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A SECOND LIMIT FORMULA FOR HIGHER RANK TWISTED EPSTEIN ZETA FUNCTIONS AND SOME APPLICATIONS

By

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Abstract. In this paper, we give the second limit formula and an analogue of the Chowla-Selberg formula for the twisted Epstein zeta functions of rank $n \geq 2$. As an application, we compute the determinant of the Euclidean Laplacian on the space of asymmetrically automorphic functions on $\mathbf{R}^n$ by using our second limit formula.

1. Introduction

Let Q be a positive definite symmetric $n \times n$ matrix, $u, v \in \mathbf{R}^n$, and let s be a complex variable with $\mathrm{Re}(s) > n/2$. Epstein [7] considered a kind of zeta function defined by

$$\zeta_n(s, u, v, Q) = \sum_{m \in \mathbf{Z}^n, m+v \neq 0} e^{2\pi i {}^t m \cdot u} Q[m+v]^{-s}$$

where $Q[x] := {}^t x Q x$ for $x \in \mathbf{R}^n$. This function admits the analytic continuation to the whole s-plane, and satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta_n(s, u, v, Q) = e^{-2\pi i {}^t u \cdot v} |Q|^{-1/2} \pi^{-(n/2-s)} \Gamma\left(\frac{n}{2} - s\right) \zeta_n\left(\frac{n}{2} - s, v, -u, Q^{-1}\right)$$

where $|Q| := \det(Q)$. Further, he obtained so called the Kronecker limit formula for $\zeta_n(s; Q) := \zeta_n(s, 0, 0, Q)$. This is the computation of the constant term of the Laurent expansion of $\zeta_n(s; Q)$ around $s = n/2$ (see also Terras [12]). There are many generalizations of the Epstein zeta function. For example, Siegel [11] defined a generalized Epstein zeta function by

$$\zeta(s, u, v, Q, P) = \sum_{m \in \mathbf{Z}^n, m+v \neq 0} e^{2\pi i {}^t m \cdot u} \frac{P(m+v)}{Q[m+v]^{s+g/2}}$$

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for $\mathrm{Re}(s) > n/2$. Here, $u, v \in \mathbf{R}^n$ are column vectors, Q is a positive definite symmetric $n \times n$ matrix, and $P(x) = P(x_1, \dots, x_n)$ is a homogeneous polynomial of degree g satisfying

$$\sum_{1 \le i,j \le n} q_{ij}^* \frac{\partial^2 P(x)}{\partial x_i \partial x_j} = 0$$

where $(q_{ij}^*) = Q^{-1}$. He proved that this function admits the analytic continuation to the whole s-plane and satisfies some functional equation. He introduced many examples of applications of the Kronecker limit formula to algebraic number theory, mainly on the quadratic fields.

On the other hand, Chowla and Selberg [4] obtained another important formula called the Chowla-Selberg formula. Let a and c be positive numbers and b be real. Assume that $d = b^2 - 4ac < 0$. Then the Epstein zeta function defined by

$$Z(s) = \frac{1}{2} \sum_{(m,n) \in \mathbf{Z}^2 \setminus \{(0,0)\}} (am^2 + bmn + cn^2)^{-s} \quad (\mathrm{Re}(s) > 1)$$

satisfies the following identity:

$$Z(s) = a^{-s}\zeta(2s) + a^{-s}\sqrt{\pi}\frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}\left(\frac{\sqrt{|d|}}{2a}\right)^{1-2s}\zeta(2s-1) + R_Q(s),$$

$$R_Q(s) = \frac{4a^{-s}}{\pi^{-s}\Gamma(s)}\left(\frac{\sqrt{|d|}}{2a}\right)^{-s+1/2}\sum_{n=1}^{\infty} n^{s-1/2}\left(\sum_{d|n} d^{1-2s}\right)K_{s-1/2}\left(\frac{\pi n\sqrt{|d|}}{a}\right)\cos\left(\frac{n\pi b}{a}\right)$$

for any $s \in \mathbf{C}$, where $K_\nu(z)$ is the K-Bessel function. There are a lot of applications of this formula in number theory, for example, to investigate the distribution of zeros of $Z(s)$.

In Section 2, we consider the Epstein zeta function defined by

$$\zeta_n(s, u, 0, Q) = \sum_{m \in \mathbf{Z}^n \setminus \{0\}} e^{2\pi i {}^t m \cdot u} Q[m]^{-s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right)$$

for general $n \ge 2$. It is known that if $u \notin \mathbf{Z}^n$, $\zeta_n(s, u, 0, Q)$ is entire and the identity which expresses $\zeta_n(n/2, u, 0, Q)$ is known as the second limit formula. The case of $n = 2$ is classical, called the second Kronecker limit formula. In [6], Efrat obtained the second limit formula for $\zeta_3(s, u, 0, Q)$ and applied this formula to compute the determinant of the Dirac operators and Laplacians. We generalize his method to general $n \geq 2$ case, and obtain the second limit formula for $\zeta_n(s, u, 0, Q)$, where u is the element of $\mathbf{R}^n \backslash \mathbf{Z}^n$ (Theorem 2.1). As a corollary, we obtain a certain generalized Dedekind η-function which has some modular properties (Corollary 2.2). Further, we give the K-Bessel expansion of $\zeta_n(s, u, 0, Q)$ with $u \in \mathbf{R}^n$ (Theorem 2.3), which is an analogue of the Chowla-Selberg formula.

In Section 3, we compute the determinant of Laplacian on the space of asymmetrically automorphic functions by using our second limit formula, which is also a generalization of Efrat's result. This corresponds to the bosonic and fermionic string theory, as explained in [1].

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2. Twisted Epstein Zeta Functions

2.1. Definition of the Twisted Epstein Zeta Functions

Let $n \geq 2$ be an integer and $Q \in GL(n, \mathbf{R})$ be a positive definite symmetric matrix. Let $u, v \in \mathbf{R}^n$. In this paper, any vector in $\mathbf{R}^n$ is regarded as a column vector in principle. For $a \in \mathbf{R}^n$, we write $Q[a] := {}^t a Q a$ where ${}^t a$ is the transpose of a. Then the Epstein zeta function $\zeta_n(s, u, v, Q)$ is defined by

$$\zeta_n(s, u, v, Q) = \sum_{m \in \mathbf{Z}^n, m+v \neq 0} e^{2\pi i {}^t m \cdot u} Q[m+v]^{-s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right). \tag{2.1}$$

It is known that $\zeta_n(s, u, v, Q)$ admits an analytic continuation to the whole plane and has a simple pole at $s = n/2$ if $u \in \mathbf{Z}^n$ and is entire if $u \notin \mathbf{Z}^n$. In this paper, we call $\zeta_n(s, u, v, Q)$ the twisted Epstein zeta function if $u \notin \mathbf{Z}^n$. $\zeta_n(s, u, v, Q)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta_n(s, u, v, Q) = e^{-2\pi i {}^t u \cdot v}|Q|^{-1/2}\pi^{-(n/2-s)}\Gamma\left(\frac{n}{2} - s\right)\zeta_n\left(\frac{n}{2} - s, v, -u, Q^{-1}\right) \tag{2.2}$$

where $|Q| := \det(Q)$. Throughout this paper we assume that $u \notin \mathbf{Z}^n$, $v = 0$ except for the subsection 2.4 of Section 2.

2.2. Formulation of the Twisted Epstein Zeta Functions

Let $Z(\mathbf{R})$ be the center of $GL(n, \mathbf{R})$ which is isomorphic to $\mathbf{R}^\times$ and $H_n = GL(n, \mathbf{R})/O(n)Z(\mathbf{R})$ be the upper half plane of degree n. By Iwasawa decomposition, the element $\tau \in H_n$ is uniquely expressed by

$$\tau = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & y_1 y_2 \cdots y_{n-2} x_{1,2} & y_1 y_2 \cdots y_{n-3} x_{1,3} & \cdots & x_{1,n} \\ 0 & y_1 y_2 \cdots y_{n-2} & y_1 y_2 \cdots y_{n-3} x_{2,3} & \cdots & x_{2,n} \\ 0 & 0 & y_1 y_2 \cdots y_{n-3} & \cdots & x_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.3}$$

with $x_{i,j} \in \mathbf{R}$ $(1 \le i < j \le n)$, $y_i > 0$ $(i = 1, 2, \dots n-1)$. Let $Q \in GL(n, \mathbf{R})$ be the positive definite symmetric matrix. Then there exists a unique $\tau \in H_n$ above such that

$$Q = |Q|^{1/n} (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{-2/n} \tau \cdot {}^t\tau. \tag{2.4}$$

Since the i-th component of ${}^t\tau \cdot m$ is $y_1 y_2 \cdots y_{n-i}(x_{1,i}m_1 + x_{2,i}m_2 + \cdots + x_{i-1,i}m_{i-1} + m_i)$ (if $i = n$, $y_1 y_2 \cdots y_{n-i} := 1$), we have

$$\begin{aligned} Q[m] &= |Q|^{1/n} (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{-2/n} ({}^t m \cdot \tau)({}^t\tau \cdot m) \\ &= |Q|^{1/n} (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{-2/n} \\ &\quad \cdot \sum_{i=1}^{n} y_1^2 y_2^2 \cdots y_{n-i}^2 (x_{1,i}m_1 + x_{2,i}m_2 + \cdots + x_{i-1,i}m_{i-1} + m_i)^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\zeta_n(s, u, 0, Q) \\ &\quad = |Q|^{-s/n} (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{2s/n} \\ &\quad \cdot \sum_{m \in \mathbf{Z}^n \setminus \{0\}} e^{2\pi i {}^t m \cdot u} \left\{ \sum_{i=1}^{n} y_1^2 y_2^2 \cdots y_{n-i}^2 \right. \\ &\qquad \left. \times (x_{1,i}m_1 + x_{2,i}m_2 + \cdots + x_{i-1,i}m_{i-1} + m_i)^2 \right\}^{-s} \end{aligned} \tag{2.5}$$

for $\mathrm{Re}(s) > n/2$.

2.3. Second Limit Formula for $\zeta_n(s, u, 0, Q)$

In [11], Siegel asked whether one can obtain the second limit formula for $\zeta_n(s, u, v, Q)$, i.e., evaluate this function at $s = n/2$ when $u \notin \mathbf{Z}^n$, in analogy with the $n = 2$ situation which is called the Kronecker limit formula. In [6], Efrat answered this question in case of $n = 3$ and gave the second limit formula for $\zeta_3(s, u, 0, Q)$. The following theorem is the answer in case of general $n \geq 2$ and $v = 0$.

THEOREM 2.1. *We define $A_1, A_2, \dots, A_{n-1}$ inductively by*

$$A_1 = -u_n,$$

$$A_{k+1} = -u_{n-k} - \sum_{i=1}^{k} (A_i + m_{n+1-i}) x_{n-k, n+1-i} \quad (k \geq 1).$$

For Q in (2.4) *with τ in* (2.3), *we put*

$$f(Q, u, m_2, \dots, m_n) = 2\pi i u_1 + 2\pi i \sum_{j=1}^{n-1} (m_{n+1-j} + A_j) x_{1, n+1-j}$$
$$- 2\pi y_1 y_2 \cdots y_{n-1} \left(\sum_{j=1}^{n-1} \frac{1}{y_1^2 \cdots y_{j-1}^2} (m_{n+1-j} + A_j)^2 \right)^{1/2}.$$

For $u = {}^t(u_1, u_2, \dots, u_n) \in \mathbf{R}^n$, we put $u' = {}^t(u_2, u_3, \dots, u_n) \in \mathbf{R}^{n-1}$.

For $\tau \in H_n$ in (2.3), *we define $\tau' \in H_{n-1}$ by*

$$\tau' = \begin{pmatrix} y_1 y_2 \cdots y_{n-2} & y_1 y_2 \cdots y_{n-3} x_{2,3} & y_1 y_2 \cdots y_{n-4} x_{2,4} & \cdots & x_{2,n} \\ 0 & y_1 y_2 \cdots y_{n-3} & y_1 y_2 \cdots y_{n-4} x_{3,4} & \cdots & x_{3,n} \\ 0 & 0 & y_1 y_2 \cdots y_{n-4} & \cdots & x_{4,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.6}$$

For $Q \in GL(n, \mathbf{R})$ in (2.4), *we define $Q' \in GL(n-1, \mathbf{R})$ by*

$$Q' = (y_1^{n-2} y_2^{n-3} \cdots y_{n-2})^{-2/(n-1)} \tau' \cdot {}^t\tau'. \tag{2.7}$$

Then we have

$$\zeta_n\left(\frac{n}{2}, u, 0, Q\right) = |Q|^{-1/2} y_1^{1/(n-1)} y_2^{2/(n-1)} \cdots y_{n-1}^{(n-1)/(n-1)} \zeta_{n-1}\left(\frac{n}{2}, u', 0, Q'\right)$$
$$- 2\pi^{n/2} \Gamma\left(\frac{n}{2}\right)^{-1} |Q|^{-1/2} \log \prod_{(m_2, \dots, m_n) \in \mathbf{Z}^{n-1}} |1 - e^{f(Q, u, m_2, \dots, m_n)}|. \tag{2.8}$$

PROOF. We have

$$\pi^{-s}\Gamma(s)\zeta_n(s,u,0,Q)$$
$$= \pi^{-s}\Gamma(s)|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}$$
$$\cdot \sum_{m\in \mathbf{Z}^n\setminus\{0\}} e^{2\pi i\,{}^t m\cdot u}\left\{\sum_{i=1}^{n} y_1^2 y_2^2\cdots y_{n-i}^2 \times (x_{1,i}m_1+x_{2,i}m_2+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\right\}^{-s}. \quad (2.9)$$

For $m = {}^t(m_1, m_2, \dots, m_n) \in \mathbf{Z}^n$, we put $m' = {}^t(m_2, m_3, \dots, m_n) \in \mathbf{Z}^{n-1}$. Firstly, the part of $m_1 = 0$ terms in (2.9) is equal to

$$\pi^{-s}\Gamma(s)|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}$$
$$\cdot \sum_{m'\in \mathbf{Z}^{n-1}\setminus\{0\}} e^{2\pi i\,{}^t m'\cdot u'}\left\{\sum_{i=2}^{n} y_1^2 y_2^2\cdots y_{n-i}^2(x_{2,i}m_2+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\right\}^{-s}$$
$$= \pi^{-s}\Gamma(s)|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}$$
$$\cdot \sum_{m'\in \mathbf{Z}^{n-1}\setminus\{0\}} e^{2\pi i\,{}^t m'\cdot u'}\left\{\sum_{i=1}^{n-1} y_1^2 y_2^2\cdots y_{(n-1)-i}^2 \times (x_{2,i+1}m_2+\cdots+x_{i,i+1}m_i+m_{i+1})^2\right\}^{-s}$$
$$= \pi^{-s}\Gamma(s)|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}$$
$$\times (y_1^{n-2}y_2^{n-3}\cdots y_{n-2})^{-2s/(n-1)}\zeta_{n-1}(s,u',0,Q'). \quad (2.10)$$

Next, we compute the part of $m_1 \neq 0$ terms in (2.9). Since

$$\frac{\pi^{-s}\Gamma(s)}{\alpha^s} = \int_0^\infty e^{-\pi\alpha t} t^s \frac{dt}{t} \quad (\alpha > 0),$$

The part of $m_1 \neq 0$ terms in (2.9) is expressed by

$$|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}$$
$$\cdot \sum_{m_1\in \mathbf{Z}\setminus\{0\},\, m'\in \mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1+\cdots m_nu_n)}$$
$$\cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^n y_1^2y_2^2\cdots y_{n-i}^2(x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}} t^s \frac{dt}{t}. \quad (2.11)$$

We transform the summation of (2.11) by applying the Poisson summation formula

$$\sum_{n\in\mathbf{Z}} e^{-\pi t(n+\alpha)^2} = \frac{1}{\sqrt{t}}\sum_{n\in\mathbf{Z}} e^{2\pi i n\alpha - \pi n^2/t} \tag{2.12}$$

to the summations in $m_n, m_{n-1}, \dots, m_2$. Firstly, we pick up the summation in m_n. Since

$$\begin{aligned}
&2\pi i u_n m_n - \pi t(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1} + m_n)^2 \\
&\quad = -\pi t\left(m_n + x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1} - \frac{iu_n}{t}\right)^2 \\
&\qquad - 2\pi i u_n(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1}) - \frac{\pi u_n^2}{t},
\end{aligned}$$

by applying the Poisson summation formula to the summation in m_n in (2.11), we have

$$\begin{aligned}
&\sum_{m_n\in\mathbf{Z}} e^{2\pi i u_n m_n - \pi t(x_{1,n}m_1+\cdots+x_{n-1,n}m_{n-1}+m_n)^2} \\
&\quad = e^{-2\pi i u_n(x_{1,n}m_1+\cdots+x_{n-1,n}m_{n-1}) - \pi u_n^2/t} \\
&\qquad \cdot \sum_{m_n\in\mathbf{Z}} e^{-\pi t\left(m_n + x_{1,n}m_1+\cdots+x_{n-1,n}m_{n-1} - iu_n/t\right)^2} \\
&\quad = e^{-2\pi i u_n(x_{1,n}m_1+\cdots+x_{n-1,n}m_{n-1}) - \pi u_n^2/t} \\
&\qquad \cdot \frac{1}{\sqrt{t}}\sum_{m_n\in\mathbf{Z}} e^{2\pi i(x_{1,n}m_1+\cdots+x_{n-1,n}m_{n-1} - iu_n/t)m_n - \pi m_n^2/t}.
\end{aligned}$$

Therefore, by rewriting the summation in m_n, $m_1 \neq 0$ part (2.11) is rewritten as follows:

$$\begin{aligned}
&|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n} \\
&\quad \times \sum_{m_1\in\mathbf{Z}\backslash\{0\},\, m'\in\mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1+\cdots+m_{n-1}u_{n-1}) - 2\pi i u_n(x_{1,n}m_1+\cdots+x_{n-1.n}m_{n-1})} \\
&\quad \cdot e^{2\pi i(x_{1,n}m_1+\cdots+x_{n-1,n}m_{n-1})m_n} \\
&\quad \cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^{n-1} y_1^2\cdots y_{n-i}^2(x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}} \cdot e^{-(\pi/t)(-u_n+m_n)^2}\, t^{s-1/2}\frac{dt}{t}.
\end{aligned} \tag{2.13}$$

Similarly, by applying the Poisson summation formula to the summations in $m_{n-1}, m_{n-2}, \dots, m_{n-(k-1)}$ $(1 \le k \le n-1)$, we have the following identity.

$$
\begin{aligned}
&|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n} \\
&\quad\cdot \sum_{m_1\in\mathbf{Z}\setminus\{0\},\, m'\in\mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1+\cdots m_nu_n)} \\
&\quad\cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^n y_1^2y_2^2\cdots y_{n-i}^2(x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}}t^s\frac{dt}{t} \\
&= |Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}y_1^{-(k-1)}y_2^{-(k-2)}\cdots y_{k-1}^{-1} \\
&\quad\times \sum_{m_1\in\mathbf{Z}\setminus\{0\},\, m'\in\mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1+\cdots+m_{n-k}u_{n-k})}\cdot e^{2\pi i(m_n+A_1)(x_{1,n}m_1+x_{2,n}m_2+\cdots+x_{n-k,n}m_{n-k})} \\
&\quad\cdot e^{2\pi i(m_{n-1}+A_2)(x_{1,n-1}m_1+x_{2,n-1}m_2+\cdots+x_{n-k,n-1}m_{n-k})} \\
&\qquad\vdots \\
&\quad\cdot e^{2\pi i(m_{n+1-k}+A_k)(x_{1,n-k+1}m_1+x_{2,n-k+1}m_2+\cdots+x_{n-k,n-k+1}m_{n-k})} \\
&\quad\cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^{n-k} y_1^2\cdots y_{n-i}^2(x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}} \\
&\quad\cdot e^{-(\pi/ty_1^2\cdots y_{k-1}^2)(m_{n+1-k}+A_k)^2-(\pi/ty_1^2\cdots y_{k-2}^2)(m_{n+2-k}+A_{k-1})^2-\cdots-(\pi/t)(m_n+A_1)^2} \\
&\quad\cdot t^{s-k/2}\frac{dt}{t} \quad (1\le k\le n-1).
\end{aligned}
\tag{2.14}
$$

We prove (2.14) by induction with respect to k. We have already known that (2.14) holds if $k=1$. Suppose that (2.14) holds for $k\in\mathbf{Z}$ such that $1\le k\le n-2$. By applying the Poisson summation formula (2.12), the summation in m_{n-k} is

$$
\begin{aligned}
&\sum_{m_{n-k}\in\mathbf{Z}} e^{2\pi iu_{n-k}m_{n-k}+2\pi i(m_n+A_1)x_{n-k,n}m_{n-k}+\cdots+2\pi i(m_{n+1-k}+A_k)x_{n-k,n-k+1}m_{n-k}} \\
&\qquad\cdot e^{-\pi ty_1^2\cdots y_k^2(x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1}+m_{n-k})^2} \\
&\quad= e^{2\pi iA_{k+1}(x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1})-\pi A_{k+1}^2/ty_1^2\cdots y_k^2} \\
&\qquad\cdot \sum_{m_{n-k}\in\mathbf{Z}} e^{-\pi ty_1^2\cdots y_k^2(m_{n-k}+x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1}+iA_{k+1}/ty_1^2\cdots y_k^2)^2}
\end{aligned}
$$

$$= e^{2\pi i A_{k+1}(x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1})-\pi A_{k+1}^2/ty_1^2\cdots y_k^2}$$

$$\cdot \frac{1}{\sqrt{t}y_1\cdots y_k}\sum_{m_{n-k}\in\mathbf{Z}} e^{2\pi i(x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1}+iA_{k+1}/ty_1^2\cdots y_k^2)m_{n-k}-\pi m_{n-k}^2/ty_1^2\cdots y_k^2}.$$

Therefore, by rewriting the summation in m_{n-k}, the right hand side of (2.14) becomes

$$|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}y_1^{-k}y_2^{-(k-1)}\cdots y_k^{-1}$$

$$\times \sum_{m_1\in\mathbf{Z}\setminus\{0\},\,m'\in\mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1+\cdots+m_{n-k-1}u_{n-k-1})}$$

$$\cdot e^{2\pi i(m_n+A_1)(x_{1,n}m_1+x_{2,n}m_2+\cdots+x_{n-k-1,n}m_{n-k-1})}$$

$$\cdot e^{2\pi i(m_{n-1}+A_2)(x_{1,n-1}m_1+x_{2,n-1}m_2+\cdots+x_{n-k-1,n-1}m_{n-k-1})}$$

$$\vdots$$

$$\cdot e^{2\pi i(m_{n+1-k}+A_k)(x_{1,n-k+1}m_1+x_{2,n-k+1}m_2+\cdots+x_{n-k-1,n-k+1}m_{n-k-1})}$$

$$\cdot e^{2\pi i(m_{n-k}+A_{k+1})(x_{1,n-k}m_1+x_{2,n-k}m_2+\cdots+x_{n-k-1,n-k}m_{n-k-1})}$$

$$\cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^{n-k-1}y_1^2\cdots y_{n-i}^2(x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}}\cdot e^{-(\pi/ty_1^2\cdots y_k^2)(m_{n-k}+A_{k+1})^2}$$

$$\cdot e^{-(\pi/ty_1^2\cdots y_{k-1}^2)(m_{n+1-k}+A_k)^2-(\pi/ty_1^2\cdots y_{k-2}^2)(m_{n+2-k}+A_{k-1})^2-\cdots-(\pi/t)(m_n+A_1)^2}t^{s-(k+1)/2}\frac{dt}{t}.$$

This is what we changed k to $k+1$ in the right hand side of (2.14). Therefore, (2.14) holds for any $k\in\mathbf{Z}$ such that $1\le k\le n-1$. By inserting $k=n-1$ in the identity (2.14), (2.11) is equal to

$$|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1}$$

$$\times \sum_{m_1\ne 0,\,m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1u_1+2\pi i m_1\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}}$$

$$\cdot \int_0^\infty e^{-\pi t y_1^2\cdots y_{n-1}^2m_1^2-(\pi/ty_1^2\cdots y_{n-2}^2)(m_2+A_{n-1})^2-\cdots-(\pi/t)(m_n+A_1)^2}t^{s-(n-1)/2}\frac{dt}{t}. \tag{2.15}$$

For $a,b>0$, K-Bessel function $K_s(a,b)$ is defined by

$$K_s(a,b)=\int_0^\infty e^{-(a^2t+b^2/t)}t^s\frac{dt}{t}. \tag{2.16}$$

The value at $s = 1/2$ is given by

$$K_{1/2}(a,b) = \frac{\sqrt{\pi}}{a} e^{-2ab}.$$

Since the integral in (2.15) is expressed by $K_{s-(n-1)/2}(a,b)$, where

$$a = \sqrt{\pi}|m_1|y_1 \cdots y_{n-1},$$

$$b = \sqrt{\pi}\biggl(\frac{1}{y_1^2 \cdots y_{n-2}^2}(m_2 + A_{n-1})^2 + \frac{1}{y_1^2 \cdots y_{n-3}^2}(m_3 + A_{n-2})^2 + \cdots + (m_n + A_1)^2\biggr)^{1/2},$$

we have

$$\begin{aligned}
&(\text{The integral in } (2.15))|_{s=n/2} \\
&\quad = \frac{1}{|m_1|y_1 \cdots y_{n-1}} e^{-2\pi|m_1|y_1\cdots y_{n-1}((1/y_1^2\cdots y_{n-2}^2)(m_2+A_{n-1})^2+\cdots+(m_n+A_1)^2)^{1/2}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(2.15)|_{s=n/2} &= |Q|^{-1/2} \sum_{m_1 \in \mathbf{Z}\setminus\{0\}, m' \in \mathbf{Z}^{n-1}} e^{2\pi i m_1 u_1 + 2\pi i m_1 \sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}} \\
&\quad \cdot \frac{1}{|m_1|} e^{-2\pi|m_1|y_1\cdots y_{n-1}((1/y_1^2\cdots y_{n-2}^2)(m_2+A_{n-1})^2+\cdots+(m_n+A_1)^2)^{1/2}} \\
&= |Q|^{-1/2} \sum_{m' \in \mathbf{Z}^{n-1}} 2 \operatorname{Re} \sum_{m_1=1}^{\infty} \frac{1}{m_1} e^{m_1 f(Q,u,m_2,\dots,m_n)} \\
&= -2|Q|^{-1/2} \log \prod_{m' \in \mathbf{Z}^{n-1}} |1 - e^{f(Q,u,m_2,\dots,m_n)}|. \qquad (2.17)
\end{aligned}$$

In the above computation, we used the identity

$$\operatorname{Re} \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log|1 - z|.$$

By combining $m_1 = 0$ part (2.10) and $m_1 \neq 0$ part (2.17), we obtain the second limit formula (2.8). □

We fix $u \in \mathbf{Z}^n \setminus \{0\}$. We define a function $\eta_n(\tau, u)$ of $\tau \in H_n$ by

$$\eta_n(\tau,u) = \exp\left(-\frac{1}{2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right) y_1^{1/(n-1)} y_2^{2/(n-1)} \cdots y_{n-1}^{(n-1)/(n-1)} \zeta_{n-1}\left(\frac{n}{2}, u', 0, Q'\right)\right) \times \prod_{m' \in \mathbf{Z}^{n-1}} (1 - e^{f(Q,u,m_2,\dots,m_n)}). \tag{2.18}$$

Since $\zeta_{n-1}\left(\frac{n}{2}, u', 0, Q'\right)$ and $f(Q, u, m_2, \dots, m_n)$ may be regarded as functions of $\tau \in H_n$ (not functions of Q), this is well-defined. This is one of the generalizations of the Dedekind η-function

$$\eta(z) = e^{\pi i z/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z}) \quad (z = x + iy, \, y > 0).$$

Some modular properties of $|\eta(z)|$ are obtained from the original Kronecker limit formula. In the same way, we can obtain some properties of $|\eta_n(\tau, u)|$ by using our second limit formula. From the second limit formula (2.8), we have

$$-\frac{1}{2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)|Q|^{1/2}\zeta_n\left(\frac{n}{2}, u, 0, Q\right) = \log|\eta_n(\tau, u)|. \tag{2.19}$$

Let

$$\iota : GL(n, \mathbf{R}) \to GL(n, \mathbf{R})/O(n) \cdot Z(\mathbf{R}) = H_n$$

be the canonical projection and define the action of $GL(n, \mathbf{R})$ on H_n by

$$\tau \mapsto g \circ \tau := \iota(g\tau) \quad (g \in GL(n, \mathbf{R}), \tau \in H_n).$$

Note that this is a group action. By the definition of Epstein zeta function, $\zeta_n(s, u, 0, Q)$ satisfies

$$\zeta_n(s, u, 0, gQ{}^tg) = \zeta_n(s, g^{-1}u, 0, Q)$$

for $\forall g \in \mathrm{GL}(n, \mathbf{Z})$. By combining this and the relation (2.19), as a corollary of Theorem 2.1, we obtain the following formula for $|\eta_n(\tau, u)|$:

Corollary 2.2. *The function* $|\eta_n(\tau, u)|$ *satisfies*

$$|\eta_n(g \circ \tau, u)| = |\eta_n(\tau, g^{-1}u)| \quad (\forall g \in \mathrm{GL}(n, \mathbf{Z})). \tag{2.20}$$

2.4. An Analogue of Chowla-Selberg Formula

We return to the identity (2.15) in the proof of the Theorem 2.1. From the definition of $A_1, \dots, A_{n-1}$, we can easily verify that the condition $m_2 + A_{n-1} =$

$m_3 + A_{n-3} = \cdots = m_n + A_1 = 0$ is equivalent to $u' = (u_2, \dots, u_n) \in \mathbf{Z}^{n-1}$ and $m_2 = u_2, \dots, m_n = u_n$.

Firstly, we consider the case of $u' \notin \mathbf{Z}^{n-1}$. The integral in (2.15) is expressed by

$$K_{s-(n-1)/2}\left(\sqrt{\pi}|m_1|y_1\cdots y_{n-1}, \sqrt{\pi}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{1/2}\right)$$

where $K_s(a,b)$ is the K-Bessel function defined by (2.16). Since $u' \notin \mathbf{Z}^{n-1}$, for any $(m_2, \dots, m_n) \in \mathbf{Z}^{n-1}$, we have $\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)^2/y_1^2\cdots y_{j-1}^2 > 0$. For $c > 0$, we define another K-Bessel function $K_s(c)$ by

$$K_s(c) = \int_0^\infty e^{-(c/2)(t+1/t)}t^s\frac{dt}{t}. \tag{2.21}$$

The relation between $K_s(c)$ and $K_s(a,b)$ is given by

$$K_s(a,b) = \left(\frac{b}{a}\right)^s K_s(2ab) \quad (a, b > 0). \tag{2.22}$$

Therefore, the integral in (2.15) is expressed by

$$\left\{\frac{1}{|m_1|y_1\cdots y_{n-1}}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{1/2}\right\}^{s-(n-1)/2}$$
$$\cdot K_{s-(n-1)/2}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{1/2}\right). \tag{2.23}$$

Thus the right hand side of (2.15) is expressed by

$$|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})(y_1y_2\cdots y_{n-1})^{-s+(n-1)/2}$$
$$\times \sum_{m_1\neq 0, m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1 u_1 + 2\pi i m_1\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}}$$
$$\cdot |m_1|^{-s+(n-1)/2}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{(1/2)(s-(n-1)/2)}$$
$$\cdot K_{s-(n-1)/2}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{1/2}\right) \tag{2.24}$$

when $\mathrm{Re}(s) > n/2$. Since this K-Bessel expansion converges absolutely for any $s \in \mathbf{C}$, this becomes the entire function. Therefore, by combining $m_1 = 0$ terms given by (2.10) and $m_1 \neq 0$ terms given by (24), we have the following formula.

$$\begin{aligned}
&\pi^{-s}\Gamma(s)\zeta_n(s,u,0,Q) \\
&\quad = \pi^{-s}\Gamma(s)|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n} \\
&\qquad \times (y_1^{n-2}y_2^{n-3}\cdots y_{n-2})^{-2s/(n-1)}\zeta_{n-1}(s,u',0,Q') \\
&\qquad + |Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1}) \\
&\qquad \times (y_1y_2\cdots y_{n-1})^{-s+(n-1)/2} \\
&\qquad \times \sum_{m_1\neq 0, m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1 u_1 + 2\pi i m_1 \sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}} \\
&\qquad \cdot |m_1|^{-s+(n-1)/2}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{(1/2)(s-(n-1)/2)} \\
&\qquad \cdot K_{s-(n-1)/2}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{1/2}\right). \qquad (2.25)
\end{aligned}$$

By the theory of analytic continuation, the right hand side of (2.25) represents $\pi^{-s}\Gamma(s)\zeta_n(s,u,0,Q)$ over the whole s-plane.

Next, we consider the case of $u' = (u_2,\ldots,u_n) \in \mathbf{Z}^{n-1}$. The computation of $m' = (m_2,\ldots,m_n) \neq (u_2,\ldots,u_n)$ part is the same, that is, this part is expressed by the series of K-Bessel functions which converges absolutely for general $s \in \mathbf{C}$ and becomes the entire function. The computation of $(m_2,\ldots,m_n) = (u_2,\ldots,u_n)$ part in the right hand side of (2.15) is given by

$$\begin{aligned}
&|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1}) \\
&\qquad \cdot \sum_{m_1\neq 0} e^{2\pi i m_1 u_1}\cdot\int_0^\infty e^{-\pi t y_1^2\cdots y_{n-1}^2 m_1^2}t^{s-(n-1)/2}\frac{dt}{t} \\
&\quad = 2\pi^{-(s-(n-1)/2)}\Gamma\left(s-\frac{n-1}{2}\right)|Q|^{-s/n} \\
&\qquad \cdot (y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})(y_1y_2\cdots y_{n-1})^{-2s+n-1} \\
&\qquad \cdot \sum_{m_1=1}^{\infty}\frac{\cos(2\pi m_1 u_1)}{m_1^{2s-(n-1)}}. \qquad (2.26)
\end{aligned}$$

Now, the function $\sum_{m_1=1}^{\infty}\cos(2\pi m_1 u_1)/m_1^{2s-(n-1)}$ $(\mathrm{Re}(s)>n/2)$ in (2.26) is expressed by $\{\mathrm{Li}_{2s-(n-1)}(e^{2\pi i u_1})+\mathrm{Li}_{2s-(n-1)}(e^{-2\pi i u_1})\}/2$, where $\mathrm{Li}_s(z)$ is the polylogarithm originally defined by

$$\mathrm{Li}_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}$$

for $\mathrm{Re}(s)>1$ and $|z|\le 1$. It is known that as a function of s, this function has the analytic continuation to the whole s-plane. Therefore, we have the following formula:

$$\begin{aligned}
&\pi^{-s}\Gamma(s)\zeta_n(s,u,0,Q)\\
&\quad=\pi^{-s}\Gamma(s)|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}\\
&\qquad\times(y_1^{n-2}y_2^{n-3}\cdots y_{n-2})^{-2s/(n-1)}\zeta_{n-1}(s,u',0,Q')\\
&\qquad+\pi^{-(s-(n-1)/2)}\Gamma\left(s-\frac{n-1}{2}\right)|Q|^{-s/n}\\
&\qquad\cdot(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})(y_1y_2\cdots y_{n-1})^{-2s+n-1}\\
&\qquad\cdot\{\mathrm{Li}_{2s-(n-1)}(e^{2\pi i u_1})+\mathrm{Li}_{2s-(n-1)}(e^{-2\pi i u_1})\}\\
&\qquad+|Q|^{-s/n}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{2s/n}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})\\
&\qquad\times(y_1y_2\cdots y_{n-1})^{-s+(n-1)/2}\\
&\qquad\times\sum_{m_1\neq 0,\,m'\in\mathbf{Z}^{n-1}\setminus\{u'\}}e^{2\pi i m_1u_1+2\pi i m_1\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}}\\
&\qquad\cdot|m_1|^{-s+(n-1)/2}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{(1/2)(s-(n-1)/2)}\\
&\qquad\cdot K_{s-(n-1)/2}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{1/2}\right). \qquad (2.27)
\end{aligned}$$

for any $s\in\mathbf{C}$. Summing up, we have the following theorem.

THEOREM 2.3. *The twisted Epstein zeta function $\zeta_n(s,u,0,Q)$ satisfies the following identity.*

1) *If $u'=(u_2,\dots,u_n)\notin\mathbf{Z}^{n-1}$, we have*

$$\begin{aligned}
&\zeta_n(s,u,0,Q) \\
&\quad = |Q|^{-s/n}\zeta_{n-1}(s,u',0,Q')\prod_{i=1}^{n-1} y_i^{2is/n(n-1)} \\
&\qquad + \pi^s\Gamma(s)^{-1}|Q|^{-s/n}\prod_{i=1}^{n-1} y_i^{((n-2i)/n)s-(n-2i-1)/2} \\
&\qquad \times \sum_{m_1\neq 0,\, m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1 u_1 + 2\pi i m_1 \sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}} \\
&\qquad \cdot |m_1|^{-s+(n-1)/2}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{(1/2)(s-(n-1)/2)} \\
&\qquad \cdot K_{s-(n-1)/2}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{1/2}\right). \qquad (2.28)
\end{aligned}$$

2) *If* $u' = (u_2,\ldots,u_n)\in\mathbf{Z}^{n-1}$, *we have*

$$\begin{aligned}
&\zeta_n(s,u,0;Q) \\
&\quad = |Q|^{-s/n}\zeta_{n-1}(s,u',0,Q')\prod_{i=1}^{n-1} y_i^{2is/n(n-1)} \\
&\qquad + \pi^{(n-1)/2}\Gamma(s)^{-1}\Gamma\left(s-\frac{n-1}{2}\right)|Q|^{-s/n} \\
&\qquad \cdot \{\mathrm{Li}_{2s-(n-1)}(e^{2\pi i u_1}) + \mathrm{Li}_{2s-(n-1)}(e^{-2\pi i u_1})\}\prod_{i=1}^{n-1} y_i^{-2is/n+i} \\
&\qquad + \pi^s\Gamma(s)^{-1}|Q|^{-s/n}\prod_{i=1}^{n-1} y_i^{((n-2i)/n)s-(n-2i-1)/2} \\
&\qquad \times \sum_{m_1\neq 0,\, m'\in\mathbf{Z}^{n-1}\setminus\{u'\}} e^{2\pi i m_1 u_1 + 2\pi i m_1 \sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}} \\
&\qquad \cdot |m_1|^{-s+(n-1)/2}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{(1/2)(s-(n-1)/2)} \\
&\qquad \cdot K_{s-(n-1)/2}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{1/2}\right). \qquad (2.29)
\end{aligned}$$

Theorem 2.3 gives the relations between $\zeta_n(s,u,0,Q)$ and $\zeta_{n-1}(s,u',0,Q')$. Assume that $u=(u_1,\dots,u_n)=(0,\dots,0)$. In this case, the polylogarithms above become the Riemann zeta function. By using (2.29) inductively, we can express $\zeta_n(s;Q)=\zeta_n(s,0,0,Q)$ by Riemann zeta function and K-Bessel series. To do this, we need some notations.

For $\tau\in H_n$ in (2.3), we define $\tau^{(j)}\in H_{n-j}$ by

$$\tau^{(j)}=\begin{pmatrix} y_1y_2\cdots y_{n-1-j} & y_1y_2\cdots y_{n-2-j}x_{j+1,j+2} & \cdots & x_{j+1,n} \\ 0 & y_1y_2\cdots y_{n-2-j} & \cdots & x_{j+2,n} \\ 0 & 0 & \cdots & x_{j+3,n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \ddots & x_{n-1,n} \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

(that is, we remove first j column vectors and j row vectors from τ) $(j=0,1,\dots,n-2)$ and $\tau^{(n-1)}:=(1)$. For Q in (2.4), we define $(n-j)\times(n-j)$ positive definite symmetric matrix $Q^{(j)}$ by

$$Q^{(j)}=(y_1^{n-j-1}y_2^{n-j-2}\cdots y_{n-1-j})^{-2/(n-j)}\tau^{(j)}\,{}^t\tau^{(j)}.$$

For $m=(m_1,\dots,m_n)\in\mathbf{Z}^n$, we define $m^{(j)}\in\mathbf{Z}^{n-j}$ by $m^{(j)}=(m_{j+1},\dots,m_n)$ $(j=1,\dots,n-1)$. For $\tau\in H_l$ in (2.3) (we replace n by l), we define $A_1^{(l)},A_2^{(l)},\dots,A_{l-1}^{(l)}$ inductively by

$$A_1^{(l)}=0,\quad A_{k+1}^{(l)}=-\sum_{i=1}^{k}(A_i^{(l)}+m_{l+1-i})x_{l-k,l+1-i}\quad (k\ge 1)$$

$(l=2,3,\dots,n)$. For $s\in\mathbf{C}$, $m_1,\dots,m_l\in\mathbf{Z}$, $\tau\in H_l$ in (2.3), we define an entire function $K_l(s,m_1,\dots,m_l,\tau)$ by

$$\begin{aligned} K_l(s,m_1,\dots,m_l,\tau) &= e^{2\pi i m_1\sum_{j=1}^{l-1}(m_{l+1-j}+A_j^{(l)})x_{1,l+1-j}} \\ &\quad\cdot|m_1|^{-s+(l-1)/2}\left(\sum_{j=1}^{l-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{l+1-j}+A_j^{(l)})^2\right)^{(1/2)(s-(l-1)/2)} \\ &\quad\cdot K_{s-(l-1)/2}\left(2\pi|m_1|y_1\cdots y_{l-1}\left(\sum_{j=1}^{l-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{l+1-j}+A_j^{(l)})^2\right)^{1/2}\right)\end{aligned}$$

$(l = 2, 3, \dots, n)$. Finally, for $s \in \mathbf{C}$, $y_1, \dots, y_{l-1} > 0$, we define three functions $a_l(s, y_1, \dots, y_{l-1})$, $b_l(s, y_1, \dots, y_{l-1})$, $c_l(s, y_1, \dots, y_{l-1})$ by

$$a_l(s, y_1, \dots, y_{l-1}) = \prod_{i=1}^{l-1} y_i^{2is/l(l-1)}, \quad b_l(s, y_1, \dots, y_{l-1}) = \prod_{i=1}^{l-1} y_i^{-2is/l+i},$$

$$c_l(s, y_1, \dots, y_{l-1}) = \prod_{i=1}^{l-1} y_i^{((l-2i)/l)s-(l-2i-1)/2}.$$

Then the identity (2.26) becomes

$$\begin{aligned}|Q|^{s/n}\zeta_n(s; Q) &= a_n(s, y_1, \dots, y_{n-1})\zeta_{n-1}(s; Q^{(1)}) \\ &\quad + 2\pi^{(n-1)/2}\Gamma(s)^{-1}\Gamma\left(s - \frac{n-1}{2}\right)b_n(s, y_1, \dots, y_{n-1})\zeta(2s-(n-1)) \\ &\quad + \pi^s\Gamma(s)^{-1}c_n(s, y_1, \dots, y_{n-1}) \sum_{m_1 \neq 0,\, m^{(1)} \in \mathbf{Z}^{n-1}\setminus\{0\}} K_n(s, m_1, \dots, m_n, \tau).\end{aligned}$$

Note that $\zeta_{n-1}(s; Q^{(1)})$ also satisfies a similar identity. By using this formula inductively, we obtain the following identity:

$$\begin{aligned}|Q|^{s/n}\zeta_n(s; Q) &= \left\{\prod_{i=1}^{k} a_{n-(i-1)}(s, y_1, \dots, y_{n-i})\right\}\zeta_{n-k}(s; Q^{(k)}) \\ &\quad + 2\sum_{j=1}^{k} \pi^{(n-j)/2}\Gamma(s)^{-1}\Gamma\left(s - \frac{n-j}{2}\right)\left\{\prod_{i=1}^{j-1} a_{n-(i-1)}(s, y_1, \dots, y_{n-i})\right\} \\ &\quad \cdot b_{n-(j-1)}(s, y_1, \dots, y_{n-j})\zeta(2s-(n-j)) \\ &\quad + \pi^s\Gamma(s)^{-1}\sum_{j=1}^{k}\left\{\prod_{i=1}^{j-1} a_{n-(i-1)}(s, y_1, \dots, y_{n-i})\right\} \\ &\quad \cdot c_{n-(j-1)}(s, y_1, \dots, y_{n-j}) \sum_{m_j \neq 0,\, m^{(j)} \in \mathbf{Z}^{n-j}\setminus\{0\}} \\ &\quad \times K_{n-(j-1)}(s, m_j, \dots, m_n, \tau^{(j-1)}). \end{aligned} \tag{2.30}$$

$(1 \le k \le n-1)$. In particular, since $\zeta_1(s; Q^{(n-1)}) = 2\zeta(2s)$, we have the following expansion formula by letting $k = n-1$.

Corollary 2.4. *The Epstein zeta function* $\zeta_n(s; Q) = \zeta_n(s, 0, 0, Q)$ *satisfies the following identity*:

$$\begin{aligned}|Q|^{s/n}\zeta_n(s;Q) = {} & 2\left\{\prod_{i=1}^{n-1} a_{n-(i-1)}(s, y_1, \dots, y_{n-i})\right\}\zeta(2s) \\ & + 2\sum_{j=1}^{n-1} \pi^{(n-j)/2}\Gamma(s)^{-1}\Gamma\left(s - \frac{n-j}{2}\right)\left\{\prod_{i=1}^{j-1} a_{n-(i-1)}(s, y_1, \dots, y_{n-i})\right\} \\ & \cdot b_{n-(j-1)}(s, y_1, \dots, y_{n-j})\zeta(2s-(n-j)) \\ & + \pi^s\Gamma(s)^{-1}\sum_{j=1}^{n-1}\left\{\prod_{i=1}^{j-1} a_{n-(i-1)}(s, y_1, \dots, y_{n-i})\right\} \\ & \cdot c_{n-(j-1)}(s, y_1, \dots, y_{n-j}) \sum_{m_j \neq 0,\, m^{(j)} \in \mathbf{Z}^{n-j}\setminus\{0\}} \\ & \times K_{n-(j-1)}(s, m_j, \dots, m_n, \tau^{(j-1)}). \end{aligned} \tag{2.31}$$

Remark 2.5. The auther thinks this formula is useful to investigate the real zeros of $\zeta_n(s;Q)$, since the term of K-Bessel series becomes sufficiently small when $y_1, \dots, y_{n-1} > 0$ are sufficiently large. For example, Bateman and Grosswald [3] established a sufficient condition for the existence of real zeros of Epstein zeta functions in case of $n = 2$ by using the Chowla-Selberg formula.

3. Determinant of Laplacian

Let Δ be a positive self-adjoint elliptic operator on L^2 functions of some closed connected Riemannian manifold with non-zero eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$. (The eigenvalues 0 are excluded if they exist). For such Δ, we associate the Dirichlet series

$$\zeta_\Delta(s) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s}.$$

It is shown in [9] that this series converges when $\mathrm{Re}(s)$ is sufficiently large, and has a meromorphic continuation to the entire plane, which is regular at $s = 0$. Formally, we have

$$-\log \prod_{i=1}^{\infty} \lambda_i = \frac{d}{ds}\zeta_\Delta(s)|_{s=0}.$$

Therefore, we define the determinant of Δ by

$$\det(\Delta) = e^{-\zeta_\Delta'(0)}.$$

We compute the determinant of the Euclidean Laplacian Δ on some functional space dependent on $u \in \mathbf{R}^n \backslash \mathbf{Z}^n$ by using the second limit formula we obtained. Let Γ be a lattice in $\mathbf{R}^n$ with basis $a_1, \dots, a_n \in \mathbf{R}^n$ and $T^n = \mathbf{R}^n/\Gamma$ be a n-dimensional torus. We put $M = (a_1 \cdots a_n) \in GL(n, \mathbf{R})$. For $u = (u_1, \dots, u_n) \in \mathbf{R}^n \backslash \mathbf{Z}^n$, we define a space of asymmetrically automorphic functions on $\mathbf{R}^n$ by

$$S^u(T^n) = \{f : \mathbf{R}^n \to \mathbf{C},\ \text{smooth}\ | f(x + a_j) = e^{2\pi i u_j} f(x),\ j = 1, \dots, n\}.$$

For any $m \in \mathbf{Z}^n$, we define $\alpha_m \in \mathbf{R}^n$ by $\alpha_m = {}^tM^{-1} \cdot (m + u)$. Then the set $\{e^{2\pi i {}^t\alpha_m \cdot x} \mid m \in \mathbf{Z}^n\}$ forms a basis of $S^u(T^n)$. Moreover, these $e^{2\pi i {}^t\alpha_m \cdot x}$ are the eigenfunctions of the Laplacian

$$\Delta := -\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$$

since

$$\begin{aligned} \Delta e^{2\pi i {}^t\alpha_m \cdot x} &= 4\pi^2 \, {}^t\alpha_m \cdot \alpha_m e^{2\pi i {}^t\alpha_m \cdot x} \\ &= 4\pi^2 ({}^tM \cdot M)^{-1}[m + u] e^{2\pi i {}^t\alpha_m \cdot x}. \end{aligned}$$

Therefore, if we put $Q = {}^tM \cdot M$ and define the zeta function attached to Δ by

$$\zeta_\Delta(s) := \sum_{m \in \mathbf{Z}^n} Q^{-1}[m + u]^{-s} = \zeta_n(s, 0, u, Q^{-1}),$$

the determinant of Laplacian Δ on $S^u(T^n)$ is given by $e^{-\zeta'_\Delta(0)}$. (We normalized Q by $1/4\pi^2$, which does not change the value $\zeta'_\Delta(0)$). From the functional equation (2.2), we have

$$\pi^{-s}\Gamma(s)\zeta_n(s, 0, u, Q^{-1}) = |Q|^{1/2}\pi^{-(n/2-s)}\Gamma\left(\frac{n}{2} - s\right)\zeta_n\left(\frac{n}{2} - s, u, 0, Q\right). \quad (3.1)$$

Since the right hand side of (3.1) is regular at $s = 0$, the left hand side of (3.1) is also regular at $s = 0$. But since $\Gamma(s)$ has a simple pole at $s = 0$ ($\Gamma(s) = 1/s + O(1)$), the function $\zeta_n(s, 0, u, Q^{-1})$ must vanish at $s = 0$. (This is the reason why the value $\zeta'_\Delta(0)$ is not changed by the normalization of Q). In other words, $\zeta_n(s, 0, u, Q^{-1})$ has a Taylor expansion

$$\zeta_n(s, 0, u, Q^{-1}) = \zeta'_n(0, 0, u, Q^{-1})s + O(s^2) \quad (3.2)$$

around $s = 0$. Therefore, by using the functional equation (3.1) and the second limit formula (2.8), we have

$$\begin{aligned}\zeta'_\Delta(0) &= \lim_{s\to 0} \frac{1}{s}\zeta_n(s,0,u,Q^{-1}) \\ &= \lim_{s\to 0}\ \pi^{-s}\Gamma(s)\zeta_n(s,0,u,Q^{-1}) \\ &= |Q|^{1/2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)\zeta_n\left(\frac{n}{2},u,0,Q\right) \\ &= \pi^{-n/2}\Gamma\left(\frac{n}{2}\right)y_1^{1/(n-1)}y_2^{2/(n-1)}\cdots y_{n-1}^{(n-1)/(n-1)}\zeta_{n-1}\left(\frac{n}{2},u',0,Q'\right) \\ &\quad - 2\log\prod_{m'\in\mathbf{Z}^{n-1}}|1-e^{f(Q,u,m_2,\dots,m_n)}|.\end{aligned}$$

Therefore, we have the following result.

THEOREM 3.1. *The determinant of Laplacian* $\Delta = -\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)$ *on* $S^u(T^n)$ *is given by*

$$\begin{aligned}\det(\Delta) = \exp\left\{-\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)y_1^{1/(n-1)}y_2^{2/(n-1)}\cdots y_{n-1}^{(n-1)/(n-1)}\zeta_{n-1}\left(\frac{n}{2},u',0,Q'\right)\right\} \\ \cdot\prod_{m'\in\mathbf{Z}^{n-1}}|1-e^{f(Q,u,m_2,\dots,m_n)}|^2. \qquad (3.3)\end{aligned}$$

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