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JACOBI OPERATORS ALONG THE STRUCTURE FLOW ON REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

By

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Abstract. Let M be a real hypersurface of a complex space form with almost contact metric structure (ϕ, ξ, η, g). In this paper, we study real hypersurfaces in a complex space form whose structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is ξ-parallel. In particular, we prove that the condition $\nabla_\xi R_\xi = 0$ characterizes the homogeneous real hypersurfaces of type A in a complex projective space or a complex hyperbolic space when $R_\xi \phi S = S\phi R_\xi$ holds on M, where S denotes the Ricci tensor of type $(1,1)$ on M.

1. Introduction

Let $(M_n(c), J, \tilde{g})$ be a complex n-dimensional complex space form with Kähler structure $(J, \tilde{g})$ of constant holomorphic sectional curvature $4c$ and let M be an orientable real hypersurface in $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from $(J, \tilde{g})$.

It is known that there are no real hypersurface with parallel Ricci tensors in a nonflat complex space form (see [6], [8]). This result say that there does not exist locally symmetric real hypersurfaces in a nonflat complex space form. The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Cho and the first author started the study on real hypersurfaces in a complex space form by using the operator R_ξ in [3], [4] and [5]. Recently Ortega, Pérez and Santos [12] have proved that there are no real hypersurfaces in $P_n\mathbf{C}$, $n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [13] due to them.

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Now in this paper, motivated by results mentioned above we consideer the parallelism of the structure Jacobi operator R_ξ in the direction of the structure vector field, that is $\nabla_\xi R_\xi = 0$.

In 1970's, the third author [14], [15] classified the homogeneous real hypersurfaces of $P_n\mathbf{C}$ into six types. On the other hand, Cecil and Ryan [2] extensively studied a Hopf hypersurface, which is realized as tubes over certain submanifolds in $P_n\mathbf{C}$, by using its focal map. By making use of those results and the mentioned work of Takagi, Kimura [10] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbf{C}$ whose all principal curvatures are constant. For the case $H_n\mathbf{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's list, a particular type of tubes over totally geodesic $P_k\mathbf{C}$ or $H_k\mathbf{C}$ $(0 \le k \le n-1)$ adding a horosphere in $H_n\mathbf{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [11] (resp. Montiel and Romero [10]) showed that a real hypersurface in $P_n\mathbf{C}$ (resp. $H_n\mathbf{C}$) is locally congruent to one of real hypersurfaces of type A if and only if the Reeb flow ξ is isometric or equivalently the structure operator ϕ commutes with the shape operator H.

Among the results related R_ξ we mention the following ones.

THEOREM 1 (Cho and Ki [5]). *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi H = HR_\xi$. Then M is a Hopf hypersurface in $M_n(c)$. Further, M is locally congruent to one of the following hypersurfaces*:

(1) *In cases that $M_n(c) = P_n\mathbf{C}$ with $\eta(H\xi) \neq 0$,*
 - (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
 - (A_2) *a tube of radius r over a totally geodesic $P_k\mathbf{C}$ $(1 \le k \le n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

(2) *In cases $M_n(c) = H_n\mathbf{C}$,*
 - (A_0) *a horosphere;*
 - (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbf{C}$;*
 - (A_2) *a tube over a totally geodesic $H_k\mathbf{C}$ $(1 \le k \le n-2)$.*

In this paper we study a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi\phi S = S\phi R_\xi$, where S denotes the Ricci tensor of the hypersurface. We give another characterization of real hypersurfaces of type A in $M_n(c)$ by above two conditions. The main purpose of the present paper is to establish Main Thoerem stated in section 5. We note that the condition $R_\xi\phi S = S\phi R_\xi$ is a much weaker condition. Indeed, every Hopf hypersurface always satisfies this condition.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

We denote by $M_n(c)$, $c \neq 0$ be a nonflat complex space form with the Fubini-Study metric $\tilde{g}$ of constant holomorphic sectional curvature $4c$ and Levi-Civita connection $\tilde{\nabla}$. For an immersed $(2n-1)$-dimensional Riemannian manifold $\tau : M \to M_n(c)$, the Levi-Civita connection ∇ of induced metric and the shape operator H of the immersion are characterized

$$\tilde{\nabla}_X Y = \nabla_X Y + g(HX, Y)v, \quad \tilde{\nabla}_X v = -HX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$ and v a unit normal vector on M. In the sequel the indeces $i, j, k, l, \dots$ run over the range $\{1, 2, \dots, 2n-1\}$ unless otherwise stated. For a local orthonormal frame field $\{e_i\}$ of M, we denote the dual 1-forms by $\{\theta_i\}$. Then the connection forms θ_{ij} are defined by

$$d\theta_i + \sum_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0.$$

Then we have

$$\nabla_{e_i} e_j = \sum_k \theta_{kj}(e_i)e_k = \sum_k \Gamma_{kij} e_k,$$

where we put $\theta_{ij} = \sum_k \Gamma_{ijk}\theta_k$. The structure tensor $\phi = \sum_i \phi_i e_i$ and the structure vector $\xi = \sum_i \xi_i e_i$ satisfy

$$\sum_k \phi_{ik}\phi_{kj} = \xi_i\xi_j - \delta_{ij}, \quad \sum_j \xi_j\phi_{ij} = 0, \quad \sum_i \xi_i^2 = 1, \quad \phi_{ij} + \phi_{ji} = 0,$$

$$d\phi_{ij} = \sum_k (\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki} - \xi_i h_{jk}\theta_k + \xi_j h_{ik}\theta_k), \tag{2.1}$$

$$d\xi_i = \sum_j \xi_j\theta_{ji} - \sum_{j,k} \phi_{ji}h_{jk}\theta_k.$$

We denote the components of the shape operator or the second fundamental tensor H of M by h_{ij}. The components $h_{ij;k}$ of the covariant derivative of H are given by $\sum_k h_{ij;k}\theta_k = dh_{ij} - \sum_k h_{ik}\theta_{kj} - \sum_k h_{jk}\theta_{ki}$. Then we have the equation of Gauss and Codazzi

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.2}$$

$$h_{ij;k} - h_{ik;j} = c(\xi_k\phi_{ij} + \xi_i\phi_{kj} - \xi_j\phi_{ik} - \xi_i\phi_{jk}), \tag{2.3}$$

respectively.

From (2.2) the structure Jacobi operator $R_\xi = (\Xi_{ij})$ is given by

$$\Xi_{ij} = \sum_{k,l} h_{ik}h_{jl}\xi_k\xi_l - \sum_{k,l} h_{ij}h_{kl}\xi_k\xi_l + c\xi_i\xi_j - c\delta_{ij}, \tag{2.4}$$

From (2.2) the Ricci tensor $S = (S_{ij})$ is given by

$$S_{ij} = (2n+1)c\delta_{ij} - 3c\xi_i\xi_j + hh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2.5}$$

where $h = \sum_i h_{ii}$.

First we remark

LEMMA 1. *Let U be an open set in M and F a smooth function on U. We put $dF = \sum_i F_i\theta_i$. Then we have*

$$F_{ij} - F_{ji} = \sum_k F_k\Gamma_{kij} - \sum_k F_k\Gamma_{kji}.$$

PROOF. Taking the exterior derivate of $dF = \sum_i F_i\theta_i$, we have the formula immediately. □

Now we retake a local orthonormal frame field e_i in such a way that (1) $e_1 = \xi$, (2) e_2 is in the direction of $\sum_{i=2}^{2n-1} h_{1i}e_i$ and (3) $e_3 = \phi e_2$. Then we have

$$\xi_1 = 1, \quad \xi_i = 0 \quad (i \geq 2), \quad h_{1j} = 0 \quad (j \geq 3) \quad \text{and} \quad \phi_{32} = 1. \tag{2.6}$$

We put $\alpha := h_{11}$, $\beta := h_{12}$, $\gamma := h_{22}$, $\varepsilon := h_{23}$ and $\delta := h_{33}$.

PROMISE. Hereafter the indeces $p, q, r, s, \ldots$ run over the range $\{4, 5, \ldots, 2n-1\}$ unless otherwise stated.

Since $d\xi_i = 0$, we have

$$\theta_{12} = \varepsilon\theta_2 + \delta\theta_3 + \sum_p h_{3p}\theta_p,$$

(2.7) $$\theta_{13} = -\beta\theta_1 - \gamma\theta_2 - \varepsilon\theta_3 - \sum_p h_{2p}\theta_p,$$

$$\theta_{1p} = \sum_q \phi_{qp}h_{q2}\theta_2 + \sum_q \phi_{qp}h_{q3}\theta_3 + \sum_{q,r} \phi_{qp}h_{qr}\theta_r.$$

We put

(2.8) $$\theta_{23} = \sum_i X_i\theta_i, \quad \theta_{2p} = \sum_i Y_{pi}\theta_i, \quad \theta_{3p} = \sum_i Z_{pi}\theta_i.$$

Then it follows from $d\phi_{2i} = 0$ that $Y_{pi} = -\sum_q \phi_{pq}Z_{qi}$ or $Z_{pi} = \sum_q \phi_{pq}Y_{qi}$. The equations (2.4) and (2.5) are rewritten as

(2.9) $$\Xi_{ij} = -\alpha h_{ij} + h_{1i}h_{1j} + c\delta_{i1}\delta_{j1} - c\delta_{ij},$$

(2.10) $$S_{ij} = hh_{ij} - \sum_k h_{ik}h_{jk} - 3c\delta_{i1}\delta_{j1} + (2n+1)c\delta_{ij},$$

respectively.

3. Real Hypersurfaces Satisfying $\nabla_\xi R_\xi = 0$ and $R_\xi\phi S = S\phi R_\xi$

First we assume that $\nabla_\xi R_\xi = 0$. The components $\Xi_{ij;k}$ of the covariant derivativation of $R_\xi = (\Xi_{ij})$ is given by

$$\sum_k \Xi_{ij;k}\theta_k = d\Xi_{ij} - \sum_k \Xi_{kj}\theta_{ki} - \sum_k \Xi_{ik}\theta_{kj}.$$

Substituting (2.9) into the above equation we have

(3.1) $$\sum_k \Xi_{ij;k}\theta_k = -(d\alpha)h_{ij} - \alpha dh_{ij} + (dh_{1i})h_{1j} + h_{1i}(dh_{1j})$$
$$+ \alpha\sum_k h_{kj}\theta_{ki} - \alpha h_{1j}\theta_{1i} - \beta h_{1j}\theta_{2i} - c\delta_{j1}\theta_{1i}$$
$$+ \alpha\sum_k h_{ik}\theta_{kj} - \alpha h_{1i}\theta_{1j} - \beta h_{1i}\theta_{2j} - c\delta_{i1}\theta_{1j}.$$

In the following, we assume that $\beta \neq 0$.

Our assumption $\nabla_\xi R_\xi = 0$ is equivalent to $\Xi_{ij;1} = 0$, which can be stated as follows:

(3.2) $\varepsilon = 0, \quad \alpha\delta + c = 0, \quad h_{3p} = 0,$

(3.3) $(\beta^2 - \alpha\gamma)_1 - 2\alpha\sum_p h_{2p} Y_{p1} = 0,$

(3.4) $(\beta^2 - \alpha\gamma - c)X_1 + \alpha\sum_p h_{2p} Z_{p1} = 0,$

(3.5) $(\alpha h_{2p})_1 + \alpha\sum_q h_{pq} Y_{q1} + (\beta^2 - \alpha\gamma) Y_{p1} - \alpha\sum_q h_{2q}\Gamma_{qp1} = 0,$

(3.6) $\alpha h_{2p} X_1 - \sum_q (\alpha h_{qp} + c\delta_{pq}) Z_{q1} = 0,$

(3.7) $-(\alpha h_{pq})_1 + \alpha h_{2q} Y_{p1} + \alpha\sum_r h_{rq}\Gamma_{rp1} + \alpha h_{2p} Y_{q1} + \alpha\sum_r h_{pr}\Gamma_{rq1} = 0.$

Hereafter we shall use (3.2) without quoting.

Furthermore we assume that $R_\xi\phi S = S\phi R_\xi$. Under the assumption $\nabla_\xi R_\xi = 0$, we have the following additional equations

(3.8) $(h\delta - \delta^2 + (2n+1)c)h_{2p} = 0,$

(3.9) $\widetilde{R_\xi}\tilde{\phi}A = 0,$

(3.10) $\widetilde{R_\xi}\tilde{\phi}\tilde{S} = \tilde{S}\tilde{\phi}\widetilde{R_\xi}.$

where $A = {}^t(h_{24}, h_{25}, \dots, h_{2,2n-1})$, $\widetilde{R_\xi} = (\Xi_{pq})$, $\tilde{\phi} = (\phi_{pq})$, $\tilde{S} = (S_{pq})$.

Now, properly speaking, we should denote the equation (2.3) by, e.g., $(23)_{ijk}$. In this paper we denote it by (ijk) simply. Then we have the following equations (112)–$(q1p)$.

(112) $\alpha_2 - \beta_1 = 0,$

(212) $\beta_2 - \gamma_1 - 2\sum_p h_{2p} Y_{p1} = 0,$

(312) $(\alpha - \delta)\gamma - \beta X_2 + (\gamma - \delta)X_1 - \beta^2 - \sum_p h_{2p} Z_{p1} = -c,$

(113) $\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0,$

(213) $\beta_3 - \alpha\delta + \gamma\delta + (\gamma - \delta)X_1 - \beta^2 - \sum_p h_{2p} Z_{p1} = c,$

(313) $\beta X_3 + \delta_1 = 0,$

(223) $\gamma_3 - 2\beta\delta + 2\sum_p h_{2p} Y_{p3} + (\gamma - \delta) X_2 - \beta\gamma - \sum_p h_{2p} Z_{p2} = 0,$

(323) $\sum_p h_{2p} Z_{p3} - \delta_2 - (\gamma - \delta) X_3 = 0,$

(1p1) $\alpha_p + \beta Y_{p1} = 0,$

(12p) $\beta_p + 2\sum_{q,r} h_{2q}\phi_{rq} h_{rp} + \beta Y_{p2} + \alpha \sum_q \phi_{qp} h_{2q} = 0,$

(13p) $-2\delta h_{2p} + \beta Y_{p3} + \alpha h_{2p} - \beta X_p = 0,$

(22p) $\gamma_p + 2\sum_q h_{2q} Y_{qp} - h_{2p2} - \sum_q h_{qp} Y_{q2} + \beta \sum_q \phi_{qp} h_{2q} + \gamma Y_{p2} + \sum_q h_{2q}\Gamma_{qp2} = 0,$

(23p) $\delta X_p + \beta h_{2p} - \gamma X_p + \sum_q h_{2q} Z_{qp} - h_{2p3} - \sum_q h_{qp} Y_{q3} + \gamma Y_{p3} + \sum_q h_{2q}\Gamma_{qp3} = 0,$

(33p) $\delta_p + h_{2p} X_3 - \sum_q h_{qp} Z_{q3} + \delta Z_{p3} = 0,$

(21p) $\beta_p + \sum_{q,r} h_{2q}\phi_{rq} h_{rp} - h_{2p1} - \sum_q h_{qp} Y_{q1} + \gamma Y_{p1} + \sum_q h_{2q}\Gamma_{qp1} = 0,$

(31p) $-\delta h_{2p} + \alpha h_{2p} - \beta X_p + h_{2p} X_1 - \sum_q h_{qp} Z_{q1} + \delta Z_{p1} = 0,$

(32p) $\delta X_p + \beta h_{2p} - \gamma X_p + \sum_q h_{2q} Z_{qp} + h_{2p} X_2 - \sum_q h_{pq} Z_{q2} + \delta Z_{p2} = 0,$

(1pq) $2\sum_{r,s} h_{rp}\phi_{sr} h_{sq} - \alpha \sum_r \phi_{rp} h_{rq} + \alpha \sum_r \phi_{rq} h_{rp} - \beta Y_{pq} + \beta Y_{qp} = -2c\phi_{pq},$

(2pq)
$$h_{2pq} + \sum_r h_{rp} Y_{rq} - \beta \sum_r \phi_{rp} h_{rq} - \gamma Y_{pq} - \sum_r h_{2r}\Gamma_{rpq} - h_{2qp}$$
$$- \sum_r h_{rq} Y_{rp} + \beta \sum_r \phi_{rq} h_{rp} + \gamma Y_{qp} + \sum_r h_{2r}\Gamma_{rqp} = 0,$$

(q1p)
$$\sum_{r,s} h_{rq}\phi_{sr} h_{sp} - \alpha \sum_r \phi_{rq} h_{rp} - \beta Y_{qp} - h_{pq1}$$
$$+ h_{2q} Y_{p1} + \sum_r h_{rq}\Gamma_{rp1} + h_{2p} Y_{q1} + \sum_r h_{rp}\Gamma_{rq1} = c\phi_{pq},$$

(q3p)
$$h_{3qp} - \varepsilon Y_{qp} - \delta Z_{qp} - \sum_r h_{3r}\Gamma_{rqp} - h_{2q} X_p + \sum_r h_{qr} Z_{rp} - h_{qp3}$$
$$+ h_{q2} Y_{p3} + h_{q3} Z_{p3} + \sum_r h_{qr}\Gamma_{rp3} + h_{p2} Y_{q3} + h_{p3} Z_{q3} + \sum_r h_{pr}\Gamma_{rq3} = 0.$$

REMARK. We did not write $(p2q)$, $(3pq)$ and (pqr) since we need not use them.

4. Formulas and Lemmas

PROMISE. In the following, we shall abbreviate the expression "take account of the coefficient of θ_i in the exterior derivative of $\cdots$" to "see θ_i of d of $\cdots$".

In this section we study the crucial case where $\beta \neq 0$. By (3.6) and $(31p)$ we have

$$\beta X_p = (\alpha - \delta) h_{2p}. \tag{4.1}$$

This and $(13p)$ imply that

$$\beta Y_{p3} = \delta h_{2p}. \tag{4.2}$$

The equation (3.9) can be rewrittened as

$$\sum_{q,r} (\alpha h_{pq} + c\delta_{pq}) \phi_{qr} h_{r2} = 0, \tag{4.3}$$

which, together with (4.2), implies

$$\beta \sum_{q,r} (h_{pq} - \delta\delta_{pq}) Z_{q3} = \delta \sum_{q,r} (h_{pq} - \delta\delta_{pq}) \phi_{qr} Y_{r3} = 0.$$

Hence it follows from $(33p)$ and $(1p1)$ that

$$\delta_p = -h_{2p} X_3 \quad \text{and} \quad \alpha_p = -\beta Y_{p1}. \tag{4.4}$$

Thus since (4.4) and $\alpha_p \delta + \alpha \delta_p = 0$ obtained from (3.2) we have

$$\beta\delta Y_{p1} = -\alpha h_{2p} X_3, \tag{4.5}$$

and so $\sum_p h_{2p} Z_{p1} = 0$. By (4.2), we have

$$\sum_p h_{2p} Z_{p3} = \sum_{p,q} h_{2p} \phi_{pq} Y_{q3} = \frac{\delta}{\beta} \sum_{p,q} h_{2p} \phi_{pq} h_{2q} = 0. \tag{4.6}$$

From (3.6), (4.3) and (4.5) we have

$$h_{2p} X_1 = 0. \tag{4.7}$$

Now we shall prove the following key lemma.

LEMMA 2. $H(e_2) \in \operatorname{span}\{e_1, e_2\}$.

Proof. Suppose that $h_{2p} \neq 0$. Then from (4.7) we have $X_1 = 0$. We can select the vector e_4 so that $h_{24} \neq 0$ and $h_{25} = \cdots = h_{2,2n-1} = 0$. We put $e_5 := \phi e_4$ and $\rho := h_{24} (\neq 0)$. Note that $\phi_{54} = 1$. Then by (4.3) we have

$$h_{55} = \delta, \quad h_{p5} = 0 \quad (p \neq 5).$$

Put $p = 5$ in $(32p)$. Then by above equation and (4.1) we have $X_5 = 0$ and so $Z_{45} = 0$. Thus we have $Y_{55} = 0$. Furthermore, put $p = q = 5$ in $(q1p)$. Then, since $\Gamma_{551} = Y_{55} = 0$, we have

$$\alpha_1 = \delta_1 = 0. \tag{4.8}$$

Thus, from (313), (323), (4.6) and (112) we have

$$X_3 = 0, \tag{4.9}$$

$$\alpha_2 = \delta_2 = 0, \tag{4.10}$$

$$\beta_1 = 0. \tag{4.11}$$

By (4.4) and (4.9) we have $\alpha_p = \delta_p = 0$. Thus it follows from $(1p1)$ that

$$\alpha_p = \delta_p = Y_{p1} = Z_{p1} = 0. \tag{4.12}$$

Now we put $F = \alpha$, $i = 1$ and $j = p$ in Lemma 1. Then, from (2.7), (4.8), (4.10) and (4.12) we have

$$0 = \alpha_{1p} - \alpha_{p1} = \sum_k \alpha_k \Gamma_{k1p} - \sum_k \alpha_k \Gamma_{kp1} = \alpha_3(\Gamma_{31p} - \Gamma_{3p1}) = \alpha_3 h_{2p}.$$

Thus we have $\alpha_3 = 0$. Hence it follows from (4.8), (4.10) and (4.12) that α and δ are constant, which, together with (113), imply

$$\alpha = 3\delta. \tag{4.13}$$

On the other hand, seeing $\theta_1 \wedge \theta_3$ of d of θ_{23}, we have

$$X_2 = -2\beta. \tag{4.14}$$

Thus, from (312) and (4.13) we have

$$2\delta\gamma + \beta^2 = -c. \tag{4.15}$$

Seeing θ_1 and θ_2 of d of (4.15) and taking account of (4.8), (4.11) and (212), we have

$$\gamma_1 = 0, \quad \beta_2 = 0 \quad \text{and} \quad \gamma_2 = 0. \tag{4.16}$$

Moreover, seeing θ_5 of d of (4.15), we have

$$\delta\gamma_5 + \beta\beta_5 = 0. \tag{4.17}$$

From (3.5) and (4.12) we have

$$h_{2p1} - \sum_q h_{2q}\Gamma_{qp1} = 0.$$

This, together with $(21p)$ and $(12p)$, implies

$$\beta_p + \rho h_{5p} = 0,$$

$$\beta_p + 2\rho h_{5p} + \alpha\rho\phi_{4p} + \beta Y_{p2} = 0.$$

Put $p = 4, 5, 6, \ldots, 2n-1$ in above two equations to get

$$\beta_p = \begin{cases} 0 & (p \neq 5) \\ -\rho\delta & (p = 5) \end{cases}, \quad Y_{p2} = \begin{cases} 0 & (p \neq 5) \\ \rho(\alpha - \delta)/\beta & (p = 5) \end{cases}, \tag{4.18}$$
$$Z_{p2} = \begin{cases} 0 & (p \neq 4) \\ -\rho(\alpha - \delta)/\beta & (p = 4) \end{cases}.$$

Hence from (4.1), (4.2), (4.17) and (4.18) we have

$$X_p = \begin{cases} 0 & (p \neq 4) \\ \rho(\alpha - \delta)/\beta & (p = 4) \end{cases}, \quad Y_{p3} = \begin{cases} 0 & (p \neq 4) \\ -\rho\delta/\beta & (p = 4) \end{cases}, \tag{4.19}$$
$$Z_{p3} = \begin{cases} 0 & (p \neq 5) \\ -\rho\delta/\beta & (p = 5) \end{cases}, \quad \gamma_p = \begin{cases} 0 & (p \neq 5) \\ -\rho\beta & (p = 5) \end{cases}.$$

Now, by (213), (223), (4.15) and (4.19) we have

$$\beta_3 = \beta^2 - \gamma\delta = -\alpha\gamma - c = 3\delta(\delta - \gamma), \tag{4.20}$$
$$\gamma_3 = 3\beta\gamma - 4\rho^2\delta/\beta.$$

On the other hand, if we put $F = \beta$ and γ in Lemma 1, then from (4.11), (4.12), (4.15), (4.16), (4.18) and (4.19) we have

$$\gamma\beta_3 + \rho\beta_5 = 0, \tag{4.21}$$
$$\gamma\gamma_3 + \rho\gamma_5 = 0.$$

Eliminating β_3, β_5, γ_3, γ_5, ρ and β from (4.17), (4.18), (4.20) and (4.21), we have

$$4\gamma^2 - 6\gamma\delta - c = 0.$$

Consequently, γ is constant, which contradicts $\gamma_5 = \rho\beta$. □

Owing to Lemma 2 the matrix (h_{pq}) is diagonalizable, that is, for a suitable choice of a orthonormal frame field $\{e_p\}$ we can set

$$h_{pq} = \lambda_p \delta_{pq}.$$

Then it is easy to see

$$\tilde{R}_\xi = -((\alpha\lambda_p + c)\delta_{pq}), \quad \tilde{S} = (\{h\lambda_p - (\lambda_p)^2 + K\}\delta_{pq}), \tag{4.22}$$

where we put $K = (2n+1)c$.

Here we shall sum up all equations obtained from Lemma 2.

From (4.1), (4.2) and (4.4) we have

$$X_p = Y_{p1} = Z_{p1} = Y_{p3} = Z_{p3} = 0, \quad \alpha_p = \delta_p = 0. \tag{4.23}$$

This, together with (3.3) and (3.4), imply

$$(\beta^2 - \alpha\gamma)_1 = 0, \tag{4.24}$$

$$(\beta^2 - \alpha\gamma - c)X_1 = 0. \tag{4.25}$$

Put $p = q$ in (3.7). Then we have

$$(\alpha\lambda_p)_1 = 0. \tag{4.26}$$

Moreover, from (112)–(32p) we have

$$\alpha_2 - \beta_1 = 0, \tag{4.27}$$

$$\beta_2 - \gamma_1 = 0, \tag{4.28}$$

$$(\alpha - \delta)\gamma - \beta X_2 + (\gamma - \delta)X_1 - \beta^2 = -c, \tag{4.29}$$

$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{4.30}$$

$$\beta_3 - \alpha\delta + \gamma\delta + (\gamma - \delta)X_1 - \beta^2 = c, \tag{4.31}$$

$$\delta_1 + \beta X_3 = 0, \tag{4.32}$$

$$\gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 - \beta\gamma = 0, \tag{4.33}$$

$$\delta_2 + (\gamma - \delta)X_3 = 0, \tag{4.34}$$

$$\beta_p = 0, \tag{4.35}$$

$$Y_{p2} = 0, \quad Z_{p2} = 0, \tag{4.36}$$

$$\gamma_p = 0. \tag{4.37}$$

It follows from $(q1p)$ and (3.7) that

$$\alpha\beta Y_{qp} = \alpha\lambda_p\lambda_q\phi_{pq} - \alpha^2\lambda_p\phi_{pq} + \alpha_1\lambda_p\delta_{pq} - c\alpha\phi_{pq}. \tag{4.38}$$

From this, $(2pq)$ and $(q3p)$ we have

$$\begin{aligned}\beta^2(\lambda_p + \lambda_q)\phi_{pq} &- (\lambda_p - \gamma)(\lambda_p\lambda_q - \alpha\lambda_q - c)\phi_{pq} \\ &- (\lambda_q - \gamma)(\lambda_p\lambda_q - \alpha\lambda_p - c)\phi_{pq} = 0,\end{aligned} \tag{4.39}$$

$$(\lambda_q - \delta)[\alpha\{(\lambda_q)^2 - \alpha\lambda_q - c\}\delta_{pq} + \alpha_1\lambda_q\phi_{pq}] - \alpha\beta\{h_{qp3} + (\lambda_p - \lambda_q)\Gamma_{qp3}\} = 0. \tag{4.40}$$

If $p = q$ in above equation, then we have

$$(\lambda_p - \delta)\{(\lambda_p)^2 - \alpha\lambda_p - c\} - \beta(\lambda_p)_3 = 0. \tag{4.41}$$

5. Proof of Main Theorem

In this section we prove

MAIN THEOREM. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ which satisfies $\nabla_\xi R_\xi = 0$. Then M holds $R_\xi\phi S = S\phi R_\xi$ if and only if M is locally congruent to one of the following*:

- (I) *in case that $M_n(c) = P_n\mathbf{C}$ with $\eta(H\xi) \neq 0$,*
  - (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
  - (A_2) *a tube of radius r over a totally geodesic $P_k\mathbf{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
- (II) *in case that $M_n(c) = H_n\mathbf{C}$,*
  - (A_0) *a horosphere,*
  - (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbf{C}$,*
  - (A_2) *a tube over a totally geodesic $H_k\mathbf{C}$ $(1 \leq k \leq n-2)$.*

PROOF. FIRST STEP. We prove $\beta = 0$.

Suppose that $\beta \neq 0$. It follows from (4.22) that (3.10) is equivalent to

$$(\rho_p\sigma_q - \sigma_p\rho_q)\phi_{pq} = 0,$$

where $\rho_p = \alpha\lambda_p + c$, $\sigma_p = h\lambda_p - (\lambda_p)^2 + K$. Therefore if $\phi_{qp} \neq 0$, then we have

$$(\lambda_p - \lambda_q)\{-ch + \alpha\lambda_p\lambda_q + c(\lambda_p + \lambda_q) + \alpha K\} = 0. \tag{5.1}$$

Here we assert that if $\phi_{pq} \neq 0$, then $\lambda_p = \lambda_q$. To prove this, we assume that there exist indices p and q such that

$$\phi_{pq} \neq 0, \quad \lambda_p - \lambda_q \neq 0.$$

First we prepare three Lemmas.

Lemma 3. $(K\alpha^2 - c\alpha h)_1 = 0$.

Proof. From (5.1) we have

$$(\alpha^2 K - \alpha hc) + (\alpha\lambda_p)(\alpha\lambda_q) + c(\alpha\lambda_p + \alpha\lambda_q) = 0.$$

Lemma 3 follows from this and (4.26). □

Lemma 4. $4n\alpha\alpha_1 - (\alpha\gamma)_1 = 0$.

Proof. From (4.26) we have $(\alpha \sum_p \lambda_p)_1 = 0$. Combining this equation with $h = \alpha + \gamma + \delta + \sum_p \lambda_p$, we have

$$(\alpha(h - \alpha - \gamma - \delta))_1 = 0.$$

Eliminate h from this and Lemma 3. □

Lemma 5. $(\gamma - \delta - 2n\alpha)\alpha_1 = 0$ *and* $(\gamma - \delta - 2n\alpha)\beta_1 = 0$.

Proof. From (4.24) we have $2\beta\beta_1 - (\alpha\gamma)_1 = 0$. Hence it follows from Lemma 4 that

$$2n\alpha\alpha_1 - \beta\beta_1 = 0. \tag{5.2}$$

On the other hand, by (4.32) and (4.34) we have $(\gamma - \delta)\delta_1 - \beta\delta_2 = 0$, and therefore $(\gamma - \delta)\alpha_1 - \beta\alpha_2 = 0$. Thus Lemma 5 follows from (4.27) and (5.2). □

We need to consider four cases.

Case I. Suppose that $\alpha_1 \neq 0$ and $X_1 = 0$. Owing to Lemma 5, we have $\gamma - \delta - 2n\alpha = 0$. Seeing θ_3 of d of this equation and making use of (4.29), (4.30) and (4.33), we have

$$2n\alpha^2(2n\alpha^2 - \delta^2 + 2nc) + \beta^2\{3\delta^2 + (6n + 4)c - 2n\alpha^2\} = 0. \tag{5.3}$$

Seeing θ_1 of d of (5.3) and taking account of (3.2) and (5.2), we have

$$4n^2\alpha^4 + 2n\alpha^2\{3\delta^2 + (8n+4)c\} - \beta^2(3\delta^2 + 2n\alpha^2) = 0. \tag{5.4}$$

Eliminating β from (5.3) and (5.4), we have a polynomial of degree four with respect to δ containing the term $12n\alpha^2\delta^4 \neq 0$. This shows that δ is constant since $\alpha\delta + c = 0$, which contradicts the assumption of Case I.

Case II. Suppose that $\alpha_1 \neq 0$ and $X_1 \neq 0$. By (4.25) we have

$$\beta^2 - \alpha\gamma - c = 0.$$

Then from (4.39) we have

$$(-\lambda_p\lambda_q + 2c)(\lambda_p + \lambda_q) + 2(\alpha + \gamma)\lambda_p\lambda_q - 2c\gamma = 0.$$

Multiply above equation by α^3 and see θ_1 of d of this equation. Then, from Lemma 4 and (4.26) we have

$$c(\alpha\lambda_p + \alpha\lambda_q - \alpha\gamma) + (2n+1)(\alpha\lambda_p)(\alpha\lambda_q) - 2cn\alpha^2 = 0.$$

Again, seeing θ_1 of d of above equation, we have $cn\alpha\alpha_1 = 0$, which is a contradiction.

Case III. Suppose that $\alpha_1 = 0$ and $\beta^2 - \alpha\gamma - c \neq 0$. From (4.24), (4.25), (4.27), (4.28), (4.32) and (4.34) we have

$$\delta_1 = \alpha_2 = \delta_2 = X_3 = \beta_1 = \gamma_1 = \beta_2 = X_1 = 0. \tag{5.5}$$

Seeing $\theta_2 \wedge \theta_3$ of d of θ_{23} we have $\beta_3 - 2\beta^2 = \gamma\delta + 2c$, which, together with (4.31) and (5.5), imply

$$\alpha\delta - \gamma\delta - \beta^2 = \gamma\delta + c.$$

Substituting of (4.14) and (5.5) into (4.29) we have

$$\alpha\gamma - \gamma\delta + \beta^2 = -c. \tag{5.6}$$

Eliminating β from above two equations, we have

$$\alpha\delta - 3\gamma\delta + \alpha\gamma = 0. \tag{5.7}$$

Seeing θ_2 of d of (5.6) and (5.7), we have $(\alpha - \delta)\gamma_2 = 0$ and $(\alpha - 3\delta)\gamma_2 = 0$. Hence we have $\gamma_2 = 0$.

Now put $F = \alpha, \beta, \gamma$ and $i = 1$, $j = 2$ in Lemma 1. Then, we have

$$\alpha_3\gamma = \beta_3\gamma = \gamma_3\gamma = 0.$$

If $\gamma \neq 0$, then from (4.14) and (4.33) we have a contradiction. Thus $\gamma = 0$, which contradicts (5.7).

Case IV. Suppose that

$$\alpha_1 = 0, \tag{5.8}$$

$$\beta^2 - \alpha\gamma - c = 0. \tag{5.9}$$

Seeing θ_2 of d of (5.9), we have

$$(\beta^2 - \alpha\gamma)_3 = 2\beta\beta_3 - \gamma\alpha_3 - \alpha\gamma_3 = 0. \tag{5.10}$$

From (4.29)–(4.31), (4.33) and (5.9) we have the following:

$$-\delta\gamma - \beta X_2 + (\gamma - \delta)X_1 = 0, \tag{5.11}$$

$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{5.12}$$

$$\beta_3 + (\gamma - \delta)X_1 + \gamma\delta - \alpha\gamma - c = 0, \tag{5.13}$$

$$\gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 + \beta\gamma = 0. \tag{5.14}$$

Substituting of (5.12)–(5.14) into (5.10) we have

$$(\delta - \gamma)(X_1 - 4\alpha) = 0,$$

by virtue of (5.11). If $\delta = \gamma$, then by (5.9) we have a contradiction. Thus

$$X_1 = 4\alpha. \tag{5.15}$$

Substituting of this equation into (5.11)–(5.13) we have

$$\beta X_2 = 4\alpha(\gamma - \delta) - \delta\gamma, \tag{5.16}$$

$$\alpha_3 + 3\beta\delta + 3\alpha\beta = 0, \tag{5.17}$$

$$\beta_3 + 3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0. \tag{5.18}$$

It follows from (4.33), (5.9) and (5.16) that

$$\alpha\gamma_3 + \beta(3\alpha\gamma - 6\alpha\delta - \gamma\delta) = 0. \tag{5.19}$$

From (4.32), (5.2) and (5.8) we have $X_3 = 0$ and $\beta_1 = 0$ and therefore $\alpha_2 = \delta_2 = 0$ because of (4.27). Hence, seeing θ_1 of d of (5.9), we have $\gamma_1 = 0$, and so $\beta_2 = 0$.

Now put $F = \alpha$ and β in Lemma 1. Then we have

$$\alpha_3(\gamma + X_1) = 0, \quad \beta_3(\gamma + X_1) = 0.$$

If $\gamma + X_1 \neq 0$, then we have $\alpha_3 = \beta_3 = 0$. It follows from (4.23) and (4.35) that α, β and δ are constant and that $\alpha_i = \beta_i = 0$ for $i = 1, 2$. Furthermore, by (5.9) we see that γ is constant. Thus from (5.17)–(5.19) we have

$$\alpha + \delta = 0,$$

$$3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0,$$

$$3\alpha\gamma - 6\alpha\delta - \gamma\delta = 0.$$

Hence, by (3.2) and (5.9) we have $\alpha^2 - c = 0$ and $2\beta^2 + c = 0$, which is a contradiction. Therefore $X_1 = -\gamma$, which, together with (5.15), implies $\gamma = -X_1 = -4\alpha$. Thus it follows from (5.17) that $\gamma_3 = -4\alpha_3 = 12\beta(\delta + \alpha)$. Hence from (5.19) we have a contradiction $\alpha\delta = 0$.

Consequently, for all p, q such that $\phi_{pq} \neq 0$, we have $\lambda_p = \lambda_q$. We take p, q such that $\phi_{pq} \neq 0$. Then by (4.39) we have

$$\beta^2\lambda_p - (\lambda_p - \gamma)\{(\lambda_p)^2 - \alpha\lambda_p - c\} = 0. \tag{5.20}$$

Furthermore, from $(q3p)$, (4.38) and (4.26) we have

$$(\lambda_p)_1(\lambda_p - \delta) = 0.$$

If $(\lambda_p)_1 = 0$, then (4.26) implies $\alpha_1 = \delta_1 = 0$. Thus it follows from (4.32), (4.34) and (4.27) that $X_3 = \alpha_2 = \delta_2 = \beta_1 = 0$. Seeing θ_1 of d of (5.20), we have $\{(\lambda_p)^2 - \alpha\lambda_p - c\}\gamma_1 = 0$. If $(\lambda_p)^2 - \alpha\lambda_p - c = 0$, then from (5.20), we have $\lambda_p = 0$, which contradicts the assumption. Hence we have $\gamma_1 = 0$. Thus, from (4.28) we have $\beta_2 = 0$. If $X_1 = 0$, then by the same argument as that in Case III, we have a contradiction. Thus we have $X_1 \neq 0$ and therefore $\beta^2 - \alpha\gamma - c = 0$ because of (4.25). By the same argument as that in Case IV, we have contradiction. Hence we have $\lambda_p = \delta$. From (4.41) and (113) we have $(\lambda_p)_3 = \delta_3 = \alpha_3 = 0$ and $X_1 = \alpha - 3\delta_p$. Thus by (4.25) we have $(\beta^2 - \alpha\gamma - c)(\alpha - 3\delta) = 0$. If $\alpha - 3\delta = 0$, then α and δ are constant and therefore by the argument as above, we have a contradiction. Thus $\beta^2 - \alpha\gamma - c = 0$. From (5.20) we have $(\alpha + \delta)(\delta - \gamma) = 0$. If $\alpha + \delta = 0$, then α and δ are constant, which is also a contradiction. Hence $\delta - \gamma = 0$. However from (5.20) we have $\beta = 0$, which is a contradiction. Consequantly we proved $\beta = 0$.

SECOND STEP. Since (2.6) and $\beta = 0$, we see that α is constant in M (see [7]). Thus from (3.1) our assumption $\Xi_{ij;1} = 0$ is equivalent to $\alpha h_{ij;1} = 0$. Put $j = 1$ in

(2.3). Then by above equation we have $\alpha h_{i1;k} = -c\alpha\phi_{ik}$. Therefore since (2.1) and $d\xi_i = 0$, we have

$$\alpha \sum_{k,l} h_{ik}\phi_{lk}h_{kj} + \alpha^2 \sum_{k} \phi_{ki}h_{kj} = -\alpha h_{i1;j} = c\alpha\phi_{ij},$$

which implies that $\alpha^2(\phi H - H\phi) = 0$.

Here, we note the case $\alpha = 0$ corresponds to the case of tube of radius $\pi/4$ in $P_n\mathbf{C}$ (see [2]). However, in the case of $H_n\mathbf{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf. [1]). Owing to Okumura's work or Montiel and Romero's work stated in the Introduction, we complete the proof of our Main Theorem. □

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