Some topics in topological dynamics

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abstract

In this paper we study about topological dynamics, especially dynamical decomposition theorems and generalized inverse limits.

In the first half of this thesis, we discuss dynamical decomposition theorems. It is well known that a space X has at most dimension n ($n \in \{0\} \cup \mathbb{N}$) (i.e. $\dim X \leq n$) if and only if X can be represented as a union of $(n+1)$ zero-dimensional subspaces of X. Here we introduce "dark spaces" and "bright spaces", and prove that if $f : X \to X$ is a homeomorphism of an n-dimensional separable metric space X with the zero-dimensional set of periodic points, then X can be decomposed into an zero-dimensional bright space of f except n times and a $(n-1)$-dimensional dark space of f except n times. Also we give some dynamical decomposition theorems by using these spaces.

In the second half, we study topological structures of inverse limits with upper semi-countinuous set-valued functions. In 2004, W.S. Mahavier [19] started studies of inverse limits with subset of $[0,1] \times [0,1]$. Since then, many topological properties of inverse limits of upper semi-continuous set-valued functions have been studied by many authors. In this paper, we introduce new indexes $\tilde{I}(X_i, f_{i,i+1})$ and $\tilde{W}(X_i, f_{i,i+1})$ for an inverse sequence $\{X_i, f_{i,i+1}\}$ with upper semi-countinuous set-valued functions, and new space "dimensionally stepwise space". By using them we investigate some topological structures of such inverse limits.

Contents

1 Introduction

In chapter 2, we define some notations and study some basic properties of topological dimension, dynamical systems, and inverse limits of mapping.

In chapter 3, we discuss dynamical decomposition. A union $X = A_0 \cup A_1 \cup \cdots \cup A_n$ is called a decomposition if A_i are pairwise disjoint. It is well known that a space X has at most dimension n ($n \in \{0\} \cup \mathbb{N}$) (i.e. $\dim X \leq n$) if and only if X can be represented as a union of $(n+1)$ zero-dimensional subspaces of X. In [1], J. M. Aarts, J. Fokkink, and J. Vermeer discussed some dynamical decomposition theorems. Here we introduce new notions of‘ bright spaces ’and‘ dark spaces ’of homeomorphisms except n times, and by use of the notions we will find some dynamical decomposition theorems of spaces related to given homeomorphisms. Finally, as a special case we consider the case that given homeomorphism is a continuum-wise expansive homeomorphism.

In chapter 4, we discuss what compactum can be obtained as an inverse limit with set-valued functions. Inverse limits have played very important roles in the development of continuum theory and topological dynamics. In 2004, W.S. Mahavier [19] started studies of inverse limits with subset of [0, 1] × [0, 1] and in 2006, W.T. Ingram and W.S. Mahavier [12] started inverse limits of upper semi-continuous set valued functions. Since then, many topological properties of inverse limits of upper semi-continuous set valued functions have been studied by many authors. In [14] and [13] W. T. Ingram and W. S. Mahavier discussed several results concerning connectedness of such inverse limits. [21] V. Nall showed other sufficient conditions of connectedness. In [9] A. Illane proved that a simple closed curve can not be obtained as an inverse limit on $[0, 1]$ with a single upper semi-continuous function. In [22] V. Nall showed that the arc is the only finite graph that is an inverse limit on $[0, 1]$ with a single upper semi-continuous function. Also, Nall showed any inverse limit on $[0, 1]$ with a single upper semi-continuous function can not be n-cell for $n > 1$. In [4] and [15] properties of shape of inverse limits was discussed. In first section, we study dimension of such inverse limits. It is well-known that inverse limits of sequences of single-valued continuous functions have dimension bounded by the dimensions of the factor spaces. V. Nall [20] proved that inverse limits of sequences of upper semi-continuous set-valued functions with 0-dimensional values have dimension bounded by the dimensions of the factor spaces and I. Banic [2]discussed previous case of finite-dimensional valued functions. H. Kato [15] generalized these results by using ”expand-contract sequences”. Here we introduce ”inverse expand-contract sequences”

In second section, we introduce ”stepwise spaces” and give a sufficient condition of step wiseness. As a corollary, we obtain that any n-dimensional manifold can not be represented as any inverse limit with single upper semi-continuous bonding function on $[0, 1]$ for $n > 1$.

In final section, we study ANR properties of generalized inverse limits. H. Kato [16] introduced ”weak homotopically trivial within small neighborhoods” and used it to prove ANR of a given space. By using this idea, we discuss ANR properties of generalized inverse limits.

2 Preliminaries

We assume that all spaces X are **separable metric spaces** in this paper. Also, let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and the set of integers, respectively. For a subset A of X, $|A|$ denotes the cardinality of a set A, $\mathrm{cl}(A)$ denotes the closure, $\mathrm{bd}(A)$ denotes the boundary, and $\mathrm{int}(A)$ denotes the interior. Also, $\mathrm{diam}A$ is a diameter of A, i.e. $\mathrm{diam}A = \sup\{d(x,y) \mid x, y \in A\}$, where d is a metric of X.

A **compactum** is a compact metric space.

A **continuum** is a nonempty compact connected metric space. A **subcontinuum** is a continuum which is a subset of a space. If a continuum that contains more than one point, we call it nondegenerate.

An **arc** means a continuum which is homeomorphic to the closed interval $[0,1]$. I means the unit interval $[0,1]$.

A **graph** is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points.

A **Tree** means a graph which contains no simple closed curve.

A subspace J of a graph G is a **free arc** in G if J is homeomorphic to the unit interval and $J \setminus \{e, e'\}$ is an open set of G, where e and e' are the two end points of J. Similarly, A simplex $\triangle$ in a polyhedron P is a **free simplex** of P if the interior $\triangle \setminus \partial\triangle$ of $\triangle$ is an open set of P, i.e., $\triangle$ is not a face of any other simplex.

Let X be a space and A be a subspace of X. We say A is a **retract** of X if there is a map $r : X \to A$ such that $r|A$ is the identity map on A. Such a r is called **retraction**.

A space X is an **absolute neighborhood retract** (abbrev. ANR) if for each metric space M containing X as a closed set, there is a retraction $r : U \to A$ from some open neighborhood $U \subset M$ in X. If $U = M$, X is called an **absolute retract** (abbrev. AR).

A compact space X is an **fundamental absolute retract** (abbrev. FAR) if there exists a decreasing sequence of compact ARs X_i such that $X = \bigcap X_i$

2.1 Topological dimension

Let X be a space and $\mathcal{U}$, $\mathcal{V}$ be two covers of X. Then $\mathcal{V}$ is a *refinement* of $\mathcal{U}$, if for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subset U$.

Definition 1. [5] *Let X be a space and let $\mathcal{U}$ be a family of subsets of X. Then, we define order of $\mathcal{U}$ as follows;*

$$\mathrm{ord}(\mathcal{U}) = \sup\{\mathrm{ord}_x(\mathcal{U}) |\ x \in X\},$$

where $\mathrm{ord}_x(\mathcal{U})$ is the number of $\mathcal{U}$ which contains x.

Definition 2. [5] *For a space X we define the topological dimension of X, denoted by $\dim X$, which is an integer n larger than or equal to -1, or ∞, the definition of the dimension function $\dim$ consists in the following conditions;*

- $\dim X \leq n$ *if every finite open cover of X has a finite open refinement of order $\leq n+1$,*
- $\dim X = n$ *if* $\dim X \leq n$ *and* $\dim X > n-1$,
- $\dim X = \infty$ *if* $\dim X > n$ *for all* $n = -1, 0, 1, \ldots$

Theorem 1. [5]*(the countable sum theorem) If a space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that $\dim F_i \leq n$ for each $i \in \mathbb{N}$, then $\dim X \leq n$.*

Theorem 2. [5] *A space X satisfies* $\dim X \le n$ *if and only if X can be represented as the union of two subspaces Y and Z such that* $\dim Y \le n-1$ *and* $\dim Z \le 0$.

Theorem 3. [5] *A space X satisfies* $\dim X \le n$ *if and only if X can be represented as the union of* $(n+1)$ *subspaces* $Z_1, Z_2, \cdots Z_{n+1}$ *such that* $\dim Z_i \le 0$ *for* $i = 1, 2, \cdots, n+1$.

Theorem 4. [5] *If* $f : X \to Y$ *is a closed mapping between two spaces and there is* $k \ge 0$ *such that* $\dim f^{-1}(y) \le k$ *for each* $y \in Y$, *then* $\dim X \le \dim Y + k$.

Theorem 5. [5] *Suppose that* $f : X \to Y$ *is a closed mapping between two spaces. If* $\dim Y \le n$ *and* $\dim D_i(f^{-1}) \le n-i$ *for each* $i = 1, 2, \ldots, n+1$, *then* $\dim X \le n$, *where* $D_i(f^{-1}) = \{y \in Y \mid \dim f^{-1}(y) \ge i\}$. *In particular, if* $f : X \to Y$ *is a closed mapping such that* $\dim Y \le 1$ *and* $\dim D_1(f^{-1}) + \dim f^{-1}(y) \le 1$ *for each* $y \in Y$, *then* $\dim X \le 1$.

2.2 Dynamical systems

A **dynamical system** is a pair of a space X and a homeomorphism $f : X \to X$.

For a homeomorphism $f : X \to X$ of a space X and $k \in \mathbb{N}$, let $P_k(f)$ denote the set of **periodic** points of period $\le k$. Also, $P\big(f\big)$ denotes the set of all periodic points of f.

For a point $x \in X$, $O_f(x) = \{f^p(x) \mid p \in \mathbb{Z}\}$ denotes the **orbit** of x.

Let $f : X \to X$ be a homeomorphism, $L \subset X$ be a subset of X, and $j = 0, 1, 2, \ldots$. Then $A_f(L, j)$ denotes the set of all points $x \in X$ whose orbit appears in L just j times, i.e.

$$A_f(L, j) = \{x \in X \mid |\{p \in \mathbb{Z} \mid f^p(x) \in L\}| = j\}.$$

Note that $P(f) \subset A_f(L, 0)$ and $A_f(L, j)$ is f-invariant for each $j = 0, 1, 2, \ldots,$ i.e. $f(A_f(L, j)) = A_f(L, j)$. If $i \ne j$, then $A_f(L, i) \cap A_f(L, j) = \emptyset$.

Definition 3. *A homeomorphism* $f : X \to X$ *of a compact metric space* (X, d) *is* **expansive** *if there is* $c > 0$ *such that for any* $x, y \in X$ *with* $x \ne y$, *there is an integer* $k \in \mathbb{Z}$ *such that* $d(f^k(x), f^k(y)) \ge c$.

Definition 4. *A homeomorphism* $f : X \to X$ *of a compact metric space* (X, d) *is* **continuum-wise expansive** *if there exists* $c > 0$ *such that for any nondegenerate subcontinuum A of X, there exists an integer* $k \in \mathbb{Z}$ *such that* $\text{diam } f^k(A) \ge 0$.

Note that every expansive homeomorphism is continuum-wise expansive. Such $c > 0$ is called an **expansive constant** for f.

Proposition 1. [17] *Let* $f : X \to X$ *be a homeomorphism of a compact metric space X. Then the following conditions are equivalent.*

1. *f is a continuum-wise expansive homeomorphism.*
2. *There is* $\delta > 0$ *such that if $\mathcal{C}$ is any finite open cover of X with* $\text{mesh}(\mathcal{C}) < \delta$ *and any* $\gamma > 0$, *there is a sufficiently large natural number N such that if* $A, B \in \mathcal{C}$, *each component of* $f^{-n}(\text{cl}(A)) \cap f^n(\text{cl}(B))$ *has diameter less than γ for each* $n \ge N$.

A subset Z of X is a **bright space** of f except n times $\big(n \in \{0\} \cup \mathbb{N}\big)$ if for any $x \in X$,

$$|\{p \in \mathbb{Z} | f^p(x) \notin Z\}| \le n.$$

Also we say that $L = X \setminus Z$ is a **dark space** of f except n times. Note that for any $x \in X$, $|O_f(x) \cap L| \le n$.

2.3 Inverse limits

Definition 5. *For each $i \in \mathbb{N}$, let X_i be a space and let $f_{i,i+1} : X_{i+1} \to X_i$ be a continuous function. Then the inverse limit, denoted $\varprojlim\{X_i, f_{i,i+1}\}$, is the set of all $\big(x_1, x_2, \cdots\big) \in \prod_{i=1}^{\infty} X_i$ such that $x_i = f_{i,i+1}(x_{i+1})$ for each i.*

Proposition 2. *If X_i is a continuum for every $i \in \mathbb{N}$, then $\varprojlim\{X_i, f_{i,i+1}\}$ is also continuum.*

Theorem 6. [5] *If for every i, let X_i is a compact metric space with $\dim X_i \leq n$ and $f_{i,i+1} : X_{i+1} \to X_i$ be a continuous function, then $\dim \varprojlim\{X_i, f_{i,i+1}\} \leq n$.*

3 Dynamical decomposition theorems

3.1 Some dynamical results

In [1], J. M. Aarts, J. Fokkink, and J. Vermeer proved an interesting theorem.

Theorem 7. *Suppose that X is a space with $\dim X \leq n$ and $f : X \to X$ is a homeomorphism. Then there exists an f-invariant zero-dimensional dense G_δ-set Z of X such that*

$$X = Z \cup f(Z) \cup f^2(Z) \cup \cdots \cup f^n(Z)$$

if and only if $\dim P_k(f) < k$ for each $1 \leq k \leq n$.

Proposition 3. *Suppose that X is a space with $\dim X \leq n$ and $f : X \to X$ is a homeomorphism. Then there exist f-invariant zero-dimensional dense G_δ-sets $Ef(j) (j = 0, 1, 2, \ldots, n)$ of X such that*

$$X = Ef(0) \cup Ef(1) \cup \cdots \cup Ef(n).$$

Proof. First, we prove the following claim (I);

for each zero-dimensional f-invariant set A of X, there is an zero-dimensional f-invariant G_δ-set A' with $A \subset A'$.

To prove the claim (I), choose an zero-dimensional G_δ-set A' with $A \subset A'$, i.e. $A' = \bigcap_{i\in\mathbb{N}} U_i$, where U_i is an open set of X. Then the set $A' = \bigcap\{f^p(U_i) \mid i \in \mathbb{N},\ p \in \mathbb{Z}\}$ is the desired G_δ-set.

Since X is separable, there is a countable dense set D' of X and we put $D = \bigcup\{f^p(D') \mid p \in \mathbb{Z}\}$. Then D is countable, dense and f-invariant.

We will show the following claim (II);

there is a zero-dimensional f-invariant dense set E_0 of X such that $D \subset E_0$ and $\dim\big[(X \setminus E_0) \cup D\big] \leq n - 1 = \dim X - 1$.

To prove the claim (II), choose a countable open base $\{U_i \mid i \in \mathbb{N}\}$ of X such that $\mathrm{bd}(U_i) \cap D =$ and $\dim(U_i) \leq n - 1$ for each $i \in \mathbb{N}$ (see [2]). Put

$$L = \bigcup\{f^p(\mathrm{bd}(U_i)) \mid i \in \mathbb{N},\ p \in \mathbb{Z}\}.$$

Note that L is an f -invariant F_σ -set of X such that $L \cap D = \emptyset$. By countable sum theorem of dimension, $\dim(L \cup D) \leq n - 1$. Put $E_0 = X \setminus L$. Then E_0 satisfies the desired conditions.

Next, we consider the $(n-1)$-dimensional space $X_1 = (X \setminus E_0) \cup D$. If we apply the above claim (II) to $f|X_1 : X_1 \to X_1$, we obtain a zero-dimensional f-invariant subset E_1 of X_1 such that $D \subset E_1$ and $\dim\big[(X_1 \setminus E_1) \cup D\big] \leq n - 2$. If we continue this procedure, we obtain subsets $E_j (j = 0, 1, \ldots, n)$ of X such that each E_j is a dense f-invariant zero-dimensional subsets of X and

$$X = \bigcup\{E_j \mid j = 0,\ 1,\ 2,\ \ldots,\ n\}.$$

Finally, by use of the claim (I), we obtain f-invariant zero-dimensional dense G_δ-sets $Ef(j) (j = 0, 1, 2, \ldots, n)$ of X, which satisfy the desired conditions.

□

3.2 Dynamical decomposition theorems using dark spaces and bright spaces

In this section, we show other types of dynamical decomposition theorems using the technique in [10].

Lemma 1. *Suppose that X is a space with $\dim X \le n$ and $f : X \to X$ is a homeomorphism with $\dim P(f) \le 0$. Let F be an F_σ-set of X with $\dim F \le 0$. Then for each $j \in \mathbb{N}$, there is a locally finite countable open cover $\mathcal{C}(j) = \{C(j)_\alpha \mid \alpha \in \mathbb{N}\}$ of X such that*

1. $\mathrm{mesh}(\mathcal{C}(j)) < \frac{1}{j}$,
2. $\mathrm{ord}(G) \le n$, *where* $G = \{f^p(\mathrm{bd}(C(j)_\alpha)) \mid \alpha \in \mathbb{N},\ j \in \mathbb{N},\ \textit{and}\ p \in \mathbb{Z}\}$, *and*
3. $F \cap L = \phi$, *where* $L = \bigcup\{(\mathrm{bd}(\mathcal{C}(j)_\alpha)) \mid \alpha \in \mathbb{N},\ j \in \mathbb{N}\}$.

Proof. Since F is an F_σ-set of X with $\dim F \le 0$, we can put $F = \bigcup_{j\in\mathbb{N}} F_j$, where F_j is an zero-dimensional closed set in X. For each $j \in \mathbb{N}$, we choose a locally finite countable open cover $\mathcal{D}(j)$ of X such that $\mathrm{mesh}(\mathcal{D}(j)) < \frac{1}{j}$. We put $\mathcal{D}(j) = \{D(j)_\alpha \mid \alpha \in \mathbb{N}\}$. Note that $D(j)_\alpha$ may be an empty set. Take an open shrinking $\mathcal{B}(j) = \{B(j)_\alpha \mid \alpha \in \mathbb{N}\}$ of $\mathcal{D}(j)$ such that $\mathcal{B}(j) = \{\mathrm{cl}(B(j)_\alpha) \mid \alpha \in \mathbb{N}\}$ is a closed shrinking of $\mathcal{D}(j)$. For each $j \in \mathbb{N}$ and each $k \in \mathbb{N}$ with $k \ge j$, we can find an open shrinking $\mathcal{D}(j,k) = \{D(j,k)_\alpha \mid \alpha \in \mathbb{N}\}$ of $\mathcal{D}(j)$ and a closed shrinking $\mathcal{B}(j,k) = \{B(j,k)_\alpha \mid \alpha \in \mathbb{N}\}$ of $\mathcal{D}(j,k)$ such that

$\big[a\big]\mathcal{D}(j,j) = \mathcal{D}(j), \mathcal{B}(j,j) = \mathcal{B}(j)$,

$\big[b\big]\mathrm{cl}(B(j)_\alpha) = B(j,j)_\alpha \subset B(j,j+1)_\alpha \subset \cdots \subset D(j,j+1)_\alpha \subset D(j,j)_\alpha = D(j)_\alpha$,

$\big[c\big]\mathrm{ord}\{cl(f^p(D(j,k+1)_\alpha \setminus B(j,k+1)_\alpha)) \mid \alpha \in \mathbb{N},\ 1 \le j \le k\ \textit{and}\ |p| \le k\} \le n$, and

$\big[d\big]\big[D(j,k+1)_\alpha \setminus B(j,k+1)_\alpha\big] \cap \bigcup_{j=1}^{k+1} F_j = \emptyset$.

We put

$$\mathcal{C}(j)_\alpha = \mathrm{int}\big[\textstyle\bigcap_{k=j}^{\infty} D(j,k)_\alpha\big] \text{ and } \mathcal{C} = \{C(j)_\alpha \mid \alpha \in \mathbb{N}\}.$$

Then $\mathcal{C}(j) = \{C(j)_\alpha \mid \alpha, \in \mathbb{N}\}$ $(j \in \mathbb{N})$ is the desired open cover of X. ☐

Theorem 8. *Suppose that X is a space with $\dim X = n$ and $f : X \to X$ is a homeomorphism. Then there exists a bright space Z of f except n times such that Z is an zero-dimensional dense G_δ-set of X and the dark space $L = X \setminus Z$ of f is an $(n-1)$-dimensional F_σ-set of X if and only if $\dim P(f) \le 0$.*

Proof. Suppose $\dim P(f) \le 0$. Since X is separable, there is a dense countable set D of X. Also we choose a zero-dimensional F_σ-set H of X with $\dim(X \setminus H) \le n-1$ (see [2]). Then the set $F = D \cup H$ is also a zero-dimensional F_σ-set of X. By Lemma 1, we have a countable base $\{B_i \mid i \in \mathbb{N}\}$ of X such that $\mathrm{ord}(G) \le n$ and $L \cap F = \emptyset$, where $G = \{f^p(\{\mathrm{bd}(B_i)) \mid i \in \mathbb{N},\ p \in \mathbb{Z}\}$ and $L = \bigcup\{\mathrm{bd}(B_i) \mid i \in \mathbb{N}\}$. Put $Z = X \setminus L$. Note that $D \subset Z$ and $L \subset X \setminus F$. Then Z is dense in X and $\dim L \le n-1$ and hence Z and L are the desired spaces.

Conversely, we assume that there exists an zero-dimensional bright space Z of f except n times. Then we see $P(f) \subset Z$, which implies that $\dim P(f) \le 0$. ☐

Corollary 1. *Suppose that X is a space with $\dim X = n$ and $f : X \to X$ is a homeomorphism. Then there exists an zero-dimensional dense G_δ-set Z of X such that for any $(n+1)$ integers $k_0 < k_1 < \cdots < k_n$,*

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z)$$

if and only if $\dim P(f) \leq 0$.

Proof. Suppose that $\dim P(f) \leq 0$. By above Theorem, there exists an zero-dimensional bright space Z of f except n times such that Z is a dense G_δ-set of X. Let $x \in X$ and let $k_0 < k_1 < \cdots < k_n$ be any $(n+1)$ integers. Then we can find some k_i such that $f^{-k_i}(x)(= y) \in Z$. Then $x = f^{k_i}(y) \in f^{k_i}(Z)$, which implies that $X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z)$.

Conversely, we assume the existence of Z satisfying the above condition. We will show that $P(f) \subset Z$. Let $x \in P(f)$. Then there is $k \in N$ such that $f^k(x) = x$. Consider $(n+1)$ integers $k_i = i \cdot k (i = 0, 1, 2, \ldots, n)$. Since $x \in Z \cup f^k(Z) \cup f^{2k}(Z) \cup \cdots \cup f^{n \cdot k}(Z)$, $x \in f^{i \cdot k}(Z)$ for some $i \in \{0, 1, 2, \ldots, n\}$. Then $x = f^{-i \cdot k}(x) \in Z$. Since $P(f) \subset Z$ and Z is zero-dimensional, hence $\dim P(f) \leq 0$. □

Theorem 9. *Suppose that X is a space with $\dim X = n$ and $f : X \to X$ is a homeomorphism with $\dim P(f) \leq 0$. If L is a dark space of f except n times such that L is an F_σ-set of X and $\dim(X \setminus L) \leq 0$, then $\dim A_f(L, j) = 0$ for each $j = 0, 1, \ldots, n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the dark space L:*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Proof. Note that $A_f(L, 0) \subset X \setminus L(= Z)$ and hence $A_f(L, 0)$ is an f-invariant zero dimensional subset of X. We will prove that $\dim A_f(L, j) = 0$ for each $j = 1, 2, \ldots, n$. Since L is an F_σ-set of X, we can put $L = \bigcup_{i \in \mathbb{N}} L_i$, where L_i is a closed subset of X. Let $1 \leq j \leq n$. For any j integers $k_1 < k_2 < \cdots < k_j$ and natural numbers $i_1, i_2, \ldots, i_j$, we consider the set

$$A(k_1, k_2, \ldots, k_j : L_{i_1}, L_{i_2}, \ldots, L_{i_j}) = \{x \in A_f(L, j) \mid f^{k_p}(x) \in L_{i_p} (p = 1,\ 2, \ldots,\ j)\}.$$

Then we can easily see that $A(k_1, k_2, \ldots, k_j : L_{i_1}, L_{i_2}, \ldots, L_{i_j})$ is closed in the subspace $A_f(L, j)$. Note that if $k \neq k_p (p = 1, 2, \ldots, j)$, then

$$f^k(A(k_1, k_2, \ldots, k_j : L_{i_1}, Li_2, \ldots, Li_j)) \subset Z$$

and hence $A(k_1, k_2, \ldots, k_j : L_{i_1}, L_{i_2}, \ldots, L_{i_j})$ is zero-dimensional. Also note that

$$A_f(L, j) = \bigcup \{A(k_1, k_2, \ldots, k_j : L_{i_1}, L_{i_2}, \ldots, L_{i_j}) | k_1 < k_2 < \cdots < k_j (\in \mathbb{Z}) \textit{ and } i_1, i_2, \ldots i_j \in \mathbb{N}\}.$$

By the countable sum theorem of dimension, we see that $\dim A_f(L, j) = 0$. □

3.3 Dynamical decomposition theorems of continuum-wise expansive homeomorphisms

As a special case, we consider the case that f is a continuum-wise expansive homeomorphism of a compact metric space X.

Theorem 10. *Suppose that X is a compact metric space with $\dim X = n$ and $f : X \to X$ is a continuum-wise expansive homeomorphism. Then there exists a compact $(n-1)$-dimensional dark space L of f except n times such that If L is a dark space of f except n times such that $\dim A_f(L, j) = 0$ for each $j = 0, 1, \ldots, n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the compact dark space L;*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Proof. Since f is a continuum-wise expansive homeomorphism, we have a positive number δ as in (2) of Proposition 1. Since $\dim P(f) \leq 0$, by Lemma 1 there is a finite open cover $\mathcal{C}$ of X such that

1. $\mathrm{mesh}(\mathcal{C}) < \delta$,
2. $\mathrm{ord}(\mathcal{G}) \leq n$, where $\mathcal{G} = \{f^p(\mathrm{bd}(C)) | C \in \mathcal{C},\ p \in \mathbb{Z}\}$,
3. $\mathrm{bd}(C) \cap P(f) = \emptyset$ for each $C \in \mathcal{C}$, and
4. $\dim H \leq n-1$, where $H = \bigcup\{\mathrm{bd}(C) \mid C \in \mathcal{C}\}$.

Let $\mathcal{C} = \{C_1,\ C_2, \ldots,\ C_m\}$ and put

$$C_1' = \mathrm{cl}(C_1), C_{i+1}' = \mathrm{cl}(\mathrm{int}\big[C_{i+1} \setminus \bigcup_{k \leq i} C_k\big])\ (1 \leq i \leq m-1).$$

Then $\mathcal{C}' = \{C_1',\ C_2', \ldots,\ C_m'\}$ is a finite closed partition of X. Let $L = \{\mathrm{bd}(C') \mid C' \in \mathcal{C}'\}$. Then $L \subset H$ and we can easily see that L is a compact $(n-1)$-dimensional dark space of f except n times. We will show that $\dim A_f(L, j) = 0$ for each $j = 0, 1, 2, \ldots, n$. Let $1 \leq j \leq n$. For any j integers $k_1 < k_2 < \cdots < k_j$, we consider the set

$$A(k_1, k_2, \ldots, k_j) = \{x \in A_f(L, j) \mid f^{k_p}(x) \in L\ (p = 1,\ 2, \ldots,\ j)\}.$$

Then we see that the space $A(k_1, k_2, \ldots, k_j)$ is closed in the subspace $A_f(L, j)$. We will show that $\dim A(k_1, k_2, \ldots, k_j) = 0$. Let $x \in A(k_1, k_2, \ldots, k_j)$ and let $\gamma > 0$ be any positive number. Then there is a sufficiently large natural number N such that $N > |k_i|$ $(i = 1, 2, \ldots, j)$ and N satisfies the condition (2) of Proposition 1. We can choose $1 \leq \alpha, \beta \leq m$ such that $f^{-N}(x) \in \mathrm{int}(C_\alpha')$ and $f^N(x) \in \mathrm{int}(C_\beta')$. Then the diameters of components of the compactum $f^N(C_\alpha') \cap f^{-N}(C_\beta')$ are less than γ. Since $f^N(C_\alpha') \cap f^{-N}(C_\beta')$ can be covered by finite mutually disjoint open sets of X whose diameters are less than γ, there is a closed and open neighborhood V of x in the subspace $A(k_1, k_2, \ldots, k_j)$ such that $V \subset f^N(\mathrm{int}(C_\alpha')) \cap f^{-N}(\mathrm{int}(C_\beta'))$ and $\mathrm{diam} V < \gamma$. This implies that $\dim A(k_1, k_2, \ldots, k_j) = 0$. Note that

$$A_f(L, j) = \bigcup\{A(k_1,\ k_2, \ldots,\ k_j) \mid k_1\ <\ k_2\ <\ \ldots\ < k_j\ (\in \mathbb{Z})\}.$$

By countable sum theorem of dimension theory, we see that $\dim A_f(L, j) = 0$. By the similar arguments to the case $j \geq 1$, we see that the case $j = 0$ is true, i.e. $\dim A_f(L, 0) = 0$. □

4 Inverse limits with set-valued functions

For a space X, 2^X denotes the collection of nonempty closed subsets of X.

Definition 6. *Let $f : X \to 2^Y$ be a set-valued function and let A be a subset of X. Then we define*

$$f(A) = \bigcup\{f(x) \mid x \in A\}.$$

Also for a subset B of Y, we define

$$f^{-1}(B) = \bigcup\{x \in X \mid f(x) \cap B \neq \emptyset\}.$$

f is called **surjective** *if $f(X) = Y$.*

Definition 7. *Let $f : X \to 2^Y$ and $g : Y \to 2^Z$ be set-valued functions. Then*

$$gf(x) = g(f(x)) = \bigcup\{g(y) \mid y \in f(x)\}.$$

Also, for $i \leq j$, we define $f_{i,j} : X_j \to 2^{X_i}$ by $f_{i,j} = f_{i,i+1} f_{i+1,i+2} \cdots f_{j-1,j}$.

Definition 8. *Let* $f : X \to 2^Y$ *be a set-valued function. Then* f *is called* **upper semi-continuous** *if for each point* x *of* X *and each open neighborhood* V *of* $f(x)$ *there is an open neighborhood* U *of* x *such that for every* $y \in U, f(y) \subset V$.

Definition 9. *Let* $X_i (i \in \mathbb{N})$ *be a sequence of spaces and let* $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ *be an upper semi-continuous function for each* $i \in \mathbb{N}$. *Then* **the inverse limit with upper semi-continuous functions***, denoted* $\varprojlim\{X_i, f_{i,i+1}\}$*, is the space*

$$\varprojlim\{X_i, f_{i,i+1}\} = \{(x_1, x_2, \cdots) \in \prod_{i=1}^{\infty} X_i \mid x_i \in f(x_{i+1}) \text{ for each } i\},$$

which has the topology inherited as a subspace of the product space $\prod_{i=1}^{\infty} X_i$. *The function* $f_{i,i+1}$ *is called a* **bonding map** *and* X_i *is called a* **factor space***.*

If $f : X \to 2^X$ *is an upper semi-continuous function, we consider the inverse sequence* $\{X, f\} = \{X_i, f_{i,i+1}\}$*, where* $X_i = X, f_{i,i+1} = f$ $(i \in \mathbb{N})$. *We put*

$$\varprojlim\{X, f\} = \{(x_i)_{i=1}^{\infty} \mid x_i \in f(x_{i+1}) \text{ for each } i \in \mathbb{N}\}.$$

Definition 10. *Let* $\{X_i, f_{i,i+1}\}$ *be an inverse sequence with set-valued functions. For* $m \leq n$*, we put*

$$G(f; m, m+1, \ldots, n) = \{(x_i) \in \prod_{i=m}^{n} X_i \mid x_i \in f_{i,i+1}(x_{i+1}) \text{ for each } m \leq i \leq n-1\}.$$

In particular,

$$G(f_{1,2}) = G(f; 1, 2) = \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 \in f_{1,2}(x_2)\}$$

is the **graph** *of* $f_{1,2}$.

Definition 11. *Let* $\varprojlim\{X_i, f_{i,i+1}\}$ *be an inverse limit of a sequence* $\{X_i, f_{i,i+1}\}$ *of spaces and upper semi-continuous functions. Then the function*

$$\pi_{[m,n]} : \varprojlim\{X_i, f_{i,i+1}\} \to G(f; m, m+1, \ldots, n),$$

defined by $\pi_{[m,n]}(x_1, x_2, \ldots, x_m, \ldots, x_n, x_{n+1}, \ldots) = (x_m, \ldots, x_n)$*, is called the* **natural projection***.*

Proposition 4. *Let* X *and* Y *be compact spaces and let* $f : X \to 2^Y$ *be a set-valued function. Then* f *is upper semi-continuous if and only if the graph of* f *is closed in* $X \times Y$.

4.1 dimension of inverse limits with set-valued functions

Recall that if each bonding map is a continuous mapping, the dimension of the inverse limit does not exceed dimensions of the factor spaces. But in the case that each bonding map is set-valued, the fact is not true.

Example 1. [14] *Let* $f : I \to 2^I$ *be the upper semi-continuous function defined by* $f(x) = I$ *for every* $x \in I$. *Then* $\varprojlim\{I, f\}$ *is the Hilbert cube.*

In Theorem 5.3 of [20] Nall proved the following theorem.

Theorem 11. [20] *If for every* i*, let* X_i *be a space with* $\dim X_i \leq n$ *and let* $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ *be an upper semi-continuous function such that* $\dim f_{i,i+1}(x) \leq 0$ *for every* $x \in X_{i+1}$*, then* $\dim \varprojlim\{X_i, f_{i,i+1}\} \leq n$.

In [2] Banič studied dimension of special types of inverse limits.

Theorem 12. [2] *Let A be a closed subset in I, let $f : I \to I$ be a continuous function, and let $\tilde{f} : I \to 2^I$ be an upper semi-continuous function defined by*
$\tilde{f}(x) = \{ \begin{array}{ll} I & (x \in A) \\ f(x) & (otherwise), \end{array}$
then $\dim \varprojlim \{I, \tilde{f}\} = 1$ *or* ∞.

On the other hands, there is a 2-dimensional inverse limit whose factor space is 1-dimensional.

Example 2. [14]*(Example 139) Let $f : I \to 2^I$ be given by $f(x) = 0$ $(0 \le x \le \frac{1}{2})$, $f(\frac{1}{2}) = \left[0, \frac{1}{2}\right]$, $f(x) = \frac{1}{2}$ $(\frac{1}{2} < x < 1)$, and $f(1) = \left[\frac{1}{2}, 1\right]$. Then $\varprojlim\{I, f\}$ is the union of a 2-cell and an arc intersecting only one point.*

In [15] H. Kato showed more generalized theorem by using "expand-constant sequence". To define this sequence, we consider the following conditions;

Definition 12. *For a function $f : X \to 2^Y$, put*

$$D_1(f) = \{x \in X \mid \dim f(x) \ge 1\}$$

$$D_1(f^{-1}) = \{y \in Y \mid \dim f^{-1}(y) \ge 1\},$$

respectively.

Definition 13. [15] *Let X_i be a sequence of spaces and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function. For $y \in X_i$ and $x \in X_{i'}$ $(i \le i')$, we consider the following symbols:*

$$y \leftarrow x \Longleftrightarrow y \in f_{i,i'}(x)$$

$$x \lhd \Longleftrightarrow x \in D_1(f^{-1}_{i',i'+1})$$

$$\rhd y \Longleftrightarrow i \ge 2 \text{ and } y \in D_1(f_{i-1,i})$$

Also, for $x \in X_i$ and $y \in X_{i'}$ $(i + 2 \le i')$, we consider the following symbols:

$$x \prec \rhd y \Longleftrightarrow y \in D_1(f_{i'-1,i'}) \text{ and } \dim[f^{-1}_{i',i'+1}(x) \cap f_{i'-1,i'}(y)] \ge 1$$

In particular,

$$x \diamond y \Longleftrightarrow i' = i + 2, x \in D_1(f^{-1}_{i,i+1}),\ y \in D_1(f_{i+1,i+2}), \text{ and } \dim[f^{-1}_{i,i+1}(x) \cap f_{i+1,i+2}(y)] \ge 1.$$

Definition 14. [15] *For each $x_i \in X_i$ with $x_i \in D_1(f^{-1}_{i,i+1})$, we consider the following sequence:*

$$\rhd y_{m_1} \prec \rhd y_{m_2} \prec \rhd y_{m_3} \prec \cdots \prec \rhd y_{m_{k-1}} \prec \rhd y_{m_k} \leftarrow x_i \lhd,$$

where $2 \le m_1, m_k \le i$, $m_j + 2 \le m_{j+1} (j = 1, 2, \ldots, i - 1)$, and $y_{m_j} \in X_{m_j}$ $(j = 1, 2, \ldots, i)$. In this case, we say the sequence $\{y_{m_j},\ x_i |\ 1 \le j \le k\}$ is an **expand-contract sequence** *in $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ with length k.*

Definition 15. [15]*For any expand-contract sequence*

$$S\ :\ \rhd y_{m_1} \prec \rhd y_{m_2} \prec \rhd y_{m_3} \prec \cdots \prec \rhd y_{m_{k-1}} \prec \rhd y_{m_k} \leftarrow x_i \lhd,$$

we put $d(S) = \sum_{j=1}^{k} \dim f_{m_j-1,m_j}(y_{m_j})$*. We define the index as follows;*

$$\tilde{J}(\{X_i, f_{i,i+1}\}) = \sup\{d(S) |\ S \textit{ is an expand-contract sequence in } \{X_i, f_{i,i+1}\}\}$$

If there is no expand-contract sequence in $\{X_i, f_{i,i+1}\}$*, we put* $\tilde{J}(\{X_i, f_{i,i+1}\}) = 0$*.*

Theorem 13. [15] *Let* X_i *be a sequence of compacta and let* $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ *be an upper semi-continuous function for each* $i \in \mathbb{N}$*. Suppose* $\dim D_1(f_{i,i+1}) \leq 0$*, then*

$$\dim \varprojlim\{X_i, f_{i,i+1}\} \leq \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i |\ i \in \mathbb{N}\}.$$

Theorem 14. [15] *Let* X_i *be a sequence of 1-dimensional compacta and let* $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ *be a surjective upper semi-continuous function for each* $i \in \mathbb{N}$*. Suppose that for each* $i \geq 2$*,* Z_i *is a zero-dimensional closed subset of* X_i *such that* $f_{i,i+1}|_{X_{i+1}\setminus Z_{i+1}} : (X_{i+1} \setminus Z_{i+1}) \to X_i$ *is a mapping for each* $x \in X_{i+1} \setminus Z_{i+1}$ *and* $i \in \mathbb{N}$*. If* $\tilde{J}(\{X_i, f_{i,i+1}\}) = k$*, then*

$$k \leq \dim \varprojlim\{X_i, f_{i,i+1}\} \leq k + 1.$$

Now we will define another index $\tilde{I}(X_i, f_{i,i+1}\})$.

Definition 16. [15] *Let* $x \in X_m$ *and* $y \in X_{m'}$*, where* $m' \geq m + 2$*. Then we consider the following condition*

$$x \lhd \succ y\ \ : x \in D_1(f^{-1}_{m,m+1}), \textit{ and } \dim[f^{-1}_{m,m+1}(x) \cap f_{m+1,m'}(y)] \geq 1$$

Note that $x \diamond y$ implies $x \prec \rhd y$ and $x \lhd \succ y$.

Definition 17. [7] *For each* $x_n \in X_n$ *with* $x_n \in D_1(f_{n-1,n})$*, we consider the following sequence:*

$$\rhd x_n \leftarrow y_{m_1} \lhd \succ y_{m_2} \lhd \succ y_{m_3} \lhd \succ \cdots \lhd \succ y_{m_{k-1}} \lhd \succ y_{m_k} \lhd,$$

where $n \leq m_1, m_i + 2 \leq m_i$ *for* $i = 1, 2, \ldots, k-1$ *and* $y_{m_i} \in X_{m_i}$ *for* $i = 1, 2, \ldots, k$*. In this case we say that the sequence* $\{x_n,\ y_n \mid 1 \leq i \leq k\}$ *is an* **inverse expand-contract sequence** *in* $\{X_i,\ f_{i,i+1}\}_{i=1}^{\infty}$ *with length* k*.*

For any inverse expand-contract sequence, consider

$$S\ :\ \rhd x_n \leftarrow y_{m_1} \lhd \succ y_{m_2} \lhd \succ y_{m_3} \lhd \succ \cdots \lhd \succ y_{m_{k-1}} \lhd \succ y_{m_k} \lhd.$$

We put $d(S) = \sum_{i=1}^{k} \dim f^{-1}_{m_i,m_{i+1}}(y_{m_i})$. We define the index $\tilde{I}(\{X_i, f_{i,i+1}\})$ as follows;

$$\tilde{I}(\{X_i, f_{i,i+1}\}) = sup\{d(S) |\ S \textit{ is an inverse expand-contract sequence in } \{X_i, f_{i,i+1}\}\}$$

If there is no inverse expand-contract sequence in $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$, we put $\tilde{I}(\{X_i, f_{i,i+1}\}) = 0$. Note that for any upper semi-continuous function $f : X \to 2^X$, $\tilde{I}(\{X, f\}) = \tilde{J}(\{X, f^{-1}\})$.

We will define a weak inverse expand-contract sequence.

Definition 18. [7] *We consider the following sequence:*

$$y_{m_1} \lhd \succ y_{m_2} \lhd \succ y_{m_3} \lhd \succ \cdots \lhd \succ y_{m_{k-1}} \lhd \succ y_{m_k} \lhd.$$

We say the sequence $\{y_{m_i} \mid 1 \leq i \leq k\}$ *is a* **weak inverse expand-contract sequence** *in* $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ *with length* k.

Also we put $d(S) = \sum_{i=1}^{k} \dim f^{-1}_{m_i,m_{i+1}}(y_{m_i})$ and the index

$$\tilde{W}(\{X_i, f_{i,i+1}\}) = \sup\{d(S)|\ S \text{ is a weak inverse expand-contract sequence in } \{X_i, f_{i,i+1}\}\}.$$

If each X_i is 1-dimensional, then $\tilde{I}(\{X_i, f_{i,i+1}\})$ (resp. $\tilde{W}(\{X_i, f_{i,i+1}\})$) is the maximal length of all (resp. weak) inverse expand-contract sequences in $\{X_i, f_{i,i+1}\}$. Note that $\tilde{I}(\{X_i, f_{i,i+1}\}) \leq \tilde{W}(\{X_i, f_{i,i+1}\})$). In general, $\tilde{I}(\{X, f\})$ is not equal to $\tilde{J}(\{X, f\})$ and $\tilde{I}(\{X, f\})$ is not equal to $\tilde{W}(\{X, f\})$.

Example 3. [7] *Let* C *be a Cantor set in* $\left[\frac{1}{2}, \frac{3}{4}\right]$ *with* $\{\frac{1}{2}, \frac{3}{4}\} \subset C$. *Let* $f : I \to 2^I$ *be the surjective upper semi-continuous function defined by as follows:* $f(\left[0, \frac{1}{4}\right]) = 0$, $f(\frac{1}{4}) = \left[0, \frac{1}{4}\right]$, $f((\frac{1}{4}, \frac{1}{2})) = \frac{1}{4}$ *and* $f|\left[\frac{1}{2}, \frac{3}{4}\right] : \left[\frac{1}{2}, \frac{3}{4}\right] \to \left[\frac{1}{4}, \frac{1}{2}\right]$ *is a map with* $f(C) = \left[\frac{1}{4}, \frac{1}{2}\right]$, $f(\frac{3}{4}) = C$, $f((\frac{3}{4}, 1)) = \frac{3}{4}$, $f(1) = \left[\frac{3}{4}, 1\right]$. *Then*

$$\rhd \tfrac{1}{4} \leftarrow \tfrac{1}{4} \lhd \succ \tfrac{3}{4} \lhd$$

is a maximal inverse expand-contract sequence in $\{I, f\}$. *Note that there is no* $x \in I$ *such that*

$$\rhd \tfrac{1}{4} \prec x \leftarrow \tfrac{3}{4} \lhd.$$

Also

$$\rhd \tfrac{1}{4} \leftarrow \tfrac{3}{4} \lhd \ \text{ and } \ \rhd \tfrac{1}{4} \leftarrow \tfrac{1}{4} \lhd$$

are maximal expand-contract sequences in $\{I, f\}$. *Hence* $\tilde{J}(\{I, f\}) = 1 < 2 = \tilde{I}(I, f\})$. *Let* $g = f^{-1}$, *then* $\tilde{I}(\{I, g\}) = \tilde{J}(\{I, g\}) = 1 < 2 = \tilde{I}(I, f\}) = \tilde{J}(\{I, g\})$. *Consider the map* $h : J = \left[\frac{1}{4}, 1\right] \to 2^J$ *defined by* $h(x) = f(x) \cap \left[\frac{1}{4}, 1\right]$. *Then* $\tilde{I}(J, h\}) = 0 < 1 = \tilde{W}(\{J, h\})$.

Theorem 15. [7] *Let* X_i *be a sequence of compacta and let* $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ *be an upper semi-continuous function for each* $i \in \mathbb{N}$. *Suppose* $\dim D_1(f^{-1}_{i,i+1}) \leq 0$, *then*

$$\dim \varprojlim\{X_i, f_{i,i+1}\} \leq \tilde{I}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Proof. We consider the inverse $f^{-1}_{i,i+1} : X_i \to 2^{X_{i+1}}$ of $f_{i,i+1}$ and the sequence $\{X_i, f^{-1}_{i,i+1} :\}$. From the proof of [15], we get

$$\dim G(f; 1, 2, \ldots, i) = \dim G(f^{-1}; 1, 2, \ldots, i) \leq \tilde{I}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}$$

for each $i \in \mathbb{N}$. Hence $\dim \varprojlim\{X_i, f_{i,i+1}\} \leq \tilde{I}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}$.

□

4.2 Dimensionally stepwise spaces and inverse limits with set-valued functions

An important question with inverse limits is what structures of the inverse limit are determined by the factor spaces and the bonding maps. For this problem, Nall [20] showed the following theorem;

Theorem 16. [20] *Suppose* $f : \left[0, 1\right] \to 2^{\left[0,1\right]}$ *is a surjective upper semi-continuous function. Then* $\varprojlim\{\left[0, 1\right], f\}$ *is not an* n*-manifold for any* $n > 1$.

We will use an idea of Nall in the proof of above theorem. Recall that an n-dimensional **Cantor manifold** is an n-dimensional compact space such that for each representation of X as the union of two non-empty closed proper subsets A and B, $\dim[A \cap B] \geq n-1$. Note that every n-manifold is a Cantor n-manifold.

Proposition 5. [7] *Let G be a graph and let $f : G \to 2^G$ be an upper semi-continuous function. If there is a point $x \in G$ such that $\dim \pi_1^{-1}(x) = m$, then there is a free arc J of G such that $\dim \pi_1^{-1}(\mathrm{int}(J)) = m$ and $\dim \pi_1^{-1}(z) \geq m-1$ for each $z \in J$. In particular, there is an open set U_m of $\varprojlim\{G, f\}$ such that $\dim U_m = m$.*

Proof. Let $\sigma_f : \varprojlim\{G, f\} \to \varprojlim\{G, f\}$ be the shift map defined by

$$\sigma_f(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

Note that if H is a subset of $\varprojlim\{G, f\}$ such that $\pi_{[1,k]}(H)$ is degenerate, then $(\sigma_f)^i|H : H \to \varprojlim\{G, f\}$ is injective for $1 \leq i \leq k$ (see the proof of Nall in [13], Theorem 5.5.).

Let $x = x_1$ be a point of G such that $\dim \pi_1^{-1}(x_1) = m$. Then we can choose a Cantor m-dimensional manifold H in $\pi_1^{-1}(x_1)$. Let $k \in \mathbb{N}$ such that $\pi_k(H)$ is nondegenerate and $\pi_i(H)$ is degenerate for each $1 \leq i < k$. Let J be a free arc of G with $J \subset \mathrm{int}(\pi_k(H))$. Note that there do not exist two points $z, z' \in J$ ($z \neq z'$) such that $\dim \pi_1^{-1}(\{z, z'\}) \leq m-2$, because $\pi_1^{-1}(\{z, z'\})$ separates the continuum $H' = \sigma_f^{(k-1)}(H)$ which is homeomorphic to the Cantor m-dimensional manifold H. Hence we can choose a small free arc J such that $\dim \pi_1^{-1}(z) \geq m-1$ for each $z \in J$. Put

$$K = \{(x_i)_{i=1}^{\infty} \in \varprojlim\{G, f\} \mid x_i = \pi_i(H) \text{ } for \text{ } 1 \leq i < k \text{ } and \text{ } x_k \in J\}.$$

Then $K \subset \pi_1^{-1}(x_1)$ and K contains a nonempty open set of H, hence K is m-dimensional. Let

$$K' = \pi_1^{-1}(J) \ (= \{(y_i)_{i=1}^{\infty} \in \varprojlim\{G, f\} \mid y_1 \in J\}) \ (= \sigma_f^{(k-1)}(K)).$$

Since K and K' are homeomorphic, K' is m-dimensional. Put $U_m = \pi_1^{-1}(\mathrm{int}(J)) \subset K'$. Note that U_m contains a nonempty open set of $\sigma_f^{(k-1)}(H)$. Then $\dim U_m = m$. □

Lemma 2. [7] *Let G be a graph and let $f : G \to 2^G$ be an upper semi-continuous function such that $\dim D_1(f^{-1}) \leq 0$ and $\tilde{W}(\{G, f\}) < \infty$. Then $\varprojlim\{G, f\}$ is finite dimensional and for any $1 \leq n < \dim \varprojlim\{G, f\}$ there is a point $y \in G$ such that $\dim \pi_1^{-1}(y) = n$.*

Proof. Since $\tilde{I}(\{G, f\}) \leq \tilde{W}(\{G, f\}) < \infty$, by theorem 15 we see that $\varprojlim\{G, f\}$ is finite dimensional.

For any natural number $m \geq 2$, we will prove the following claim $C(m)$.

$C(m)$: If there is a point $y \in G$ such that $\dim \pi_1^{-1}(y) = m$, then there is a point $y' \in G$ such that $\dim \pi_1^{-1}(y') = m-1$.

Suppose, on the contrary, that for any $x \in G$, $\dim \pi_1^{-1}(x) \neq m-1$. Let $y_1 \in G$ such that $\dim \pi_1^{-1}(y_1) = m$. We choose a Cantor m-dimensional manifold H in $\dim \pi_1^{-1}(y_1)$. Let $m_1 \in \mathbb{N}$ such that $\pi_{m_1+1}(H)$ is nondegenerate and $\pi_i(H)$ is degenerate for each $1 \leq i \leq m_1$, i.e., $\pi_{[1,m_1]}(H)$ is degenerate and $\pi_{[1,m_1+1]}(H)$ is nondegenerate. Put $\pi_{[1,m_1]}(H) = (y_1, y_2, \dots, y_{m_1})$. Let J_1 be a free arc in $\mathrm{int}(\pi_{m_1+1}(H))$. Then we may assume that

$$\dim \pi_{[1,m_1+1]}^{-1}(y_1, y_2, \dots, y_{m_1}, x) = \dim \pi_1^{-1}(x) \geq m-1$$

for each $x \in J_1$ (see above Proposition), and hence by the assumption,

$$\dim \pi_{[1,m_1+1]}^{-1}(y_1, y_2, \dots, y_{m_1}, x) = m.$$

Let $\mathcal{L} = \{L_j \mid j \in \mathbb{N}\}$ be a countable family of arcs in G satisfying that for any nonempty open set V of G, there is $L_j \in \mathcal{L}$ with $L_j \subset V$. For $k,\ j \in \mathbb{N}$, let $J(k,j)$ be the set of all $x \in J_1$ such that there is a Cantor m-dimensional manifold H_x of $\pi^{-1}_{[1,m_1+1]}(y_1, y_2, \ldots, y_{m_1}, x)$ ($\cong \pi_1^{-1}(x)$), and $\pi_i(H_x)$ is degenerate for $1 \leq i < k$, $\pi_k(H_x)$ contains L_j. Note that

$$J_1 = \bigcup_{k,j\in\mathbb{N}} J(k,j) = \bigcup_{k,j\in\mathbb{N}} \mathrm{cl}(J(k,j)).$$

By the Baire Category theorem, we can choose $k, j \in \mathbb{N}$ such that $\mathrm{cl}(J(k,j))$ contains a nonempty open set, hence $\dim \mathrm{cl}(J(k,j)) = 1$. Put $m'_1 = k$ and we can choose a point $y_{m'} \in \mathrm{int}(L_j)$ such that $\dim \pi_1^{-1}(y_{m'_1}) \geq m - 1$ (see the proof of above Proposition). By the assumption, $\dim \pi_1^{-1}(y_{m'_1}) = m$. Then $f^{(m'_1-(m_1+1))}(y_{m'_1}) \supset J(k,j)$ and hence $f^{(m'_1-(m_1+1))}(y_{m'_1}) \supset \mathrm{cl}(J(k,j))$. Then we can choose $y_{m_2} \in G$ such that $m'_1 \leq m_2$,

$$y_{m_1} \lhd \succ y_{m'_1} \leftarrow y_{m_2} \lhd,$$

and there is a free arc J_2 in $f^{-1}(y_{m_2})$ such that $\dim \pi_1^{-1}(z) = m$ for each $z \in J_2$. If we continue this procedure, we obtain a sequence of natural numbers

$$m_1 < m'_1 \leq m_2 < m'_2 \leq m_2 < \ldots,$$

and an infinite weak inverse expand-contract sequence

$$y_{m_1} \lhd \succ y_{m_2} \lhd \succ \cdots \lhd \succ y_{m_{k-1}} \lhd \succ y_{m_k} \lhd \cdots,$$

in $\{G, f\}$. Then $\tilde{W}(\{G, f\}) = \infty$. This is a contradiction. Consequently, the claim $C(m)$ is true. Consider the map $\pi_1 : \varprojlim\{G, f\} \to G$. By Theorem 4, we can find for any $1 \leq n < \dim \varprojlim\{G, f\}$ there is a point $y \in G$ such that $\dim \pi_1^{-1}(y) = n$. □

Definition 19. *Let X be a space with $\dim X < \infty$. Then X is a* **dimensionally stepwise space** *if for any $1 \leq m \leq \dim X$, there is an open subset U_m of X such that $\dim U_m = m$.*

Note that any zero-dimensional spaces and one-dimensional spaces are dimensionally stepwise spaces.

Theorem 17. *Suppose that G is a graph and $f : G \to 2^G$ is an upper semi-continuous function such that $\dim D_1(f^{-1}) \leq 0$ and $\tilde{W}(\{G, f\}) < \infty$. Then $X = \varprojlim\{G, f\}$ is a dimensionally stepwise space.*

Proof. This theorem follows from Proposition 5 and Lemma 2. □

Theorem 18. [7] *Suppose that G is a graph and $f : G \to 2^G$ is a surjective upper semi-continuous function. If the inverse limit $\varprojlim\{G, f\}$ is homeomorphic to a polyhedron P, then P is a dimensionally stepwise space.*

Proof. Since P is a polyhedron, the following condition $(*_i)$ is true:

$(*_i)$ If U is an open set of P with $\dim U = i$ $(i \geq 1)$, then U can not contain uncountable mutually disjoint i-dimensional subsets.

Let $\dim P = m$. We may assume that $m \geq 2$. Consider the map $\pi_1 : P \to G$. By the Theorem 4, we can find a point $y \in G$ such that $\dim \pi_1^{-1}(y) \geq m-1$. If $\dim \pi_1^{-1}(y) = m$, by Proposition 5 there is a free arc J_1 of G such that $\dim \pi_1^{-1}(\mathrm{int}(J)) = m$ and $\dim \pi_1^{-1}(z) \geq m-1$ for each $z \in J_1$. By the condition $(*_i)$, we can find a point $y_1 \in J_1$ such that $\dim \pi_1^{-1}(y_1) = m-1$. Also, by Proposition 5 we can find a free arc J_2 such that $\dim \pi_1^{-1}(\mathrm{int}(J_2)) = m-1$ and $\dim \pi_1^{-1}(z) \geq m-2$ for each $z \in J_2$. By $(*_{m-1})$, we can find a point $y_2 \in J_2$ such that $\dim \pi_1^{-1}(y_2) = m-2$. If we continue this procedure, we can prove that for any $1 \leq i < \dim P = m$ there is a point $z \in G$ such that $\dim \pi_1^{-1}(z) = i$. Then the theorem follows from Proposition 5. □

Corollary 2. *Let G be a graph and let $f : G \to 2^G$ be an upper semi-continuous function. Suppose that the inverse limit $\varprojlim\{G, f\} = X$ satisfies the condition that $\dim X < \infty$ and if U is any open set of X with $\dim U = i \geq 1$, U can not contain uncountable mutually disjoint i-dimensional subsets. Then X is a dimensionally stepwise space.*

Corollary 3. *No inverse limit with a single upper semi-continuous bonding function on a graph can be an n-cell ($n \geq 2$).*

Example 4. *Let $f : I \to C(I)$ be the surjective upper semi-continuous function defined by $f(x) = 0$ ($x \in [0, \frac{1}{3})$), $f(\frac{1}{3}) = [0, \frac{1}{3}]$, $f(x) = \frac{1}{3}$($x \in (\frac{1}{3}, \frac{2}{3})$), $f(\frac{2}{3}) = [\frac{1}{3}, \frac{2}{3}]$, $f(x) = \frac{2}{3}$ ($x \in (\frac{2}{3}, 1)$), and $f(1) = [\frac{2}{3}, 1]$. Note that*

$$0\Diamond\tfrac{1}{3}\Diamond\tfrac{2}{3}\lhd$$

is a maximal weak inverse expand-contract sequence in $\{I, f\}$, and

is a maximal (inverse) expand-contract sequence in $\{I, f\}$. We see that $\tilde{I}(\{I, f\}) = 2 = \tilde{J}(\{I, f\})$ and $\tilde{W}(\{I, f\}) = 3$. Also $\{I, f\}$ satisfies the condition of Theorem 18. Hence $\varprojlim\{I, f\}$ is a 3-dimensional, dimensionally stepwise space. In fact, $\varprojlim\{I, f\}$ is a 3-cell with a fin.

Example 5. *Let $f : I \to C(I)$ be the surjective upper semi-continuous function defined by $f(0) = I$ and $f(x) = 0$ ($x \in (0, 1]$). In this case, we have the inverse expand-contract sequence with infinite length as follows;*

$$\rhd 0\Diamond 0\Diamond 0\Diamond \ldots$$

Note that $\tilde{W}(\{I, f\}) = \infty$. We see that $\varprojlim\{I, f\}$ is the Hilbert cube. Note that $\varprojlim\{I, f\}$ has no finite dimensional nondegenerate open sets and hence it is not a dimensionally stepwise space.

4.3 ANR of inverse limits with set-valued functions

In this section, we study ANR properties of inverse limits with set-valued functions.

Definition 20. [16] *Let X be a continuum contained in a metric space M. Then X is* **weak homotopically trivial within small neighborhoods** *of M provided that if $f : S^n \to X$ is any map from the n-sphere S^n ($n \geq 0$) to X, then f is null-homotopic in any neighborhood of X in M.*

Note that if X is an FAR, then X is weak homotopically trivial within small neighborhoods of any ANR M.

We consider the following property $(*)$;

there is a sequence $\{\mathcal{V}_n\}_{n \geq 0}$ of finite closed coverings of X such that

1. $\mathcal{V}_0 = \{X\}$, and $X = \bigcup\{\text{int}_X V \mid V \in \mathcal{V}_n\}$ for each n,
2. $\lim_{n\to\infty} \text{mesh}(\mathcal{V}_n) = 0$, and
3. if $V_\alpha \in \mathcal{V} = \bigcup_n \mathcal{V}_n$ and $\bigcap_\alpha V_\alpha \neq \emptyset$, then $\bigcap_\alpha V_\alpha$ is weak homotopically trivial within small neighborhoods of M.

Also we consider the following property local $(*)$;

there is a sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ of finite closed coverings of X such that

1. $X = \bigcup\{\text{int}_X V \mid V \in \mathcal{V}_n\}$ for each n,
2. $\lim_{n\to\infty} \text{mesh}(\mathcal{V}_n) = 0$, and
3. if $V_\alpha \in \mathcal{V} = \bigcup_n \mathcal{V}_n$ and $\bigcap_\alpha V_\alpha \neq \emptyset$, then $\bigcap_\alpha V_\alpha$ is weak homotopically trivial within small neighborhoods of M.

We need the following propositions in Lemma 3.2. of [16]

Proposition 6. [16] *Suppose that X_i $(i \in \mathbb{N})$ is a continuum contained in a metric space M. If X has the property local $(*)$, then X is an ANR. Moreover, if X has the property $(*)$, then X is an AR.*

Proposition 7. [16] *Suppose that X_i is a finite dimensional compactum and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be a surjective upper semi-continuous function for each $i \in \mathbb{N}$ such that $f_{i,i+1}^{-1}$ is cell-like (i.e., $f_{i,i+1}^{-1}(x_{i+1})$ is an FAR). Then the inverse limit $\varprojlim\{X_i, f_{i,i+1}\}$ is shape equivalent to X_1. Moreover if X_1 is an FAR, then $\varprojlim\{X_i, f_{i,i+1}\}$ is also an FAR.*

Proof. Consider the inverse sequence

$$X_1 \leftarrow G(f;1,2) \leftarrow G(f;1,2,3) \leftarrow \cdots$$

whose bonding maps $p_{n,n+1} : G(f;1,2,\ldots,n+1) \to G(f;1,2,\ldots,n)$ are natural projections defined by $p_{n,n+1}(x_1,x_2,\ldots,x_n,x_{n+1}) = (x_1,x_2,\ldots,x_n)$. Since the projections $p_{n,n+1}^{-1}$ are cell-like, $p_{n,n+1} : G(f;1,2,\ldots,n+1) \to G(f;1,2,\ldots,n)$ induces a shape equivalence. Hence we see that the inverse $\varprojlim\{G(f;1,2,\ldots,i), p_{i,i+1}\} = \varprojlim\{X_i, f_{i,i+1}\}$ is shape equivalent to X_1. If X_1 is an FAR, then $\varprojlim\{X_i, f_{i,i+1}\}$ is also an FAR. □

In [13], Ingram gave many examples of inverse sequences of the unit interval I with upper semi-continuous set-valued functions whose inverse limits are dendrites. We need the following condition. Let $f : X \to 2^Y$ be an upper semi-continuous function. Consider the condition $Z(f)$ for f.

Z(f): For any $x \in X$ and $y \in Y$ with $y \in f(x)$, any closed neighborhood A' of x in X and any closed neighborhood B' of y in Y, there are a closed neighborhood A of x in X and a closed connected neighborhood B of y in Y such that $A \subset A'$, $B \subset B'$, and the pair (B, A) satisfies the condition; for any subcontinuum K of A with $x \in K$, the set $C(B, A;\ K) = \{z \in B \mid f^{-1}(z) \cap K \neq \emptyset\}$ $(= f(K) \cap B)$ is connected.

Remark. Let K be any finite simplicial complex in $I \times I$ and let $f; I \to 2^I$ be the upper semi-continuous function defined by $G(f) = |K|$. Then f satisfies the condition $Z(f)$.

The main theorem of this section is the following.

Theorem 19. *Let G_i ($i \in \mathbb{N}$) be a graph and let $f_{i,i+1} : G_{i+1} \to 2^{G_i}$ be a surjective upper semi-continuous function for each $i \in \mathbb{N}$ such that f^{-1} is cell-like. Suppose that each $f_{i,i+1} : G_{i+1} \to 2^{G_i}$ satisfies the condition $Z(f_{i,i+1})$. Then the inverse limit $\varprojlim\{G_i, f_{i,i+1}\}$ of the inverse sequence $\{G_i, f_{i,i+1}\}$ is an ANR which is homotopic to G_1. Moreover, if G_1 is a tree, then $\varprojlim\{G_i, f_{i,i+1}\}$ is an AR. Especially, if $\dim D_1(f_{i,i+1}^{-1}) \leq 0$ ($i \in \mathbb{N}$) and $\tilde{I}(\{G_i, f_{i,i+1}\}) = 0$, then $\{G_i, f_{i,i+1}\}$ is a dendrite.*

Proof. In the proof, we use the fact that the intersection of continua (= trees) contained in a tree is an empty set or a tree.

Suppose that $\varepsilon > 0$ is a very small positive number. Let $n \in \mathbb{N}$ and $(x_1, x_2, \ldots, x_n) \in G(f; 1, 2, \ldots, n)$. Since $f_{i,i+1}^{-1}(x_i)$ ($i = 1, 2, \ldots, n-1$) is a tree in G_{i+1}, we choose a closed neighborhood T_{i+1} of $f_{i,i+1}^{-1}(x_i)$ in G_{i+1} such that T_{i+1} is a tree. Also, we choose a closed neighborhood B_i ($i = 1, 2, \ldots, n$) of x_i in G_i such that B_i is a tree such that $B_{i+1} \subset T_{i+1}$ ($i = 1, 2, \ldots, n-1$), $\mathrm{diam} B_i \leq \varepsilon$, and $f_{i,i+1}^{-1}(B_i) \subset T_{i+1}$ for each $i = 1, 2, \ldots, n-1$. Put

$$V(x_1, x_2, \ldots, x_n; B_1, B_2, \ldots, B_n; \varepsilon) = \{(z_i) \in \varprojlim\{G_i, f_{i,i+1}\} \mid z_i \in B_i \ (i = 1,\ 2, \ldots,\ n)\}.$$

Moreover, by use of the property $Z(f_{i,i+1})$, we can choose closed neighborhoods $B_n, B_{n-1}, \ldots, B_1$ such that $V(x_1, x_2, \ldots, x_n; B_1, B_2, \ldots, B_n; \varepsilon)$ is an FAR. First, we choose a small closed connected neighborhood B_n of x_n in G_n which is a tree and a small closed connected neighborhood B'_{n-1} of x_{n-1} in G_{n-1} such that the pair (B'_{n-1}, B_n) satisfies the condition $c(B'_{n-1}, B_n)$. Inductively, we have pairs (B'_{i-1}, B_i) ($i = n-1, n-2, \ldots, 2$) such that B'_i and B_i are small closed connected neighborhoods of x_i in G_i, $B_i \subset B'_i$ ($i = n-1, n-2, \ldots, 2$) and the pair (B'_{i-1}, B_i) satisfies the condition $c(B'_{i-1}, B_i)$. Put $B_1 = B'_1$. Let $C_n = B_n$ and let $C_{n-1} = C(B'_{n-1}, B_n; C_n) \cap B_{n-1}, C_{n-2} = C(B'_{n-2}, B_{n-1}; C_{n-1}) \cap B_{n-2}$. If we continue this procedure inductively, we have the sequence C_i ($i = n, n-1, n-2, \ldots, 1$) of trees such that $x_i \in C_i$. We will show that

$$V(x_1, x_2, \ldots, x_n; B_1, B_2, \ldots, B_n; \varepsilon) = \varprojlim\{Y_i, g_{i,i+1}\},$$

where $Y_1 = C_1, Y_2 = f_{1,2}^{-1}(Y_1) \cap C_2, Y_3 = f_{2,3}^{-1}(Y_2) \cap C_3, \ldots, Y_n = f_{n-1,n}^{-1}(Y_{n-1}) \cap C_n, Y_i = f_{n,i}^{-1}(Y_n)$ ($i \geq n$) and $g_{i,i+1} : Y_{i+1} \to 2^{Y_i}$ is the set-valued function defined by $g_{i,i+1}(z) = Y_i \cap f_{i,i+1}(z)$ for $z \in Y_{i+1}$. By the definitions, we see that $V(x_1, x_2, \ldots, x_n; B_1, B_2, \ldots, B_n; \varepsilon)) \supset \varprojlim\{Y_i, g_{i,i+1}\}$. We will show the converse inclusion. Let

$$y = (y_i) \in V(x_1, x_2, \ldots, x_n; B_1, B_2, \ldots, B_n; \varepsilon).$$

Since $y_n \in B_n = C_n$, then $y_{n-1} \in C(B'_{n-1}, B_n; C_n) \cap B_{n-1} = C_{n-1}$. Since $y_{n-1} \in C_{n-1}$, then $y_{n-2} \in C(B'_{n-2}, B_{n-1}; C_{n-1}) \cap B_{n-2} = C_{n-2}$. If we continue this procedure, we see that $y_i \in C_i$ and hence $y_i \in Y_i$ for $i \in \mathbb{N}$. This implies that $y \in \varprojlim\{Y_i, g_{i,i+1}\}$. Hence

$$V(x_1, x_2, \ldots, x_n; B_1, B_2, \ldots, B_n; \varepsilon) = \varprojlim\{Y_i, g_{i,i+1}\}.$$

Note that for $x \in Y_i$ ($i = 1, 2, \ldots, n-1$), $g_{i,i+1}^{-1}(x) = f_{i,i+1}^{-1}(x) \cap Y_{i+1}$ ($\subset T_{i+1}$). Hence $g_{i,i+1}^{-1}$ is cell-like for $i \in \mathbb{N}$. Since $Y_1 = C_1$ is a tree, by Proposition 7, $\varprojlim\{Y_i, g_{i,i+1}\}$ is an FAR. Hence $V(x_1, x_2, \ldots, x_n; B_1, B_2, \ldots, B_n; \varepsilon)$ is an FAR.

Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots$ be a sequence of positive numbers with $\lim_{i \to \infty} \varepsilon_i = 0$. For $n \in \mathbb{N}$, there is a finite set F_n of $G(g : 1, 2, \ldots, n)$ such that

$$\varprojlim\{G_i, f_{i,i+1}\} = \bigcup\{V(x_1,\ x_2, \ldots,\ x_n; B_1,\ B_2, \ldots,\ B_n; \varepsilon_n) \mid (x_1,\ x_2, \ldots,\ x_n) \in F_n\}.$$

Put

$$\mathcal{V}_n = \{V(x_1,\ x_2,\dots,\ x_n; B_1,\ B_2,\dots,\ B_n;\varepsilon_n) \mid (x_1,\ x_2,\dots,\ x_n) \in F_n\}.$$

By the definitions of $V(x_1, x_2, \dots, x_n; B_1, B_2, \dots, B_n;\varepsilon_n)$, we see that the sequence $\{\mathcal{V}_n\}_{n=1,2,\dots}$ is a family of finite closed coverings of $\varprojlim\{G_i, f_{i,i+1}\}$ satisfying the conditions (i) and (ii) of local $(*)$. Note that
if $V = V(x_1, x_2, \dots, x_n; B_1, B_2, \dots, B_n;\varepsilon_n) \in \{\mathcal{V}_n\}_{n=1,2,\dots}$,
then V can be represented by the inverse limit $\varprojlim\{Y_i, g_{i,i+1}\}$ as above. If $n \le n'$,

$$V(x_1, x_2, \dots, x_n; B_1, B_2, \dots, B_n;\varepsilon_n),$$
and
$$V(x'_1, x'_2, \dots, x'_{n'}; B'_1, B'_2, \dots, B'_{n'};\varepsilon_{n'}) \in \{\mathcal{V}_n\}_{n=1,2,\dots},$$

then we see that

$$\begin{aligned} &V(x_1, x_2, \dots, x_n; B_1, B_2, \dots, B_n;\varepsilon_n) \cap V(x'_1, x'_2, \dots, x'_{n'}; B'_1, B'_2, \dots, B'_{n'};\varepsilon_{n'}) \\ &= \{(z_i) \in \varprojlim\{G_i, f_{i,i+1}\} \mid z_i \in B_i \cap B'_i\ (i = 1,\ 2,\dots,\ n) \text{ and } z_j \in B_j\ (j = n+1,\dots,\ n')\} \end{aligned}$$

is an empty set or an FAR, because that it can be represented by an inverse limit $\varprojlim\{Z_i, g_{i,i+1}\}$, where Z_1 is a tree and $g_{i,i+1}$ is cell-like. Note that the intersection of decreasing sequence of FARs is also an FAR. By using these arguments, moreover we see that $\{\mathcal{V}_n\}_{n=1,2,\dots}$ also satisfies the condition (iii) of local $(*)$. By Proposition 6, $\varprojlim\{G_i, f_{i,i+1}\}$ is an ANR. By Proposition 7, we see that the inverse limit $\varprojlim\{G_i, f_{i,i+1}\}$ is shape equivalent to G_1 and hence it is homotopy equivalent to G_1. Moreover, if G_1 is a tree, then $\varprojlim\{G_i, f_{i,i+1}\}$ is a contractible ANR and hence AR. If $\dim D_1(f^{-1}_{i,i+1}) \le 0$ $(i \in \mathbb{N})$ and $\tilde{I}(\{G_i, f_{i,i+1}\}) = 0$, then $\varprojlim\{G_i, f_{i,i+1}\}$ is 1-dimensional and hence it is a dendrite. This completes the proof. □

Corollary 4. *Let I_i $(i \in \mathbb{N})$ be a sequence of the unit interval I and let $f_{i,i+1} : I_{i+1} \to 2^{I_i}$ be a surjective upper semi-continuous function for each $i \in \mathbb{N}$ such that $f^{-1}_{i,i+1}$ is monotone and f satisfies $Z(f_{i,i+1})$. Then $\varprojlim\{I_i, f_{i,i+1}\}$ is an AR. Moreover, if $\dim D_1(f^{-1}_{i,i+1}) \le 0$ $(i \in \mathbb{N})$ and $\tilde{I}(\{I_i, f_{i,\ i+1}\}) = 0$, then $\varprojlim\{I_i, f_{i,i+1}\}$ is a dendrite.*

Corollary 5. *Let I_i $(i \in \mathbb{N})$ be a sequence of the unit interval I and let K_i be a finite simplicial complex in $I_i \times I_{i+1}$ satisfying that for any $x \in I_{i+1}$, $(I_i \times \{x\}) \cap |K_i| \neq \emptyset$ and for any $y \in I_i$, $(\{y\} \times I_{i+1}) \cap |K_i|$ is a nonempty connected set (=a closed interval). Let $f_{i,i+1} : I_{i+1} \to 2^{I_i}$ be the surjective upper semi-continuous function defined by $G(f_{i,i+1}) = |K_i|$. Then $\varprojlim\{I_i, f_{i,i+1}\}$ is an AR. Moreover, if $\dim |K_i| \le 1$ $(i \in \mathbb{N})$ and $\tilde{I}(\{I_i, f_{i,i+1}\}) = 0$, then $\varprojlim\{I_i, f_{i,i+1}\}$ is a dendrite.*

Corollary 6. *If $f : G \to 2^G$ is a surjective upper semi-continuous function such that f^{-1} is a tree, $\dim D_1(f^{-1}) \le 0$, and $\tilde{W}(\{G, f\}) < \infty$ and f satisfies $Z(f)$, then the inverse limit $\varprojlim\{G, f\}$ with the single upper semi-continuous bonding function f is a dimensionally stepwise ANR-space which is homotopic to G.*

Example 6. [7] *Let $g : I \to I$ be the map defined by*

$$g(x) = \tfrac{x}{2}(1 + \sin\tfrac{\pi}{2x})$$

for $x \in (0,1]$ and $g(0) = 0$. Let $f = g^{-1} : I \to 2^I$ and $h : I \to 2^I$ be the surjective upper semi-continuous function defined by $h(x) = 0$ $(x \in [0,1))$ and $h(1) = I$. Consider the inverse sequence $\{I_i, f_{i,i+1}\}$ defined by $f_{1,2} = f$, $f_{2,3} = h$, $f_{i,i+1} = id$ $(i \ge 3)$. Note that $f^{-1}_{i,i+1}$ is cell-like, each graph $G(f_{i,i+1})$ is homeomorphic to an arc, and hence locally connected. But it does not satisfies the condition $Z(f_{1,2})$. For the points $x = 0$, $y = 0$, the set $C(B, A; K(= \{0\})) = \{z \in B \mid f^{-1}(z) \cap \{0\} \neq \emptyset\}$ is not connected for *any neighborhood A of $x = 0$ and any neighborhood B of $y = 0$. In fact, we see that $\varprojlim\{I_i, f_{i,i+1}\}$ is homeomorphic to the following set X in the Euclidean 3-space $\mathbb{R}^3$;*

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \in I,\ y = g(x)\} \cup S \times \big[0, 1\big]\ (\subset \mathbb{R}^2 \times \mathbb{R}),$$

where $S = \{(x, 0) \mid x \in I,\ g(x) = 0\}$. Note that $\varprojlim\{I_i, f_{i,i+1}\}$ is not locally connected and hence not an ANR.

Example 7. [13] *Let $f : I \to 2^I$ be the upper semi-continuous function defined by $f(x) = \{0, 1\}$ $(x \in I)$. Note that f is not surjective, f satisfies the condition $Z(f)$ and $f^{-1}(0)$, $f^{-1}(1)$ are arcs. But $\varprojlim\{I, f\}$ is a Cantor set and hence not an ANR.*

Example 8. [13] *Let $f : I \to C(I)$ be the surjective upper semi-continuous function defined by $f(x) = \{0, x\}$ $(x \in I)$. Note that f satisfies the condition $Z(f)$, Hence $\varprojlim\{I, f\}$ is a dendrite. In fact, it is a simple fan.*

Example 9. *Let $f : I \to 2^I$ be the surjective upper semi-continuous function defined by $f(x) = \{0, 1\}$ $(x \neq \frac{1}{2})$ and $f(\frac{1}{2}) = I$. Note that f satisfies the condition $Z(f)$, f^{-1} is cell-like, $\dim D_1(f^{-1}) \leq 0$ and $\tilde{I}(\{I, f\}) = 0$. Hence $\varprojlim\{I, f\}$ is a dendrite. In fact, it is a dendrite with a Cantor set of endpoints.*

Example 10. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $f : I \to C(I)$ be the surjective upper semi-continuous function defined by $f(x) = 0$ $(x \in \big[0, \frac{1}{n}))$ and for $1 \leq i \leq n-1$, $f(\frac{1}{n}) = \big[(\frac{(i-1)}{n}, \frac{1}{n}\big]$, $f(x) = \frac{i}{n}$ $(x \in (\frac{i}{n}, \frac{(i+1)}{n}))$, $f(1) = \big[\frac{(n-1)}{n}, 1\big]$. Then*

$$0 \diamond \tfrac{1}{n} \diamond \tfrac{2}{n} \diamond \cdots \diamond \tfrac{(n-1)}{n} \triangleleft$$

is a maximal weak inverse expand-contract sequence in $\varprojlim\{I, f\}$. Note that $\tilde{I}(\{I, f\}) = J(\{I, f\}) = n - 1$, $\tilde{W}(\{I, f\}) = n$, f^{-1} is cell-like and f satisfies the condition $Z(f)$. We see that $\varprojlim\{I, f\}$ is n-dimensional and a dimensionally stepwise AR. In fact, the space is a polyhedron.

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