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LORENTZIAN STATIONARY SURFACES IN 4-DIMENSIONAL SPACE FORMS OF INDEX 2

By

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Abstract. We discuss the necessary and sufficient conditions for the existence of Lorentzian stationary surfaces in 4-dimensional space forms of index 2, and isometric stationary deformations preserving normal curvature.

1. Introduction

Let $N_p^n(c)$ denote the n-dimensional semi-Riemannian space form of constant curvature c and index p. Namely, it is the n-dimensional semi-Euclidean space R_p^n of index p, the n-dimensional pseudo-sphere $S_p^n(c)$ of constant curvature $c > 0$ and index p, or the n-dimensional pseudo-hyperbolic space $H_p^n(c)$ of constant curvature $c < 0$ and index p. We write $N^n(c)$ if $p = 0$. A surface in $N_p^n(c)$ is called Lorentzian if the induced metric is Lorentzian. We shall say that a Lorentzian surface in $N_p^n(c)$ is stationary if the mean curvature vector is identically zero.

For a minimal surface in $N^4(c)$, the Gaussian curvature $K(\leq c)$ and the normal curvature K_ν satisfy $(K - c)^2 - K_\nu^2 \geq 0$, where the equality holds at isotropic points. In [12] Tribuzy and Guadalupe give the necessary and sufficient conditions for the existence of minimal surfaces in $N^4(c)$ in terms of the metric and the normal curvature, and discuss isometric minimal deformations preserving normal curvature. Also, for a spacelike maximal surface in $N_2^4(c)$, $K(\geq c)$ and K_ν satisfy $(K - c)^2 - K_\nu^2 \geq 0$, where the equality holds at isotropic points. In a previous paper [7], we give the necessary and sufficient conditions for the existence of spacelike maximal surfaces in $N_2^4(c)$, and discuss isometric maximal deformations preserving normal curvature.

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For a Lorentzian stationary surface in $N_2^4(c)$, the signs of $K-c$ and $(K-c)^2-K_\nu^2$ are not fixed, and it seems that there are many different situations compared with the case of minimal surfaces in $N^4(c)$ or spacelike maximal surfaces in $N_2^4(c)$. In this paper, we will discuss the necessary and sufficient conditions for the existence of Lorentzian stationary surfaces in $N_2^4(c)$, and isometric stationary deformations preserving normal curvature.

The results are stated as follows:

Theorem 1.1. (i) *Let M be a Lorentzian stationary surface in $N_2^4(c)$ with Gaussian curvature K, normal curvature K_ν and Laplacian Δ. If $(K-c)^2-K_\nu^2 \neq 0$, then*

$$\Delta \log|K-c+K_\nu| = 2(2K+K_\nu), \tag{1}$$

$$\Delta \log|K-c-K_\nu| = 2(2K-K_\nu). \tag{2}$$

(ii) *Let M be a 2-dimensional simply connected Lorentzian manifold with Gaussian curvature K and Laplacian Δ. If K_ν is a function on M satisfying $(K-c)^2-K_\nu^2>0$ and* (1), (2), *then there exists an isometric stationary immersion of M into $N_2^4(c)$ with normal curvature K_ν.*

Theorem 1.2. *Let $f: M \to N_2^4(c)$ be an isometric stationary immersion of a 2-dimensional simply connected Lorentzian manifold M into $N_2^4(c)$ with Gaussian curvature K and normal curvature K_ν.*

(i) *There exist two one-parameter families of isometric stationary immersions $f_\theta, \bar{f}_\theta : M \to N_2^4(c)$ $(\theta \in R)$ with the same normal curvature K_ν.*

(ii) *Assume that $(K-c)^2-K_\nu^2>0$. If $\hat{f}: M \to N_2^4(c)$ is an arbitrary isometric stationary immersion with the same normal curvature K_ν, then there exists $\theta \in R$ such that $\hat{f}$ coincides with f_θ or $\bar{f}_\theta$ up to congruence.*

Remark. The theorems and their proof imply that Lorentzian stationary surfaces in $N_1^3(c)$ and $N_2^3(c)$ with $K \neq c$ are intrinsically characterized by the same condition $\Delta \log|K-c| = 4K$, and each of them has a one-parameter family of isometric stationary immersions into $N_1^3(c)$ and $N_2^3(c)$, respectively. So, viewing $N_1^3(c)$ and $N_2^3(c)$ as subspaces of $N_2^4(c)$, a Lorentzian stationary surface in $N_1^3(c)$ or $N_2^3(c)$ with $K \neq c$ has two one-parameter families of isometric stationary immersions with zero normal curvature into $N_2^4(c)$. Theorem 1.2 is a natural generalization of this situation.

The theorems say that, in the case where $(K-c)^2-K_\nu^2>0$, Lorentzian stationary surfaces in $N_2^4(c)$ have similar properties to minimal surfaces in $N^4(c)$

or spacelike maximal surfaces in $N_2^4(c)$ (cf. [12] and [7]), except for the existence of two kinds of isometric stationary deformations preserving normal curvature. But, in the case where $(K-c)^2 - K_\nu^2 \leq 0$, we do not know how is a Lorentzian stationary surface in $N_2^4(c)$ determined by the metric and the normal curvature. As will be noted in the last section, the crucial different point is that a certain symmetric linear transformation of the normal bundle can be diagonalized or not.

The study of Lorentzian stationary surfaces in $N_2^4(c)$ may be seen as a special case of that of real parakähler submanifolds in $N_p^n(c)$, namely, isometric immersions of parakähler manifolds into $N_p^n(c)$, in particular, in the case of zero mean curvature. The results in this paper suggest that the geometry of real parakähler submanifolds may be quite different from that of real Kähler submanifolds (cf. [4], [3], [2] and their references).

2. Preliminaries

In this section, we recall the method of moving frames for Lorentzian surfaces in $N_2^4(c)$. Unless otherwise stated, we use the following convention on the ranges of indices:

$$1 \leq A, B, \ldots \leq 4, \quad 1 \leq i, j, \ldots \leq 2, \quad 3 \leq \alpha, \beta, \ldots \leq 4.$$

Let $\{e_A\}$ be a local orthonormal frame field in $N_2^4(c)$, and $\{\omega^A\}$ be the dual coframe field, so that the metric of $N_2^4(c)$ is given by

$$d\sigma^2 = (\omega^1)^2 - (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2.$$

The connection forms $\{\omega_B^A\}$ are defined by

$$de_B = \sum_A \omega_B^A e_A.$$

Then, $\omega_B^A = -\omega_A^B$ if $|A-B|$ is even, and $\omega_B^A = \omega_A^B$ if $|A-B|$ is odd. The structure equations are given by

$$d\omega^A = -\sum_B \omega_B^A \wedge \omega^B, \tag{3}$$

$$d\omega_B^A = -\sum_C \omega_C^A \wedge \omega_B^C + \frac{1}{2}\sum_{C,D} R_{BCD}^A \omega^C \wedge \omega^D, \tag{4}$$

$$R_{BCD}^A = c\varepsilon_B(\delta_C^A \delta_{BD} - \delta_D^A \delta_{BC}), \tag{5}$$

where $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_4 = -1$.

Let M be a Lorentzian surface in $N_2^4(c)$. We choose the frame $\{e_A\}$ so that $\{e_i\}$ are tangent to M. Then $\omega^\alpha = 0$ on M. In the following our argument will be restricted to M. By (3),

$$0 = -\sum_i \omega_i^\alpha \wedge \omega^i.$$

So there is a symmetric tensor $\{h_{ij}^\alpha\}$ such that

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j, \tag{6}$$

where h_{ij}^α are the components of the second fundamental form h of M.

The Gaussian curvature K and the normal curvature K_ν of M are given by

$$d\omega_2^1 = -K\omega^1 \wedge \omega^2, \quad d\omega_4^3 = -K_\nu \omega^1 \wedge \omega^2. \tag{7}$$

Then by (4), (5) and (6), we have

$$K = c - h_{11}^3 h_{22}^3 + (h_{12}^3)^2 + h_{11}^4 h_{22}^4 - (h_{12}^4)^2, \tag{8}$$

and

$$K_\nu = h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4 - h_{12}^3 h_{22}^4 + h_{22}^3 h_{12}^4. \tag{9}$$

The mean curvature vector H of M is defined by

$$H = \frac{1}{2}\sum_\alpha (h_{11}^\alpha - h_{22}^\alpha) e_\alpha.$$

We say that M is stationary if $H = 0$ on M.

In the following we assume that M is stationary. Then by (8) and (9),

$$K = c - (h_{11}^3)^2 + (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2, \tag{10}$$

and

$$K_\nu = 2(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4). \tag{11}$$

We can see that

$$\begin{aligned}(K-c)^2 - K_\nu^2 &= \{-(h_{11}^3)^2 + (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2\}^2 - 4(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4)^2 \\ &= \{(h_{11}^3)^2 + (h_{12}^3)^2 - (h_{11}^4)^2 - (h_{12}^4)^2\}^2 - 4(h_{11}^3 h_{12}^3 - h_{11}^4 h_{12}^4)^2 \\ &= \{(h_{11}^3)^2 - (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2\}^2 - 4(h_{11}^3 h_{11}^4 - h_{12}^3 h_{12}^4)^2. \end{aligned} \tag{12}$$

3. Some Examples

In this section, we give some examples of Lorentzian stationary surfaces in $N_2^4(c)$.

Example 3.1. Let $\{x_1, x_2, x_3, x_4\}$ be the standard coordinate system for R_2^4 with metric

$$d\sigma^2 = dx_1^2 - dx_2^2 + dx_3^2 - dx_4^2.$$

Let $\bar{J}$ be the paracomplex structure on R_2^4 given by

$$\bar{J}\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2}, \quad \bar{J}\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_1}, \quad \bar{J}\left(\frac{\partial}{\partial x_3}\right) = \frac{\partial}{\partial x_4}, \quad \bar{J}\left(\frac{\partial}{\partial x_4}\right) = \frac{\partial}{\partial x_3}.$$

Then $(\bar{J}, d\sigma^2)$ is a flat parakähler structure on R_2^4.

Let M be a paracomplex surface in R_2^4, that is, $\bar{J}(T_pM) = T_pM$ for each $p \in M$. Then, by Corollary 3.1 of [1], M is a Lorentzian stationary surface in R_2^4. For example, set

$$f(u,v) = \begin{pmatrix} Q_1(u+v) + Q_2(u-v) \\ Q_1(u+v) - Q_2(u-v) \\ Q_3(u+v) + Q_4(u-v) \\ Q_3(u+v) - Q_4(u-v) \end{pmatrix},$$

and assume that

$$Q_1'(u+v)Q_2'(u-v) + Q_3'(u+v)Q_4'(u-v) > 0,$$

where $Q_1(z)$, $Q_2(z)$, $Q_3(z)$ and $Q_4(z)$ are smooth functions. Then it gives a paracomplex surface in R_2^4. See [9] and [11] for a relation between paracomplex surfaces and minimal lightlike submanifolds.

Example 3.2. For a constant $k > 0$ and a smooth function $Q(u)$ with $Q'(u) > 0$, let M be a surface in R_2^4 given by

$$f(u,v) = (Q(u)\cosh v, Q(u)\sinh v, u, kv),$$

where R_2^4 has the same metric as in Example 3.1. It is a deformation in R_2^4 of a Lorentzian surface of revolution in R_1^3 with spacelike axis of revolution (cf. [14, p. 350], [5, p. 520]). Set

$$e_1 = \frac{1}{\sqrt{1+Q'^2}} f_u = \frac{1}{\sqrt{1+Q'^2}}(Q'\cosh v, Q'\sinh v, 1, 0),$$

$$e_2 = \frac{1}{\sqrt{Q^2+k^2}} f_v = \frac{1}{\sqrt{Q^2+k^2}} (Q \sinh v, Q \cosh v, 0, k),$$

$$e_3 = \frac{1}{\sqrt{1+Q'^2}} (\cosh v, \sinh v, -Q', 0),$$

$$e_4 = \frac{1}{\sqrt{Q^2+k^2}} (k \sinh v, k \cosh v, 0, -Q).$$

Then $\{e_A\}$ is an orthonormal frame field along M with signature $(+,-,+,-)$, and $\{e_i\}$ are tangent to M. The components of the second fundamental form h are given by

$$h_{11}^3 = \frac{Q''}{(1+Q'^2)^{3/2}}, \quad h_{12}^3 = 0, \quad h_{22}^3 = \frac{Q}{(Q^2+k^2)\sqrt{1+Q'^2}},$$

$$h_{11}^4 = 0, \quad h_{12}^4 = \frac{kQ'}{(Q^2+k^2)\sqrt{1+Q'^2}}, \quad h_{22}^4 = 0.$$

Thus M is stationary if and only if

$$\frac{Q''}{1+Q'^2} = \frac{Q}{Q^2+k^2}.$$

Multiplying by $2Q'$ and integrating, we may obtain

$$Q' = \sqrt{c_1^2 Q^2 + c_1^2 k^2 - 1}, \quad (c_1 > 0).$$

If $c_1 k = 1$, then $Q(u) = c_2 e^{u/k}$. If $c_1 k < 1$, then

$$Q(u) = \frac{\sqrt{1 - c_1^2 k^2}}{c_1} \cosh(c_1 u + c_2).$$

If $c_1 k > 1$, then

$$Q(u) = \frac{\sqrt{c_1^2 k^2 - 1}}{c_1} \sinh(c_1 u + c_2).$$

EXAMPLE 3.3. Let M be a 2-dimensional simply connected Lorentzian manifold with Gaussian curvature K and Laplacian Δ. Suppose that $K > c$ and

$$\Delta \log(K - c) = 6K - 2c.$$

Then by Theorem 1 of [8], there exists an isometric stationary immersion of M into $N_2^4(c)$. This is an isotropic-like example.

Example 3.4. Let $P(u)$, $Q(v)$ be null curves in R_2^4, and assume that $\langle P'(u), Q'(v)\rangle \neq 0$. Set $f(u,v) = P(u) + Q(v)$. Then it gives a Lorentzian stationary surface in R_2^4. See [13, Chap. 8] for such a representation in R_1^3. See [10] for the geometry of null curves in R_2^4.

4. Proof of Theorem 1.1

Proof of Theorem 1.1. (i) As M is a Lorentzian stationary surface in $N_2^4(c)$, using the notations in Section 2, we may write

$$\omega_1^3 = s\omega^1 + t\omega^2, \quad \omega_2^3 = t\omega^1 + s\omega^2, \quad \omega_1^4 = u\omega^1 + v\omega^2, \quad \omega_2^4 = v\omega^1 + u\omega^2. \tag{13}$$

By (10) and (11),

$$K = c - s^2 + t^2 + u^2 - v^2, \quad K_v = 2(sv - tu). \tag{14}$$

Using (3), (4), (5) and (13), we have

$$\begin{aligned} d\omega_1^3 &= ds \wedge \omega^1 - s\omega_2^1 \wedge \omega^2 + dt \wedge \omega^2 - t\omega_1^2 \wedge \omega^1 \\ &= -\omega_2^3 \wedge \omega_1^2 - \omega_4^3 \wedge \omega_1^4 \\ &= -(t\omega^1 + s\omega^2) \wedge \omega_1^2 - \omega_4^3 \wedge (u\omega^1 + v\omega^2). \end{aligned}$$

Using the notation like

$$ds = s_1\omega^1 + s_2\omega^2, \quad dt = t_1\omega^1 + t_2\omega^2,$$

$$\omega_2^1 = (\omega_2^1)_1\omega^1 + (\omega_2^1)_2\omega^2, \quad \omega_4^3 = (\omega_4^3)_1\omega^1 + (\omega_4^3)_2\omega^2,$$

we get

$$2s(\omega_2^1)_1 - 2t(\omega_2^1)_2 - v(\omega_4^3)_1 + u(\omega_4^3)_2 = t_1 - s_2. \tag{15}$$

Similarly, from the exterior derivatives of ω_2^3, ω_1^4 and ω_2^4,

$$2s(\omega_2^1)_2 - 2t(\omega_2^1)_1 - v(\omega_4^3)_2 + u(\omega_4^3)_1 = t_2 - s_1, \tag{16}$$

$$2u(\omega_2^1)_1 - 2v(\omega_2^1)_2 - t(\omega_4^3)_1 + s(\omega_4^3)_2 = v_1 - u_2, \tag{17}$$

$$2u(\omega_2^1)_2 - 2v(\omega_2^1)_1 - t(\omega_4^3)_2 + s(\omega_4^3)_1 = v_2 - u_1. \tag{18}$$

Using (14) we can see that

$$\begin{vmatrix} s & -t & -v & u \\ -t & s & u & -v \\ u & -v & -t & s \\ -v & u & s & -t \end{vmatrix} = -\{(K-c)^2 - K_v^2\} \neq 0.$$

So the simultaneous linear equations (15)–(18) for $2(\omega_2^1)_1$, $2(\omega_2^1)_2$, $(\omega_4^3)_1$ and $(\omega_4^3)_2$ can be solved uniquely. From (15)–(18) we have

$$2s\omega_2^1 - 2t(*\omega_2^1) - v\omega_4^3 + u(*\omega_4^3) = dt - (*ds), \tag{19}$$

$$-2t\omega_2^1 + 2s(*\omega_2^1) + u\omega_4^3 - v(*\omega_4^3) = (*dt) - ds, \tag{20}$$

$$2u\omega_2^1 - 2v(*\omega_2^1) - t\omega_4^3 + s(*\omega_4^3) = dv - (*du), \tag{21}$$

$$-2v\omega_2^1 + 2u(*\omega_2^1) + s\omega_4^3 - t(*\omega_4^3) = (*dv) - du. \tag{22}$$

Here $*$ is the Lorentzian Hodge star operator on M given by $*\omega^1 = \omega^2$ and $*\omega^2 = \omega^1$.

By $(19) \times (-s) + (20) \times (-t) + (21) \times u + (22) \times v$ and $(19) \times v + (20) \times u + (21) \times (-t) + (22) \times (-s)$, together with (14), we can get

$$2(K-c)\omega_2^1 + K_v\omega_4^3 = -\frac{1}{2}(*d(K-c)) + t\,ds - s\,dt - v\,du + u\,dv, \tag{23}$$

and

$$2K_v\omega_2^1 + (K-c)\omega_4^3 = -\frac{1}{2}(*dK_v) - u\,ds + v\,dt + s\,du - t\,dv. \tag{24}$$

Set

$$X = s^2 + t^2 - u^2 - v^2, \quad Y = 2(st - uv),$$

$$Z = s^2 - t^2 + u^2 - v^2, \quad W = 2(su - tv).$$

By (12) we have

$$(K-c)^2 - K_v^2 = X^2 - Y^2 = Z^2 - W^2. \tag{25}$$

Using (14) we can compute that

$$(K-c)(t\,ds - s\,dt - v\,du + u\,dv) - K_v(-u\,ds + v\,dt + s\,du - t\,dv)$$
$$= \frac{1}{2}(X\,dY - Y\,dX), \tag{26}$$

and

$$-K_v(t\,ds - s\,dt - v\,du + u\,dv) + (K-c)(-u\,ds + v\,dt + s\,du - t\,dv)$$
$$= -\frac{1}{2}(Z\,dW - W\,dZ). \tag{27}$$

By (23), (24), (25), (26) and (27), we can get

$$2\omega_2^1 = -\frac{1}{4} * d \log|(K-c)^2 - K_v^2| + \frac{1}{2} \cdot \frac{X\,dY - Y\,dX}{X^2 - Y^2}, \tag{28}$$

and

$$\omega_4^3 = \frac{1}{4} * d \log\left|\frac{K-c-K_v}{K-c+K_v}\right| - \frac{1}{2} \cdot \frac{Z\,dW - W\,dZ}{Z^2 - W^2}. \tag{29}$$

The Laplacian Δ on M is given by

$$d * df = (\Delta f)\omega^1 \wedge \omega^2$$

for a smooth function f on M. By the exterior derivative of (28) and (29), together with (7), we may obtain

$$\Delta \log|(K-c)^2 - K_v^2| = 8K, \tag{30}$$

and

$$\Delta \log\left|\frac{K-c-K_v}{K-c+K_v}\right| = -4K_v. \tag{31}$$

By (30)$\pm$(31), we have the equations (1) and (2).

(ii) As $(K-c)^2 - K_v^2 > 0$, we have $K \neq c$. By the anti-isometry from $N_2^4(c)$ to $N_2^4(-c)$ (cf. [6, p. 110]), M remains Lorentzian, the Gaussian curvature, the normal curvature and the Laplacian turn to $-K$, $-K_v$ and $-\Delta$, respectively. So it suffices to consider only the case where $K > c$.

We may assume that M is a small neighborhood. Let $d\sigma^2$ be the induced metric on M. By (1)+(2) we have

$$\Delta \log\{(K-c)^2 - K_v^2\} = 8K,$$

which implies that the metric

$$d\tilde{\sigma}^2 = \{(K-c)^2 - K_v^2\}^{1/4}\, d\sigma^2$$

is flat. So there exists a coordinate system $\{x^1, x^2\}$ such that

$$d\sigma^2 = \{(K-c)^2 - K_v^2\}^{-1/4}\{(dx^1)^2 - (dx^2)^2\}.$$

Set

$$\omega^i = \{(K-c)^2 - K_v^2\}^{-1/8}\, dx^i, \tag{32}$$

so that $\{\omega^i\}$ is an orthonormal coframe field dual to $\{e_i\}$. By

$$d\omega^1 = -\omega_2^1 \wedge \omega^2, \quad d\omega^2 = -\omega_1^2 \wedge \omega^1,$$

we can find that the connection form $\omega_2^1 = \omega_1^2$ is given by

$$\omega_2^1 = \omega_1^2 = -\frac{1}{8} * d \log\{(K-c)^2 - K_v^2\}.$$

As $(K-c)^2 - K_v^2 > 0$ and $K > c$, there exist smooth functions t and u so that

$$t^2 + u^2 = K - c, \quad tu = -\frac{1}{2}K_v.$$

In fact, letting

$$q = \frac{\sqrt{K-c-K_v} + \sqrt{K-c+K_v}}{2}, \tag{33}$$

and

$$r = \frac{\sqrt{K-c-K_v} - \sqrt{K-c+K_v}}{2}, \tag{34}$$

we have $(t,u) = \pm(q,r)$, or $(t,u) = \pm(r,q)$.

Let E be a 2-plane bundle over M with metric $\langle\, , \rangle$ and orthonormal sections $\{e_3, e_4\}$ of signature $(+,-)$. Let h be a symmetric section of $\mathrm{Hom}(TM \times TM, E)$ such that

$$(h_{ij}^3) = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},$$

and set

$$\omega_1^3 = -\omega_3^1 = t\omega^2, \quad \omega_2^3 = \omega_3^2 = t\omega^1, \quad \omega_1^4 = \omega_4^1 = u\omega^1, \quad \omega_2^4 = -\omega_4^2 = u\omega^2.$$

We define a compatible connection ${}^\perp\nabla$ of E so that

$${}^\perp\nabla e_3 = \omega_3^4 e_4, \quad {}^\perp\nabla e_4 = \omega_4^3 e_3,$$

where

$$\omega_4^3 = \omega_3^4 = \frac{1}{4} * d \log\left(\frac{K-c-K_v}{K-c+K_v}\right).$$

Now, almost reversing the argument in (i) for $s = v = 0$, we can see that $\{\omega_B^A\}$ satisfy the structure equations:

$$d\omega_2^1 = -\omega_3^1 \wedge \omega_2^3 - \omega_4^1 \wedge \omega_2^4 - c\omega^1 \wedge \omega^2,$$

$$d\omega_1^3 = -\omega_2^3 \wedge \omega_1^2 - \omega_4^3 \wedge \omega_1^4, \quad d\omega_2^3 = -\omega_1^3 \wedge \omega_2^1 - \omega_4^3 \wedge \omega_2^4,$$

$$d\omega_1^4 = -\omega_2^4 \wedge \omega_1^2 - \omega_3^4 \wedge \omega_1^3, \quad d\omega_2^4 = -\omega_1^4 \wedge \omega_2^1 - \omega_3^4 \wedge \omega_2^3,$$

$$d\omega_4^3 = -\omega_1^3 \wedge \omega_4^1 - \omega_2^3 \wedge \omega_4^2,$$

which are the integrability conditions. Hence, by the fundamental theorem, there exists an isometric immersion of M into $N_2^4(c)$, which is stationary and has normal curvature K_ν. □

5. Proof of Theorem 1.2

Proof of Theorem 1.2. (i) For $f : M \to N_2^4(c)$, let s, t, u, v and ω_B^A be as in the proof of Throem 1.1 (i). For each $\theta \in R$, let $h(\theta)$ be a symmetric section of $\mathrm{Hom}(TM \times TM, T^\perp M)$ such that

$$(h_{ij}^3(\theta)) = \begin{pmatrix} s \cosh 2\theta + t \sinh 2\theta & s \sinh 2\theta + t \cosh 2\theta \\ s \sinh 2\theta + t \cosh 2\theta & s \cosh 2\theta + t \sinh 2\theta \end{pmatrix},$$

$$(h_{ij}^4(\theta)) = \begin{pmatrix} u \cosh 2\theta + v \sinh 2\theta & u \sinh 2\theta + v \cosh 2\theta \\ u \sinh 2\theta + v \cosh 2\theta & u \cosh 2\theta + v \sinh 2\theta \end{pmatrix},$$

and correspondingly, set

$$\omega_1^3(\theta) = -\omega_3^1(\theta) = \omega_1^3 \cosh 2\theta + \omega_2^3 \sinh 2\theta,$$

$$\omega_2^3(\theta) = \omega_3^2(\theta) = \omega_1^3 \sinh 2\theta + \omega_2^3 \cosh 2\theta,$$

$$\omega_1^4(\theta) = \omega_4^1(\theta) = \omega_1^4 \cosh 2\theta + \omega_2^4 \sinh 2\theta,$$

$$\omega_2^4(\theta) = -\omega_4^2(\theta) = \omega_1^4 \sinh 2\theta + \omega_2^4 \cosh 2\theta.$$

Let $\omega_2^1(\theta) = \omega_1^2(\theta) = \omega_2^1$ and $\omega_4^3(\theta) = \omega_3^4(\theta) = \omega_4^3$ for convenience. Then by the computation, we can see that $\{\omega_B^A(\theta)\}$ satisfy the structure equations. Thus, for each $\theta \in R$, there exists an isometric immersion $f_\theta : M \to N_2^4(c)$, which is stationary and has the same normal curvature K_ν.

Next, let $\bar{h}$ be a symmetric section of $\mathrm{Hom}(TM \times TM, T^\perp M)$ such that

$$(\bar{h}_{ij}^3) = \begin{pmatrix} v & u \\ u & v \end{pmatrix}, \quad (\bar{h}_{ij}^4) = \begin{pmatrix} t & s \\ s & t \end{pmatrix},$$

and correspondingly, set

$$\bar{\omega}_1^3 = -\bar{\omega}_3^1 = \omega_2^4, \quad \bar{\omega}_2^3 = \bar{\omega}_3^2 = \omega_1^4,$$
$$\bar{\omega}_1^4 = \bar{\omega}_4^1 = \omega_2^3, \quad \bar{\omega}_2^4 = -\bar{\omega}_4^2 = \omega_1^3.$$

Let $\bar{\omega}_2^1 = \bar{\omega}_1^2 = \omega_2^1$ and $\bar{\omega}_4^3 = \bar{\omega}_3^4 = \omega_4^3$ for convenience. By the computation, we can see that $\{\bar{\omega}_B^A\}$ satisfy the structure equations. So there exists an isometric immersion $\bar{f}: M \to N_2^4(c)$, which is stationary and has the same normal curvature K_ν.

Combining the above two methods, we get two one-parameter families of isometric stationary immersions $f_\theta, \bar{f}_\theta : M \to N_2^4(c)$ $(\theta \in R)$ with the same normal curvature K_ν. □

Before proving the part (ii), we shall prepare a lemma. Let M be a Lorentzian stationary surface in $N_2^4(c)$ satisfying $(K-c)^2 - K_\nu^2 > 0$. As in the proof of Theorem 1.1 (ii), we may assume that $K > c$. As $(K-c)^2 - K_\nu^2 > 0$, by (12), we may choose a smooth function φ so that

$$\{(h_{11}^3)^2 - (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2\} \sinh 2\varphi + 2(h_{11}^3 h_{11}^4 - h_{12}^3 h_{12}^4) \cosh 2\varphi = 0.$$

Set

$$\bar{e}_3 = e_3 \cosh\varphi + e_4 \sinh\varphi, \quad \bar{e}_4 = e_3 \sinh\varphi + e_4 \cosh\varphi,$$

and let $\{\bar{h}_{ij}^\alpha\}$ be the components of h with respect to $\{e_i, \bar{e}_\alpha\}$. Then we may have

$$\bar{h}_{11}^3 \bar{h}_{11}^4 - \bar{h}_{12}^3 \bar{h}_{12}^4 = 0, \tag{35}$$

which is independent of the choice of $\{e_i\}$.

Set

$$\hat{e}_1 = e_1 \cosh\theta + e_2 \sinh\theta, \quad \hat{e}_2 = e_1 \sinh\theta + e_2 \cosh\theta$$

for a smooth function θ, and let $\{\hat{h}_{ij}^\alpha\}$ be the components of h with respect to the frame $\{\hat{e}_i, \bar{e}_\alpha\}$. Then we have

$$\hat{h}_{11}^3 = \bar{h}_{11}^3 \cosh 2\theta + \bar{h}_{12}^3 \sinh 2\theta, \quad \hat{h}_{12}^3 = \bar{h}_{11}^3 \sinh 2\theta + \bar{h}_{12}^3 \cosh 2\theta,$$
$$\hat{h}_{11}^4 = \bar{h}_{11}^4 \cosh 2\theta + \bar{h}_{12}^4 \sinh 2\theta, \quad \hat{h}_{12}^4 = \bar{h}_{11}^4 \sinh 2\theta + \bar{h}_{12}^4 \cosh 2\theta. \tag{36}$$

As we assume that $K > c$, we have by (10),

$$(\bar{h}_{11}^4)^2 - (\bar{h}_{12}^4)^2 > (\bar{h}_{11}^3)^2 - (\bar{h}_{12}^3)^2.$$

So $(\bar{h}_{11}^4)^2 - (\bar{h}_{12}^4)^2 > 0$, or $(\bar{h}_{11}^3)^2 - (\bar{h}_{12}^3)^2 < 0$. When $(\bar{h}_{11}^4)^2 - (\bar{h}_{12}^4)^2 > 0$, we may choose the smooth function θ so that $\hat{h}_{12}^4 = 0$. Then $\hat{h}_{11}^4 \neq 0$, and by (35), $\hat{h}_{11}^3 = 0$. Similarly, when $(\bar{h}_{11}^3)^2 - (\bar{h}_{12}^3)^2 < 0$, we may choose the smooth function θ so that $\hat{h}_{11}^3 = 0$. Then $\hat{h}_{12}^3 \neq 0$, and by (35), $\hat{h}_{12}^4 = 0$.

Thus, with respect to the frame $\{\hat{e}_i, \bar{e}_\alpha\}$, we have

$$\hat{\omega}_1^3 = t\hat{\omega}^2, \quad \hat{\omega}_2^3 = t\hat{\omega}^1, \quad \hat{\omega}_1^4 = u\hat{\omega}^1, \quad \hat{\omega}_2^4 = u\hat{\omega}^2,$$

and

$$K - c = t^2 + u^2, \quad K_\nu = -2tu.$$

Let q and r be defined as in (33) and (34). Then we have $(t,u) = \pm(q,r)$, or $(t,u) = \pm(r,q)$.

Hence we get the following:

Lemma 5.1. *Let M be a Lorentzian stationary surface in $N_2^4(c)$ satisfying $(K-c)^2 - K_\nu^2 > 0$ and $K > c$. Then we may choose the frame $\{e_A\}$ so that*

$$\omega_1^3 = q\omega^2, \quad \omega_2^3 = q\omega^1, \quad \omega_1^4 = r\omega^1, \quad \omega_2^4 = r\omega^2,$$

or

$$\omega_1^3 = r\omega^2, \quad \omega_2^3 = r\omega^1, \quad \omega_1^4 = q\omega^1, \quad \omega_2^4 = q\omega^2.$$

Proof of Theorem 1.2. (ii) We may assume that $K > c$. For the Lorentzian stationary immersion f, we choose the frame $\{e_A\}$ as in Lemma 5.1. Let $\hat{f} : M \to N_2^4(c)$ be an arbitrary isometric stationary immersion with the same normal curvature K_ν. By Lemma 5.1, we may choose the frame $\{\hat{e}_A\}$ so that

$$\hat{\omega}_1^3 = q\hat{\omega}^2, \quad \hat{\omega}_2^3 = q\hat{\omega}^1, \quad \hat{\omega}_1^4 = r\hat{\omega}^1, \quad \hat{\omega}_2^4 = r\hat{\omega}^2,$$

or

$$\hat{\omega}_1^3 = r\hat{\omega}^2, \quad \hat{\omega}_2^3 = r\hat{\omega}^1, \quad \hat{\omega}_1^4 = q\hat{\omega}^1, \quad \hat{\omega}_2^4 = q\hat{\omega}^2.$$

Then, as in (28) and (29), we have $\hat{\omega}_2^1 = \omega_2^1$ and $\hat{\omega}_4^3 = \omega_4^3$.

Also as in (32), there exist coordinate systems $\{x^1, x^2\}$ and $\{\hat{x}^1, \hat{x}^2\}$ such that

$$\omega^i = \{(K-c)^2 - K_\nu^2\}^{-1/8}\, dx^i,$$

and

$$\hat{\omega}^i = \{(K-c)^2 - K_\nu^2\}^{-1/8}\, d\hat{x}^i.$$

We may write

$$\frac{\partial}{\partial \hat{x}^1} = \cosh\theta \frac{\partial}{\partial x^1} - \sinh\theta \frac{\partial}{\partial x^2}, \quad \frac{\partial}{\partial \hat{x}^2} = -\sinh\theta \frac{\partial}{\partial x^1} + \cosh\theta \frac{\partial}{\partial x^2}$$

for a smooth function θ. As $[\partial/\partial\hat{x}^1, \partial/\partial\hat{x}^2] = 0$, we find that θ is constant. We note that

$$e_1 = (\cosh\theta)\hat{e}_1 + (\sinh\theta)\hat{e}_2, \quad e_2 = (\sinh\theta)\hat{e}_1 + (\cosh\theta)\hat{e}_2.$$

Using (36) in this situation, we can see that the components of the second fundamental form of $\hat{f}$ with respect to the frame $\{e_i, \hat{e}_\alpha\}$ are the same as those of f_θ or $\bar{f}_\theta$ with respect to $\{e_i, e_\alpha\}$. Also, with respect to those frames, $\hat{\omega}_4^3 = \omega_4^3 = \omega_4^3(\theta) = \bar{\omega}_4^3(\theta)$, that is, $\hat{f}$, f_θ and $\bar{f}_\theta$ have the same normal connection. Therefore, $\hat{f}$ coincides with f_θ or $\bar{f}_\theta$ up to congruence. □

Remark. In the case where $(K-c)^2 - K_v^2 \le 0$, we may not choose the frame so that the equation (35) is satisfied. That is, $(T^{\alpha\beta})$ given by

$$T^{\alpha\beta} = h_{11}^\alpha h_{11}^\beta - h_{12}^\alpha h_{12}^\beta$$

may not be diagonalized. It should be a crucial different point compared with the case where $(K-c)^2 - K_v^2 > 0$.

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