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THE CONSTRUCTION OF THE UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATOR

By

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Abstract. For a one-parameter exponential family of distributions, a method to find the uniformly minimum variance unbiased (UMVU) estimator based on the complete sufficient statistic is given in Jani and Dave [1] by change of the expression of the unbiasedness condition. But, it heavily depends on the concrete form of the distribution of the statistic in obtaining indeed the UMVU estimator. In this paper, from the different point of view, the construction of the UMVU estimator for a one-parameter exponential families of distributions and certain two-parameter family of distributions is discussed. Some examples are also given.

1. Introduction

In the problem of estimating a function of parameter, there are some ways to obtain the uniformly minimum variance unbiased (UMVU) estimator (see, e.g. Lehmann and Casella [4], Zacks [7]). One of them is to use the information inequality like Cramér-Rao one which gives the lower bound for the variance of unbiased estimators. Another way is to obtain an unbiased estimator which is a function of the complete sufficient statistic if it exists (see, e.g. Voinov and Nikulin [6], Jani and Dave [1], Kim and Akahira [3]). To do so, it is important to consider how to construct the UMVU estimator. Jani and Dave [1] proposed a way to find the UMVU estimator based on the complete sufficient statistic for a one-parameter exponential family of distributions by change of the expression of the unbiasedness condition. But, in order to obtain indeed the UMVU estimator, we need the concrete form of the distribution of the statistic. It is not, in

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general, so easy to obtain the distribution, but, in special cases, it is given (see, e.g. [6]).

In this paper, from the different viewpoint based on the complete sufficient statistic from [1], we consider the construction of the UMVU estimator of a function of parameter for a one-parameter exponential families of distributions and certain two-parameter family of distributions. Some examples also are given.

2. A Construction of the UMVU Estimator in a One-Parameter Exponential Family of Distributions

Suppose that $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) random variables with a probability density function (p.d.f.) (with respect to the Lebesgue measure)

$$p(x,\theta) = \alpha(x)\beta(\theta)e^{c(\theta)d(x)}, \quad x \in \mathscr{X} \subset \boldsymbol{R}^1;\ \theta \in \Theta \subset \boldsymbol{R}^1 \tag{2.1}$$

where $\alpha(x) > 0$ for all $x \in \mathscr{X}$, $c(\cdot)$ is a real-valued function and $c(\theta) \neq 0$ for some $\theta \in \Theta$ and

$$\beta(\theta) = \left\{ \int_{\mathscr{X}} \alpha(x) e^{c(\theta)d(x)}\, dx \right\}^{-1}.$$

Denote $(X_1, \dots, X_n)$ by $\boldsymbol{X}$. Here, we assume that the range of $c(\cdot)$ involves an open interval in $\boldsymbol{R}^1$. Then it is well known that the statistic $Z(\boldsymbol{X}) := \sum_{i=1}^n d(X_i)$ is a complete sufficient statistic for θ and has the p.d.f.

$$f_Z(z,\theta) = \tilde{B}_n(z)\{\psi_n(\theta)\}^{-1} e^{c(\theta)z}, \quad z \in \mathscr{Z} \subset \boldsymbol{R}^1;\ \theta \in \Theta \subset \boldsymbol{R}^1,$$

where

$$\psi_n(\theta) = \left\{ \int_{\mathscr{X}} \alpha(x) e^{c(\theta)d(x)}\, dx \right\}^n$$

and $\tilde{B}_n(\cdot)$ satisfies

$$\psi_n(\theta) = \int_{\mathscr{Z}} \tilde{B}_n(z) e^{c(\theta)z}\, dz.$$

(see, e.g. Jani and Dave [1]).

Let $T = T(\boldsymbol{X}) = k_n Z(\boldsymbol{X})$ with a constant $k_n \neq 0$, and suppose that its range is an interval (a, b), where a and b are independent of θ and may include $-\infty$ and ∞, respectively. Then the p.d.f. of T (with respect to the Lebesgue measure) is given by the form of

$$f_T(t,\theta) = B_n(t)\{\psi_n(\theta)\}^{-1} e^{h_n(\theta)t} \chi_{(a,b)}(t), \quad t \in \boldsymbol{R}^1;\ \theta \in \Theta \tag{2.2}$$

Here, for a natural number k, let $B_n(t)$ be k-times differentiable in t and assume the following condition.

(C1) For $j = 0, 1, \dots, k-1$

$$\lim_{t \to a+0} B_n^{(j)}(t) e^{h_n(\theta)t} = 0, \quad \lim_{t \to b-0} B_n^{(j)}(t) e^{h_n(\theta)t} = 0,$$

where $B_n^{(j)}(t) = (d^j/dt^j) B_n(t)$ for $j = 1, \dots, k-1$, and $B_n^{(0)}(t) = B_n(t)$.

Then, we have the following.

Theorem 2.1. *Assume that the condition* (C1) *holds. Then*

$$\hat{h}_{k,n}(T) := (-1)^k \frac{B_n^{(k)}(T)}{B_n(T)} \chi_{(a,b)}(T)$$

is an unbiased estimator of $\{h_n(\theta)\}^k$.

Proof. Using integration by parts and the condition (C1), we obtain for each n and each $\theta \in \Theta$

$$\begin{aligned} E_\theta[\hat{h}_{k,n}(T)] &= E_\theta\left[(-1)^k \frac{B_n^{(k)}(T)}{B_n(T)} \chi_{(a,b)}(T)\right] \\ &= (-1)^k \{\psi_n(\theta)\}^{-1} \int_a^b B_n^{(k)}(t) e^{h_n(\theta)t}\, dt. \end{aligned} \tag{2.3}$$

Here,

$$\begin{aligned} J_{n,k}(\theta) &:= \int_a^b B_n^{(k)}(t) e^{h_n(\theta)t}\, dt \\ &= [B_n^{(k-1)}(t) e^{h_n(\theta)t}]_a^b - \int_a^b B_n^{(k-1)}(t) h_n(\theta) e^{h_n(\theta)t}\, dt \\ &= -h_n(\theta) J_{n,k-1}(\theta) \\ &= (-1)^k \{h_n(\theta)\}^k J_{n,0}(\theta) \\ &= (-1)^k \{h_n(\theta)\}^k \int_a^b B_n(t) e^{h_n(\theta)t}\, dt. \end{aligned}$$

Then it follows from (2.2) and (2.3) that for each $\theta \in \Theta$

$$E_\theta[\hat{h}_{k,n}(T)] = \{\psi_n(\theta)\}^{-1} \{h_n(\theta)\}^k \int_a^b B_n(t) e^{h_n(\theta)t}\, dt = \{h_n(\theta)\}^k.$$

Hence, $\hat{h}_{k,n}(T)$ is an unbiased estimator of $\{h_n(\theta)\}^k$. □

Remark 2.1. For $k = 1$, Suzuki [5] also discussed the above, and extended to the multiparameter case.

Corollary 2.1. *If the condition* (C1) *holds, then* $\hat{h}_{k,n}(T)$ *is a UMVU estimator of* $\{h_n(\theta)\}^k$.

The proof is straightforward from Theorem 2.1, since T is a complete sufficient statistic for θ.

Corollary 2.2. *Assume that, for each* $k = 1, \dots, K$, *the condition* (C1) *holds. Let* $g_K(\theta) := \sum_{k=1}^{K} c_k\{h_n(\theta)\}^k$ *with constants* $c_k \neq 0$ $(k = 1, \dots, K)$. *Then* $\sum_{k=1}^{K} c_k\hat{h}_{k,n}(T)$ *is a UMVU estimator of* $g_K(\theta)$.

The proof is straightforward from Corollary 2.1.

Remark 2.2. Let infinite series $\sum_{k=1}^{\infty} c_k\{h_n(\theta)\}^k$, $\sum_{k=1}^{\infty} c_k\hat{h}_{k,n}(T)$ and $\sum_{k=1}^{\infty} |c_k|$ be convergent. Put $h_{K,n}(T) := \sum_{k=1}^{K} c_k\hat{h}_{k,n}(T)$, and assume that there is a measurable function h_n^* such that $|h_{K,n}(t)| \leq h_n^*(t)$ for all $K = 1, 2, \dots$ and all $t \in \boldsymbol{R}^1$ and $E[h_n^*(T)] < \infty$. If the condition (C1) holds for all $k = 1, 2, \dots,$ then the result of Corollary 2.2 is extended to the case when $K \to \infty$.

Example 2.1. Let $X_1, \dots, X_n$ be i.i.d. random variables from the normal distribution $N(\theta, \sigma_0^2)$, with σ_0^2 known. Since the statistic $T = \overline{X} := (1/n)\sum_{i=1}^{n} X_i$ has the normal distribution $N(\theta, \sigma_0^2/n)$, the p.d.f. of T is given by

$$f_T(t, \theta) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{n\theta^2}{2\sigma_0^2}\right) \exp\left(-\frac{nt^2}{2\sigma_0^2}\right) \exp\left(\frac{n\theta}{\sigma_0^2} t\right).$$

Now, letting

$$h_n(\theta) = \frac{n\theta}{\sigma_0^2}, \quad B_n(t) = \exp\left(-\frac{nt^2}{2\sigma_0^2}\right),$$

in the form (2.2), we have

$$B_n^{(1)}(t) = -\frac{nt}{\sigma_0^2} \exp\left(-\frac{nt^2}{2\sigma_0^2}\right),$$

$$B_n^{(2)}(t) = \left(-\frac{n}{\sigma_0^2} + \frac{n^2t^2}{\sigma_0^4}\right) \exp\left(-\frac{nt^2}{2\sigma_0^2}\right),$$

$$B_n^{(3)}(t) = \left(\frac{3n^2t}{\sigma_0^4} - \frac{n^3t^3}{\sigma_0^6}\right) \exp\left(-\frac{nt^2}{2\sigma_0^2}\right),$$

and so on. Hence, for $j = 0, 1, \dots, k-1$

$$\lim_{t \to \pm\infty} B_n^{(j)}(t) e^{h_n(\theta)t} = \lim_{t \to \pm\infty} P_{j,n}(t) \exp\left(-\frac{nt^2}{2\sigma_0^2} + \frac{n\theta t}{\sigma_0^2}\right) = 0,$$

where $P_{j,n}(t)$ is certain j-polynomial of t for each $j = 0, 1, \dots, k-1$, which implies that the condition (C1) is satisfied. Also, since T is a complete sufficient statistic, it follows from Corollary 2.1 that $\hat{h}_{k,n}(T)$ is a UMVU estimator of $\{h_n(\theta)\}^k = (n\theta/\sigma_0^2)^k$. For example, in the case when $k = 3$,

$$T^3 - \frac{3\sigma_0^2}{n} T$$

is seen to be a UMVU estimator of θ^3.

Example 2.2. Let $X_1, \dots, X_n$ be i.i.d. random variables from the normal distribution $N(0, \theta)$ for $\theta > 0$, where $n \geq 3$. Since the p.d.f. of the statistic $T := \sum_{i=1}^n X_i^2$ is given by

$$f_T(t, \theta) = \begin{cases} \frac{t^{(n/2)-1}}{2^{n/2}\Gamma(n/2)} \frac{e^{-t/(2\theta)}}{\theta^{n/2}} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

letting

$$h_n(\theta) = -\frac{1}{2\theta}, \quad B_n(t) = t^{(n/2)-1},$$

in the form (2.2), we have

$$B_n^{(k)}(t) = \left(\frac{n}{2} - 1\right) \cdots \left(\frac{n}{2} - k\right) t^{(n/2)-(k+1)}.$$

Then, for a natural number k with $k < n/2$,

$$\lim_{t \to 0+0} B_n^{(j)}(t) e^{h_n(\theta)t} = \lim_{t \to 0+0} \left(\frac{n}{2} - 1\right) \cdots \left(\frac{n}{2} - j\right) t^{(n/2)-(j+1)} e^{-t/(2\theta)} = 0,$$

$$\lim_{t \to \infty} B_n^{(j)}(t) e^{h_n(\theta)t} = \lim_{t \to \infty} \left(\frac{n}{2} - 1\right) \cdots \left(\frac{n}{2} - j\right) t^{(n/2)-(j+1)} e^{-t/(2\theta)} = 0$$

for $0 \leq j \leq k-1$, hence, the condition (C1) is satisfied. Also, since T is a complete sufficient statistic, it follows from Corollary 2.1 that

$$(n-2) \cdots (n-2k) T^{-k} \chi_{(0,\infty)}(T)$$

is a UMVU estimator of $1/\theta^k$.

3. A Construction of the UMVU Estimator in a Two-Parameter Exponential Family of Distributions

Let $T_1 = T_1(\boldsymbol{X})$ and $T_2 = T_2(\boldsymbol{X})$ be statistics based on a random vector $\boldsymbol{X} := (X_1, \dots, X_n)$, and $\mathscr{T} = (a_1(\theta), b_1(\theta)) \times (a_2(\theta), b_2(\theta))$ be the range of $T := (T_1, T_2)$, where $a_1(\theta)$ and $a_2(\theta)$ may include $-\infty$, and $b_1(\theta)$ and $b_2(\theta)$ may include ∞. Now, define the p.d.f. of T as

$$\begin{aligned} f_T(t, \theta) = B_n(t)\{\psi_n(\theta)\}^{-1}[\exp\{c_{11}(\theta)t_1^2 + c_{12}(\theta)t_1t_2 + c_{22}(\theta)t_2^2 \\ + c_{10}(\theta)t_1 + c_{01}(\theta)t_2 + c_{00}(\theta)\}]\chi_{\mathscr{T}}(t) \\ \text{for } t := (t_1, t_2) \in \boldsymbol{R}^2;\ \theta := (\theta_1, \theta_2) \in \Theta \subset \boldsymbol{R}^2, \end{aligned} \tag{3.1}$$

where $B_n(t)$ is twice differentiable with respect to t_2 and $B_n(t) > 0$ in $\mathscr{T}$, $\psi_n(\theta) > 0$ in Θ, and $c_{11}(\theta)$, $c_{12}(\theta)$, $c_{22}(\theta)$, $c_{10}(\theta)$, $c_{01}(\theta)$, $c_{00}(\theta)$ are functions of θ. Let

$$A_\theta(t) := c_{11}(\theta)t_1^2 + c_{12}(\theta)t_1t_2 + c_{22}(\theta)t_2^2 + c_{10}(\theta)t_1 + c_{01}(\theta)t_2 + c_{00}(\theta),$$

$$B_n^{(0,1)}(t) := \frac{\partial}{\partial t_2} B_n(t)$$

and, further, assume the following condition.

(C2) For $\theta \in \Theta$ and $t_1 \in (a_1(\theta), b_1(\theta))$

$$\lim_{t_2 \to a_2(\theta)+0} B_n(t) \exp\{A_\theta(t)\} = 0, \qquad \lim_{t_2 \to b_2(\theta)-0} B_n(t) \exp\{A_\theta(t)\} = 0.$$

Then, we have the following.

THEOREM 3.1. *Assume that the condition* (C2) *holds. Then for any* $\theta \in \Theta$

$$E_\theta\left[\frac{B_n^{(0,1)}(T)}{B_n(T)}\right] = -c_{12}(\theta)E_\theta(T_1) - 2c_{22}(\theta)E_\theta(T_2) - c_{01}(\theta),$$

provided that all the expectations on both sides exist.

PROOF. By the condition (C2), we obtain for any $\theta \in \Theta$

$$\begin{aligned} E_\theta\left[\frac{B_n^{(0,1)}(T)}{B_n(T)}\right] &= \{\psi_n(\theta)\}^{-1} \int_{a_1(\theta)}^{b_1(\theta)} \int_{a_2(\theta)}^{b_2(\theta)} \left\{\frac{\partial}{\partial t_2} B_n(t)\right\} \exp\{A_\theta(t)\}\, dt_2 dt_1 \\ &= \{\psi_n(\theta)\}^{-1} \left(\int_{a_1(\theta)}^{b_1(\theta)} [B_n(t) \exp\{A_\theta(t)\}]_{t_2=a_2(\theta)}^{t_2=b_2(\theta)}\, dt_1 \right. \\ &\quad \left. - \int_{a_1(\theta)}^{b_1(\theta)} \int_{a_2(\theta)}^{b_2(\theta)} B_n(t) \left\{\frac{\partial}{\partial t_2} A_\theta(t)\right\} \exp\{A_\theta(t)\}\, dt_2 dt_1 \right) \end{aligned}$$

$$= -\{\psi_n(\theta)\}^{-1} \int_{a_1(\theta)}^{b_1(\theta)} \int_{a_2(\theta)}^{b_2(\theta)} B_n(t)$$

$$\times \{c_{12}(\theta)t_1 + 2c_{22}(\theta)t_2 + c_{01}(\theta)\} \exp\{A_\theta(t)\}\, dt_2 dt_1$$

$$= -c_{12}(\theta)E_\theta(T_1) - 2c_{22}(\theta)E_\theta(T_2) - c_{01}(\theta). \qquad \square$$

Next we assume the following condition.
(C3) For $\theta \in \Theta$ and $t_1 \in (a_1(\theta), b_1(\theta))$

$$\lim_{t_2 \to a_2+0} B_n^{(0,1)}(t) \exp\{A_\theta(t)\} = 0, \qquad \lim_{t_2 \to b_2-0} B_n^{(0,1)}(t) \exp\{A_\theta(t)\} = 0.$$

Here we put

$$B_n^{(0,2)}(t) := \frac{\partial^2}{\partial t_2^2} B_n(t).$$

Then we have the following.

THEOREM 3.2. *Assume that the condition* (C3) *holds. Then for any* $\theta \in \Theta$

$$E_\theta\left[\frac{B_n^{(0,2)}(T)}{B_n(T)}\right] = -c_{12}(\theta)E_\theta\left[T_1 \frac{B_n^{(0,1)}(T)}{B_n(T)}\right] - 2c_{22}(\theta)E_\theta\left[T_2 \frac{B_n^{(0,1)}(T)}{B_n(T)}\right]$$

$$+ c_{01}(\theta)\{c_{12}(\theta)E_\theta(T_1) + 2c_{22}(\theta)E_\theta(T_2) + c_{01}(\theta)\},$$

provided that all the expectations on both sides exist.

PROOF. By the condition (C3), we obtain for any $\theta \in \Theta$

$$E_\theta\left[\frac{B_n^{(0,2)}(T)}{B_n(T)}\right] = \{\psi_n(\theta)\}^{-1} \int_{a_1(\theta)}^{b_1(\theta)} \int_{a_2(\theta)}^{b_2(\theta)} \left\{\frac{\partial^2}{\partial t_2^2} B_n(t)\right\} \exp\{A_\theta(t)\}\, dt_2 dt_1$$

$$= \{\psi_n(\theta)\}^{-1} \Bigg(\int_{a_1(\theta)}^{b_1(\theta)} \left[\left\{\frac{\partial}{\partial t_2} B_n(t)\right\} \exp\{A_\theta(t)\}\right]_{t_2=a_2(\theta)}^{t_2=b_2(\theta)} dt_1$$

$$- \int_{a_1(\theta)}^{b_1(\theta)} \int_{a_2(\theta)}^{b_2(\theta)} \left\{\frac{\partial}{\partial t_2} B_n(t)\right\}\left\{\frac{\partial}{\partial t_2} A_\theta(t)\right\} \exp\{A_\theta(t)\}\, dt_2 dt_1 \Bigg)$$

$$= -\{\psi_n(\theta)\}^{-1} \int_{a_1(\theta)}^{b_1(\theta)} \int_{a_2(\theta)}^{b_2(\theta)} \left\{\frac{\partial}{\partial t_2} B_n(t)\right\}$$

$$\times \{c_{12}(\theta)t_1 + 2c_{22}(\theta)t_2 + c_{01}(\theta)\} \exp\{A_\theta(t)\}\, dt_2 dt_1.$$

From Theorem 3.1 we have

$$E_\theta\left[\frac{B_n^{(0,2)}(T)}{B_n(T)}\right] = -c_{12}(\theta)E_\theta\left[T_1\frac{B_n^{(0,1)}(T)}{B_n(T)}\right]$$

$$- 2c_{22}(\theta)E_\theta\left[T_2\frac{B_n^{(0,1)}(T)}{B_n(T)}\right] - c_{01}(\theta)E_\theta\left[\frac{B_n^{(0,1)}(T)}{B_n(T)}\right]$$

$$= -c_{12}(\theta)E_\theta\left[T_1\frac{B_n^{(0,1)}(T)}{B_n(T)}\right] - 2c_{22}(\theta)E_\theta\left[T_2\frac{B_n^{(0,1)}(T)}{B_n(T)}\right]$$

$$+ c_{01}(\theta)\{c_{12}(\theta)E_\theta(T_1) + 2c_{22}(\theta)E_\theta(T_2) + c_{01}(\theta)\}. \qquad \square$$

Example 3.1. Let $X_1, \dots, X_n$ be i.i.d. random variables from the normal distribution $N(\mu, \sigma^2)$, where $n \geq 6$. Assume that μ and σ^2 are unknown. Now, let

$$T_1 = \bar{X} = \frac{1}{n}\sum_{i=1}^n X_i, \quad T_2 = S^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2.$$

Then, the p.d.f. of $T = (T_1, T_2)$ is given by

$$f_T(t,\theta) = \frac{n^{n/2}t_2^{(n-3)/2}}{\sqrt{\pi}\Gamma((n-1)/2)(2\sigma^2)^{n/2}} \exp\left[-\frac{n}{2\sigma^2}\{(t_1-\mu)^2 + t_2\}\right],$$

$$\text{for } t = (t_1, t_2) \in \boldsymbol{R}^1 \times \boldsymbol{R}_+;\ \theta := (\mu, \sigma^2) \in \Theta := \boldsymbol{R}^1 \times \boldsymbol{R}_+$$

where $\boldsymbol{R}_+ = (0, \infty)$. In the form (3.1),

$$c_{01}(\theta) = -\frac{n}{2\sigma^2}, \quad c_{12}(\theta) = c_{22}(\theta) = 0, \tag{3.2}$$

and also

$$E_\theta(T_1) = \mu, \quad E_\theta(T_2) = \left(1 - \frac{1}{n}\right)\sigma^2.$$

On the other hand, since

$$\begin{aligned} A_\theta(t) &= -\frac{n}{2\sigma^2}\{(t_1-\mu)^2 + t_2\}, \\ B_n(t) &= t_2^{(n-3)/2}, \end{aligned} \tag{3.3}$$

we have for $\theta \in \Theta$ and $t_1 \in \boldsymbol{R}^1$,

$$\lim_{t_2 \to 0+0} B_n(t) \exp\{A_\theta(t)\} = \lim_{t_2 \to 0+0} t_2^{(n-3)/2} \exp\left[-\frac{n}{2\sigma^2}\{(t_1-\mu)^2 + t_2\}\right] = 0,$$

$$\lim_{t_2 \to \infty} B_n(t) \exp\{A_\theta(t)\} = 0,$$

which implies that the condition (C2) is satisfied. By Theorem 3.1, we obtain

$$E_\theta\left[\frac{B_n^{(0,1)}(T)}{B_n(T)}\right] = \frac{n}{2\sigma^2}. \tag{3.4}$$

Since

$$B_n^{(0,1)}(t) = \frac{\partial}{\partial t_2} B_n(t) = \frac{n-3}{2} t_2^{(n-5)/2},$$

it is seen from (3.3) that

$$\frac{B_n^{(0,1)}(t)}{B_n(t)} = \frac{n-3}{2t_2}. \tag{3.5}$$

Hence, it follows from (3.4) and (3.5) that for any $\theta \in \Theta$

$$E_\theta\left[\frac{n-3}{nT_2}\right] = \frac{1}{\sigma^2},$$

that is, $(n-3)/(nT_2)$ is an unbiased estimator of $1/\sigma^2$. Also, since T is a complete sufficient statistic for θ, $(n-3)/(nT_2)$ is a UMVU estimator of $1/\sigma^2$. Since, for $\theta \in \Theta$ and $t_1 \in \boldsymbol{R}^1$,

$$\lim_{t_2 \to 0+0} B_n^{(0,1)}(t) \exp\{A_\theta(t)\} = \lim_{t_2 \to 0+0} \frac{n-3}{2} t_2^{(n-5)/2} \exp\left[-\frac{n}{2\sigma^2}\{(t_1-\mu)^2 + t_2\}\right] = 0,$$

$$\lim_{t_2 \to \infty} B_n^{(0,1)}(t) \exp\{A_\theta(t)\} = 0,$$

the condition (C3) is satisfied. From Theorem 3.2 and (3.2) we have

$$E_\theta\left[\frac{B_n^{(0,2)}(T)}{B_n(T)}\right] = \frac{n^2}{4\sigma^4}. \tag{3.6}$$

Since, by (3.3),

$$B_n^{(0,2)}(t) = \frac{\partial^2}{\partial t_2^2} B_n(t) = \frac{1}{4}(n-3)(n-5) t_2^{(n-7)/2},$$

it follows from (3.3) that

$$\frac{B_n^{(0,2)}(T)}{B_n(T)} = \frac{(n-3)(n-5)}{4T_2^2}. \tag{3.7}$$

Hence it is seen from (3.6) and (3.7) that $(n-3)(n-5)/(n^2T_2^2)$ is a UMVU estimator of $1/\sigma^4$.

EXAMPLE 3.2. Let $X_1, \dots, X_n$ be i.i.d. random variables from the exponential distribution with the p.d.f.

$$p(x;\mu,\sigma) = \begin{cases} \frac{1}{\sigma}e^{-(x-\mu)/\sigma} & \text{for } x > \mu, \\ 0 & \text{for } x \le \mu, \end{cases}$$

where $n \ge 4$, and $\theta := (\mu,\sigma) \in \Theta := \boldsymbol{R}^1 \times \boldsymbol{R}_+$. Assume that μ and σ are unknown. Then, the maximum likelihood estimator of θ is given by

$$\hat{\theta} := (\hat{\mu}, \hat{\sigma}) = (X_{(1)}, \overline{X} - X_{(1)})$$

where $X_{(1)} = \min_{1 \le i \le n} X_i$ and $\overline{X} = (1/n)\sum_{i=1}^n X_i$. Here, the p.d.f. of $T_1 := \hat{\mu}$ is given by

$$f_{T_1}(t_1;\mu,\sigma) = \begin{cases} \frac{n}{\sigma}\exp\{-\frac{n}{\sigma}(t_1-\mu)\} & \text{for } t_1 > \mu, \\ 0 & \text{for } t_1 \le \mu, \end{cases}$$

and the p.d.f. of $T_2 := \hat{\sigma}$ is given by

$$f_{T_2}(t_2;\mu,\sigma) = \begin{cases} \frac{1}{\Gamma(n-1)}\left(\frac{n}{\sigma}\right)^{n-1} t_2^{n-2}e^{-(n/\sigma)t_2} & \text{for } t_2 > 0, \\ 0 & \text{for } t_2 \le 0. \end{cases}$$

Now, since T_1 and T_2 are independent (see Johnson et al. [2], pp. 506, 507), the p.d.f. of $T = (T_1, T_2) = (\hat{\mu}, \hat{\sigma})$ is given by

$$f_T(t,\theta) = \begin{cases} \frac{1}{\Gamma(n-1)}\left(\frac{n}{\sigma}\right)^n t_2^{n-2}\exp\left(-\frac{n}{\sigma}t_1 - \frac{n}{\sigma}t_2 + \frac{n\mu}{\sigma}\right) & \text{for } (t_1,t_2) \in \mathscr{T}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathscr{T} = (\mu,\infty) \times (0,\infty)$, $\theta \in \Theta$. Let

$$c_{01}(\theta) = -\frac{n}{\sigma}, \quad c_{12}(\theta) = c_{22}(\theta) = 0 \tag{3.8}$$

in the form (3.1). Also, we have

$$E_\theta(T_1) = \mu + \frac{\sigma}{n}, \quad E_\theta(T_2) = \frac{n-1}{n}\sigma.$$

On the other hand, since

$$B_n(t) = t_2^{n-2}, \tag{3.9}$$

for $\theta \in \Theta$ and $t_1 \in (\mu, \infty)$, we have

$$\lim_{t_2 \to 0+0} B_n(t) \exp\{A_\theta(t)\} = \lim_{t_2 \to 0+0} t_2^{n-2} \exp\left(-\frac{n}{\sigma}t_1 - \frac{n}{\sigma}t_2 + \frac{n\mu}{\sigma}\right) = 0,$$

$$\lim_{t_2 \to \infty} B_n(t) \exp\{A_\theta(t)\} = 0,$$

hence, the condition (C2) is satisfied. By Theorem 3.1 and (3.8) we obtain for any $\theta \in \Theta$

$$E_\theta\left[\frac{B_n^{(0,1)}(T)}{B_n(T)}\right] = \frac{n}{\sigma}. \tag{3.10}$$

Since,

$$B_n^{(0,1)}(t) = \frac{\partial}{\partial t_2} B_n(t) = (n-2)t_2^{n-3},$$

it follows from (3.9) that

$$\frac{B_n^{(0,1)}(t)}{B_n(t)} = \frac{n-2}{t_2}.$$

Therefore, by (3.10), $(n-2)/(nT_2)$ is seen to be an unbiased estimator of $1/\sigma$. Also, since T is a complete sufficient statistic for θ, $(n-2)/(nT_2)$ is a UMVU estimator of $1/\sigma$.

The above discussion may be extended to the equality on $E_\theta[B_n^{(i,j)}(T)/B_n(T)]$ where $B_n^{(i,j)}(t) := (\partial^{i+j}/\partial t_1^i \partial t_2^j)B_n(t)$ $(i, j = 0, 1, 2, \ldots)$. A similar one may be done.

4. Conclusion

In the previous sections, we argue about the construction of the UMVU estimator in one-paremeter and two-parameter exponential families of distributions. In the one-parameter exponential family case, the form of the UMVU estimator is directly obtained from the factor of the p.d.f. independent of θ. In the two-parameter family case, the UMVU estimator is similarly derived from the equation based on the factor.

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