Application of homotopy analysis method to option pricing under Lévy processes

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Abstract

Option pricing under the Lévy process has been considered an important research direction in the field of financial engineering, where a closed-form expression for the standard European option is available due to the existence of analytically tractable characteristic function according to the Lévy-Khinchin representation. However this approach cannot be applied to exotic derivatives (such as barrier options) directly, although a large volume of exotic derivatives are actively traded in the current options market. An alternative approach is to solve the corresponding partial integro-differential equation (PIDE) numerically, which is, in fact, time-consuming and is not computationally tractable in general.

In this paper, we apply the so-called homotopy analysis method (HAM) to solve the corresponding PIDE in a *semi analytic form*, being obtained from the following three steps: (1) Apply the Fourier transform to convert the PIDE to an ordinal differential equitation (ODE), and construct a differential system of ODEs. (2) Solve the system of ODEs, where each differential equation is shown to have an analytical solution. (3) Express the option price using the sum of infinite series, where each term may be expressed analytically and derived by applying Steps (1) and (2) recursively. To illustrate our technique more precisely, we take the variance gamma model as an example and provide the semi-analytic form. Numerical examples demonstrate a fast convergence of our proposed method to the prices of European and down-and-out call options with a few number of terms. Note that this method is easy to implement and can be applied to other types of options under general Lévy processes.

Keywords: Barrier options, Homotopy analysis method, Lévy processes, Variance gamma model.

1 Introduction

One of the main objectives in financial engineering is to build fast yet accurate methodology for pricing financial derivatives. Although the Black-Scholes model [2] has been popular among many financial industries, such a standard pricing model may not describe some important behaviors of skew and smile effects in empirical options market. To incorporate these behaviors, option pricing under Lévy process has been considered an important research direction in financial engineering. For computing European type options under Lévy processes, the Fast Fourier Transform (FFT) method introduced by Carr and Madan [4] may be used, where analytic expression is available according to the Lévy-Khinchin representation [14]. However, the application to exotic derivatives with general Lévy processes may still be difficult, in spite of the fact that a large volume of exotic derivatives (such as barrier options) are actively traded in current options market. A typical approach under the Lévy case involves to solve partial integro-differential equations (PIDEs) or the Monte-Carlo simulations, although those methods are generally time-consuming.

In this paper, we apply a new approach by applying the so-called homotopy analysis method to option pricing under Lévy processes. The HAM is a general framework initially proposed by Ortega and Rheinboldt [13], and has widely been applied to solving non-linear differential equations (e.g., [1] and [11]). In the field of financial engineering, it was first applied in [16] for American options under the standard Black-Scholes assumptions. Then, Zhao and Wong [15] extended to the case where the volatility of the underlying is a function of time and showed a faster convergence of option prices by using the Padé approximation.

But to best of our knowledge, no one has applied for general Lévy processes yet, where our approach may involve an extension to Barrier options. Moreover, we demonstrate that a convenient series expansion formula may be derived under the Variance Gamma (VG) model [12] and that the individual terms of the expansion are represented analytically.

This paper is organized as follows. In section 2 we introduce the underlying stock model as a one dimensional Lévy processes. In section 3, we demosntrate the HAM. In section 4, we apply the HAM to European option under VG model. In section 5, we extend our method to barrier options. Section 6 provides numerical examples. Section 7 offers some concluding remark.

2 Underlying stock price dynamics

We assume that the underlying stock price process as $S_t = S_0 e^{X(t)}$ under the risk-neutral measure, where $X(t)$ is a one-dimensional Lévy process expressed by the following Lévy-Ito decomposition [14]:

$$X(t) = \mu t + \sigma W(t) + \int_0^t \int_{|y|\geq 1} y\, h(dy \times ds) + \lim_{\epsilon\to 0} \int_0^t \int_{\epsilon \leq |y| < 1} y\,[h(dy \times ds) - \nu(dy \times ds)],$$

where $W(t)$ is the standard Brownian motion, $h(dyds)$ is the Poisson random measure and $\nu(dyds)$ is its compensator. Denoting the value of an option at time t as $v_t = E_t^Q[e^{-r(T-t)}\varphi]$, where φ is pay-off at maturity one can show that $e^{r(T-t)}v_t$ satisfies PIDE [5]:

$$\partial_\tau v + Lv = 0, Lv := \mu \frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + \int_{\mathbb{R}} [v(x+y) - v(x) - y\frac{\partial v}{\partial x} 1_{|y|<1}]\, \nu(dy). \quad (1)$$

Let $\mathbf{F}$ be the Fourier transform operator s.t.

$$\mathbf{F}v = \int_{-\infty}^{+\infty} e^{-i\omega x} v(x)\, dx.$$

Then by applying the Fourier transform for (1) we have the following ordinal differential equitation (ODE)

$$\partial_t \hat{v} + \Phi(\omega) \cdot \hat{v} = 0, \tag{2}$$

where

$$\hat{v}(\omega, t) := \mathbf{F}v, \quad \hat{\varphi}(\omega) := \mathbf{F}\varphi,$$

and $\Phi(\omega)$ is the characteristic exponent of the Lévy process. Note that an analytical expression of $\Phi(\omega)$ is available for most Lévy processes.

In this paper, we apply the homotopy analysis method (HAM) based on the Fourier transform formulations of PIDE (1). To explain our approach, we next introduce the HAM in the following section.

3 Homotopy Analysis Method

The HAM is a general framework initially proposed by Ortega and Rheinboldt [13], and has widely been applied to solving non-linear differential equations (e.g., [1] and [11]). The basic idea comes from the Topology, and the objective is to built a deformation process such that a simple initial function chosen at parameter $p = 0$ gradually approaches to an unknown solution (that we want to obtain) at $p = 1$. The final solution derived at $p = 1$ is given as an infinite series of functions which can be calculated analytically.

Suppose that we would like to find a function V such that

$$A(V(x,t)) = 0 \tag{3}$$

with a given differential operator A. To solve this equation, let A_0 be another differential operator and $\bar{V}_0(x,t)$ be a function. Then consider a function $\bar{V}(x,t,p)$ satisfying the following differential systems,

$$(1-p)[A_0(\bar{V}(x,t,p)) - A_0(\bar{V}_0(x,t))] = -p \cdot A(\bar{V}(x,t,p)). \tag{4}$$

Plugging $p = 0$ gives

$$A_0(\bar{V}(x,t,0)) - A_0(\bar{V}_0(x,t)) = 0$$

and it is obvious that

$$\bar{V}(x,t,0) = \bar{V}_0(x,t)$$

holds. On the other hand, plugging $p = 1$ gives

$$A(\bar{V}(x,t,1)) = 0$$

which provides the solution to the original differential equation (3) of , i.e., $V(x,t) = \bar{V}(x,t,1)$. Next we consider the following Taylor's expansion of $V(x,t) = \bar{V}(x,t,1)$ with respect to p as

$$\bar{V}(x,t,p) = V_0(x,t) + V_1(x,t)p + \frac{1}{2}V_2(x,t)p^2 + \cdots = \sum_{n=0}^{\infty} \frac{V_n(x,t)}{n!} p^n,$$

where $V_n(x,t) = \frac{\partial^n}{\partial p^n}\bar{V}(x,t,p)|_{p=0}$. We want to compute each term of the expansion. Differentiation of both sides of equation (4) with respect to p yields

$$-[A_0(\bar{V}(x,t,p)) - A_0(\bar{V}_0(x,t))] + (1-p)\frac{\partial}{\partial p}A_0(\bar{V}(x,t,p)) + A(\bar{V}(x,t,p)) + p \cdot \frac{\partial}{\partial p}A(\bar{V}(x,t,p)) = 0.$$

Plugging $p = 0$ gives

$$A_0(V_1(x,t)) + [A(V_0(x,t)) - A_0(V_0(x,t))] = 0. \tag{5}$$

In (5), we typically choose initial conditions such that $A_0(\bar{V}_0(x,t)) = 0$ is satisfied. Similarly, it can be confirmed that

$$A_0(V_n(x,t)) + n \cdot [A(V_{n-1}(x,t)) - A_0(V_{n-1}(x,t))] = 0$$

holds for the general case of $n \geq 1$. We see that each Taylor coefficient, $V_n(x,t),\ n \geq 1$, is a solution of a differential equation that may be solved recursively for given $V_{n-1}(x,t)$. Notice that we can choose any initial operator A_0 and initial function V_0, although a poor choice may lead to a slower convergence of the Taylor expansion.

4 Applying the HAM for European options

Now, we apply the HAM to derive semi-analytical formula of a European call, where the underlying stock is assumed to follow a VG model [12]. Note that our methodology may be easily applied to other Lévy models, although we omit to explain the detail for brevity.

Let

$$\Phi_{VG}(\omega) := \gamma i\omega - \frac{1}{\kappa}\ln\left(1 - i\mu\kappa\omega + \frac{\sigma^2\kappa\omega^2}{2}\right)$$

and $\hat{\mathcal{A}} := \partial_t + \Phi_{VG}(\omega)$. We will solve the following differential equation

$$\hat{\mathcal{A}}(\omega) \cdot \hat{v}(\omega,t) = 0 \tag{6}$$

under the VG model with the terminal condition $\hat{v}(\omega, t = T) = \hat{\varphi}(\omega)$, where

$$\hat{\varphi}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x}[S_0e^x - K]^+\,dx.$$

Because the payoff function $[S_0e^x - K]^+$ is not L^1-integrable, we use the idea explained in [4], where we first specify α such that the Fourier transform of $e^{\alpha x}v(x,t)$ exists and modify $e^{-\alpha x}$ afterward so that

$$v(x,t) = e^{-\alpha x}\mathbf{F}^{-1}[\mathbf{F}[e^{\alpha x}v(x,t)]]$$

holds. Noting that, in the Fourier-domain, the operation of $v(x,t) \to e^{\alpha x}v(x,t)$ corresponds to $\hat{v}(\omega,t) \to \hat{v}(\omega+\alpha i,t)$, (6) can be rewritten as

$$\hat{\mathcal{A}}(\omega+\alpha i) \cdot \hat{v}_S(\omega,t) = 0, \quad \hat{v}_S(\omega,T) = \hat{\varphi}(\omega+\alpha i),$$

where $\hat{v}_{Sn}(\omega,t) = \hat{v}_n(\omega+\alpha i,t)$ and $\hat{\varphi}(\omega+\alpha i) = K\left(\frac{K}{S_0}\right)^\alpha \frac{e^{-i\omega\ln\frac{K}{S_0}}}{(-i\omega+\alpha+1)(-i\omega+\alpha)}$ with $\alpha < -1$. Notice that the solution is just

$$\hat{v}_S(\omega,t) = e^{-\Phi(\omega+\alpha i)(T-t)}\hat{\varphi}(\omega+\alpha i)$$

and

$$v(x,t) = \frac{e^{-r(T-t)}e^{-\alpha x}}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}e^{\Phi_{VG}(\omega+\alpha i)(T-t)}\hat{\varphi}(\omega+\alpha i)\,d\omega. \tag{7}$$

We use the HAM to derive the approximation formula of (7). Let $\hat{\mathcal{A}}_0$ and $\hat{v}_{S0}(\omega,t)$ be a given linear operator and a initial function satisfying

$$\hat{\mathcal{A}}_0(\omega+\alpha i) \cdot \hat{v}_{S0}(\omega,t) = 0, \quad \hat{v}_{S0}(\omega,T) = \hat{\varphi}(\omega+\alpha i).$$

Then we construct the following differential system for a parameter $p \in [0,\ 1]$:

$$\begin{gathered}(1-p)[\hat{\mathcal{A}}_0(\omega+\alpha i)\cdot\bar{V}_S(\omega,t,p) - \hat{\mathcal{A}}_0(\omega+\alpha i)\cdot\bar{V}_{S0}(\omega,t)] = -p\cdot\hat{\mathcal{A}}_0(\omega+\alpha i)\cdot\bar{V}_S(\omega,t,p)\\ \bar{V}_S(\omega,T,p) = \hat{\varphi}(\omega+\alpha i)\end{gathered}$$

where $\bar{V}_S(\omega, t, p)$ is a function satisfying

$$\bar{V}_S(\omega, t, 0) = \hat{v}_{S0}(\omega, t), \quad \bar{V}_S(\omega, t, 1) = \hat{v}_S(\omega, t)$$

and

$$\bar{v}_S(x, t, p) := \mathbf{F}^{-1}\bar{V}_S(\omega, t, p).$$

By applying the HAM, we get the following recursive formula for $n \geq 1$:

$$\begin{aligned} \hat{\mathcal{A}}_0(\omega + \alpha i) \cdot \hat{v}_{Sn}(\omega, t) &= n \cdot [\hat{\mathcal{A}}(\omega + \alpha i) - \hat{\mathcal{A}}_0(\omega + \alpha i)] \cdot \hat{v}_{Sn-1}(\omega, t) \\ \hat{v}_{Sn}(\omega, T) &= 0 \end{aligned} \tag{8}$$

where

$$\hat{v}_n(\omega, t) := \left.\frac{\partial^n \bar{V}}{\partial p^n}\right|_{p=0}, \quad v_n(x, t) := \left.\frac{\partial^n \bar{v}}{\partial p^n}\right|_{p=0}.$$

To solve the above system of equations, we need to choose $\hat{\mathcal{A}}_0(\omega + \alpha i)$ so that we can derive analytical solutions for each n. Since the Taylor expansion of

$$\begin{aligned} \ln\left(1 - i\mu\kappa(\omega + \alpha i) + \frac{\sigma^2\kappa(\omega + \alpha i)^2}{2}\right) &= \ln\left[\left(1 + \mu\kappa\alpha - \frac{\sigma^2\kappa\alpha^2}{2}\right) + [\sigma^2\kappa\alpha - \mu\kappa]\omega i + \frac{\sigma^2\kappa\omega^2}{2}\right] \\ &:= \ln(z_0 + z_1\omega i + z_2\omega^2) \end{aligned}$$

at $\omega = 0$ is given by $\ln z_0 + \frac{z_1}{z_0}\omega i + \frac{1}{2}\left(\frac{2z_0z_2+z_1^2}{z_0^2}\right)\omega^2 + \cdots$, Φ_{VG} may be rewritten as

$$\begin{aligned} \Phi_{VG}(\omega + \alpha i) &= \gamma\,(\omega + \alpha i)\,i - \frac{1}{\kappa}\left[\ln z_0 + \frac{z_1}{z_0}\omega i + \frac{1}{2}\left(\frac{2z_0z_2 + z_1^2}{z_0^2}\right)\omega^2 + \cdots\right] \\ &= -\gamma\alpha - \frac{1}{\kappa}\ln z_0 + \left(\gamma - \frac{z_1}{\kappa z_0}\right)\omega i - \frac{1}{2\kappa}\left(\frac{2z_0z_2 + {z_1}^2}{{z_0}^2}\right)\omega^2 + \cdots \\ &:= Z_0 + Z_1\omega i + Z_2\omega^2 + \cdots. \end{aligned}$$

Noting that the analytical formula for the price of European call under the Black-Scholes model is available by solving the corresponding diffusion equation, we choose a diffusion approximation of $\Phi_{VG}(\omega + \alpha i)$ as

$$\hat{\mathcal{A}}_0(\omega + \alpha i) = \partial_t + (Z_0 + Z_1\omega i + Z_2\omega^2).$$

Furthermore because

$$Z_0 + Z_1\omega i + Z_2\omega^2 = Z_2\left(\omega + \frac{Z_1}{2Z_2}i\right)^2 + \frac{{Z_1}^2}{4Z_2} + Z_0 := A_1(\omega + A_2 i)^2 + A_3,$$

we solve the following differential system by defining $\hat{v}^c_{Sn}(\omega, t') := e^{\mathrm{A}_3 t'}\hat{v}_{Sn}(\omega - \mathrm{A}_2 i, t')$ and $t' = T - t$,

$$[\partial_{t'} - \mathrm{A}_1\omega^2] \cdot \hat{v}^c_{S0}(\omega, t') = 0, \quad \hat{v}^c_{S0}(\omega, 0) = \hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i).$$

The solution is obtained as

$$\hat{v}_{S0}(\omega, t) = e^{\mathrm{A}_1\omega^2 t'}\hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i),$$

where the inverse Fourier transform provides

$$v_0(x, t) = e^{\mathrm{A}_2 x}e^{\mathrm{A}_3 t}\frac{e^{-\alpha x}}{2\pi}\int_{-\infty}^{+\infty} e^{A_1\omega^2 \cdot t'}\hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)\,d\omega. \tag{9}$$

Similarly for $n \geq 1$,

$$[\partial_{t'} - \mathrm{A}_1\omega^2] \cdot \hat{v}^c_{Sn}(\omega, t') = n \cdot [G^c(\omega - \mathrm{A}_2 i)] \cdot \hat{v}^c_{Sn-1}(\omega, t'), \hat{v}^c_{Sn}(\omega, 0) = 0,$$

with $G^c(\omega - \mathrm{A}_2 i) := \Phi_{VG}(\omega + \alpha i - \mathrm{A}_2 i) - \mathrm{A}_1\omega^2$. The solution is obtained as

$$\hat{v}^c_{Sn}(\omega, t') = n \cdot \int_0^{t'} e^{\mathrm{A}_1\omega^2(t'-\tau)} G^c(\omega - \mathrm{A}_2 i)\hat{v}^c_{Sn-1}(\omega, \tau) d\tau. \tag{10}$$

By solving the above equation recursively, we have

$$\hat{v}^c_{Sn}(\omega, t') = n \cdot t'^n e^{\mathrm{A}_1\omega^2(t'-\tau)}[G^c(\omega - \mathrm{A}_2 i)]^n \hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)$$

for $n \geq 2$. Finally the value of European call with maturity T at $t = 0$ is attained as by $\sum_{n=0}^{\infty} e^{-rT} \frac{v_n(x,0)}{n!}$, where

$$v_n(x, 0) = n \cdot \frac{e^{\mathrm{A}_2 x + \mathrm{A}_3 T - \alpha x}}{2\pi} \int_{-\infty}^{+\infty} e^{\mathrm{A}_1\omega^2 T} \cdot T^n G^c(\omega - \mathrm{A}_2 i)^n \hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)\, d\omega \tag{11}$$

and $x = \ln(S/S_0)$.

5 Applying the HAM for Barrier Options

In this section we apply the HAM for barrier option under VG model. For simplicity we only consider the down-and-out call whose barrier price B is less than the strike price K; this method can be easily extended to other type of barrier options.
Adding the corresponding boundary condition to European case in the previous section, the differential system to solve is

$$[\partial_{t'} - \mathrm{A}_1\omega^2] \cdot \hat{v}^c_{S0}(\omega, t') = 0, \hat{v}^c_{S0}(\omega, 0) = \hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i), v_{S0}(x \leq b, t') = 0,$$

where $b := ln(\frac{B}{S_0})$ and for $n \geq 1$,

$$[\partial_{t'} - \mathrm{A}_1\omega^2] \cdot \hat{v}^c_{Sn}(\omega, t') = n \cdot G^c(\omega - \mathrm{A}_2 i) \cdot \hat{v}^c_{Sn-1}(\omega, t'), \hat{v}^c_{Sn}(\omega, 0) = 0, v^c_{Sn}(x \leq b, t') = 0.$$

Now we solve the system of equations for each n. For the case $n = 0$, with the reflection principle the solution is given by $v^c_{S0}(x, t') = v^c_{E0}(x, t') - v^c_{E0}(2b - x, t')$ for $x \geq b$, where

$$\begin{aligned} v^c_{E0}(x, t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} e^{\mathrm{A}_1\omega^2 t'} \hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)\, d\omega, & (12) \\ v^c_{E0}(2b - x, t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(-x)} e^{2i\omega b} e^{\mathrm{A}_1\omega^2 t'} \hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)\, d\omega. & (13) \end{aligned}$$

$v^c_{E0}(x, t')$ is the solution of the same equation with no boundary condition. Then we get

$$\begin{aligned} v_0(x, t') = {} & e^{\mathrm{A}_2 x} e^{\mathrm{A}_3 t} \frac{e^{-\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} e^{\mathrm{A}_1\omega^2 t'} \hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)\, d\omega \\ & - e^{\mathrm{A}_2 x} e^{\mathrm{A}_3 t} \frac{e^{-\alpha(-x)}}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(-x)} e^{2i\omega b} e^{\mathrm{A}_1\omega^2 t'} \hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)\, d\omega. \end{aligned}$$

Next for the case $n = 1$, we solve the following

$$\begin{gathered} [\partial_{t'} - \mathrm{A}_1\omega^2]\hat{v}^c_{S1}(\omega, t') = G^c(\omega - \mathrm{A}_2 i) \cdot \mathbf{F}[1_{(b,\infty)} \cdot v^c_{S0}(x, t')], \qquad (14) \\ \hat{v}^c_{S1}(\omega, 0) = 0, v^c_{S1}(x \leq b, t') = 0. \end{gathered}$$

The equation (14) has a form of

$$[\partial_{t'} - A_1\omega^2]\hat{f}(\omega,t') = \hat{h}(\omega,t'), \hat{f}(\omega,0) = 0, f(x \le b, t') = 0 \tag{15}$$

and the solution is given by the Duhamel Principle (see [8] for example) as follows

$$\hat{f}(\omega,t') = \int_0^{t'} \hat{\psi}(\omega,t';\tau)\,d\tau, \tag{16}$$

where $\hat{\psi}(\omega,t';\tau)$ satisfies

$$\begin{gathered}[\partial_{t'} - A_1\omega^2]\hat{\psi}(\omega,t';\tau) = 0, \\ \hat{\psi}(\omega,\tau;\tau) = \hat{h}(\omega,\tau), \psi(x \le b, t', \tau) = 0\end{gathered} \tag{17}$$

for $t' \ge \tau$. Therefore

$$\begin{aligned}\hat{v}^c_{S1}(\omega,t') &= G^c(\omega - \mathrm{A}_2 i)\int_0^{t'} e^{\mathrm{A}_1\omega^2(t'-\tau)}\mathbf{F}[1_{(b,\infty)}\cdot v^c_{S0}(x,\tau)]\,d\tau \\ &\quad - e^{-2i\omega b}G^c(-\omega - \mathrm{A}_2 i)\int_0^{t'} e^{\mathrm{A}_1\omega^2(t'-\tau)}\mathbf{F}[1_{(b,\infty)}\cdot v^c_{S0}(x,\tau)]\,d\tau\end{aligned}$$

and finally we get

$$\begin{aligned}v_1(x,t') &= e^{\mathrm{A}_2 x}e^{\mathrm{A}_3 t}\frac{e^{-\alpha x}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}G^c(\omega - \mathrm{A}_2 i)e^{\mathrm{A}_1\omega^2 t'}[\int_0^{t'} e^{-\mathrm{A}_1\omega^2\tau}\mathbf{F}[1_{(b,\infty)}\cdot v^c_{S0}(x,\tau)]\,d\tau]\,d\omega \\ &\quad - e^{\mathrm{A}_2 x}e^{\mathrm{A}_3 t}\frac{e^{-\alpha(-x)}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(-x)}G^c(\omega - \mathrm{A}_2 i)e^{\mathrm{A}_1\omega^2 t'}[\int_0^{t'} e^{-\mathrm{A}_1\omega^2\tau}\mathbf{F}[1_{(b,\infty)}\cdot v^c_{S0}(x,\tau)]\,d\tau]\,d\omega.\end{aligned}$$

Repeating the same argument, the value of down-and-out call with maturity T at $t = 0$ is approximated as $\sum_{n=0}^{\infty} e^{-rT}\frac{v_n(x,0)}{n!}$, where

$$\begin{aligned}v_0(x,0) &= e^{\mathrm{A}_2 x + \mathrm{A}_3 T}[\frac{e^{-\alpha x}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}e^{\mathrm{A}_1\omega^2 T}\hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)\,d\omega \\ &\quad - \frac{e^{\alpha x}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(-x)}e^{2i\omega b}e^{\mathrm{A}_1\omega^2 T}\hat{\varphi}(\omega + \alpha i - \mathrm{A}_2 i)\,d\omega],\end{aligned} \tag{18}$$

$$\begin{aligned}v_n(x,0) &= n\cdot e^{\mathrm{A}_2 x + \mathrm{A}_3 T}[\frac{e^{-\alpha x}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}G^c(\omega - \mathrm{A}_2 i)e^{\mathrm{A}_1\omega^2 T}\int_0^T e^{-\mathrm{A}_1\omega^2\tau}\mathbf{F}[1_{(b,\infty)}\cdot v^c_{Sn-1}(x,\tau)]\,d\tau\,d\omega \\ &\quad - \frac{e^{\alpha x}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(-x)}G^c(\omega - \mathrm{A}_2 i)e^{\mathrm{A}_1\omega^2 T}\int_0^T e^{-\mathrm{A}_1\omega^2\tau}\mathbf{F}[1_{(b,\infty)}\cdot v^c_{Sn-1}(x,\tau)]\,d\tau\,d\omega].\end{aligned} \tag{19}$$

In addition, the following formula (see [9] for derivation) can be applied to express $\mathbf{F}[1_{(b,\infty)}\cdot v^c_{Sn-1}(x,\tau)]$ analytically; for example given $f(x)$,

$$\mathbf{F}[1_{(b,\infty)}\cdot f(x)] = \frac{1}{2}\hat{f}(\omega) - \frac{i}{2}e^{-i\omega b}H(e^{i\eta b}\hat{f}(\eta))(\omega) \tag{20}$$

where the Hilbert transform $H(\hat{f}(\eta))(\omega)$ is defined as

$$H(\hat{f}(\eta))(\omega) := \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{\hat{f}(\eta)}{\omega - \eta}\,d\eta.$$

But in this case we need to compute the Hilbert transform numerically. Therefore for simplicity, we compute the Fourier transform $\mathbf{F}[1_{(b,\infty)}\cdot v^c_{Sn-1}(x,\tau)]$ directly in the following numerical experiments.

Remark 1 *Furthermore we can apply the approximated expansion in case of the down-and-out call to compute the credit default swaps (CDS) under the Lévy model. If we assume S_t as a firm's value, the fair spread C of Credit Default Swaps (CDS) under structural model is given in [3] as*

$$C = (1 - R)\left(\frac{BDOB(B,T)}{\int_0^T BDOB(B,t)\,dt} - r\right), \tag{21}$$

where $BDOB(B,T)$ is price of binary down-and-out option with maturity T, r is the risk-free rate, R is the recovery rate and B is the barrier (the default is assumed to occur if $S_t \leq B$). Cariboni and Schoutens [3] applied a finite-difference method to to compute each term in (21). On the other hand, the analytical approximation is directly available by integrating our analytical approximation with respect to T.

Remark 2 *Theoretically barrier and lookback options can be expressed in terms of Wiener-Hopf factors under Lévy models. However, as pointed out in [5], the factors themselves are not available explicitly in most case and the pricing algorithm may require the inversions of the Laplace and the Fourier transforms, which may not be so tractable. Jeannin and Pistorius [9] derived an analytical formula of barrier option prices in terms of the Laplace transform as an example of this method, although the densities must be approximated by hyper-exponential Lévy densities and the parameters have to be computed via root mean squares minimizations.*

6 Numerical Examples

This section gives numerical examples to examine the efficiency of pricing options by the HAM. We compare our formula (11) with a reference price computed by analytical expressions of Carr and Madan [4] for a European call. We use the following parameter set given in [10] : $\sigma = 0.19071$, $\kappa = 0.49083$, $\mu = -0.28113$ and other parameters are set as $S = 100$, $K = 100$, $r = 0.0549$, $q = 0.011$, $T = 0.1$, $\alpha = -12.8$. Figure 1 is the numerical result. The horizontal axis in the top figure represents the number of terms to approximate the infinite Taylor expansion in HAM and the vertical line represents option prices. The horizontal axis in the bottom figure represents the number of terms to approximate the infinite Taylor expansion in HAM and the vertical axis represents the difference between option price computed by HAM and the reference price. Both figures indicate that only the first five terms seem to give sufficient approximation.

The second case is down-and-out call price and we compare our formula with a reference price computed by the finite difference method [7] where the price at a grid point $S = 100.01116$ is chosen for comparison. For simplicity, the integral $\int_0^T e^{-\mathrm{A}_1\omega^2\tau}\mathbf{F}[1_{(b,\infty)} \cdot v^c_{Sn-1}(x,\tau)]\,d\tau$ is approximated by the following simple trapezoid rule:

$$\frac{T}{2}[\mathbf{F}[1_{(b,\infty)} \cdot v^c_{Sn-1}(x,0)] + e^{-\mathrm{A}_1\omega^2 T}\mathbf{F}[1_{(b,\infty)} \cdot v^c_{Sn-1}(x,T)]].$$

We use the same parameter set as that for the above example except $\alpha = -12.87$ and $B = 85$ for the barrier. Figure 2 shows an numerical result. The horizontal axis in the top figure represents the number of terms to approximate the infinite Taylor expansion in HAM and the vertical axis represents option prices. The horizontal axis in the bottom figure represents the number of terms to approximate the infinite Taylor expansion in HAM and the vertical axis represents the difference between option price computed by HAM and the reference price. Similar to the case of European call option, both figures indicate that only the first five terms seem to provide a sufficient approximation. Therefore we conclude that Figure 1 and 2 demonstrate the efficiency of HAM in pricing these options.

7 Conclusion

In this paper, we present general methodology of applying the homotopy analysis method to European and barrier options under Lévy processes, and derive the sum of infinite series whose individual term may be calculated analytically. As an example of Lévy processes whose characteristic functions are available analytically, we used the Variance Gamma model. Our numerical examples demonstrates that HAM gives a sufficient approximation of original option price with the low orders of Taylor coefficients. Therefore HAM is efficient and applicable in practice. Moreover each term in the infinite series is expressed in terms of Fourier transform so the method of Fast Fourier transform can be used to achieve faster computation of each term. Once the price is computed, we can calculate option sensitivities such as delta and gamma easily as well. The method is easy to understand and can be applied to other type of options.

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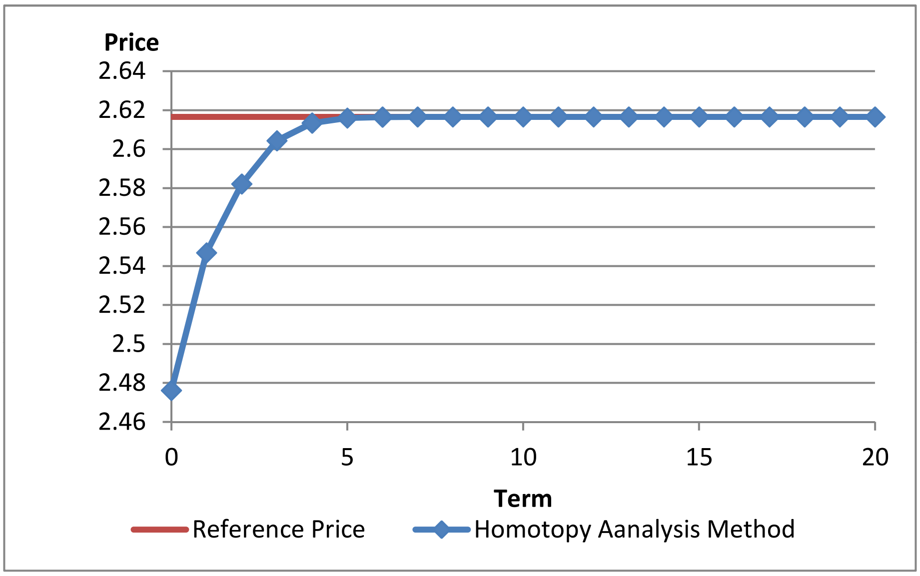


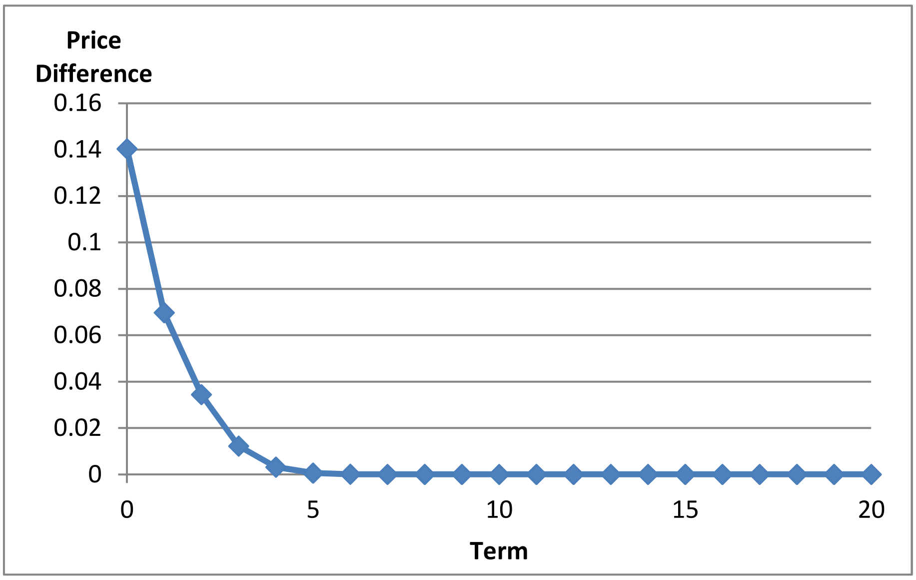
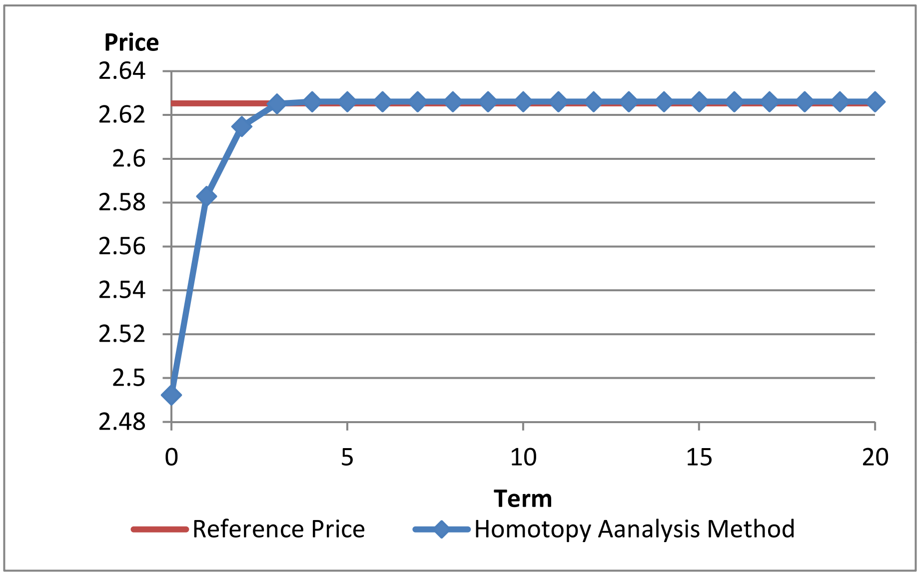


Figure 1: European call under VG.



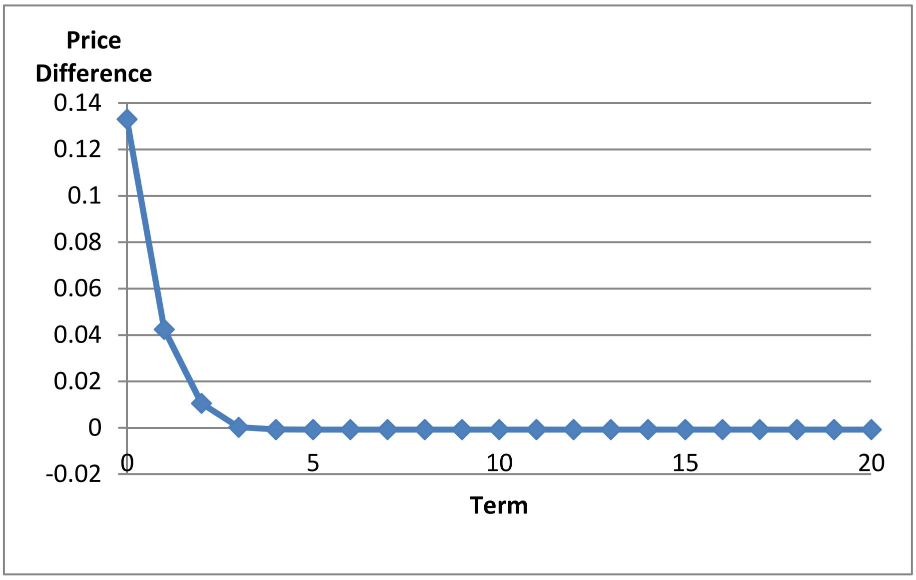


Figure 2: Down-and-out call under VG.