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Abstract

The Amoroso kernel density estimator (Igarashi and Kakizawa (2017)) for nonnegative data is boundary-bias-free and has the mean integrated squared error (MISE) of order $O(n^{-4/5})$, where n is the sample size. In this paper, we construct a linear combination of the Amoroso kernel density estimator and its derivative with respect to the smoothing parameter. Also, we propose a related multiplicative estimator. We show that the MISEs of these bias-reduced estimators achieve the convergence rates $n^{-8/9}$, if the underlying density is four times continuously differentiable. We illustrate the finite sample performance of the proposed estimators, through the simulations.

Keywords: nonparametric density estimation; boundary bias problem; asymmetric kernel; Amoroso kernel; bias reduction;

MSC: 62G07; 62G20

1. Introduction

The kernel density estimation, introduced by Rosenblatt (1956), is perhaps the most popular among the nonparametric approaches, and various asymptotic results have been well-established when the support $\mathcal{S}$ of the underlying density is $\mathbb{R}$ (see, e.g., Silverman (1986) and Wand and Jones (1995)). However, if $\mathcal{S}$ is a closed interval or semi-infinite interval, the standard kernel density estimator is, in general, inconsistent, due to the bias that is $O(1)$ near the boundary. To remove (or avoid) the boundary bias, there have been a variety of important methods; renormalization, reflection, generalized jackknifing, and so on. See, e.g., Jones (1993) for a review. Note that, in the standard kernel density estimation using a symmetric kernel K and bandwidth $h > 0$, the location-scale function $K((x-\cdot)/h)$, at the point x near the boundary, has a mass outside the support $\mathcal{S}$. Probably, this fact causes the boundary bias problem. Over the last two decades, there is a growing interest in the use of a varying asymmetric kernel whose support matches the support $\mathcal{S}$ of the density to be estimated. To the best of our knowledge, Silverman (1986; page 28) first mentioned a possible application of gamma or log-normal (LN) density (rather than a location-scale symmetric density), and, concretely, Chen

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The authors preliminarily reported the Amoroso kernel density estimator and its bias-reduced estimators, at the Japanese Joint Statistical Meeting (2016, September).

(1999, 2000) developed beta and gamma kernel density estimators assuming $\mathcal{S} = [0, 1]$ and $[0, \infty)$, respectively, in such a way that the kernel shape varies according to the point $x \in \mathcal{S}$ and a smoothing parameter $b = b_n > 0$, where n is the sample size.

1.1. Asymmetric kernel density estimation

On the basis of a parametric density $K_\theta^{(AK)}$ with support $[0, \infty)$ and a finite dimensional parameter θ, several estimators in the form of $\widehat{f}_b^{(AK)}(x) = n^{-1} \sum_{i=1}^n K_{\theta_1(x,b),\theta_2}^{(AK)}(X_i)$, $x \geq 0$, have been suggested, where a subcomponent of θ; θ_1 (say) is chosen to be $\theta_1 = \theta_1(x, b)$ as a function of (x, b), for nonnegative data $X_1, \ldots, X_n$. The existing estimators are (i) gamma kernel density estimator (Chen (2000) and Igarashi and Kakizawa (2014b)), (ii) (weighted) LN kernel density estimator (Jin and Kawczak (2003) and Igarashi (2016b)), (iii) Birnbaum–Saunders (BS), inverse Gaussian (IG), and reciprocal inverse Gaussian (RIG) kernel density estimators (Jin and Kawczak (2003), Scaillet (2004), and Igarashi and Kakizawa (2014b)), (iv) inverse gamma kernel density estimator (Koul and Song (2013) and Kakizawa and Igarashi (2017)), and (v) generalized BS and skew BS kernel density estimators (Marchant et al. (2013) and Saulo et al. (2013))[1]. Note that Igarashi and Kakizawa (2014b) applied a generalized inverse Gaussian density (in their paper, it was renamed as a modified Bessel density) and then treated the IG, RIG, and BS kernel density estimators in a unified way (the resulting estimator was referred to as a mixture of IG (MIG) kernel density estimator).

Recently, Igarashi and Kakizawa (2017) considered an application of a family of Amoroso densities, with parameters $\alpha, \beta > 0$ and $\gamma \neq 0$ (Amoroso (1925) and Stacy and Mihram (1965))

$$K_{\alpha,\beta,\gamma}^{(A)}(s) = \frac{|\gamma| s^{\alpha\gamma-1} e^{-(s/\beta)^\gamma}}{\beta^{\alpha\gamma}\Gamma(\alpha)}$$

(see Hirukawa and Sakudo (2015) for an application of Stacy (1962)'s generalized gamma density with parameters $\alpha, \beta, \gamma > 0$). Here, if $\alpha\gamma \geq 1$, then, $K_{\alpha,\beta,\gamma}^{(A)}(0) > 0$; if $\gamma < 0$, then, $K_{\alpha,\beta,\gamma}^{(A)}(0)$ is understood as $\lim_{s\to 0+} K_{\alpha,\beta,\gamma}^{(A)}(s) = 0$ (the remaining case $0 < \alpha\gamma < 1$ is not considered here, due to the unboundedness of the density at the origin). In this paper, we focus on the Amoroso kernel density estimator for every constant $\gamma \neq 0$ (Igarashi and Kakizawa (2017)), as follows:

$$\widehat{f}_{b,c,\gamma}(x) = \frac{1}{n} \sum_{i=1}^n K_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}^{(A)}(X_i), \quad x \geq 0, \tag{1}$$

[1]The second author reported symmetrical-based IG, RIG, and BS kernel density estimators, that is an extension of Igarashi and Kakizawa (2014b), at the Mathematical Society of Japan (2016 Spring Meeting) and the Japanese Joint Statistical Meeting (2016, September). He also studied log-symmetrical kernel density estimator, including a reformulation of the previous estimators due to Marchant et al. (2013) and Saulo et al. (2013).

where $c \geq 1$ is a constant, and α_γ and β_γ are infinitely differentiable functions on $(0, \infty)$, defined by

$$\alpha_\gamma(\rho) = \begin{cases} \dfrac{\rho}{\gamma}, & \gamma > 0, \\ \dfrac{\rho + 1}{|\gamma|}, & \gamma < 0, \end{cases} \quad \beta_\gamma(\rho) = \rho \frac{\Gamma(\alpha_\gamma(\rho))}{\Gamma(\alpha_\gamma(\rho) + 1/\gamma)} = \begin{cases} \rho \dfrac{\Gamma(\rho/\gamma)}{\Gamma((\rho+1)/\gamma)}, & \gamma > 0, \\ \rho \dfrac{\Gamma((\rho+1)/|\gamma|)}{\Gamma(\rho/|\gamma|)}, & \gamma < 0 \end{cases} \tag{2}$$

(both $\alpha_\gamma(\rho)$ and $\alpha_\gamma(\rho) + 1/\gamma$ are positive when $\rho > 0$). Note that the Amoroso kernel density estimator $\widehat{f}_{b,c,\gamma}$ is differentiable with respect to b, and that the squared kernel is easily tractable, i.e.,

$$\{K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho), \gamma}(s)\}^2 = b^{-1}|\gamma| v_\gamma(\rho) K^{(A)}_{2\alpha_\gamma(\rho) - 1/\gamma, b\beta_\gamma(\rho)/2^{1/\gamma}, \gamma}(s),$$

where v_γ is an infinitely differentiable function on $(1/2, \infty)$, defined by[2]

$$v_\gamma(\rho) = \frac{\Gamma(2\alpha_\gamma(\rho) - 1/\gamma)\Gamma(\alpha_\gamma(\rho) + 1/\gamma)}{2^{2\alpha_\gamma(\rho) - 1/\gamma} \rho \Gamma^3(\alpha_\gamma(\rho))}.$$

1.2. General methodology of bias reduction

Now, let us consider any density estimator $\widehat{g}_\beta$ for an unknown density f, with the support $\mathcal{S}$, where $\beta = \beta_n > 0$ is a smoothing parameter. Suppose that $\widehat{g}_\beta$ is differentiable with respect to β, and that $E[\widehat{g}_\beta(x)] = f(x) + \sum_{i=1}^2 \beta^{iq} B^{[i]}(x) + o(\beta^{2q})$ for some constant $q > 0$ and functions $B^{[i]}$ $(i = 1, 2)$, independent of β. The bias of $\widehat{g}_\beta(x)$ may be reduced from $O(\beta^q)$ to $O(\beta^{2q})$, in the following ways:

$$\text{additive.} \quad \widehat{g}_\beta(x) - \frac{\beta}{q}\frac{\partial}{\partial \beta}\widehat{g}_\beta(x), \tag{3}$$

$$\text{multiplicative.} \quad \{\widehat{g}_\beta(x) + \epsilon\} \exp\Big\{\frac{\widehat{g}_\beta(x) - \frac{\beta}{q}\frac{\partial}{\partial \beta}\widehat{g}_\beta(x)}{\widehat{g}_\beta(x) + \epsilon} - 1\Big\} \quad (\text{assume } \widehat{g}_\beta(x) \geq 0), \tag{4}$$

where the introduction of a small parameter $\epsilon > 0$ enables us to avoid dividing by zero. The idea behind these methods is simple. Ignoring the remainder term of $E[\widehat{g}_\beta(x)]$ and differentiating under the expectation sign, we formally obtain

$$E\Big[\widehat{g}_\beta(x) - \frac{\beta}{q}\frac{\partial}{\partial \beta}\widehat{g}_\beta(x)\Big] \approx f(x) + \sum_{i=1}^2 \beta^{iq} B^{[i]}(x) - \frac{\beta}{q}\frac{\partial}{\partial \beta}\Big\{f(x) + \sum_{i=1}^2 \beta^{iq} B^{[i]}(x)\Big\} = f(x) - \beta^{2q} B^{[2]}(x).$$

Also, assuming $f(x) > 0$, the multiplicative estimator (4) admits the stochastic expansion

$$\{\widehat{g}_\beta(x) + \epsilon\} \exp\Big\{\frac{\widehat{g}_\beta(x) - \frac{\beta}{q}\frac{\partial}{\partial \beta}\widehat{g}_\beta(x)}{\widehat{g}_\beta(x) + \epsilon} - 1\Big\} \approx \widehat{g}_\beta(x) - \frac{\beta}{q}\frac{\partial}{\partial \beta}\widehat{g}_\beta(x) + \frac{1}{2f(x)}\Big\{\frac{\beta}{q}\frac{\partial}{\partial \beta}\widehat{g}_\beta(x) + \epsilon\Big\}^2,$$

[2] By definition (see (2)), we see that

$$2\alpha_\gamma(\rho) - 1/\gamma = \begin{cases} \dfrac{2\rho - 1}{\gamma}, & \gamma > 0, \\ \dfrac{2\rho + 3}{|\gamma|}, & \gamma < 0 \end{cases}$$

is positive when $\rho > 1/2$.

which yields (3), except for the additional quadratic term $\{(\beta/q)\partial\widehat{g}_\beta(x)/\partial\beta+\epsilon\}^2/\{2f(x)\}$. Of course, the above-mentioned approximations must be validated. Note that the additive estimator (3) is a linear combination of $\widehat{g}_\beta(x)$ and $(\partial/\partial\beta)\widehat{g}_\beta(x)$, as in a generalized jackknifing estimator (Jones and Foster (1993; Example 2.3)) for the standard kernel density estimator ($\mathcal{S}=\mathbb{R}$). See also Igarashi and Kakizawa (2015) for the gamma/MIG/weighted LN kernel density estimators ($\mathcal{S}=[0,\infty)$), and Igarashi (2016a) for the beta kernel density estimator ($\mathcal{S}=[0,1]$).

In principle, these estimators (3) and (4) have other motivations. For each $a\in(0,1)$, one may construct additive/multiplicative estimators

$$\widehat{g}_\beta^{(SS_a)}(x)=\frac{1}{1-a^q}\widehat{g}_\beta(x)-\frac{a^q}{1-a^q}\widehat{g}_{\beta/a}(x), \tag{5}$$

$$\widehat{g}_\beta^{(TS_a)}(x)=\frac{\{\widehat{g}_\beta(x)+\epsilon\}^{1/(1-a^q)}}{\{\widehat{g}_{\beta/a}(x)+\epsilon/a^q\}^{a^q/(1-a^q)}}, \tag{6}$$

$$\widehat{g}_\beta^{(JF_a)}(x)=\{\widehat{g}_\beta(x)+\epsilon\}\exp\left\{\frac{\widehat{g}_\beta^{(SS_a)}(x)}{\widehat{g}_\beta(x)+\epsilon}-1\right\} \tag{7}$$

by means of the Schucany–Sommers (SS), Terrell–Scott (TS), and Jones–Foster (JF) bias reduction methods, respectively, since Schucany and Sommers (1977), Terrell and Scott (1980), and Jones and Foster (1993) originally developed these techniques (with $\epsilon=0$) for the standard kernel density estimator ($\mathcal{S}=\mathbb{R}$). By definition, the estimators (5)–(7) are not well-defined when $a=1$; however, if ϵ is independent of a, taking the limits as $a\to1$ yields the estimators (3) and (4), i.e.,

$$\lim_{a\to1}\widehat{g}_\beta^{(SS_a)}(x)=\widehat{g}_\beta(x)-\frac{\beta}{q}\frac{\partial}{\partial\beta}\widehat{g}_\beta(x),$$

$$\lim_{a\to1}\widehat{g}_\beta^{(JF_a)}(x)=\lim_{a\to1}\widehat{g}_\beta^{(TS_a)}(x)=\{\widehat{g}_\beta(x)+\epsilon\}\exp\left\{\frac{\widehat{g}_\beta(x)-\frac{\beta}{q}\frac{\partial}{\partial\beta}\widehat{g}_\beta(x)}{\widehat{g}_\beta(x)+\epsilon}-1\right\}.$$

These limiting estimators ($a\to1$) are denoted by $\widehat{g}_\beta^{(SS_1)}(x)$ and $\widehat{g}_\beta^{(JF_1)}(x)=\widehat{g}_\beta^{(TS_1)}(x)$, respectively. It is interesting that the TS_a type (6) is linked with the JF_a type (7), through the estimator (4).

1.3. Overview of the paper

The contribution of this paper is the application of the bias reduction methods (3) and (4) to the Amoroso kernel density estimator (1). We show that the limiting $\mathrm{SS}_1/\mathrm{JF}_1(=\mathrm{TS}_1)$ type bias-reduced Amoroso kernel density estimators have the mean integrated squared errors (MISEs) of order $O(n^{-8/9})$, whose convergence rates are faster than the rate $n^{-4/5}$ of the MISE of the estimator (1). We found that the asymptotic MISE (AMISE)-efficiency of the limiting $\mathrm{SS}_1/\mathrm{TS}_1$ type bias-reduced Amoroso kernel density estimator relative to the $\mathrm{SS}_a/\mathrm{TS}_a$ type bias-reduced Amoroso kernel density estimator, for each $a \in (0,1)$, is given by $(27/16)^{8/9}/\{\lambda^4(a)/a\}^{2/9} < 1$, where

$$\lambda(a) = \frac{1}{(1-a)^2}\left\{1 + a^{5/2} - 2a\left(\frac{2a}{a+1}\right)^{1/2}\right\}$$

and $\lim_{a\to 1}\{\lambda^4(a)/a\}^{2/9} = (27/16)^{8/9}$. It turns out that $a = 1$ is the best choice for the $\mathrm{SS}_a/\mathrm{TS}_a$ types. On the other hand, the corresponding result does not hold for the JF_a type. Consequently, we conclude that the best implemented (with respect to $a \in (0,1]$) JF_a type bias reduction is superior to the TS_a type bias reduction, in the AMISE sense.

The rest of this paper is organized as follows. In Section 2, we introduce new bias-reduced Amoroso kernel density estimators by applying two techniques (3) and (4), together with a brief description of some asymptotic properties of the (uncorrected) estimator (1). Section 3 is devoted to the study of the bias, variance, (weak/strong) consistency, asymptotic normality, and MISE of the resulting new estimators, under suitable assumptions. In Section 4, we conduct simulation studies to investigate the finite sample performance of the proposed estimators. All proofs of Theorems are given in Appendix.

Notation For the notational simplicity, the dependency on the sample size n is suppressed (e.g., the smoothing parameter is denoted by b, instead of b_n), but, unless otherwise stated, the limits will be taken as n goes to infinity.

2. Amoroso kernel density estimation for nonnegative data

In what follows, we always assume that

A1. $\mathcal{X}^{(n)} = \{X_1, \ldots, X_n\}$ is a random sample from an unknown density f with support $[0, \infty)$.

A2. $b > 0$ is a smoothing parameter satisfying $b \to 0$ and $nb \to \infty$.

If the density f has the support $[\delta, \infty)$, whose (finite) left boundary point δ is known, then, x and X_i in the definition (1) (see also (10) and (11)) should read as $x-\delta$ and $X_i-\delta$, respectively. It is important to consider the case where δ is unknown. Probably, the plug-in approach, with $\widehat{\delta} = \min(X_1, \ldots, X_n)$, would be a solution. However, we do not pursue this topic here.

2.1. Amoroso kernel density estimator (uncorrected case)

We begin with a brief description of the mean squared error (MSE) and MISE properties of the (uncorrected) Amoroso kernel density estimator recently suggested by Igarashi and Kakizawa (2017). As usual, we use the notation $MISE[\widehat{f}] = \int_0^\infty MSE[\widehat{f}(x)]dx$ for the MISE of any estimator $\widehat{f}$, where $MSE[\widehat{f}(x)] = E[\{\widehat{f}(x) - f(x)\}^2]$. Here, we impose the following additional assumptions:

A3. (i) f is twice continuously differentiable on $[0,\infty)$. (ii) f'' is Hölder continuous, i.e., there exist $L_2>0$ and $\eta_2\in(0,1]$ such that $|f''(s)-f''(t)|\le L_2|s-t|^{\eta_2}$ for any $s,t\ge 0$. (iii) f, f', and f'' are bounded.

A4. $\int_0^\infty\{f'(x)\}^2dx$ and $\int_0^\infty\{xf''(x)\}^2dx$ are finite.

A5. $\int_0^\infty x^{k_2+1}f(x)dx$ is finite for some $k_2>(\eta_2+6)/\eta_2$, where $\eta_2\in(0,1]$ is given in A3.

Given $\gamma\ne 0$, choose $c\ge 1$ when $\gamma>0$ or $c>1$ when $\gamma<0$. Igarashi and Kakizawa (2017) gave the bias and variance approximations

$$Bias[\widehat{f}_{b,c,\gamma}(x)]=\begin{cases} b\dfrac{B_{c|\gamma|}(x)}{|\gamma|}+O(b^2+(bx)^{1+\eta_2/2}), & \dfrac{x}{b}\to\infty,\\ bcf'(0)+O(b^2), & \dfrac{x}{b}\to\kappa,\end{cases}$$

$$V[\widehat{f}_{b,c,\gamma}(x)]=\begin{cases} n^{-1}b^{-1/2}|\gamma|^{1/2}V(x)\{1+O(bx^{-1})\}+O(n^{-1}), & \dfrac{x}{b}\to\infty,\\ n^{-1}b^{-1}|\gamma|f(0)\{v_\gamma(\kappa+c)+o(1)\}+O(n^{-1}), & \dfrac{x}{b}\to\kappa\ (x\ne 0),\\ n^{-1}b^{-1}|\gamma|v_\gamma(c)f(0)+O(n^{-1}), & x=0\end{cases}$$

(here and subsequently, $\kappa\ge 0$ is a constant), where

$$B_{c|\gamma|}(x)=c|\gamma|f'(x)+x\frac{f''(x)}{2},\quad V(x)=\frac{f(x)}{2\sqrt{\pi x}}.$$

Despite of the different rate phenomenon

$$MSE[\widehat{f}_{b,c,\gamma}(x)]=\begin{cases} O(n^{-4/5}) & \text{for fixed } x>0 \text{ (using } b\propto n^{-2/5}),\\ O(n^{-2/3}) & \text{for } x/b\to\kappa \text{ (using } b\propto n^{-1/3}),\end{cases}\tag{8}$$

Igarashi and Kakizawa (2017) showed rigorously that $MISE[\widehat{f}_{b,c,\gamma}]=AMISE[\widehat{f}_{b,c,\gamma}]+o(b^2+n^{-1}b^{-1/2})$, where

$$AMISE[\widehat{f}_{b,c,\gamma}]=b^2\int_0^\infty\left\{\frac{B_{c|\gamma|}(x)}{|\gamma|}\right\}^2dx+n^{-1}b^{-1/2}\int_0^\infty|\gamma|^{1/2}V(x)dx.$$

The AMISE of the estimator (1) is minimized at

$$b=|\gamma|\left\{\frac{\int_0^\infty V(x)dx}{4\int_0^\infty B^2_{c|\gamma|}(x)dx}\right\}^{2/5}n^{-2/5},$$

when $B_{c|\gamma|}(x)\not\equiv 0$, i.e., the optimal AMISE is given by

$$\min_{b>0}AMISE[\widehat{f}_{b,c,\gamma}]=\frac{5}{4^{4/5}}\left\{\int_0^\infty B^2_{c|\gamma|}(x)dx\right\}^{1/5}\left\{\int_0^\infty V(x)dx\right\}^{4/5}n^{-4/5}.\tag{9}$$

2.2. New bias-reduced Amoroso kernel density estimators

This paper primarily aims at improving the above-mentioned rates (8) and (9). We can apply the bias reduction methods (3) and (4) to the Amoroso kernel density estimator (1), i.e., we set $q=1$ to define the new estimators as

$$\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) = \widehat{f}_{b,c,\gamma}(x) - b\frac{\partial}{\partial b}\widehat{f}_{b,c,\gamma}(x) = \frac{1}{n}\sum_{i=1}^{n} K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(X_i)H^{(A)}_{b,c,\gamma,x/b+c}(X_i), \tag{10}$$

$$\widehat{f}^{(JF_1)}_{b,c,\gamma}(x) = \widehat{f}^{(TS_1)}_{b,c,\gamma}(x) = \{\widehat{f}_{b,c,\gamma}(x)+\epsilon\}\exp\Big\{\frac{\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)}{\widehat{f}_{b,c,\gamma}(x)+\epsilon} - 1\Big\} \tag{11}$$

for $x \geq 0$, where

$$\begin{aligned} H^{(A)}_{b,c,\gamma,\rho}(s) = 1 &+ \frac{1}{|\gamma|}(\rho - c)\Big[\log\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^{\gamma} - \psi(\alpha_\gamma(\rho))\Big] \\ &+\gamma\Big[\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^{\gamma} - \alpha_\gamma(\rho)\Big]\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\big\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho)+1/\gamma)\big\}\Big] \end{aligned}$$

($\psi(z) = \Gamma'(z)/\Gamma(z)$ is known as the digamma function). Equivalently, these estimators are viewed as the limiting case ($a \to 1$) of the SS_a and $\mathrm{TS}_a/\mathrm{JF}_a$ type estimators (Igarashi and Kakizawa (2017))

$$\begin{aligned} \widehat{f}^{(SS_a)}_{b,c,\gamma}(x) &= \frac{1}{1-a}\widehat{f}_{b,c,\gamma}(x) - \frac{a}{1-a}\widehat{f}_{b/a,c,\gamma}(x), \\ \widehat{f}^{(TS_a)}_{b,c,\gamma}(x) &= \frac{\{\widehat{f}_{b,c,\gamma}(x)+\epsilon\}^{1/(1-a)}}{\{\widehat{f}_{b/a,c,\gamma}(x)+\epsilon/a\}^{a/(1-a)}}, \\ \widehat{f}^{(JF_a)}_{b,c,\gamma}(x) &= \{\widehat{f}_{b,c,\gamma}(x)+\epsilon\}\exp\Big\{\frac{\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)}{\widehat{f}_{b,c,\gamma}(x)+\epsilon} - 1\Big\} \end{aligned}$$

for $x \geq 0$, if $\epsilon > 0$ is independent of $a \in (0,1)$. Note that the estimator (10) is written as

$$\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) = \frac{1}{n}\sum_{i=1}^{n} K^{(A_{SS_1})}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(X_i), \quad x \geq 0,$$

where $K^{(A_{SS_1})}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s) = K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)$.

The kernels $K^{(A)}_{\alpha_\gamma(x/b+1),b\beta_\gamma(x/b+1),\gamma}$ and $K^{(A_{SS_1})}_{\alpha_\gamma(x/b+1),b\beta_\gamma(x/b+1),\gamma}$, where $\gamma = -1.5, -1, -0.5, 0.5, 1, 1.5$, are displayed for $x = 0, 2, 5$ and $b = 0.25, 1$ (see Figures 1 and 2). Both kernels concentrate at $s = x$, as $b \to 0$, and the latter kernel $K^{(A_{SS_1})}_{\alpha_\gamma(x/b+1),b\beta_\gamma(x/b+1),\gamma}$ becomes sharper, though it loses the nonnegativity, to a very small extent. Also, by construction, the shapes of these kernels vary according to the position $x \geq 0$ where the density estimation is made.

2.3. Some comments on the Amoroso kernel density estimators

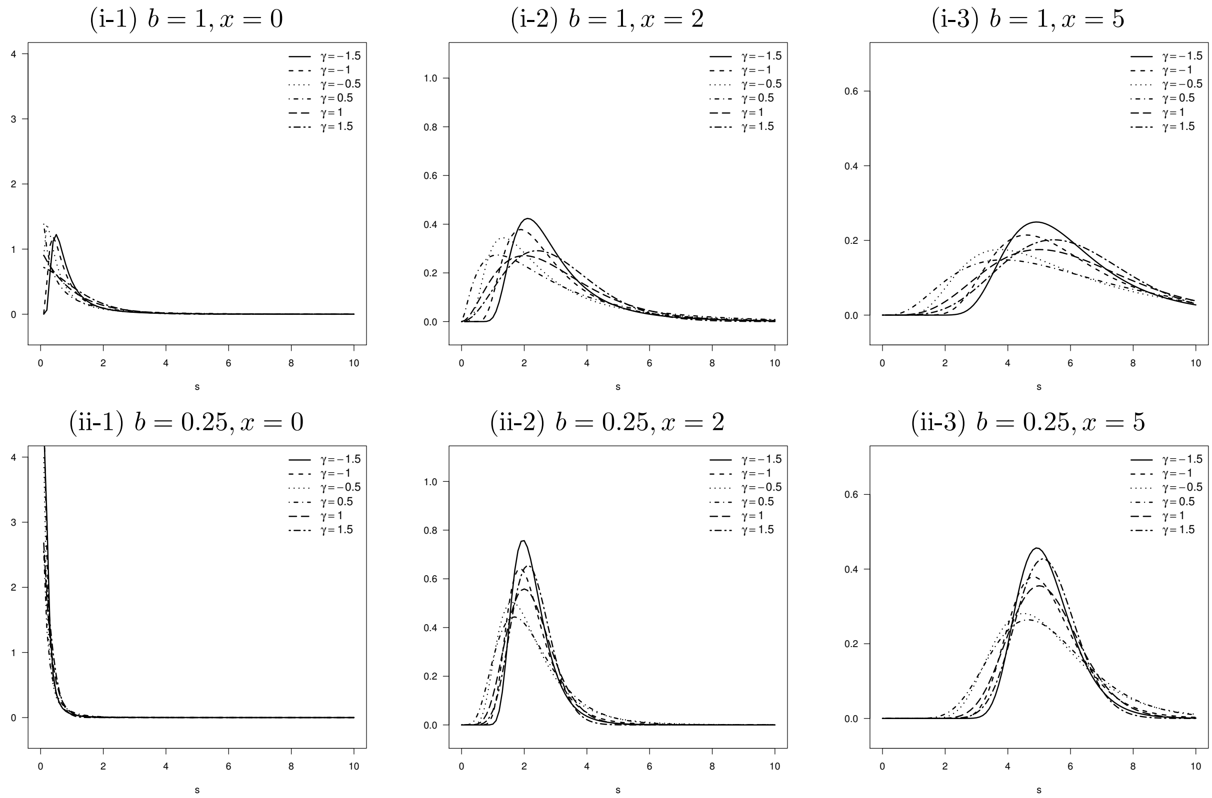
It may be true that the definition (2), depending on the sign of $\gamma > 0$ or $\gamma < 0$, is possibly inconvenient. But, we emphasize that, by construction,

$$\int_0^\infty sK^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)ds = b\rho \quad \text{for any } \gamma \neq 0 \text{ and } \rho > 0.$$

When $\gamma > 0$, the following integral exists for any $j_1 \geq 0$ and nonnegative integer j_2:

$$\int_0^\infty s^{j_1}K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)\Big\{\log\Big(\frac{s}{b\beta_\gamma(\rho)}\Big)^{\gamma}\Big\}^{j_2}ds = (b\rho)^{j_1}\frac{\Gamma^{j_1-1}(\rho/\gamma)\Gamma^{(j_2)}((\rho+j_1)/\gamma)}{\Gamma^{j_1}((\rho+1)/\gamma)} \quad \text{if } \rho > 0,$$

Figure 1: Shapes of the kernels $K^{(A)}_{\alpha_\gamma(x/b+1),b\beta_\gamma(x/b+1),\gamma}$, $\gamma = -1.5, -1, -0.5, 0.5, 1, 1.5$.



but the resulting kernel $K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}$ is bounded if $\rho \geq 1$, i.e.,

$$\sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s) = \frac{\{(\rho-1)/\gamma\}^{(\rho-1)/\gamma} e^{-(\rho-1)/\gamma}\Gamma((\rho+1)/\gamma)}{b\Gamma(\rho/\gamma)\Gamma(\rho/\gamma+1)}$$

(0^0 is understood to be 1). On the other hand, when $\gamma < 0$, we always have

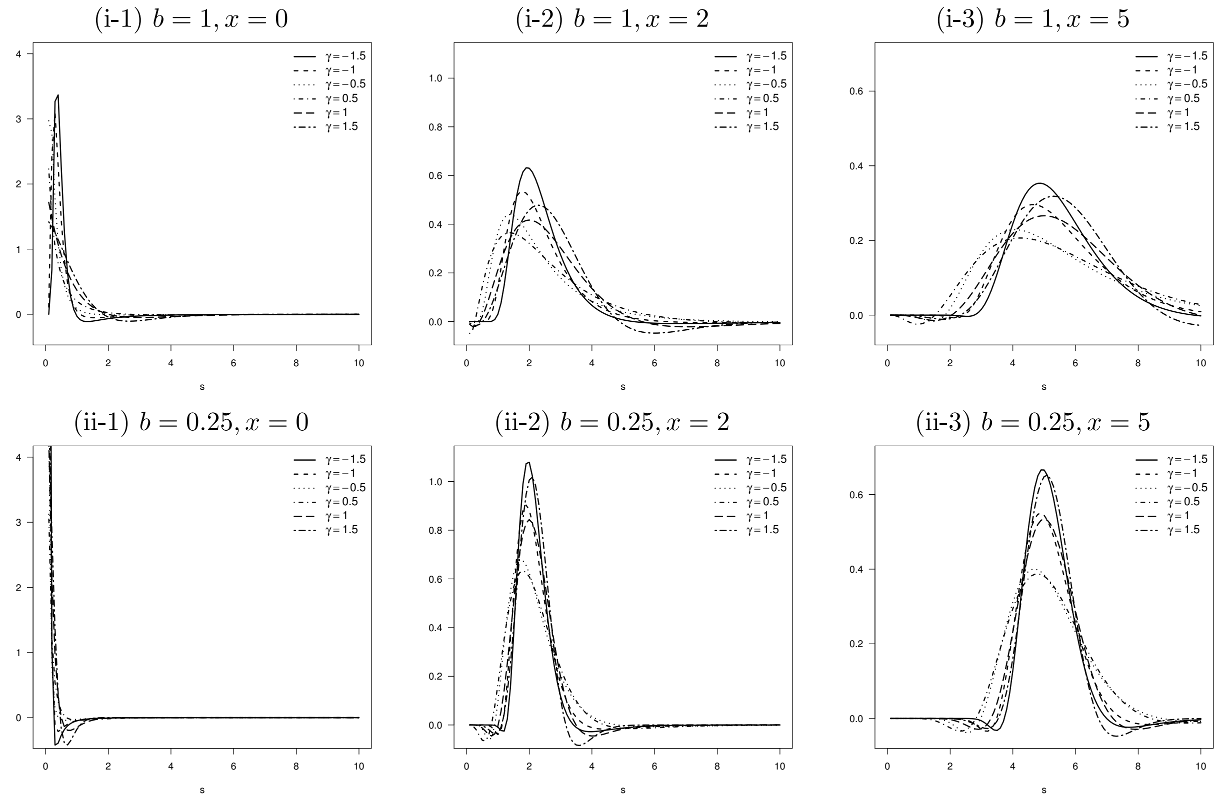
$$\sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s) = \frac{|\gamma|\{(\rho+2)/|\gamma|\}^{(\rho+2)/|\gamma|} e^{-(\rho+2)/|\gamma|}\Gamma(\rho/|\gamma|)}{b\rho\Gamma^2((\rho+1)/|\gamma|)} \quad \text{if } \rho > 0,$$

but we must pay attention to the fact that, for any nonnegative integer j_2,

$$\int_0^\infty s^{j_1} K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)\Big\{\log\Big(\frac{s}{b\beta_\gamma(\rho)}\Big)^\gamma\Big\}^{j_2} ds = (b\rho)^{j_1}\frac{\Gamma^{j_1-1}((\rho+1)/|\gamma|)\Gamma^{(j_2)}((\rho+1-j_1)/|\gamma|)}{\Gamma^{j_1}(\rho/|\gamma|)}$$

is well-defined if $j_1 < \rho + 1$; hence, setting $\rho = x/b + c$ ($c > 0$), a sufficient condition for the existence of this integral, for each $x \geq 0$, is $j_1 < c + 1 = \min_{x\geq 0}(x/b + c) + 1$. Since some arguments rely on the case $j_1 = 2, \ldots, \ell$, where $\ell \in \mathbb{N}$, a further restriction $c > \ell - 1$ is globally required when $\gamma < 0$. Therefore, we often impose, in Section 3, that "$c \geq 1$ when $\gamma > 0$ or $c > \ell - 1$ when $\gamma < 0$", except that, whenever $x/b \to \infty$, there is no restriction on c, even when $\gamma < 0$.

Figure 2: Shapes of the kernels $K^{(A_{SS_1})}_{\alpha_\gamma(x/b+1), b\beta_\gamma(x/b+1),\gamma}$, $\gamma = -1.5, -1, -0.5, 0.5, 1, 1.5$.



Remark 1 Igarashi and Kakizawa (2017) gave the following uniform/non-uniform bounds for any $b > 0$:

$$\text{(i).}\quad \sup_{\rho\geq 1}\sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s) \leq \frac{\widetilde{L}_\gamma}{b},$$

$$\text{(ii).}\quad \sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s) \leq \frac{|\gamma|^{1/2}\widetilde{L}_\gamma}{b\sqrt{2\pi}(\rho-1)^{1/2}} \quad \text{for any } \rho > 1,$$

where

$$\widetilde{L}_\gamma = \begin{cases} 1, & \gamma \geq 1, \\ \dfrac{\Gamma(2/\gamma)}{\Gamma(1/\gamma)\Gamma(1/\gamma+1)}, & 0 < \gamma < 1, \\ \dfrac{3\Gamma(1/|\gamma|)\Gamma(3/|\gamma|)}{\Gamma^2(2/|\gamma|)}, & \gamma < 0. \end{cases}$$

These bounds (i) and (ii) were the keys to get the (pointwise) strong consistency and asymptotic normality of the estimator (1) under Assumptions A1, A2, and A3 (i) and (iii), i.e., given $\gamma \neq 0$ and $c \geq 1$, it was shown that

- $\widehat{f}_{b,c,\gamma}(x) \stackrel{a.s.}{\to} f(x)$ for fixed $x \geq 0$, provided that $nb/\log n \to \infty$,
- $(nb^{1/2})^{1/2}\{\widehat{f}_{b,c,\gamma}(x) - E[\widehat{f}_{b,c,\gamma}(x)]\} \stackrel{d}{\to} N(0, |\gamma|^{1/2}V(x))$ for fixed $x > 0$ ($b \to 0$ and $nb^{1/2} \to \infty$ are

sufficient), and $(nb)^{1/2}\{\widehat{f}_{b,c,\gamma}(0) - E[\widehat{f}_{b,c,\gamma}(0)]\} \xrightarrow{d} N(0, |\gamma| v_\gamma(c) f(0))$; the statement via Slutsky's lemma, using the bias approximation, is omitted here.

3. Main results: asymptotic properties

3.1. Limiting SS_1 type bias-reduced Amoroso kernel density estimator

In this subsection, we study the asymptotic properties of the limiting estimator (10). For this purpose, instead of Assumptions A3–A5, we make the following assumptions:

A3′. (i) f is four times continuously differentiable on $[0, \infty)$. (ii) $f^{(4)}$ is Hölder continuous, i.e., there exist $L_4 > 0$ and $\eta_4 \in (0, 1]$ such that $|f^{(4)}(s) - f^{(4)}(t)| \le L_4 |s - t|^{\eta_4}$ for any $s, t \ge 0$. (iii) f, f', f'', $f^{(3)}$, and $f^{(4)}$ are bounded, i.e., $C_0 = \sup_{x \ge 0} f(x)$ and $C_i = \sup_{x \ge 0} |f^{(i)}(x)|$, $i = 1, 2, 3, 4$ are finite.

A4′. $\int_0^\infty \{f''(x)\}^2 dx$, $\int_0^\infty \{x f^{(3)}(x)\}^2 dx$, and $\int_0^\infty \{x^2 f^{(4)}(x)\}^2 dx$ are finite.

A5′. $\int_0^\infty x^{k_4+1} f(x) dx$ is finite for some $k_4 > (3\eta_4 + 20)/\eta_4$, where $\eta_4 \in (0, 1]$ is given in A3′.

Additionally, assumptions on the decay $b \to 0$, if necessary, will be imposed for various results. Assumption A3′ is required for the bias approximation (Theorem 1), and Assumption A5′ is imposed to validate the asymptotic expansion for the MISE (see the comment before Theorem 4); the details are included in Appendix.

Theorem 1 *Given $\gamma \neq 0$, choose $c \ge 1$ when $\gamma > 0$ or $c > 2$ when $\gamma < 0$ (see Subsection 2.3; $\ell = 3$). Under Assumptions A1, A2, and A3′, we have*

$$
Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \begin{cases} -b^2 \dfrac{B^{[2]}_{c,\gamma}(x)}{\gamma^2} + \mathcal{E}^{(SS_1)}_{b,c,\gamma}(x), & \dfrac{x}{b} \to \infty, \\ -b^2 \zeta^{(SS_1)}_{c,\gamma}(\kappa) \dfrac{f''(0)}{2} + o(b^2), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ -b^2 \zeta^{(SS_1)}_{c,\gamma}(0) \dfrac{f''(0)}{2} + O(b^3), & x = 0, \end{cases}
$$

$$
V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \begin{cases} n^{-1} b^{-1/2} \dfrac{27}{16} |\gamma|^{1/2} V(x) \{1 + O(bx^{-1})\} + O(n^{-1}), & \dfrac{x}{b} \to \infty, \\ n^{-1} b^{-1} |\gamma| f(0) \{v^{(SS_1)}_{c,\gamma}(\kappa) + o(1)\} + O(n^{-1}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ n^{-1} b^{-1} |\gamma| f(0) v^{(SS_1)}_{c,\gamma}(0) + O(n^{-1}), & x = 0, \end{cases}
$$

with $\mathcal{E}^{(SS_1)}_{b,c,\gamma}(x) = O(b^3 x^{-1} + \{b(1+x)\}^{2+\eta_4/2})$ *for* $x/b \to \infty$*, where*

$$\begin{aligned}
B^{[2]}_{c,\gamma}(x) &= \delta^{[2]}_{c,\gamma}\frac{f''(x)}{2} + \delta^{[3]}_{c,\gamma}x\frac{f^{(3)}(x)}{6} + 3x^2\frac{f^{(4)}(x)}{24},\\
\zeta^{(SS_1)}_{c,\gamma}(\kappa) &= -(\kappa+c)^2\frac{\Gamma(\alpha_\gamma(\kappa+c))\Gamma(\alpha_\gamma(\kappa+c)+2/\gamma)}{\Gamma^2(\alpha_\gamma(\kappa+c)+1/\gamma)}\left\{\frac{\kappa}{\kappa+c} + 2\mathcal{H}_{c,\gamma,1}(\kappa+c) - \mathcal{H}_{c,\gamma,2}(\kappa+c)\right\} + \kappa^2,\\
v^{(SS_1)}_{c,\gamma}(\kappa) &= v_\gamma(\kappa+c)\left[\left\{1 - \frac{1}{2}\mathcal{H}_{c,\gamma,1}(\kappa+c) + \mathcal{H}_{c,\gamma}(\kappa+c)\right\}^2 + \frac{\gamma\kappa}{|\gamma|}\mathcal{H}_{c,\gamma,1}(\kappa+c)\right.\\
&\qquad \left. + \frac{\gamma^2}{2}\left\{\alpha_\gamma(\kappa+c) - \frac{1}{2\gamma}\right\}\mathcal{H}^2_{c,\gamma,1}(\kappa+c) + \frac{\kappa^2}{\gamma^2}\psi'(2\alpha_\gamma(\kappa+c) - 1/\gamma)\right].
\end{aligned}$$

Here, $\delta^{[2]}_{c,\gamma}$ *and* $\delta^{[3]}_{c,\gamma}$ *are coefficients, given by*

$$\delta^{[2]}_{c,\gamma} = \begin{cases} \frac{1}{2}\{(2c^2+1)\gamma^2 + 2(c-1)\gamma + 1\}, & \gamma > 0,\\ \frac{1}{2}\{(2c^2+1)\gamma^2 + 2c|\gamma| + 1\}, & \gamma < 0,\end{cases} \qquad \delta^{[3]}_{c,\gamma} = \begin{cases} (3c-1)\gamma + 3, & \gamma > 0,\\ (3c+1)|\gamma| + 3, & \gamma < 0,\end{cases}$$

and $\mathcal{H}_{c,\gamma,j}$ $(j = 1, 2)$ *and* $\mathcal{H}_{c,\gamma}$ *are infinitely differentiable functions on* $[c, \infty)$*, defined by*[3]

$$\begin{aligned}
\mathcal{H}_{c,\gamma,j}(\rho) &= -\frac{c}{\rho} + \frac{\rho - c}{|\gamma|}\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + j/\gamma)\},\\
\mathcal{H}_{c,\gamma}(\rho) &= \frac{\rho - c}{|\gamma|}\{\psi(2\alpha_\gamma(\rho) - 1/\gamma) - \log 2 - \psi(\alpha_\gamma(\rho))\}.
\end{aligned}$$

Remark 2 The following statements hold under Assumptions A1, A2, and A3 (i) and (iii); the results (12)–(15) for $\widehat{f}_{b,c,\gamma}(x)$ are reproduced from Igarashi and Kakizawa (2017), for ease of reference.

(i). Given $\gamma \neq 0$, choose $c \geq 1$. We have

$$Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \begin{cases} O(bx), & \frac{x}{b} \to \infty,\\ O(b), & \frac{x}{b} \to \kappa,\end{cases} \qquad Bias[\widehat{f}_{b,c,\gamma}(x)] = \begin{cases} O(b+bx), & \frac{x}{b} \to \infty,\\ O(b), & \frac{x}{b} \to \kappa,\end{cases} \tag{12}$$

$$\sup_{x\geq 0} V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = O(n^{-1}b^{-1}), \qquad \sup_{x\geq 0} V[\widehat{f}_{b,c,\gamma}(x)] = O(n^{-1}b^{-1}). \tag{13}$$

(ii). Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 1$ when $\gamma < 0$ (see Subsection 2.3; $\ell = 2$). We have

$$Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \begin{cases} O(b^2 + b^2x^2), & \frac{x}{b} \to \infty,\\ O(b^2), & \frac{x}{b} \to \kappa,\end{cases} \qquad Bias[\widehat{f}_{b,c,\gamma}(x)] = \begin{cases} b\frac{B_{c|\gamma|}(x)}{|\gamma|} + O(b^2 + b^2x^2), & \frac{x}{b} \to \infty,\\ bcf'(0) + O(b^2), & \frac{x}{b} \to \kappa\end{cases} \tag{14}$$

(note that (14) when $x/b \to \infty$ hold under Assumptions A1, A2, and A3′ (i) and (iii)). Also,

$$\sup_{x\in[0,b^\tau]}|Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]| = O(b^{2\tau}), \qquad \sup_{x\in[0,b^\tau]}|Bias[\widehat{f}_{b,c,\gamma}(x)]| = O(b^{\min(1,2\tau)}) \quad \text{for any } \tau \in (0,1). \tag{15}$$

[3]By definition (see (2)), we see that, in addition to the footnote [2],

$$\alpha_\gamma(\rho) + 2/\gamma = \begin{cases} \frac{\rho+2}{\gamma}, & \gamma > 0,\\ \frac{\rho-1}{|\gamma|} & \gamma < 0\end{cases}$$

is positive when $\rho \geq c$, provided that the parameter c satisfies "$c \geq 1$ when $\gamma > 0$ or $c > 1$ when $\gamma < 0$".

From Theorem 1, the estimator (10) is (pointwise) weak consistent, i.e.,

$$MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \begin{cases} b^4\Big\{\dfrac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 + n^{-1}b^{-1/2}\dfrac{27}{16}|\gamma|^{1/2}V(x) + O(b^{4+\eta_4/2}+n^{-1}) & \text{for fixed } x>0, \\ b^4\Big\{\zeta^{(SS_1)}_{c,\gamma}(0)\dfrac{f''(0)}{2}\Big\}^2 + n^{-1}b^{-1}|\gamma|v^{(SS_1)}_{c,\gamma}(0)f(0) + O(b^5+n^{-1}) & \text{for } x=0 \end{cases}$$

tends to zero (for fixed $x>0$, $b\to 0$ and $nb^{1/2}\to\infty$ are sufficient in Assumption A2).

The (pointwise) strong consistency and asymptotic normality of the estimator (10) can be proved.

Theorem 2 *Given $\gamma \neq 0$, choose $c>1$. Suppose that Assumptions* A1, A2, *and* A3 (i) *and* (iii) *hold. Then, $\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) \overset{a.s.}{\to} f(x)$ for fixed $x\geq 0$, provided that $nb^2/\log n\to\infty$ (for $x=0$, $nb/\log n\to\infty$ is sufficient).*

Remark 3 The case $c=\gamma=1$ is exceptional; if Assumptions A1, A2, and A3 (i) and (iii) hold and $nb/\log n\to\infty$, then, $\widehat{f}^{(SS_1)}_{b,1,1}(x)\overset{a.s.}{\to} f(x)$ for fixed $x\geq 0$. Actually, we can see that

$$\text{(i).}\quad \sup_{s\geq 0} K^{(A)}_{\alpha_1(x/b+1),b\beta_1(x/b+1),1}(s)|H^{(A)}_{b,1,1,x/b+1}(s)| \leq 2b^{-1},$$

$$\text{(ii).}\quad \int_0^\infty \{K^{(A)}_{\alpha_1(x/b+1),b\beta_1(x/b+1),1}(s)H^{(A)}_{b,1,1,x/b+1}(s)\}^2 f(s)ds \leq 2b^{-1}C_0$$

(see Igarashi and Kakizawa (2015)), which yield the exponential convergence of the two-sided tail probability of $\widehat{f}^{(SS_1)}_{b,1,1}(x) - E[\widehat{f}^{(SS_1)}_{b,1,1}(x)]$, as in (A7) (see also Remark A.1 (ii)). The detail is omitted.

Theorem 3 *Given $\gamma\neq 0$, choose $c\geq 1$. Suppose that Assumptions* A1 *and* A2 *hold, and that $C_0=\sup_{x\geq 0} f(x)$ is finite. Then,*
(i). *$(nb^{1/2})^{1/2}\{\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) - E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\} \overset{d}{\to} N(0,(27/16)|\gamma|^{1/2}V(x))$ for fixed $x>0$ (in this case, $b\to 0$ and $nb^{1/2}\to\infty$ are sufficient),*
(ii). *$(nb)^{1/2}\{\widehat{f}^{(SS_1)}_{b,c,\gamma}(0) - E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(0)]\} \overset{d}{\to} N(0,|\gamma|v^{(SS_1)}_{c,\gamma}(0)f(0))$.*

Theorem 3′ *Suppose that Assumptions* A1, A2, *and* A3′ *hold.*
(i). *Given $\gamma\neq 0$, choose $c\geq 1$. If $nb^{1/2}\to\infty$ and $nb^{9/2+\eta_4}\to 0$, where $\eta_4\in(0,1]$ is given in Assumption* A3′, *then, for fixed $x>0$,*

$$(nb^{1/2})^{1/2}\Big\{\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) - f(x) + b^2\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\} \overset{d}{\to} N(0,\frac{27}{16}|\gamma|^{1/2}V(x)),$$

hence, if, in addition, $nb^{9/2}\to 0$, then, $(nb^{1/2})^{1/2}\{\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) - f(x)\} \overset{d}{\to} N(0,(27/16)|\gamma|^{1/2}V(x))$.
(ii). *Given $\gamma\neq 0$, choose $c\geq 1$ when $\gamma>0$ or $c>2$ when $\gamma<0$ (see Subsection* 2.3; *$\ell=3$). If $nb^7\to 0$, then,*

$$(nb)^{1/2}\Big\{\widehat{f}^{(SS_1)}_{b,c,\gamma}(0) - f(0) - b^2\zeta^{(SS_1)}_{c,\gamma}(0)\frac{f''(0)}{2}\Big\} \overset{d}{\to} N(0,|\gamma|v^{(SS_1)}_{c,\gamma}(0)f(0)),$$

hence, if, in addition, $nb^5\to 0$, then, $(nb)^{1/2}\{\widehat{f}^{(SS_1)}_{b,c,\gamma}(0) - f(0)\} \overset{d}{\to} N(0,|\gamma|v^{(SS_1)}_{c,\gamma}(0)f(0))$.

We notice that the convergence rate of the MSE of the estimator (10) near the boundary is slower than that in the interior, i.e.,

$$MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \begin{cases} O(n^{-8/9}) & \text{for fixed } x > 0 \text{ (using } b \propto n^{-2/9}), \\ O(n^{-4/5}) & \text{for } x/b \to \kappa \text{ (using } b \propto n^{-1/5}). \end{cases} \tag{16}$$

However, (13) and (15) yield $\int_0^{b^{\tau_1}} MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]dx = O(b^{5\tau_1} + n^{-1}b^{\tau_1 - 1}) = o(b^4 + n^{-1}b^{-1/2})$ if $\tau_1 \in (4/5, 1)$, and, as will be shown in Appendix A2, $\int_{b^{-\tau_2}}^{\infty} MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]dx$ is indeed asymptotically negligible, with a suitable choice $\tau_2 \in (0,1)$ under Assumption A5′; such a different rate phenomenon (16) has negligible impact on the MISE.

Theorem 4 *Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 1$ when $\gamma < 0$ (see Remark* 2 (ii))*. Under Assumptions* A1*,* A2*, and* A3′–A5′*, we have*

$$MISE[\widehat{f}^{(SS_1)}_{b,c,\gamma}] = AMISE[\widehat{f}^{(SS_1)}_{b,c,\gamma}] + o(b^4 + n^{-1}b^{-1/2}),$$

where

$$AMISE[\widehat{f}^{(SS_1)}_{b,c,\gamma}] = b^4 \int_0^\infty \left\{ \frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2} \right\}^2 dx + n^{-1}b^{-1/2}\frac{27}{16}\int_0^\infty |\gamma|^{1/2} V(x)dx.$$

The AMISE of the estimator (10) is minimized at

$$b^{(SS_1)} = |\gamma| \left(\frac{27}{16}\right)^{2/9} \left[\frac{\int_0^\infty V(x)dx}{8\int_0^\infty \{B^{[2]}_{c,\gamma}(x)\}^2 dx} \right]^{2/9} n^{-2/9},$$

when $B^{[2]}_{c,\gamma}(x) \not\equiv 0$, i.e., the optimal AMISE is given by

$$\min_{b>0} AMISE[\widehat{f}^{(SS_1)}_{b,c,\gamma}] = \frac{9}{8^{8/9}} \left(\frac{27}{16}\right)^{8/9} \left[\int_0^\infty \{B^{[2]}_{c,\gamma}(x)\}^2 dx \right]^{1/9} \left\{ \int_0^\infty V(x)dx \right\}^{8/9} n^{-8/9}, \tag{17}$$

whose convergence rate is faster than the rate $n^{-4/5}$ of the optimal AMISE (9). This, together with $\min_{b>0} AMISE[\widehat{f}^{(SS_a)}_{b,c,\gamma}]$, $a \in (0,1)$, studied in Igarashi and Kakizawa (2017), yields the following corollary.

Corollary 5 *Under the assumptions in Theorem* 4*, the SS_1 type estimator* (10) *is best among the SS_a type estimators for $a \in (0,1)$, in the sense of the AMISE-efficiency*

$$\frac{\min_{b>0} AMISE[\widehat{f}^{(SS_1)}_{b,c,\gamma}]}{\min_{b>0} AMISE[\widehat{f}^{(SS_a)}_{b,c,\gamma}]} = \frac{(27/16)^{8/9}}{\{\lambda^4(a)/a\}^{2/9}} < 1 \quad \textit{with} \quad \lim_{a\to 1}\left\{\frac{\lambda^4(a)}{a}\right\}^{2/9} = \left(\frac{27}{16}\right)^{8/9}.$$

3.2. Limiting JF$_1$(=TS$_1$) type bias-reduced Amoroso kernel density estimator

It should be remarked that, unlike the SS$_1$ type estimator (10), the JF$_1$(=TS$_1$) estimator (11) retains the nonnegativity, by definition. As mentioned in Subsection 1.2, one may understand that asymptotic properties of the estimator (11), when $f(x) > 0$, are similar to those of the estimator (10); heuristically, for some results in Subsection 3.1, $B_{c,\gamma}^{[2]}(x)/\gamma^2$ and $\zeta_{c,\gamma}^{(SS_1)}(\kappa)f''(0)/2$ should read as $B_{c,\gamma}^{(JF_1)}(x)/\gamma^2$ and $\zeta_{c,\gamma}^{(JF_1)}(\kappa)$, respectively, where

$$B_{c,\gamma}^{(JF_1)}(x) = -\frac{B_{c|\gamma|}^2(x)}{2f(x)} + B_{c,\gamma}^{[2]}(x), \quad \zeta_{c,\gamma}^{(JF_1)}(\kappa) = -\frac{c^2\{f'(0)\}^2}{2f(0)} + \zeta_{c,\gamma}^{(SS_1)}(\kappa)\frac{f''(0)}{2}.$$

The additional term $B_{c|\gamma|}^2(x)/\{2\gamma^2 f(x)\}$ when $x/b \to \infty$ (or $c^2\{f'(0)\}^2/\{2f(0)\}$ when $x/b \to \kappa$) comes from the expectation of the quadratic term $\mathcal{Q}(x)/\{2f(x)\}$ in the stochastic expansion of $\widehat{f}_{b,c,\gamma}^{(JF_1)}(x)$;

$$\widehat{f}_{b,c,\gamma}^{(JF_1)}(x) = \widehat{f}_{b,c,\gamma}^{(SS_1)}(x) + \frac{\mathcal{Q}(x)}{2f(x)} + \mathcal{R}(x), \tag{18}$$

where $\mathcal{Q}(x) = \{\widehat{f}_{b,c,\gamma}(x) - \widehat{f}_{b,c,\gamma}^{(SS_1)}(x) + \epsilon\}^2$, and $\mathcal{R}(x)$ is the remainder term, defined by

$$\begin{aligned}\mathcal{R}(x) = \frac{f(x)}{2}\int_0^1 \sum_{\ell=0}^{3} {}_3C_\ell \left\{\frac{\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)}{f(x)}\right\}^{3-\ell}\left\{\frac{\widehat{f}_{b,c,\gamma}^{(SS_1)}(x) - f(x)}{f(x)}\right\}^{\ell} \\ \times g_{3-\ell,\ell}\left(\frac{\theta\{\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)\}}{f(x)}, \frac{\theta\{\widehat{f}_{b,c,\gamma}^{(SS_1)}(x) - f(x)\}}{f(x)}\right)(1-\theta)^2 d\theta,\end{aligned}$$

with

$$g_{i,j}(t,v) = \frac{\partial^{i+j}}{\partial t^i \partial v^j}\left\{(1+t)\exp\left(\frac{1+v}{1+t} - 1\right)\right\}.$$

We know that

$$|\mathcal{R}(x)| \le \left\{\frac{2^3 3^2 e^2}{f^2(x)}\right\}\{|\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)| + |\widehat{f}_{b,c,\gamma}^{(SS_1)}(x) - f(x)|\}^3 \tag{19}$$

on the event

$$\widetilde{\mathcal{S}}_{x,b} = \left\{\mathcal{X}^{(n)} \;\middle|\; \frac{1}{f(x)}|\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)| \le \frac{1}{2} \text{ and } \frac{1}{f(x)}|\widehat{f}_{b,c,\gamma}^{(SS_1)}(x) - f(x)| \le \frac{1}{2}\right\} \quad \text{(say)},$$

noting

$$\begin{aligned}&\max_{|t|\le 1/2, |v| \le 1/2}\left|\int_0^1 \sum_{\ell=0}^{3} {}_3C_\ell t^{3-\ell} v^\ell g_{3-\ell,\ell}(\theta t, \theta v)(1-\theta)^2 d\theta\right| \\ &\le e^2(2^4 3^3|t|^3 + 2^2 3^4 t^2|v| + 2^3 3^2 |t| v^2 + 2^2|v|^3)\int_0^1 (1-\theta)^2 d\theta \\ &\le 2^4 3^2 e^2(|t| + |v|)^3\end{aligned}$$

(a similar argument was made by Igarashi and Kakizawa (2014a, 2015)). In Appendix A3, we will rigorously estimate $E[|\mathcal{R}(x)|^j \chi_{S_{x,b}}] + E[|\mathcal{R}(x)|^j \chi_{S_{x,b}^c}]$ for $j \ge 2/3$ (the event $S_{x,b}$ ($\subset \widetilde{\mathcal{S}}_{x,b}$) is found in Proof of Lemma A.7), where χ_S and S^c denote the indicator function and complement of a set S, respectively. Technically, however, we change the usual (unweighted) criterion to the weighted criterion $MISE_w[\widehat{f}] = \int_0^\infty w(x)MSE[\widehat{f}(x)]dx$, where, unless otherwise stated, we assume that the weight function w is nonnegative, bounded, and continuous except for a finite number of discontinuities (we write $\overline{w} = \sup_{x\geq 0} w(x)$).

Remark 4 If possible, it will be better for us not to use such a weighted criterion. At present, we do not yet realize whether or not the valid asymptotic expansion

$$MISE[\widehat{f}_{b,c,\gamma}^{(JF_1)}] = b^4 \int_0^\infty \left\{ \frac{B_{c,\gamma}^{(JF_1)}(x)}{\gamma^2} \right\}^2 dx + n^{-1}b^{-1/2}\frac{27}{16}\int_0^\infty |\gamma|^{1/2}V(x)dx + o(b^4 + n^{-1}b^{-1/2})$$

can be obtained for the case $w(x) \equiv 1$.

Modifying the argument in Igarashi and Kakizawa (2015), we introduce a set of pairs (q, ι_0)[4], defined by

$$\widetilde{\mathcal{S}} = \{(0,0)\} \cup \{(q,\iota_0) \mid 0 < q < \eta_4/(4+\eta_4) \text{ and } 0 < \iota_0 < 1/4 - q\} \tag{20}$$

($\eta_4 \in (0,1]$ is given in Assumption A3$'$), and consider a set of the points x, as follows:

$$\mathcal{I}_{q,\iota_0}[r_b] = \{x \in [0, r_b] \mid r_b = O(b^{-q}) \text{ and } f(x) \geq \varrho b^{\iota_0}\}$$

for some $r_b \equiv r$ or $r_b \to \infty$ according to $(q,\iota_0) = (0,0)$ or $(q,\iota_0) \in \widetilde{\mathcal{S}}\backslash\{(0,0)\}$. Here and subsequently, $\varrho, r > 0$ are some constants, unless otherwise stated. The present setting $\mathcal{I}_{q,\iota_0}[r_b]$ is preferable to the previous setting in Igarashi and Kakizawa (2015), since the former enables us to define the speed $r_b \to \infty$ more concretely.

In order to study asymptotic properties of the estimator (11), we make the following assumptions:

A6. Given a pair $(q,\iota_0) \in \widetilde{\mathcal{S}}$ (see (20); we write $p_0 = q + \iota_0$), $b \propto n^{-\iota_1}$ and $\epsilon \propto b^{\iota_2}$ for some $(\iota_1, \iota_2) \in \{(\iota_1,\iota_2) \mid 0 < \iota_1 < 1/\{2(2+p_0)\} \text{ and } 1 + p_0 < \iota_2 < \iota_1^{-1} - 3 - p_0\}$.

A7. Given $r_b \equiv r$ or $r_b \to \infty$, the density f satisfies (i) $\min_{x\in[0,r_b]} f(x) \geq \varrho b^{\iota_0}$ for some constant ι_0 (see (20); note that $\iota_0 = 0$ or $\iota_0 > 0$ according to $r_b \equiv r$ or $r_b \to \infty$), and w is a weight function, independent of b, such that (ii) $\int_{r_b}^\infty w(x)dx \propto \exp(-b^{-A})$ for some constant $A > 1 + \iota_2$[5], where ι_2 is given in A6, and that (iii) $w(x)\{B_{c,\gamma}^{(JF_1)}(x)\}^2$ is integrable (when $r_b \equiv r$, the requirement (ii) holds iff w is a truncated weight function, with $w(y) = 0$ for any $|y| > r$).

[4]We assume $\iota_0 < 1/4 - q$ so that $b \propto n^{-2/9}$ (i.e., $\iota_1 = 2/9$) is indeed feasible in Assumption A6.

[5]For the TS$_a$/JF$_a$ type estimators, where $a \in (0,1)$, "$\int_{r_b}^\infty w(x)dx \propto \exp(-b^{-A})$ for some constant $A > 0$" was sufficient; see the companion paper (Igarashi and Kakizawa (2017)).

Note that, if $b \propto n^{-\iota_1}$ for some $\iota_1 \in (0,1)$, then, Assumption A2 holds; Assumptions A4$'$ and A5$'$ do not have to be imposed here for the derivation of the weighted MISE.

Theorem 6 *Given $\gamma \neq 0$, choose $c > 1$ when $\gamma > 0$ or $c > 2$ when $\gamma < 0$ (see Subsection 2.3; $\ell = 3$). Suppose that Assumptions A1, A3$'$, and A6 hold. Then, the bias and variance of the estimator (11) on $\mathcal{I}_{q,\iota_0}[r_b]$ are given by*

$$Bias[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] = \begin{cases} -b^2 \dfrac{B^{(JF_1)}_{c,\gamma}(x)}{\gamma^2} + \mathcal{E}^{(JF_1)}_{b,c,\gamma}(x), & \dfrac{x}{b} \to \infty, \\ -b^2 \zeta^{(JF_1)}_{c,\gamma}(\kappa) + o(b^2) + O(n^{-1}b^{-(1+\iota_0)}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ -b^2 \zeta^{(JF_1)}_{c,\gamma}(0) + O(b^{\min(3-2\iota_0, 1+\iota_2-\iota_0)} + n^{-1}b^{-(1+\iota_0)}), & x = 0, \end{cases}$$

and

$$\begin{aligned} &V[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] \\ &= \begin{cases} n^{-1}b^{-1/2}\dfrac{27}{16}|\gamma|^{1/2}V(x) + \widetilde{\mathcal{E}}^{(JF_1)}_{b,c,\gamma}(x), & \dfrac{x}{b} \to \infty, \\ n^{-1}b^{-1}|\gamma| f(0) v^{(SS_1)}_{c,\gamma}(\kappa) + O(b^{5-2\iota_0}) + o(n^{-1}b^{-1}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ n^{-1}b^{-1}|\gamma| f(0) v^{(SS_1)}_{c,\gamma}(0) + O(b^{5-2\iota_0} + (b^{1-2\iota_0} + n^{-1/2}b^{-(1/2+\iota_0)})n^{-1}b^{-1}), & x = 0, \end{cases} \end{aligned}$$

with

$$\begin{aligned} \mathcal{E}^{(JF_1)}_{b,c,\gamma}(x) &= O(b^3 x^{-1} + b^2 \omega_{b,\iota_0}(x) + n^{-1}b^{-\iota_0}\{b^{-1/2}V(x) + 1\}), \\ \widetilde{\mathcal{E}}^{(JF_1)}_{b,c,\gamma}(x) &= O(b^{5-2\iota_0}(1+x)^3 + \{\widetilde{\omega}_{b,\iota_0}(x) + bx^{-1}\}n^{-1}b^{-1/2}V(x) + n^{-1}) \end{aligned}$$

for $x/b \to \infty$, where

$$\begin{aligned} \omega_{b,\iota_0}(x) &= b^{\eta_4/2}(1+x)^{2+\eta_4/2} + b^{1-2\iota_0}(1+x)^3 + b^{\iota_2-(\iota_0+1)}(1+x), \\ \widetilde{\omega}_{b,\iota_0}(x) &= b^{1-2\iota_0}(1+x)^3 + n^{-1/2}b^{-(1/2+\iota_0)}. \end{aligned}$$

From Theorem 6 (set $r_b \equiv r$ and $q = \iota_0 = 0$), the estimator (11) is (pointwise) weak consistent, i.e.,

$$\begin{aligned} &MSE[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] \\ &= \begin{cases} b^4 \Big\{ \dfrac{B^{(JF_1)}_{c,\gamma}(x)}{\gamma^2} \Big\}^2 + n^{-1}b^{-1/2}\dfrac{27}{16}|\gamma|^{1/2}V(x) + \mathcal{D}^{(JF_1)}_{b,c,\gamma}(x) & \text{for fixed } x \in \mathcal{I}_{0,0}[r]\backslash\{0\}, \\ b^4 \Big[-\dfrac{c^2\{f'(0)\}^2}{2f(0)} + \zeta^{(SS_1)}_{c,\gamma}(0)\dfrac{f''(0)}{2} \Big]^2 + n^{-1}b^{-1}|\gamma| v^{(SS_1)}_{c,\gamma}(0) f(0) + \mathcal{D}^{(JF_1)}_{b,c,\gamma}(x) & \text{for } x = 0 \end{cases} \end{aligned}$$

tends to zero (suppose that $f(0) > 0$), since $b \propto n^{-\iota_1}$ for some $\iota_1 \in (0, 1/4)$ implies that $b \to 0$ and $nb \to \infty$ (hence, $nb^{1/2} \to \infty$), where

$$\mathcal{D}^{(JF_1)}_{b,c,\gamma}(x) = \begin{cases} O(b^{\min(4+\eta_4/2, 3+\iota_2)} + n^{-1} + n^{-3/2}b^{-1}) & \text{for fixed } x \in \mathcal{I}_{0,0}[r]\backslash\{0\}, \\ O(b^{\min(5, 3+\iota_2)} + n^{-1} + n^{-3/2}b^{-3/2}) & \text{for } x = 0. \end{cases}$$

The (pointwise) strong consistency and asymptotic normality of the estimator (11) can be proved.

Theorem 7 *Given $\gamma \neq 0$, choose $c > 1$ or choose $c = \gamma = 1$. Suppose that Assumptions* A1, A2, *and* A3 (i) *and* (iii) *hold. If $nb^2/\log n \to \infty$ and $\epsilon \to 0$, then, $\widehat{f}^{(JF_1)}_{b,c,\gamma}(x) \overset{a.s.}{\to} f(x)$ for fixed $x \geq 0$, provided that $f(x) > 0$ (for $x = 0$, $nb/\log n \to \infty$ is sufficient).*

Theorem 8 *Given $\gamma \neq 0$, choose $c > 1$. Suppose that Assumptions* A1 *and* A3 (i) *and* (iii) *hold, and that $b \propto n^{-\iota_1}$ and $\epsilon \propto b^{\iota_2}$ for some $(\iota_1, \iota_2) \in \{(\iota_1, \iota_2) \mid 2/13 < \iota_1 < 1/4$ and $1 < \iota_2 < \iota_1^{-1} - 3\}$ or $(\iota_1, \iota_2) \in \{(\iota_1, \iota_2) \mid 1/7 < \iota_1 < 1/4$ and $1 < \iota_2 < \iota_1^{-1} - 3\}$ according to fixed $x \in \mathcal{I}_{0,0}[r] \backslash \{0\}$ or $x = 0$. Then,*

(i). $(nb^{1/2})^{1/2}\{\widehat{f}^{(JF_1)}_{b,c,\gamma}(x) - E[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)]\} \overset{d}{\to} N(0, (27/16)|\gamma|^{1/2}V(x))$ *for fixed $x \in \mathcal{I}_{0,0}[r] \backslash \{0\}$,*

(ii). *Suppose that $f(0) > 0$. Then, $(nb)^{1/2}\{\widehat{f}^{(JF_1)}_{b,c,\gamma}(0) - E[\widehat{f}^{(JF_1)}_{b,c,\gamma}(0)]\} \overset{d}{\to} N(0, |\gamma| v^{(SS_1)}_{c,\gamma}(0) f(0))$.*

Theorem 8′ *Suppose that Assumptions* A1 *and* A3′ *hold, and that $b \propto n^{-\iota_1}$ and $\epsilon \propto b^{\iota_2}$ for some $(\iota_1, \iota_2) \in \{(\iota_1, \iota_2) \mid 0 < \iota_1 < 1/4$ and $1 < \iota_2 < \iota_1^{-1} - 3\}$ (require more stringent exponents ι_1 and ι_2 for the statements below).*

(i). *Given $\gamma \neq 0$, choose $c > 1$. If $2/\min(9 + 2\eta_4, 5 + 4\iota_2) < \iota_1 < 1/4$, where $\eta_4 \in (0, 1]$ is given in Assumption* A3′ *(i.e., the feasible region of (ι_1, ι_2) is given by $2/(9 + 2\eta_4) < \iota_1 < 1/4$ and* $\max\{1, (2\iota_1^{-1} - 5)/4\} < \iota_2 < \iota_1^{-1} - 3)$*, then, for fixed $x \in \mathcal{I}_{0,0}[r] \backslash \{0\}$,*

$$(nb^{1/2})^{1/2}\Big\{\widehat{f}^{(JF_1)}_{b,c,\gamma}(x) - f(x) + b^2 \frac{B^{(JF_1)}_{c,\gamma}(x)}{\gamma^2}\Big\} \overset{d}{\to} N(0, \frac{27}{16}|\gamma|^{1/2}V(x)),$$

hence, if, in addition, $\iota_1 \in (2/9, 1/4)$, then, $(nb^{1/2})^{1/2}\{\widehat{f}^{(JF_1)}_{b,c,\gamma}(x) - f(x)\} \overset{d}{\to} N(0, (27/16)|\gamma|^{1/2}V(x))$.

(ii). *Given $\gamma \neq 0$, choose $c > 1$ when $\gamma > 0$ or $c > 2$ when $\gamma < 0$ (see Subsection* 2.3; $\ell = 3$). *Suppose that $f(0) > 0$. If $1/\min(7, 3 + 2\iota_2) < \iota_1 < 1/4$ (i.e., the feasible region of (ι_1, ι_2) is given by $1/7 < \iota_1 < 1/4$ and* $\max\{1, (\iota_1^{-1} - 3)/2\} < \iota_2 < \iota_1^{-1} - 3)$*, then,*

$$(nb)^{1/2}\Big[\widehat{f}^{(JF_1)}_{b,c,\gamma}(0) - f(0) + b^2\Big\{-\frac{c^2\{f'(0)\}^2}{2f(0)} + \zeta^{(SS_1)}_{c,\gamma}(\kappa)\frac{f''(0)}{2}\Big\}\Big] \overset{d}{\to} N(0, |\gamma| v^{(SS_1)}_{c,\gamma}(0) f(0)),$$

hence, if, in addition, $\iota_1 \in (1/5, 1/4)$, then, $(nb)^{1/2}\{\widehat{f}^{(JF_1)}_{b,c,\gamma}(0) - f(0)\} \overset{d}{\to} N(0, |\gamma| v^{(SS_1)}_{c,\gamma}(0) f(0))$.

The following theorem says that the different MSE rate phenomenon

$$MSE[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] = \begin{cases} O(n^{-8/9}) & \text{for fixed } x \in \mathcal{I}_{0,0}[r] \backslash \{0\} \text{ (using } b \propto n^{-2/9}), \\ O(n^{-4/5}) & \text{for } x/b \to \kappa \text{ (using } b \propto n^{-1/5}) \text{ if } f(0) > 0 \end{cases} \tag{21}$$

has negligible impact on the weighted MISE of the estimator (11); note that $b \propto n^{-\iota_1}$ ($\iota_1 \in (0, 2/9]$) is feasible, at least, under the settings in Theorem 6 (see also Theorem 9).

Theorem 9 *Given $\gamma \neq 0$, choose $c > 1$. Suppose that Assumptions* A1, A3$'$, A6, *and* A7 *hold. Then,*

$$MISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}] = AMISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}] + o(b^4 + n^{-1}b^{-1/2}),$$

where

$$AMISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}] = b^4 \int_0^\infty w(x)\Big\{\frac{B^{(JF_1)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 dx + n^{-1}b^{-1/2}\frac{27}{16}\int_0^\infty w(x)|\gamma|^{1/2}V(x)dx.$$

The AMISE of the estimator (11) is minimized at

$$b^{(JF_1)}_w = |\gamma|\Big(\frac{27}{16}\Big)^{2/9}\bigg[\frac{\int_0^\infty w(x)V(x)dx}{8\int_0^\infty w(x)\{B^{(JF_1)}_{c,\gamma}(x)\}^2 dx}\bigg]^{2/9} n^{-2/9},$$

when $\sqrt{w(x)}B^{(JF_1)}_{c,\gamma}(x) \not\equiv 0$, i.e.,

$$\min_{b>0} AMISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}] = \frac{9}{8^{8/9}}\Big(\frac{27}{16}\Big)^{8/9}\bigg[\int_0^\infty w(x)\{B^{(JF_1)}_{c,\gamma}(x)\}^2 dx\bigg]^{1/9}\bigg\{\int_0^\infty w(x)V(x)dx\bigg\}^{8/9} n^{-8/9}. \tag{22}$$

This, together with $\min_{b>0} AMISE_w[\widehat{f}^{(\#_a)}_{b,c,\gamma}]$, $\# = TS, JF$, $a \in (0,1)$, studied in Igarashi and Kakizawa (2017), yields the following corollary.

Corollary 10 *Suppose that the same assumptions as in Theorem 9 hold.*
(i). *The $JF_1(=TS_1)$ type estimator* (11) *is best among the TS_a type estimators for $a \in (0,1)$, in the sense of the AMISE-efficiency*

$$\frac{\min_{b>0} AMISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}]}{\min_{b>0} AMISE_w[\widehat{f}^{(TS_a)}_{b,c,\gamma}]} = \frac{(27/16)^{8/9}}{\{\lambda^4(a)/a\}^{2/9}} < 1 \quad \text{with} \quad \lim_{a\to 1}\Big\{\frac{\lambda^4(a)}{a}\Big\}^{2/9} = \Big(\frac{27}{16}\Big)^{8/9}.$$

(ii). *The AMISE-efficiency of the estimator $\widehat{f}^{(JF_1)}_{b,c,\gamma}$ relative to the estimator $\widehat{f}^{(JF_a)}_{b,c,\gamma}$, where $a \in (0,1)$, is given by*

$$\frac{\min_{b>0} AMISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}]}{\min_{b>0} AMISE_w[\widehat{f}^{(JF_a)}_{b,c,\gamma}]} = \frac{(27/16)^{8/9}\big[\int_0^\infty w(x)\{B^{(JF_1)}_{c,\gamma}(x)\}^2 dx\big]^{1/9}}{\{\lambda^4(a)/a\}^{2/9}\big[\int_0^\infty w(x)\{B^{(JF_a)}_{c,\gamma}(x)\}^2 dx\big]^{1/9}},$$

where $B^{(JF_a)}_{c,\gamma}(x) = -aB^2_{c|\gamma|}(x)/\{2f(x)\} + B^{[2]}_{c,\gamma}(x)$. Consequently, the best implemented (with respect to $a \in (0,1]$) JF_a type estimator is superior to any TS_a type estimator in the AMISE sense, i.e.,

$$\begin{aligned}\min_{a\in(0,1]}\min_{b>0} AMISE_w[\widehat{f}^{(JF_a)}_{b,c,\gamma}] &\leq \min_{b>0} AMISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}] \\ &= \min_{b>0} AMISE_w[\widehat{f}^{(TS_1)}_{b,c,\gamma}] = \min_{a\in(0,1]}\min_{b>0} AMISE_w[\widehat{f}^{(TS_a)}_{b,c,\gamma}].\end{aligned}$$

Here are some examples that we can apply Theorem 9.
(a). For a truncated weight function w, with $w(y) = 0$ for any $y > r$, Theorem 9 is applicable, whenever $\min_{x\in[0,r]} f(x) > 0$ (choose $r_b \equiv r$ and $q = \iota_0 = 0$).

(b). Suppose that there exist constants $c_0 > 1$[6] and $c_1 > 0$ such that $w(x) \propto x^{c_0-1}\exp\{x^{c_0} - \exp(x^{c_0})\}$ for sufficiently large x, and that $\min_{x\geq 0} f(x)\exp(c_1 x) > 0$ (in this case, we see that $w(x)\{B_{c,\gamma}^{(JF_1)}(x)\}^2$ is integrable). Choosing $r_b = (\iota_0/c_1)\log(1/b)$, Assumption A7 (i) and (ii) can be verified:

- $\min_{x\in[0,r_b]} f(x) \geq \varrho b^{\iota_0}$, where $\varrho = \min_{x\geq 0} f(x)\exp(c_1 x)$,
- $\int_{r_b}^{\infty} w(x)dx \propto \exp(-b^{-(\iota_0/c_1)^{c_0}\{\log(1/b)\}^{c_0-1}})$; hence, we can choose any constant $A > 0$ for all sufficiently large n, noting that $\lim_{n\to\infty}(\iota_0/c_1)^{c_0}\{\log(1/b)\}^{c_0-1} = \infty$ (we assume $b \to 0$).

(c). Suppose that $w(x) \propto \exp\{x - \exp(x)\}$ (say)[7] for sufficiently large x, and that there exists a constant $c_1 > 1$ such that $\min_{x\geq 0} f(x)(1+x)^{c_1} > 0$ (in this case, we see that $w(x)\{B_{c,\gamma}^{(JF_1)}(x)\}^2$ is integrable). We choose $r_b = b^{-\iota_0/c_1} - 1$ $(= O(b^{-q}))$, where the possible pair (q, ι_0), depending on $\eta_4 \in (0, 1]$ (see Assumption A3′), is pre-determined[8] according to the inequalities $0 < q < \eta_4/(4+\eta_4)$, $0 < \iota_0 < 1/4 - q$, and $\iota_0 \leq c_1 q$; more precisely,

- if $\eta_4 \in (0, 4/(3+4c_1)]$, then, $(q, \iota_0) \in \widetilde{\mathcal{S}}_1 \subset \widetilde{\mathcal{S}}$, where

$$\widetilde{\mathcal{S}}_1 = \{(q, \iota_0) \mid 0 < q < \eta_4/(4+\eta_4) \text{ and } 0 < \iota_0 \leq c_1 q\},$$

- if $\eta_4 \in (4/(3+4c_1), 1]$, then, $(q, \iota_0) \in \bigcup_{j=2}^{3} \widetilde{\mathcal{S}}_j \subset \widetilde{\mathcal{S}}$, where

$$\widetilde{\mathcal{S}}_2 = \{(q, \iota_0) \mid 0 < q < 1/\{4(1+c_1)\} \text{ and } 0 < \iota_0 \leq c_1 q\},$$
$$\widetilde{\mathcal{S}}_3 = \{(q, \iota_0) \mid 1/\{4(1+c_1)\} \leq q < \eta_4/(4+\eta_4) \text{ and } 0 < \iota_0 < 1/4 - q\}.$$

Then, Assumption A7 (i) and (iii) can be verified:

- $\min_{x\in[0,r_b]} f(x) \geq \varrho b^{\iota_0}$, where $\varrho = \min_{x\geq 0} f(x)(1+x)^{c_1}$,
- $\int_{r_b}^{\infty} w(x)dx \propto \exp\{-\exp(b^{-\iota_0/c_1} - 1)\}$; hence, we can choose any constant $A > 0$ for all sufficiently large n, noting $\exp(b^{-\iota_0/c_1} - 1) = b^{-A}\exp(b^{-\iota_0/c_1} - 1 + A\log b)$ and $\lim_{n\to\infty}(b^{-\iota_0/c_1} + A\log b) = \infty$ (we assume $b \to 0$).

Remark 5 Theorems 6, 8, and 9 (set $q = \iota_0 = 0$ for Theorem 8) remain valid even when $c = \gamma = 1$ (see Remark 3 (i)), with relaxed conditions for $(q, \iota_0, \iota_1, \iota_2)$[9], i.e., for Theorems 6 and 9, we impose

[6]For the $\mathrm{TS}_a/\mathrm{JF}_a$ type estimators, where $a \in (0, 1)$, "$c_0 \geq 1$" (rather than $c_0 > 0$) was sufficient; see the companion paper (Igarashi and Kakizawa (2017)).

[7]For the $\mathrm{TS}_a/\mathrm{JF}_a$ type estimators, where $a \in (0, 1)$, a larger weight function "$w(x) \propto \exp(-x)$" was sufficient; see the companion paper (Igarashi and Kakizawa (2017)).

[8]For the $\mathrm{TS}_a/\mathrm{JF}_a$ type estimators, where $a \in (0, 1)$, a wider pair (q, ι_0), depending on $\eta_4 \in (0, 1]$ (see Assumption A3′) was pre-determined according to the inequalities $0 < q < \eta_4/(4+\eta_4)$, $0 < \iota_0 < (1-3q)/2$, and $\iota_0 \leq c_1 q$ (see Igarashi and Kakizawa (2017)).

[9]For the $\mathrm{TS}_a/\mathrm{JF}_a$ type estimators, where $a \in (0, 1)$, studied in Igarashi and Kakizawa (2017), the results remain valid under the same conditions as the case $c = \gamma = 1$.

that "given $(q, \iota_0) \in \{(0,0)\} \cup \{(q, \iota_0) \mid 0 < q < \eta_4/(4+\eta_4)$ and $0 < \iota_0 < (1-3q)/2\}$, $b \propto n^{-\iota_1}$ and $\epsilon \propto b^{\iota_2}$ for some $(\iota_1, \iota_2) \in \{(\iota_1, \iota_2) \mid 0 < \iota_1 < 1/(1+2\iota_0)$ and $\iota_2 > 1 + p_0\}$, where $\eta_4 \in (0,1]$ is given in Assumption A3$'$, and $p_0 = q + \iota_0$"; for Theorem 8, we impose that "$b \propto n^{-\iota_1}$ and $\epsilon \propto b^{\iota_2}$ for some $(\iota_1, \iota_2) \in \{(\iota_1, \iota_2) \mid 2/13 < \iota_1 < 1$ and $\iota_2 > 1\}$ or $(\iota_1, \iota_2) \in \{(\iota_1, \iota_2) \mid 1/7 < \iota_1 < 1$ and $\iota_2 > 1\}$".

4. Simulation studies

We illustrate, through the simulations, the finite sample performance of the bias-reduced Amoroso kernel density estimators (10) and (11) (and the uncorrected estimator (1)), to check the usefulness of the bias reductions. We generated 1000 replicate samples of $n = 100, 200, 500$ from the four densities:

$$\text{A.}\quad f(x) = \frac{1}{2}\Big(\frac{e^{-x/3}}{3} + \frac{xe^{-x/3}}{9}\Big),$$

$$\text{B.}\quad f(x) = \frac{e^{-x/3}}{3},$$

$$\text{C.}\quad f(x) = \frac{1}{2}\Big(\frac{e^{-x/10}}{10} + xe^{-x}\Big),$$

$$\text{D.}\quad f(x) = \frac{1}{2}\Big[\frac{1}{\sqrt{2\pi}0.8x}\exp\Big\{-\frac{(\log x - 1)^2}{2(0.8)^2}\Big\} + \frac{1}{\sqrt{2\pi}0.4x}\exp\Big\{-\frac{(\log x - 2)^2}{2(0.4)^2}\Big\}\Big].$$

For the kth sample, let $ISE_k = \int_0^\infty \{\widehat{f}^{[k]}(x) - f(x)\}^2 dx$ be the integrated squared error (ISE), where $\widehat{f}^{[k]}$ is a density estimator using the (leave-one-out) least squared cross-validated smoothing parameter b (see, e.g., Wand and Jones (1995; Chapter 3)). We then calculated the average ISEs; $(1/1000)\sum_{k=1}^{1000} ISE_k$ (and the corresponding standard deviations) for each estimator. Here, for the estimators (1) and (10), we used $c = 1$ or $c = 1.1$ according to $\gamma > 0$ or $\gamma < 0$, whereas we used $c = 1.1$ for the estimator (11) (we further chose $\epsilon = (0.1)^6 b^{1.25}$ for the estimator (11)).

Tables 1–4 show that the average ISEs decreased, as the sample size n increased. Overall, for the case A (B), the limiting SS_1 and $\mathrm{JF}_1(=\mathrm{TS}_1)$ type estimators (10) and (11) with $\gamma = 1.5$ ($\gamma = 2$) worked well, and outperformed the estimator (1), whereas, for the case C (D), the limiting SS_1 and $\mathrm{JF}_1(=\mathrm{TS}_1)$ type estimators (10) and (11) with $\gamma = 0.5$ ($\gamma = -0.5$) worked well, but, some limiting estimators underperformed the estimator (1). We guess that the undesired results were caused by the small sample size n. Also, Figure 3 indicates that the SS_1 type may be superior (inferior) to the $\mathrm{JF}_1(=\mathrm{TS}_1)$ type, depending on the parameter $\gamma \neq 0$ at which $[\int_0^\infty \{B_{c,\gamma}^{[2]}(x)\}^2 dx / \int_0^\infty \{B_{c,\gamma}^{(JF_1)}(x)\}^2 dx]^{1/9}$ is negative (positive). Our simulation results, except for the case A, seemed to be consistent with such a finding from the asymptotic results in Section 3.

In summary, the selection of $\gamma \neq 0$ depends on f, as expected. We can say that, when $f(0)$ is small or zero, the limiting estimators (10) and (11) using $\gamma < 0$ have better performance.

Table 1: Case A. The average ISEs$\times 10^6$ of $\widehat{f}^{(\#)}_{b,c,\gamma,1}$ ($\# = SS, JF$) and $\widehat{f}_{b,c,\gamma}$ ($c = 1$ for $\widehat{f}^{(SS_1)}_{b,c,\gamma}$ and $\widehat{f}_{b,c,\gamma}$ with $\gamma > 0$ or $c = 1.1$ for $\widehat{f}^{(SS_1)}_{b,c,\gamma}$ and $\widehat{f}_{b,c,\gamma}$ with $\gamma < 0$, and $\widehat{f}^{(JF_1)}_{b,c,\gamma}$).

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

	$n = 100$			$n = 200$			$n = 500$		
	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$
$\gamma = 2$	3389	2289	**2120**	2107	1310	**1224**	1000	617	**603**
	(2946)	(2684)	(2487)	(2285)	(1727)	(1340)	(727)	(638)	(537)
1.5	3263	2289	**2022**	2005	1281	**1204**	919	595	**590**
	(3216)	(3121)	(2488)	(2353)	(1820)	(1743)	(708)	(660)	(646)
1	3006	2211	**2123**	1802	1233	**1207**	829	568	**552**
	(3144)	(2861)	(2945)	(2171)	(1735)	(1795)	(690)	(676)	(627)
0.5	2914	2393	**2353**	1690	1317	**1305**	790	621	**612**
	(3206)	(2847)	(2879)	(2077)	(1728)	(1728)	(722)	(675)	(667)
0.25	3802	3568	**3531**	2143	1928	**1878**	1024	906	**882**
	(3457)	(3394)	(3401)	(1932)	(1657)	(1620)	(851)	(846)	(801)
-0.25	3656	**3471**	3609	2044	**1825**	1915	983	**885**	934
	(3254)	(3038)	(3073)	(1776)	(1574)	(1604)	(795)	(775)	(809)
-0.5	3201	**2504**	2630	1831	1477	**1475**	847	**707**	716
	(3235)	(2255)	(2393)	(2098)	(1672)	(1534)	(689)	(648)	(641)
-1	3392	2544	**2414**	2048	1439	**1324**	970	680	**626**
	(2898)	(2305)	(2463)	(2044)	(1295)	(1316)	(699)	(508)	(478)
-1.5	3735	**2963**	2973	2283	1661	**1539**	1106	777	**669**
	(2674)	(2452)	(2682)	(2018)	(1373)	(1512)	(732)	(536)	(548)
-2	4135	**3471**	3674	2489	1875	**1866**	1209	891	**781**
	(2941)	(2733)	(3054)	(1913)	(1321)	(1564)	(691)	(590)	(641)

Table 2: Case B. The average ISEs$\times 10^6$ of $\widehat{f}^{(\#)}_{b,c,\gamma,1}$ ($\# = SS, JF$) and $\widehat{f}_{b,c,\gamma}$ ($c = 1$ for $\widehat{f}^{(SS_1)}_{b,c,\gamma}$ and $\widehat{f}_{b,c,\gamma}$ with $\gamma > 0$ or $c = 1.1$ for $\widehat{f}^{(SS_1)}_{b,c,\gamma}$ and $\widehat{f}_{b,c,\gamma}$ with $\gamma < 0$, and $\widehat{f}^{(JF_1)}_{b,c,\gamma}$).

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

	$n = 100$			$n = 200$			$n = 500$		
	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$
$\gamma = 2$	6452	4088	**3458**	3726	2456	**2058**	1924	1225	**996**
	(5871)	(4789)	(4123)	(2917)	(2695)	(2268)	(1512)	(1261)	(1021)
1.5	6480	4297	**3656**	3531	2400	**2111**	1793	1162	**1032**
	(7440)	(5453)	(4990)	(2122)	(2110)	(2783)	(1537)	(1286)	(1299)
1	6049	4368	**4023**	3222	2317	**2117**	1599	1095	**1023**
	(7483)	(5791)	(5803)	(3304)	(2758)	(2741)	(1412)	(1262)	(1319)
0.5	5578	4659	**4447**	2882	2339	**2240**	1419	1075	**1025**
	(7131)	(6653)	(6333)	(2891)	(2729)	(2690)	(1358)	(1155)	(1119)
0.25	7108	6643	**6509**	3684	3415	**3305**	1774	1542	**1475**
	(9159)	(6904)	(9476)	(3293)	(3243)	(3297)	(1509)	(1338)	(1280)
-0.25	6629	**6176**	6420	3467	**3311**	3452	1656	**1492**	1531
	(6872)	(6399)	(6469)	(2990)	(2984)	(3028)	(1338)	(1334)	(1312)
-0.5	5924	**4673**	4700	3084	2455	**2442**	1535	1129	**1103**
	(6366)	(5257)	(5270)	(2702)	(2440)	(2372)	(1355)	(1079)	(1014)
-1	6652	4693	**4159**	3701	2644	**2265**	1804	1208	**1114**
	(6154)	(4184)	(4240)	(3209)	(2298)	(1999)	(1323)	(1017)	(1013)
-1.5	7505	5334	**4815**	4098	2999	**2614**	2059	1465	**1293**
	(6561)	(4556)	(4835)	(2766)	(2182)	(2171)	(1363)	(1106)	(1059)
-2	7983	5969	**5667**	4515	3561	**3090**	2268	1730	**1469**
	(6147)	(4762)	(5169)	(2800)	(2481)	(2385)	(1391)	(1195)	(1081)

Table 3: Case C. The average ISEs$\times 10^6$ of $\widehat{f}^{(\#)}_{b,c,\gamma,1}$ ($\# = SS, JF$) and $\widehat{f}_{b,c,\gamma}$ ($c = 1$ for $\widehat{f}^{(SS_1)}_{b,c,\gamma}$ and $\widehat{f}_{b,c,\gamma}$ with $\gamma > 0$ or $c = 1.1$ for $\widehat{f}^{(SS_1)}_{b,c,\gamma}$ and $\widehat{f}_{b,c,\gamma}$ with $\gamma < 0$, and $\widehat{f}^{(JF_1)}_{b,c,\gamma}$).

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

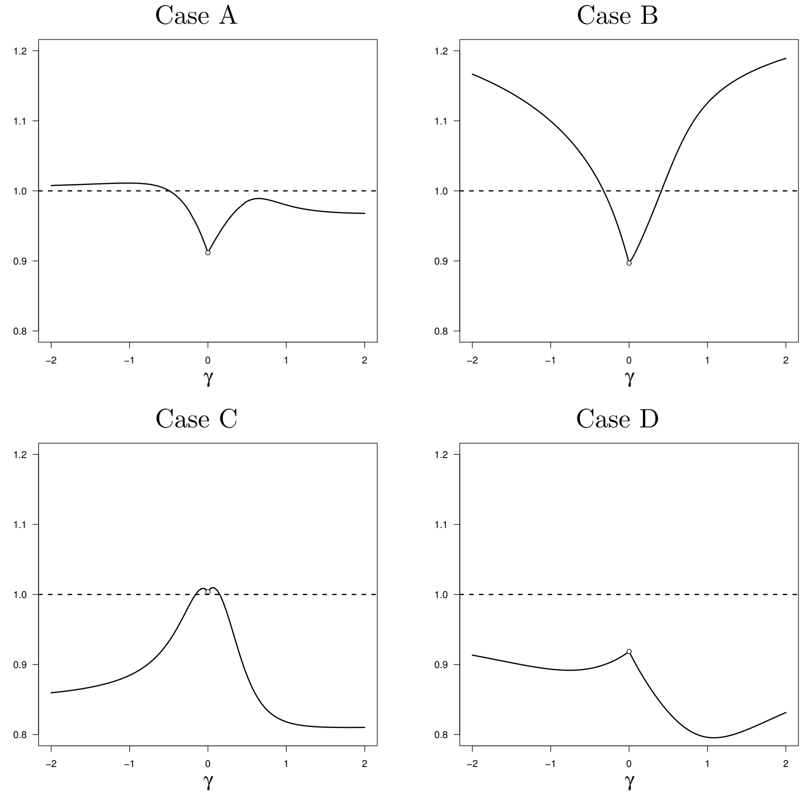
	$n = 100$			$n = 200$			$n = 500$		
	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$
$\gamma = 2$	**7178**	7796	9043	**4241**	4447	5215	2062	**2024**	2243
	(4902)	(4918)	(4567)	(2773)	(2726)	(3182)	(1142)	(1288)	(1376)
1.5	**6656**	7204	8059	**3911**	3994	4396	1853	**1767**	1930
	(4722)	(4930)	(5084)	(2770)	(2593)	(2859)	(1083)	(1174)	(1209)
1	**6045**	6295	6805	3492	**3424**	3761	1623	**1490**	1605
	(4604)	(4578)	(4782)	(2659)	(2314)	(2791)	(1026)	(1034)	(1068)
0.5	5411	**5308**	5372	3048	**2813**	2872	1388	**1248**	1274
	(4325)	(4407)	(4385)	(2492)	(2014)	(2071)	(935)	(926)	(932)
0.25	5587	5515	**5355**	3164	2942	**2859**	1437	1299	**1248**
	(4417)	(4605)	(4519)	(2662)	(2203)	(2129)	(990)	(928)	(897)
-0.25	5637	5612	**5506**	3173	3004	**2959**	1433	1332	**1312**
	(4267)	(4427)	(4362)	(2593)	(2101)	(2101)	(962)	(924)	(940)
-0.5	5763	5847	**5720**	3224	3024	**3006**	1444	**1322**	1324
	(4565)	(4453)	(4379)	(2634)	(2194)	(2213)	(955)	(935)	(941)
-1	**6688**	7164	6704	**3774**	4310	4320	1712	**1681**	1891
	(4967)	(4503)	(4576)	(2799)	(3168)	(3210)	(1033)	(1516)	(1924)
-1.5	**7454**	8343	7870	**4248**	5758	5667	**1994**	2662	3050
	(4944)	(4943)	(5235)	(2842)	(3955)	(4300)	(1117)	(2641)	(3148)
-1.5	**8060**	9218	8864	**4696**	6740	6440	**2238**	3682	4057
	(4936)	(5446)	(6167)	(2942)	(4336)	(4770)	(1175)	(3425)	(3938)

Table 4: Case D. The average ISEs$\times 10^6$ of $\widehat{f}^{(\#)}_{b,c,\gamma,1}$ ($\# = SS, JF$) and $\widehat{f}_{b,c,\gamma}$ ($c = 1$ for $\widehat{f}^{(SS_1)}_{b,c,\gamma}$ and $\widehat{f}_{b,c,\gamma}$ with $\gamma > 0$ or $c = 1.1$ for $\widehat{f}^{(SS_1)}_{b,c,\gamma}$ and $\widehat{f}_{b,c,\gamma}$ with $\gamma < 0$, and $\widehat{f}^{(JF_1)}_{b,c,\gamma}$).

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

	$n = 100$			$n = 200$			$n = 500$		
	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$	$\widehat{f}_{b,c,\gamma}$	$\widehat{f}^{(SS_1)}_{b,c,\gamma}$	$\widehat{f}^{(JF_1)}_{b,c,\gamma}$
$\gamma = 2$	**4665**	5195	5734	**2896**	3156	3619	1376	**1364**	1554
	(2851)	(2818)	(2956)	(1845)	(1931)	(1992)	(710)	(762)	(813)
1.5	**4230**	4601	5099	**2615**	2734	3056	1225	**1184**	1332
	(2744)	(2964)	(3097)	(1780)	(2013)	(2065)	(652)	(680)	(706)
1	**3669**	3841	4175	2286	**2260**	2491	1071	**1001**	1117
	(2491)	(2719)	(2793)	(1677)	(1758)	(1849)	(594)	(586)	(617)
0.5	3168	**3133**	3275	1988	**1954**	2022	948	**900**	933
	(2295)	(2333)	(2355)	(1564)	(1591)	(1616)	(563)	(548)	(552)
0.25	3180	**3174**	3197	1992	1968	**1956**	938	**892**	894
	(2331)	(2209)	(2328)	(1561)	(1442)	(1420)	(576)	(547)	(566)
-0.25	3195	**3178**	3214	1986	**1962**	1965	939	896	**890**
	(2334)	(2225)	(2324)	(1555)	(1616)	(1620)	(582)	(561)	(548)
-0.5	3155	**3047**	3174	1976	**1879**	1937	942	**881**	906
	(2316)	(2384)	(2426)	(1548)	(1597)	(1592)	(558)	(529)	(553)
-1	**3800**	3825	3933	2311	**2139**	2277	1089	**940**	1034
	(2659)	(2911)	(2883)	(1724)	(1833)	(1830)	(618)	(552)	(590)
-1.5	**4600**	4617	4665	**2683**	2878	3001	1259	**1223**	1324
	(3011)	(3037)	(2985)	(1828)	(2350)	(2286)	(680)	(1111)	(1058)
-2	5371	**5182**	5334	**3055**	3692	3708	**1414**	1900	1923
	(3326)	(3100)	(3206)	(2007)	(2517)	(2513)	(720)	(1961)	(1777)

Figure 3: Graph of $[\int_0^\infty \{B_{c,\gamma}^{[2]}(x)\}^2 dx / \int_0^\infty \{B_{c,\gamma}^{(JF_1)}(x)\}^2 dx]^{1/9}$ $(w(x)=1)$.



5. Concluding remark

In this paper, we have studied, under appropriate assumptions, the asymptotic properties of new limiting $\mathrm{SS}_1/\mathrm{JF}_1(=\mathrm{TS}_1)$ type bias-reduced Amoroso kernel density estimators. It turns out that the asymptotic MISE (MSE) convergence rates (17) and (22) ((16) and (21)) for the bias-reduced estimators (10) and (11) are faster than that of (9) ((8)) for the estimator (1). We have shown that, in terms of the AMISE, (i) the limiting $\mathrm{SS}_1/\mathrm{TS}_1$ type bias-reduced Amoroso kernel density estimators are superior to the $\mathrm{SS}_a/\mathrm{TS}_a$ type bias-reduced Amoroso kernel density estimators, respectively, and (ii) the best implemented (with respect to $a \in (0,1]$) JF_a type bias-reduced Amoroso kernel density estimator outperforms any TS_a type bias-reduced Amoroso kernel density estimator. We have illustrated the finite sample performance of the proposed estimators, through the simulation studies, using the least squared cross-validated smoothing parameter.

Surprisingly, the factor $\{\lambda^4(a)/a\}^{2/9}$ (see Corollaries 5 and 10 (i)) is common even for other $\mathrm{SS}_a/\mathrm{TS}_a$ type bias-reduced MIG/weighted LN/beta kernel density estimators (Igarashi and Kakizawa (2015) for the case $\mathcal{S} = [0,\infty)$ and Igarashi (2016a) for the case $\mathcal{S} = [0,1]$), as well as the standard kernel density estimator using the Gaussian-based fourth-order kernel for the case $\mathcal{S} = \mathbb{R}$ (Wand and Schucany

(1990)). It would be interesting to discuss the conditions under which the factor appears in the AMISE of the asymmetric kernel density estimation, but it is left as a topic for future work, though, as in Koul and Song (2013), the approximation to the Gaussian kernel may be related to this issue.

The first author found, in his master's thesis (2012, Graduate School of Economics and Business Administration, Hokkaido University, in Japanese), that the AMSE of the TS_a type bias-reduced standard kernel density estimator using the Gaussian kernel for the case $\mathcal{S} = \mathbb{R}$ is minimized at $a = 1$. Sakhanenko (2017) discussed the AMISE of the SS_a type bias-reduced standard kernel density estimator using some kernels for the case $\mathcal{S} = \mathbb{R}$, and gave the respective optimal a.

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Appendix A: Proofs of the results in Section 3

A1 Technical lemmas

Throughout this appendix, we denote by $\xi_{\alpha,\beta,\gamma}$ the random variable that is distributed according to the Amoroso density $K^{(A)}_{\alpha,\beta,\gamma}$, where $\alpha, \beta > 0$ and $\gamma \neq 0$. It is easy to see that

$$\sup_{s\geq 0} K^{(A)}_{\alpha,\beta,\gamma}(s) = \frac{|\gamma|}{\beta\Gamma(\alpha)}(\alpha - 1/\gamma)^{\alpha-1/\gamma}e^{-(\alpha-1/\gamma)} \quad \text{when } \alpha - 1/\gamma \geq 0. \tag{A1}$$

We now recall the definition (2). Then, we have, for $j > 0$,

$$E[\xi^j_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}] = (b\rho)^j\frac{\Gamma^{j-1}(\alpha_\gamma(\rho))\Gamma(\alpha_\gamma(\rho)+j/\gamma)}{\Gamma^j(\alpha_\gamma(\rho)+1/\gamma)} = \begin{cases} (b\rho)^j\dfrac{\Gamma^{j-1}(\rho/\gamma)\Gamma((\rho+j)/\gamma)}{\Gamma^j((\rho+1)/\gamma)}, & \gamma > 0, \\ (b\rho)^j\dfrac{\Gamma^{j-1}((\rho+1)/|\gamma|)\Gamma((\rho+1-j)/|\gamma|)}{\Gamma^j(\rho/|\gamma|)}, & \gamma < 0 \end{cases} \tag{A2}$$

(this moment, for $\rho > 0$, always exists when $\gamma > 0$, whereas, when $\gamma < 0$, the restriction $\rho > \max(0, j-1)$ is required).

For ease of reference, we reproduce the following results (Claims A.1 and A.2).

Claim A.1 (Igarashi and Kakizawa (2017)) (i). Given $\gamma \neq 0$ and $c > 0$, we have, for $x \geq 0$,

$$E[\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma} - x] = bc.$$

(ii). Given $\gamma \neq 0$ and $c > 0$, we have, for $x/b \to \infty$,

$$E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma} - x)^j] = \begin{cases} \dfrac{1}{|\gamma|}bx + \dfrac{\delta_{c,\gamma}^{[2]}}{\gamma^2}b^2 + O(b^3x^{-1}), & j = 2, \\ \dfrac{\delta_{c,\gamma}^{[3]}}{\gamma^2}b^2x + O(b^3), & j = 3, \\ \dfrac{3}{\gamma^2}b^2x^2 + O(b^3x), & j = 4, \\ O(b^3x^3), & j = 6. \end{cases}$$

Also, we have, for $x/b \to \kappa$,

$$E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma} - x)^2] = \begin{cases} b^2\eta_\gamma(\kappa, \kappa + c) + o(b^2), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ b^2\eta_\gamma(0, c), & x = 0, \end{cases}$$

provided that $x/b + c > 1$ when $\gamma < 0$ (see (A2)), where

$$\eta_\gamma(\kappa, \kappa + c) = (\kappa + c)^2\frac{\Gamma(\alpha_\gamma(\kappa + c))\Gamma(\alpha_\gamma(\kappa + c) + 2/\gamma)}{\Gamma^2(\alpha_\gamma(\kappa + c) + 1/\gamma)} - 2\kappa(\kappa + c) + \kappa^2.$$

(iii). Given $\gamma \neq 0$ and $c > 1/2$, we have

$$v_\gamma(x/b + c) = \begin{cases} \dfrac{b^{1/2}|\gamma|^{-1/2}}{2(\pi x)^{1/2}}\{1 + O(bx^{-1})\}, & \dfrac{x}{b} \to \infty, \\ v_\gamma(\kappa + c) + o(1), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ v_\gamma(c), & x = 0. \end{cases}$$

Claim A.2 (Igarashi and Kakizawa (2017)) For any $z, p > 0$, we have

$$\max\Big\{(z + p)^p\Big(1 - \frac{6p^2 + 6p + 1}{12z}\Big), 0\Big\} \leq \frac{\Gamma(z + p)}{\Gamma(z)} \leq (z + p)^p.$$

First of all, we prepare the the following approximations (or bounds) of some expectations involving the random variables $\xi_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}$ and $\xi_{2\alpha_\gamma(\rho)-1/\gamma,b\beta_\gamma(\rho)/2^{1/\gamma},\gamma}$ (Lemmas A.1–A.3).

Lemma A.1 *Let $\gamma \neq 0$ and $c > 0$.*

(i). *We have, for $x \geq 0$,*

$$E[H_{b,c,\gamma,x/b+c}^{(A)}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})] = 1,$$
$$E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma} - x)H_{b,c,\gamma,x/b+c}^{(A)}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})] = 0.$$

(ii). *We have, for $x/b \to \infty$,*

$$E[(\xi_{b,x/b+c} - x)^j H_{b,c,\gamma,x/b+c}^{(A)}(\xi_{b,x/b+c})] = \begin{cases} -\dfrac{\delta_{c,\gamma}^{[2]}}{\gamma^2}b^2 + O(b^3x^{-1}), & j = 2, \\ -\dfrac{\delta_{c,\gamma}^{[3]}}{\gamma^2}b^2x + O(b^3), & j = 3, \\ -\dfrac{3}{\gamma^2}b^2x^2 + O(b^3x), & j = 4, \end{cases}$$
$$E[(\xi_{b,x/b+c} - x)^{2i}\{H_{b,c,\gamma,x/b+c}^{(A)}(\xi_{b,x/b+c})\}^2] = O((bx)^i), \quad i = 1, 2.$$

Also, we have, for $x/b \to \kappa$,

$$E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma} - x)^2 H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})] = \begin{cases} -b^2\zeta^{(SS_1)}_{c,\gamma}(\kappa) + o(b^2), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ -b^2\zeta^{(SS_1)}_{c,\gamma}(0), & x = 0, \end{cases}$$

provided that $x/b + c > 1$ when $\gamma < 0$ (see (A2)).

Proof It is straightforward to see that

$$\int_0^\infty s^j K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)\{H^{(A)}_{b,c,\gamma,\rho}(s)\}^i ds = \mathcal{G}^{[i]}_j(\rho) E[\xi^j_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}], \quad i = 1, 2, \quad j \geq 0 \tag{A3}$$

(if these integrals exist), where

$$\begin{aligned}
\mathcal{G}^{[1]}_j(\rho) &= 1 - \frac{jc}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho) + j/\gamma) - j\psi(\alpha_\gamma(\rho) + 1/\gamma) + (j-1)\psi(\alpha_\gamma(\rho))\}, \\
\mathcal{G}^{[2]}_j(\rho) &= 1 + \gamma^2\Big\{\alpha_\gamma(\rho) + \frac{j^2 + j\gamma}{\gamma^2}\Big\}\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}\Big]^2 \\
&\quad + \frac{1}{\gamma^2}(\rho - c)^2\big[\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + j/\gamma)\}^2 + \psi'(\alpha_\gamma(\rho) + j/\gamma)\big] \\
&\quad + 2j\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}\Big] - \frac{2}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + j/\gamma)\} \\
&\quad + \frac{2\gamma}{|\gamma|}(\rho - c)\Big[1 - \frac{j}{\gamma}\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + j/\gamma)\}\Big]\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}\Big].
\end{aligned}$$

The result (i) follows by noting that $\mathcal{G}^{[1]}_0(\rho) \equiv 1$, $\mathcal{G}^{[1]}_1(\rho) = (\rho - c)/\rho$, and $E[\xi_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}] = b\rho$. Also, note that

$$\mathcal{G}^{[1]}_2(\rho) = \frac{\rho - c}{\rho} + 2\mathcal{H}_{c,\gamma,1}(\rho) - \mathcal{H}_{c,\gamma,2}(\rho),$$

hence,

$$\begin{aligned}
&E[(\xi_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma} - x)^2 H^{(A)}_{b,c,\gamma,\rho}(\xi_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma})] \\
&= \mathcal{G}^{[1]}_2(\rho)E[\xi^2_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}] - 2x\mathcal{G}^{[1]}_1(\rho)E[\xi_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}] + x^2\mathcal{G}^{[1]}_0(\rho) \\
&= \Big\{\frac{\rho - c}{\rho} + 2\mathcal{H}_{c,\gamma,1}(\rho) - \mathcal{H}_{c,\gamma,2}(\rho)\Big\}E[\xi^2_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}] - b^2(\rho - c)^2 + \{b(\rho - c) - x\}^2.
\end{aligned}$$

After some tedious calculations (the details are omitted), we have the asymptotic expansions when $\rho \to \infty$:

$$\begin{aligned}
&E[\xi^j_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}] \\
&= \begin{cases} b^j\rho^j + \dfrac{j(j-1)}{2\gamma}b^j\rho^{j-1} + \dfrac{j(j-1)\{6\gamma^2 - 4(j+1)\gamma + 3j(j-1)\}}{24\gamma^2}b^j\rho^{j-2} + O(b^j\rho^{j-3}), & \gamma > 0, \\ b^j\rho^j + \dfrac{j(j-1)}{2|\gamma|}b^j\rho^{j-1} + \dfrac{j(j-1)\{6\gamma^2 + 4(j-2)|\gamma| + 3j(j-1)\}}{24\gamma^2}b^j\rho^{j-2} + O(b^j\rho^{j-3}), & \gamma < 0, \end{cases} \\
&(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + j/\gamma)\} \\
&= \begin{cases} -j - \dfrac{j(\gamma - j - 2c)}{2\rho} - \dfrac{j(\gamma - j)(\gamma - 2j - 3c)}{6\rho^2} + O(\rho^{-3}), & \gamma > 0, \\ j - \dfrac{j(\gamma - j + 2c + 2)}{2\rho} - \dfrac{j\{(\gamma - j)(\gamma - 2j + 3c + 6) + 6c + 6\}}{6\rho^2} + O(\rho^{-3}), & \gamma < 0, \end{cases}
\end{aligned} \tag{A4}$$

$$\begin{aligned}
&(\rho - c)^2\psi'(\alpha_\gamma(\rho) + j/\gamma) \\
&= \begin{cases} \gamma(\rho - c) + \dfrac{\gamma^2 - 2\gamma(j + c)}{2} + \dfrac{\gamma^3 - 6\gamma^2(j + c) + 6\gamma(j + c)^2}{6\rho} + O(\rho^{-2}), & \gamma > 0, \\ -\gamma(\rho - c) + \dfrac{\gamma^2 - 2\gamma(j - c - 1)}{2} - \dfrac{\gamma^3 - 6\gamma^2(j - c - 1) + 6\gamma(j - c - 1)^2}{6\rho} + O(\rho^{-2}), & \gamma < 0 \end{cases}
\end{aligned} \tag{A5}$$

(see Proof of Lemma A.1 of Igarashi and Kakizawa (2017) and Abramowitz and Stegun (1972; 6.3.18 and 6.4.12)). The result (ii) follows by letting $\rho = x/b + c$. $\square$

Lemma A.2 (i). *Given $\gamma \neq 0$ and $j \geq 0$, let $c > 0$ when $\gamma > 0$ or $c > \max(0, j-1)$ when $\gamma < 0$. For any $b > 0$ and $\rho \geq c$, there exists a constant $M_{c,\gamma,j} > 0$, independent of b and ρ, such that*

$$\int_0^\infty s^j K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s)\{H^{(A)}_{b,c,\gamma,\rho}(s)\}^2 ds \leq M_{c,\gamma,j} E[\xi^j_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}].$$

(ii). *Let $\gamma \neq 0$ and $c > 0$. For any $b > 0$, there exists a constant $M_{c,\gamma} > 0$, independent of b, such that*

$$\sup_{\rho \geq c} \int_0^\infty K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s)\{H^{(A)}_{b,c,\gamma,\rho}(s)\}^4 ds \leq M_{c,\gamma}.$$

Proof In view of the proof of Lemma A.1, the function $\mathcal{G}^{[2]}_j$ is continuous on $[c, \infty)$, and, using (A4) and (A5), $\mathcal{G}^{[2]}_j(\rho) = 3/2 + O(\rho^{-1})$ as $\rho \to \infty$ (hence, $\sup_{\rho \geq c} \mathcal{G}^{[2]}_j(\rho)$ is bounded). The result (i) follows from (A3).

On the other hand, we define

$$\begin{aligned}\mathcal{G}^{[3]}(\rho) &= \gamma^4\{3\alpha^2_\gamma(\rho) + 6\alpha_\gamma(\rho)\}\mathcal{H}^4(\rho) + 4\gamma|\gamma|(\rho - c)\{3\alpha_\gamma(\rho) + 2\}\mathcal{H}^3(\rho) + 6(\rho - c)^2\{2 + \alpha_\gamma(\rho)\psi'(\alpha_\gamma(\rho))\}\mathcal{H}^2(\rho) \\ &\quad + 12\frac{1}{\gamma|\gamma|}(\rho - c)^3\psi'(\alpha_\gamma(\rho))\mathcal{H}(\rho) + \frac{1}{\gamma^4}(\rho - c)^4[\psi'''(\alpha_\gamma(\rho)) + 3\{\psi'(\alpha_\gamma(\rho))\}^2]\end{aligned}$$

(the function $\mathcal{G}^{[3]}$ is continuous on $[c, \infty)$), where

$$\mathcal{H}(\rho) = -\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}.$$

Using the asymptotic expansions when $\rho \to \infty$;

$$\mathcal{H}(\rho) = \begin{cases} -\dfrac{1}{\gamma} - \dfrac{(\gamma-1)(2c+1)}{2\gamma\rho} - \dfrac{(\gamma-1)(\gamma-3c-2)}{6\gamma\rho^2} + O(\rho^{-3}), & \gamma > 0, \\ -\dfrac{1}{\gamma} - \dfrac{(\gamma-1)(2c-1)-2}{2\gamma\rho} + \dfrac{(\gamma-1)(\gamma+3c+4)+6c+6}{6\gamma\rho^2} + O(\rho^{-3}), & \gamma < 0, \end{cases} \tag{A6}$$

$$(\rho - c)^2\psi'(\alpha_\gamma(\rho)) = \begin{cases} \gamma(\rho - c) + \dfrac{\gamma^2 - 2c\gamma}{2} + \dfrac{\gamma^3 - 6c\gamma^2 + 6c^2\gamma}{6\rho} + O(\rho^{-2}), & \gamma > 0, \\ -\gamma(\rho - c) + \dfrac{\gamma^2 + 2(c+1)\gamma}{2} - \dfrac{\gamma^3 + 6(c+1)\gamma^2 + 6(c+1)^2\gamma}{6\rho} + O(\rho^{-2}), & \gamma < 0, \end{cases}$$

$$(\rho - c)^4\psi'''(\alpha_\gamma(\rho)) = \begin{cases} 2\gamma^3(\rho - c) + 3\gamma^3(\gamma - 2c) + O(\rho^{-1}), & \gamma > 0, \\ -2\gamma^3(\rho - c) + 3\gamma^3(\gamma + 2c + 2) + O(\rho^{-1}), & \gamma < 0 \end{cases}$$

(see (A4), (A5), and Abramowitz and Stegun (1972; 6.4.14)), it is shown that $\mathcal{G}^{[3]}(\rho) = 15/4 + O(\rho^{-1})$ as $\rho \to \infty$ (hence, $\sup_{\rho \geq c} \mathcal{G}^{[3]}(\rho)$ is bounded). The result (ii) follows from

$$\int_0^\infty K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s)\{H^{(A)}_{b,c,\gamma,\rho}(s)\}^4 ds \leq 2^3 \int_0^\infty K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s)[1 + \{H^{(A)}_{b,c,\gamma,\rho}(s) - 1\}^4] ds = 2^3\{1 + \mathcal{G}^{[3]}(\rho)\}. \quad \square$$

Lemma A.3 *Given $\gamma \neq 0$ and $c > 1/2$, we have*

$$E[\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{2\alpha_\gamma(x/b+c)-1/\gamma, b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma})\}^2] = \begin{cases} \dfrac{27}{16} + O(bx^{-1}), & \dfrac{x}{b} \to \infty, \\ \dfrac{v^{(SS_1)}_{c,\gamma}(\kappa)}{v_\gamma(\kappa + c)} + o(1), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ \dfrac{v^{(SS_1)}_{c,\gamma}(0)}{v_\gamma(c)}, & x = 0 \end{cases}$$

and

$$E[(\xi_{2\alpha_\gamma(x/b+c)-1/\gamma,b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma}-x)^2\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{2\alpha_\gamma(x/b+c)-1/\gamma,b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma})\}^2]=\begin{cases}O(bx), & \frac{x}{b}\to\infty,\\ O(b^2), & \frac{x}{b}\to\kappa.\end{cases}$$

Proof It is straightforward to see that

$$\int_0^\infty s^j K^{(A)}_{2\alpha_\gamma(\rho)-1/\gamma,b\beta_\gamma(\rho)/2^{1/\gamma},\gamma}(s)\{H^{(A)}_{b,c,\gamma,\rho}(s)\}^2ds=\mathcal{G}^{[4]}_j(\rho)E[\xi^j_{2\alpha_\gamma(\rho)-1/\gamma,b\beta_\gamma(\rho)/2^{1/\gamma},\gamma}]$$

(if $c>1/2$, this integral is well-defined at least for $j=0,1,2$, since, by definition (see (2)), $2\alpha_\gamma(\rho)+(j-1)/\gamma>0$, $j=0,1,2$), where

$$\begin{aligned}\mathcal{G}^{[4]}_j(\rho)&=1+\frac{\gamma^2}{4}\Big\{2\alpha_\gamma(\rho)+\frac{(j-1)^2}{\gamma^2}+\frac{j-1}{\gamma}\Big\}\mathcal{H}^2(\rho)\\&+\frac{1}{\gamma^2}(\rho-c)^2\big[\psi'(2\alpha_\gamma(\rho)+(j-1)/\gamma)+\{\psi(2\alpha_\gamma(\rho)+(j-1)/\gamma)-\psi(\alpha_\gamma(\rho))-\log 2\}^2\big]\\&+(j-1)\mathcal{H}(\rho)+\frac{\gamma}{|\gamma|}(\rho-c)\Big[1+\frac{j-1}{\gamma}\{\psi(2\alpha_\gamma(\rho)+(j-1)/\gamma)-\psi(\alpha_\gamma(\rho))-\log 2\}\Big]\mathcal{H}(\rho)\\&+\frac{2}{|\gamma|}(\rho-c)\{\psi(2\alpha_\gamma(\rho)+(j-1)/\gamma)-\psi(\alpha_\gamma(\rho))-\log 2\}.\end{aligned}$$

In addition to (A6), we have the asymptotic expansions when $\rho\to\infty$:

$$E[\xi^j_{2\alpha_\gamma(\rho)-1/\gamma,b\beta_\gamma(\rho)/2^{1/\gamma},\gamma}]=(b\rho)^j\{1+O(\rho^{-1})\},$$

$$(\rho-c)^2\psi'(2\alpha_\gamma(\rho)+(j-1)/\gamma)=\begin{cases}\frac{\gamma(\rho-c)}{2}+\frac{\gamma\{\gamma-2(j-1)-2c\}}{4}+O(\rho^{-1}), & \gamma>0,\\ -\frac{\gamma(\rho-c)}{2}+\frac{\gamma\{\gamma-2(j-1)+2(c+1)\}}{4}+O(\rho^{-1}), & \gamma<0,\end{cases}$$

$$(\rho-c)\{\psi(2\alpha_\gamma(\rho)+(j-1)/\gamma)-\psi(\alpha_\gamma(\rho))-\log 2\}=\begin{cases}\frac{\gamma+2(j-1)}{4}+O(\rho^{-1}), & \gamma>0,\\ -\frac{\gamma+2(j-1)}{4}+O(\rho^{-1}), & \gamma<0\end{cases}$$

(see Igarashi and Kakizawa (2017) and Abramowitz and Stegun (1972; 6.3.18 and 6.4.12)). The result follows by letting $\rho=x/b+c$. □

Next, we establish the following lemma (Lemma A.4), which gives the bound of $K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)|H^{(A)}_{b,c,\gamma,\rho}(s)|^j$ and the inequality of $K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)H^{(A)}_{b,c,\gamma,\rho}(s)$. Utilizing Remark 1 (i) and Lemmas A.2 (i) and A.4 (i), we can readily obtain the (nonasymptotic) bounds of the two-sided tail probabilities and absolute moments of $\overline{\Delta}_{b,x/b+c}$ and $\overline{\Delta}^{(SS_1)}_{b,x/b+c}$ (Lemma A.5). Here, as usual, we rewrite $\widehat{f}_{b,c,\gamma}(x)-E[\widehat{f}_{b,c,\gamma}(x)]$ and $\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)-E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]$ as the averages $\overline{\Delta}_{b,x/b+c}=n^{-1}\sum_{i=1}^n\Delta_{b,x/b+c,i}$ and $\overline{\Delta}^{(SS_1)}_{b,x/b+c}=n^{-1}\sum_{i=1}^n\Delta^{(SS_1)}_{b,x/b+c,i}$ of independent zero mean random variables $\Delta_{b,x/b+c,i}$ and $\Delta^{(SS_1)}_{b,x/b+c,i}$, $i=1,\ldots,n$, where

$$\begin{aligned}\Delta_{b,\rho,i}&=K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(X_i)-E[K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(X_i)],\\ \Delta^{(SS_1)}_{b,\rho,i}&=K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(X_i)H^{(A)}_{b,c,\gamma,\rho}(X_i)-E[K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(X_i)H^{(A)}_{b,c,\gamma,\rho}(X_i)].\end{aligned}$$

Lemma A.4 *Let $\gamma\neq 0$ and $c>1$.*

(i). *For any $j \geq 1$ and $b > 0$, there exists a constant $\widetilde{L}_{c,\gamma,j} > 0$, independent of b, such that*

$$\sup_{\rho \geq c} \sup_{s \geq 0} \big[\{1 + (\rho - c)^j\}^{-1} K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s) |H^{(A)}_{b,c,\gamma,\rho}(s)|^j\big] \leq \frac{\widetilde{L}_{c,\gamma,j}}{b}.$$

(ii). *For any $b > 0$, there exists a constant $\widetilde{L}_{c,\gamma} > 0$, independent of b, such that*

$$\sup_{\rho \geq c} \sup_{s \geq 0} \big\{K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s) H^{(A)}_{b,c,\gamma,\rho}(s)\big\} \leq \frac{\widetilde{L}_{c,\gamma}}{b}.$$

Proof of Lemma A.4 is postponed to Appendix B.

Lemma A.5 *Given $\gamma \neq 0$, choose $c \geq 1$ for $\overline{\Delta}_{b,x/b+c}$ or $c > 1$ for $\overline{\Delta}^{(SS_1)}_{b,x/b+c}$. Under Assumption* A1 (*assume that $C_0 = \sup_{x \geq 0} f(x)$ is finite*), *we have, for any $n \in \mathbb{N}$, $b, t > 0$, $x \geq 0$, and $j \geq 2$,*

- *exponential bounds of the two-sided tail probabilities*

$$P[|\overline{\Delta}_{b,x/b+c}| \geq t] \leq 2 \exp\Big\{-\frac{nbt^2}{\widetilde{L}_\gamma(2C_0 + t)}\Big\}, \tag{A7}$$

$$P[|\overline{\Delta}^{(SS_1)}_{b,x/b+c}| \geq t] \leq 2 \exp\Big[-\frac{nb^2t^2}{2\{C_0 \widetilde{L}_\gamma M_{c,\gamma,0} b + \widetilde{L}_{c,\gamma,1}(b+x)t\}}\Big], \tag{A8}$$

- *bounds of absolute moments*

$$\begin{aligned} E[|\overline{\Delta}_{b,x/b+c}|^j] &\leq n^{-j} C(j) \{nE[|\Delta_{b,x/b+c,1}|^j] + (nE[\Delta^2_{b,x/b+c,1}])^{j/2}\} \\ &\leq 2C(j)(n^{-2}b^{-2}\widetilde{L}^2_\gamma + n^{-1}b^{-1}C_0\widetilde{L}_\gamma)^{(j-2)/2} V[\widehat{f}_{b,c,\gamma}(x)], \end{aligned} \tag{A9}$$

$$\begin{aligned} E[|\overline{\Delta}^{(SS_1)}_{b,x/b+c}|^j] &\leq n^{-j} C(j) \{nE[|\Delta^{(SS_1)}_{b,x/b+c,1}|^j] + (nE[\{\Delta^{(SS_1)}_{b,x/b+c,1}\}^2])^{j/2}\} \\ &\leq 2C(j)\{4n^{-2}b^{-4}\widetilde{L}^2_{c,\gamma,1}(b+x)^2 + n^{-1}b^{-1}C_0\widetilde{L}_\gamma M_{c,\gamma,0}\}^{(j-2)/2} V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)], \end{aligned} \tag{A10}$$

where the constant $C(j)$ depends only on j.

Proof Using Remark 1 (i) and Lemmas A.2 (i) and A.4 (i), we can see that, for $i = 1, \ldots, n$,

$$\begin{aligned} |\Delta_{b,x/b+c,i}| &\leq \frac{\widetilde{L}_\gamma}{b}, \quad |\Delta^{(SS_1)}_{b,x/b+c,i}| \leq 2\frac{\widetilde{L}_{c,\gamma,1}}{b^2}(b+x), \\ V[\Delta_{b,x/b+c,i}] &\leq \int_0^\infty \{K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)\}^2 f(x) ds \leq b^{-1}C_0\widetilde{L}_\gamma, \\ V[\Delta^{(SS_1)}_{b,x/b+c,i}] &\leq \int_0^\infty \{K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s)\}^2 f(x) ds \leq b^{-1}C_0\widetilde{L}_\gamma M_{c,\gamma,0}. \end{aligned}$$

Bennett's inequality and Rosenthal's inequality immediately yield the results. □

Remark A.1 (i). Remark 1 (i) and Lemma A.4 (i) yield

$$\widehat{f}^{(JF_1)}_{b,c,\gamma}(x) \leq \Big(\frac{\widetilde{L}_\gamma}{b} + \epsilon\Big) \exp\Big(\frac{\widetilde{L}_{c,\gamma}}{b\epsilon}\Big), \quad \widehat{f}^{(SS_1)}_{b,c,\gamma}(x) \leq \frac{\widetilde{L}_{c,\gamma,1}}{b^2}(b+x), \quad \mathcal{Q}(x) \leq 3\Big\{\frac{\widetilde{L}^2_{c,\gamma,1}}{b^4}(b+x)^2 + \frac{\widetilde{L}^2_\gamma}{b^2} + \epsilon^2\Big\}.$$

These bounds will be used in Appendix A3.

(ii). We do not yet realize, at present, whether or not $\sup_{\rho\geq c}\sup_{s\geq 0}\{bK^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)|H^{(A)}_{b,c,\gamma,\rho}(s)|\}$ is uniformly bounded for b, due to the complexity of the function $H^{(A)}_{b,c,\gamma,\rho}$, except for the case $c=\gamma=1$ (see Remark 3 (i)). Nonetheless, we managed to prove Lemma A.4 (i). Making use of the non-uniform bound $\widetilde{L}_{c,\gamma,1}b^{-2}(b+x)$, rather than $\widetilde{L}_\gamma b^{-1}$, is a technical reason why (A8) and (A10) are more cumbersome than (A7) and (A9), respectively. Note that they are asymptotically equivalent for an exceptional case $x=O(b)$.

Finally, we prepare the following lemma (a slight extension of Lemma A.4 in Igarashi and Kakizawa (2017)), which is crucial to ensure that $\int_{b^{-\tau_2}}^\infty MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]dx$ is indeed asymptotically negligible, with a suitable choice of $\tau_2\in(0,1)$ under Assumption A5$'$; this argument is the key to prove Theorem 4.

Lemma A.6 *Let $\gamma\neq 0$ and $c>0$. For any $\tau\in(0,1)$, $j\geq 0$, $k>0$, and sufficiently small $b>0$, we have*

$$\int_{b^{-\tau}}^\infty x^j K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)dx = O(b^{\tau(k+1)}s^{j+k+1}),\quad s>0.$$

Proof It is easy to see that

$$K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s) = \begin{cases} \dfrac{b\rho(\rho+1)}{s^2}|\gamma|G_{[s/\{b\beta_\gamma(\rho)\}]^\gamma|\gamma|/(\rho+1)}((\rho+1)/|\gamma|), & \gamma>0,\\ \dfrac{\rho+1}{s}|\gamma|G_{[s/\{b\beta_\gamma(\rho)\}]^\gamma|\gamma|/(\rho+1)}((\rho+1)/|\gamma|), & \gamma<0,\end{cases}$$

where, given $q>0$,

$$G_q(u)=\frac{(qu)^u e^{-qu}}{u\Gamma(u)}=\frac{e^{u(1-q+\log q)}}{u^{1-u}e^u\Gamma(u)}$$

is strictly decreasing for $u>0$ (see Theorem 3.2 (2) of Anderson et al. (1995)), and by the definition of (2),

$$\frac{1}{\{\beta_\gamma(\rho)\}^\gamma}=\frac{1}{\rho^\gamma}\left\{\frac{\Gamma((\rho+1)/|\gamma|)}{\Gamma(\rho/|\gamma|)}\right\}^{|\gamma|}.$$

Using Claim A.2 (we set $z=\rho/|\gamma|$ and $p=1/|\gamma|$), we have

$$\frac{\rho+1}{\rho^\gamma|\gamma|}(1-c_\gamma\rho^{-1})^{|\gamma|}\leq\frac{1}{\{\beta_\gamma(\rho)\}^\gamma}\leq\frac{\rho+1}{\rho^\gamma|\gamma|}\quad(\text{if } \rho>c_\gamma),$$

where $c_\gamma=(6+6|\gamma|+\gamma^2)/(12|\gamma|)$. We can see that, for sufficiently small $b>0$, if $\rho\geq b^{-(\tau+1)}$, then,

$$\begin{aligned}
&|\gamma|G_{[s/\{b\beta_\gamma(\rho)\}]^\gamma|\gamma|/(\rho+1)}((\rho+1)/|\gamma|)\\
&\leq|\gamma|G_{[s/\{b\beta_\gamma(\rho)\}]^\gamma|\gamma|/(\rho+1)}(b^{-(\tau+1)}/|\gamma|)\\
&\leq\left\{\left(\frac{s}{b\rho}\right)^\gamma\frac{b^{-(\tau+1)}}{|\gamma|}\right\}^{b^{-(\tau+1)}/|\gamma|}\frac{1}{\Gamma(b^{-(\tau+1)}/|\gamma|+1)}\exp\left\{-\left(\frac{s}{b\rho}\right)^\gamma(1-c_\gamma\rho^{-1})^{|\gamma|}\frac{b^{-(\tau+1)}}{|\gamma|}\right\}\\
&\leq s^{-\gamma}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}}K^{(A)}_{b^{-(\tau+1)}/|\gamma|+1,s^{-\gamma},1}((b\rho)^{-\gamma}(1-c_\gamma b^{\tau+1})^{|\gamma|}b^{-(\tau+1)}/|\gamma|).
\end{aligned}$$

It follows that, if $\gamma > 0$, then, for sufficiently small $b > 0$,

$$
\begin{aligned}
&\int_{b^{-\tau}}^{\infty} x^j K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)dx \\
&\le b^{-1}s^{-2-\gamma}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}}(1+b^{\tau+1}) \\
&\quad \times \int_{b^{-\tau}}^{\infty} (x+bc)^{j+2} K^{(A)}_{b^{-(\tau+1)}/|\gamma|+1, s^{-\gamma},1}((x+bc)^{-\gamma}(1-c_\gamma b^{\tau+1})^{|\gamma|}b^{-(\tau+1)}/\gamma)dx \\
&\le b^{-(j+3)(\tau+1)/\gamma-1}s^{-2-\gamma}\gamma^{-(j+3)/\gamma-1}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}+j+3}(1+b^{\tau+1}) \\
&\quad \times \int_0^{(b^{-\tau}+bc)^{-\gamma}(1-c_\gamma b^{\tau+1})^\gamma b^{-(\tau+1)}/\gamma} y^{-(j+3)/\gamma-1} K^{(A)}_{b^{-(\tau+1)}/\gamma+1, s^{-\gamma},1}(y)dy \\
&\le b^{-(j+k+3)(\tau+1)/\gamma+k\tau-1}s^{-2-\gamma}\gamma^{-(j+k+3)/\gamma-1}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}+j+k+3}(1+b^{\tau+1}) \\
&\quad \times E[(\xi_{b^{-(\tau+1)}/\gamma+1, s^{-\gamma},1})^{-(j+k+3)/\gamma-1}] \\
&= b^{-(j+k+3)(\tau+1)/\gamma+k\tau-1}s^{j+k+1}\gamma^{-(j+k+3)/\gamma-1}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}+j+k+3}(1+b^{\tau+1}) \\
&\quad \times \frac{\Gamma(b^{-(\tau+1)}/\gamma-(j+k+3)/\gamma)}{\Gamma(b^{-(\tau+1)}/\gamma+1)} \\
&\le b^{\tau(k+1)}s^{j+k+1}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}+j+k+3}(1+b^{\tau+1})\left\{1-\frac{(j+k+3)c_{\gamma/(j+k+3)}b^{\tau+1}}{1-(j+k+3)b^{\tau+1}}\right\}^{-1},
\end{aligned}
$$

where we used Claim A.2 with $z = b^{-(\tau+1)}/\gamma - (j+k+3)/\gamma$ and $p = (j+k+3)/\gamma$ to get the last inequality.

Similarly, if $\gamma < 0$, then, for sufficiently small $b > 0$,

$$
\begin{aligned}
&\int_{b^{-\tau}}^{\infty} x^j K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)dx \\
&\le b^{-1}s^{-1-\gamma}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}}(1+b^{\tau+1}) \\
&\quad \times \int_{b^{-\tau}}^{\infty} (x+bc)^{j+1} K^{(A)}_{b^{-(\tau+1)}/|\gamma|+1, s^{-\gamma},1}((x+bc)^{-\gamma}(1-c_\gamma b^{\tau+1})^{|\gamma|}b^{-(\tau+1)}/|\gamma|)dx \\
&\le b^{(j+2)(\tau+1)/|\gamma|-1}s^{-1-\gamma}|\gamma|^{(j+2)/|\gamma|-1}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}-(j+2)}(1+b^{\tau+1}) \\
&\quad \times \int_{(b^{-\tau}+bc)^{|\gamma|}(1-c_\gamma b^{\tau+1})^{|\gamma|}b^{-(\tau+1)}/|\gamma|}^{\infty} y^{(j+2)/|\gamma|-1} K^{(A)}_{b^{-(\tau+1)}/|\gamma|+1, s^{-\gamma},1}(y)dy \\
&\le b^{(j+k+2)(\tau+1)/|\gamma|+k\tau-1}s^{-1-\gamma}|\gamma|^{(j+k+2)/|\gamma|-1}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}-(j+k+2)}(1+b^{\tau+1}) \\
&\quad \times E[(\xi_{b^{-(\tau+1)}/|\gamma|+1, s^{-\gamma},1})^{(j+k+2)/|\gamma|-1}] \\
&= b^{(j+k+2)(\tau+1)/|\gamma|+k\tau-1}s^{j+k+q+1}|\gamma|^{(j+k+2)/|\gamma|-1}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}-(j+k+2)}(1+b^{\tau+1}) \\
&\quad \times \frac{\Gamma(b^{-(\tau+1)}/|\gamma|+(j+k+2)/|\gamma|)}{\Gamma(b^{-(\tau+1)}/|\gamma|+1)} \\
&\le b^{\tau(k+1)}s^{j+k+1}(1-c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}-(j+k+2)}(1+b^{\tau+1})\{1+(j+k+2)b^{\tau+1}\}^{(j+k+2)/|\gamma|},
\end{aligned}
$$

where we used Claim A.2 with $z = b^{-(\tau+1)}/|\gamma|$ and $p = (j+k+2)/|\gamma|$ to get the last inequality. □

A2 Proofs of Theorems 1–4 and Remark 2

Proof of Theorem 1 Using Lemma A.1 (i), we have, for $x/b \to \infty$,

$$
\begin{aligned}
&E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] \\
&= \int_0^{\infty} K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s) f(s) ds
\end{aligned}
$$

$$= f(x) + \sum_{j=2}^{4} \frac{1}{j!} f^{(j)}(x) E[(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma} - x)^j H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})]$$
$$+ \frac{1}{6} \int_0^\infty (s-x)^4 \int_0^1 \{f^{(4)}(x+\theta(s-x)) - f^{(4)}(x)\}(1-\theta)^3 d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s) ds,$$

where

$$\left| \int_0^\infty (s-x)^4 \int_0^1 \{f^{(4)}(x+\theta(s-x)) - f^{(4)}(x)\}(1-\theta)^3 d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s) ds \right|$$
$$\leq \frac{L_4}{4} E[|\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma} - x|^{4+\eta_4} |H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})|]$$
$$\leq \frac{L_4}{4} \big\{ E[(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma} - x)^6] \big\}^{(2+\eta_4)/6}$$
$$\times \big\{ E[(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma} - x)^4 \{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})\}^2] \big\}^{1/2}$$
$$= O((bx)^{2+\eta_4/2}) \quad \text{(we used Claim A.1 (ii) and Lemma A.1 (ii))}.$$

Similarly, we have, for $x/b \to \kappa$,

$$E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = f(x) + \frac{1}{2} f''(x) E[(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma} - x)^2 H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})]$$
$$+ \frac{1}{2} \int_0^\infty (s-x)^3 \int_0^1 f^{(3)}(x+\theta(s-x))(1-\theta)^2 d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s) ds,$$

where

$$\left| \int_0^\infty (s-x)^3 \int_0^1 f^{(3)}(x+\theta(s-x))(1-\theta)^2 d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s) ds \right|$$
$$\leq \frac{C_3}{3} E[|\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma} - x|^3 |H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})|]$$
$$\leq \frac{C_3}{3} \big[E[|\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma} - x|^3] E[|\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma} - x|^3 \{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})\}^2] \big]^{1/2}$$
$$\leq \frac{4C_3}{3} (M_{c,\gamma,0} + M_{c,\gamma,3})^{1/2} \Big\{ (x+bc)^3 \frac{\Gamma^2(\alpha_\gamma(x/b+c))\Gamma(\alpha_\gamma(x/b+c)+3/\gamma)}{\Gamma^3(\alpha_\gamma(x/b+c)+1/\gamma)} + x^3 \Big\}$$
$$\leq \frac{4C_3}{3} (M_{c,\gamma,0} + M_{c,\gamma,3})^{1/2} \Big\{ (x+bc)^3 \frac{\Gamma^2(\alpha_\gamma(c))\Gamma(\alpha_\gamma(c)+3/\gamma)}{\Gamma^3(\alpha_\gamma(c)+1/\gamma)} + x^3 \Big\}$$
$$= O(b^3),$$

using Lemma A.2 (i) and noting that, given $p > 0$, $\Gamma^2(z)\Gamma(z+3p)/\Gamma^3(z+p)$ and $\Gamma(z)\Gamma^2(z+3p)/\Gamma^3(z+2p)$ are strictly decreasing for $z > 0$ (see Theorem 10 of Alzer (1997)). The bias follows from Lemma A.1 (ii); note that $b^3(1+x) \leq b^{1-\eta_4/2}\{b(1+x)\}^{2+\eta_4/2} = o(\{b(1+x)\}^{2+\eta_4/2})$.

On the other hand, we can see that

$$V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = n^{-1} \int_0^\infty \{K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s)\}^2 f(s) ds + O(n^{-1})$$
$$= n^{-1} b^{-1} |\gamma| v_\gamma(x/b+c) \int_0^\infty K^{(A)}_{2\alpha_\gamma(x/b+c)-1/\gamma, b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma}(s) \{H^{(A)}_{b,c,\gamma,x/b+c}(s)\}^2 f(s) ds + O(n^{-1}).$$

The variance follows from Claim A.1 (iii) and Lemma A.3, since

$$\begin{aligned}
&\left|\int_0^\infty (s-x)\int_0^1 f'(x+\theta(s-x))d\theta K^{(A)}_{2\alpha_\gamma(x/b+c)-1/\gamma, b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma}(s)\{H^{(A)}_{b,c,\gamma,x/b+c}(s)\}^2 ds\right| \\
&\le C_1 E[|\xi_{2\alpha_\gamma(x/b+c)-1/\gamma, b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma}-x|\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{2\alpha_\gamma(x/b+c)-1/\gamma, b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma})\}^2] \\
&\le C_1\{E[\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{2\alpha_\gamma(x/b+c)-1/\gamma, b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma})\}^2] \\
&\quad \times E[(\xi_{2\alpha_\gamma(x/b+c)-1/\gamma, b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma}-x)^2\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{2\alpha_\gamma(x/b+c)-1/\gamma, b\beta_\gamma(x/b+c)/2^{1/\gamma},\gamma})\}^2]\}^{1/2} \\
&= \begin{cases} O((bx)^{1/2}), & \dfrac{x}{b}\to\infty, \\ O(b), & \dfrac{x}{b}\to\kappa. \quad \square \end{cases}
\end{aligned}$$

Proof of Remark 2 (i). Remark 1 (i) and Lemma A.2 (i) yield

$$V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] \le n^{-1}\int_0^\infty \{K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)\}^2 f(s)ds \le n^{-1}b^{-1}C_0\widetilde{L}_\gamma M_{c,\gamma,0}.$$

On the other hand, as in the proof of Theorem 1, we have different expressions

$$E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = f(x) + \int_0^\infty (s-x)^2\int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)ds$$

and

$$E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = f(x) + \int_0^\infty (s-x)\int_0^1 f'(x+\theta(s-x))d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)ds.$$

Then, for $x/b\to\infty$,

$$\begin{aligned}
&\left|\int_0^\infty (s-x)^2\int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)ds\right| \\
&\le \frac{C_2}{2}E[(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}-x)^2|H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})|] \\
&\le \frac{C_2}{2}\{E[(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}-x)^2]E[(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}-x)^2\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})\}^2]\}^{1/2} \\
&= O(bx) \quad \text{(we used Claim A.1 (ii) and Lemma A.1 (ii))},
\end{aligned}$$

and, for $x/b\to\kappa$,

$$\begin{aligned}
&\left|\int_0^\infty (s-x)\int_0^1 f'(x+\theta(s-x))d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)ds\right| \\
&\le C_1 E[|\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}-x||H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})|] \\
&\le C_1\left[E[|\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}-x|]E[|\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}-x|\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})\}^2]\right]^{1/2} \\
&\le C_1(M_{c,\gamma,0}+M_{c,\gamma,1})^{1/2}(E[\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}]+x) \quad \text{(we used Lemma A.2 (i))} \\
&= C_1(M_{c,\gamma,0}+M_{c,\gamma,1})^{1/2}(2x+bc) \\
&= O(b).
\end{aligned}$$

(ii). Similarly, we have, for $x/b\to\infty$,

$$\begin{aligned}
E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] &= f(x) + \sum_{j=2}^3 \frac{1}{j!}f^{(j)}(x)E[(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}-x)^j H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma})] \\
&\quad + \frac{1}{6}\int_0^\infty (s-x)^4\int_0^1 f^{(4)}(x+\theta(s-x))(1-\theta)^3 d\theta K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)ds,
\end{aligned}$$

where

$$\left|\int_0^\infty (s-x)^4 \int_0^1 f^{(4)}(x+\theta(s-x))(1-\theta)^3 d\theta K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s)ds\right|$$
$$\leq \frac{C_4}{4} E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^4 |H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})|]$$
$$\leq \frac{C_4}{4}\big\{E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^4]E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^4\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})\}^2]\big\}^{1/2}$$
$$= O(b^2x^2) \quad \text{(we used Claim A.1 (ii) and Lemma A.1 (ii))},$$

and, for $x/b \to \kappa$,

$$E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = f(x) + \int_0^\infty (s-x)^2 \int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s)ds,$$

where

$$\left|\int_0^\infty (s-x)^2 \int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s)ds\right|$$
$$\leq \frac{C_2}{2} E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^2 |H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})|]$$
$$\leq \frac{C_2}{2}\big[E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^2]E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^2\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})\}^2]\big]^{1/2}$$
$$\leq C_2(M_{c,\gamma,0}+M_{c,\gamma,2})^{1/2}\Big\{(x+bc)^2\frac{\Gamma(\alpha_\gamma(x/b+c))\Gamma(\alpha_\gamma(x/b+c)+2/\gamma)}{\Gamma^2(\alpha_\gamma(x/b+c)+1/\gamma)} + x^2\Big\}$$
$$\leq C_2(M_{c,\gamma,0}+M_{c,\gamma,2})^{1/2}\Big\{(x+bc)^2\frac{\Gamma(\alpha_\gamma(c))\Gamma(\alpha_\gamma(c)+2/\gamma)}{\Gamma^2(\alpha_\gamma(c)+1/\gamma)} + x^2\Big\}$$
$$= O(b^2),$$

using Lemma A.2 (i) and noting that, given $p > 0$, $\Gamma(z)\Gamma(z+2p)/\Gamma^2(z+p)$ is strictly decreasing for $z > 0$ (see Theorem 10 of Alzer (1997)). Also, we obtain

$$\sup_{x\in[0,b^\tau]}\left|\int_0^\infty (s-x)^2 \int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s) H^{(A)}_{b,c,\gamma,x/b+c}(s)ds\right| = O(b^{2\tau}). \quad \square$$

Proof of Theorem 2 (A8) and the Borel-Cantelli lemma immediately yield $\overline{\Delta}^{(SS_1)}_{b,x/b+c} \overset{a.s.}{\to} 0$, if $nb^2/\log n \to \infty$. This, together with (12), yields $\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) \overset{a.s.}{\to} f(x)$ for fixed $x \geq 0$. $\square$

Proof of Theorem 3 Recall $\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) - E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \sum_{i=1}^n n^{-1}\Delta^{(SS_1)}_{b,x/b+c,i}$. Using Lemma A.2 (ii), we have, for any $\delta \in (0,2]$,

$$\begin{aligned}E[K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(X_1)|H^{(A)}_{b,c,\gamma,x/b+c}(X_1)|^{2+\delta}] &= \int_0^\infty K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)|H^{(A)}_{b,c,\gamma,x/b+c}(s)|^{2+\delta}f(s)ds\\ &\leq C_0 E[|H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})|^{2+\delta}]\\ &\leq C_0\big\{E[\{H^{(A)}_{b,c,\gamma,x/b+c}(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma})\}^4]\big\}^{(2+\delta)/4}\\ &\leq C_0 M^{(2+\delta)/4}_{c,\gamma}.\end{aligned}$$

This, together with Remark 1 (i) and (ii), yields

$$
\begin{aligned}
E[|\Delta^{(SS_1)}_{b,x/b+c,i}|^{2+\delta}] &\le 2^{2+\delta}E[|K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(X_i)H^{(A)}_{b,c,\gamma,x/b+c}(X_i)|^{2+\delta}] \\
&\le \begin{cases} 2^{2+\delta}\Big(\dfrac{|\gamma|^{1/2}\widetilde{L}_\gamma}{b^{1/2}\sqrt{2\pi x}}\Big)^{1+\delta}C_0M_{c,\gamma}^{(2+\delta)/4} & \text{for fixed } x>0, \\ 2^{2+\delta}\Big(\dfrac{\widetilde{L}_\gamma}{b}\Big)^{1+\delta}C_0M_{c,\gamma}^{(2+\delta)/4} & \text{for } x=0. \end{cases}
\end{aligned}
$$

Using Theorem 1, i.e.,

$$
\lim_{n\to\infty} nb^{1/2}V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \frac{27}{16}|\gamma|^{1/2}V(x) \text{ for fixed } x>0, \quad \lim_{n\to\infty} nbV[\widehat{f}^{(SS_1)}_{b,c,\gamma}(0)] = |\gamma|v^{(SS_1)}_{c,\gamma}(0)f(0) \qquad \text{(A11)}
$$

(for fixed $x>0$, $b\to 0$ and $nb^{1/2}\to\infty$ are sufficient in Assumption A2), it follows that, for fixed $x>0$,

$$
\frac{\sum_{i=1}^n E\big[\big|n^{-1}\Delta^{(SS_1)}_{b,x/b+c,i}\big|^{2+\delta}\big]}{\{V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\}^{1+\delta/2}} \le \frac{2^{2+\delta}C_0M_{c,\gamma}^{(2+\delta)/4}}{(nb^{1/2})^{\delta/2}\{nb^{1/2}V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\}^{1+\delta/2}}\Big(\frac{|\gamma|^{1/2}\widetilde{L}_\gamma}{\sqrt{2\pi x}}\Big)^{1+\delta} = O((nb^{1/2})^{-\delta/2}) = o(1),
$$

and

$$
\frac{\sum_{i=1}^n E\big[\big|n^{-1}\Delta^{(SS_1)}_{b,c,i}\big|^{2+\delta}\big]}{\{V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(0)]\}^{1+\delta/2}} \le \frac{2^{2+\delta}\widetilde{L}_\gamma^{1+\delta}C_0M_{c,\gamma}^{(2+\delta)/4}}{(nb)^{\delta/2}\{nbV[\widehat{f}^{(SS_1)}_{b,c,\gamma}(0)]\}^{1+\delta/2}} = O((nb)^{-\delta/2}) = o(1).
$$

Therefore, Lyapunov's central limit theorem enables us to see that

$$
\frac{\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) - E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]}{\{V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\}^{1/2}} \xrightarrow{d} N(0,1).
$$

The result follows from (A11) and Slutsky's lemma. □

Proof of Theorem 3′ Use Theorems 1 and 3 and Slutsky's lemma to get the result. □

Proof of Theorem 4 We have

$$
MISE[\widehat{f}^{(SS_1)}_{b,c,\gamma}] = \Big(\int_0^{b^{\tau_1}} + \int_{b^{\tau_1}}^{b^{-\tau_2}} + \int_{b^{-\tau_2}}^\infty\Big) MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]dx,
$$

where $\tau_1 \in (4/5, 1)$, $\tau_2 \in (4/(k_4+1), \eta_4/(\eta_4+5)) \subset (5/\{2(k'+1)\}, 1/2)$ for some $k_4 > (3\eta_4+20)/\eta_4$ and $k' = k_4 - 2$ (see Assumption A5′). Using (13) and (15), we can see that

$$
\int_0^{b^{\tau_1}} MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]dx = o(b^4 + n^{-1}b^{-1/2}).
$$

Lemmas A.2 (i), A.4 (i), and A.6 yield

$$
\begin{aligned}
&\int_{b^{-\tau_2}}^\infty MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]dx \\
&\le 2\int_{b^{-\tau_2}}^\infty\Big[\Big\{\int_0^\infty K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)f(s)ds\Big\}^2 + f^2(x)\Big]dx \\
&\quad + n^{-1}\int_{b^{-\tau_2}}^\infty\int_0^\infty\{K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)H^{(A)}_{b,c,\gamma,x/b+c}(s)\}^2f(s)dsdx \\
&\le 2C_0\Big\{M_{c,\gamma,0}\int_0^\infty\int_{b^{-\tau_2}}^\infty K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)dxf(s)ds + b^{\tau_2(k_4+1)}\int_{b^{-\tau_2}}^\infty x^{k_4+1}f(x)dx\Big\} \\
&\quad + n^{-1}b^{-3}\widetilde{L}_{c,\gamma,2}\int_0^\infty\int_{b^{-\tau_2}}^\infty(b^2+x^2)K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)dxf(s)ds \\
&= O(b^{\tau_2(k_4+1)} + n^{-1}b^{\tau_2(k'+1)-3}) \\
&= o(b^4 + n^{-1}b^{-1/2}).
\end{aligned}
$$

Also, in view of Theorem 1 (with $x \geq b^{\tau_1}$), we have

$$\Big|V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] - n^{-1}b^{-1/2}\frac{27}{16}|\gamma|^{1/2}V(x)\Big| = o(n^{-1}b^{-1/2}V(x)) + O(n^{-1}),$$
$$\left|\{Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\}^2 - b^4\Big\{\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2\right| \leq 2b^2\frac{|B^{[2]}_{c,\gamma}(x)|}{\gamma^2}|\mathcal{E}^{(SS_1)}_{b,c,\gamma}(x)| + \{\mathcal{E}^{(SS_1)}_{b,c,\gamma}(x)\}^2,$$

where $\{\mathcal{E}^{(SS_1)}_{b,c,\gamma}(x)\}^2 = O(b^6x^{-2} + \{b(1+x)\}^{4+\eta_4})$. It follows that

$$\begin{aligned}
&\Big|\int_{b^{\tau_1}}^{b^{-\tau_2}} V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]dx - n^{-1}b^{-1/2}\frac{27}{16}\int_0^\infty |\gamma|^{1/2}V(x)dx\Big| \\
&\leq o(n^{-1}b^{-1/2}) + O(n^{-1}b^{-\tau_2}) + n^{-1}b^{-1/2}\frac{27}{16}\Big(\int_0^{b^{\tau_1}} + \int_{b^{-\tau_2}}^\infty\Big)|\gamma|^{1/2}V(x)dx \\
&= o(n^{-1}b^{-1/2})
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_{b^{\tau_1}}^{b^{-\tau_2}} \{Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\}^2dx - b^4\int_0^\infty\Big\{\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2dx\Big| \\
&\leq 2b^2\Big[\int_{b^{\tau_1}}^{b^{-\tau_2}}\Big\{\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2dx\int_{b^{\tau_1}}^{b^{-\tau_2}}\{\mathcal{E}^{(SS_1)}_{b,c,\gamma}(x)\}^2dx\Big]^{1/2} + \int_{b^{\tau_1}}^{b^{-\tau_2}}\{\mathcal{E}^{(SS_1)}_{b,c,\gamma}(x)\}^2dx + b^4\Big(\int_0^{b^{\tau_1}} + \int_{b^{-\tau_2}}^\infty\Big)\Big\{\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2dx \\
&= o(b^4),
\end{aligned}$$

since $\int_{b^{\tau_1}}^{b^{-\tau_2}}\{\mathcal{E}^{(SS_1)}_{b,c,\gamma}(x)\}^2dx = O(b^{6-\tau_1} + b^{4+\eta_4-\tau_2(5+\eta_4)}) = o(b^4)$. □

A3 Proofs of Theorems 6–9

Assuming $f(x) > 0$, we recall the stochastic expansion (18), from which we have

$$E[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] = E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] + \frac{E[\mathcal{Q}(x)]}{2f(x)} + E[\mathcal{R}(x)], \tag{A12}$$

and, using $V[\mathcal{Q}(x)/\{2f(x)\} + \mathcal{R}(x)] \leq 2\{V[\mathcal{Q}(x)]/\{4f^2(x)\} + E[\mathcal{R}^2(x)]\} = 2\mathcal{J}(x)$ (say),

$$\big|V[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] - V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\big| \leq 2\mathcal{J}(x) + 2\big\{2\mathcal{J}(x)V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\big\}^{1/2}. \tag{A13}$$

Before proving Theorems 6–9, we prepare the following lemma.

Lemma A.7 *Given $\gamma \neq 0$, choose $c > 1$. Suppose that Assumptions* A1 *and* A3′ *hold, and that $b \propto n^{-\iota_1}$ and $\epsilon \propto b^{\iota_2}$ for some $\iota_1 \in (0, 1/3]$ and $\iota_2 > 1$. Let $j \geq 2/3$.*

(i). *Define $\mathcal{I}_{\iota_0}[r_b] = \{x \in [0, r_b] \mid f(x) \geq \varrho b^{\iota_0}$ and $b^{1-\iota_0}r_b = o(1)\}$ for some $r_b \equiv r$ or $r_b \to \infty$ according to $\iota_0 = 0$ or $\iota_0 \in (0,1)$. We have, on $\mathcal{I}_{\iota_0}[r_b]$,*

$$E\Big[\frac{\mathcal{Q}(x)}{f(x)}\Big] = \begin{cases} b^2\dfrac{B^2_{c|\gamma|}(x)}{\gamma^2f(x)} + O(b^{-\iota_0}[b^3(1+x)^3 + b^{1+\iota_2}(1+x) + n^{-1}\{b^{-1/2}V(x)+1\}]), & \dfrac{x}{b} \to \infty, \\ b^2\dfrac{c^2\{f'(0)\}^2}{f(0)} + O(b^{-\iota_0}(b^{3-\iota_0} + b^{1+\iota_2} + n^{-1}b^{-1})), & \dfrac{x}{b} \to \kappa\ (x \neq 0), \\ b^2\dfrac{c^2\{f'(0)\}^2}{f(0)} + O(b^{-\iota_0}(b^3 + b^{1+\iota_2} + n^{-1}b^{-1})), & x = 0, \end{cases}$$

$$V\Big[\frac{\mathcal{Q}(x)}{f(x)}\Big] = \begin{cases} O(\{b^2(1+x)^2 + n^{-1}b^{-1}\}n^{-1}b^{-2\iota_0}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b} \to \infty, \\ O((b^2 + n^{-1}b^{-1})n^{-1}b^{-(1+2\iota_0)}), & \dfrac{x}{b} \to \kappa, \end{cases}$$

and, assuming $n^{-1}b^{-(3+\iota_0+\iota_2)}r_b = o(1)$,

$$E[|\mathcal{R}(x)|^j] = \begin{cases} O(b^{(3-2\iota_0)j}(1+x)^{3j} + (n^{-1}b^{-1})^{3j/2}b^{1-2\iota_0 j}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b} \to \infty, \\ O(b^{(3-2\iota_0)j} + (n^{-1}b^{-1})^{3j/2}b^{-2\iota_0 j}), & \dfrac{x}{b} \to \kappa. \end{cases}$$

(ii). *Suppose that* $f(0) > 0$ (*in this case, due to the continuity, there exists a* $\delta > 0$ *such that* $x \in [0, \delta]$ *implies* $f(x) > f(0)/2$ (*say*)). *For any* $\tau \in [1/2, 1)$, *we have*

$$\sup_{x\in[0,b^\tau]} E\Big[\frac{\mathcal{Q}(x)}{f(x)}\Big] = O(b^2 + n^{-1}b^{-1}), \qquad \sup_{x\in[0,b^\tau]} V\Big[\frac{\mathcal{Q}(x)}{f(x)}\Big] = O((b^2 + n^{-1}b^{-1})n^{-1}b^{-1}),$$

and, assuming $n^{-1}b^{-(3+\iota_2)+\tau} = o(1)$,

$$\sup_{x\in[0,b^\tau]} E[|\mathcal{R}(x)|^j] = O(b^{3j} + (n^{-1}b^{-1})^{3j/2}).$$

Proof of Lemma A.7 is postponed to Appendix C.

In proving Theorems 6, 8, and 9, the following observations are useful: First, under the same conditions in Lemma A.7,

- we have, on $\mathcal{I}_{\iota_0}[r_b]$,

$$\mathcal{J}(x) = \begin{cases} O(b^{6-4\iota_0}(1+x)^6 + \{b^{2(1-\iota_0)}(1+x)^2 + n^{-1}b^{-(1+2\iota_0)}\}n^{-1}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b} \to \infty, \\ O(b^{6-4\iota_0} + (b^{2(1-\iota_0)} + n^{-1}b^{-(1+2\iota_0)})n^{-1}b^{-1}), & \dfrac{x}{b} \to \kappa \end{cases} \tag{A14}$$

  (see Lemma A.7 (i)),

- for any $\tau \in [1/2, 1)$ (we assume $f(0) > 0$; in this case, due to the continuity, there exists a $\delta > 0$ such that $x \in [0, \delta]$ implies $f(x) > f(0)/2$ (say)), we have

$$\begin{aligned} \sup_{x\in[0,b^\tau]} MSE[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] &\le 3 \sup_{x\in[0,b^\tau]} \Big[MSE[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] + \frac{\{E[\mathcal{Q}(x)]\}^2}{4f^2(x)} + \mathcal{J}(x)\Big] \\ &= O(b^{4\tau} + n^{-1}b^{-1}) \quad \text{if } n^{-1}b^{-(3+\iota_2)+\tau} = o(1) \end{aligned} \tag{A15}$$

  (we used (13), (15), and Lemma A.7 (ii)).

Second, $b^q r_b = O(1)$ for some $q \in [0, \eta_4/(4+\eta_4))$, where $\eta_4 \in (0, 1]$ is given in Assumption A3$'$, implies that

$$\begin{aligned} b^{1-\iota_0}r_b &= O(b^{1-p_0}) = o(1) \quad (\text{hence, } \mathcal{I}_{q,\iota_0}[r_b] \subset \mathcal{I}_{\iota_0}[r_b]), \\ n^{-1}b^{-(3+\iota_0+\iota_2)}r_b &= O(n^{-1}b^{-(3+p_0+\iota_2)}) = o(1), \\ \omega_{b,\iota_0}(r_b) + \widetilde{\omega}_{b,\iota_0}(r_b) &= O(b^{\eta_4/2-q(2+\eta_4/2)} + b^{1-(3q+2\iota_0)} + b^{\iota_2-(1+p_0)} + n^{-1/2}b^{-(1/2+\iota_0)}) = o(1) \end{aligned}$$

for $0 \le \iota_0 < 1/4 - q$ (hence, $0 \le \iota_0 < (1-3q)/2 < 1-q$), $0 < \iota_1 < 1/\{2(2+p_0)\}$ (hence, $0 < \iota_1 < 1/(1+2\iota_0)$), and $1 + p_0 < \iota_2 < \iota_1^{-1} - 3 - p_0$ (we write $p_0 = q + \iota_0$); note that $n^{-1}b^{-(1+2\iota_0)} = o(1)$ and, for any $\tau \in [1/2, 1)$, $n^{-1}b^{-(3+\iota_2)+\tau} = o(1)$.

Proof of Theorem 6 Let $x \in \mathcal{I}_{q,\iota_0}[r_b] \subset \mathcal{I}_{\iota_0}[r_b]$. Recall (A12) and (A13). The bias follows from Theorem 1, Lemma A.7 (i), and $n^{-1}b^{-(1+2\iota_0)} = o(1)$. The variance follows from (A14), Theorem 1, and $\widetilde{\omega}_{b,\iota_0}(r_b) = o(1)$, since

$$\big\{\mathcal{J}(x)V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\big\}^{1/2} = \begin{cases} O(b^{5-2\iota_0}(1+x)^3 + n^{-1}\widetilde{\omega}_{b,\iota_0}(x)\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b} \to \infty, \\ O(b^{5-2\iota_0} + n^{-1}b^{-1}(b^{1-2\iota_0} + n^{-1/2}b^{-(1/2+\iota_0)})), & \dfrac{x}{b} \to \kappa \end{cases}$$

(we also used the fact that, on $\mathcal{I}_{q,\iota_0}[r_b]$, $b^{1-2\iota_0}(1+x)^3 \le b^{1-2\iota_0}(1+r_b)^3 = o(1)$ when $x/b \to \infty$). □

Proof of Theorem 7 Use Remark 1, Theorem 2, and Slutsky's lemma to get the result. □

Proof of Theorem 8 Suppose that $\iota_1 \in (2/13, 1/4)$ or $\iota_1 \in (1/7, 1/4)$ according to $x \in \mathcal{I}_{0,0}[r]\backslash\{0\} \subset \mathcal{I}_0[r]\backslash\{0\}$ or $x = 0$. Then, (A14) (we set $q = \iota_0 = 0$) yields

$$nb^{1/2}\mathcal{J}(x) = O(nb^{13/2} + b^2 + n^{-1}b^{-1}) = o(1) \quad \text{for fixed } x \in \mathcal{I}_{0,0}[r]\backslash\{0\},$$
$$nb\mathcal{J}(0) = O(nb^7 + b^2 + n^{-1}b^{-1}) = o(1) \quad \text{(we suppose } f(0) > 0).$$

It follows from (18) that

$$(nb^{1/2})^{1/2}\{\widehat{f}^{(JF_1)}_{b,c,\gamma}(x) - E[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)]\} = (nb^{1/2})^{1/2}\{\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) - E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\} + o_p(1) \quad \text{for fixed } x \in \mathcal{I}_{0,0}[r]\backslash\{0\},$$
$$(nb)^{1/2}\{\widehat{f}^{(JF_1)}_{b,c,\gamma}(0) - E[\widehat{f}^{(JF_1)}_{b,c,\gamma}(0)]\} = (nb)^{1/2}\{\widehat{f}^{(SS_1)}_{b,c,\gamma}(0) - E[\widehat{f}^{(SS_1)}_{b,c,\gamma}(0)]\} + o_p(1).$$

The result is a consequence of Theorem 3. □

Proof of Theorem 8′ Use Theorems 6 and 8 and Slutsky's lemma to get the result. □

Proof of Theorem 9 Choosing $\tau_1 \in (4/5, 1)$, we have

$$\big|MISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}] - AMISE_w[\widehat{f}^{(JF_1)}_{b,c,\gamma}]\big| \le \sum_{j=1}^{4} I_j,$$

where

$$I_1 = \overline{w}\int_0^{b^{\tau_1}} MSE[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)]dx, \quad I_2 = \int_{r_b}^{\infty} w(x)MSE[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)]dx,$$
$$I_3 = b^4\Big(\int_0^{b^{\tau_1}} + \int_{r_b}^{\infty}\Big)w(x)\Big\{\frac{B^{(JF_1)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2dx + n^{-1}b^{-1/2}\frac{27}{16}\overline{w}\Big(\int_0^{b^{\tau_1}} + \int_{r_b}^{\infty}\Big)|\gamma|^{1/2}V(x)dx,$$
$$I_4 = \int_{b^{\tau_1}}^{r_b} w(x)\Big|MSE[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] - b^4\Big\{\frac{B^{(JF_1)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 - n^{-1}b^{-1/2}\frac{27}{16}|\gamma|^{1/2}V(x)\Big|dx.$$

We can see that $I_1 = o(b^4 + n^{-1}b^{-1/2})$, $I_2 \le 2\big\{\big(b^{-1}\widetilde{L}_\gamma + \epsilon\big)^2\exp(2b^{-1}\epsilon^{-1}\widetilde{L}_{c,\gamma}) + C_0^2\big\}\int_{r_b}^{\infty} w(x)dx = o(b^4)$, and $I_3 = o(b^4 + n^{-1}b^{-1/2})$; use (A15) and Remark A.1 (i) for I_1 and I_2, respectively. Also, noting that $n^{-1}b^{-(1+2\iota_0)} = o(1)$ and $\omega_{b,\iota_0}(r_b) + \widetilde{\omega}_{b,\iota_0}(r_b) = o(1)$, Theorem 6 (with $x \in [b^{\tau_1}, r_b]$) yields

$$w(x)\Big|V[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)] - n^{-1}b^{-1/2}\frac{27}{16}|\gamma|^{1/2}V(x)\Big| = o(b^4w(x) + n^{-1}b^{-1/2}V(x)) + O(n^{-1}w(x)),$$
$$w(x)\Big|\{Bias[\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)]\}^2 - b^4\Big\{\frac{B^{(JF_1)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2\Big| \le w(x)\Big[2b^2\frac{|B^{(JF_1)}_{c,\gamma}(x)|}{\gamma^2}|\mathcal{E}^{(JF_1)}_{b,c,\gamma}(x)| + \{\mathcal{E}^{(JF_1)}_{b,c,\gamma}(x)\}^2\Big],$$

where $\int_{b^{\tau_1}}^{r_b} w(x)\{\mathcal{E}_{b,c,\gamma}^{(JF_1)}(x)\}^2 dx = O(b^{6-\tau_1}) + o(b^4 + n^{-1}b^{-1/2}) = o(b^4 + n^{-1}b^{-1/2})$, since

$$w(x)\{\mathcal{E}_{b,c,\gamma}^{(JF_1)}(x)\}^2 = O(b^6 x^{-2}) + o((b^4 + n^{-1})w(x) + n^{-1}b^{-1/2}V(x)).$$

It follows that

$$\begin{aligned} I_4 &\leq 2b^2 \left[\int_{b^{\tau_1}}^{r_b} w(x) \Big\{ \frac{B_{c,\gamma}^{(JF_1)}(x)}{\gamma^2} \Big\}^2 dx \int_{b^{\tau_1}}^{r_b} w(x)\{\mathcal{E}_{b,c,\gamma}^{(JF_1)}(x)\}^2 dx \right]^{1/2} + \int_{b^{\tau_1}}^{r_b} w(x)\{\mathcal{E}_{b,c,\gamma}^{(JF_1)}(x)\}^2 dx \\ &\quad + \int_{b^{\tau_1}}^{r_b} w(x) \left| V[\widehat{f}_{b,c,\gamma}^{(JF_1)}(x)] - n^{-1}b^{-1/2}\frac{27}{16}|\gamma|^{1/2}V(x) \right| dx \\ &= o(b^4 + n^{-1}b^{-1/2}). \quad \square \end{aligned}$$

Appendix B: Proof of Lemma A.4

Before proving Lemma A.4, we prepare following lemmas (Lemmas B.1–B.3).

Lemma B.1 *Let $\gamma \neq 0$ and $c > 0$.*

(i). *We have*

$$\sup_{\rho \geq c} (\rho - c)|\psi(\alpha_\gamma(\rho) + 1/\gamma) - \psi(\alpha_\gamma(\rho))| \leq \lceil 1/|\gamma| \rceil |\gamma|,$$

where $\lceil y \rceil$ is the smallest integer greater than or equal to y, i.e., $y \leq \lceil y \rceil < y + 1$.

(ii). *There exists a constant $\widetilde{M}_{c,\gamma} > 0$ such that*

$$\sup_{\rho \geq c} \alpha_\gamma^2(\rho) \left| \psi(\alpha_\gamma(\rho) + 1/\gamma) - \psi(\alpha_\gamma(\rho)) - \frac{1}{\gamma \alpha_\gamma(\rho)} \right| \leq \widetilde{M}_{c,\gamma}.$$

Proof Let $\rho \geq c$.

(i). We know that $\log \Gamma(z)$ is convex (hence, $\psi(z)$ is strictly increasing for $z > 0$), and that $\psi(z+1) = \psi(z) + z^{-1}$, hence, for any $\Delta > 0$, $0 < \psi(z + \Delta) - \psi(z) \leq \psi(z + \lceil \Delta \rceil) - \psi(z) = \sum_{j=0}^{\lceil \Delta \rceil - 1} (z + j)^{-1} \leq \lceil \Delta \rceil / z$. Then, for $\gamma > 0$,

$$(\rho - c)|\psi(\alpha_\gamma(\rho) + 1/\gamma) - \psi(\alpha_\gamma(\rho))| = (\rho - c)\{\psi(\alpha_\gamma(\rho) + 1/\gamma) - \psi(\alpha_\gamma(\rho))\} \leq \rho \frac{\lceil 1/\gamma \rceil}{\rho/\gamma} = \lceil 1/\gamma \rceil \gamma,$$

and, for $\gamma < 0$,

$$(\rho - c)|\psi(\alpha_\gamma(\rho) + 1/\gamma) - \psi(\alpha_\gamma(\rho))| = (\rho - c)\{\psi(\alpha_\gamma(\rho) - 1/|\gamma| + 1/|\gamma|) - \psi(\alpha_\gamma(\rho) - 1/|\gamma|)\} \leq \rho \frac{\lceil 1/|\gamma| \rceil}{\rho/|\gamma|} = \lceil 1/|\gamma| \rceil |\gamma|.$$

(ii). It is easy to see that $\mathcal{G}(\rho) = \alpha_\gamma^2(\rho)\{\psi(\alpha_\gamma(\rho) + 1/\gamma) - \psi(\alpha_\gamma(\rho)) - 1/(\gamma\alpha_\gamma(\rho))\}$ is continuous on $[c, \infty)$, and that, using (A4), $\mathcal{G}(\rho) = (\gamma - 1)/(2\gamma^2) + O(\rho^{-1})$ as $\rho \to \infty$ (hence, $\sup_{\rho \geq c} |\mathcal{G}(\rho)|$ is bounded). $\square$

Lemma B.2 *Let $\gamma \neq 0$ and $c > 1$ (in this case, by definition (see (2)), $\alpha_\gamma(\rho) + j/\gamma > 0$, $j = -1, 0$). For $q > -(c-1)/|\gamma|$ (in this case, if $\rho \geq c$, then, $\alpha_\gamma(\rho) + q > \alpha_\gamma(\rho) + q - 1/\gamma \geq (c-1)/\gamma + q > 0$ for $\gamma > 0$, and $\alpha_\gamma(\rho) + q - 1/\gamma > \alpha_\gamma(\rho) + q > (c+1)/|\gamma| + q > 2/|\gamma|$ for $\gamma < 0$), we have*

$$\sup_{\rho \geq c} \frac{\{\alpha_\gamma(\rho) + q - 1/\gamma\}\Gamma(\alpha_\gamma(\rho) + q - 1/\gamma)\Gamma(\alpha_\gamma(\rho))}{\{\alpha_\gamma(\rho) - 1/\gamma\}\Gamma(\alpha_\gamma(\rho) + q)\Gamma(\alpha_\gamma(\rho) - 1/\gamma)} \leq \widetilde{m}_{\gamma,q},$$

where

$$\widetilde{m}_{\gamma,q} = (1 + q|\gamma|\chi_{\{q>0\}}) \Big[1 + \Big\{ \frac{\Gamma(1 + |q| + 1/|\gamma|)}{\Gamma(1 + |q|)\Gamma(1 + 1/|\gamma|)} - 1 \Big\} \chi_{\{\gamma q < 0\}} \Big].$$

Proof Let $\rho \geq c$. The case $q = 0$ is obvious. For $q \neq 0$, we have

$$\begin{aligned}\frac{\{\alpha_\gamma(\rho) + q - 1/\gamma\}\Gamma(\alpha_\gamma(\rho) + q - 1/\gamma)\Gamma(\alpha_\gamma(\rho))}{\{\alpha_\gamma(\rho) - 1/\gamma\}\Gamma(\alpha_\gamma(\rho) + q)\Gamma(\alpha_\gamma(\rho) - 1/\gamma)} &= \Big\{1 + \frac{q}{\alpha_\gamma(\rho)}\Big\}\frac{\Gamma(\alpha_\gamma(\rho) + 1 + q - 1/\gamma)\Gamma(\alpha_\gamma(\rho) + 1)}{\Gamma(\alpha_\gamma(\rho) + 1 + q)\Gamma(\alpha_\gamma(\rho) + 1 - 1/\gamma)} \\ &\leq (1 + q|\gamma|\chi_{\{q>0\}})\frac{\Gamma(\alpha_\gamma(\rho) + 1 + q - 1/\gamma)\Gamma(\alpha_\gamma(\rho) + 1)}{\Gamma(\alpha_\gamma(\rho) + 1 + q)\Gamma(\alpha_\gamma(\rho) + 1 - 1/\gamma)}.\end{aligned}$$

We know that, given $p_1, p_2 > 0$,

Fact 1. $\Gamma(z)\Gamma(z + p_1 + p_2)/\{\Gamma(z + p_1)\Gamma(z + p_2)\}$, is strictly decreasing for $z > 0$ (see Theorem 10 of Alzer (1997)).

Fact 2. $\Gamma(z)\Gamma(z + p_1 + p_2)/\{\Gamma(z + p_1)\Gamma(z + p_2)\} \geq 1$, $z > 0$ (see Alzer (1997; page 386)).

When $\gamma q < 0$, Facts 1 and 2 yield

$$1 \leq \frac{\Gamma(\alpha_\gamma(\rho) + 1 + q - 1/\gamma)\Gamma(\alpha_\gamma(\rho) + 1)}{\Gamma(\alpha_\gamma(\rho) + 1 + q)\Gamma(\alpha_\gamma(\rho) + 1 - 1/\gamma)} \leq \frac{\Gamma(1 + |q| + 1/|\gamma|)}{\Gamma(1 + |q|)\Gamma(1 + 1/|\gamma|)}$$

(set $z = \alpha_\gamma(\rho) + 1 + q - 1/\gamma$, $p_1 = 1/\gamma$, and $p_2 = |q|$ for $\gamma > 0$ and $q < 0$, or set $z = \alpha_\gamma(\rho) + 1$, $p_1 = 1/|\gamma|$, and $p_2 = q$ for $\gamma < 0$ and $q > 0$), and, when $\gamma q > 0$, Fact 2 yields

$$\frac{\Gamma(\alpha_\gamma(\rho) + 1 + q - 1/\gamma)\Gamma(\alpha_\gamma(\rho) + 1)}{\Gamma(\alpha_\gamma(\rho) + 1 + q)\Gamma(\alpha_\gamma(\rho) + 1 - 1/\gamma)} \leq 1$$

(set $z = \alpha_\gamma(\rho) + 1 - 1/\gamma$, $p_1 = 1/\gamma$, and $p_2 = q$ for $\gamma > 0$ and $q > 0$, or set $z = \alpha_\gamma(\rho) + 1 + q$, $p_1 = 1/|\gamma|$, and $p_2 = |q|$ for $\gamma < 0$ and $q < 0$). $\square$

Lemma B.3 *Given $\gamma \neq 0$ and $c > 1$, let $q > -(c-1)/|\gamma|$ (note that $c > 1$ allows q to be negative). For any $b > 0$, we have*

$$\sup_{\rho \geq c}\sup_{s \geq 0} K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho), \gamma}(s)\frac{1}{\alpha^q_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^{q\gamma} \leq b^{-1}\widetilde{m}_{c,\gamma,q}\widetilde{m}_{\gamma,q}\widetilde{L}_\gamma,$$

where

$$\widetilde{m}_{c,\gamma,q} = \begin{cases}(1 + q|\gamma|)^q, & q \geq 0,\\ \{1 + (q + \lceil |q| \rceil)|\gamma|\}^{q + \lceil |q| \rceil}c^{\lceil |q| \rceil}, & q \in (-(c-1)/|\gamma|, 0).\end{cases}$$

Proof Rewrite

$$K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho), \gamma}(s)\frac{1}{\alpha^q_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^{q\gamma} = \frac{\Gamma(\alpha_\gamma(\rho) + q)}{\alpha^q_\gamma(\rho)\Gamma(\alpha_\gamma(\rho))}K^{(A)}_{\alpha_\gamma(\rho)+q, b\beta_\gamma(\rho), \gamma}(s) \quad \text{(assume } \alpha_\gamma(\rho) + q > 0\text{)}.$$

We can show that, for $q > -(c-1)/|\gamma|$,

$$\sup_{\rho \geq c}\frac{\Gamma(\alpha_\gamma(\rho) + q)}{\alpha^q_\gamma(\rho)\Gamma(\alpha_\gamma(\rho))} \leq \widetilde{m}_{c,\gamma,q}.$$

Actually, the case $q = 0$ is obvious, and Claim A.2 yields, for $q > 0$,

$$\frac{\Gamma(\alpha_\gamma(\rho) + q)}{\alpha^q_\gamma(\rho)\Gamma(\alpha_\gamma(\rho))} \leq \Big\{1 + \frac{q}{\alpha_\gamma(\rho)}\Big\}^q \leq (1 + q|\gamma|)^q,$$

and, for $q \in (-(c-1)/|\gamma|, 0)$,

$$\begin{aligned}
\frac{\Gamma(\alpha_\gamma(\rho)+q)}{\alpha_\gamma^q(\rho)\Gamma(\alpha_\gamma(\rho))} &= \frac{\Gamma(\alpha_\gamma(\rho)+q+\lceil |q| \rceil)}{\alpha_\gamma^q(\rho)\Gamma(\alpha_\gamma(\rho))} \prod_{j=0}^{\lceil |q| \rceil -1} \{\alpha_\gamma(\rho)+q+j\}^{-1} \\
&\le \{1+(q+\lceil |q| \rceil)|\gamma|\}^{q+\lceil |q| \rceil} \prod_{j=0}^{\lceil |q| \rceil -1} \Big\{1+\frac{q+j}{\alpha_\gamma(\rho)}\Big\}^{-1} \\
&\le \{1+(q+\lceil |q| \rceil)|\gamma|\}^{q+\lceil |q| \rceil} c^{\lceil |q| \rceil}, \quad \text{since } -(c-1)/|\gamma| < q \le q+j < q+|q| = 0.
\end{aligned}$$

Now, we know that, given $p > 0$, $\Gamma(z)\Gamma(z+2p)/\Gamma^2(z+p)$ is strictly decreasing for $z > 0$ (see Theorem 10 of Alzer (1997)), and that $z^{1-z}e^z\Gamma(z) > 1$ is strictly increasing for $z > 0$ (see Theorem 3.2 (2) of Anderson et al. (1995)). Then, for any $\rho \ge c$ and $\gamma \ne 0$ (in this case, $\alpha_\gamma(\rho)+q-1/\gamma > 0$), we use (A1) and Lemma B.2 to get

$$\begin{aligned}
\sup_{s \ge 0} K^{(A)}_{\alpha_\gamma(\rho)+q, b\beta_\gamma(\rho), \gamma}(s) &= \frac{|\gamma|\{\alpha_\gamma(\rho)+q-1/\gamma\}^{\alpha_\gamma(\rho)+q-1/\gamma} e^{-\{\alpha_\gamma(\rho)+q-1/\gamma\}}}{b\beta_\gamma(\rho)\Gamma(\alpha_\gamma(\rho)+q)} \\
&\le \frac{|\gamma|\{\alpha_\gamma(\rho)+q-1/\gamma\}\Gamma(\alpha_\gamma(\rho)+1/\gamma)\Gamma(\alpha_\gamma(\rho)+q-1/\gamma)}{b\rho\Gamma(\alpha_\gamma(\rho))\Gamma(\alpha_\gamma(\rho)+q)} \\
&\le \frac{\widetilde{m}_{\gamma,q}|\gamma|\{\alpha_\gamma(\rho)-1/\gamma\}\Gamma(\alpha_\gamma(\rho)+1/\gamma)\Gamma(\alpha_\gamma(\rho)-1/\gamma)}{b\rho\Gamma^2(\alpha_\gamma(\rho))},
\end{aligned}$$

where

$$\frac{|\gamma|\{\alpha_\gamma(\rho)-1/\gamma\}\Gamma(\alpha_\gamma(\rho)+1/\gamma)\Gamma(\alpha_\gamma(\rho)-1/\gamma)}{\rho\Gamma^2(\alpha_\gamma(\rho))} \le \widetilde{L}_\gamma$$

(see Proof of Lemma A.2 of Igarashi and Kakizawa (2017)). □

Proof of Lemma A.4 Let $\rho \ge c$. Using $(2z)^{-1} < \log z - \psi(z) < z^{-1}$ for $z > 0$ (see Theorem 3.1 of Anderson et al. (1995) or (2.2) of Alzer (1997)), we have

$$0 \le \frac{1}{|\gamma|}(\rho-c)\big\{\log\alpha_\gamma(\rho) - \psi(\alpha_\gamma(\rho))\big\} \le \alpha_\gamma(\rho)\big\{\log\alpha_\gamma(\rho) - \psi(\alpha_\gamma(\rho))\big\} \le 1. \qquad \text{(B1)}$$

This, together with Lemma B.1 (i), yields

$$\begin{aligned}
|H^{(A)}_{b,c,\gamma,\rho}(s)| &= \Bigg|1 + \frac{1}{|\gamma|}(\rho-c)\log\frac{1}{\alpha_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma + \frac{1}{|\gamma|}(\rho-c)\big\{\log\alpha_\gamma(\rho)-\psi(\alpha_\gamma(\rho))\big\} \\
&\quad +\gamma\Big[\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma - \alpha_\gamma(\rho)\Big]\Big[-\frac{c}{\rho} - \frac{1}{|\gamma|}(\rho-c)\big\{\psi(\alpha_\gamma(\rho)+1/\gamma)-\psi(\alpha_\gamma(\rho))\big\}\Big]\Bigg| \\
&\le 1 + \frac{1}{|\gamma|}(\rho-c)\Big|\log\frac{1}{\alpha_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma\Big| + \frac{1}{|\gamma|}(\rho-c)\big\{\log\alpha_\gamma(\rho)-\psi(\alpha_\gamma(\rho))\big\} \\
&\quad +|\gamma|\Big[\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma + \alpha_\gamma(\rho)\Big]\Big\{1 + \frac{1}{|\gamma|}(\rho-c)\big|\psi(\alpha_\gamma(\rho)+1/\gamma)-\psi(\alpha_\gamma(\rho))\big|\Big\} \\
&\le 2 + \frac{1}{|\gamma|}(\rho-c)\Big|\log\frac{1}{\alpha_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma\Big| + |\gamma|\Big[\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma + \alpha_\gamma(\rho)\Big](1+\lceil 1/|\gamma| \rceil).
\end{aligned}$$

Note that, for any $\epsilon' > 0$ and $z > 0$, $|\log z|/(z^{\epsilon'}+z^{-\epsilon'}) \le (\epsilon')^{-1}|\log(z^{\epsilon'}+z^{-\epsilon'})|/(z^{\epsilon'}+z^{-\epsilon'}) \le (\epsilon' e)^{-1}$. The result (i) follows from Lemma B.3 with $q = 0, 1, \pm\epsilon'$.

On the other hand, we can see that, for $\rho \geq c$,

$$
\begin{aligned}
H^{(A)}_{b,c,\gamma,\rho}(s) &= 1 + \frac{1}{|\gamma|}(\rho - c)\big\{\log \alpha_\gamma(\rho) - \psi(\alpha_\gamma(\rho))\big\} + \frac{1}{|\gamma|}(\rho - c)\Big[1 - \frac{1}{\alpha_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma + \log \frac{1}{\alpha_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma\Big] \\
&\quad + \gamma\Big[\frac{1}{\alpha_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma - 1\Big]\Big[-\frac{c\alpha_\gamma(\rho)}{\rho} - \frac{1}{|\gamma|}(\rho - c)\alpha_\gamma(\rho)\Big\{\psi(\alpha_\gamma(\rho) + 1/\gamma) - \psi(\alpha_\gamma(\rho)) - \frac{1}{\gamma\alpha_\gamma(\rho)}\Big\}\Big] \\
&\leq 2 + |\gamma|\Big[\frac{1}{\alpha_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma + 1\Big]\Big\{\frac{c\alpha_\gamma(\rho)}{\rho} + \frac{1}{|\gamma|}(\rho - c)\alpha_\gamma(\rho)\Big|\psi(\alpha_\gamma(\rho) + 1/\gamma) - \psi(\alpha_\gamma(\rho)) - \frac{1}{\gamma\alpha_\gamma(\rho)}\Big|\Big\} \\
&\leq 2 + |\gamma|\Big[\frac{1}{\alpha_\gamma(\rho)}\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma + 1\Big]\Big(\frac{c+1}{|\gamma|} + \widetilde{M}_{c,\gamma}\Big),
\end{aligned}
$$

using (B1), $1 - z + \log z \leq 0$ for $z > 0$, and Lemma B.1 (ii). The result (ii) follows from Lemma B.3 with $q = 0, 1$. $\square$

Appendix C: Proof of Lemma A.7

Proof of Lemma A.7 Rewrite

$$
\mathcal{Q}(x) = \big\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] + \epsilon + \overline{\Delta}_{b,x/b+c} - \overline{\Delta}^{(SS_1)}_{b,x/b+c}\big\}^2.
$$

Note that, if $b \propto n^{-\iota_1}$ ($\iota_1 \in (0, 1/3]$) and $\tau > 0$, then, $n^{-2}b^{-2} = o(n^{-1}b^{-1})$ and $n^{-2}b^{2\tau-4} = (n^{-1}b^{-3})n^{-1}b^{2\tau-1} = o(n^{-1}b^{-1})$; these facts and the assumption $\epsilon \propto b^{\iota_2}$ ($\iota_2 > 1$), as well as

$$
V[\widehat{f}_{b,c,\gamma}(x)] + V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] = \begin{cases} O(n^{-1}\{b^{-1/2}V(x) + 1\}), & \dfrac{x}{b} \to \infty, \\ O(n^{-1}b^{-1}), & \dfrac{x}{b} \to \kappa \end{cases}
$$

(see Theorem 1 and the variance approximation of $\widehat{f}_{b,c,\gamma}(x)$ in Subsection 2.1), will be repeatedly used without mentioning them explicitly, throughout this proof.

Firstly, it is easy to see that

$$
E[\mathcal{Q}(x)] = \{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] + \epsilon\}^2 + E[(\overline{\Delta}_{b,x/b+c} - \overline{\Delta}^{(SS_1)}_{b,x/b+c})^2],
$$

where $E[(\overline{\Delta}_{b,x/b+c} - \overline{\Delta}^{(SS_1)}_{b,x/b+c})^2] \leq 2\{V[\widehat{f}_{b,c,\gamma}(x)] + V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\}$, and

$$
\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] + \epsilon\}^2 = \begin{cases} b^2\dfrac{B^2_{c|\gamma|}(x)}{\gamma^2} + O(b^3(1+x)^3 + b^{1+\iota_2}(1+x)), & \dfrac{x}{b} \to \infty, \\ b^2c^2\{f'(0)\}^2 + O(b^3 + b^{1+\iota_2}), & \dfrac{x}{b} \to \kappa, \end{cases}
$$

using (12) and (14), i.e.,

$$
\begin{aligned}
\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] + \epsilon\}^2 &= b^2\frac{B^2_{c|\gamma|}(x)}{\gamma^2} + \Big\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] + \epsilon + b\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\} \\
&\qquad \times\Big\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] + \epsilon - b\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\} \\
&= \begin{cases} b^2\dfrac{B^2_{c|\gamma|}(x)}{\gamma^2} + \{\epsilon + O(b + bx)\}\{\epsilon + O(b^2 + b^2x^2)\}, & \dfrac{x}{b} \to \infty, \\ b^2c^2\{f'(0)\}^2 + O(b^3) + \{\epsilon + O(b)\}\{\epsilon + O(b^2)\}, & \dfrac{x}{b} \to \kappa. \end{cases}
\end{aligned}
$$

Note that (15), i.e.,

$$\sup_{x\in[0,b^\tau]}\{|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]|\}=O(b^{\min(1,2\tau,\iota_2)})=O(b) \tag{C1}$$

(we assume $\tau\in[1/2,1)$), together with (13), yield $\sup_{x\in[0,b^\tau]}E[\mathcal{Q}(x)]=O(b^2+n^{-1}b^{-1})$.

Secondly, using (12) and $\epsilon=o(b)$, i.e.,

$$|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]|=\begin{cases}O(b+bx), & \dfrac{x}{b}\to\infty,\\ O(b), & \dfrac{x}{b}\to\kappa,\end{cases} \tag{C2}$$

(A9) and (A10) yield

$$\begin{aligned}
&V[\mathcal{Q}(x)]\\
&=V\big[2\{Bias[\widehat{f}_{b,c,\gamma}(x)]-Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]+\epsilon\}(\overline{\Delta}_{b,x/b+c}-\overline{\Delta}^{(SS_1)}_{b,x/b+c})+(\overline{\Delta}_{b,x/b+c}-\overline{\Delta}^{(SS_1)}_{b,x/b+c})^2\big]\\
&\le 2^4\big[\{Bias[\widehat{f}_{b,c,\gamma}(x)]-Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]+\epsilon\}^2\{E[\overline{\Delta}^2_{b,x/b+c}]+E[(\overline{\Delta}^{(SS_1)}_{b,x/b+c})^2]\}+\{E[\overline{\Delta}^4_{b,x/b+c}]+E[(\overline{\Delta}^{(SS_1)}_{b,x/b+c})^4]\}\big]\\
&=\begin{cases}O(\{b^2(1+x)^2+n^{-2}b^{-4}x^2+n^{-2}b^{-2}+n^{-1}b^{-1}\}n^{-1}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b}\to\infty,\\ O((b^2+n^{-2}b^{-2}+n^{-1}b^{-1})n^{-1}b^{-1}), & \dfrac{x}{b}\to\kappa\end{cases}
\end{aligned}$$

(we assumed $\iota_1\in(0,1/3]$ to get $n^{-2}b^{-4}x^2\le(n^{-1}b^{-3})^2b^2(1+x)^2=O(b^2(1+x)^2)$ when $x/b\to\infty$). Note that (A9) and (A10), together with (13) and (C1), yield

$$\sup_{x\in[0,b^\tau]}V[\mathcal{Q}(x)]=O((b^2+n^{-2}b^{2\tau-4}+n^{-2}b^{-2}+n^{-1}b^{-1})n^{-1}b^{-1}).$$

Thirdly, we estimate $|\mathcal{R}(x)|^j$ for $j\ge 2/3$, in the spirit of Chen et al. (2009) (see also Igarashi and Kakizawa (2014, 2015)). Consider the following event:

$$S_{x,b}=\Big\{\mathcal{X}^{(n)}\ \Big|\ \frac{|\overline{\Delta}_{b,x/b+c}|}{f(x)}\le\frac{1}{4}\ \text{and}\ \frac{|\overline{\Delta}^{(SS_1)}_{b,x/b+c}|}{f(x)}\le\frac{1}{4}\Big\}\quad\text{(say)}.$$

Assuming $b^{1-\iota_0}r_b=o(1)$, it is easy to see that, for $\tau\in(1/2,1)$,

$$\begin{aligned}
&\sup_{x\in\mathcal{I}_{\iota_0}[r_b]}\Big[\frac{1}{f(x)}\{|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]|\}\Big]\\
&\le\frac{1}{\varrho b^{\iota_0}}\sup_{x\in[0,r_b]}\{|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]|\}\\
&=\frac{1}{\varrho b^{\iota_0}}\max\Big[\sup_{x\in(b^\tau,r_b]}\{|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]|\},\sup_{x\in[0,b^\tau]}\{|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]|\}\Big]\\
&=O(b^{-\iota_0}(b+br_b))\\
&=o(1)
\end{aligned}$$

(see (12) and (15); note that $x\in(b^\tau,r_b]$ implies $x/b\to\infty$). Thus, for all sufficiently large n, on $\mathcal{I}_{\iota_0}[r_b]$, we have $S_{x,b}\subset\widetilde{\mathcal{S}}_{x,b}$, hence,

$$E[|\mathcal{R}(x)|^j\chi_{S_{x,b}}]\le\Big\{\frac{2^33^2e^2}{(\varrho b^{\iota_0})^2}\Big\}^jE[\{|\widehat{f}_{b,c,\gamma}(x)+\epsilon-f(x)|+|\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)-f(x)|\}^{3j}]\quad\text{(we used (19))}.$$

We know that, for a random variable Y, $E[|Y+C|^{3j}] \leq 2^{3j-1}\{|E[Y]+C|^{3j}+E[|Y-E[Y]|^{3j}]\}$, $j \geq 2/3$, $C \in \mathbb{R}$. Combining them with (A9), (A10), and (C2), it follows that, on $\mathcal{I}_{\iota_0}[r_b]$,

$$E[|\mathcal{R}(x)|^j \chi_{S_{x,b}}] = \begin{cases} O(\{b^{3j}(1+x)^{3j}+(n^{-1}b^{-2}x)^{3j-2}V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]\}b^{-2\iota_0 j} \\ \quad +(n^{-2}b^{-2}+n^{-1}b^{-1})^{(3j-2)/2}n^{-1}b^{-2\iota_0 j}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b}\to\infty, \\ O(b^{(3-2\iota_0)j}+(n^{-2}b^{-2}+n^{-1}b^{-1})^{(3j-2)/2}n^{-1}b^{-(1+2\iota_0 j)}), & \dfrac{x}{b}\to\kappa \end{cases}$$

(we assumed $\iota_1 \in (0,1/3]$ to get $(n^{-1}b^{-2}x)^{3j-2}V[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)] \leq (n^{-1}b^{-3})^{3j-1}b^{3j}(1+x)^{3j} = O(b^{3j}(1+x)^{3j})$ when $x/b\to\infty$, using (13)). On the other hand, using (A7) and (A8), there exist constants $\mathcal{L},\mathcal{L}'>0$, independent of n, b, and x, such that

$$E[\chi_{S^c_{x,b}}] \leq P\Big[|\overline{\Delta}_{b,x/b+c}| > \frac{\varrho b^{\iota_0}}{4}\Big] + P\Big[|\overline{\Delta}^{(SS_1)}_{b,x/b+c}| > \frac{\varrho b^{\iota_0}}{4}\Big] \leq \begin{cases} 4\exp(-nb^{2+\iota_0}r_b^{-1}\mathcal{L}), & \dfrac{x}{b}\to\infty, \\ 4\exp(-nb^{1+2\iota_0}\mathcal{L}'), & \dfrac{x}{b}\to\kappa. \end{cases}$$

Then, it follows that, on $\mathcal{I}_{\iota_0}[r_b]$,

$$\begin{aligned} E[|\mathcal{R}(x)|^j\chi_{S^c_{x,b}}] &= E\Big[\Big|\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)-\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)-\frac{\mathcal{Q}(x)}{2f(x)}\Big|^j\chi_{S^c_{x,b}}\Big] \\ &\leq O(b^{-j}e^{jb^{-1}\epsilon^{-1}\widetilde{L}_{c,\gamma}}+b^{-2j}(b+x)^j+b^{-(4+\iota_0)j}(b+x)^{2j})E[\chi_{S^c_{x,b}}] \quad \text{(see Remark A.1 (i))} \\ &= \begin{cases} o(b^{(3-2\iota_0)j}(1+x)^{3j}), & \dfrac{x}{b}\to\infty, \\ o(b^{(3-2\iota_0)j}), & \dfrac{x}{b}\to\kappa, \end{cases} \end{aligned}$$

if $b \propto n^{-\iota_1}$, $\epsilon \propto b^{\iota_2}$, and $n^{\iota_1(3+\iota_0+\iota_2)-1}r_b = o(1)$, where $r_b \equiv r$ or $r_b\to\infty$ according to $\iota_0 = 0$ or $\iota_0 \in (0,1)$. Note that, under $f(0)>0$ (due to the continuity, there exists a $\delta>0$ such that $x\in[0,\delta]$ implies $f(x)>f(0)/2$ (say)), we have $\sup_{x\in[0,b^\tau]}\{|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)]|\}/f(x) \leq 2O(b)/f(0)$ (see (C1)); for all sufficiently large n, it follows that, on $[0,b^\tau]$, $S_{x,b}\subset\widetilde{\mathcal{S}}_{x,b}$, hence,

$$E[|\mathcal{R}(x)|^j\chi_{S_{x,b}}] \leq \Big[\frac{2^3 3^2 e^2}{\{f(0)/2\}^2}\Big]^j E[\{|\widehat{f}_{b,c,\gamma}(x)+\epsilon-f(x)|+|\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)-f(x)|\}^{3j}]$$

(we used (19)), and consequently, for $j\geq 2/3$,

$$\begin{aligned} \sup_{x\in[0,b^\tau]} E[|\mathcal{R}(x)|^j\chi_{S_{x,b}}] &\leq \Big[\frac{2^3 3^2 e^2}{\{f(0)/2\}^2}\Big]^j \sup_{x\in[0,b^\tau]} E[\{|\widehat{f}_{b,c,\gamma}(x)+\epsilon-f(x)|+|\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)-f(x)|\}^{3j}] \\ &= O(b^{3j}+(n^{-2}b^{2\tau-4}+n^{-2}b^{-2}+n^{-1}b^{-1})^{(3j-2)/2}n^{-1}b^{-1}) \end{aligned}$$

(we used (A9) and (A10), together with (13) and (C1)). Also, we obtain

$$\begin{aligned} \sup_{x\in[0,b^\tau]} E[|\mathcal{R}(x)|^j\chi_{S^c_{x,b}}] &= \sup_{x\in[0,b^\tau]} E\Big[\Big|\widehat{f}^{(JF_1)}_{b,c,\gamma}(x)-\widehat{f}^{(SS_1)}_{b,c,\gamma}(x)-\frac{\mathcal{Q}(x)}{2f(x)}\Big|^j\chi_{S^c_{x,b}}\Big] \\ &\leq O(b^{-j}e^{jb^{-1}\epsilon^{-1}\widetilde{L}_{c,\gamma}}+b^{-(2-\tau)j}+b^{-2(2-\tau)j})\sup_{x\in[0,b^\tau]}E[\chi_{S^c_{x,b}}] \quad \text{(see Remark A.1 (i))} \\ &= o(b^{3j}), \quad \text{provided that } b\propto n^{-\iota_1},\ \epsilon\propto b^{\iota_2}, \text{ and } n^{-1}b^{-(3+\iota_2)+\tau}=o(1), \end{aligned}$$

since, using (A7) and (A8), there exists a constant $\mathcal{L}''>0$, independent of n, b, and x, such that

$$\sup_{x\in[0,b^\tau]} E[\chi_{S^c_{x,b}}] \leq \sup_{x\in[0,b^\tau]} P\Big[|\overline{\Delta}_{b,x/b+c}| > \frac{f(0)}{8}\Big] + \sup_{x\in[0,b^\tau]} P\Big[|\overline{\Delta}^{(SS_1)}_{b,x/b+c}| > \frac{f(0)}{8}\Big] \leq 4\exp(-nb^{2-\tau}\mathcal{L}''). \quad \square$$

Appendix D: Technical details

We always assume that $\alpha, \beta > 0$ and $\gamma \neq 0$. The following facts are used repeatedly:

- $s^v K^{(A)}_{\alpha,\beta,\gamma}(s) = \beta^v \{\Gamma(\alpha + v/\gamma)/\Gamma(\alpha)\} K^{(A)}_{\alpha+v/\gamma,\beta,\gamma}(s)$, provided that $\alpha + v/\gamma > 0$.
- We know that, for $z > 0$ and natural number ℓ,

$$\Gamma^{(\ell)}(z) = \int_0^\infty t^{z-1}\{\log(t)\}^\ell e^{-t} dt = \int_0^\infty |\gamma| s^{\gamma z-1}\{\gamma \log(s)\}^\ell e^{-s^\gamma} ds,$$

hence,

$$E\Big[\Big\{\gamma \log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)\Big\}^\ell\Big] = \int_0^\infty \frac{|\gamma|(s/\beta)^{\alpha\gamma-1}}{\beta\Gamma(\alpha)}\{\gamma\log(s/\beta)\}^\ell e^{-(s/\beta)^\gamma} ds = \frac{\Gamma^{(\ell)}(\alpha)}{\Gamma(\alpha)}$$

can be expressed in terms of the digamma function $\psi(z) = (d/dz)\log\Gamma(z) = \Gamma'(z)/\Gamma(z)$. For example, we have

$$E\Big[\gamma \log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)\Big] = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \psi(\alpha)$$

and

$$E\Big[\Big\{\gamma \log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha)\Big\}^2\Big] = \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - \psi^2(\alpha) = \psi'(\alpha).$$

D1: Functions $\mathcal{G}^{[1]}_j$ and $\mathcal{G}^{[2]}_j$, $j \geq 0$, related to Lemmas A.1 and A.2 (i)

We list the following formulas: For any $j \geq 0$ satisfying $\alpha + j/\gamma > 0$, we have

$$E[\xi^j_{\alpha,\beta,\gamma}] = \beta^j \frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)}, \tag{D1}$$

$$\begin{aligned} E\Big[\xi^j_{\alpha,\beta,\gamma}\Big\{\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^\gamma - \alpha\Big\}\Big] &= \{(\alpha + j/\gamma) - \alpha\}\beta^j\frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)} \\ &= \frac{j}{\gamma}\beta^j \frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)}, \end{aligned} \tag{D2}$$

$$E\Big[\xi^j_{\alpha,\beta,\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha)\Big\}\Big] = \beta^j\frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)}\{\psi(\alpha+j/\gamma) - \psi(\alpha)\}, \tag{D3}$$

$$\begin{aligned} E\Big[\xi^j_{\alpha,\beta,\gamma}\Big\{\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^\gamma - \alpha\Big\}^2\Big] &= \{(\alpha+1+j/\gamma)(\alpha+j/\gamma) - 2\alpha(\alpha+j/\gamma) + \alpha^2\}\beta^j\frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)} \\ &= \Big(\alpha + \frac{j}{\gamma} + \frac{j^2}{\gamma^2}\Big)\beta^j\frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)}, \end{aligned} \tag{D4}$$

$$\begin{aligned} &E\Big[\xi^j_{\alpha,\beta,\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha)\Big\}^2\Big] \\ &= E\Big[\xi^j_{\alpha,\beta,\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha+j/\gamma)\Big\}^2\Big] \\ &\quad +2\{\psi(\alpha+j/\gamma) - \psi(\alpha)\}E\Big[\xi^j_{\alpha,\beta,\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha+j/\gamma)\Big\}\Big] + \{\psi(\alpha+j/\gamma) - \psi(\alpha)\}^2 E[\xi^j_{\alpha,\beta,\gamma}] \\ &= [\psi'(\alpha+j/\gamma) + \{\psi(\alpha+j/\gamma) - \psi(\alpha)\}^2]\beta^j\frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha+1/\gamma)}, \end{aligned} \tag{D5}$$

$$\begin{aligned} &E\Big[\xi^j_{\alpha,\beta,\gamma}\Big\{\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^\gamma - \alpha\Big\}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha)\Big\}\Big] \\ &= \Big[(\alpha+j/\gamma)\Big\{\frac{1}{\alpha+j/\gamma} + \psi(\alpha+j/\gamma) - \psi(\alpha)\Big\} - \alpha\{\psi(\alpha+j/\gamma) - \psi(\alpha)\}\Big]\beta^j\frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)} \\ &= \Big[1 + \frac{j}{\gamma}\{\psi(\alpha+j/\gamma) - \psi(\alpha)\}\Big]\beta^j\frac{\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)} \quad \text{(use the recurrence relation } \psi(z+1) = \psi(z) + z^{-1}\text{).} \end{aligned} \tag{D6}$$

Utilizing (D1)–(D3) immediately yields, for $j \geq 0$,

$$\int_0^\infty s^j K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s) H^{(A)}_{b,c,\gamma,\rho}(s)ds = \mathcal{G}_j^{[1]}(\rho) E[\xi^j_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}],$$

provided that $\alpha_\gamma(\rho) + 1/\gamma > 0$ and $\alpha_\gamma(\rho) + j/\gamma > 0$, where

$$\mathcal{G}_j^{[1]}(\rho) = 1 + j\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}\Big] + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho) + j/\gamma) - \psi(\alpha_\gamma(\rho))\}.$$

Also, using (D1)–(D6), we have, for $j \geq 0$,

$$\int_0^\infty s^j K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)\{H^{(A)}_{b,c,\gamma,\rho}(s)\}^2 ds = \mathcal{G}_j^{[2]}(\rho) E[\xi^j_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}],$$

provided that $\alpha_\gamma(\rho) + 1/\gamma > 0$ and $\alpha_\gamma(\rho) + j/\gamma > 0$, where

$$\begin{aligned}
\mathcal{G}_j^{[2]}(\rho) = {} & 1 + 2j\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}\Big] + \frac{2}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho) + j/\gamma) - \psi(\alpha_\gamma(\rho))\} \\
& + \gamma^2\Big(\alpha + \frac{j}{\gamma} + \frac{j^2}{\gamma^2}\Big)\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}\Big]^2 \\
& + \frac{2\gamma}{|\gamma|}(\rho - c)\Big[1 + \frac{j}{\gamma}\{\psi(\alpha_\gamma(\rho) + j/\gamma) - \psi(\alpha_\gamma(\rho))\}\Big]\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}\Big] \\
& + \frac{1}{\gamma^2}(\rho - c)^2\big[\psi'(\alpha_\gamma(\rho) + j/\gamma) + \{\psi(\alpha_\gamma(\rho) + j/\gamma) - \psi(\alpha_\gamma(\rho))\}^2\big].
\end{aligned}$$

D2: Function $\mathcal{G}^{[3]}$, related to Lemma A.2 (ii)

Note that, for $z > 0$,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \psi'(z) = \frac{\Gamma''(z)}{\Gamma(z)} - \frac{\{\Gamma'(z)\}^2}{\Gamma^2(z)}, \quad \psi''(z) = \frac{\Gamma'''(z)}{\Gamma(z)} - \frac{3\Gamma'(z)\Gamma''(z)}{\Gamma^2(z)} + 2\frac{\{\Gamma'(z)\}^3}{\Gamma^3(z)},$$

$$\psi'''(z) = \frac{\Gamma''''(z)}{\Gamma(z)} - \frac{3\{\Gamma''(z)\}^2}{\Gamma^2(z)} - \frac{4\Gamma'(z)\Gamma'''(z)}{\Gamma^2(z)} + 12\frac{\{\Gamma'(z)\}^2\Gamma''(z)}{\Gamma^3(z)} - 6\frac{\{\Gamma'(z)\}^4}{\Gamma^4(z)}.$$

In addition to

$$E\Big[\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha)\Big] = 0, \quad E\Big[\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha)\Big\}^2\Big] = \psi'(\alpha),$$

it is straightforward to see that

$$\begin{aligned}
E\Big[\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha)\Big\}^3\Big] &= \frac{\Gamma'''(\alpha)}{\Gamma(\alpha)} - 3\psi(\alpha)\frac{\Gamma''(\alpha)}{\Gamma(\alpha)} + 2\psi^3(\alpha) \\
&= \psi''(\alpha), \\
E\Big[\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big) - \psi(\alpha)\Big\}^4\Big] &= \frac{\Gamma''''(\alpha)}{\Gamma(\alpha)} - 4\psi(\alpha)\frac{\Gamma'''(\alpha)}{\Gamma(\alpha)} + 6\psi^2(\alpha)\frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - 3\psi^4(\alpha) \\
&= \psi'''(\alpha) + \frac{3\{\Gamma''(\alpha)\}^2}{\Gamma^2(\alpha)} + \frac{4\Gamma'(\alpha)\Gamma'''(\alpha)}{\Gamma^2(\alpha)} - 12\frac{\{\Gamma'(\alpha)\}^2\Gamma''(\alpha)}{\Gamma^3(\alpha)} + 6\frac{\{\Gamma'(\alpha)\}^4}{\Gamma^4(\alpha)} \\
&\quad -4\frac{\Gamma'(\alpha)\Gamma'''(\alpha)}{\Gamma^2(\alpha)} + 6\frac{\{\Gamma'(\alpha)\}^2\Gamma''(\alpha)}{\Gamma^3(\alpha)} - 3\frac{\{\Gamma'(\alpha)\}^4}{\Gamma^4(\alpha)} \\
&= \psi'''(\alpha) + \frac{3\{\Gamma''(\alpha)\}^2}{\Gamma^2(\alpha)} - 6\frac{\{\Gamma'(\alpha)\}^2\Gamma''(\alpha)}{\Gamma^3(\alpha)} + 3\frac{\{\Gamma'(\alpha)\}^4}{\Gamma^4(\alpha)} \\
&= \psi'''(\alpha) + 3\{\psi'(\alpha)\}^2. \qquad \text{(D7)}
\end{aligned}$$

After some algebra, we have

$$E\Big[\Big\{\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}-\alpha\Big\}^4\Big]=(\alpha+3)(\alpha+2)(\alpha+1)\alpha-4(\alpha+2)(\alpha+1)\alpha^2+6(\alpha+1)\alpha^3-3\alpha^4$$
$$=3\alpha^2+6\alpha, \tag{D8}$$

and, using the recurrence relation

$$\psi^{(\ell)}(z+1)=\psi^{(\ell)}(z)+\frac{(-1)^{\ell}\ell!}{z^{\ell+1}}$$

for nonnegative integer ℓ,

$$\begin{aligned}
&E\Big[\Big\{\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}-\alpha\Big\}^3\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}\Big]\\
&=(\alpha+2)(\alpha+1)\alpha\Big(\frac{1}{\alpha+2}+\frac{1}{\alpha+1}+\frac{1}{\alpha}\Big)-3(\alpha+1)\alpha^2\Big(\frac{1}{\alpha+1}+\frac{1}{\alpha}\Big)+3\alpha^3\frac{1}{\alpha}\\
&=\{(\alpha+1)\alpha+(\alpha+2)\alpha+(\alpha+2)(\alpha+1)\}-3\{\alpha^2+(\alpha+1)\alpha\}+3\alpha^2\\
&=3\alpha+2,
\end{aligned}\tag{D9}$$

$$\begin{aligned}
&E\Big[\Big\{\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}-\alpha\Big\}^2\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}^2\Big]\\
&=E\Big[\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{2\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}^2\Big]\\
&\quad-2\alpha E\Big[\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}^2\Big]+\alpha^2E\Big[\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}^2\Big]\\
&=(\alpha+1)\alpha\big[\psi'(\alpha+2)+\{\psi(\alpha+2)-\psi(\alpha)\}^2\big]-2\alpha^2\big[\psi'(\alpha+1)+\{\psi(\alpha+1)-\psi(\alpha)\}^2\big]+\alpha^2\psi'(\alpha)\\
&=(\alpha+1)\alpha\Big\{-\frac{1}{(\alpha+1)^2}-\frac{1}{\alpha^2}+\psi'(\alpha)+\Big(\frac{1}{\alpha+1}+\frac{1}{\alpha}\Big)^2\Big\}-2\alpha^2\Big\{-\frac{1}{\alpha^2}+\psi'(\alpha)+\frac{1}{\alpha^2}\Big\}+\alpha^2\psi'(\alpha)\\
&=2+\alpha\psi'(\alpha),
\end{aligned}\tag{D10}$$

$$\begin{aligned}
&E\Big[\Big\{\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}-\alpha\Big\}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}^3\Big]\\
&=E\Big[\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)-\psi(\alpha+1)\Big\}^3\Big]\\
&\qquad+3\{\psi(\alpha+1)-\psi(\alpha)\}E\Big[\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{b\beta}\Big)-\psi(\alpha+1)\Big\}^2\Big]\\
&\qquad+3\{\psi(\alpha+1)-\psi(\alpha)\}^2E\Big[\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}\Big\{\gamma\log\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)-\psi(\alpha+1)\Big\}\Big]\\
&\qquad+\{\psi(\alpha+1)-\psi(\alpha)\}^3E\Big[\Big(\frac{\xi_{\alpha,\beta,\gamma}}{\beta}\Big)^{\gamma}\Big]-\alpha\psi''(\alpha)\\
&=\alpha\big[\psi''(\alpha+1)-\psi''(\alpha)+3\psi'(\alpha+1)\{\psi(\alpha+1)-\psi(\alpha)\}+\{\psi(\alpha+1)-\psi(\alpha)\}^3\big]\\
&=\alpha\Big[\frac{2}{\alpha^3}+3\Big\{-\frac{1}{\alpha^2}+\psi'(\alpha)\Big\}\frac{1}{\alpha}+\frac{1}{\alpha^3}\Big]\\
&=3\psi'(\alpha).
\end{aligned}\tag{D11}$$

Utilizing (D7)–(D11) immediately yields

$$\int_0^{\infty}K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)\{H^{(A)}_{b,c,\gamma,\rho}(s)-1\}^4ds=\mathcal{G}^{[3]}(\rho)\,,$$

where

$$\begin{aligned}
\mathcal{G}^{[3]}(\rho) &= \gamma^4\{3\alpha_\gamma^2(\rho)+6\alpha_\gamma(\rho)\}\Big[-\frac{c}{\rho}+\frac{1}{|\gamma|}(\rho-c)\{\psi(\alpha_\gamma(\rho))-\psi(\alpha_\gamma(\rho)+1/\gamma)\}\Big]^4 \\
&\quad +4\gamma|\gamma|(\rho-c)\{3\alpha_\gamma(\rho)+2\}\Big[-\frac{c}{\rho}+\frac{1}{|\gamma|}(\rho-c)\{\psi(\alpha_\gamma(\rho))-\psi(\alpha_\gamma(\rho)+1/\gamma)\}\Big]^3 \\
&\quad +6(\rho-c)^2\{2+\alpha_\gamma(\rho)\psi'(\alpha_\gamma(\rho))\}\Big[-\frac{c}{\rho}+\frac{1}{|\gamma|}(\rho-c)\{\psi(\alpha_\gamma(\rho))-\psi(\alpha_\gamma(\rho)+1/\gamma)\}\Big]^2 \\
&\quad +4\frac{1}{\gamma|\gamma|}(\rho-c)^3 3\psi'(\alpha_\gamma(\rho))\Big[-\frac{c}{\rho}+\frac{1}{|\gamma|}(\rho-c)\{\psi(\alpha_\gamma(\rho))-\psi(\alpha_\gamma(\rho)+1/\gamma)\}\Big] \\
&\quad +\frac{1}{\gamma^4}(\rho-c)^4\big[\psi'''(\alpha_\gamma(\rho))+3\{\psi'(\alpha_\gamma(\rho))\}^2\big].
\end{aligned}$$

D3: Function $\mathcal{G}_j^{[4]}$, $j \geq 0$, related to Lemma A.3

Further, we use (D1)–(D5) to obtain the following formulas: For any $j \geq 0$ satisfying $2\alpha+(j-1)/\gamma>0$ (here, we implicitly assume that $2\alpha-1/\gamma>0$), we have

$$E[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}] = \frac{\beta^j}{2^{j/\gamma}}\frac{\Gamma(2\alpha+(j-1)/\gamma)}{\Gamma(2\alpha-1/\gamma)},$$

$$\begin{aligned}
&E\Big[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta}\Big)^\gamma-\alpha\Big\}\Big] \\
&\quad = \frac{1}{2}E\Big[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)^\gamma-(2\alpha-1/\gamma)\Big\}\Big]-\frac{1}{2\gamma}E[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}] \\
&\quad = \frac{j-1}{2\gamma}\frac{\beta^j}{2^{j/\gamma}}\frac{\Gamma(2\alpha+(j-1)/\gamma)}{\Gamma(2\alpha-1/\gamma)},
\end{aligned}$$

$$\begin{aligned}
&E\Big[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}\Big] \\
&\quad = E\Big[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)-\psi(\alpha)-\log 2\Big\}\Big] \\
&\quad = E\Big[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)-\psi(2\alpha-1/\gamma)\Big\}\Big] \\
&\qquad +\{\psi(2\alpha-1/\gamma)-\psi(\alpha)-\log 2\}E[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}] \\
&\quad = \{\psi(2\alpha+(j-1)/\gamma)-\psi(\alpha)-\log 2\}\frac{\beta^j}{2^{j/\gamma}}\frac{\Gamma(2\alpha+(j-1)/\gamma)}{\Gamma(2\alpha-1/\gamma)}, \\
&E\Big[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta}\Big)^\gamma-\alpha\Big\}^2\Big] \\
&\quad = \frac{1}{4}E\Big[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)^\gamma-(2\alpha-1/\gamma)\Big\}^2\Big] \\
&\qquad -\frac{1}{2\gamma}E\Big[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)^\gamma-(2\alpha-1/\gamma)\Big\}\Big]+\frac{1}{4\gamma^2}E[\xi^j_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}] \\
&\quad = \Big\{\frac{1}{4}\Big(2\alpha+\frac{j-1}{\gamma}+\frac{j^2}{\gamma^2}\Big)-\frac{j}{2\gamma^2}+\frac{1}{4\gamma^2}\Big\}\frac{\beta^j}{2^{j/\gamma}}\frac{\Gamma(2\alpha+(j-1)/\gamma)}{\Gamma(2\alpha-1/\gamma)} \\
&\quad = \frac{1}{4}\Big\{2\alpha+\frac{j-1}{\gamma}+\frac{(j-1)^2}{\gamma^2}\Big\}\frac{\beta^j}{2^{j/\gamma}}\frac{\Gamma(2\alpha+(j-1)/\gamma)}{\Gamma(2\alpha-1/\gamma)},
\end{aligned}$$

$$
\begin{aligned}
&E\Big[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}^2\Big]\\
&\quad=E\Big[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)-\psi(\alpha)-\log 2\Big\}^2\Big]\\
&\quad=E\Big[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)-\psi(2\alpha-1/\gamma)\Big\}^2\Big]\\
&\qquad+2\{\psi(2\alpha-1/\gamma)-\psi(\alpha)-\log 2\}E\Big[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)-\psi(2\alpha-1/\gamma)\Big\}\Big]\\
&\qquad+\{\psi(2\alpha-1/\gamma)-\psi(\alpha)-\log 2\}^2E[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}]\\
&\quad=\big[\psi'(2\alpha+(j-1)/\gamma)+\{\psi(2\alpha+(j-1)/\gamma)-\psi(\alpha)-\log 2\}^2\big]\frac{\beta^j}{2^{j/\gamma}}\frac{\Gamma(2\alpha+(j-1)/\gamma)}{\Gamma(2\alpha-1/\gamma)},
\end{aligned}
$$

$$
\begin{aligned}
&E\Big[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta}\Big)^\gamma-\alpha\Big\}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta}\Big)-\psi(\alpha)\Big\}\Big]\\
&\quad=E\Big[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta}\Big)^\gamma-\alpha\Big\}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)-\psi(\alpha)-\log 2\Big\}\Big]\\
&\quad=\frac{1}{\beta^\gamma}E\Big[\xi^{j+\gamma}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)-\psi(2\alpha-1/\gamma)\Big\}\Big]\\
&\qquad-\alpha E\Big[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\gamma\log\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta/2^{1/\gamma}}\Big)-\psi(2\alpha-1/\gamma)\Big\}\Big]\\
&\qquad+\{\psi(2\alpha-1/\gamma)-\psi(\alpha)-\log 2\}E\Big[\xi^{j}_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}\Big\{\Big(\frac{\xi_{2\alpha-1/\gamma,\beta/2^{1/\gamma},\gamma}}{\beta}\Big)^\gamma-\alpha\Big\}\Big]\\
&\quad=\Big[\frac{1}{2}\{2\alpha+(j-1)/\gamma\}\Big\{\frac{1}{2\alpha+(j-1)/\gamma}+\psi(2\alpha+(j-1)/\gamma)-\psi(2\alpha-1/\gamma)\}\Big\}\\
&\qquad-\alpha\{\psi(2\alpha+(j-1)/\gamma)-\psi(2\alpha-1/\gamma)\}+\frac{j-1}{2\gamma}\{\psi(2\alpha-1/\gamma)-\psi(\alpha)-\log 2\}\Big]\frac{\beta^j}{2^{j/\gamma}}\frac{\Gamma(2\alpha+(j-1)/\gamma)}{\Gamma(2\alpha-1/\gamma)}\\
&\hspace{18em}\text{(use the recurrence relation } \psi(z+1)=\psi(z)+z^{-1})\\
&\quad=\Big[\frac{1}{2}+\frac{j-1}{2\gamma}\{\psi(2\alpha+(j-1)/\gamma)-\psi(\alpha)-\log 2\}\Big]\frac{\beta^j}{2^{j/\gamma}}\frac{\Gamma(2\alpha+(j-1)/\gamma)}{\Gamma(2\alpha-1/\gamma)}.
\end{aligned}
$$

It follows that, for $j\ge 0$,

$$
\int_0^\infty s^jK^{(A)}_{2\alpha_\gamma(\rho)-1/\gamma,b\beta_\gamma(\rho)/2^{1/\gamma},\gamma}(s)\{H^{(A)}_{b,c,\gamma,\rho}(s)\}^2ds=\mathcal{G}^{[4]}_j(\rho)E[\xi^j_{2\alpha_\gamma(\rho)-1/\gamma,b\beta_\gamma(\rho)/2^{1/\gamma},\gamma}],
$$

provided that $\alpha_\gamma(\rho)+1/\gamma>0$ and $2\alpha_\gamma(\rho)+(j-1)/\gamma>0$, where

$$
\begin{aligned}
\mathcal{G}^{[4]}_j(\rho)&=1+(j-1)\Big[-\frac{c}{\rho}+\frac{1}{|\gamma|}(\rho-c)\{\psi(\alpha_\gamma(\rho))-\psi(\alpha_\gamma(\rho)+1/\gamma)\}\Big]\\
&\quad+\frac{2}{|\gamma|}(\rho-c)\{\psi(2\alpha_\gamma(\rho)+(j-1)/\gamma)-\psi(\alpha_\gamma(\rho))-\log 2\}\\
&\quad+\frac{\gamma^2}{4}\Big\{2\alpha_\gamma(\rho)+\frac{j-1}{\gamma}+\frac{(j-1)^2}{\gamma^2}\Big\}\Big[-\frac{c}{\rho}+\frac{1}{|\gamma|}(\rho-c)\{\psi(\alpha_\gamma(\rho))-\psi(\alpha_\gamma(\rho)+1/\gamma)\}\Big]^2\\
&\quad+\frac{\gamma}{|\gamma|}(\rho-c)\Big[1+\frac{j-1}{\gamma}\{\psi(2\alpha_\gamma(\rho)+(j-1)/\gamma)-\psi(\alpha_\gamma(\rho))-\log 2\}\Big]\\
&\qquad\qquad\times\Big[-\frac{c}{\rho}+\frac{1}{|\gamma|}(\rho-c)\{\psi(\alpha_\gamma(\rho))-\psi(\alpha_\gamma(\rho)+1/\gamma)\}\Big]\\
&\quad+\frac{1}{\gamma^2}(\rho-c)^2\big[\psi'(2\alpha_\gamma(\rho)+(j-1)/\gamma)+\{\psi(2\alpha_\gamma(\rho)+(j-1)/\gamma)-\psi(\alpha_\gamma(\rho))-\log 2\}^2\big].
\end{aligned}
$$

D4: Asymptotic expansions of the digamma and polygamma functions

We list the following asymptotic expansions when $z \to \infty$ (see Abramowitz and Stegun (1972; 6.3.18, 6.4.12, and 6.4.14)):

$$
\begin{aligned}
\psi(z) &\approx \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4}, \\
\psi'(z) &\approx \frac{1}{z} + \frac{1}{z^2} + \frac{1}{6z^3}, \\
\psi'''(z) &\approx \frac{2}{z^3} + \frac{3}{z^4}.
\end{aligned}
$$

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