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String field theory solution corresponding to constant background magnetic field

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Following the method recently proposed by Erler and Maccaferri, we construct solutions to the equation of motion of Witten's cubic string field theory, which describe a constant magnetic field background. We study the boundary condition changing operators relevant to such a background, and calculate their operator product expansions. We obtain solutions whose classical action coincide with the Born–Infeld action.

Subject Index B24, B28

1. Introduction

Since the discovery of the tachyon vacuum solution [1], the classical solutions of cubic string field theory [2] have been studied quite actively [3–21]. In a recent paper, Erler and Maccaferri [22] have given a way to realize any boundary conformal field theory (BCFT) as a solution of the equation of motion.

With the method of Erler and Maccaferri, it is now possible to study various open string backgrounds from the point of view of string field theory. It has been known for many years that the open string theory in a constant magnetic field background [23] has various interesting features. It corresponds to an exactly solvable BCFT, and the action is given by the Born–Infeld action. Noncommutative geometry appears in the open string theory around the background [24,25].

What we would like to do in this paper is to construct the Erler–Maccaferri solution corresponding to the constant magnetic background. We start from the string field theory associated with a D-brane with vanishing gauge field background and would like to realize the constant magnetic background as a solution to the equation of motion. In order to do so, we study the boundary condition changing (BCC) operators which change the open string boundary condition from that of one background to another.[1]

The organization of this paper is as follows. In Sect. 2, we briefly review the Erler–Maccaferri solutions. In Sect. 3, we study the BCC operators necessary for the construction of our solution. We investigate the open string states corresponding to these operators, calculate their correlation functions, and finally obtain their operator product expansions. In Sect. 4, we construct the solutions

[1] In a different context, it was shown that the Dirac–Born–Infeld factor appears from the tachyon vacuum solution in a constant $B_{\mu\nu}$ background [26]. There is also an earlier related work [27], which seems rather formal.

of the equation of motion of string field theory using the BCC operators studied in Sect. 3. Section 5 is devoted to conclusions and discussions. In the appendices, details of calculations concerning the BCC operators are exhibited.

2. Erler–Maccaferri solutions

Let us first explain how to construct the Erler–Maccaferri solutions to the equation of motion

$$Q\Psi + \Psi^2 = 0 \tag{2.1}$$

of Witten's cubic string field theory. Here let us assume that the solution $\Psi = 0$ corresponds to a boundary conformal field theory BCFT_0. We would like to discuss a solution associated with another boundary conformal field theory BCFT_*. The solution is constructed from the string fields

$$K, B, c, \sigma, \bar{\sigma}.$$

The fields K, B, c are the ones which play crucial roles in the construction of classical solutions of cubic string field theory [1,28] and satisfy the so-called KBc algebra:

$$\begin{aligned} B^2 = c^2 = 0, [K, B] = 0, Bc + cB = 1, \\ QK = 0,\ QB = K,\ Qc = c\partial c. \end{aligned} \tag{2.2}$$

Here, the product of the string fields is the star product and $Q(\cdot)$ denotes the BRST variation. The string fields $\sigma, \bar{\sigma}$ are the ones which can be expressed by

$$\begin{aligned} \sigma &= \sigma(1)I, \\ \bar{\sigma} &= \bar{\sigma}(1)I, \end{aligned}$$

where I is the identity string field and $\sigma(s), \bar{\sigma}(s)$ are the BCC operators such that $\sigma(s)$ changes the open string boundary condition from the one corresponding to BCFT_* to the one corresponding to BCFT_0, and $\bar{\sigma}(s)$ changes in reverse, as indicated in Fig. 1. We assume that $\sigma(s)$, $\bar{\sigma}(s)$ are matter primary fields of weight 0 and satisfy the operator product expansion (OPE)

$$\begin{aligned} \bar{\sigma}(s)\sigma(0) &\sim 1, \\ \sigma(s)\bar{\sigma}(0) &\sim \frac{g_*}{g_0} \end{aligned} \tag{2.3}$$

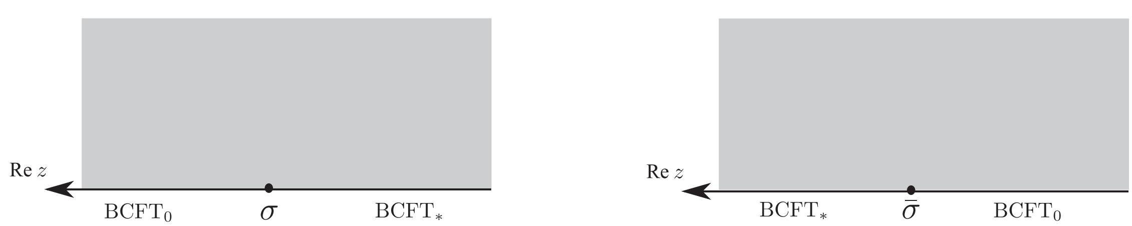

Fig. 1. $\sigma(z)$, $\bar{\sigma}(z)$. The boundary on the left of σ on the real axis corresponds to the boundary condition of BCFT_0 and the boundary on the right corresponds to that of BCFT_*, and vice versa for $\bar{\sigma}$. The order of the star product and that in the figure coincide if we take the direction of the real axis as above [4].

for $s > 0$, where g_* is the disk partition function of BCFT$_*$ and g_0 is that of BCFT$_0$. We have the following algebraic relations involving σ, $\bar{\sigma}$:

$$\begin{aligned} &[\sigma, c] = 0, [\sigma, \partial c] = 0, [\sigma, B] = 0, \\ &[\bar{\sigma}, c] = 0, [\bar{\sigma}, \partial c] = 0, [\bar{\sigma}, B] = 0, \\ &Q\sigma = c\partial\sigma, Q\bar{\sigma} = c\partial\bar{\sigma}, \\ &\bar{\sigma}\sigma = 1, \ \sigma\bar{\sigma} = \frac{g_*}{g_0}. \end{aligned} \tag{2.4}$$

With all these string fields, it is possible to construct a solution Ψ given by

$$\Psi = \Psi_{\text{tv}} - \Sigma \Psi_{\text{tv}} \bar{\Sigma}, \tag{2.5}$$

where Ψ_{tv} is the Erler–Schnabl solution [4] for the tachyon vacuum,

$$\Psi_{\text{tv}} = \frac{1}{\sqrt{1+K}} c(1+K)Bc \frac{1}{\sqrt{1+K}},$$

and Σ, $\bar{\Sigma}$ are the string fields

$$\begin{aligned} \Sigma &= Q_{\Psi_{\text{tv}}} \left(\frac{1}{\sqrt{1+K}} B\sigma \frac{1}{\sqrt{1+K}} \right), \\ \bar{\Sigma} &= Q_{\Psi_{\text{tv}}} \left(\frac{1}{\sqrt{1+K}} B\bar{\sigma} \frac{1}{\sqrt{1+K}} \right) \end{aligned} \tag{2.6}$$

with

$$Q_{\Psi_{\text{tv}}} = Q + [\Psi_{\text{tv}}, \ ].$$

Using the formula

$$\bar{\Sigma}\Sigma = 1,$$

which can be derived from Eqs. (2.2), (2.4), and (2.6), it is possible to calculate the energy and the gauge-invariant observable [29–33] as

$$\begin{aligned} E &= \frac{1}{\text{Vol}(X^0)} \left(-\frac{g_0}{2\pi^2} + \frac{g_*}{2\pi^2} \right), \\ \text{Tr}_{\mathcal{V}}[\Psi] &= \frac{1}{4\pi i} \left(\langle \mathcal{V} | c_0^- | B_0 \rangle - \langle \mathcal{V} | c_0^- | B_* \rangle \right). \end{aligned}$$

These results imply that the solution corresponds to the boundary conformal field theory BCFT$_*$.

Therefore, in order to construct a solution corresponding to BCFT$_*$, we need to construct primary fields $\sigma(s)$, $\bar{\sigma}(s)$ of weight 0 satisfying the OPE (2.3). Erler and Maccaferri considered the case of time-independent solutions and $\sigma(s)$, $\bar{\sigma}(s)$ are expressed as

$$\sigma(s) = \sigma_* e^{i\sqrt{h}X^0}(s), \ \bar{\sigma}(s) = \bar{\sigma}_* e^{-i\sqrt{h}X^0}(s).$$

Here, $\sigma_*(s)$, $\bar{\sigma}_*(s)$ are the BCC operators of weight h which satisfy

$$\begin{aligned}\bar{\sigma}_*(s)\sigma_*(0) &\sim s^{-2h},\\ \sigma_*(s)\bar{\sigma}_*(0) &\sim \frac{g_*}{g_0}s^{-2h}\end{aligned} \tag{2.7}$$

for $s > 0$, and they act as the identity operator in the time direction.

3. BCC operators for a constant background magnetic field

We would like to construct a solution to the equation of motion (2.1) which corresponds to a constant magnetic field background. Here, we consider the bosonic string theory in 26-dimensional Minkowski space-time and take BCFT_0 to be the usual one for a Dp-brane, i.e. Neumann boundary conditions for X^μ ($\mu = 0, \ldots, p$) and Dirichlet boundary conditions for X^I ($I = p+1, \ldots, 25$). We take BCFT_* to be the one with a constant magnetic field, namely

$$F_{\mu\nu} = \text{constant}, \tag{3.1}$$

with $F_{0\mu} = 0$. Our goal is to study the BCC operators in this setup and calculate the OPEs, which are necessary for the construction of the Erler–Maccaferri solution.

3.1. *Canonical quantization*

The BCC operators correspond to states of open strings, with one end satisfying the BCFT_0 boundary condition and the other end satisfying the BCFT_* boundary condition. Therefore, it is possible to deduce properties of the BCC operators by studying the open string states in such sectors. Let us consider the worldsheet theory of the open string which is given by the conformal field theory (CFT) on the strip where the Neumann boundary conditions are imposed at $\sigma = 0$ and the other boundary at $\sigma = \pi$ is coupled to the electromagnetic fields as

$$\int A_\mu(X)\,\dot{X}^\mu|_{\sigma=\pi}\,dt. \tag{3.2}$$

In the case at hand, we have a constant magnetic field (3.1) and we take

$$A_\mu = -\frac{1}{2}F_{\mu\nu}X^\nu. \tag{3.3}$$

Since $F_{\mu\nu}$ is a real antisymmetric tensor with $F_{0\mu} = 0$, we can make a rotation to put it into block diagonal form:

$$F_{\mu\nu} = \begin{pmatrix} 0 & & & & & \\ & 0 & F_{12} & & & \\ & -F_{12} & 0 & & & \\ & & & 0 & F_{34} & \\ & & & -F_{34} & 0 & \\ & & & & & \ddots \end{pmatrix}. \tag{3.4}$$

We concentrate on one of the blocks,

$$\begin{pmatrix} 0 & F_{12} \\ -F_{12} & 0 \end{pmatrix},$$

and introduce complex coordinates, $X = (X^1 + iX^2)/\sqrt{2}$ and $\tilde{X} = (X^1 - iX^2)/\sqrt{2}$, in these two dimensions. The boundary conditions for these variables turn out to be

$$X' = 0, \quad \tilde{X}' = 0 \qquad (\sigma = 0), \tag{3.5}$$

$$X' = -i\,2\pi\alpha' F_{12}\dot{X}, \quad \tilde{X}' = i\,2\pi\alpha' F_{12}\dot{\tilde{X}} \quad (\sigma = \pi). \tag{3.6}$$

The open string theory with such boundary conditions is studied in Ref. [23], the results of which are reviewed in Appendix A. The mode expansions of X and $\tilde{X}$ are given by

$$X(z,\bar{z}) = x + i\sqrt{\frac{\alpha'}{2}}\sum_{k=-\infty}^{\infty}\frac{1}{k+\lambda}(z^{-k-\lambda} + \bar{z}^{-k-\lambda})\,\alpha_{k+\lambda}, \tag{3.7}$$

$$\tilde{X}(z,\bar{z}) = \tilde{x} + i\sqrt{\frac{\alpha'}{2}}\sum_{k=-\infty}^{\infty}\frac{1}{k-\lambda}(z^{-k+\lambda} + \bar{z}^{-k+\lambda})\,\tilde{\alpha}_{k-\lambda}, \tag{3.8}$$

where $z = e^{\tau + i\sigma}$ and λ is related to the magnetic field as $2\pi\alpha' F_{12} = \tan\pi\lambda$ $(0 \le \lambda < 1)$. In order for X^1, X^2 to be real, $(\alpha_{k+\lambda})^\dagger = \tilde{\alpha}_{-k-\lambda}$ and $(x)^\dagger = \tilde{x}$. $\alpha_{k+\lambda}$, $\tilde{\alpha}_{k-\lambda}$, x, and $\tilde{x}$ satisfy the commutation relations

$$\begin{aligned}
[\alpha_{k+\lambda}, \tilde{\alpha}_{k'-\lambda}] &= (k+\lambda)\delta_{k+k',0}, \\
[\alpha_{k+\lambda}, \alpha_{k'+\lambda}] &= [\tilde{\alpha}_{k-\lambda}, \tilde{\alpha}_{k'-\lambda}] = 0,
\end{aligned} \tag{3.9}$$

$$[x, \tilde{x}] = -\frac{1}{F_{12}} = -\frac{2\pi\alpha'}{\tan\pi\lambda}. \tag{3.10}$$

Equation (3.10) implies that a noncommutative space arises due to the background gauge field.

The energy–momentum tensor is given by

$$T(z) = \lim_{z'\to z}\left[-\frac{2}{\alpha'}\,\partial X(z)\partial\tilde{X}(z') - \frac{1}{(z-z')^2}\right], \tag{3.11}$$

and we get the Virasoro generators

$$L_n = \sum_{k=-\infty}^{\infty} :\alpha_{k+\lambda}\tilde{\alpha}_{n-k-\lambda}: + \frac{1}{2}\lambda(1-\lambda)\delta_{n,0}, \tag{3.12}$$

where $:\cdot:$ denotes the normal ordering. We construct a basis of Fock space using the oscillators $\alpha_{k+\lambda}$, $\tilde{\alpha}_{k-\lambda}$.

To every ket state, there corresponds a BCC operator which may be used to construct the Erler–Maccaferri solution. In this paper, for simplicity, we restrict ourselves to the BCC operators corresponding to the ground states. From Eq. (3.12), we can see that these operators are primary fields with conformal weight

$$\frac{1}{2}\lambda(1-\lambda).$$

A ground state can be expressed as a linear combination of

$$|y\rangle$$

that satisfies

$$\begin{aligned} \alpha_{k+\lambda}\,|y\rangle &= 0 \qquad (k \geq 0), \\ \tilde{\alpha}_{k+1-\lambda}\,|y\rangle &= 0 \qquad (k \geq 0), \\ \frac{1}{\sqrt{2}i}(x-\tilde{x})\,|y\rangle &= y\,|y\rangle\,, \\ \langle y|y'\rangle &= \delta(y-y'). \end{aligned} \tag{3.13}$$

We can define the BCC operators σ_*^y, $\bar{\sigma}_*^y$ such that

$$\begin{aligned} |y\rangle &= \sigma_*^y(0)\,|0\rangle\,, \\ \langle y| &= \lim_{z\to\infty}\langle 0|\,z^{\lambda(1-\lambda)}\bar{\sigma}_*^y(z). \end{aligned}$$

Another choice of basis of the ground states is given by the eigenstates of the operator $F_{12}x\tilde{x}$ which is included in the angular momentum J associated with the rotational symmetry $X \to e^{i\phi}X$, $\tilde{X} \to e^{-i\phi}\tilde{X}$:

$$J = F_{12}\,\tilde{x}x - \frac{1}{2} - \sum_{k=0}^{\infty}\left(\frac{1}{k+\lambda}\tilde{\alpha}_{-k-\lambda}\alpha_{k+\lambda} - \frac{1}{k-\lambda+1}\alpha_{-k+\lambda-1}\tilde{\alpha}_{k-\lambda+1}\right), \tag{3.14}$$

where a normal ordering constant is determined by the parity along the x^2 direction.[2] Since the zero modes satisfy $x^\dagger = \tilde{x}$ and Eq. (3.10), the operator $F_{12}\,\tilde{x}x$ is a kind of number counting operator and its expectation value must be positive or negative definite, depending on the signature of F_{12}. So, the operator x is regarded as the annihilation operator and $\tilde{x}$ as the creation operator if $-F_{12} > 0$, and the roles of x and $\tilde{x}$ are inverted if $-F_{12} < 0$. Hence, the vacuum is defined as $x\,|\Omega\rangle = 0$ for $-F_{12} > 0$ or $\tilde{x}\,|\Omega\rangle = 0$ for $-F_{12} < 0$. We can define the BCC operators σ_*^Ω, $\bar{\sigma}_*^\Omega$ such that

$$\begin{aligned} |\Omega\rangle &= \sigma_*^\Omega(0)\,|0\rangle\,, \\ \langle\Omega| &= \lim_{z\to\infty}\langle 0|\,z^{\lambda(1-\lambda)}\bar{\sigma}_*^\Omega(z). \end{aligned}$$

The eigenstate of the angular momentum is given as $\tilde{x}^n\,|\Omega\rangle$ $(n = 0, 1, 2, \dots)$ for $-F_{12} > 0$ and its eigenvalues are $J = -(n + 1/2)$. For $-F_{12} < 0$, $x^n\,|\Omega\rangle$ is the eigenstate of J corresponding to the eigenstate $n+1/2$. Classically, this is interpreted as circular motion in a rotational direction for a fixed direction of the magnetic field and, quantum mechanically, its angular momentum is discretized.

3.2. *Toroidally compactified theory*

We can follow the same procedure as above and deal with the case where the directions X^1, X^2 are toroidally compactified as

$$X^1 \sim X^1 + 2\pi R_1, \quad X^2 \sim X^2 + 2\pi R_2. \tag{3.15}$$

[2] Under the x^2 parity, the magnetic field F_{12} is transformed to $-F_{12}$ $(\lambda \to 1-\lambda)$ and the modes are transformed as $x \leftrightarrow \tilde{x}$, $\alpha_{k+\lambda} \leftrightarrow \tilde{\alpha}_{k-\lambda+1}$. Since the angular momentum is transformed as $J \to -J$, the normal ordering constant turns out to be $1/2$.

We obtain the same mode expansions (3.7), (3.8) and the commutation relations (3.9), (3.10). In this case, we need to introduce two unitary operators

$$U = \exp\left(i\frac{x^1}{R_1}\right), \quad V = \exp\left(i\frac{x^2}{R_2}\right) \tag{3.16}$$

with

$$x^1 = \frac{1}{\sqrt{2}}(x+\tilde{x}), \quad x^2 = \frac{1}{\sqrt{2}i}(x-\tilde{x})$$

to get the representation of the zero-mode algebra (3.10) consistent with the periodicity (3.15). Since the Dirac quantization implies

$$(2\pi)^2 R_1 R_2 F_{12} = 2\pi N, \tag{3.17}$$

for some integer N, we can see that the U, V satisfy the relation

$$UV = e^{i\frac{2\pi}{N}}\, VU, \quad (N = \pm 1, \pm 2, \ldots). \tag{3.18}$$

It is well known that the algebra (3.18) has a $|N|$-dimensional representation. For $|N| = 1$, we can take $U = V = 1$. For $|N| \neq 1$, if we diagonalize the operator V, the representation is explicitly given in matrix form as

$$U = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{|N|-1} \end{bmatrix}, \quad (\omega = e^{i\frac{2\pi}{N}}). \tag{3.19}$$

Let $|k\rangle$ be the eigenstate of V corresponding to the eigenvalue ω^k ($k \in \mathbb{Z}$, $|k| \leq \frac{|N|}{2}$). We normalize the eigenstates as

$$\langle k|l\rangle = \delta_{k,l}. \tag{3.20}$$

We can define the BCC operators σ_*^k, $\bar{\sigma}_*^k$ so that

$$|k\rangle = \sigma_*^k(0)\,|0\rangle\,,$$
$$\langle k| = \lim_{z\to\infty} \langle 0|\, z^{\lambda(1-\lambda)}\bar{\sigma}_*^k(z).$$

The theory discussed in the previous subsection corresponds to the limit $R_1, R_2, |N| \to \infty$ with

$$F_{12} = \frac{N}{2\pi R_1 R_2}$$

fixed. The states $|y\rangle$ defined in Eq. (3.13) can be given by the limit

$$|y\rangle = \lim\left(\sqrt{\frac{|N|}{2\pi R_2}}\,|k\rangle\right), \tag{3.21}$$

with

$$y = \frac{2\pi R_2}{N} k$$

fixed.

3.3. *Correlation functions*

For the construction of Erler–Maccaferri solutions, it is necessary to obtain the OPEs of the BCC operators. We calculate the three-point and four-point correlation functions of the BCC operators and derive the OPE from these.

Let us first consider the theory in the compactified space (3.15) and study the three-point function of the form

$$\left\langle \bar{\sigma}_*^l(\infty)\, e^{ip\cdot X}(z,\bar{z})\, \sigma_*^k(0) \right\rangle,$$

which can be expressed as

$$\langle l|\, e^{ip\cdot X}(z,\bar{z})\, |k\rangle$$

in the operator language. As is demonstrated in Appendix B, this correlation function can be evaluated using Eqs. (3.8), (3.9), and (3.10), and we obtain

$$\left\langle \bar{\sigma}_*^l(\infty)\, e^{ip\cdot X}(z,\bar{z})\, \sigma_*^k(0) \right\rangle = \begin{cases} \dfrac{\delta^{-\frac{\alpha' p^2}{2}}}{|z|^{\alpha' p^2}} \langle l|\, e^{i(p\tilde{x}+\tilde{p}x)}\, |k\rangle & (z > 0) \\[2ex] \dfrac{\delta^{-\frac{\alpha' p^2}{2}\cos^2\pi\lambda}}{|z|^{\alpha' p^2 \cos^2\pi\lambda}} \langle l|\, e^{i(p\tilde{x}+\tilde{p}x)}\, |k\rangle & (z < 0). \end{cases} \tag{3.22}$$

For

$$p_i = \frac{n_i}{R_i} \qquad (i = 1, 2,\ n_i \in \mathbb{Z}),$$

we get

$$\langle l|\, e^{i(p\tilde{x}+\tilde{p}x)}\, |k\rangle = e^{-i\pi \frac{n_1 n_2}{N}} \langle l|\, U^{n_1} V^{n_2}\, |k\rangle = \omega^{\frac{n_1 n_2}{2} + n_2 l} \delta_{k-l,\, n_1\, (\mathrm{mod} N)}.$$

With the three-point function (3.22), the OPEs of the operators $\sigma_*^k, \bar{\sigma}_*^l$ are obtained as

$$\begin{aligned} \sigma_*^k(s)\bar{\sigma}_*^l(0) &\sim \frac{1}{g_0} \sum_{n_1,n_2} \left\langle \bar{\sigma}_*^l(\infty)\, e^{ip\cdot X}(1,1)\, \sigma_*^k(0) \right\rangle s^{-\lambda(1-\lambda)+\alpha' p^2} e^{-ip\cdot X}(0), \\ \bar{\sigma}_*^l(s)\sigma_*^k(0) &\sim \frac{1}{g_*} \sum_{n_1,n_2} \left\langle \bar{\sigma}_*^l(\infty)\, e^{ip\cdot X}(-1,-1)\, \sigma_*^k(0) \right\rangle s^{-\lambda(1-\lambda)+\alpha' p^2 \cos^2\pi\lambda} e^{-ip\cdot X}(0) \end{aligned} \tag{3.23}$$

for $s > 0$. g_0 is equal to the volume $(2\pi)^2 R_1 R_2$ of the two-dimensional space, and all we have to do is to obtain g_*, which can be derived from the four-point function. The four-point functions of the BCC operators are calculated in Appendix C, and we have

$$\begin{aligned}
&\left\langle \bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle \\
&= \frac{1}{(2\pi)^2 R_1 R_2} x^{-\lambda(1-\lambda)}(1-x)^{-\lambda(1-\lambda)} \frac{1}{F(\lambda,1-\lambda,1;x)}\,\delta_{j-i,l-k\,(\mathrm{mod}N)} \\
&\quad\times \sum_{n,m} \omega^{(j-l)m} \exp\left[-\frac{\pi\alpha'}{\sin\pi\lambda}\frac{F(\lambda,1-\lambda,1;1-x)}{F(\lambda,1-\lambda,1;x)}\left\{\frac{(j-i+nN)^2}{R_1^2}+\frac{m^2}{R_2^2}\right\}\right].
\end{aligned} \tag{3.24}$$

In order to evaluate g_*, we examine the limit $x \to 1-0$, which can be studied by rewriting it into the form

$$\begin{aligned}
&\left\langle \sigma_*^j(0)\bar{\sigma}_*^i(x)\sigma_*^k(1)\bar{\sigma}_*^l(\infty)\right\rangle \\
&= \frac{|\cos\pi\lambda|}{(2\pi)^2 R_1 R_2} x^{-\lambda(1-\lambda)}(1-x)^{-\lambda(1-\lambda)} \frac{1}{F(\lambda,1-\lambda,1;1-x)}\,\delta_{j-i,l-k\,(\mathrm{mod}N)} \\
&\quad\times \sum_{n,m} \omega^{(i-j)n} \exp\left[-\frac{\pi\alpha'}{\sin\pi\lambda}\cos^2\pi\lambda\frac{F(\lambda,1-\lambda,1;x)}{F(\lambda,1-\lambda,1;1-x)}\left\{\frac{(i-k+mN)^2}{R_1^2}+\frac{n^2}{R_2^2}\right\}\right]
\end{aligned} \tag{3.25}$$

using the Poisson resummation formula

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} e^{i\frac{2\pi}{N}(j-l)m} e^{-am^2} &= \sqrt{\frac{\pi}{a}} \sum_{m=-\infty}^{\infty} e^{-\frac{\pi^2}{N^2 a}(j-l+mN)^2}, \\
\sum_{n=-\infty}^{\infty} e^{-a(j-i+nN)^2} &= \frac{1}{|N|}\sqrt{\frac{\pi}{a}} \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi}{N}(i-j)n} e^{-\frac{\pi^2}{N^2 a}n^2}.
\end{aligned}$$

From Eq. (3.25), we find that, for $x \sim 1$,

$$\begin{aligned}
&\left\langle \bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle \\
&\sim \frac{|\cos\pi\lambda|}{(2\pi)^2 R_1 R_2} \sum_{\vec{p}} \omega^{n_1 n_2 + n_2(i-l)} \delta_{k-i,n_1\,(\mathrm{mod}N)} \delta_{l-j,n_1\,(\mathrm{mod}N)} \delta^{-\alpha' p^2} (1-x)^{-\lambda(1-\lambda)+\alpha' p^2\cos^2\pi\lambda}.
\end{aligned}$$

Comparing this with Eq. (3.23), we get

$$g_* = \frac{(2\pi)^2 R_1 R_2}{|\cos\pi\lambda|}, \tag{3.26}$$

and the OPEs of

$$\begin{aligned}
&\sigma_*^k(s)\bar{\sigma}_*^l(0) \\
&\quad\sim \frac{1}{(2\pi)^2 R_1 R_2} s^{-\lambda(1-\lambda)} \sum_{n_1,n_2} \left(s\delta^{-\frac{1}{2}}\right)^{\alpha'\left(\left(\frac{n_1}{R_1}\right)^2+\left(\frac{n_2}{R_2}\right)^2\right)} \\
&\qquad\times \omega^{\frac{n_1 n_2}{2}+n_2 l} \delta_{k-l,\,n_1\,(\mathrm{mod}N)} e^{-i\left(\frac{n_1}{R_1}X^1+\frac{n_2}{R_2}X^2\right)}(0), \\
&\bar{\sigma}_*^l(s)\sigma_*^k(0)
\end{aligned}$$

$$\sim \frac{|\cos\pi\lambda|}{(2\pi)^2 R_1 R_2} s^{-\lambda(1-\lambda)} \sum_{n_1,n_2} \left(s\delta^{-\frac{1}{2}}\right)^{\alpha'\left(\left(\frac{n_1}{R_1}\right)^2+\left(\frac{n_2}{R_2}\right)^2\right)\cos^2\pi\lambda}$$
$$\times\, \omega^{\frac{n_1 n_2}{2}+n_2 l} \delta_{k-l,\, n_1\,(\mathrm{mod}N)} e^{-i\left(\frac{n_1}{R_1}X^1+\frac{n_2}{R_2}X^2\right)}(0). \tag{3.27}$$

Since

$$\frac{1}{|\cos\pi\lambda|} = \sqrt{1+\tan^2\pi\lambda} = \sqrt{1+(2\pi\alpha' F_{12})^2}, \tag{3.28}$$

g_* can be expressed as

$$g_* = (2\pi)^2 R_1 R_2 \sqrt{\det\begin{pmatrix} 1 & 2\pi\alpha' F_{12} \\ 2\pi\alpha' F_{21} & 1 \end{pmatrix}} = \int d^2x \sqrt{\det\begin{pmatrix} 1 & 2\pi\alpha' F_{12} \\ 2\pi\alpha' F_{21} & 1 \end{pmatrix}},$$

which coincides with the contribution to the Born–Infeld action from the two dimensions we are dealing with.

The noncompact case can be dealt with in the same way. The three-point function and the four-point function of $\sigma^\Omega, \bar{\sigma}^\Omega$ can be calculated to be

$$\left\langle \sigma_*^\Omega(\infty)\, e^{ip\cdot X}(z,\bar{z})\, \bar{\sigma}_*^\Omega(0)\right\rangle = \begin{cases} \dfrac{\delta^{-\frac{\alpha' p^2}{2}}}{|z|^{\alpha' p^2}} \exp\left(-\frac{p^2}{4|F_{12}|}\right) & (z>0) \\[2ex] \dfrac{\delta^{-\frac{\alpha' p^2}{2}\cos^2\pi\lambda}}{|z|^{\alpha' p^2\cos^2\pi\lambda}} \exp\left(-\frac{p^2}{4|F_{12}|}\right) & (z<0), \end{cases} \tag{3.29}$$

$$\begin{aligned} &\left\langle \bar{\sigma}_*^\Omega(0)\sigma_*^\Omega(x)\bar{\sigma}_*^\Omega(1)\sigma_*^\Omega(\infty)\right\rangle \\ &= x^{-\lambda(1-\lambda)}(1-x)^{-\lambda(1-\lambda)} \frac{1}{F(\lambda,1-\lambda,1;x)} \\ &\quad\times \int \frac{d^2p}{(2\pi)^2} \exp\left[-\frac{\pi\alpha'}{|\tan\pi\lambda|}p^2 - \frac{\pi\alpha'}{\sin\pi\lambda}\frac{F(\lambda,1-\lambda,1;1-x)}{F(\lambda,1-\lambda,1;x)}p^2\right] \\ &= |\cos\pi\lambda|\, x^{-\lambda(1-\lambda)}(1-x)^{-\lambda(1-\lambda)} \frac{1}{F(\lambda,1-\lambda,1;1-x)} \\ &\quad\times \int \frac{d^2p}{(2\pi)^2} \exp\left[-\frac{\pi\alpha'}{|\tan\pi\lambda|}p^2 - \frac{\pi\alpha'}{\sin\pi\lambda}\cos^2\pi\lambda\frac{F(\lambda,1-\lambda,1;x)}{F(\lambda,1-\lambda,1;1-x)}p^2\right]. \end{aligned} \tag{3.30}$$

For $z\in\mathbb{R}$, the two-point functions of the BCFT_0 on the upper half plane are normalized as

$$\left\langle e^{ip\cdot X}(z,\bar{z}) e^{ip'\cdot X}(0,0)\right\rangle_{\mathrm{BCFT}_0} = (2\pi)^2\delta^2(p+p')\,|z|^{-2\alpha' p^2}, \tag{3.31}$$

and those for the BCFT_* should be

$$\left\langle e^{ip\cdot X}(z,\bar{z}) e^{ip'\cdot X}(0,0)\right\rangle_{\mathrm{BCFT}_*} = \frac{1}{|\cos\pi\lambda|}(2\pi)^2\delta^2(p+p')\,|z|^{-2\alpha' p^2\cos^2\pi\lambda}. \tag{3.32}$$

Equations (3.31) and (3.32) imply

$$g_0 = \langle 1 \rangle_{\mathrm{BCFT}_0} = (2\pi)^2 \delta^2(p+p')\big|_{p=p'=0} = \int d^2x,$$

$$g_* = \langle 1 \rangle_{\mathrm{BCFT}_*} = \int d^2x \sqrt{\det \begin{pmatrix} 1 & 2\pi\alpha' F_{12} \\ 2\pi\alpha' F_{21} & 1 \end{pmatrix}},$$

although both g_0 and g_* are infinite. We can derive the OPE

$$\begin{aligned}
\sigma_*^{\Omega}(s)\bar{\sigma}_*^{\Omega}(0) &\sim \int \frac{d^2p}{(2\pi)^2} \left\langle \bar{\sigma}_*^{\Omega}(\infty)\, e^{ip\cdot X}(1,1)\, \sigma_*^{\Omega}(0) \right\rangle s^{-\lambda(1-\lambda)+\alpha' p^2} e^{-ip\cdot X}(0) \\
&= s^{-\lambda(1-\lambda)} \int \frac{d^2p}{(2\pi)^2} \left(s\delta^{-\frac{1}{2}}\right)^{\alpha' p^2} e^{-\frac{\pi\alpha'}{2|\tan\pi\lambda|}p^2} e^{ip\cdot X}(0), \\
\bar{\sigma}_*^{\Omega}(s)\sigma_*^{\Omega}(0) &\sim |\cos\pi\lambda| \int \frac{d^2p}{(2\pi)^2} \left\langle \bar{\sigma}_*^{\Omega}(\infty)\, e^{ip\cdot X}(-1,-1)\, \sigma_*^{\Omega}(0) \right\rangle s^{-\lambda(1-\lambda)+\alpha' p^2 \cos^2\pi\lambda} e^{-ip\cdot X}(0) \\
&\sim |\cos\pi\lambda|\, s^{-\lambda(1-\lambda)} \int \frac{d^2p}{(2\pi)^2} \left(s\delta^{-\frac{1}{2}}\right)^{\alpha' p^2\cos^2\pi\lambda} e^{-\frac{\pi\alpha'}{2|\tan\pi\lambda|}p^2} e^{ip\cdot X}(0)
\end{aligned} \tag{3.33}$$

for $s > 0$. They are given as an integral of the exponential operator over the continuous momenta and it is difficult to construct BCC operators satisfying (2.3) from these.

OPEs of $\sigma_*^{y}, \bar{\sigma}_*^{y}$ can be obtained either by calculating the correlation functions or simply by taking the limit (3.21) of Eq. (3.27). We get

$$\begin{aligned}
&\sigma_*^{y}(s)\bar{\sigma}_*^{y'}(0) \\
&\quad\sim s^{-\lambda(1-\lambda)} \frac{|\tan\pi\lambda|}{2\pi\alpha'} \int \frac{dp_2}{(2\pi)^2} \left(s\delta^{-\frac{1}{2}}\right)^{\alpha'\left(\left(\frac{\tan\pi\lambda}{2\pi\alpha'}(y-y')\right)^2+(p_2)^2\right)} \\
&\qquad\times e^{ip_2\frac{y+y'}{2}} e^{-i\left(\frac{\tan\pi\lambda}{2\pi\alpha'}(y-y')X^1+p_2X^2\right)}(0), \\
&\bar{\sigma}_*^{y'}(s)\sigma_*^{y}(0) \\
&\quad\sim s^{-\lambda(1-\lambda)} \frac{|\sin\pi\lambda|}{2\pi\alpha'} \int \frac{dp_2}{(2\pi)^2} \left(s\delta^{-\frac{1}{2}}\right)^{\alpha'\left(\left(\frac{\tan\pi\lambda}{2\pi\alpha'}(y-y')\right)^2+(p_2)^2\right)\cos^2\pi\lambda} \\
&\qquad\times e^{ip_2\frac{y+y'}{2}} e^{-i\left(\frac{\tan\pi\lambda}{2\pi\alpha'}(y-y')X^1+p_2X^2\right)}(0)
\end{aligned} \tag{3.34}$$

for $s > 0$.

4. Classical solutions for constant magnetic field

With the OPEs (3.27), (3.33), and (3.34) derived in the previous section, we are able to construct the Erler–Maccaferri solution corresponding to the constant magnetic field background. All we have to do is to find BCC operators σ_*, $\bar{\sigma}_*$ satisfying Eq. (2.7).

4.1. *Toroidally compactified theory*

In the case of the theory in the toroidally compactified space, it is straightforward to construct σ_*, $\bar{\sigma}_*$. Since Eq. (3.27) imply

$$\sigma_*^k(s)\bar{\sigma}_*^l(0) \sim \frac{1}{(2\pi)^2 R_1 R_2} s^{-\lambda(1-\lambda)}\delta_{k,l},$$
$$\bar{\sigma}_*^l(s)\sigma_*^k(0) \sim \frac{|\cos\pi\lambda|}{(2\pi)^2 R_1 R_2} s^{-\lambda(1-\lambda)}\delta_{k,l}$$

for small positive s, one can take

$$\sigma_*(s) = \sqrt{\frac{(2\pi)^2 R_1 R_2}{|\cos\pi\lambda|}}\sigma_*^k(s),$$
$$\bar{\sigma}_*(s) = \sqrt{\frac{(2\pi)^2 R_1 R_2}{|\cos\pi\lambda|}}\bar{\sigma}_*^k(s) \tag{4.1}$$

for some k ($|k| \leq \frac{|N|}{2}$) which satisfy the OPE (2.7), with

$$h = \frac{1}{2}\lambda(1-\lambda), \qquad \frac{g_*}{g_0} = \frac{1}{|\cos\pi\lambda|}.$$

From these, one can construct the Erler–Maccaferri solution (2.5) which describes the D-branes with gauge field strength F_{12}. From Eq. (3.28), we can see that the action of the background is given by the difference between the Born–Infeld action and the D-brane tension.

It is also possible to construct solutions corresponding to multiple brane solutions with the F_{12} background by considering

$$\sigma_{*,k}(s) = \sqrt{\frac{(2\pi)^2 R_1 R_2}{|\cos\pi\lambda|}}\sigma_*^k(s),$$
$$\bar{\sigma}_{*,l}(s) = \sqrt{\frac{(2\pi)^2 R_1 R_2}{|\cos\pi\lambda|}}\bar{\sigma}_*^l(s), \tag{4.2}$$

from which we can construct string fields Σ_k, $\bar{\Sigma}_l$ similarly to Eq. (2.6) satisfying

$$\bar{\Sigma}_k\Sigma_l = \delta_{k,l}, \tag{4.3}$$

so that

$$\Psi = \Psi_{\rm tv} - \sum_{k=1}^{M}\Sigma_k\Psi_{\rm tv}\bar{\Sigma}_k \tag{4.4}$$

gives a solution for $0 < M \leq |N|$. This solution can be regarded as a solution corresponding to M D-branes with magnetic field condensation.[3] M cannot be greater than $|N|$ because the BCC

[3] More general solutions can be given by $\Psi = \Psi_{\rm tv} - \sum_{i,j} A_{ij}\Sigma_i\Psi_{\rm tv}\bar{\Sigma}_j$, where A_{ij} denotes a Hermitian $|N|\times|N|$ matrix satisfying $A^2 = A$. A can be considered as a projection operator.

operators we consider in this paper are those corresponding to the ground states of the Fock space in Sect. 3.1. It should be possible to deal with more general cases by introducing BCC operators corresponding to other states.

It is also possible to construct solutions with general $F_{\mu\nu}$ of the form (3.4) by considering the tensor product of the BCC operators (4.2), one for each block. It is straightforward to show that the energy of the solution is given by the difference of the Born–Infeld action and the D-brane tension.

4.2. Noncompact case

Unlike the compactified case, (3.34) and (3.33) include extra $\ln s$ dependence for small positive s, and therefore we cannot choose σ_*^y, $\bar{\sigma}_*^y$ or σ_*^Ω, $\bar{\sigma}_*^\Omega$ as the σ_*, $\bar{\sigma}_*$ satisfying the OPE (2.7). One somewhat artificial way to construct such BCC operators is to make the following linear combinations of σ_*^y, $\bar{\sigma}_*^y$:

$$\sigma_*(s) \equiv \sqrt{\frac{(2\pi)^2 a\alpha'}{|\sin\pi\lambda|}} \sum_{n=-\infty}^{\infty} \sigma_*^{na}(s),$$

$$\bar{\sigma}_*(s) \equiv \sqrt{\frac{(2\pi)^2 a\alpha'}{|\sin\pi\lambda|}} \sum_{n=-\infty}^{\infty} \sigma_*^{na}(s)$$

for $a > 0$. It is straightforward to derive the following OPEs for $s > 0$ from (3.34):

$$\bar{\sigma}_*(s)\sigma_*(0) \sim s^{-\lambda(1-\lambda)},$$

$$\sigma_*(s)\bar{\sigma}_*(0) \sim \frac{1}{|\cos\pi\lambda|} s^{-\lambda(1-\lambda)}.$$

From σ_*, $\bar{\sigma}_*$, we can construct solutions corresponding to the D-brane with the $F_{\mu\nu}$ background in the way explained in the previous subsection. Similarly, by taking different linear combinations of σ_*^y, $\bar{\sigma}_*^y$, such as

$$\sigma_{*,k}(s) \equiv \sqrt{\frac{(2\pi)^2 Ma\alpha'}{|\sin\pi\lambda|}} \sum_{n=-\infty}^{\infty} \sigma_*^{(nM+k)a}(s),$$

$$\bar{\sigma}_{*,l}(s) \equiv \sqrt{\frac{(2\pi)^2 Ma\alpha'}{|\sin\pi\lambda|}} \sum_{n=-\infty}^{\infty} \sigma_*^{(nM+l)a}(s)$$

($M \in \mathbb{N}$; $k, l = 0, 1, \ldots, M-1$), we have

$$\bar{\sigma}_{*,k}(s)\sigma_{*,l}(0) \sim s^{-\lambda(1-\lambda)}\delta_{k,l},$$

$$\sigma_{*,k}(s)\bar{\sigma}_{*,l}(0) \sim \frac{1}{|\cos\pi\lambda|} s^{-\lambda(1-\lambda)}\delta_{k,l}$$

for $s > 0$ and we can construct solutions corresponding to multiple D-branes with $F_{\mu\nu}$ from them.

5. Concluding remarks

In this paper, we have constructed the Erler–Maccaferri solutions corresponding to constant magnetic field configurations on D-branes. In order to do so, we have calculated the correlation functions of the BCC operators and obtained their OPEs in the cases of the theories in toroidally compactified and noncompact space. We have shown that the Born–Infeld action appears as the action for such backgrounds.

There are several important things to be studied about the solutions obtained in this paper. In the case of toroidally compactified space, the Chern number of the $U(1)$ gauge field is nonvanishing. Therefore the configuration should be a topologically nontrivial one, from the low energy point of view. One question is how such configurations are realized as those of the string field. Such a question may be studied by examining the position space profile of the gauge field, as was done in Ref. [22]. From the OPE of the operators σ_*, $\bar{\sigma}_*$, one can calculate the profile, which is easily seen to be a periodic function of the coordinates x^1, x^2. Therefore one expects that the gauge field profile has jump discontinuities as a function of x^1, x^2 in order for the configuration to be topologically nontrivial.[4] We leave this for future work.

With the solution constructed in this paper, we should be able to deduce all the interesting features of the open string theory around the background. In particular, the relation to the noncommutative geometry should be seen by analyzing the solution. One thing that can be done would be to get the relation between the worldsheet variables of BCFT_0 and BCFT_* by using the string field theory technique. In Ref. [22], it is pointed out that the correspondence between the states in BCFT_0 and BCFT_* can be given by the string fields Σ, $\bar{\Sigma}$. It will be interesting to explore such a correspondence and compare with the one given in Refs. [34,35].

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Appendix A. First quantization of open strings in background gauge field

In this appendix, we discuss the first quantization of open strings in background gauge fields. It has been discussed in Ref. [23] and we present their results here in order to fix our notation.

Let us consider the worldsheet theory of the variables $X, \tilde{X}$ defined in Sect. 3.1. Here, we study the general case where the charges at the ends of the strings are q_1, q_2. The action is

$$S = \frac{1}{2\pi\alpha'}\int dtd\sigma\,(\dot{X}\dot{\tilde{X}} - X'\tilde{X}') - \frac{i\theta}{4\pi\alpha'}\int dt\left[q_1(X\dot{\tilde{X}} - \tilde{X}\dot{X})\Big|_{\sigma=0} + q_2(X\dot{\tilde{X}} - \tilde{X}\dot{X})\Big|_{\sigma=\pi}\right], \quad \text{(A.1)}$$

where $\theta = -2\pi\alpha' F_{12}$. The system discussed in Sect. 3.1 corresponds to the case $q_1 = 0$, $q_2 = 1$. The canonical momentum is defined as

$$P(t,\sigma) = \frac{\partial\mathcal{L}}{\partial\dot{X}} = \frac{1}{2\pi\alpha'}\dot{\tilde{X}} + \frac{i\theta}{4\pi\alpha'}\{q_1\delta(\sigma) + q_2\delta(\pi-\sigma)\}\tilde{X}, \quad \text{(A.2)}$$

$$\tilde{P}(t,\sigma) = \frac{\partial\mathcal{L}}{\partial\dot{\tilde{X}}} = \frac{1}{2\pi\alpha'}\dot{X} - \frac{i\theta}{4\pi\alpha'}\{q_1\delta(\sigma) + q_2\delta(\pi-\sigma)\}X, \quad \text{(A.3)}$$

and they satisfy the canonical commutation relations

$$\left[X(t,\sigma),\, P(t,\sigma')\right] = \left[\tilde{X}(t,\sigma),\, \tilde{P}(t,\sigma')\right] = i\delta(\sigma-\sigma'), \quad \text{(A.4)}$$

[4] We thank T. Erler, M. Schnabl, and M. Kudrna for pointing this out to the authors.

with all other commutators vanishing. The boundary conditions are given by

$$X' = -iq_1\theta\dot{X} \quad (\sigma = 0), \quad X' = iq_2\theta\dot{X} \quad (\sigma = \pi), \tag{A.5}$$

$$\tilde{X}' = iq_1\theta\dot{\tilde{X}} \quad (\sigma = 0), \quad \tilde{X}' = -iq_2\theta\dot{\tilde{X}} \quad (\sigma = \pi). \tag{A.6}$$

First we consider the case $q_1 + q_2 \neq 0$. The variables $X, \tilde{X}$ can be expanded as

$$X(t,\sigma) = x + i\sqrt{2\alpha'}\sum_{k=-\infty}^{\infty}\frac{1}{k+\lambda_1+\lambda_2}\alpha_{k+\lambda_1+\lambda_2}\psi_k(t,\sigma), \tag{A.7}$$

$$\tilde{X}(t,\sigma) = \tilde{x} + i\sqrt{2\alpha'}\sum_{k=-\infty}^{\infty}\frac{1}{k-\lambda_1-\lambda_2}\tilde{\alpha}_{k-\lambda_1-\lambda_2}\psi^*_{-k}(t,\sigma), \tag{A.8}$$

where $q_i\theta = -\tan\pi\lambda_i$ $(i = 1, 2)$. $\psi_k(t,\sigma)$ are the mode functions which are solutions to the wave equation with boundary conditions (A.5):

$$\psi_k(t,\sigma) = e^{-i(k+\lambda_1+\lambda_2)t}\cos\left[(k+\lambda_1+\lambda_2)\sigma - \pi\lambda_1\right]. \tag{A.9}$$

Their complex conjugates $\psi^*_k(t,\sigma)$ satisfy the boundary conditions (A.6). Since $\tilde{X}$ is the Hermitian conjugate of X, the operators x, $\tilde{x}$, $\alpha_{k+\lambda_1+\lambda_2}$, and $\tilde{\alpha}_{k+\lambda_1+\lambda_2}$ obey

$$x^\dagger = \tilde{x}, \quad (\alpha_{k+\lambda_1+\lambda_2})^\dagger = \tilde{\alpha}_{-k-\lambda_1-\lambda_2}. \tag{A.10}$$

The orthogonality relations are given by

$$\int_0^\pi d\sigma\,\psi^*_k(t,\sigma)D\psi_l(t,\sigma) = \pi(k+\lambda_1+\lambda_2)\delta_{k,l}, \tag{A.11}$$

$$\int_0^\pi d\sigma D\psi_k(t,\sigma) = 0, \tag{A.12}$$

where $D = i\overrightarrow{\partial_t} - i\overleftarrow{\partial_t} + q_1\theta\delta(\sigma) + q_2\theta\delta(\pi-\sigma)$. Since $\psi_k(t,\sigma)$ and the constant mode form a complete set, the operators x, $\tilde{x}$, $\alpha_{k+\lambda_1+\lambda_2}$, and $\tilde{\alpha}_{k+\lambda_1+\lambda_2}$ are expressed as

$$\alpha_{k+\lambda_1+\lambda_2} = \frac{1}{\sqrt{2\alpha'}\pi i}\int_0^\pi d\sigma\;\psi^*_k\left[2\pi\alpha' i\tilde{P} + \left\{k+\lambda_1+\lambda_2+\frac{q_1\theta}{2}\delta(\sigma)+\frac{q_2\theta}{2}\delta(\pi-\sigma)\right\}X\right], \tag{A.13}$$

$$\tilde{\alpha}_{k-\lambda_1-\lambda_2} = \frac{1}{\sqrt{2\alpha'}\pi i}\int_0^\pi d\sigma\;\psi_{-k}\left[2\pi\alpha' iP + \left\{k-\lambda_1-\lambda_2-\frac{q_1\theta}{2}\delta(\sigma)-\frac{q_2\theta}{2}\delta(\pi-\sigma)\right\}\tilde{X}\right], \tag{A.14}$$

$$x = \frac{1}{(q_1+q_2)\theta}\int_0^\pi d\sigma\;\left[2\pi\alpha' i\tilde{P} + \left\{\frac{q_1\theta}{2}\delta(\sigma)+\frac{q_2\theta}{2}\delta(\pi-\sigma)\right\}X\right], \tag{A.15}$$

$$\tilde{x} = \frac{1}{(q_1+q_2)\theta}\int_0^\pi d\sigma\;\left[-2\pi\alpha' iP + \left\{\frac{q_1\theta}{2}\delta(\sigma)+\frac{q_2\theta}{2}\delta(\pi-\sigma)\right\}\tilde{X}\right]. \tag{A.16}$$

From these expressions and the canonical commutation relations, it follows that

$$\left[\alpha_{k+\lambda_1+\lambda_2}, \tilde{\alpha}_{l-\lambda_1-\lambda_2}\right] = (k+\lambda_1+\lambda_2)\delta_{k+l,0}, \quad [x, \tilde{x}] = \frac{2\pi\alpha'}{(q_1+q_2)\theta}. \tag{A.17}$$

The energy–momentum tensor is defined as

$$T(z) = \lim_{z'\to z}\left[-\frac{2}{\alpha'}\,\partial X(z)\partial\tilde{X}(z') - \frac{1}{(z-z')^2}\right],$$

and we get the Virasoro generator

$$L_n = \sum_{k=-\infty}^{\infty} :\alpha_{n-k+\lambda_1+\lambda_2}\tilde{\alpha}_{k-\lambda_1-\lambda_2}: + \frac{1}{2}(\lambda_1+\lambda_2)(1-\lambda_1-\lambda_2)\delta_{n,0}.$$

Let us turn to the case $q_1 + q_2 = 0$, which corresponds to a neutral string. An easy way to deal with this case is to take the limit $\lambda_1 = -\lambda + \epsilon$, $\lambda_2 = \lambda + \epsilon$, $\epsilon \to 0$ of the above results. Although the limit of the zero modes requires some care, we finally obtain the mode expansions

$$\begin{aligned} X(t,\sigma) &= x_0 + \sqrt{2\alpha'}\alpha_0\left\{t\cos\pi\lambda - i\left(\sigma - \frac{\pi}{2}\right)\sin\pi\lambda\right\} \\ &\quad + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-int}\cos(n\sigma + \pi\lambda), \end{aligned} \tag{A.18}$$

$$\begin{aligned} \tilde{X}(t,\sigma) &= \tilde{x}_0 + \sqrt{2\alpha'}\tilde{\alpha}_0\left\{t\cos\pi\lambda + i\left(\sigma - \frac{\pi}{2}\right)\sin\pi\lambda\right\} \\ &\quad + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n e^{-int}\cos(n\sigma - \pi\lambda) \end{aligned} \tag{A.19}$$

and the commutation relations

$$[x_0, \tilde{\alpha}_0] = \left[\tilde{x}_0, \alpha_0\right] = i\sqrt{2\alpha'}\cos\pi\lambda, \quad [\alpha_m, \tilde{\alpha}_m] = n\delta_{n+m,0}, \tag{A.20}$$

with all other commutators vanishing. The Virasoro generators are given by

$$L_n = \sum_k :\alpha_{n-k}\tilde{\alpha}_k: .$$

The Fock vacuum $|\vec{p}\rangle$ can be defined to satisfy

$$\begin{aligned} \alpha_0\,|\vec{p}\rangle &= \sqrt{2\alpha'}p\cos\pi\lambda\,|\vec{p}\rangle\,, \\ \tilde{\alpha}_0\,|\vec{p}\rangle &= \sqrt{2\alpha'}\tilde{p}\cos\pi\lambda\,|\vec{p}\rangle\,. \end{aligned}$$

$|\vec{p}\rangle$ corresponds to the primary field $e^{i\vec{p}\cdot\vec{X}}(z,\bar{z}) = e^{i(\tilde{p}X+p\tilde{X})}(z,\bar{z})$ with weight $\alpha' p^2\cos^2\pi\lambda$,

$$|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{X}}(0,0)\,|0\rangle\,.$$

The operator $e^{i\vec{p}\cdot\vec{X}}(0,0)$ here is normal ordered and can be expressed more precisely as

$$\begin{aligned} e^{i\vec{p}\cdot\vec{X}}(0,0) &= \lim_{\epsilon\to 0}\left\{\exp\left[\frac{i}{\pi}\int_0^{\pi}d\theta\left(p\tilde{X}(\epsilon e^{i\theta},\epsilon e^{-i\theta}) + \tilde{p}X(\epsilon e^{i\theta},\epsilon e^{-i\theta})\right)\right]\right. \\ &\quad \left.\times\,\exp\left[-\frac{\alpha' p\tilde{p}}{\pi^2}\int_0^{\pi}d\theta\int_0^{\pi}d\theta'\left(\ln\left|\epsilon e^{i\theta} - \epsilon e^{i\theta'}\right| + (\cos 2\pi\lambda)\ln\left|\epsilon e^{i\theta} - \epsilon e^{-i\theta'}\right|\right)\right]\right\}. \end{aligned} \tag{A.21}$$

Here, for regularization, we replace the local operator $X(z,\bar{z})$ by an integral along the small contour around $z = 0$. Taking the limit $\epsilon \to 0$ with the factor on the second line of Eq. (A.21), we get the operator $e^{i\vec{p}\cdot\vec{X}}(0,0)$ normal ordered. This expression is useful in the calculation of three-point functions.

Appendix B. Three-point functions

In this appendix, we show how to calculate the correlation functions of the form

$$\langle\,|\,e^{i\vec{p}\cdot\vec{X}}(z,\bar{z})\,|\,\rangle' \tag{B.1}$$

for $z \in \mathbb{R}$, which play crucial roles in deriving the OPE of the BCC operators in Sect. 3.3. Here, $|\,\rangle'$, $\langle\,|$ are states in the Fock space defined in Sects. 3.1 and 3.2, and satisfy

$$\begin{aligned}\alpha_{k+\lambda}\,|\,\rangle' &= 0,\\ \tilde{\alpha}_{k+1-\lambda}\,|\,\rangle' &= 0,\\ \langle\,|\,\alpha_{-k-1+\lambda} &= 0,\\ \langle\,|\,\tilde{\alpha}_{-k-\lambda} &= 0\end{aligned} \tag{B.2}$$

for $k \geq 0$. Such correlation functions can be evaluated by essentially following the method in Ref. [36].

For $z > 0$, we take the primary field $e^{i\vec{p}\cdot\vec{X}}$ in Eq. (B.1) to be the one corresponding to the delta function normalized ground state in the BCFT$_0$. Therefore it should coincide with Eq. (A.21) in the $\lambda = 0$ case, and we get

$$\begin{aligned}&\langle\,|\,e^{i\vec{p}\cdot\vec{X}}(z,\bar{z})\,|\,\rangle'\\ &= \lim_{\epsilon\to 0}\Bigg\{\langle\,|\,\mathrm{R}\exp\left[\frac{i}{\pi}\int_0^\pi d\theta\left(p\tilde{X}(z+\epsilon e^{i\theta},\bar{z}+\epsilon e^{-i\theta})+\tilde{p}X(z+\epsilon e^{i\theta},\bar{z}+\epsilon e^{-i\theta})\right)\right]|\,\rangle'\\ &\qquad\times\exp\left[-\frac{\alpha' p\tilde{p}}{\pi^2}\int_0^\pi d\theta\int_0^\pi d\theta'\left(\ln\left|\epsilon e^{i\theta}-\epsilon e^{i\theta'}\right|+\ln\left|\epsilon e^{i\theta}-\epsilon e^{-i\theta'}\right|\right)\right]\Bigg\},\end{aligned} \tag{B.3}$$

where $\mathrm{R}[\cdots]$ denotes the radial ordering. Substituting the mode expansions (3.7), (3.8) into Eq. (B.3), it is straightforward to calculate the expectation value on the right-hand side of Eq. (B.3) and we obtain

$$\begin{aligned}&\langle\,|\,\mathrm{R}\exp\left[\frac{i}{\pi}\int_0^\pi d\theta\left(p\tilde{X}(z+\epsilon e^{i\theta},\bar{z}+\epsilon e^{-i\theta})+\tilde{p}X(z+\epsilon e^{i\theta},\bar{z}+\epsilon e^{-i\theta})\right)\right]|\,\rangle'\\ &= \exp\Bigg[-\frac{\alpha' p\tilde{p}}{4\pi^2}\int_0^\pi d\theta\int_0^\pi d\theta'\Bigg(\frac{1}{\lambda}\left(\frac{w'}{w}\right)^\lambda F(\lambda,1,1+\lambda;\frac{w'}{w})+\frac{1}{\lambda}\left(\frac{\bar{w}'}{w}\right)^\lambda F(\lambda,1,1+\lambda;\frac{\bar{w}'}{w})\\ &\qquad+\frac{1}{\lambda}\left(\frac{w'}{\bar{w}}\right)^\lambda F(\lambda,1,1+\lambda;\frac{w'}{\bar{w}})+\frac{1}{\lambda}\left(\frac{\bar{w}'}{\bar{w}}\right)^\lambda F(\lambda,1,1+\lambda;\frac{\bar{w}'}{\bar{w}})\\ &\qquad+\frac{1}{1-\lambda}\left(\frac{w}{w'}\right)^{1-\lambda}F(1-\lambda,1,2-\lambda;\frac{w}{w'})\\ &\qquad+\frac{1}{1-\lambda}\left(\frac{\bar{w}}{w'}\right)^{1-\lambda}F(1-\lambda,1,2-\lambda;\frac{\bar{w}}{w'})\\ &\qquad+\frac{1}{1-\lambda}\left(\frac{w}{\bar{w}'}\right)^{1-\lambda}F(1-\lambda,1,2-\lambda;\frac{w}{\bar{w}'})\\ &\qquad+\frac{1}{1-\lambda}\left(\frac{\bar{w}}{\bar{w}'}\right)^{1-\lambda}F(1-\lambda,1,2-\lambda;\frac{\bar{w}}{\bar{w}'})\Bigg)\Bigg]\\ &\quad\times\langle\,|\,e^{i(p\tilde{x}+\tilde{p}x)}\,|\,\rangle',\end{aligned} \tag{B.4}$$

where $F(\alpha, \beta, \gamma; z)$ is the hypergeometric function[5] and

$$w = z + \epsilon e^{i\theta},$$
$$w' = z + \epsilon e^{i\theta'}.$$

Using the formula [37]

$$\begin{aligned} &F(\alpha, \beta, \alpha+\beta; z) \\ &\quad = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(n!)^2} \left\{2\psi(n+1) - \psi(n+\alpha) - \psi(n+\beta) - \ln(1-z)\right\}(1-z)^n \\ &\quad = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left\{-\ln(1-z) + 2\psi(1) - \psi(\alpha) - \psi(\beta)\right\} + \cdots \end{aligned}$$

for $|\arg(1-z)| < \pi$, $|1-z| < 1$, the right-hand side of Eq. (B.3) is evaluated to be

$$\frac{\delta^{-\frac{\alpha' p^2}{2}}}{|z|^{\alpha' p^2}} \langle \,|\, e^{i(p\tilde{x}+\tilde{p}x)} \,|\, \rangle',$$

where

$$\ln \delta = 2\psi(1) - \psi(\lambda) - \psi(1-\lambda), \tag{B.5}$$

and $\psi(x)$ is the digamma function.

For $z < 0$, the three-point function (B.1) can be calculated in the same way using

$$\begin{aligned} &\langle \,|\, e^{i\vec{p}\cdot\vec{X}}(z,\bar{z}) \,|\, \rangle' \\ &\quad = \lim_{\epsilon \to 0} \Bigg\{ \langle \,|\, \mathrm{R}\exp\left[\frac{i}{\pi}\int_0^{\pi} d\theta \left(p\tilde{X}(z+\epsilon e^{i\theta}, \bar{z}+\epsilon e^{-i\theta}) + \tilde{p}X(z+\epsilon e^{i\theta}, \bar{z}+\epsilon e^{-i\theta})\right)\right] \,|\, \rangle' \\ &\qquad\qquad \times \exp\left[-\frac{\alpha' p\tilde{p}}{\pi^2}\int_0^{\pi} d\theta \int_0^{\pi} d\theta' \left(\ln\left|\epsilon e^{i\theta} - \epsilon e^{i\theta'}\right| + (\cos 2\pi\lambda) \ln\left|\epsilon e^{i\theta} - \epsilon e^{-i\theta'}\right|\right)\right]\Bigg\} \end{aligned} \tag{B.6}$$

and Eq. (B.4) with $z = e^{\pi i}|z|$. We eventually obtain

$$\langle \,|\, e^{i\vec{p}\cdot\vec{X}}(z,\bar{z}) \,|\, \rangle' = \begin{cases} \dfrac{\delta^{-\frac{\alpha' p^2}{2}}}{|z|^{\alpha' p^2}} \langle \,|\, e^{i(p\tilde{x}+\tilde{p}x)} \,|\, \rangle' & (z > 0) \\[2ex] \dfrac{\delta^{-\frac{\alpha' p^2}{2}\cos^2\pi\lambda}}{|z|^{\alpha' p^2 \cos^2\pi\lambda}} \langle \,|\, e^{i(p\tilde{x}+\tilde{p}x)} \,|\, \rangle' & (z < 0). \end{cases}$$

[5] We note the relation:

$$\frac{1}{\lambda}F(\lambda, 1, 1+\lambda; z) = \frac{\pi}{\sin\pi\lambda}(-z)^{-\lambda} + \frac{1}{(1-\lambda)z} F\left(1-\lambda, 1, 2-\lambda; \frac{1}{z}\right) \quad (|\arg(-z)| < \pi\,).$$

Appendix C. Correlation functions of four BCC operators

The four-point functions of the BCC operators

$$\left\langle \bar{\sigma}_*^j(x_1)\sigma_*^i(x_2)\bar{\sigma}_*^k(x_3)\sigma_*^l(x_4)\right\rangle$$

can be calculated using the the technique developed in the orbifold CFT [38,39]. The correlation function can be given as

$$\left\langle \bar{\sigma}_*^j(x_1)\sigma_*^i(x_2)\bar{\sigma}_*^k(x_3)\sigma_*^l(x_4)\right\rangle = \sum_{\vec{p}} C_{\vec{p}} \left\langle \bar{\sigma}_*^j(x_1)\sigma_*^i(x_2)\bar{\sigma}_*^k(x_3)\sigma_*^l(x_4)\right\rangle_{\vec{p}},$$

where $\left\langle \bar{\sigma}_*^j(x_1)\sigma_*^i(x_2)\bar{\sigma}_*^k(x_3)\sigma_*^l(x_4)\right\rangle_{\vec{p}}$ is the conformal block specified by the momenta $\vec{p}$ of the operators which appear in the OPE of $\bar{\sigma}_*^j(x_1)$ and $\sigma_*^i(x_2)$.

Let us consider the following ratio of correlation functions defined on the upper half plane with the insertions of the BCC operators at $x_1 < x_2 < x_3 < x_4$:

$$\tilde{G}_{\vec{p}}(z, w, x_i) = \frac{\left\langle \partial X(z)\partial\tilde{X}(w)\bar{\sigma}_*^j(x_1)\sigma_*^i(x_2)\bar{\sigma}_*^k(x_3)\sigma_*^l(x_4)\right\rangle_{\vec{p}}}{\left\langle \bar{\sigma}_*^j(x_1)\sigma_*^i(x_2)\bar{\sigma}_*^k(x_3)\sigma_*^l(x_4)\right\rangle_{\vec{p}}}. \tag{C.1}$$

As a function of z, w, $\tilde{G}_{\vec{p}}(z, w, x_i)$ can be analytically continued to be defined on the whole complex plane, which corresponds to the double of the worldsheet. Since $\tilde{G}_{\vec{p}}(z, w, x_i)$ is a 1-form on the sphere with respect to variables z, w, it behaves like $\sim \mathcal{O}(z^{-2})$ $(z \sim \infty)$ and $\sim \mathcal{O}(w^{-2})$ $(w \sim \infty)$. From the OPE of ∂X and $\partial\tilde{X}$, it follows that

$$\tilde{G}_{\vec{p}}(z, w, x_i) = \frac{-\alpha'/2}{(z-w)^2} + \text{finite} \quad (z \sim w). \tag{C.2}$$

From the OPE between ∂X, $\partial\tilde{X}$ and the BCC operators, we get

$$\tilde{G}_{\vec{p}}(z, w, x_i) \sim \begin{cases} (z-x_i)^{-(1-\lambda)} & (z \sim x_i,\ i = 1,3) \\ (z-x_i)^{-\lambda} & (z \sim x_i,\ i = 2,4) \\ (w-x_i)^{-\lambda} & (w \sim x_i,\ i = 1,3) \\ (w-x_i)^{-(1-\lambda)} & (w \sim x_i,\ i = 2,4). \end{cases} \tag{C.3}$$

Using all these conditions, $\tilde{G}_{\vec{p}}(z, w, x_i)$ is fixed as

$$\begin{aligned} \tilde{G}_{\vec{p}}(z, w; x_i) = -\frac{\alpha'}{2}\omega(z)\tilde{\omega}(w)\Bigg[&(1-\lambda)\frac{(z-x_1)(z-x_3)(w-x_2)(w-x_4)}{(z-w)^2} \\ &+\lambda\frac{(z-x_2)(z-x_4)(w-x_1)(w-x_3)}{(z-w)^2} + A(x_i)\Bigg], \end{aligned}$$

where the functions $\omega(z)$ and $\tilde{\omega}(w)$ are given by

$$\omega(z) = (z-x_1)^{-1+\lambda}(z-x_2)^{-\lambda}(z-x_3)^{-1+\lambda}(z-x_4)^{-\lambda}, \tag{C.4}$$

$$\tilde{\omega}(w) = (w-x_1)^{-\lambda}(w-x_2)^{-1+\lambda}(w-x_3)^{-\lambda}(w-x_4)^{-1+\lambda}, \tag{C.5}$$

and $A(x_i)$ is a meromorphic function which is left undetermined. We can use $SL(2,R)$ transformation to fix $x_1=0, x_2=x, x_3=1$, and $x_4\to\infty$, and $\tilde{G}(z,\,w;\,x)$ becomes

$$\tilde{G}_{\vec{p}}(z,\,w;\,x)=-\frac{\alpha'}{2}\omega(z)\tilde{\omega}(w)\left[(1-\lambda)\frac{z(z-1)(w-x)}{(z-w)^2}+\lambda\frac{(z-x)w(w-1)}{(z-w)^2}+A(x)\right], \quad \text{(C.6)}$$

where $\omega(z)=z^{-1+\lambda}(z-x)^{-\lambda}(z-1)^{-1+\lambda}$ and $\tilde{\omega}(w)=w^{-\lambda}(w-x)^{-1+\lambda}(w-1)^{-\lambda}$.

$A(x)$ can be fixed by using the conditions

$$\oint dz\,\partial X=2\pi\alpha' p,\quad \oint dz\,\partial\tilde{X}=2\pi\alpha'\tilde{p}, \quad \text{(C.7)}$$

where the integration contours are around the branch cut on the interval $(0,\,x)$. p and $\tilde{p}$ are the momenta of the operators which appear in the OPE of $\bar{\sigma}^j_*(0)$ and $\sigma^i_*(x)$. Integrating (C.6) around the branch cut in the interval $(0,\,x)$ and using the condition (C.7), we obtain

$$\oint dz\,\tilde{G}_{\vec{p}}(z,\,w,\,x)=2ie^{\pi i\lambda}\sin\pi\lambda\int_0^x dz\,\tilde{G}_{\vec{p}}(z,\,w,\,x)=2\pi\alpha' p\frac{\left\langle\partial\tilde{X}(w)\bar{\sigma}^j_*(0)\sigma^i_*(x)\bar{\sigma}^k_*(1)\sigma^l_*(\infty)\right\rangle_{\vec{p}}}{\left\langle\bar{\sigma}^j_*(0)\sigma^i_*(x)\bar{\sigma}^k_*(1)\sigma^l_*(\infty)\right\rangle_{\vec{p}}}. \quad \text{(C.8)}$$

The quantity on the rightmost side of Eq. (C.8) can be seen to be proportional to $\tilde{\omega}(w)$, and the proportionality factor can be determined by imposing Eq. (C.7) once more. As a result, we get

$$\int_0^x dz\,\tilde{G}_{\vec{p}}(z,\,w;\,x)=\frac{\pi^2\alpha'^2p^2}{2\sin^2\pi\lambda}\frac{\tilde{\omega}(w)}{\int_0^x dw\,\tilde{\omega}(w)}. \quad \text{(C.9)}$$

Taking the limit $w\to\infty$, we find that $A(x)$ is given by

$$A(x)=-\lambda\frac{\int_0^x dz\,(z-x)\omega(z)}{\int_0^x dz\,\omega(z)}-\frac{\pi^2\alpha' p^2}{\sin^2\pi\lambda}\frac{1}{\int_0^x dz\,\omega(z)\int_0^x dw\,\tilde{\omega}(w)}. \quad \text{(C.10)}$$

All the integrals which appear on the right-hand side can be expressed in terms of the hypergeometric function:

$$\int_0^x dz\,\omega(z)=-\frac{\pi}{\sin\pi\lambda}F(\lambda,1-\lambda,1;x), \quad \text{(C.11)}$$

$$\int_0^x dw\,\tilde{\omega}(w)=-\frac{\pi}{\sin\pi\lambda}F(\lambda,1-\lambda,1;x), \quad \text{(C.12)}$$

$$\int_0^x dz\,(z-x)\omega(z)=\frac{\pi}{\lambda\sin\pi\lambda}x(1-x)\frac{d}{dx}F(\lambda,1-\lambda,1;x), \quad \text{(C.13)}$$

$$\frac{1}{\int_0^x dz\,\omega(z)\int_0^x dw\,\tilde{\omega}(w)}=-\frac{\sin\pi\lambda}{\pi}x(1-x)\frac{d}{dx}\frac{F(\lambda,1-\lambda,1;1-x)}{F(\lambda,1-\lambda,1;x)}. \quad \text{(C.14)}$$

Here we have used the following formula for the hypergeometric function: Ref. [37],

$$F(\alpha,\beta,\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt, \quad \text{(C.15)}$$

$$\frac{d}{dz}\left[(1-z)^{\alpha+\beta-\gamma}F(\alpha,\beta,\gamma;z)\right]=\frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma}(1-z)^{\alpha+\beta-\gamma-1}F(\alpha,\beta,\gamma+1;z), \tag{C.16}$$

$$\frac{d}{dz}[z^{\gamma-1}F(\alpha,\beta,\gamma;z)]=(\gamma-1)z^{\gamma-2}F(\alpha,\beta,\gamma-1;z), \tag{C.17}$$

$$\frac{d}{dz}\left[\frac{F(\alpha,1-\alpha,1;\,1-z)}{F(\alpha,1-\alpha,1;\,z)}\right]=-\frac{\sin\pi\alpha}{\pi}\frac{z^{-1}(1-z)^{-1}}{F(\alpha,1-\alpha,1;\,z)^2}. \tag{C.18}$$

Thus, we eventually find

$$\begin{aligned}\tilde{G}_{\vec{p}}(z,w;x)=-\frac{\alpha'}{2}\omega(z)\tilde{\omega}(w)\Bigg[&(1-\lambda)\frac{z(z-1)(w-x)}{(z-w)^2}+\lambda\frac{(z-x)w(w-1)}{(z-w)^2}\\&+x(1-x)\frac{d}{dx}\ln F(\lambda,1-\lambda,1;x)+\frac{\pi\alpha'p^2}{\sin\pi\lambda}x(1-x)\frac{d}{dx}\frac{F(\lambda,1-\lambda,1;1-x)}{F(\lambda,1-\lambda,1;x)}\Bigg].\end{aligned} \tag{C.19}$$

From this correlation function, we can derive

$$\begin{aligned}&\frac{\left\langle T(z)\bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle_{\vec{p}}}{\left\langle\bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle_{\vec{p}}}\\&=\frac{1}{2}\lambda(1-\lambda)\left(\frac{1}{z}+\frac{1}{z-1}-\frac{1}{z-x}\right)^2\\&\quad+\frac{x(1-x)}{z(z-1)(z-x)}\frac{d}{dx}\left[\ln F(\lambda,1-\lambda,1;x)+\frac{\pi\alpha'p^2}{\sin\pi\lambda}\frac{F(\lambda,1-\lambda,1;1-x)}{F(\lambda,1-\lambda,1;x)}\right]\end{aligned}$$

using Eq. (3.11). Considering the limit $z\to x$, we get

$$\begin{aligned}&\frac{d}{dx}\ln\left\langle\bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle_{\vec{p}}\\&=-\lambda(1-\lambda)\left(\frac{1}{x}+\frac{1}{x-1}\right)\\&\quad-\frac{d}{dx}\left[\ln F(\lambda,1-\lambda,1;x)+\frac{\pi\alpha'p^2}{\sin\pi\lambda}\frac{F(\lambda,1-\lambda,1;1-x)}{F(\lambda,1-\lambda,1;x)}\right].\end{aligned}$$

Therefore, we find that the four-point function can be expressed as

$$\begin{aligned}&\left\langle\bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle\\&=\sum_{\vec{p}}C_{\vec{p}}\,x^{-\lambda(1-\lambda)}(1-x)^{-\lambda(1-\lambda)}\frac{1}{F(\lambda,1-\lambda,1;x)}\exp\left[-\frac{\pi\alpha'}{\sin\pi\lambda}p^2\frac{F(\lambda,1-\lambda,1;1-x)}{F(\lambda,1-\lambda,1;x)}\right],\end{aligned} \tag{C.20}$$

where the constant $C_{\vec{p}}$ is determined by taking the limit $x\to 0$ in Eq. (C.20). Taking the limit of the right-hand side of Eq. (C.20), we get

$$\left\langle\bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle\sim\sum_{\vec{p}}C_p\delta^{-\alpha'p^2}x^{-\lambda(1-\lambda)+\alpha'p^2}, \tag{C.21}$$

where δ is the one which appears in Eq. (B.5). On the other hand, using the OPE (3.23), we obtain

$$\left\langle \bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle$$
$$\sim \frac{1}{(2\pi)^2 R_1 R_2} x^{-\lambda(1-\lambda)} \sum_{n_1,n_2} (x\delta^{-\frac{1}{2}})^{\alpha'\left(\left(\frac{n_1}{R_1}\right)^2+\left(\frac{n_2}{R_2}\right)^2\right)} \omega^{\frac{n_1 n_2}{2}+n_2 j} \delta_{i-j,n_1\,(\mathrm{mod}N)} \left\langle \bar{\sigma}_*^k \middle| e^{-ip\cdot X(1,1)} \middle| \sigma_*^l \right\rangle, \tag{C.22}$$

where $p_i = n_i/R_i$ $(i=1,2)$. Substituting Eq. (3.22) into Eq. (C.22) and comparing it with Eq. (C.21), we can derive

$$C_p = \frac{1}{(2\pi)^2 R_1 R_2} \omega^{n_1 n_2 + n_2(j-k)} \delta_{i-j,n_1\,(\mathrm{mod}N)} \delta_{l-k,-n_1\,(\mathrm{mod}N)} \tag{C.23}$$

and

$$\left\langle \bar{\sigma}_*^j(0)\sigma_*^i(x)\bar{\sigma}_*^k(1)\sigma_*^l(\infty)\right\rangle$$
$$= \frac{1}{(2\pi)^2 R_1 R_2} x^{-\lambda(1-\lambda)} (1-x)^{-\lambda(1-\lambda)} \frac{1}{F(\lambda,1-\lambda,1;x)} \delta_{j-i,l-k\,(\mathrm{mod}N)}$$
$$\times \sum_{n,m} \omega^{(j-l)m} \exp\left[-\frac{\pi\alpha'}{\sin\pi\lambda} \frac{F(\lambda,1-\lambda,1;1-x)}{F(\lambda,1-\lambda,1;x)} \left\{\frac{(j-i+nN)^2}{R_1^2} + \frac{m^2}{R_2^2}\right\}\right]. \tag{C.24}$$

Here, the sums over n, m correspond to those over the momenta in the x^1 and x^2 directions. The momenta in these directions are treated asymmetrically because of the choice of the states corresponding to the BCC operators.

The noncompact case can be dealt with in the same way. The sum over the intermediate momenta in Eq. (C.20) becomes an integral over them and we get Eq. (3.30).

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