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ON BOUNDARIES OF COXETER GROUPS AND TOPOLOGICAL FRACTAL STRUCTURES

By

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Abstract. In this paper, based on research on rank-one isometries by W. Ballmann and M. Brin and recent research on rank-one isometries of Coxeter groups by P. Caprace and K. Fujiwara, we study a topological fractal structure of boundaries of Coxeter groups. We also show that the limit-point set is dense in a boundary of a Coxeter group and introduce some observations on boundaries of CAT(0) groups with rank-one isometries.

1. Introduction

In this paper, we study boundaries of Coxeter groups, where we suppose that Coxeter groups are finitely generated and infinite. A Coxeter group acts geometrically (i.e. properly and cocompactly by isometries) on a Davis complex which is a CAT(0) space [28] and every Coxeter group is a CAT(0) group. Details of Coxeter groups and Coxeter systems are found in [5], [7], [13], [23] and [31], and details of CAT(0) spaces, CAT(0) groups and their boundaries are found in [6], [9] and [16].

Now we suppose that an infinite group G acts geometrically on a proper CAT(0) space X and G is non-elementary (hence $|\partial X| > 2$).

A hyperbolic isometry g of a proper CAT(0) space X is said to be *rank-one*, if some (any) axis for g does not bound a flat half-plane. In [1, Theorem A], W. Ballmann and M. Brin have proved that if there exists a rank-one isometry $g \in G$ of X then for any two non-empty open subsets U and V of ∂X, there exists

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an element $g \in G$ such that $g(\partial X - U) \subset V$ and $g^{-1}(\partial X - V) \subset U$ where it is possible to choose g to be rank-one (cf. [8], [18]).

This statement implies that if there exists a rank-one isometry $g \in G$ of X then we can say that the boundary ∂X has a *topological fractal structure*; that is, for any proper closed subset F of ∂X and any non-empty open subset U of ∂X, there exists $g \in G$ such that $gF \subset U$.

We first note that if G is hyperbolic then G contains a rank-one isometry (because X does not contain a flat-half plane) and the boundary ∂X has a topological fractal structure.

In particular, if G is hyperbolic and the boundary ∂X is an n-sphere then the boundary $\partial X \approx \mathbf{S}^n$ has a topological fractal structure. This case is the most simple case of boundaries of CAT(0) groups with rank-one isometries. In general, the boundary ∂X with a topological fractal structure is very complex.

In [15], H. Fischer has investigated the boundary $\partial \Sigma$ of the Davis complex of a right-angled Coxeter group whose nerve is a connected closed orientable PL-manifold. These boundaries are typical examples of boundaries with topological fractal structures.

In such a case that G contains a rank-one isometry and ∂X is not an n-sphere, then the boundary ∂X seems to be a topological fractal.

This fractal structure seems to be suggested in some research on boundaries of CAT(0) groups by M. Bestvina (cf. [4]) and some research on cohomology of boundaries of Coxeter groups (cf. [3], [11], [14], [19]).

If the boundary ∂X has a topological fractal structure, then (the action of G on) ∂X is *minimal*; that is, every orbit $G\alpha$ is dense in the boundary ∂X. Indeed if we take $F = \{\alpha\}$ then for any open subset U of ∂X, $gF \subset U$ for some $g \in G$.

Also then (the action of G on) ∂X is *scrambled*; that is, for any two points $\alpha, \beta \in \partial X$ with $\alpha \neq \beta$,

$$\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > 0 \quad \text{and}$$
$$\liminf\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} = 0$$

(cf. [21]). Indeed $\limsup\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} > 0$ always holds ([21, Theorem 3.1]) and if we take $F = \{\alpha, \beta\}$ then for any small open subset U of ∂X, $gF \subset U$ for some $g \in G$, hence $\liminf\{d_{\partial X}(g\alpha, g\beta) \mid g \in G\} = 0$.

Thus if the boundary ∂X is a topological fractal, then ∂X is minimal and scrambled.

We can find recent research on minimality and scrambled sets of boundaries of Coxeter groups in [20] and [21].

From recent research on rank-one isometries of Coxeter groups by P. Caprace and K. Fujiwara [8, Proposition 4.5], we obtain that for a Coxeter system (W, S) such that S is finite and W is infinite and non-elementary, if (W, S) is irreducible and non-affine then the Coxeter group W contains a rank-one isometry of the Davis complex Σ defined by (W, S). Hence a finitely generated, infinite and non-elementary Coxeter group W contains a rank-one isometry if and only if W does not contain a finite-index subgroup which splits as a product $W_1 \times W_2$ where W_1 and W_2 are infinite.

By the observation above, we obtain the following theorem.

Theorem 1.1. *Let (W, S) be a Coxeter system such that W is infinite and non-elementary and S is finite. For the Davis complex Σ of (W, S) and any proper $CAT(0)$ space X on which W acts geometrically, the following statements are equivalent.*

(1) *$(W_{\tilde{S}}, \tilde{S})$ is irreducible and non-affine.*
(2) *W contains a rank-one isometry of Σ.*
(3) *W contains a rank-one isometry of X.*
(4) *$\partial\Sigma$ has a topological fractal structure.*
(5) *$\partial\Sigma$ is minimal.*
(6) *$\partial\Sigma$ is scrambled.*
(7) *∂X has a topological fractal structure.*
(8) *∂X is minimal.*
(9) *∂X is scrambled.*
(10) *Σ does not contain a quasi-dense subspace which splits as a product $\Sigma_1 \times \Sigma_2$ of two unbounded subspaces.*
(11) *X does not contain a quasi-dense subspace which splits as a product $X_1 \times X_2$ of two unbounded subspaces.*
(12) *W does not contain a finite-index subgroup which splits as a product $W_1 \times W_2$ of two infinite subgroups.*

Here $W_{\tilde{S}}$ is the minimum finite-index parabolic subgroup of (W, S) ([13], cf. [20], [21]).

Thus if (W, S) is an irreducible Coxeter system, then W is finite, W is affine or W contains a rank-one isometry.

Hence for any Coxeter system (W, S) and the irreducible decomposition of (W, S) as

$$W = W_{S_1} \times \cdots \times W_{S_k} \times W_{S_{k+1}} \times \cdots \times W_{S_n},$$

each W_{S_i} is finite, affine or contains a rank-one isometry.

It is known that the following problem is open.

Question. Suppose that a group G acts geometrically on a proper CAT(0) space X. Then is it the case that the limit-point set $\{g^\infty \mid g \in G, o(g) = \infty\}$ is dense in the boundary ∂X?

Here g^∞ is the limit-point of the boundary ∂X to which the sequence $\{g^i x_0 \mid i \in \mathbf{N}\} \subset X$ converges in $X \cup \partial X$, where x_0 is a point of X and the limit-point g^∞ is not depend on x_0. We note that any element g of a CAT(0) group G with the order $o(g) = \infty$ is a hyperbolic isometry.

We obtain a positive answer to this question for Coxeter groups.

Theorem 1.2. *Suppose that a finitely generated infinite Coxeter group W acts geometrically on a proper CAT(0) space X. Then the limit-point set $\{w^\infty \mid w \in W, o(w) = \infty\}$ is dense in the boundary ∂X.*

Finally, we introduce some observations on boundaries of CAT(0) groups with rank-one isometries in Section 4, which relates to local properties of boundaries of CAT(0) groups.

2. Rank-one Isometries of Coxeter Groups and Topological Fractal Structures of Their Boundaries

We prove Theorem 1.1.

Proof of Theorem 1.1. We first obtain the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (12) from [8, Proposition 4.5] and the observation in Section 1. Also (2) $\Leftrightarrow$ (3) holds by [1, Theorem B].

From the observation in Section 1 on rank-one isometries and topological fractal structures of boundaries, we obtain (2) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6), also, (3) $\Rightarrow$ (7), (7) $\Rightarrow$ (8) and (7) $\Rightarrow$ (9).

Concerning scrambled sets of boundaries, [21, Theorem 5.5] implies (6) $\Rightarrow$ (10) and (9) $\Rightarrow$ (11).

Also concerning minimality of boundaries, [20, Theorem 6.4] implies (5) $\Rightarrow$ (12) and (8) $\Rightarrow$ (12).

By splitting theorems (cf. [22], [27]), we obtain (10) $\Rightarrow$ (12) and (11) $\Rightarrow$ (12) (cf. [20, Proposition 6.3]).

Therefore the statements (1)–(12) are equivalent. □

3. On Limit-Point Sets of Boundaries of Coxeter Groups

We prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that a finitely generated infinite Coxeter group W acts geometrically on a proper CAT(0) space X.

Here there exists $S \subset W$ such that (W, S) is a Coxeter system. Now we consider the irreducible decomposition of (W, S) as

$$W = W_{S_1} \times \cdots \times W_{S_k} \times W_{S_{k+1}} \times \cdots \times W_{S_n}$$

where each (W_i, S_i) is irreducible and we may suppose that W_{S_i} is infinite for any $i = 1, \ldots, k$ and W_{S_i} is finite for any $i = k+1, \ldots, n$. Let $W' = W_{S_1} \times \cdots \times W_{S_k}$. Then W' is a finite-index subgroup of W and acts geometrically on the CAT(0) space X (where W' is the minimum finite-index parabolic subgroup of (W, S)).

Here we note that every Coxeter group has finite center. Hence by the splitting theorem [22, Theorem 2] and [27, Corollary 10], X contains a closed convex W'-invariant quasi-dense subspace X' which splits as a product $X' = X_1 \times \cdots \times X_k$ where the action of $W' = W_{S_1} \times \cdots \times W_{S_k}$ on $X' = X_1 \times \cdots \times X_k$ splits and W_{S_i} acts geometrically on X_i for each $i = 1, \ldots, k$.

Then every irreducible infinite Coxeter group W_{S_i} is either affine or contains a rank-one isometry by [8, Proposition 6.5] and the observation in Section 1.

If W_{S_i} is affine, then W_{S_i} contains a finite-index subgroup which isomorphic to $\mathbf{Z}^{n_i}$ and X_i contains a quasi-dense subspace which isometric to $\mathbf{R}^{n_i}$. Hence the limit-point set $\{w_i^\infty \mid w_i \in W_i, o(w_i) = \infty\}$ is dense in the boundary ∂X_i.

Also if W_{S_i} contains a rank-one isometry, then the action of W_{S_i} on the boundary ∂X_i is minimal. Then for some (any) $w \in W_{S_i}$ with $o(w) = \infty$,

$$W_{S_i} w^\infty = \{a w^\infty \mid a \in W_{S_i}\} = \{(a w a^{-1})^\infty \mid a \in W_{S_i}\}$$

is dense in the boundary ∂X_i (cf. [20, Proposition 6.2]). Hence the limit-point set $\{w_i^\infty \mid w_i \in W_i, o(w_i) = \infty\}$ is dense in the boundary ∂X_i.

Therefore, by a similar argument to the proof of [20, Proposition 6.5], we obtain that the limit-point set $\{w^\infty \mid w \in W, o(w) = \infty\}$ is dense in the boundary ∂X. □

4. Observations on Boundaries of CAT(0) Groups with Rank-One Isometries

We introduce some observations on boundaries of CAT(0) groups with rank-one isometries.

Now we suppose that a group G acts geometrically on a proper CAT(0) space X and suppose that G contains a rank-one isometry (hence the boundary ∂X has a topological fractal structure).

Let V be a non-empty open subset of ∂X whose closure cl V is a proper subset of ∂X. Then there exists a rank-one isometry $g \in G$ as $g^\infty \in V$, because the limit-point set of rank-one isometries in G is dense in ∂X. Indeed ∂X is minimal and

$$Gg^\infty = \{ag^\infty \mid a \in G\} = \{(aga^{-1})^\infty \mid a \in G\}$$

is dense in the boundary ∂X.

Every rank-one isometry acts with *north-south dynamics* on the boundary ∂X (cf. [18, p. 7]). Hence, since g is a rank-one isometry of X and $g^\infty \in V$, the set $\{g^i V \mid i \in \mathbf{N}\}$ is a neighborhood basis for g^∞ in ∂X. Here all $g^i V$ are homeomorphic to V.

Thus if there exists a non-empty open subset V of ∂X whose closure cl V is a proper subset of ∂X such that V has some *topological property* (P), then ∂X has the locally *topological property* (P) at the limit-point g^∞.

Also for any rank-one isometry $h \in G$, we can consider the limit-point $h^\infty \in \partial X$. Then Gh^∞ is dense in ∂X, since ∂X is minimal. Hence $ah^\infty \in V$ for some $a \in G$. Then $h^\infty \in a^{-1}V$ and $a^{-1}V$ is homeomorphic to V. Thus the boundary ∂X has the locally *topological property* (P) at the limit-point h^∞ of all rank-one isometries $h \in G$.

As one example, if there exists a non-empty *connected* open subset V of ∂X whose closure cl V is a proper subset of ∂X, then ∂X is locally *connected* at the limit-points g^∞ of all rank-one isometries $g \in G$.

Moreover if ∂X is non-locally connected at some point $\alpha \in \partial X$, then ∂X is non-locally connected at $g\alpha$ for all $g \in G$. Here $G\alpha$ is also dense in ∂X.

It seems that these arguments relate to research on local connectivity of boundaries of CAT(0) groups by M. Mihalik, K. Ruane and S. Tschantz ([25], [26]) and research on cut-points and limit-points of boundaries of CAT(0) groups by P. Papasoglu and E. L. Swenson ([29], [30]).

Also as one application, we obtain the following theoreom by a similar argument to the proof of [24, Theorem 4.4].

Theorem 4.1. *If a $CAT(0)$ group G with a rank-one isometry acts geometrically on a proper $CAT(0)$ space X, then the following statements are equivalent*:

(i) *the boundary* ∂X *is an* n*-manifold,*
(ii) *the boundary* ∂X *of* X *contains some closed neighborhood* U *which is homeomorphic to an* n*-ball,*
(iii) *the boundary* ∂X *is homeomorphic to an* n*-sphere.*

Proof. We first note that the implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are obvious. Hence now we show the implication (ii) $\Rightarrow$ (iii).

Suppose that (iii) holds; that is, the boundary ∂X of X contains some closed neighborhood U which is homeomorphic to an n-ball. For a point $\alpha \in \partial X - U$, there exists $g \in G$ such that $g\alpha \in \operatorname{Int} U$, since the action of G on ∂X is minimal. Then $V := g^{-1}U$ is a neighborhood of α which is homeomorphic to an n-ball. Let U' and V' be a proper subsets of $\operatorname{Int} U$ and $\operatorname{Int} V$ respectively such that U' and V' are homeomorphic to an n-ball and $U' \cap V' = \emptyset$. Let $F = \partial X - \operatorname{Int} U'$. Then there exists $g' \in G$ such that $g'F \subset V'$, because the boundary ∂X has a topological fractal structure. Then $g'U' \cup V' = \partial X$ and $g'U'$ and V' are homeomorphic to an n-ball. (Moreover, $g'U \cup V = \partial X$ and $g'U$ and V are homeomorphic to an n-ball.) Using some argument on bicollars of n-disks by the generalized Schoenflies theorem, we obtain that ∂X is homeomorphic to an n-sphere (cf. [24, Theorem 4.4]). □

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