TSUKUBA J. MATH.
Vol. 35 No. 2 (2011), 203–213

A GENERALIZATION OF SHELAH'S OMITTING TYPES THEOREM

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Abstract. This note gives generalizations of Shelah's omitting types theorem and Lopez–Escobar's Theorem.

1. Introduction

The omitting types theorem states that for a given countable set S of nonisolated types in T, there is a model of T omitting all the members of S, where T is a theory of a countable language. If L is uncountable, it is easy to construct an L-theory, that is, a counter example to the omitting types theorem. So, we are always interested in a theory with a countable language. There are many generalizations of the theorem. Among these, Shelah's omitting types theorem is of special interest.

Theorem (Shelah). *Let T be a theory of a countable language L. Let R be a set of nonisolated complete types such that $|R| < 2^{\omega}$. Then there is a model $M \models T$ omitting all the members of R.*

If we assume Martin's Axiom, we can omit $< 2^{\omega}$ nonisolated types. Newelski studied the maximum cardinal κ such that we can omit $< \kappa$ nonisolated types. It is known that there is a model of ZFC+$\neg$CH such that $\kappa = \omega_1$ (see [4]). So, we cannot omit the assumption of the completeness of types in Shelah's omitting types theorem.

One of the main theorems in this paper is the following; it simultaneously generalizes the usual omitting types theorem and Shelah's omitting types theorem, and is proved in section 3.

AMS 2000 *subject classification*: 03C68, 03C95.
Keywords and phrases: Shelah's omitting types theorem, nonelementary class, uncountably many types, the omitting types theorem.
Received February 1, 2011.
Revised May 23, 2011.

Theorem. *Let T be a theory formulated in a countable language L and L_0 a sublanguage of L. Let R be a set of nonisolated complete L_0-types such that $|R| < 2^{\omega}$. Let S be a countable set of nonisolated L-types. Then there is a model $M \models T$ omitting all the members of $R \cup S$.*

In section 4, we apply the above theorem to another version of the omitting types theorem, the Lopes–Escobar theorem [3]. The Lopes–Escobar theorem is as follows.

Theorem (Lopez-Escobar). *Let T be a theory formulated in a countable language L having a binary relation $<$. Let S be a countable set of L-types. Suppose that for any $\alpha < \omega_1$, there is a model M_α of T omitting S and with the order type α. Then there is a model $N \models T$ omitting S and with the order type* $\mathbf{Q}$.

This theorem has already been generalized for uncountably many complete types by Tsuboi [8]. We generalize the theorem to our situation.

Theorem. *Let T be a theory formulated in a countable language L and L_0 a sublanguage of L, which have a binary relation $<$. Let R be a set of nonisolated complete L_0-types such that $|R| < 2^{\omega}$. Let S be a countable set of L-types. Suppose that for any $\alpha < \omega_1$, there is a model M_α of T omitting all the members of $R \cup S$ and with the order type α. Then there is a model $N \models T$ omitting all the members of $R \cup S$ and with the order type* $\mathbf{Q}$.

The omitting types theorem is also studied in nonclassical logics, $L_{\omega_1,\omega}$ [2], $L(Q)$ [5], etc. Our generalization of the omitting types theorem implies that Shelah's omitting types theorem holds in PC_δ-classes. Precise definitions are given in section 5.

2. Preliminaries and Notations

Throughout, L is a countable language and T is a countable first-order theory formulated in L. (T may be incomplete.) We always work under T. L-formulas are denoted by $\varphi, \psi, \theta, \chi, \ldots$. We fix a sublanguage $L_0 \subset L$. L_0-formulas are denoted by $\xi, \ldots$. We assume that $\varphi, \psi, \ldots, \xi, \ldots$ are satisfiable unless otherwise noted. Types are (possibly inocomplete) L-types over the empty set. We say a type $p(\bar{x})$ is a complete L_0-type if p consists of only L_0-formulas, and if for every $\xi(\bar{x}) \in L_0$, ξ or $\neg\xi$ is in p.

EXAMPLE 1. Let $L = \{R_i(x) \mid i < \omega\}$. Consider an L-theory T, where for every finite subset $F, G \subset \omega$ with $F \cap G = \emptyset$, there is an element x satisfying $\bigwedge_{i \in F} R_i(x) \wedge \bigwedge_{j \in G} \neg R_j(x)$. Then T is complete and not small. Moreover, there is no isolated complete type. Let R be a set of complete types with $|R| < 2^\omega$. Shelah's omitting types theorem says that there is a model $M \models T$ omitting all the members of R. Take an infinite–coinfinite subset $S \subset \omega$. Set $\Sigma(x) = \{R_i(x) \mid i \in S\}$. Then

1. Σ is a nonprincipal type,
2. Σ has continuum many extensions to nonprincipal complete types.

So, it is not clear that there is a model $N \models T$ omitting all the members of $R \cup \{\Sigma\}$.

3. A Proof of the Theorem

The main idea of the proof is simple; construct continuum many models such that each type in $R \cup S$ is omitted by almost models. Then, there must be a model that omits all the members of $R \cup S$ because $|R \cup S| < 2^\omega$. To prove the theorem, we make the following definitions.

DEFINITION 2. Let $L_0 \subset L$ and $\varphi_i(\bar{x}) \in L$.

1. We say that two L-formulas $\varphi_0(\bar{x})$ and $\varphi_1(\bar{x})$ are L_0-separable in $\bar{x}' \subset \bar{x}$ if there are L_0-formulas $\xi_0(\bar{x}')$ and $\xi_1(\bar{x}')$ such that $T \models \varphi_k(\bar{x}) \to \xi_k(\bar{x}')$ $(k = 0, 1)$, and ξ_0 and ξ_1 are incompatible in T.
2. We say $\varphi_0(\bar{x})$ and $\varphi_1(\bar{x})$ are essentially L_0-separable in $\bar{x}' \subset \bar{x}$ if there are L-formulas $\varphi'_k(\bar{x})$ $(k = 0, 1)$ with $T \models \varphi'_k(\bar{x}) \to \varphi_k(\bar{x})$ $(k = 0, 1)$ such that φ'_0 and φ'_1 are L_0-separable in $\bar{x}'$.
3. Let $\Phi = \varphi_0(\bar{x}), \ldots, \varphi_n(\bar{x})$ be a sequnece of L-formulas. We say that Φ is maximally L_0-separated if for each $i \neq j$ and each subsequence $\bar{x}' \subset \bar{x}$, whenever $\varphi'_i(\bar{x})$ and $\varphi'_j(\bar{x})$ are essentially L_0-separable in $\bar{x}'$ then they are L_0-separable in $\bar{x}'$.

   A maximally L_0-separated sequence $\Phi' = \varphi'_0(\bar{x}), \ldots, \varphi'_n(\bar{x})$ will be called a maximal L_0-separation of Φ if $T \models \varphi'_i(\bar{x}) \to \varphi_i(\bar{x})$ $(i = 0, \ldots, n)$.

LEMMA 3. *Let $\Phi = \varphi_0(\bar{x}), \ldots, \varphi_n(\bar{x})$ be L-formulas. Then there are L-formulas $\varphi'_i(\bar{x})$ $(i \leq n)$ such that $\Phi' = \varphi'_0(\bar{x}), \ldots, \varphi'_n(\bar{x})$ is a maximal L_0-separation of Φ.*

PROOF. Let $\bar{y} \subset \bar{x}$ and suppose that $\varphi_i(\bar{y})$ and $\varphi_j(\bar{y})$ are essentially L_0-separable in $\bar{y}$. Choose an L-formula $\varphi'_i(\bar{x})$ and an L-formula $\varphi'_j(\bar{x})$ witnessing the essential L_0-separability. Then we replace $\varphi_i(\bar{x})$ and $\varphi_j(\bar{x})$ by $\varphi_i'(\bar{x})$ and $\varphi_j'(\bar{x})$, respectively. We repeat this process (finitely many times) and finally we get a desired maximal L_0-separation.

DEFINITION 4. Let $\psi(x_1, \dots, x_n)$ be an L-formula and $s(\bar{y})$ an L-type. We say $\psi(x_1, \dots, x_n)$ totally omits $s(\bar{y})$ if whenever $M \models T$ and $a_1, \dots, a_n \in M$ satisfy $\psi(\bar{x})$ then no tuple from $\{a_1, \dots, a_n\}$ realizes $s(\bar{y})$. Let Σ be a finite set of formulas. We simply say that Σ totally omits s if $\bigwedge \Sigma$ totally omits s.

REMARK 5.
- Let $s(\bar{x})$ be a nonisolated type. Then for every L-formula $\varphi(\bar{x})$ there is an L-formula $\varphi'(\bar{x})$ with $T \models \varphi'(\bar{x}) \to \varphi(\bar{x})$ such that φ' and s are inconsistent.
- It is easy to check that for every L-formula $\varphi(\bar{x})$ and nonisolated type $s(\bar{y})$, there is an L-formula $\psi(\bar{x})$ with $T \models \psi \to \phi$ such that ψ totally omits s.

Next lemma is easy but important for our proof of the theorem.

LEMMA 6. *Let $\varphi_0(\bar{x})$ and $\varphi_1(\bar{x})$ be L-formulas such that they are not essentially L_0-separable in $\bar{x}' \subset \bar{x}$. Then φ_0 and φ_1 isolate the same complete L_0-type $p(\bar{x}')$.*

PROOF. Suppose otherwise. Then it is easy to find an L_0-formula $\xi(\bar{x}')$ such that both $\varphi_0 \wedge \xi$ and $\varphi_1 \wedge \neg\xi$ are satisfiable. Two L-formulas $\varphi_0 \wedge \xi$ and $\varphi_1 \wedge \neg\xi$ are L_0-separable in $\bar{x}'$. Since $T \models \varphi_0 \wedge \xi \to \varphi_0$ and $T \models \varphi_1 \wedge \neg\xi \to \varphi_1$, this means that φ_0 and φ_1 are essentiall L_0-separable. A contradiction.

THEOREM 7. *Let R be a set of nonisolated complete L_0-types such that $|R| < 2^\omega$. Let S be a countalbe set of nonisolated L-types. Then there is a model $M \models T$ omitting all the members of $R \cup S$.*

PROOF. Suppose $Z = \{z_i \mid i < \omega\}$ is a fixed countable set of new variables. We denote a sequence $z_0, z_1, \dots, z_{i-1}$ by $\bar{z}_i$. Enumerate S as $S = \{s_i(\bar{x}_i) : i \in \omega\}$. We may assume that for each $s_n(\bar{x}_n)$, $|\bar{x}_n| \le n$. Let $\{\theta_i(\bar{z}_i, z_i)\}$ be an enumeration of the L-formulas having the form $\exists x \varphi(\bar{z}_i, x) \to \varphi(\bar{z}_i, z_i)$.

By induction, we construct a binary tree $\{\Sigma_\eta(\bar{z}_{len(\eta)}) \mid \eta \in 2^{<\omega}\}$ of finite sets of L-formulas with the following properties: For every $n \in \omega$ and every $\eta \in 2^n$,

1. If $m < n$ then $\Sigma_{\eta|m} \subset \Sigma_{\eta|n}$;
2. $\{\bigwedge \Sigma_\sigma(\bar{z}_n)\}_{\sigma \in 2^n}$ is maximally separated;

3. Σ_η is consistent;
4. Σ_η contains θ_n;
5. Σ_η totally omits each of s_i $(i \le n)$.

Let $\Sigma_{\langle\ \rangle} = \emptyset$ and suppose $\Sigma_\sigma(\bar{z}_n)$ has been defined for every $\sigma \in 2^n$. Take two copies of $\Sigma_\sigma(\bar{z}_n)$ and set

$$\Sigma_\sigma^{0,k}(\bar{z}_n) = \Sigma_\sigma(\bar{z}_n) \quad (k = 0, 1).$$

Then, by Lemma 3, there is a set $\{\psi_{\sigma,k}(\bar{z}_n)\}_{\sigma \in 2^n, k=0,1}$ which is a maximal L_0-separation of $\{\bigwedge \Sigma_\sigma^{0,k}(\bar{z}_n)\}_{\sigma \in 2^n, k=0,1}$. Set

$$\Sigma_\sigma^{1,k}(\bar{z}_n) = \Sigma_\sigma^{0,k}(\bar{z}_n) \cup \{\psi_{\sigma,k}(\bar{z}_n)\}.$$

Next, for each $\sigma \in 2^n$, take an L-formula $\chi_{\sigma,k}(\bar{z}_n) \models \Sigma_\sigma^{1,k}(\bar{z}_n)$ such that $\chi_{\sigma,k}$ totally omits $s_i(\bar{x}_i)$ for every $i \le n$. (Such formula exists by Remark 5.) Set

$$\Sigma_\sigma^{2,k}(\bar{z}_n) = \Sigma_\sigma^{1,k}(\bar{z}_n) \cup \{\chi_{\sigma,k}(\bar{z}_n)\}.$$

Finally set $\Sigma_{\sigma\hat{\ }k} = \Sigma_\sigma^{2,k}(\bar{z}_n) \cup \{\theta_n(\bar{z}_n, z_n)\}$. It is easy to check that $\{\Sigma_\eta(\bar{z}_{n+1})\}_{\eta \in 2^{n+1}}$ satisfies the required conditions 1–5 (with n replaced by $n+1$). So we have succeeded to construct all Σ_η's. Now, for a path $\eta \in 2^\omega$, we define $\Sigma_\eta(Z)$ by $\Sigma_\eta = \bigcup_{n \in \omega} \Sigma_{\eta|n}$. Recall that θ_n has the form $\exists x \varphi(\bar{z}_n, x) \to \varphi(\bar{z}_n, z_n)$. So, by the condition 4, every M_η realizing $\Sigma_\eta(Z)$ is a model of T. By the condition 5, M_η omits all types in S.

Claim A. *For each $p \in R$, $\{\eta \in 2^\omega \mid M_\eta \models \exists \bar{x} p(\bar{x})\}$ is countable.*

We fix $p(\bar{x}) \in R$ and $\bar{z} \subset Z$ with $|\bar{x}| = |\bar{z}|$. Suppose $\Sigma_\eta(Z) \cup p(\bar{z})$ is consistent. Take any $\eta' \neq \eta$. If $\Sigma_{\eta'}(Z) \cup p(\bar{z})$ is also consistent, then $\Sigma_{\eta|n}$ and $\Sigma_{\eta'|n}$ are not essentially L_0-separable in $\bar{z}$, where n is chosen so that $\bar{z} \subset \bar{z}_n$. Hence p must be isolated by a L-formula, by Lemma 6. But R is a set of nonisolated types, a contradiction. So, for each $p \in R$ and $\bar{z} \subset Z$, $\{\eta \in 2^\omega \mid \Sigma_\eta(Z) \cup p(z) \text{ consistant}\}$ has at most one element. This proves the claim, since there are only countably many possible choices of $\bar{z} \subset Z$. (End of Proof of Claim)

Finally, by the claim above and the assumption that $|R| < 2^\omega$, we can find a path $\eta \in 2^\omega$ such that M_η omits R.

Corollary 8. *Suppose $\alpha < 2^\omega$. Let T_0 be a complete L-theory and $p, q_i \in S(T_0)$ $(i < \alpha)$. If for every $i < \alpha$ there is a model M_i such that M_i omits q_i and M_i realizes p, then there is a model N such that N omits all q_i's but N realizes p.*

4. Another Version of Omitting Types Theorem with Uncountably Many Types

Recall that L is a countable language and L_0 a sublanguage of L. In this section we show the following,

Theorem 9. *Let T be a (possibly incomplete) L-theory. Let R be a set of complete L_0-types with $|R| < 2^{\omega}$ and S a countable set of L-types. Fix an L-formula $\chi(x, y)$. Suppose that for any $\alpha < \omega_1$, there is a model M_α of T containing a set $A_\alpha \subset M_\alpha$ such that*

- $A_\alpha = \{a_i^\alpha \mid i \leq \alpha\}$,
- $M_\alpha \models \chi(a_i^\alpha, a_j^\alpha)$ *if and only if* $i < j$,
- M_α *omits all the members of* $R \cup S$.

Then there is a model $N \models T$ with a subset $A \subset N$ such that

- $A = \{a_q \mid q \in \mathbf{Q}\}$,
- $N \models \chi(a_q, a_{q'})$ *if and only if* $q < q'$,
- N *omits all the members of* $R \cup S$.

In the rest of this section, we denote $\chi(x, y)$ by $x < y$. For a tuple $\bar{a}$, the ith element of $\bar{a}$ is denoted by $(\bar{a})_i$. We also denote the βth element $a_{i+\beta}^\gamma$ from a_i^γ in A_γ by $a_i^\gamma + \beta$.

Note that if, with new constants c_q ($q \in \mathbf{Q}$), $T \cup \{c_q < c_{q'} \mid q < q', q, q' \in \mathbf{Q}\}$ isolates no type in $R \cup S$ then the theorem is clear by theorem 7. But, in general, $T \cup \{c_q < c_{q'}\}_{q,q'}$ may isolate some types. (Notice that $T \cup \{c_q < c_{q'}\}_{q,q'}$ may not be complete.) So we need find a theory $T' \supset T$ that isolates no type in $R \cup S$. To construct T', we need some definitions. The following definitions are taken from the proof of Lopez–Escobar's theorem in [2].

Definition 10. 1. An m-sequence is a sequence of tuples of length m.

2. We say an ascending tuple $\bar{b} \in A_\gamma$ of length $m + 1$ is a k-extension ($k \leq m$) of an ascending tuple $\bar{a} \in A_\gamma$ of length m if $(\bar{b})_1 = (\bar{a})_1, \ldots, (\bar{b})_k = (\bar{a})_k$, $(\bar{b})_{k+2} = (\bar{a})_{k+1}, \ldots, (\bar{b})_{m+1} = (\bar{a})_m$.
3. Let Γ be a subset of ω_1. We say that the m-sequence $\{\bar{a}^\gamma \mid \gamma \in \Gamma\}$ is an unbounded m-sequence if
   - Γ is unbounded in ω_1,
   - $\bar{a}^\gamma$ is an ascending tuple of length m of elements of A_γ,

  - for any $\beta \in \omega_1$ there is a $\gamma \in \Gamma$ such that $a^\gamma_\beta < (\bar{a}^\gamma)_1$, $(\bar{a}^\gamma)_1 + \beta < (\bar{a}^\gamma)_2$, $(\bar{a}^\gamma)_2 + \beta < (\bar{a}^\gamma)_3, \dots, (\bar{a}^\gamma)_m + \beta < a^\gamma_\gamma$.

4. Let Γ be a subset of ω_1. Let $X = \{\bar{a}^\gamma \mid \gamma \in \Gamma\}$ be an unbounded m-sequence and $Y = \{\bar{b}^\gamma \mid \gamma \in \Gamma\}$ an unbounded $(m+1)$-sequence. We say Y is a k-extension of X $(0 \le k \le m)$ if for all $\gamma \in \Gamma$, $\bar{b}^\gamma$ is a k-extension of $\bar{a}^\gamma$.
5. We consider the unbounded 0-sequence, the empty sequence. Every unbounded 1-sequence is a 0-extension of the unbounded 0-sequence.

Lemma 11. 1. *There is an unbounded* 1-*sequence.*
2. *Let X be an unbounded m-sequence and $k \le m$. Then there are an unbounded $(m+1)$-sequence Y and an unbounded m-sequence X' such that X' is an unbounded m-sequence, $X' \subset X$, and Y is a k-extension of X'. This condition will be denoted as $X \lhd_k Y$.*

Proof. We show the second with $m = 1$ and $k = 0$, and the other cases are similar. Let $X = \{a^\gamma\}_{\gamma \in \Gamma}$ be an unbounded 1-sequence. Then for any $\beta + \beta + \beta \in \omega_1$ there is a $\gamma \in \Gamma$ such that $a^\gamma_{\beta \cdot 3} < a^\gamma$ (Recall $a^\gamma_{\beta \cdot 3}$ is the $\beta \cdot 3$-th element of A_γ). So we have a 0-extension $\bar{b}^\gamma = a^\gamma_\beta + 1$, a^γ of a^γ. Collect such 0-extension $\bar{b}^\gamma$ of a^γ for every $\beta \in \omega$, then it is a required 2-sequence.

Take a set $C = \{c_q \mid q \in \mathbf{Q}\}$ of new constant symbols. To prove the theorem, it is enough to show that there is an L-theory $T' \supset T \cup \{c_q < c_{q'} \mid q < q'$ and $q, q' \in \mathbf{Q}\}$ such that all the members of R and S are nonisolated types in T', by theorem 7. We fix an enumeration $\{c_{q_n} \mid n < \omega\}$ of C. Let $\bar{c}_n$ be the sequence consisting $c_{q_0}, c_{q_1}, \dots, c_{q_{n-1}}$ with the order of $\mathbf{Q}$ (e.g. if $q_0, q_1, q_2 = 0.5, -1, 0$ then $\bar{c}_3$ is the sequence c_{-1}, c_0, $c_{0.5}$). Most ideas of the following definitions are from [8]. We adapt it to our situation.

Definition 12. Let $X = \{\bar{a}^\gamma \mid \gamma \in \Gamma\}$ be an unbounded m-sequence with $\Gamma \subset \omega_1$ and $\varphi(\bar{x}, \bar{c})$ an $L(\bar{c})$-formula.

1. We say X is $\varphi(\bar{x}, \bar{c})$-uniform if for every L_0-formula $\xi(\bar{x})$ and $\gamma, \gamma' \in \Gamma$, $M_\gamma \models \exists \bar{x}(\varphi(\bar{x}, \bar{a}^\gamma) \wedge \xi(\bar{x}))$ if and only if $M_{\gamma'} \models \exists \bar{x}(\varphi(\bar{x}, \bar{a}^{\gamma'}) \wedge \xi(\bar{x}))$
2. We say X is essentially $\varphi(\bar{x}, \bar{c})$-uniform if there is an unbounded subset $\Gamma' \subset \Gamma$ such that X' is $\varphi(\bar{x}, \bar{c})$-uniform where $X' = \{\bar{a}^\gamma \in X \mid \gamma \in \Gamma'\}$.

Lemma 13. *Let $X = \{\bar{a}^\gamma \mid \gamma \in \Gamma\}$ be an unbounded m-sequence with $\Gamma \subset \omega_1$ and $\varphi(\bar{x}, \bar{c})$ an $L(\bar{c})$-formula. If X is not essentially $\varphi(\bar{x}, \bar{c})$-uniform then there is an*

L_0-formula $\xi(\bar{x})$ such that $X_{\varphi\wedge\xi}$ and $X_{\varphi\wedge\neg\xi}$ are unbounded m-sequences where $X_{\theta(\bar{x},\bar{c})} := \{\bar{a}^{\gamma} \mid M_{\gamma} \models \exists\bar{x}\theta(\bar{x},\bar{a}^{\gamma})\}$.

Proof. Suppose that $X_{\varphi\wedge\xi}$ or $X_{\varphi\wedge\neg\xi}$ is bounded for every L_0-formula $\xi(\bar{x})$. Notice that if $X_{\varphi\wedge\xi}$ is bounded then $X_{\varphi\wedge\xi}$ is countable. So, the union Y of all bounded $X_{\varphi\wedge\xi}$'s is also countable, because L_0 is countable. Set $X' = X \setminus Y$. Then X' is an unbounded m-sequence, and X' is $\varphi(\bar{x},\bar{c})$-uniform by the definition of X'. This means that X is essentially φ-uniform.

Lemma 14. *Let Y and Y' be unbounded m-sequences. Suppose they are not essentially $\varphi_i(\bar{x})$-uniform for $i \le n$. Then there is an L_0-formula $\xi_i(\bar{x})$, $X \subset Y$ and $X' \subset Y'$ such that $X_{\varphi_i\wedge\xi_i} = X$, $X'_{\varphi_i\wedge\neg\xi_i} = X'$ and X, X' are unbounded m-sequences, for each $i \le n$.*

Proof. We show by induction on n. The case $n = 0$ is trivial. Let $n = k+1$. By induction hypothesis, we have $Z \subset Y$ and $Z' \subset Y'$ such that $Z_{\varphi_i\wedge\xi_i} = Z$, $Z'_{\varphi_i\wedge\neg\xi_i} = Z'$ for each $i \le k$. Let ξ^0 be an L_0-formula dividing Z into two uncountable sets $Z_{\varphi_{k+1}\wedge\xi^0}$, $Z_{\varphi_{k+1}\wedge\neg\xi^0}$. By shrinking Z', if necessary, we may assume $Z'_{\varphi_{k+1}\wedge\neg\xi^0} = Z'$. Then, let ξ^1 be an L_0-formula dividing Z' into two uncountable sets $Z'_{\varphi_{k+1}\wedge\xi^1}$, $Z'_{\varphi_{k+1}\wedge\neg\xi^1}$. Either $Z_{\varphi_{k+1}\wedge\xi^0\wedge\xi^1}$ or $Z_{\varphi_{k+1}\wedge\xi^0\wedge\neg\xi^1}$ is uncountable, we can take $\xi^0\wedge\xi^1$ or $\xi^0\wedge\neg\xi^1$ as ξ_{k+1}. Then put $X = Z_{\varphi_{k+1}\wedge\xi_{k+1}}$ and $X' = Z'_{\varphi_{k+1}\wedge\neg\xi_{k+1}}$.

Let $\{\varphi_n(\bar{x},\bar{c}_n) \mid n < \omega\}$ be an enumeration of all $L(C)$-formulas. Also enumerate S as $S = \{s_n(\bar{x}_n) \mid n < \omega\}$. We can assume that for every tuple $(\varphi, s) \in L(C) \times S$, there is n such that $(\varphi, s) = (\varphi_n, s_n)$. So, each member of $L(C)$, S appears infinitely many times in the enumerations. By induction, we construct a binary tree $\{T^{\sigma}(\bar{c}_{len(\sigma)}) \mid \sigma \in 2^{<\omega}\}$ of sets of $L(C)$-formulas and unbounded $len(\sigma)$-sequence $X^{\sigma} = \{\bar{a}_{\gamma} \mid \gamma \in \Gamma_{\sigma}\}$ with the following properties: For every $\sigma, \sigma' \in 2^{<\omega}$ and $n \le len(\sigma)$,

1. $T^{\sigma}(\bar{c}_{len(\sigma)}) \cup \{c_q < c_{q'} \mid c_q, c_{q'} \in \bar{c}_{len(\sigma)}$ and $q < q'\}$ is consistent,
2. $\sigma \subset \sigma'$ then $T^{\sigma} \subset T^{\sigma'}$,
3. M_{γ}, $\bar{a}_{\gamma} \models T^{\sigma}(\bar{c}_{len(\sigma)})$ for uncountably many $\gamma \in \Gamma_{\sigma}$,
4. T^{σ} contains $\exists\bar{x}\varphi_{len(\sigma)}$ or $\neg\exists\bar{x}\varphi_{len(\sigma)}$,
5. T^{σ} contains $\exists\bar{x}(\varphi_{len(\sigma)}(\bar{x},\bar{c}_{len(\sigma)}) \wedge \neg\psi(\bar{x}))$ for some $\psi \in s_{len(\sigma)}$,
6. if X^{σ} is essentially φ_n-uniform then it is φ_n-uniform,
7. if X^{σ} is not essentially φ_n-uniform then $\exists\bar{x}(\xi(\bar{x}) \wedge \varphi_n(\bar{x},\bar{c}_{len(\sigma)})) \in T^{\sigma\hat{\ }0}$ and $\exists\bar{x}(\neg\xi(\bar{x}) \wedge \varphi_n(\bar{x},\bar{c}_{len(\sigma)})) \in T^{\sigma\hat{\ }1}$ for some $\xi \in L_0$.

Let $T^{\langle\,\rangle} = \emptyset$, $X^{\langle\,\rangle}$ the unbounded 0-sequence and suppose $T^{\sigma}(\bar{c}_n)$ and X^{σ} are defined for every $\sigma \in 2^n$. Suppose $\bar{c}_{n+1}$ is a k-extension of $\bar{c}_n$. Take an unbounded $(n+1)$-sequence $Y^{\sigma} \rhd_k X^{\sigma}$ (Lemma 11). If $Y^{\sigma}_{\varphi_{n+1}}$ is uncountable, set

$$Y^{\sigma,0} = Y^{\sigma}_{\varphi_{n+1}}$$

and

$$T^{\sigma,0} = T^{\sigma} \cup \{\exists \bar{x} \varphi_{n+1}(\bar{x}, \bar{c}_{n+1})\}$$

otherwise

$$Y^{\sigma,0} = Y^{\sigma}_{\neg\varphi_{n+1}},$$

$$T^{\sigma,0} = T^{\sigma} \cup \{\neg\exists \bar{x} \varphi_{n+1}(\bar{x}, \bar{c}_{n+1})\}.$$

Recall that $s_{n+1}(\bar{x}_{n+1})$ is countable and M_{α} omits $s_{n+1}(\bar{x}_{n+1})$ for every α. Hence, we can find $\psi(\bar{x}_{n+1}) \in s_{n+1}$ such that $Y^{\sigma,0}_{\varphi_{n+1}\wedge\neg\psi}$ is uncountable. Set

$$Y^{\sigma,1} = Y^{\sigma,0}_{\varphi_{n+1}\wedge\neg\psi},$$

$$T^{\sigma,1} = T^{\sigma,0} \cup \{\exists \bar{x}(\varphi_{n+1}(\bar{x}, \bar{c}_{n+1}) \wedge \psi(\bar{x}))\}.$$

Then, if $Y^{\sigma,1}$ is essentially φ_{n+1}-uniform, by shrinking it, we may assume $Y^{\sigma,1}$ is φ_{n+1}-uniform.

Finally, we consider the φ_j-uniformity of $Y^{\sigma,1}$ $(j \leq N+1)$. If $Y^{\sigma,1}$ is φ_j-uniform for every $j \leq n+1$ then set $X^{\sigma\hat{}0} = X^{\sigma\hat{}1} = Y^{\sigma,1}$ and $T^{\sigma\hat{}0} = T^{\sigma\hat{}1} = T^{\sigma,1}$. Otherwise, assume $Y^{\sigma,1}$ is not essentially φ_j-uniform for some j. Then, take an unbounded $(n+1)$-sequence $X^{\sigma\hat{}i} \subset Y^{\sigma,1}$ $(i = 0,1)$ such that for all $j \leq n$, if $Y^{\sigma,1}$ is not essentially φ_j-uniform, then $X^{\sigma\hat{}0}_{\varphi_j\wedge\xi_j} = X^{\sigma\hat{}0}$ and $X^{\sigma\hat{}1}_{\varphi\wedge\neg\xi_j} = X^{\sigma\hat{}1}$ for some $\xi_j(\bar{x}) \in L_0$ (See lemma 14). We set

$$T^{\sigma\hat{}0} = T^{\sigma,1} \cup \{\exists \bar{x}(\varphi_j(\bar{x}, \bar{c}_j) \wedge \xi_j(\bar{x})\}_j,$$

$$T^{\sigma\hat{}1} = T^{\sigma,1} \cup \{\exists \bar{x}(\varphi_j(\bar{x}, \bar{c}_j) \wedge \neg\xi_j(\bar{x})\}_j.$$

It is easy to check that they satisfy the required conditions. At the end of this inductive construction, we have 2^{ω} complete $L(C)$-theories T^{η} $(\eta \in 2^{\omega})$. By condition 5. and the way of enumerations of $L(C)$ and S, every member of S is not isolated in T^{η}.

Claim A. *The set* $\{\eta \in 2^{\omega} \mid p$ *is isolated in* $T^{\eta}\}$ *is countable for every* $p \in R$.

Suppose $p(\bar{x})$ is isolated by an $L(C)$-formula $\varphi_n(\bar{x}, \bar{c}_n)$ in T^{η} and $T^{\eta'}$. If $X^{\eta|n} = \{\bar{a}_{\gamma} \mid \gamma \in \Gamma_{\eta|n}\}$ is φ_n-uniform then M_{γ} $(\gamma \in \Gamma_{\eta|n})$ realizes $p(\bar{x})$. So, $X^{\eta|n}$ is not essentially φ_n-uniform. We can assume that $\eta|n \neq \eta'|n$, because, by condition 7., if $\eta|n = \eta'|n$, φ_n cannot isolates same complete L_0-type in T^η, $T^{\eta'}$. Therefore, $\{\eta \in 2^\omega \mid p$ is isolated in $T^\eta\}$ is countable. (End of proof of claim)

Hence, there is an $\eta \in 2^\omega$ such that every member of R is nonisolated in T^η. By theorem 7, we have a required model. (End of proof of theorem 9)

5. Omitting Types Theorem with Nonelementary Classes

In this section, we look at the definitions of some nonelementary classes. Then we have Shelah's omitting types theorem for such classes.

Definition 15. Let $\mathscr{K}$ be a class of L-structures. We say $\mathscr{K}$ is an $EC(\aleph_0, \aleph_0)$-class if

- L is countable,
- there is a countable set S of types and an L-theory T such that $M \in \mathscr{K}$ if and only if $M \models T$, and M omits all the members of S.

$\mathscr{K}$ is denoted by $EC(T, S)$.

More general definitions and properties of $EC(\kappa, \lambda)$ can be found in [1]. It is well known that every $EC(\aleph_0, \aleph_0)$-class can be translated to a class defined by an $L_{\omega_1,\omega}$-sentence, and vice versa (see [7], for example). Next, we introduce a PC_δ-class. This is defined by Keisler in [2] with $L_{\omega_1,\omega}$. The following definition of a PC_δ-class is given without $L_{\omega_1,\omega}$. Note that Shelah and Baldwin use other notations, e.g., $PC(\aleph_0, \aleph_0)$, $PC\Gamma(\aleph_0, \aleph_0)$ (see [1]).

Definition 16. Let $\mathscr{K}$ be a class of L-structures. We say that $\mathscr{K}$ is a PC_δ-class if there is a countable language $L' \supset L$ and a class of L'-structures $\mathscr{K}'$ such that

- K' is an $EC(\aleph_0, \aleph_0)$-class, and
- $M' \mid L \in \mathscr{K}$ if and only if $M' \in \mathscr{K}'$ for every L'-structure M'.

To generalize the omitting types theorem for a PC_δ-class, we need definitions of types and isolated types.

Definition 17. Let $\mathscr{K}$ be a class of L-structures. A type $\Sigma(\bar{x})$ in $\mathscr{K}$ is a set of L-formulas with free variables $\bar{x}$ such that there is a structure $M \in \mathscr{K}$ having a realization of $\Sigma(\bar{x})$.

DEFINITION 18. Let $\mathscr{K}$ be either $EC(T,S)$ or the PC_δ-class obtained from $EC(T,S)$ by restricting the language. An isolated type in $\mathscr{K}$ is a type in $\mathscr{K}$ which is isolated in T in the usual sense.

The above definitions and theorem 7 immediately give Shelah's omitting types theorem for PC_δ-classes.

THEOREM 19. *Let $\mathscr{K}$ be a PC_δ-class. Let R be a set of nonisolated complete types in $\mathscr{K}$ such that $|R| < 2^\omega$. Then there is a model $M \models T$ omitting all the members of R.*

We also have Lopez-Escobar's theorem (with uncountably many types) for PC_δ-classes.

THEOREM 20. *Let $\mathscr{K}$ be a PC_δ-class with a countable language L having a binary relation $<$. Let R be a set of complete L-types such that $|R| < 2^\omega$. Suppose that for any $\alpha < \omega_1$, there is a model $M_\alpha \in \mathscr{K}$ omitting R and with the order type α. Then there is a model $N \in \mathscr{K}$ omitting R and with the order type* $\mathbf{Q}$.

References

[1] Baldwin, J. T., Categoricity, Amer Mathematical Society, 2009.
[2] Keisler, H. J., Model theory for Infinitatry Logic. North-Holland, 1971.
[3] Lopez-Escobar, E. G. K., "On definable well-orderings," Fundamenta Mathmatica, vol. 59 (1966), pp. 13–21, 299–300.
[4] Newelski, L., "Omitting types and the real line," The Journal of Symbolic Logic, vol. 52 (1987), pp. 1020–1026.
[5] Shelah, S., "Models with second order properties. III. Omitting types for $L(Q)$" Archiv fur Math Logik und Grundlagenforschung, vol. 21 (1981), pp. 1–11 GgSh:83.
[6] Shelah, S., Classification Theory and the Number of Nonisomorphic Models, North-Holland, 1991. second edition.
[7] Takeuchi, K., "Completeness and The Number of Types for Infinitary Logic," RIMS Kokyuroku, vol. 1718 (2010) Reserch Institute for Mathmatical Sciences, Kyoto University, pp. 75–80.
[8] Tsuboi, A., "Models omitting Given Complete Types," Notre Dame Journal of Formal Logic, vol. 49 (2008), number 4, pp. 393–399.

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