TSUKUBA J. MATH.
Vol. 35 No. 1 (2011), 1–12

THE GROMOV-HAUSDORFF DISTANCES BETWEEN ALEXANDROV SPACES OF CURVATURE BOUNDED BELOW BY 1 AND THE STANDARD SPHERES

By

Ayato MITSUISHI

Abstract. Main result in the present paper is the following: If an n-dimensional Alexandrov spaces X^n of curvature ≥ 1 has radius greater than $\pi - \varepsilon$, then the Gromov-Hausdorff distance between X^n and the standard sphere $\mathbf{S}^n$ is less than $\tau(\varepsilon)$. Here, $\tau(\varepsilon)$ is an explicit positive function depending only on ε such that $\lim_{\varepsilon \to 0} \tau(\varepsilon) = 0$. We prove this by using quasigeodesics on Alexandrov spaces.

1. Introduction and Main Results

Alexandrov spaces are metric spaces having a generalized notion of sectional curvature bounds. Let X be an Alexandrov space with curvature bounded from below by 1 possibly of infinite dimensional. It is known by [3] that X has diameter less than or equal to π. For a metric space A, its radius is defined by $\operatorname{rad} A = \inf_{a \in A} \sup_{a' \in A} d(a, a')$. Obviously, $\frac{1}{2} \operatorname{diam} A \leq \operatorname{rad} A \leq \operatorname{diam} A$. The maximum radius theorem states that if X has radius π then X is isometric to the unit sphere of some Hilbert space with the angle metric ([8], cf. [2]). When we write X^n for a natural number n, this denotes an n-dimensional X. Perelman-Petrunin [14], (cf. [11]) and Grove-Petersen [5] proved, in different ways, that if X^n has radius greater than $\frac{\pi}{2}$, then X^n is homeomorphic to $\mathbf{S}^n$.

Let us consider the Riemannian case. Shiohama and Yamaguchi [16] proved

THEOREM 1.1 ([16]). *Let M^n be an n-dimensional Riemannian manifold of sectional curvature greater than or equal to 1. If M^n has radius greater than $\pi - \varepsilon$, then the Gromov-Hausdorff distance $d_{GH}(M^n, \mathbf{S}^n)$ between M^n and $\mathbf{S}^n$ is less than $\tau(\varepsilon)$. Here, $\tau(\varepsilon)$ is an explicit positive function depending only on ε such that $\lim_{\varepsilon \to 0} \tau(\varepsilon) = 0$.*

AMS 2000 subject classifications: 53C20, 53C21.
Keywords and phrases: Alexandrov space, Sphere theorem, quasigeodesic.
Received January 8, 2010.

Furthermore, if ε is sufficiently small depending on n, then M^n must be diffeomorphic to $\mathbf{S}^n$ with a diffeomorphism having the norm of its differential $\tau_n(\varepsilon)$-close to 1. Here, $\tau_n(\varepsilon)$ is an explicit positive function depending on ε and n such that $\lim_{\varepsilon\to 0}\tau_n(\varepsilon)=0$.

Now, a natural question must emerge. Is it true that if an Alexandrov space X^n of curvature ≥ 1 has radius greater than $\pi-\varepsilon$, then the Gromov-Hausdorff distance $d_{GH}(X^n,\mathbf{S}^n)$ is less than $\tau(\varepsilon)$? In the present paper, we solve the above problem affirmatively. The main results are

THEOREM 1.2. *Let n be a nonnegative integer. Let X^n be an n-dimensional Alexandrov space of curvature ≥ 1. If its radius* rad X^n *is greater than $\pi-\varepsilon$ then $d_{GH}(X^n,\mathbf{S}^n)$ is less than $\tau(\varepsilon)$. Here, $\tau(\varepsilon)$ is an explicit function depending only on ε such that $\lim_{\varepsilon\to 0}\tau(\varepsilon)=0$.*

In particular, for a sequence X_i^n of n-dimensional Alexandrov spaces of curvature ≥ 1, rad $X_i^n\to\pi$ *as $i\to\infty$ is equivalent to $X_i^n\to\mathbf{S}^n$ as $i\to\infty$ in the Gromov-Hausdorff topology.*

Also, the stability theorems in [3, Theorem 9.5] or [17, Corollary 0.4], together with Theorem 1.2 implies

COROLLARY 1.3. *For any nonnegative integer n, there is positive constant $\varepsilon_0(n)$ such that if an n-dimensional Alexandrov space X^n of curvature ≥ 1 having radius $>\pi-\varepsilon$ for $\varepsilon\leq\varepsilon_0(n)$, then X is $\tau_n(\varepsilon)$-isometric to the standard sphere $\mathbf{S}^n$, i.e. there is a bijective map $f: X\to\mathbf{S}^n$ such that*

$$\sup_{x\neq y\in X}\left|\frac{d(f(x),f(y))}{d(x,y)}-1\right|<\tau_n(\varepsilon).$$

Here, $\tau_n(\varepsilon)$ is an explicit function depending on ε and n such that $\lim_{\varepsilon\to 0}\tau_n(\varepsilon)=0$.

Now we define $\mathscr{M}(n,\varepsilon)$ as the set of all isometry classes of n-dimensional Alexandrov spaces of curvature ≥ 1 having radius $>\pi-\varepsilon$, and

$$\theta_n(\varepsilon):=\max\{d_{GH}(X^n,\mathbf{S}^n)\mid X^n\in\mathscr{M}(n,\varepsilon)\}.$$

In this notation, Theorem 1.2 states the existence of an explicit estimate: $\theta_n(\varepsilon)\leq\tau(\varepsilon)$, which is not depending on n. Grove and Petersen essentially proved

$\lim_{\varepsilon\to 0}\theta_n(\varepsilon)=0$, in the proof of [5, Theorem 3] (cf. [2, Exercise 10. 9. 15]). Their proof was done by using the stability theorem for the unit spheres [3, Theorem 9.5] (cf. [10]) and the maximal radius theorem, and the filling radius [4].

To prove Theorem 1.1, Shiohama and Yamaguchi used the geodesic completeness of complete Riemannian manifolds. But on an Alenxadrov space, a geodesic can not be extended to a complete geodesic. To avoid this difficulty, we will use quasigeodesics on Alexandrov spaces instead of geodesics.

Quasigeodesics on convex hypersurfaces in $\mathbf{R}^3$ were introduced by A. D. Alexandrov [1]. The precise history of them were written in [14], [13]. For finite dimensional Alexandrov spaces, Perelman-Petrunin formulated them as curves satisfying some comparison inequality (2.3.1) which is similar to that for a geodesic on an Alexandrov space. (cf. Definition 2.3). We will use the following Theorem 1.4 instead of the geodesic completeness.

THEOREM 1.4 ([13], [14]). *Any finite dimensional Alexandrov space X satisfies the following property* (1.4.1).

(1.4.1) For any $p\in X$ and $\xi\in\Sigma_p$, there is a quasigeodesic $\gamma:[0,a)\to X$ such that $\gamma(0)=p$ and $\gamma^+(0)=\xi$ and any quasigeodesic can be extended to a complete quasigeodesic.

Since $\tau(\varepsilon)$ in Theorem 1.2 does not depend on n, we expect to prove that an infinite dimensional Alexandrov space of curvature ≥ 1 having radius $>\pi-\varepsilon$ is $\tau(\varepsilon)$-close to the unit sphere in a Hilbert space in the sense of Gromov-Hausdorff distance.

We can have a weak answer for the above expectation. Under the assumption (1.4.1), we can prove the next statement.

THEOREM 1.5. *Let X be an Alexandrov space of curvature ≥ 1 possibly of infinite dimensional. Assume that X satisfies the property* (1.4.1). *If* $\operatorname{rad} X>\pi-\varepsilon$ *then the Gromov-Hausdorff distance $d_{GH}(X,\Sigma\Sigma_p)$ between X and the suspension $\Sigma\Sigma_p$ of the space of directions Σ_p at any $p\in X$ is less than $\tau(\varepsilon)$.*

In particular, if Σ_p is isometric to the unit sphere $\mathbf{S}_{\mathscr{H}}$ of a Hilbert space $\mathscr{H}$ for some $p\in X$, then $d_{GH}(X,\mathbf{S}_{\mathscr{H}})<\tau(\varepsilon)$.

A point p in an Alexandrov space X^n is regular if its space of the directions Σ_p is isometric to $\mathbf{S}^n$. Burago-Gromov-Perelman [3] and Otsu-Shioya [9] proved the next theorem.

THEOREM 1.6 ([3], [9]). *Let X^n be an n-dimensional Alexandrov space of curvature $\geq \kappa$. Then a subset R_X consisting of all regular points of X^n has full n-dimensional Hausdorff measure. In particular, R_X is dense in X^n.*

Theorem 1.6 together with Theorem 1.4 and Theorem 1.5 implies Theorem 1.2. Therefore we have only to prove Theorem 1.5. This will be done in the last section.

Finally, in this section, we provide a few problems.

PROBLEM 1.7. (1) Is Theorem 1.4 valid without the assumption of finiteness of dimension of X?

(2) Is there a regular point p in an infinite dimensional Alexandrov space X such that Σ_p is isometric to the unit sphere of a Hilbert space? Moreover, are there regular points densely in X?

(3) Is there a universal constant $\varepsilon_0 > 0$ instead of $\varepsilon_0(n)$ such that Corollary 1.3 is valid? Can we take $\tau(\varepsilon)$ instead of $\tau_n(\varepsilon)$ in Corollary 1.3? (Even if X^n is a Riemannian manifold, these problems are still open.)

(4) For an infinite dimensional Alexandrov space X of curvature ≥ 1 having radius rad $X > \pi - \varepsilon$, is it true that X and $\Sigma\Sigma_p$ for some $p \in X$ are $\tau(\varepsilon)$-isometric? Moreover, is it true that X and the unit sphere $\mathbf{S}_{\mathscr{H}}$ for some Hilbert space $\mathscr{H}$ are $\tau(\varepsilon)$-isometric?

2. Settings

First of all, we recall the definitions and properties around Alexandrov spaces. The reader may refer to [2], [3], [15] for more informations.

For a while, A denotes a metric space, and fix p, $q \in A$ and $\kappa \in \mathbf{R}$. The distance between p and $q \in A$ is denoted by $d(p,q)$, $d_X(p,q)$, or $|pq|$. The length of a curve γ is denoted by $L(\gamma)$. The symbol κ_+ denotes $\max\{\kappa, 0\}$, and M_κ denotes the κ-plane which is a simply-connected surface of constant curvature κ. Geodesics are always parametrized by arclength. pq denotes a geodesic from p to q. $\Sigma_p = \Sigma_p X$ denotes the space of directions at p. The distance function on Σ_p will be denoted by $\angle$. $C_p^{(\kappa)} = C^{(\kappa)}\Sigma_p$ denotes the (κ-)tangent cone at p which is the κ-cone over Σ_p. Here, the κ-cone over $A = (A, d_A)$ is the completion of a warped product pseudo-metric space $(A, \min\{d_A, \pi\}) \times_{\mathrm{sn}_\kappa} \left[0, \frac{\pi}{\sqrt{\kappa_+}}\right]$ with the warping function sn_κ. Here, $\mathrm{sn}_\kappa : \left[0, \frac{\pi}{\sqrt{\kappa_+}}\right] \to \mathbf{R}$ is the solution of the ODE: $\mathrm{sn}_\kappa'' + \kappa\, \mathrm{sn}_\kappa = 0$, $\mathrm{sn}_\kappa(0) = 0$, and $\mathrm{sn}_\kappa'(0) = 1$. An element $(a,t) \in C^{(\kappa)}A$ is often denoted by ta, and $1a$ is simply denoted by a. Set $|(a,t)| = t$ for $(a,t) \in C^{(\kappa)}A$. The 1-cone is called the suspension over A denoted by ΣA. A point $\Sigma_p \times \{0\}$ (respectively $\Sigma_p \times \{\pi\}$) in $\Sigma\Sigma_p$ is denoted by o_p (respectively $-o_p$).

For three points a_0, a_1, a_2 in a metric space with $a_0 \neq a_1, a_2$ and $|a_0a_1| + |a_1a_2| + |a_2a_0| < \frac{2\pi}{\sqrt{\kappa_+}}$, there are points $\bar{a}_i$ in M_κ, uniquely up to isometry, such that $|a_ia_j| = |\bar{a}_i\bar{a}_j|$ for $i, j = 1, 2, 3$, We denote $\tilde{\angle}a_1a_0a_2$ by $\angle\bar{a}_1\bar{a}_0\bar{a}_2$, and call it a comparison angle of $a_1a_0a_2$.

DEFINITION 2.1. A metric space X is called *an Alexandrov space of curvature* $\geq \kappa$ if X satisfies the following two properties:

(1) The distance $d(x, y)$ between any two points x, y in X is given by the infimum of the lengths $L(\gamma)$ of curves γ connecting x and y.

(2) There is a open covering $\{U_\alpha\}$ of X such that for any α and x_0, x_1, x_2, x_3 in U_α with $x_0 \neq x_i$ and $|x_0x_i| + |x_0x_j| + |x_ix_j| < \frac{2\pi}{\sqrt{\kappa_+}}$ for $i, j \in \{1, 2, 3\}$, we have

$$\tilde{\angle}x_1x_0x_2 + \tilde{\angle}x_2x_0x_3 + \tilde{\angle}x_3x_0x_1 \leq 2\pi.$$

There are many other equivalent definitions for Alexandrov spaces. From the definition, any geodesic in an Alexandrov space does not branch.

A geodesic pq is called almost extendable if for all $\varepsilon > 0$ there is x such that $\bar{\angle}qpx > \pi - \varepsilon$. A subset in a topological space is called G_δ if it is intersection of countable collection of open subsets. Plaut proved the following theorem [15, Theorem 1.4].

THEOREM 2.2 ([15]). *Let X be an Alexandrov space of* (*locally*) *curvature bounded from below and $p \in X$. Then there is a dense G_δ subset J_p such that for all $q \in J_p$ there is a unique, almost extendable geodesic pq.*

For a function $f : X \to \mathbf{R}$, its absolute gradient $|\nabla_x f|$ at $x \in X$ is defined as

$$|\nabla_x f| = \max\left\{\limsup_{y \to x} \frac{f(y) - f(x)}{d(x, y)}, 0\right\}.$$

In this notation, J_p is precisely given by (cf. [15])

$$J_p = \{x \in X \mid |\nabla_x \operatorname{dist}_p| = 1\}.$$

For a geodesic $\gamma = pq$ on X with $\gamma(0) = p$, $\gamma^+(0) = \uparrow_p^q \in \Sigma_p$ denotes the direction of pq at p and, we put $\log_p q = (\uparrow_p^q, d(p, q)) \in C_p$. For a curve $\gamma : [0, a] \to X$ on an Alexandrov space X, a (forward) direction $v = \gamma^+(0) \in C_p$ of γ at $p = \gamma(0)$ is defined as

$$v = \lim_{t_j \to 0} \log_p x_j$$

if its limit exists, where $x_j \in J_p$ is any sequence with $\lim_{t_j \to 0} \frac{d(x_j, \gamma(t_j))}{t_j} = 0$.

For two curves $\alpha : [0, a) \to X$ and $\beta : [0, b) \to X$ having the directions at $t = 0$ with $\alpha(0) = \beta(0)$, the angle $\angle(\alpha, \beta)$ between these curves at $t = 0$ is defined by $\angle\left(\frac{\alpha^+(0)}{|\alpha^+(0)|}, \frac{\beta^+(0)}{|\beta^+(0)|}\right)$ in $\Sigma_{\alpha(0)}$. For two geodesics pq and pr, denotes $\angle qpr$ by the angle between pq and pr.

Perelman and Petrunin formulated quasigeodesics on Alexandrov spaces as follows.

DEFINITION 2.3 ([13], [14]). A curve parametrized by arclength $\gamma : [0, a] \to X$ is called (κ-)*quasigeodesic* on an Alexandrov space X of curvature $\geq \kappa$ if there is a unique forward direction $\gamma^+(t) \in C_{\gamma(t)}$ at any $t \in [0, a)$, and for any $p \in J_{\gamma(t)}$ with $d(\gamma(s), p) < \pi/\sqrt{\kappa_+}$ for all $s \in [t, a]$, we have

$$d(\gamma(s), p) \leq d(\overline{\gamma}(s), \overline{p}) \quad \text{for any } s \in [t, a], \tag{2.3.1}$$

where $\overline{\gamma} : [t, a] \to M_\kappa$ is a geodesic with length $a - t$ such that

$$d(\gamma(t), p) = d(\overline{\gamma}(t), \overline{p}) \quad \text{and} \quad \angle(\gamma^+(t), \uparrow^p_{\gamma(t)}) = \angle(\overline{\gamma}, \overline{\gamma}(t)\overline{p}). \tag{2.3.2}$$

The pair of geodesics $(\overline{\gamma}, \overline{\gamma}(t)\overline{p})$ satisfying (2.3.2) as above is called *a comparison hinge* of the hinge $(\gamma, \gamma(t)p)$. A quasigeodesic with infinite length is called a *complete quasigeodesic*.

There are many other equivalent definition of quasigeodesics in [13] and [14]. From the definition, geodesics on an Alexandrov space of curvature $\geq \kappa$ are (κ-)quasigeodesics. If the dimension of X is finite, then the assumption of the existence of forward directions in a quasigedesic γ can be taken off in Definition 2.3. (cf. [14], [13]).

REMARK 2.4. Quasigeodesics may branch. Set $C = C^{(0)}\Sigma$ the Euclidean cone over an Alexandrov space Σ of curvature ≥ 1. For $\xi \in \Sigma$, the corresponding ray γ_ξ in C is naturally defined as $\gamma_\xi(t) = (\xi, t)$ for $t \in [0, \infty)$. Assume that Σ has the diameter $\leq \pi/2$. Then for any ξ, $\eta \in \Sigma$ we have concatenating curve $\gamma : \mathbf{R} \to C$ such that $\gamma(-t) = \gamma_\xi(t)$ and $\gamma(t) = \gamma_\eta(t)$ for any $t \geq 0$ which is a quasigeodesic starting at $\gamma(-1) = \gamma_\xi(1) = (\xi, 1) \in C$ tangent to the direction $\gamma^+_\xi(1 - t)|_{t=1} \in \Sigma_{(\xi, 1)}$ and branch at the origin.

Quasigeodesics can not branch in geodesic directions:

LEMMA 2.5. *Let X be an Alexandrov space with curvature $\geq \kappa$ and $p \in X$. Suppose that there is a geodesic $\gamma : [0, a] \to X$ with initial vector $\xi = \gamma^+(0) \in \Sigma_p$ and*

$p = \gamma(0)$. *Then any quasigeodesic* σ *starting at* $\sigma(0) = p$ *with initial vector* $\sigma^+(0) = \xi$ *coincides with* γ *on* $[0.a]$.

PROOF. Take any $t \in [0, a]$. By the assumption, the comparison hinge $(\bar{\gamma}, \bar{\sigma})$ of (γ, σ) has angle $\angle(\bar{\gamma}, \bar{\sigma}) = \angle(\xi, \xi) = 0$. Thus by the hinge comparison (2.3.1), we have $\gamma(t) = \sigma(t)$. □

Now, we summarize the fundamental propositions for Alexandrov spaces which can be found in [3]. We will implicitly use those propositions in the proof of main results.

PROPOSITION 2.6 ([3]). *Let* X *be an Alexandrov space of curvature* $\geq \kappa$. *The following are true.*

(1) *For any* $x_0, x_1, x_2, x_3 \in X$ *with* $|x_0x_i| + |x_0x_j| + |x_ix_j| < \frac{2\pi}{\sqrt{\kappa_+}}$ *for* $i, j \in \{1, 2, 3\}$, *we have* $\sum_{1 \leq i < j \leq 3} \tilde{\angle} x_i x_0 x_j \leq 2\pi$.

(2) *For any* ξ_1, ξ_2, ξ_3 *in the space of directions* Σ_p *of any* $p \in X$, *we have* $|\xi_1\xi_2| + |\xi_2\xi_3| + |\xi_3\xi_1| \leq 2\pi$.

(3) *If* X *has curvature* ≥ 1 *then for any* $x_1, x_2, x_3 \in X$, *we have* $|x_1x_2| + |x_2x_3| + |x_3x_1| \leq 2\pi$.

3. Proof of Main Results

We start to prove Theorem 1.5 (and hence 1.2) in this section. Its proof will be done along the same line as in [16]. However, as expressed in the introduction, we make use of the existence property (1.4.1) for quasigeodesics instead of the geodesic completeness.

Now, we fix an Alexandrov space X of curvature ≥ 1 having radius $> \pi - \varepsilon$ satisfying the property (1.4.1). For any $p \in X$, we denote p^* a point which satisfies $d(p, p^*) > \pi - \varepsilon$. Since we assume radius $> \pi - \varepsilon$, we can have p^*.

LEMMA 3.1. *The radius of* $\Sigma\Sigma_p$ *is greater than* $\pi - \varepsilon$ *for any* $p \in X$.

PROOF. For any v in $\Sigma\Sigma_p \backslash \{o_p, -o_p\}$, take a quasigeodesic $\gamma : [0, |v|] \to X$ such that $\gamma(0) = p$ and $\gamma^+(0) = \frac{v}{|v|}$. Put $x = \gamma(|v|)$ and take x^* in J_p. Take geodesics $\bar{\gamma}$ from o_p to v, and $\bar{\sigma}$ from o_p to $\log_p(x^*)$ in $\Sigma\Sigma_p$. By the construction, $(\bar{\gamma}, \bar{\sigma})$ configurates a comparison hinge of (γ, px^*) on $\Sigma\Sigma_p$. By (2.3.1) we have

$$d(v, \log_p(x^*)) \geq d(x, x^*) > \pi - \varepsilon.$$

Therefore it completes the proof of the lemma. □

REMARK 3.2. An argument similar to the proof of [14, Proposition] or [11, Proposition 3.1] can be applied to our X, to prove that the radius of Σ_p is greater than $\pi - \varepsilon$. This also implies Lemma 3.1.

We will use the next notation. A function $\tau(\delta|\varepsilon)$ is an explicit positive function depending only on δ and ε defined on $\{(\delta, \varepsilon) \,|\, \delta > 0 \text{ and } 0 < \varepsilon < \tau(\delta)\}$ for some $\tau(\delta)$ such that $\lim_{\varepsilon \to 0} \tau(\delta|\varepsilon) = 0$ for fixing δ.

LEMMA 3.3. *Let γ be a quasigeodesic on X starting at p ending at x. Let σ be a geodesic starting at p ending x^*. If $\delta < |px| \leq L(\gamma) < \pi - \delta$, then we have $\pi - \angle(\gamma, \sigma) < \tau(\delta|\varepsilon)$.*

PROOF. Take a comparison hinge $(\bar{\gamma}, \bar{\sigma})$ of (γ, σ). By the assumption, $\delta < L(\bar{\gamma}) = L(\gamma) < \pi - \delta$, $L(\bar{\sigma}) = L(\sigma) = |px^*| \geq |xx^*| - |px| > \delta - \varepsilon$, and $L(\bar{\sigma}) \leq 2\pi - |xx^*| - |px| < \pi - \delta + \varepsilon$. The hinge comparison inequality (2.3.1) implies

$$|\bar{x}\overline{x^*}| \geq |xx^*| > \pi - \varepsilon,$$

where $\bar{x}$ and $\overline{x^*}$ are corresponding points for x and x^* in the comparison hinge. Thus by the elementary spherical geometry, $\angle(\gamma, \sigma) = \angle(\bar{\gamma}, \bar{\sigma}) > \pi - \tau(\delta|\varepsilon)$. □

LEMMA 3.4. *Let $\gamma : [0, \pi] \to X$ be a quasigeodesic. Then $d(\gamma(0), \gamma(\pi)) > \pi - 2\varepsilon$.*

Moreover, we have

$$|\,|t - s| - d(\gamma(t), \gamma(s))| < 2\varepsilon \tag{3.4.1}$$

for any $t, s \in [0, \pi]$.

PROOF. Set $p = \gamma(0)$ and take $p^* \in J_p$. Let σ be a geodesic from p to p^*. Then the definition of quasigeodesic implies $d(p^*, \gamma(\pi)) < \varepsilon$. Hence, $d(p, \gamma(\pi)) > d(p, p^*) - d(p^*, \gamma(\pi)) > \pi - 2\varepsilon$.

To show the second assertion, we consider the following formula

$$\begin{aligned} &(t - |\gamma(0)\gamma(t)|) + (s - t - |\gamma(t)\gamma(s)|) + (\pi - s - |\gamma(s)\gamma(\pi)|) \\ &\quad = \pi - (|\gamma(0)\gamma(t)| + |\gamma(t)\gamma(s)| + |\gamma(s)\gamma(\pi)|) \\ &\quad \leq \pi - |\gamma(0)\gamma(\pi)| \\ &\quad < 2\varepsilon. \end{aligned} \tag{3.4.2}$$

Since a quasigeodesic is 1-Lipschitz, all three terms in (3.4.2) are nonnegative. Therefore, we have (3.4.1). □

LEMMA 3.5. *Let $p, x, y \in X$ be points such that $d(p,x) > \delta$, and γ be a quasigeodesic from p to x, and σ be a geodesic from p to y. Suppose that $L(\gamma) < \pi - \delta$. If y is sufficiently near x, then the angle between γ and σ at p is less than $\tau(\delta|\varepsilon)$.*

PROOF. Since y is sufficiently near x, we can take $x^* = y^*$ in J_p. The assumption: $\delta < |px| \le L(\gamma) < \pi - \delta$ and Lemma 3.3 implies $\angle(px^*, \gamma) > \pi - \tau(\delta|\varepsilon)$ and $\angle(py^*, \sigma) > \pi - \tau(\delta|\varepsilon)$. Since $\angle(px^*, \gamma) + \angle(px^*, \sigma) + \angle(\gamma, \sigma) \le 2\pi$, we have $\angle(\gamma, \sigma) < \tau(\delta|\varepsilon)$. □

Now, we begin to fix a quasigeodesics-map from $\Sigma\Sigma_p$ to X like as the exponential map at p. First, we already define $\log_p : J_p \to \Sigma\Sigma_p$ by

$$J_p \ni x \mapsto (\uparrow_p^x, |px|) \in \Sigma_p \times [0, \pi].$$

Then we can define an exponential map $\exp_p : \log(J_p) \to X$ as the inverse of $\log_p$. And next, for any $\xi \in \Sigma_p$, we fix a quasigeodesic $\gamma_\xi = \gamma(\xi, \cdot) : [0, \pi] \to X$ with initial direction $\gamma_\xi^+(0) = \xi$. By Lemma 2.5, we have $\gamma(\uparrow_p^x, d(p,x)) = x$ for any $x \in J_p$. Namely, $\gamma(\uparrow_p^x, \cdot)|_{[0, d(p,x)]}$ is a geodesic and the quasigeodesics-map

$$f : \Sigma\Sigma_p \ni (\xi, t) \mapsto \gamma(\xi, t) \in X$$

is an extension of the exponential map. Of course, $\gamma_\xi(\pi)$ is depending on $\xi \in \Sigma_p$. Thus f is multivalued at $-o_p = \Sigma_p \times \{\pi\} \in \Sigma\Sigma_p$. $f(\pi\xi)$ is not simply f of $-o_p$, we absolutely regard $f(\pi\xi)$ as $\gamma_\xi(\pi)$ for $\xi \in \Sigma_p$. Lemma 3.4 implies $|f(\pi\xi)f(\pi\eta)| < 4\varepsilon$ for every $\xi, \eta \in \Sigma_p$.

LEMMA 3.6. *For any $v_1, v_2 \in \Sigma\Sigma_p$, $|d(v_1, v_2) - d(f(v_1), f(v_2))| < \tau(\varepsilon)$.*

PROOF. For $i = 1, 2$, put $\xi_i = \frac{v_i}{|v_i|}$, and $\gamma_i : [0, |v_i|] \to X$ quasigeodesics defined by $\gamma_i(t) = f(\xi_i, t)$, and $x_i = f(v_i)$.

First, we assume that $|v_i| \le \delta$ for some $i = 1, 2$. Without loss of generality, we can do that $|v_1| \le \delta$. Since quasigeodesics are 1-Lipschitz, $|px_1| \le \delta$. Thus, we have

$$| \, |x_1x_2| - |px_2| \, | \le |px_1| \le \delta,$$

$$| \, |v_1v_2| - |v_2| \, | \le |v_1| \le \delta.$$

This together with Lemma 3.4 implies

$$|\,|v_1v_2| - |x_1x_2|\,| \leq 2\delta + |\,|v_2| - |px_2|\,| < 2\delta + 2\varepsilon.$$

Next, we may assume that $|v_1| \geq \pi - \delta$. By Lemma 3.4, we can estimate the following value for $v \in \Sigma\Sigma_p \backslash \{\pm o_p\}$ and $f(v) = x$.

$$\begin{aligned} |d(p^*, x) - d(-o_p, v)| &\leq |d(p^*, x) - (\pi - d(p, x))| + |\pi - d(p, x) - d(-o_p, v)| \\ &< 2\varepsilon + |-d(p, x) + |v|\,| \\ &< 4\varepsilon. \end{aligned}$$

By assumption, we have

$$|x_1p^*| \leq |f(v_1)f(\pi\xi_1)| + |f(\pi\xi_1)p^*| < \pi - |v_1| + 3\varepsilon \leq \delta + 3\varepsilon.$$

Thus we have

$$\begin{aligned} |\,|x_1x_2| - |v_1v_2|\,| &\leq |\,|x_1x_2| - |x_2p^*|\,| + |\,|x_2p^*| - |-o_pv_2|\,| + |\,|-o_pv_2| - |v_1v_2|\,| \\ &< |x_1p^*| + 4\varepsilon + |-o_pv_1| \\ &\leq 2\delta + 10\varepsilon. \end{aligned}$$

Lastly, assume that $\delta < |v_i| < \pi - \delta$ for any $i = 1, 2$. By the proof of Lemma 3.1, we can take $\log_p(x_1^*) \in \Sigma\Sigma_p$. Put $v_3 = \log_p(x_1^*)$ and $x_3 = x_1^* = f(v_3)$. For $i = 1, 2, 3$, set geodesics $\overline{\gamma}_i$ from o_p to v_i and quasigeodesics $\gamma_i = f \circ \overline{\gamma}_i$ from p to x_i. Remark that γ_3 is geodesic.

Take a point $y_1 \in J_p$ sufficiently close to x_1 and put a geodesic σ_1 from p to y_1. Then by Lemma 3.5 we have $\angle(\gamma_1, \sigma_1) < \tau(\delta|\varepsilon)$. It follows that

$$|\angle(\sigma_1, \gamma_2) - \angle(\overline{\gamma}_1, \overline{\gamma}_2)| < \tau(\delta|\varepsilon).$$

Therefore, by the hinge comparison (2.3.1), we have

$$d(x_1, x_2) \leq d(v_1, v_2) + \tau(\delta|\varepsilon).$$

On the other hand, using (2.3.1) for (γ_3, γ_2) at (3.6.1), we have

$$\begin{aligned} |x_1x_2| &\geq |x_1x_1^*| - |x_1^*x_2| \\ &> \pi - \varepsilon - |v_3v_2| \\ &\geq \pi - \varepsilon - (2\pi - |v_3v_1| - |v_1v_2|) \\ &> |v_1v_2| - 2\varepsilon. \end{aligned} \tag{3.6.1}$$

Therefore we have in any case,

$$| \, |x_1x_2| - |v_1v_2| \, | < \tau(\delta|\varepsilon) + \tau(\delta).$$

Since δ is arbitrary, if we now take a suitable function $\delta = \delta(\varepsilon)$ in the explicit forms of $\tau(\delta|\varepsilon)$ and $\tau(\delta)$, (e.g. put $\delta(\varepsilon) = \sqrt{\varepsilon}$), then we complete the proof of the lemma. □

From the definition of the map f, the closure of the image of f is X. Then it follows together with Lemma 3.6 that the Gromov-Hausdorff distance between X and $\Sigma\Sigma_p$ is less than $\tau(\varepsilon)$. It completes the proof of Theorems 1.5 and 1.2.

Acknowledgments

The author would like to express his thanks to Professor Takao Yamaguchi for his encouragement, and comments. Also the author would like to thank Koichi Nagano and Takumi Yokota for their comments. This work was partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

References

[1] A. D. Alexandrov. Curves on convex surfaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47** (1945), 315–317.
[2] D. Burago, Y. Burago, and S. Ivanov. A Course in Metric Geometry, Graduate Studies in Mathematics 33, American Mathematical Society, Providence, 2001.
[3] Y. Burago, M. Gromov, and G. Perel'man. A. D. Alexandrov spaces with curvatures bounded below. Uspekhi Mat. Nauk. (Russian), Russian Math. Surveys (English), **47**(2):3–51 (Russian), 1–58 (English), 1992.
[4] M. Gromov. Filling Riemannian manifolds. J. Differential Geom. **18** (1983), no. 1, 1–147.
[5] K. Grove and P. Petersen. A radius sphere theorem. Invent. Math. 112, no. 3, 577–583, 1993.
[6] K. Grove and K. Shiohama. A generalized sphere theorem. Ann. Math. (2) **106** (1977), no. 2, 201–211.
[7] V. Kapovitch. Perelman's stability theorem, Surveys of Differential Geometry, Metric and Comparison Geometry, vol. XI, International press, (2007), 103–136.
[8] A. Mitsuishi. A splitting theorem for infinite dimensional Alexandrov spaces with nonnegative curvature and its applications. Geom. Dedicata. to appear.
[9] Y. Otsu and T. Shioya. The Riemannian structure of Alexandrov spaces. J. Differential Geom. **39** (1994), no. 3, 629–658.
[10] Y. Otsu, K. Shiohama and T. Yamaguchi. A new version of differentiable sphere theorem. Invent. Math. **98** (1989), no. 2, 219–228.
[11] A. Petrunin. Application of quasigeodesics and gradient curves. Comparison geometry (Berkeley, CA, 1993–94), 203–219, Math. Sci. Res. Inst. Publ., 30, Cambridge Univ. Press, Cambridge, 1997.
[12] A. Petrunin. Semiconcave functions in Alexandrov's geometry. Surveys in differential geometry. Vol. XI, 137–201, Surv. Differ. Geom., 11, Int. Press, Somerville, MA, 2007.

[13] A. Petrunin. Quasigeodesics in multidimensional Alexandrov spaces. Ph.D. thesis, Steklov Institute at St. Petersburg and University of Illinois at Urbana-Champaign, 1995.
[14] G. Perel'man, and A. Petrunin. Quasigeodesics and gradient curves in Alexandrov spaces. preprint.
[15] C. Plaut, Spaces of Wald-Berestovskii curvature bounded below, J. Geom. Anal. **6** (1996), 113–134.
[16] K. Shiohama, and T. Yamaguchi. Positively curved manifolds with restricted diameter. Geometry of manifolds (Matsumoto, 1988), 345–350, Perspect. Math., 8, Academic Press, Boston, MA, 1989.
[17] T. Yamaguchi. A convergence theorem in the geometry of Alexandrov spaces. Actes de la Table Ronde de Geometrie Differentielle (Luminy, 1992), 601–642, Semin. Congr., 1, Soc. Math. France, Paris, 1996.

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki, 305-8571, Japan
E-mail address: ayato@math.tsukuba.ac.jp