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A CLASSIFICATION OF REDUCTIVE PREHOMOGENEOUS VECTOR SPACES WITH TRIVIAL REPRESENTATION FREE

Dedicated to Professor Tatsuo Kimura on his 60th birthday

By

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Abstract. In this paper, we classify k-simple prehomogeneous vector spaces of type

$$(GL_1^l \times G_1 \times \cdots \times G_k, \rho_1^{(1)} \otimes \cdots \otimes \rho_k^{(1)} + \cdots + \rho_1^{(l)} \otimes \cdots \otimes \rho_k^{(l)})$$

where for any i, j, each $\rho_j^{(i)}$ is a nontrivial irreducible representation of a simple algebraic group G_j (i.e., $\rho_j^{(i)} \neq 1$) with $k \geq 3$ and $l \geq 2$ under full scalar multiplications. We consider everything over the complex number field $\mathbf{C}$.

Introduction

Let G be a connected linear algebraic group, V a finite dimensional vector space ($\dim V \geq 1$), and ρ a rational representation of G on V, all defined over the complex number field $\mathbf{C}$. If V has a Zariski-dense G-orbit, the triplet (G, ρ, V) is called a *prehomogeneous vector space* (abbrev. PV). In this paper, we assume that $G = GL_1^l \times G_1 \times \cdots \times G_k$ where each G_i is a simple algebraic group $(1 \leq i \leq k)$. Then any representation (σ, V) of $G_1 \times \cdots \times G_k$ is the direct sum of irreducible representations $(\sigma, V) = (\sigma_1, V_1) \oplus \cdots \oplus (\sigma_l, V_l)$ where each σ_i is of the form $\sigma_1^{(i)} \otimes \cdots \otimes \sigma_k^{(i)}$ and $\sigma_j^{(i)}$ is an irreducible representation of G_j $(1 \leq j \leq k)$. Let ρ be the composition of scalar multiplications GL_1^l on each V_i $(1 \leq i \leq l)$ and σ. We call such a PV (G, ρ, V) a k-simple PV. When $k = 1$ ([K]) or $k = 2$ ([KKIY], [KKTI]), they are completely classified. When $k = 3$, it has been classified under some condition. In this paper, we classify all k-simple

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PVs under the condition that $\sigma_j^{(i)} \neq 1$ for $1 \leq i \leq l$ and $1 \leq j \leq k$. For any representation ρ of any group G and any natural number n satisfying $\deg \rho \leq n$, a triplet $(G \times GL_n, \rho \otimes \Lambda_1, V \otimes V(n)) \cong (GL_1 \times G \times SL_n, \Lambda_1 \otimes \rho \otimes \Lambda_1, V(1) \otimes V \otimes V(n))$ is always a PV. We call such a PV a trivial PV.

In section 1, we give, as preliminaries, a notion of castling transform. Our solution of the classification problem consists of the explicit description of the process of castling transform and list of reduced prehomogeneous vector spaces. We also give basic propositions. In particular, the unpublished result of T. Kimura (Proposition 1.9) is essential for our classification in this paper. We generalize it in section 2.

In section 2, first we give some lemmas. By using them, we classify k-simple PVs of type (A) which is

$$\left(GL_1^l \times G_1 \times \cdots \times G_k, \bigoplus_{i=1}^{l} \rho_1^{(i)} \otimes \cdots \otimes \rho_k^{(i)}, \bigoplus_{i=1}^{l} V(m_1^{(i)}) \otimes \cdots \otimes V(m_k^{(i)})\right) \cdots (A)$$

where for any i, j, each $\rho_j^{(i)}$ is a nontrivial irreducible representation of G_j on $V(m_j^{(i)})$ with $k \geq 3$ and $l \geq 2$ under full scalar multiplications with $m_j^{(i)} \geq 2$ for $1 \leq i \leq l$ and $1 \leq j \leq k$.

In section 3, we give the list of k-simple prehomogeneous vector spaces of type (A).

Notation. As usual, $\mathbf{C}$ stands for the field of complex numbers. For positive integers m, n, we denote by $M(m,n)$ the totality of $m \times n$ matrices over $\mathbf{C}$. If $m = n$, we simply write $M(n)$ instead of $M(n,n)$. Two triplets are called isomorphic and denotes by $(G, \rho, V) \cong (G', \rho', V')$ if there exists a group isomorphism $\sigma : \rho(G) \to \rho(G')$ and an isomorphism $\tau : V \to V'$ of vector spaces satisfying $\tau(\rho(g)(v)) = (\sigma\rho(g))\tau(v)$ for all $g \in G$ and $v \in V$. In our classification, we identify isomorphic triplets.

We denote by Λ_1 the standard representation of GL_n on $\mathbf{C}^n$. More generally, Λ_k $(k = 1, \ldots, r)$ denotes the fundamental irreducible representation of a simple algebraic group of rank r. In general, we denote by ρ^* the dual representation of a rational representation ρ. Note that $(G, \rho, V) \cong (G, \rho^*, V^*)$ if G is reductive. Also if G_1 and G_2 are reductive, then we have $(G_1 \times G_2, \rho_1^{(*)} \otimes \rho_2^{(*)}) \cong (G_1 \times G_2, \rho_1 \otimes \rho_2)$ where $\rho^{(*)}$ stands for ρ or its dual ρ^*.

We denote $\rho_1 + \cdots + \rho_l$ by $\bigoplus_{i=1}^{l} \rho_i$. When there is no confusion, we sometimes write (G, ρ) instead of (G, ρ, V).

For $l \geq 2$, we do not write the action of GL_1^l for simplicity. Namely we write

$(GL_1^l \times G, \sigma_1 + \cdots + \sigma_l, V_1 \oplus \cdots \oplus V_l)$ instead of $(GL_1^l \times G, \Lambda_1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \Lambda_1 \otimes \sigma_l, V_1 \oplus \cdots \oplus V_l)$ when $l \geq 2$.

1. Preliminaries

For our classifications of PVs, the following propositions and theorems are basic.

Proposition 1.1 ([SK, Proposition 1 in section 3]). *If (G, ρ, V) is a PV, then we have dim $G \geq$ dim V.*

Proposition 1.2 (Castling transform) ([SK, Proposition 7 in section 2]). *Let ρ be a representation of an algebraic group H on an m-dimensional vector space V. For any n satisfying $m > n \geq 1$, the following conditions are equivalent.*

1. *$(H \times GL_n, \rho \otimes \Lambda_1, V \otimes V(n))$ is a PV.*
2. *$(H \times GL_{m-n}, \rho^* \otimes \Lambda_1, V \otimes V(m-n))$ is a PV.*
3. *$(H \times GL_{m-n}, \rho \otimes \Lambda_1, V \otimes V(m-n))$ is a PV if H is reductive.*

We say the triplets 1, 2 (resp. 1, 3 if H is reductive) in Proposition 1.2 are castling transforms of each other. Two triplets are called castling-equivalent if one obtained from the other by a finite number of castling transformations.

Theorem 1.3. *Let $\rho_i : G \to GL(V_i)$ $(i = 1, 2)$ be finite-dimensional rational representations of an algebraic group G, and let $n \geq max\{dim\ V_1, dim\ V_2\}$ be an arbitrary natural number. Then the following conditions are equivalent.*

1. *$(G \times GL_n, \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, V_1 \otimes V(n) + V_2 \otimes V(n)^*)$ is a PV.*
2. *$(G, \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ is a PV.*

Proof. See [K3, Proposition 7.8, P229]. □

The following Theorem is a modification of Theorem 1.3 with respect to scalar multiplication.

Theorem 1.4. *Let $\sigma_i : H \to GL(V_i)$ $(i = 1, 2)$ be finite-dimensional rational representations of an algebraic group H, and let $n \geq max\{dim\ V_1, dim\ V_2\}$ be an arbitrary natural number. Then the following conditions are equivalent.*

1. $(GL_1^2 \times H \times SL_n, \sigma_1 \otimes \Lambda_1 + \sigma_2 \otimes \Lambda_1^*, V_1 \otimes V(n) + V_2 \otimes V(n)^*)$ *is a PV.*
2. $(GL_1 \times H, \Lambda_1 \otimes \sigma_1 \otimes \sigma_2, V_1 \otimes V_2)$ *is a PV.*

PROOF. If we put $G = GL_1 \times H$ and $\rho_i = \Lambda_1 \otimes \sigma_i$ $(i = 1, 2)$ in Theorem 1.3, we have $(G \times GL_n, \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*) \cong (GL_1^2 \times H \times SL_n, \sigma_1 \otimes \Lambda_1 + \sigma_2 \otimes \Lambda_1^*)$ and $(G, \rho_1 \otimes \rho_2) \cong (GL_1 \times H, \Lambda_1 \otimes (\sigma_1 \otimes \sigma_2))$. Hence we have our result by Theorem 1.3. □

Now we shall introduce some classifications of reductive PVs.

THEOREM 1.5 ([SK, Section 7]). *Any irreducible PV* (G, ρ, V) *is castling-equivalent to one of the following PVs*:

(I) *Regular PVs*
  (1) *A trivial PV, i.e.,* $(H \times GL_n, \rho \otimes \Lambda_1, M(n))$ *where* ρ *is an n-dimensional irreducible representation of a connected semisimple algebraic group* H.
  (2) (GL_n, ρ) *where* $\rho = 2\Lambda_1$; $3\Lambda_1$ $(n = 2)$; Λ_2 $(n = even)$; Λ_3 $(n = 6, 7, 8)$.
  (3) $(SL_3 \times GL_2, 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2))$.
  (4) $(SL_6 \times GL_2, \Lambda_2 \otimes \Lambda_1, V(15) \otimes V(2))$.
  (5) $(SL_5 \times GL_n, \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(n))$ $(n = 3, 4)$.
  (6) $(SL_3 \times SL_3 \times GL_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2))$.
  (7) $(Sp_n \times GL_{2m}, \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ $(n > m \geq 1)$.
  (8) $(GL_1 \times Sp_3, \Lambda_1 \otimes \Lambda_3, V(14))$.
  (9) $(SO_n \times GL_m, \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m))$ $(n > m \geq 1)$.
  Note that $(SO_3, \Lambda_1) \cong (SL_2, 2\Lambda_1)$, $(SO_4, \Lambda_1) \cong (SL_2 \times SL_2, \Lambda_1 \otimes \Lambda_1)$, $(SO_5, \Lambda_1) \cong (Sp_2, \Lambda_2)$, $(SO_6, \Lambda_1) \cong (SL_4, \Lambda_2)$.
  (10) $(Spin_7 \times GL_n,$ *the spin rep.* $\otimes \Lambda_1)$ $(1 \leq n \leq 3)$.
  (11) $(GL_1 \times Spin_n,$ $\Lambda_1 \otimes$ *the spin rep.*$)$ $(n = 9, 11)$.
  (12) $(Spin_{10} \times GL_n,$ *a half-spin rep.* $\otimes \Lambda_1)$ $(n = 2, 3)$.
  (13) $(GL_1 \times Spin_n,$ $\Lambda_1 \otimes$ *a half-spin rep.*$)$ $(n = 12, 14)$.
  (14) $(G_2 \times GL_n, \Lambda_2 \otimes \Lambda_1, V(7) \otimes V(n))$ $(n = 1, 2)$.
  (15) $(E_6 \times GL_n, \Lambda_1 \otimes \Lambda_1, V(27) \otimes V(n))$ $(n = 1, 2)$.
  (16) $(GL_1 \times E_7, \Lambda_1 \otimes \Lambda_1, V(56))$.

(II) *Non-regular PVs*
  (1) $(GL_1 \times Sp_n \times SO_3, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(3))$ $(n \geq 2)$.

(2) $((GL_1 \times)H \times SL_n, (\Lambda_1 \otimes)\rho \otimes \Lambda_1, V(m) \otimes V(n))$ *where* ρ *is an* m-*dimensional irreducible representation of a semisimple algebraic group* H *with* $1 \le m < n$.

(3) $((GL_1 \times)SL_{2m+1}, (\Lambda_1 \otimes)\Lambda_2, V(m(2m+1)))$ $(m \ge 2)$.

(4)

$$((GL_1 \times)SL_{2m+1} \times SL_2, (\Lambda_1 \otimes)\Lambda_2 \otimes \Lambda_1, V(m(2m+1)) \otimes V(2)) \quad (m \ge 2).$$

(5)

$$((GL_1 \times)Sp_n \times SL_{2m+1}, (\Lambda_1 \otimes)\Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m+1)) \quad (n \ge 2m+1 \ge 1).$$

(6) $((GL_1 \times)Spin_{10}, (\Lambda_1 \otimes)$ *a half-spin rep.*, $V(16))$.

Theorem 1.6 ([K, Section 3]). *All non-irreducible simple PVs with scalar multiplications are given as follows*:

(1) $(GL_1^l \times SL_n, \Lambda_1 \overbrace{+ \cdots +}^{l-1} \Lambda_1 + \Lambda_1^{(*)})$ $(2 \le l \le n+1, n \ge 2)$.
(2) $(GL_1^l \times SL_n, \Lambda_2 + \Lambda_1^{(*)} \overbrace{+ \cdots +}^{l-1} \Lambda_1^{(*)})$ $(2 \le l \le 4, n \ge 4)$
except $(GL_1^4 \times SL_n, \Lambda_2 + \Lambda_1 + \Lambda_1 + \Lambda_1^*)$ *with* $n = odd$.
(3) $(GL_1^2 \times SL_{2m+1}, \Lambda_2 + \Lambda_2)$ $(m \ge 2)$.
(4) $(GL_1^2 \times SL_n, 2\Lambda_1 + \Lambda_1^{(*)})$.
(5) $(GL_1^3 \times SL_5, \Lambda_2 + \Lambda_2 + \Lambda_1^*)$.
(6) $(GL_1^2 \times SL_n, \Lambda_3 + \Lambda_1^{(*)})$ $(6 \le n \le 7)$.
(7) $(GL_1^3 \times SL_6, \Lambda_3 + \Lambda_1 + \Lambda_1)$.
(8) $(GL_1^l \times Sp_n, \Lambda_1 \overbrace{+ \cdots +}^{l} \Lambda_1)$ $(2 \le l \le 3, n \ge 2)$.
(9) $(GL_1^2 \times Sp_2, \Lambda_2 + \Lambda_1)$.
(10) $(GL_1^2 \times Sp_3, \Lambda_3 + \Lambda_1)$.
(11) $(GL_1^2 \times Spin_7$, *the spin rep.* + *the vector rep.*).
(12) $(GL_1^2 \times Spin_n$, *a half-spin rep.* + *the vector rep.*) $(n = 8, 10, 12)$.
(13) $(GL_1^2 \times Spin_{10}, \Lambda + \Lambda)$, *where* $\Lambda =$ *the even half-spin representation.*

Theorem 1.7 ([KKIY, Section 3] and [KKTI, Section 5]). *All non-irreducible 2-simple PVs are classified in* [*KKIY*] *for type* I *and* [*KKTI*] *for type* II.

For our classification, following two propositions are basic. In particular, Proposition 1.9 is essential.

PROPOSITION 1.8 (T. Kimura). *Let* (G, ρ, V) *be a triplet such that*

$$G = GL_1^l \times SL_{n_1} \times \cdots \times SL_{n_k}, \quad \rho = (\Lambda_1 \overbrace{\otimes \cdots \otimes}^{k} \Lambda_1) \overbrace{+ \cdots +}^{l} (\Lambda_1 \overbrace{\otimes \cdots \otimes}^{k} \Lambda_1),$$

$$V = V(n_1 \cdots n_k) \overbrace{+ \cdots +}^{l} V(n_1 \cdots n_k) \quad \text{with } n_1 \geq n_2 \geq \cdots \geq n_k \geq 2,$$

$$k \geq 3 \text{ and } l \geq 2.$$

1. *If dim* $G >$ *dim* V, *then it is castling-equivalent to a trivial PV.*
2. *If dim* $G =$ *dim* V *and* $k \geq 3$, *it is castling-equivalent to a reguler simple PV* $(GL_1^l \times SL_{l-1}, \Lambda_1 \overbrace{+ \cdots +}^{l} \Lambda_1, M(l, l-1))$.
3. *If dim* $G =$ *dim* V *and* $k = 2$, *it is a non PV.*

PROOF. See [K2, Theorem 9.2 in section 9]. □

PROPOSITION 1.9 ([K4, T. Kimura's unpublished result]). *If* (G, ρ, V) *which is same as that of Proposition* 1.8 *is a PV, then* $(l-1)n_2 \cdots n_k < n_1$.

We shall generalize this proposition (See Proposition 2.4).

2. A Classification

In this section, we classify k-simple PVs of type (A) by using following lemmas.

LEMMA 2.1. *Let* (G, ρ, V) *be an irreducible triplet such that* $G = GL_1 \times G_1 \times \cdots \times G_k$, $\rho = \Lambda_1 \otimes \rho_1 \otimes \cdots \otimes \rho_k$, $V = V(m_1) \otimes \cdots \otimes V(m_k)$ *with* $m_1 \geq m_2 \geq \cdots \geq m_k \geq 2$, $k \geq 3$ *and each* ρ_j *is a nontrivial irreducible representation of a simple algebraic group* G_j *(i.e.,* $\rho_j \neq 1$*) for any* j. *If* (G, ρ, V) *is a PV, then* $(G_1, \rho_1, V(m_1)) \cong (SL_{m_1}, \Lambda_1, V(m_1))$.

PROOF. First assume that (G, ρ, V) is a reduced PV. If it is a trivial PV, it is clear. If it is a nontrivial PV, then it is also obvious since (G, ρ) is $(GL_1 \times SL_3 \times SL_3 \times SL_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ or $(GL_1 \times SL_2 \times SL_2 \times SL_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ by Theorem 1.5. Next we assume that (G, ρ, V) is a non reduced PV with $(G_1, \rho_1, V(m_1)) \not\cong (SL_{m_1}, \Lambda_1, V(m_1))$, Then there exists a castling transformation to obtain a lower dimensional PV which must be of the form (G', ρ', V') such that $G' = GL_1 \times G_1 \times \cdots \times SL_{m_j'} \times \cdots \times G_k$, $V' = V(m_1)$

$\otimes \cdots \otimes V(m_j') \otimes \cdots \otimes V(m_k)$ with $m_j' = m_1 \cdots m_{j-1} m_{j+1} \cdots m_k - m_j$ for some j satisfying $2 \leq j \leq k$. However we have $m_j' = m_1 \cdots m_{j-1} m_{j+1} \cdots m_k - m_j \geq 2m_j - m_j = m_j$ which is a contradiction. So we have our result. □

Lemma 2.2. *Let $\rho = \rho_1 \otimes \cdots \otimes \rho_k$ be an irreducible representation of $G_1 \times \cdots \times G_k$ where each ρ_j is a nontrivial irreducible representation of a simple algebraic group G_j $(1 \leq j \leq k)$. Then we have $\dim(G_1 \times \cdots \times G_k) \leq (\deg \rho_1)^2 + \cdots + (\deg \rho_k)^2 - k \leq (\deg \rho)^2 - k$.*

Proof. Since ρ_j is a nontrivial representation of a simple algebraic group, we have $\dim \rho_j(G_j) = \dim G_j$ and $\rho_j(G_j) \subset SL(\deg \rho_j)$ $(1 \leq j \leq k)$. Hence we obtain the first inequality. Since we have $t_1 \cdots t_k \geq t_1 + \cdots + t_k$ for any $t_1, \ldots, t_k \geq 2$, we obtain the second inequality. □

Lemma 2.3. *Let (G, ρ, V) be a k-simple PV of type* (A). *Then it is of the following form*:

$$\left(GL_1^l \times SL_t \times G_2 \times \cdots \times G_k, \bigoplus_{i=1}^{l} \Lambda_1^{(*)} \otimes \rho_2^{(i)} \otimes \cdots \otimes \rho_k^{(i)}, \right.$$

$$\left. \bigoplus_{i=1}^{l} V(t)^{(*)} \otimes V(m_2^{(i)}) \otimes \cdots \otimes V(m_k^{(i)}) \right)$$

with $k \geq 3$, $l \geq 2$, $\rho_j^{(i)} \neq 1$ and $m_j^{(i)} \geq 2$ for any i, j and $t \geq \max\{m_j^{(i)} \mid 1 \leq i \leq l, 2 \leq j \leq k\}$.

Proof. Assume that $m_1^{(1)} \geq \cdots \geq m_k^{(1)}$. Then we have $(G_1, \rho_1^{(1)}, V(m_1^{(1)})) \cong (SL_t, \Lambda_1, V(t))$ and hence $G_1 = SL_t$ and $\rho_1^{(1)} = \Lambda_1$, $t = m_1^{(1)}$. Now assume that $\rho_1^{(2)} \neq \Lambda_1, \Lambda_1^*$. Then we have $m_1^{(2)} = \deg \rho_1^{(2)} > \deg \Lambda_1 = \deg \rho_1^{(1)} = t$. Then by Lemma 2.1, there is some $j \geq 2$ satisfying $(G_j, \rho_j^{(2)}, V(m_j^{(2)})) \cong (SL(T), \Lambda_1, V(T))$ with $T \geq m_1^{(2)} (> t)$. Then we have $G_j = SL(T)$ and $t \geq m_j^{(1)} \geq T (> t)$, which is a contradiction. Hence we have $\rho_1^{(2)} = \Lambda_1^{(*)}$ and $m_1^{(2)} = t$. Similary we have $\rho_1^{(j)} = \Lambda_1^{(*)}$ and $m_1^{(j)} = t$ for $1 \leq j \leq k$. □

Proposition 2.4. *Let (G, ρ, V) be a k-simple PV of type* (A) *of the form in Lemma* 2.3. *Put $m_i = m_2^{(i)} \cdots m_k^{(i)}$ $(1 \leq i \leq l)$ and $M = m_1 + \cdots + m_l$. Then we have $M - m_i < t$ $(1 \leq i \leq l)$.*

PROOF. First note that the number l and $\dim G - \dim V$ are invariant under castling transformations. Put $g = \dim(G_2 \times \cdots \times G_k)$. We may assume that $m_1 \geq m_2 \geq \cdots \geq m_l (\geq 2^{k-1})$. Then, by Lemma 2.2, we have $g \leq (m_l)^2 - (k-1) \leq (m_l)^2 - 2$. If $t \geq M$, it is clear. Hence we may assume that $M > t$. By Proposition 1.1, we have $\dim G \geq \dim V$ and hence $f(t) \geq 0$ where $f(x) = x^2 - Mx + l + g - 1$. First we shall show that the discriminant D of $f(x)$ satisfies $D \geq 0$. If $D = M^2 - 4(l+g-1) < 0$, then we have $M^2/4 - l + 1 < g \leq (m_l)^2 - 2$ and hence $(M + 2m_l)(M - 2m_l) < 4(l-3)$. Since $2(l+2) \leq M + 2m_l$ and $2(l-2) \leq M - 2m_l$, we have $4(l^2 - 4) < 4(l-3)$, i.e., $l(l-1) < 1$. This is a contradiction and we have $D \geq 0$. Next we shall show that $t \geq (M + \sqrt{D})/2$. For this purpose, it is enough to show that $(M - \sqrt{D})/2 < t$. Put $m = \max\{m_2^{(l)}, \ldots, m_k^{(l)}\}(\leq t)$. Then by Lemma 2.2, we have $g \leq \{m_2^{(l)}\}^2 + \cdots + \{m_k^{(l)}\}^2 - (k-1) \leq (k-1)m^2 - k + 1$. Hence we have

$$\begin{aligned}(M - \sqrt{D})/2 &= (M^2 - D)/(2M + 2\sqrt{D}) = 2(l+g-1)/(M + \sqrt{D}) \\ &\leq 2(l+g-1)/M \leq 2(l+g-1)/(lm_l) = 2(l+g-1)/(lm_2^{(l)} \cdots m_k^{(l)}) \\ &\leq 2(l + (k-1)m^2 - k)/(2^{k-2} lm)\end{aligned}$$

which is $< m(\leq t)$ if and only if $(T =)(2^{k-2}l - 2k + 2)m^2 - 2l + 2k > 0$. Since $m \geq 2$, $l \geq 2$ and $k \geq 3$, we have $T \geq (2^k - 2)l - 6k + 8 \geq 2^{k+1} - 6k + 4 \geq 2$. Thus we have $(M - \sqrt{D})/2 < t$ and hence $t \geq (M + \sqrt{D})/2$. Now assume that $M - m_i \geq t$ for some i. Then we have $M - m_i \geq t \geq (M + \sqrt{D})/2$ and hence $(M - 2m_i)^2 \geq D = M^2 - 4(l+g-1)$, i.e., $l + g - 1 \geq m_i(M - m_i)$. Since

$$\begin{aligned}m_i(M - m_i) &= m_i(M - m_i - m_j) + m_i m_j \geq m_i(M - m_i - m_j) + (m_l)^2 \\ &\geq 2(2(l-2)) + (g+2) = 4l + g - 6,\end{aligned}$$

we have $l - 1 \geq 4l - 6$, i.e., $5 \geq 3l$ $(l \geq 2)$. This is a contradiction and we have $M - m_i < t$ for any i. □

REMARK 2.5. In particular, if $(G_j, \rho_j^{(i)}, V(m_j^{(i)})) = (SL_{n_j}, \Lambda_1, V(n_j))$ where for any i, j, $m_j^{(i)} = n_j$ with $n_1 \geq \cdots \geq n_k \geq 2$, we obtain the proof of proposition 1.9 since $M - m_i = ln_2 \cdots n_k - n_2 \cdots n_k = (l-1)n_2 \cdots n_k < t = n_1$.

LEMMA 2.6. *Let σ and τ be nontrivial irreducible representations of a simple algebraic group H. Then we have $(deg\ \sigma)(deg\ \tau) > dim\ H$.*

Proof. We may assume that $m = \deg \sigma \geq n = \deg \tau$. Then we have $mn > n^2 - 1 \geq \dim H$ by Lemma 2.2. □

Lemma 2.7. *Assume that $k \geq 2$. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ and $\tau = \tau_1 \otimes \cdots \otimes \tau_k$ be irreducible representations of $G_1 \times \cdots \times G_k$ where σ_j and τ_j are nontrivial irreducible representations of a simple algebraic group G_j $(1 \leq j \leq k)$. Then we have $(deg\ \sigma)(deg\ \tau) > 1 + dim(G_1 \times \cdots \times G_k)$. In particular $(GL_1 \times G_1 \times \cdots \times G_k, \Lambda_1 \otimes (\sigma \otimes \tau))$ is a non PV.*

Proof. By Lemma 2.6, we have $(\deg \sigma_j)(\deg \tau_j) > \dim G_j (\geq 3)$ for $1 \leq j \leq k$. Since $t_1 \cdots t_k > t_1 + \cdots + t_k$ for any $t_1, \ldots, t_k \geq 3$ with $k \geq 2$, we have

$$(\deg \sigma)(\deg \tau) = \prod_{j=1}^{k} (\deg \sigma_j)(\deg \tau_j) > \prod_{j=1}^{k} \dim G_j > \sum_{j=1}^{k} \dim G_j$$
$$= \dim(G_1 \times \cdots \times G_k).$$

Hence we have $(\deg \sigma)(\deg \tau) > 1 + \dim(G_1 \times \cdots \times G_k)$. □

Now we shall classify all k-simple PVs of type (A). By Lemma 2.3, it is of the following form:

$$\left(GL_1^l \times SL_t \times G_2 \times \cdots \times G_k, \bigoplus_{i=1}^{l} \rho_1^{(i)} \otimes \rho_2^{(i)} \otimes \cdots \otimes \rho_k^{(i)},\right.$$
$$\left.\bigoplus_{i=1}^{l} V(t) \otimes V(m_2^{(i)}) \otimes \cdots \otimes V(m_k^{(i)})\right)$$

with $k \geq 3$, $l \geq 2$, $\rho_1^{(i)} = \Lambda_1$ or Λ_1^*, $\rho_j^{(i)} \neq 1$, $m_j^{(i)} \geq 2$ $(2 \leq j \leq k)$ and $t \geq \max\{m_j^{(i)} \mid 1 \leq i \leq l, 2 \leq j \leq k\}$. Here we may assume that $\rho_1^{(1)} = \Lambda_1$ without loss of generality. First we shall show that $\rho_1^{(i)} = \Lambda_1$ for all i $(1 \leq i \leq l)$. Assume that $\rho_1^{(i)} = \Lambda_1^*$ for some $i (\geq 2)$. Then

$$(GL_1^2 \times SL_t \times G_2 \times \cdots \times G_k, \Lambda_1 \otimes \rho_2^{(1)} \otimes \cdots \otimes \rho_k^{(1)} + \Lambda_1^* \otimes \rho_2^{(i)} \otimes \cdots \otimes \rho_k^{(i)},$$
$$V(t) \otimes V(m_2^{(1)}) \otimes \cdots \otimes V(m_k^{(1)}) + V(t)^* \otimes V(m_2^{(i)}) \otimes \cdots \otimes V(m_k^{(i)}))$$

is a PV. By Lemma 2.4, we have $m_1 = m_2^{(1)} \cdots m_k^{(1)} \leq M - m_i < t$ and $m_i = m_2^{(i)} \cdots m_k^{(i)} \leq M - m_1 < t$. Hence by Theorem 1.4, it is PV-equivalent to

$$(GL_1 \times G_2 \times \cdots \times G_k, \Lambda_1 \otimes (\rho_2^{(1)} \otimes \rho_2^{(i)}) \otimes \cdots \otimes (\rho_k^{(1)} \otimes \rho_k^{(i)}),$$
$$(V(m_2^{(1)}) \otimes V(m_2^{(i)})) \otimes \cdots \otimes (V(m_k^{(1)}) \otimes V(m_k^{(i)})))$$

which is a non PV by Lemma 2.7. This is a contradiction and $\rho_1^{(i)} = \Lambda_1$ for all i. Hence our PV is of the form:

$$\left(GL_1^l \times SL_t \times (G_2 \times \cdots \times G_k), \Lambda_1 \otimes \left(\bigoplus_{i=1}^{l} \rho_2^{(i)} \otimes \cdots \otimes \rho_k^{(i)}\right),\right.$$
$$\left. V(t) \otimes (V(m_1) + \cdots + V(m_l))\right)$$

where $m_i = m_2^{(i)} \cdots m_k^{(i)}$ $(1 \le i \le l)$. If $M = m_1 + \cdots + m_l \le t$, then it is a trivial PV. Assume that $M > t$. Then by Proposition 1.2, it is castling-equivalent to

$$\left(GL_1^l \times SL(M-t) \times (G_2 \times \cdots \times G_k), \Lambda_1 \otimes \left(\bigoplus_{i=1}^{l} \rho_2^{(i)} \otimes \cdots \otimes \rho_k^{(i)}\right),\right.$$
$$\left. V(M-t) \otimes (V(m_1) + \cdots + V(m_l))\right).$$

Put $m = \max\{m_j^{(i)} \mid 1 \le i \le l, 2 \le j \le k\}$. We shall show that $m > M - t$. If $m \le M - t$, then by Proposition 2.4, we have $M - m_i < M - t$, and hence $t < m_i$. Take $j \ne i$ and applying Proposition 2.4 to the previous PV, we have $M - m_j < t$. Then we have $t < m_i \le M - m_j < t$, which is a contradiction. Hence we have $m > M - t$. Take i and j satisfying $m_j^{(i)} = m$. Then

$$(GL_1 \times G_j \times SL(M-t) \times G_2 \times \cdots G_{j-1} \times G_{j+1} \times \cdots \times G_k,$$
$$\Lambda_1 \otimes \rho_j^{(i)} \otimes \Lambda_1 \otimes \rho_2^{(i)} \otimes \cdots \otimes \rho_{j-1}^{(i)} \otimes \rho_{j+1}^{(i)} \otimes \cdots \otimes \rho_k^{(i)},$$
$$V(m) \otimes V(M-t) \otimes V(m_2^{(i)}) \otimes \cdots \otimes V(m_{j-1}^{(i)}) \otimes V(m_{j+1}^{(i)}) \otimes \cdots \otimes V(m_k^{(i)}))$$

is a PV with $m > M - t$ and $m \ge m_j^{(i)}$ $(2 \le j \le k)$. If $(G_j, \rho_j^{(i)}, V(m)) \not\cong (SL_m, \Lambda_1, V(m))$, then we have $M - t = 1$ and $k = 3$ by Lemma 2.1. This implies that our original PV is castling-equivalent to a 2-simple PV. Now assume that $(G_j, \rho_j^{(i)}, V(m)) \cong (SL_m, \Lambda_1, V(m))$. If $M - t = 1$ and $k = 3$, it is a 2-simple PV. If $M - t \ge 2$ or $(M - t = 1$ and $k \ge 4)$, then this PV is the same type as the original PV with less dimension. Therefore, by repeating this procedure, we obtain a k-simple trivial PV with $k \geq 3$ or a 2-simple PV or a simple PV. They have already been classified by Theorem 1.7 ([KKIY] for type I and [KKTI] for type II) and Theorem 1.6 ([K]). This completes the classification of PVs of type (A).

3. A List

THEOREM 3.1. *Any non-irreducible k-simple prehomogeneous vector space of type* $(GL_1^l \times G_1 \times \cdots \times G_k, \rho_1^{(1)} \otimes \cdots \otimes \rho_k^{(1)} + \cdots + \rho_1^{(l)} \otimes \cdots \otimes \rho_k^{(l)})$ *where for any* i, j, *each* $\rho_j^{(i)}$ *is a nontrivial irreducible representation of a simple algebraic group* G_j *(i.e.,* $\rho_j^{(i)} \neq 1$*) with* $k \geq 3$ *and* $l \geq 2$ *under full scalar multiplications is castling-equivalent to one of the following PVs*:

(I) *A trivial PV, i.e.,* $(GL_1^l \times G_1 \times \cdots \times G_k \times SL_t, (\rho_1 + \cdots + \rho_l) \otimes \Lambda_1)$ *where* $\rho_i = \rho_1^{(i)} \otimes \cdots \otimes \rho_k^{(i)}$ *is an irreducible representation of* $G_1 \times \cdots \times G_k$, $m_1 + \cdots + m_l \leq t$ *by putting* $m_i = \deg \rho_i$ *and each* $\rho_j^{(i)}$ *is a nontrivial irreducible representation of* G_j *for any* i, j *with* $k \geq 0$, $l \geq 2$ *and* $t \geq 2$.

(II) *A nontrivial* 2*-simple PV*

$$(GL_1^2 \times SL_4 \times SL_2, \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1).$$

(III) *Nontrivial simple PVs*

(1) $(GL_1^l \times SL_{l-1}, \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{l})$ $(l \geq 2)$.

(2) $(GL_1^l \times SL_n, \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{l-1} + \Lambda_1^*)$ $(2 \leq l \leq n+1, n \geq 2)$.

(3) $(GL_1^l \times SL_n, \Lambda_2 + \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{l-1})$ $(2 \leq l \leq 4, n \geq 4)$ *except* $(GL_1^4 \times SL_n, \Lambda_2 + \Lambda_1 + \Lambda_1 + \Lambda_1^*)$ *with* $n = odd$.

(4) $(GL_1^2 \times SL_{2m+1}, \Lambda_2 + \Lambda_2)$ $(m \geq 2)$.

(5) $(GL_1^2 \times SL_n, 2\Lambda_1 + \Lambda_1^{(*)})$ $(n \geq 2)$.

(6) $(GL_1^3 \times SL_5, \Lambda_2 + \Lambda_2 + \Lambda_1^*)$.

(7) $(GL_1^2 \times SL_n, \Lambda_3 + \Lambda_1^{(*)})$ $(n = 6, 7)$.

(8) $(GL_1^3 \times SL_6, \Lambda_3 + \Lambda_1 + \Lambda_1)$.

(9) $(GL_1^l \times Sp_n, \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{l})$ $(2 \leq l \leq 3, n \geq 2)$.

(10) $(GL_1^2 \times Sp_2, \Lambda_2 + \Lambda_1)$.

(11) $(GL_1^2 \times Sp_3, \Lambda_3 + \Lambda_1)$.

(12) $(GL_1^2 \times Spin_7$, *the spin rep.* $+$ *the vector rep.*).

(13) $(GL_1^2 \times Spin_n$, *a half-spin rep.* $+$ *the vector rep.*) $(n = 8, 10, 12)$.

(14) $(GL_1^2 \times Spin_{10}, \Lambda + \Lambda)$, *where* $\Lambda =$ *the even half-spin representation.*

REMARK 3.2. In particular, if $(G_j, \rho_j^{(i)}, V(m_j^{(i)})) = (SL_{n_j}, \Lambda_1, V(n_j))$ where for any i, j, $m_j^{(i)} = n_j$ with $n_1 \geq \cdots \geq n_k \geq 2$, the results (I) and (III) (1) are that of proposition 1.8.

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