# STRONG COINCIDENCE AND OVERLAP COINCIDENCE

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Abstract. We show that strong coincidences of a certain many choices of control points are equivalent to overlap coincidence for the suspension tiling of Pisot substitution. The result is valid for degree $\geq 2$ as well, under certain topological conditions. This result gives a converse of the paper [2] and elucidates the tight relationship between two coincidences.

## 1. Introduction

Self-affine tiling dynamical system in $\mathbb{R}^d$ is a generalization of *substitution* dynamical system on letters, which gives a nice model of self-inducing structures appear in dynamical systems, number theory and the mathematics of aperiodic order. Pure discreteness of self-affine tiling dynamics is long studied from many points of views. The idea of *coincidence*[1] appeared firstly in Kamae [8], and then in a comprehensive form in Dekking [6] for constant length substitution (see also [16]). Generalizing a pioneer work of Rauzy [17], Arnoux-Ito [3] gave a geometric realization of irreducible Pisot unit substitution of degree $d$. They defined *strong coincidence*, which ensures that their geometric substitution gives rise to a domain exchange of $\mathbb{R}^{d-1}$, which is also semi-conjugate to the toral rotation of $\mathbb{T}^d$. It is remarkable that in many cases, it is even conjugate to the total rotation, which immediately implies that the system in pure discrete (see [4, 7, 5, 18] for further developments). On the other hand, *overlap coincidence* introduced by Solomyak [19] is an equivalent condition for pure discreteness of a given self-affine tiling dynamical system. This is also described as a geometric/combinatorial condition which guarantees that the tiling and its translation by return vectors become exponentially close if we iteratively enlarge return vectors by substitution. Lee [11] showed deeper characterizations that overlap coincidence is equivalent to *algebraic coincidence*, and the fact that the corresponding point set is an inter-model set.

Until now, the relation between strong coincidence and overlap coincidence is not fully understood. Motivated by the claim of Nakaishi [15], Akiyama-Lee [2] generalized the notion of strong coincidence to $\mathbb{R}^d$ and showed that overlap coincidence implies strong coincidence, and moreover *simultaneous coincidence*, provided that the associated point set is admissible and its height group is trivial. In this paper we shall give a converse statement for the suspension tiling of Pisot substitution at the expense of assuming many strong coincidences at a time, that is, strong coincidences on a certain many choices of control points imply overlap coincidence and vice versa. If every tile is connected and the tiling is not a collection of unbounded connected identical colored patches, then the same result holds for $d \geq 2$ (Theorem 3.2). This result elucidates the tight relationship between two coincidences.

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[1] In their notation, the *column number* is one.

## 2. Terminologies

2.1. **Tiles and tilings.** We shall briefly recall basic definitions used in this paper. A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A$ is a compact set in $\mathbb{R}^d$ which is the closure of its interior, and $i = \ell(T) \in \{1, \dots, m\}$ is the *color* of $T$. We call $A$ the *support* of $T$ and denote $\mathrm{supp}(T) = A$. The *translate* of $T$ is defined by $g + T = (g + A, i)$ for $g \in \mathbb{R}^d$. Let $\mathcal{A} = \{T_1, \dots, T_m\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles*. A tiling $\mathcal{T}$ is a collection of translates of prototiles which covers $\mathbb{R}^d$ without interior overlaps. A finite collection of tiles which appear in $\mathcal{T}$ is called a *patch*. A *generalized patch* is a collection of tiles in $\mathcal{T}$ whose cardinality is not necessarily finite. Its support is defined to be the union of the supports of tiles. The diameter of a generalized patch is the supremum of Euclidean distance of two points lie within the support of the patch. A map $\Omega$ from $\mathcal{A}$ to the set of patches is called a *substitution* with a $d \times d$ expansive matrix $Q$ if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le m$ such that

$$\Omega(T_j) = \{T_i + u : \ u \in \mathcal{D}_{ij}, \ i = 1, \dots, m\} \tag{2.1}$$

with

$$QA_j = \bigcup_{i=1}^m (A_i + \mathcal{D}_{ij}) = \bigcup_{i=1}^m \bigcup_{u \in \mathcal{D}_{ij}} (A_i + u) \quad \text{for } j \le m, \tag{2.2}$$

and the last union has mutually disjoint interiors. The substitution (2.1) extends to all translates of prototiles and patches in a natural way. A *substitution tiling* of $\Omega$ is a tiling $\mathcal{T}$ that all the patches of $\mathcal{T}$ is a sub-patch of $\Omega^n(T)$ for some $n \in \mathbb{N}$ and $T \in \mathcal{T}$. A substitution tiling $\mathcal{T}$ is a *fixed point* of $\Omega$ if $\Omega(\mathcal{T}) = \mathcal{T}$ holds. We say that a substitution tiling is *primitive* if the corresponding substitution matrix $M = (\sharp \mathcal{D}_{ij})$ is primitive, and *irreducible* if the characteristic polynomial of $M$ is irreducible. We say that $\mathcal{T}$ has *finite local complexity* (FLC) if for any $R$ there are only finitely many patches of diameter less than $R$ up to translation. A tiling $\mathcal{T}$ is *repetitive* if every patch is relatively dense in $\mathcal{T}$. A FLC substitution tiling of a primitive substitution is called a *self-affine tiling*. Every self-affine tiling is repetitive, which follows from the primitivity of substitution. Let $\lambda > 1$ be the Perron-Frobenius eigenvalue of the substitution matrix $M$ and $D$ be the set of eigenvalues of $Q$. By the tiling criterion of Lagarias-Wang [9], $\lambda$ is the element of $D$ of maximum modulus. We say that $Q$ fulfills *Pisot family condition* if every algebraic conjugate $\mu$ of an element of $D$ with $|\mu| \ge 1$ is contained in $D$.

The set of all substitution tilings of $\Omega$ forms a *tiling space*. By using a fixed point $\mathcal{T}$ of $\Omega$, we can describe this space as the orbit closure of $\mathcal{T}$ under the translation action: $X_{\mathcal{T}} = \overline{\{\mathcal{T} - g : g \in \mathbb{R}^d\}}$, the closure is taken by 'local topology'. The FLC assumption implies $X_{\mathcal{T}}$ is compact and we get a topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ where $\mathbb{R}^d$ acts by translations. This system is minimal and uniquely ergodic ([19, 12]), and we are interested in the spectra of self-affine tiling dynamical systems. Tiling dynamical system $X_{\mathcal{T}}$ has *pure discrete spectrum* if the eigenfunctions for the $\mathbb{R}^d$-action forms a complete orthonormal basis of $L^2(X_{\mathcal{T}}, \mu)$ [19].

2.2. **Control points.** A Delone set is a relatively dense and uniformly discrete subset of $\mathbb{R}^d$. We say that $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is a *Delone multi-color set* in $\mathbb{R}^d$ if each $\Lambda_i$ is Delone and $\cup_{i=1}^m \Lambda_i \subset \mathbb{R}^d$ is Delone. We say that $\Lambda \subset \mathbb{R}^d$ is a *Meyer set* if it is a Delone set and $\Lambda - \Lambda$ is uniformly discrete in $\mathbb{R}^d$ [10]. $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is called a *substitution Delone multi-color set* if $\mathbf{\Lambda}$ is a Delone multi-color set and there exist an expansive matrix $Q$ and finite sets $\mathcal{D}_{ij}$ for $i, j \le m$ such that

$$\Lambda_i = \bigcup_{j=1}^{m} (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m, \tag{2.3}$$

where the union on the right side is disjoint.

Given a fixed point $\mathcal{T}$ of $\Omega$, we can associate a substitution Delone multi-color set $\mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \le m}$ of $\mathcal{T}$ by taking representative points of tiles in the relatively same positions for the same color tiles in the tiling. There is a canonical way to choose representative points, called *control points*. A tile map $\gamma = \gamma_\Omega$ is a map from $\mathcal{T}$ to itself which sends a tile $T$ to the one in $\Omega(T)$ such that $\gamma(T_1)$ and $\gamma(T_2)$ are located in the same relative position in $\Omega(T_1)$ and $\Omega(T_2)$ whenever $\ell(T_1) = \ell(T_2)$. A control point $c(T)$ of $T \in \mathcal{T}$ is defined by

$$c(T) = \bigcap_{n=1}^{\infty} Q^{-n}(\gamma^n(T)).$$

Control points are representative points, i.e., $U - c(U) = V - c(V)$ holds if $\ell(U) = \ell(V)$ with $U, V \in \mathcal{T}$. Let $\Lambda_i$ be the set of control points of color $i$. Clearly, $j = \ell(\gamma(T_i))$ implies $Q\Lambda_i \subset \Lambda_j$ and the set of control points $\mathcal{C} = \bigcup_{i=1}^{m} \Lambda_i$ is invariant under the expansion by $Q$, that is, $Q\mathcal{C} \subset \mathcal{C}$. We obtain an associated substitution Delone multi-color set $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \le m}$.

In section 3, we have to assume a lot of strong coincidences by changing control points for a given tiling $\mathcal{T}$. If we change control points of tiles of $\mathcal{T}$ by $\Lambda'_i = \Lambda_i - g_i$, then the set equation will be shifted like

$$\Lambda'_i = \bigcup_{j=1}^{m} Q\Lambda'_j + \mathcal{D}'_{ij}$$

with $\mathcal{D}'_{ij} = \{d_{ij} + Qg_j - g_i \;:\; 1 \le i, j \le m\}$. The corresponding tile equation becomes

$$QA'_j = \bigcup A'_i + \mathcal{D}'_{ij}$$

which is satisfied by $A'_j = A_j + g_j$. So we set $\text{supp}(T'_j) = A'_j$ and $\ell(T'_j) = \ell(T_j)$. To avoid heavy notation, we do not distinguish such changes of control points and use the same symbols $\Lambda_i$ and $T_i$.

2.3. **Coincidences.** The set of *return vectors* is defined by $\Xi(\mathcal{T}) = \{y \in \mathbb{R}^d \;:\; U = V - y, \text{ where } U, V \in \mathcal{T}\}$. A triple $(U, y, V)$, with $U, V \in \mathcal{T}$ and $y \in \Xi(\mathcal{T})$, is called an *overlap* if

$$(\text{supp}(U))^\circ \cap (\text{supp}(V) - y)^\circ \ne \emptyset.$$

An overlap $(U, y, V)$ is a *coincidence* if $U = V - y$. Let $\mathcal{O} = (U, y, V)$ be an overlap in $\mathcal{T}$, we define *$\ell$-th inflated overlap*

$$\Omega^\ell \mathcal{O} = \{(U', Q^\ell y, V') : U' \in \Omega^\ell(U), V' \in \Omega^\ell(V), \text{ and } (U', Q^\ell y, V') \text{ is an overlap}\}.$$

We say that a self-affine tiling $\mathcal{T}$ admits *overlap coincidence* if there exists $\ell \in \mathbb{Z}_+$ such that for each overlap $\mathcal{O}$ in $\mathcal{T}$, $\Omega^\ell \mathcal{O}$ contains a coincidence. Two overlaps $(U, y, V)$ and $(U_1, y_1, V_1)$ are equivalent, if there is $x \in \mathbb{R}^d$ that both $U_1 = U - x$ and $V_1 - y_1 = V - y - x$ hold. The equivalence class is denoted by $\widetilde{(U, y, V)}$. Hereafter we assume an important condition that $\Xi(\mathcal{T})$ forms a Meyer set. This condition is equivalent to the Pisot family condition for $Q$, if $Q$ is diagonalizable and all its eigenvalues are algebraic conjugate with the same multiplicity [13]. The number of equivalence classes of overlaps is finite, by the Meyer property of $\Xi(\mathcal{T})$. The action of $\Omega$ is well-defined on equivalence classes of overlaps. An *overlap graph with multiplicity* is a finite directed graph whose vertices are the equivalence classes of overlaps. Multiplicities of the edge from $\widetilde{(U, y, V)}$ to $\widetilde{(A, z, B)}$ is given by the number of overlaps in $\Omega((U,y,V))$ equivalent to $(A,z,B)$ (c.f. [1]). Overlap coincidence is confirmed by checking whether from each vertex of this graph there is a path leading to a coincidence. Overlap coincidence is equivalent to pure discreteness of self-affine tiling dynamical system $X_{\mathcal{T}}$ [19].

Strong coincidence on letter substitution is naturally generalized to self-similar tiling in $\mathbb{R}^d$ in [2]. We adapt this definition to control points. Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ and $\mathcal{A}=\{T_1,\cdots,T_m\}$ be the prototile set of $\mathcal{T}$. We say that the set of the control points is *admissible* if $\cap_{i\le m}(\mathrm{supp}(T_i)-c(T_i))$ has non-empty interior.

Let $\mathcal{T}$ be the fixed point of $\Omega$. Let $c(T_i)$ $(i=1,\dots,m)$ be the admissible control points and $\mathbf{\Lambda}$ be an associated substitution Delone multi-color set for which $\mathcal{T}=\{T_i-c(T_i)+u_i \mid u_i\in\Lambda_i, i\le m\}$. If for any $1\le i,j\le m$, there is a positive integer $L$ that

$$\Omega^L(T_i-c(T_i))\cap\Omega^L(T_j-c(T_j))\neq\emptyset, \tag{2.4}$$

then we say that $\mathbf{\Lambda}$ admits *strong coincidence*. In other words, strong coincidence means that for every pair of tiles $(U,V)\in\mathcal{T}^2$, $\Omega^L(U-c(U))$ and $\Omega^L(V-c(V))$ share a common tile in the same position for some $L$.

## 3. Strong coincidence and Overlap coincidence

The set of *eventually return vectors* is defined by

$$\mathcal{G}:=\bigcup_{k=0}^{\infty}Q^{-k}(\Lambda_i-\Lambda_i),\quad \text{for some } i\le m$$

which is independent of the choice of $i$, by primitivity of $\Omega$. The tiling dynamical system is invariant under replacement of the substitution rule $\Omega$ by $\Omega^n$. We consider control points of $\Omega^n$ as well. Hereafter we put $\Lambda=\bigcup_{i=1}^m\Lambda_i$ for $\mathbf{\Lambda}=\mathbf{\Lambda}_{\mathcal{T}}=(\Lambda_i)$ to distinguish the multi-color set and its union. Let $\langle\mathcal{G}\rangle$ be the additive subgroup of $\mathbb{R}^d$ generated by $\mathcal{G}$. We say that $\mathcal{T}$ satisfies *multiple strong coincidence of level* $n$ if all multi-color Delone set $\mathbf{\Lambda}$'s generated by admissible control points of $\Omega^n$ with $\Lambda-\Lambda\subset\langle\mathcal{G}\rangle$ admit strong coincidence.

Hereafter when we speak about a topological/metrical property (connected, bounded, diameter) of a generalized patch, it refers to the corresponding property of its support. A *rod* is an unbounded connected generalized patch of $\mathcal{T}$ whose tiles have an identical color. A *rod tiling* is a tiling that every tile belongs to a rod. For ease of negation, a *non-rod* tiling is a tiling which is not a rod tiling. A tiling is called *non-periodic* if there are no non-trivial period, i.e., $\{p\in\mathbb{R}^d \mid \mathcal{T}+p=\mathcal{T}\}=\{0\}$.

**Remark 3.1.** There are many examples of periodic self-affine rod tiling. Consider a tiling of $\mathbb{R}^2$ by squares $[0,1]^2+(x,y)$ with $(x,y)\in\mathbb{Z}^2$ and their colors are defined by $y \pmod 2$ or $x+y \pmod 2$. However we do not know an example of non-periodic self-affine rod tiling.

**Theorem 3.2.** *Let $\mathcal{T}$ be a non-rod self affine tiling by connected tiles such that $\Xi(\mathcal{T})$ is a Meyer set. Then there is a constant $n$ depending only on $\mathcal{T}$ that $\mathcal{T}$ satisfies multiple strong coincidence of level $n$ if and only if $\mathcal{T}$ satisfies overlap coincidence.*

Consider a substitution $\sigma$ over $m$ letters $\{1,2,\dots,m\}$ whose substitution matrix is $M_\sigma=(|\sigma(j)|_i)$, where $|w|_i$ is the number of letter $i$ in a word $w$. We say that $\sigma$ is a *Pisot* substitution, if the Perron Frobenius root $\beta$ of $M_\sigma$ is a Pisot number. The canonical *suspension tiling* $\mathcal{T}$ in $\mathbb{R}$ of $\sigma$ with an expansion factor $\beta$ is defined by associating to the letters the intervals whose lengths are given by a left eigenvector of $M_\sigma$ corresponding to $\beta$.

**Corollary 3.3.** *The statement is valid for the suspension tiling of a Pisot substitution.*

Indeed, $1 \times 1$ matrix $Q = (\beta)$ satisfies Pisot family condition, tiles are intervals and the suspension tiling can not be a rod tiling, since it has at least two translationally inequivalent tiles in $\mathbb{R}$.

**Remark 3.4.** Multiple strong coincidence of level $n$ requires many strong coincidences at a time for a fixed tiling $\mathcal{T}$ even when $n = 1$. In dimension one, the claim of Nakaishi [15] reads a single strong coincidence implies overlap coincidence. Theorem 3.2 covers general cases but the requirement is much stronger. It would be interesting is to make smaller the constant $n$ in Theorem 3.2. For e.g., can we take $n = 1$ ?

We prepare a lemma.

**Lemma 3.5.** *Let $G$ be a strongly connected finite directed graph and $C$ be a set of cycles of $G$. Then there is a subgraph $G(C)$ of $G$ with the following property.*

- *The set of vertices of $G(C)$ is equal to that of $G$.*
- *Every vertex has exactly one outgoing edge.*
- *The set of cycles of $G(C)$ is equal to $C$.*

*Proof.* Put $H_0 = C$. We inductively construct $H_i$ for $i = 0, 1, \dots$ which satisfies:

- Every vertex has exactly one outgoing edge.
- The set of cycles of $H_i$ is equal to $C$.

Assume that the induced graph $G \setminus H_i$ is non empty and take a vertex $v$ from $G \setminus H_i$. Since $G$ is strongly connected, there is a path from $v$ leading to $H_i$. So there is a vertex $u \in G \setminus H_i$ and an edge from $u$ to a vertex of $H_i$. We define $H_{i+1}$ by adding this $u$ and the outgoing edge. Then $H_{i+1}$ clearly satisfies above two conditions. Since $G$ is finite, we find $m$ that $G \setminus H_m$ is empty, i.e., the set of vertices of $G$ and $H_m$ are the same. We finish the proof by taking $G(C) = H_m$. □

*Proof of Theorem 3.2.* Theorem 4.3 of [2] shows that overlap coincidence of $\mathcal{T}$ implies multiple strong coincidence of level $n$ for any $n \geq 1$. We prove that there is a constant $n$ such that multiple strong coincidence of level $n$ implies overlap coincidence.

Assume that $\mathcal{T}$ does not admit overlap coincidence. Construct the overlap graph $G$ of $\mathcal{T}$ with multiplicity. Since $\mathcal{T}$ does not admit overlap coincidence, there is a strongly connected component[2] $S$ of $G$ such that its spectral radius is equal to $|\det(Q)|$ and from each overlap of $S$ there is no path leading to a coincidence in $G$. Without loss of generality, we may assume that the incidence matrix of $S$ is primitive[3]. Thus we can find a positive integer $n_0$ such that for every overlap $(U, y, V)$, $\Omega^{n_0}(U, y, V)$ contains an overlap equivalent to $(U, y, V)$. Since $\Xi(\mathcal{T})$ is a Meyer set, number of equivalence classes of overlaps is finite and bounded by a constant which depends only on $\mathcal{T}$. Thus there is an upper bound of $n_0$ which depends only on $\mathcal{T}$. We further assume multiple strong coincidence of level $n = n_0$ on $\mathcal{T}$ and derive a contradiction.

We claim that in the component $S$ there is an overlap $(U, y, V)$ with $\ell(U) \neq \ell(V)$ for any non-rod self-affine tiling by connected tiles. Assume on the contrary that all overlaps in $S$ are of the

[2]In this assertion, one can take either usual overlaps or potential overlaps as we like.

[3]If the incidence matrix of $S$ is irreducible but not primitive, then take a suitable power of $\Omega$ by Perron-Frobenius theory.

form $(A, z, B)$ with $\ell(A) = \ell(B)$. Since $S$ does not contain a coincidence, $z \neq 0$ for these overlaps. Taking $k$-th inflated overlap of $(A, z, B)$, we obtain of patches $P$ and $Q$, both contain large balls, say $B_p(r)$ and $B_q(r)$, that the tiles of $P$ close to $p$ and the tiles of $Q$ close to $q$ are in multiple correspondence in the following sense. Putting $x = Q^k z$, for a tile $U \in P$ close to $p$ there are several (at least two) tiles $V \in Q$ that $(U, x, V)$ are overlaps in[4] $S$ and $\mathrm{supp}(U)$ is contained in the union of $\mathrm{supp}(V - x)$, and the same statements hold after interchanging the role of $U$ and $V$. Take a tile $U$ with $p \in \mathrm{supp}(U) \subset B_p(r)$. Then overlaps $(U, x, V)$ with $V \in Q$ give rise to a patch $\mathcal{V}_1 = \bigcup V$ that $\mathrm{supp}(U) \subsetneq \mathrm{supp}(\mathcal{V}_1) - x$. By assumption, $\ell(V_1) = \ell(U)$ for every $V_1 \in \mathcal{V}_1$. By using path connectedness of tiles[5], the patch $\mathcal{V}_1$ is path connected. If $\mathrm{supp}(V_1) \subset B_q(r)$, then there is a patch $\mathcal{U}_1 = \bigcup U_1$ where $U_1 \in P$ are taken from all overlaps of the form $(U_1, x, V_1)$ with some $V_1 \in \mathcal{V}_1$. This patch is also path connected and satisfies $\mathrm{supp}(\mathcal{V}_1) - x \subsetneq \mathrm{supp}(\mathcal{U}_1)$ and each tile of $\mathcal{U}_1$ has the same color as $U$. In this manner, by taking large $r$, we obtain a long sequence of path connected patches

$$\mathrm{supp}(U) \subsetneq \mathrm{supp}(\mathcal{V}_1 - x) \subsetneq \mathrm{supp}(\mathcal{U}_1) \subsetneq \mathrm{supp}(\mathcal{V}_2 - x) \subsetneq \mathrm{supp}(\mathcal{U}_2) \subsetneq \dots.$$

The number of tiles strictly increases and all tiles appear in this sequence has the same color $\ell(U)$. This shows for any $M > 0$, there exists a ball of radius $R$ that each tile $U$ in the ball belongs to a connected patch in $\mathcal{T}$ having diameter greater than $M$, whose tiles have an identical color $\ell(U)$. Therefore by using FLC, among $X_{\mathcal{T}}$ we can choose a rod tiling. Being a rod tiling is invariant under translation and closure operation, using minimality of $X_{\mathcal{T}}$ we see that every tiling in $X_{\mathcal{T}}$ is a rod tiling. This gives a contradiction, which finishes the proof of the claim.

Consider a directed graph $\mathcal{V}$ over $\{1, \dots, m\}$ whose edge $i \to j$ is given if there are $U, V \in S$ that $V \in \Omega^n(U)$ with $i = \ell(U)$ and $j = \ell(V)$. Clearly $\mathcal{V}$ is strongly connected as well. Pick one overlap $(U, y, V)$ from $S$ that $\ell(U) \neq \ell(V)$ and select one of the overlaps equivalent to $(U, y, V)$ in $\Omega^n(U, y, V)$. We select a tile map $\gamma = \gamma_{\Omega^n}$ which sends $\gamma(U)$ to this $U$ in $(U, y, V)$, and $\gamma(V)$ to the $V$ in $(U, y, V)$, which correspond to two cycles $\ell(U) \to \ell(U)$ and $\ell(V) \to \ell(V)$ on $\mathcal{V}$. Let $C$ be the set of these two cycles and take $\mathcal{V}(C)$ by Lemma 3.5. The tile map $\gamma = \gamma_{\Omega^n}$ is chosen so that $\ell(U) \to \ell(\gamma(U))$ for $U \in \{T_1, \dots, T_m\}$ forms the set of edges of $\mathcal{V}(C)$. By the choice of the subgraph, every path of length $m$ on this subgraph must fall into one of the two cycles. Note that by this choice of $\gamma$, the control points of $U$ and $V - y$ are exactly matching, because both of them are equal to a common point $\cap_{k=1}^{\infty} Q^{-nk} \left(\gamma^k(U) \cap \gamma^k(V - y)\right)$.

We claim that by this $\gamma$, we have $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$. In fact, since every overlap in the overlap graph is of the form $(A, z, B)$ with $z \in \bigcup_{i=1}^m (\Lambda_i - \Lambda_i)$, and control points of $U$ and $V - y$ are matching on $(U, y, V)$, i.e., $c(U) = c(V) - y$, we have $c(U) - c(V) \in \mathcal{G}$. By construction of $\mathcal{V}$ for any $x, y \in \Lambda$, we have $Q^m x, Q^m y \in \Lambda_{\ell(U)} \cup \Lambda_{\ell(V)}$. For instance, if $Q^m x \in \Lambda_{\ell(U)}$ and $Q^m y \in \Lambda_{\ell(V)}$, then $Q^m x = c(U) + f, Q^m y = c(V) + g$ hold with $f \in \Lambda_{\ell(U)} - \Lambda_{\ell(U)}, g \in \Lambda_{\ell(V)} - \Lambda_{\ell(V)}$. Therefore we have $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$.

We also see that the set of control points $\Lambda = (\Lambda_i)$ associated to $\gamma$ is admissible. In fact, since $(U, y, V)$ is an overlap, $\mathrm{supp}(U) \cap \mathrm{supp}(V - y)$ has an inner point. Since $y = c(V) - c(U)$, we have $(\mathrm{supp}(U - c(U)))^\circ \cap (\mathrm{supp}(V - c(V)))^\circ \neq \emptyset$. The admissibility follows from $Q^m x \in \Lambda_{\ell(U)} \cup \Lambda_{\ell(V)}$ for any $x \in \Lambda$.

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[4]We say that an overlap belongs to $S$ if its equivalence class does.

[5]Connectedness and path connectedness are equivalent for self-affine tiles [14].

Summing up, from $\widetilde{(U, y, V)} \in S$, we have chosen a tile map $\gamma_{\Omega^n}$ which produces a substitution Delone multi-color set of admissible control points with $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$. By the assumption of multiple strong coincidence of level $n$, we know $\Omega^k(U - c(U)) \cap \Omega^k(V - c(V))$ is non empty for some $k$, which shows that $(U, y, V)$ leads to a coincidence, giving a desired contradiction. $\square$

**Remark 3.6.** We use the assumptions that each tile is connected and $\mathcal{T}$ is a non-rod tiling only to show that there is an overlap $(U, y, V) \in S$ that $\ell(U) \neq \ell(V)$, which allows us to define a tile map. It is likely that these assumptions are not necessary, i.e., every non-periodic self-affine tiling that $\Xi(\mathcal{T})$ is a Meyer set, then such overlap must appear in $S$.

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