.7.6. *Let $d \geq 2$ be an integer and $w \geq (3d+2+\sqrt{9d^2+4d+4})/2$ be a real number. Let $b \geq 1$ be an integer and $(\varepsilon_j)_{j\geq 0}$ be a sequence over the set $\{0, 1\}$. We define a sequence $(a_{n,w})_{n\geq 1}$ by*

$$a_{n,w} = \begin{cases} b+3i+2 & \text{if } n = \lfloor w^i \rfloor \text{ for some integer } i \geq 0, \\ b+3i+\varepsilon_j & \text{if } \lfloor w^i \rfloor < n < \lfloor w^{i+1} \rfloor \text{ for some integer } i \geq 0. \end{cases}$$

Set the Schneider's p-adic continued fraction $\xi_w := [a_{1,w}, a_{2,w}, \ldots] \in \mathbb{Q}_p$. *Then we have*

$$w_n^*(\xi_w) = w - 1, \quad w_n(\xi_w) = w$$

for all $2 \leq n \leq d$.

Theorem 4.7.7. *Let* $d \geq 2$ *be an integer and* $w \geq 121d^2$ *be a real number. Let* $b \geq 1$ *be an integer,* $(\varepsilon_j)_{j\geq 0}$ *be a sequence over the set* $\{0, 1\}$ *and* $0 < \eta < \sqrt{w}/d$ *be a positive number. We define a sequence* $(a_{n,w,\eta})_{n\geq 1}$ *by*

$$a_{n,w,\eta} = \begin{cases} b + 4i + 3 & \text{if } n = \lfloor w^i \rfloor \text{ for some integer } i \geq 0, \\ b + 4i + 2 & \text{if } \lfloor w^i \rfloor < n < \lfloor w^{i+1} \rfloor \text{ for some integer } i \geq 0 \text{ and } (n - \lfloor w^i \rfloor)/\lfloor \eta w^i \rfloor \in \mathbb{Z}, \\ b + 4i + \varepsilon_i & \text{if } \lfloor w^i \rfloor < n < \lfloor w^{i+1} \rfloor \text{ for some integer } i \geq 0 \text{ and } (n - \lfloor w^i \rfloor)/\lfloor \eta w^i \rfloor \notin \mathbb{Z}. \end{cases}$$

Set the Schneider's p-adic continued fraction $\xi_{w,\eta} := [a_{1,w,\eta}, a_{2,w,\eta}, \ldots] \in \mathbb{Q}_p$. *Then we have*

$$w_n^*(\xi_{w,\eta}) = \frac{2w - 2 - \eta}{2 + \eta}, \quad w_n(\xi_{w,\eta}) = \frac{2w - \eta}{2 + \eta}$$

for all $2 \leq n \leq d$. *Hence, we have*

$$w_n(\xi_{w,\eta}) - w_n^*(\xi_{w,\eta}) = \frac{2}{2 + \eta}$$

for all $2 \leq n \leq d$.

Note that the definition of Schneider's p-adic continued fractions is written in [20]. It seems that for each $d \geq 3$, the p-adic numbers ξ defined by Theorems 4.7.6 and 4.7.7 are the first explicit continued fraction examples for which $w_d(\xi)$ and $w_d^*(\xi)$ are different.

4.7.3 Rational approximation in $\mathbb{F}_q((T^{-1}))$

Lemma 4.7.8. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q$. *Set* $\xi := \sum_{n=0}^{\infty} a_n T^{-n}$. *Then we have*

$$w_1(\xi) \geq \max(1, \mathrm{Dio}(\mathbf{a}) - 1). \tag{4.40}$$

Proof. From Theorem 4.4.5, we have $w_1(\xi) \geq 1$. Therefore, we may assume that $\mathrm{Dio}(\mathbf{a}) > 1$. Take a real number δ such that $1 < \delta < \mathrm{Dio}(\mathbf{a})$. For $n \geq 1$, there exist finite words U_n, V_n and real numbers w_n such that $U_n V_n^{w_n}$ are the prefix of $\mathbf{a}$, the sequence $(|V_n^{w_n}|)_{n\geq 1}$ is strictly increasing, and $|U_n V_n^{w_n}| \geq \delta |U_n V_n|$. Put $q_n := T^{|U_n|}(T^{|V_n|} - 1)$. Then there exists $p_n \in \mathbb{F}_q[T]$ such that

$$\frac{p_n}{q_n} = \sum_{k=0}^{\infty} b_k^{(n)} T^{-k},$$

where $(b_k^{(n)})_{k\geq 0}$ is the infinite word $U_n \overline{V_n}$ by Lemma 3.4 in [35]. Since ξ and p_n/q_n have the same first $|U_n V_n^{w_n}|$-th digits, we obtain

$$\left| \xi - \frac{p_n}{q_n} \right| \leq |q_n|^{-\delta}.$$

Hence, we have (4.40). □

The following theorem is an analogue of Théorème 2.1 in [6] and Theorem 3.1.3, and is an extension of Theorem 1.2 in [35].

Theorem 4.7.9. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q$*. Set* $\xi := \sum_{n=0}^{\infty} a_n T^{-n}$*. Assume that there exist integers* $n_0 \geq 1$ *and* $\kappa \geq 2$ *such that for all* $n \geq n_0$*,*

$$p(\mathbf{a}, n) \leq \kappa n.$$

If the Diophantine exponent of $\mathbf{a}$ *is finite, then we have*

$$w_1(\xi) \leq 8(\kappa+1)^2(2\kappa+1)\,\mathrm{Dio}(\mathbf{a}) - 1. \tag{4.41}$$

Proof. For $n \geq n_0$, take finite words U_n, V_n and rational numbers w_n satisfying Lemma 2.3.10 (i)–(vi). Put $q_n := T^{|U_n|}(T^{|V_n|} - 1)$. Then there exists $p_n \in \mathbb{F}_q[T]$ such that

$$\frac{p_n}{q_n} = \sum_{k=0}^{\infty} b_k^{(n)} T^{-k},$$

where $(b_k^{(n)})_{k\geq 0}$ is the infinite word $U_n \overline{V_n}$ by Lemma 3.4 in [35]. Since ξ and p_n/q_n have the same first $|U_n V_n^{w_n}|$-th digits, we obtain

$$\left| \xi - \frac{p_n}{q_n} \right| \leq |q_n|^{-1-\frac{1}{4\kappa+2}}.$$

Take a real number δ which is greater than $\mathrm{Dio}(\mathbf{a})$. Note that $\delta > 1$. By the definition of Diophantine exponent, there exists an integer $n_1 \geq n_0$ such that for all $n \geq n_1$

$$\left| \xi - \frac{p_n}{q_n} \right| \geq |q_n|^{-\delta}.$$

We define a positive integer sequence $(n_j)_{j\geq 1}$ by $n_{j+1} = 2(\kappa+1)n_j$ for $j \geq 1$. It follows from Lemma 2.3.10 (iii) and (vi) that for $j \geq 1$

$$|q_{n_j}| < |q_{n_{j+1}}| \leq |q_{n_j}|^{4(\kappa+1)^2}.$$

Thus, by Lemma 3.2 in [35], we obtain (4.41). □

Consequently, the following corollary holds.

Corollary 4.7.10. *Let* $\mathbf{a} = (a_n)_{n\geq 0}$ *be a non-ultimately periodic sequence over* $\mathbb{F}_q$*. Set* $\xi := \sum_{n=0}^{\infty} a_n T^{-n}$*. Then the Diophantine exponent of* $\mathbf{a}$ *is finite if and only if* ξ *is not a* U_1*-number.*

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