On Second Order Hyperbolic Equations with Coefficients Degenerating at Infinity and the Loss of Derivatives and Decays

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Abstract

In this paper , we study well-posedness issues in the weighted L^2 space for the Cauchy problem for the wave equation of the form $\partial_t^2 u - a(t,x)\partial_x^2 u = 0$. We give the condition $a(t,x) > 0$ for all $(t,x) \in [0,T] \times \mathbf{R}_x$ which is between the strictly hyperbolic condition and weakly hyperbolic one, and allows the decaying coefficient $a(t,x)$ such that $\lim_{|x|\to\infty} a(t,x) = 0$ for all $t \in [0,T]$. Our concerns are the loss of derivatives and decays of the solutions.

Keywords Hyperbolic equations, Decaying coefficients, Weighted L^2 space.
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1 Introduction

We consider the Cauchy problem on $[0,T] \times \mathbf{R}_x$

$$\begin{cases} \partial_t^2 u - a(t,x)\partial_x^2 u = 0, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.1}$$

where the real coefficient $a(t,x)$ satisfies

$$a(t,x) > 0 \ \text{ for all } \ (t,x) \in [0,T] \times \mathbf{R}_x. \tag{1.2}$$

A standard hyperbolic condition is usually given as follows:

$$a(t,x) \geq c_T \ \text{ for all } \ (t,x) \in [0,T] \times \mathbf{R}_x. \tag{1.3}$$

For the cases $c_T > 0$ and $c_T = 0$, we call *strictly hyperbolic* and *weakly hyperbolic* respectively. So, the condition (1.2) is between the strictly hyperbolic condition and weakly hyperbolic one, and allows the decaying coefficient $a(t,x)$ such that $\lim_{|x|\to\infty} a(t,x) = 0$ for all $t \in [0,T]$ like an integrable function on $\mathbf{R}_x$. Then we shall assume that

$$a(t,x) \in C^1\Big([0,T] \times C^2(\mathbf{R}_x)\Big). \tag{1.4}$$

In the strictly hyperbolic case, more relaxed assumptions can be considered (see [2], [7]). But, the assumption (1.4) is not enough to show the well-posedness in the weakly hyperbolic case. Therefore, we also assume that

$$|\partial_t a(t,x)| \le M(t)a(t,x) \ \text{ with } \ M(t) \in L^1(0,T) \text{ for } (t,x) \in [0,T]\times\mathbf{R}_x. \tag{1.5}$$

Remark 1.1 *The assumption (1.5) under the (standard) weakly hyperbolic condition (1.3) with $c_T = 0$ would be strong (see [10]), but the assumption (1.5) (under the condition (1.2)) is not so restrictive. Indeed, when the coefficient has the form $a(t,x) = a_1(t)a_2(x)$ (including the case independent of t), thanks to (1.2) there exists a constant $M(\equiv M(t))$ such that*

$$|\partial_t a(t,x)| = |\partial_t a_1(t)|a_2(x) \le Ca_2(x) \le Ma_1(t)a_2(x) = Ma(t,x),$$

here we used the fact that $a_1(t)$ must satisfies $a_1(t) \ge c_T > 0$ on the compact set $[0,T]$ (since $a_1(t_0) = 0$ for some $t_0 \in [0,T]$ contradicts (1.2)).

The assumption (1.5) plays a similar role as the regularity assumption C^k under the (standard) weakly hyperbolic condition (1.3) with $c_T = 0$ (see [5], [6], [8], [9], [12], etc.). Actually, defining the energy

$$E(t)^2 := \|\partial_t u\|^2_{L^2(\mathbf{R}_x)} + \left\|a(t,x)^{1/2}\partial_x u\right\|^2_{L^2(\mathbf{R}_x)} + \delta\|u\|^2_{L^2(\mathbf{R}_x)}, \tag{1.6}$$

a standard argument gives the energy inequality $E(t)^2 \le E(0)^2$ which can be translated into the following estimate of the solution:

$$\|\partial_t u\|^2_{L^2(\mathbf{R}_x)} + \delta\|u\|^2_{L^2(\mathbf{R}_x)} \le E(t)^2 \le E(0)^2 \le \|u_1\|^2_{L^2(\mathbf{R}_x)} + (A+\delta)\|u_0\|^2_{H^1(\mathbf{R}_x)},$$

where $A := \sup_{(t,x)\in[0,T]\times\mathbf{R}_x} a(t,x)$. The usual loss of derivatives means the decreasing weight functions in the Fourier space (see [11], etc.). Though this estimate does not include a weight function, it can be regarded as the second type of loss of derivatives, since $\|u_0\|^2_{H^1(\mathbf{R}_x)}$ in the righthand side corresponds to not $\|u\|^2_{H^1(\mathbf{R}_x)}$ but $\|u\|^2_{L^2(\mathbf{R}_x)}$ in the lefthand side. In this paper we shall propose a new kind of the energy to avoid such a loss of derivatives.

Remark 1.2 *If we suppose that the initial data have (fixed) compact supports, from the finite propagation property, we may use just the energy* $E(t)^2$ *with* $\delta = 0$ *under the strictly hyperbolic condition (1.3) with* $c_T > 0$*. Then, usual loss of derivatives and the second type of loss of derivatives never occur for the strictly hyperbolic case, since there exists* $c'_T > 0$ *such that*

$$\|\partial_t u\|^2_{L^2(\mathbf{R}_x)} + c'_T \|u\|^2_{H^1(\mathbf{R}_x)} \le E(t)^2 \le E(0)^2 \le \|u_1\|^2_{L^2(\mathbf{R}_x)} + A\|u_0\|^2_{H^1(\mathbf{R}_x)}.$$

We have to investigate how the supports of the initial data can be enlarged. So, our strategy is to consider the weight functions which control the solution degenerating at infinity. Now, let us consider the following example:
Example Let $\chi_R \in C^\infty(\mathbf{R})$ be a cut off function such that $0 \le \chi_R(x) \le 1$ and $\chi_R(x) = 0$ for $x \le 0$, $= 1$ for $x \ge R(> 0)$ and the coefficient independent of t

$$a_R(t,x) = \Big\{1 - \chi_R(|x| - 2)\Big\} \cdot 1 + \chi_R(|x| - 2) \cdot (x^2 - 1)^{-1}.$$

Put $T = 1$, $u_0(x) = u_1(x) = e^{-x^2/2}$ and $a(t,x) = a_1(t,x)$ (with $R = 1$) satisfying (1.2), (1.4), (1.5), that is, the Cauchy problem on $[0,1] \times \mathbf{R}_x$

$$\begin{cases} \partial_t^2 u - a_1(t,x)\partial_x^2 u = 0, \\ u(0,x) = e^{-x^2/2},\ \partial_t u(0,x) = e^{-x^2/2}. \end{cases} \tag{1.7}$$

Then, we find the following (i) and (ii):

(i) The coefficient $a_1(t,x)$ satisfies $A_1 := \sup_{(t,x)\in[0,1]\times\mathbf{R}_x} a_1(t,x) \le 1$. More precisely, $a_1(t,x)$ has a polynomial decay such as

$$c\langle x\rangle^{-2} \le a_1(t,x) \le C\langle x\rangle^{-2}\ (\le 1) \quad \text{for all} \quad (t,x) \in [0,1] \times \mathbf{R}_x.$$

(ii) For the initial data u_0, u_1 having a super-exponential decay, there exists a unique solution u having a super-exponential decay such that the loss of derivatives does not occur.

(i) is trivial. In order to show (ii), we prepare the auxiliary Cauchy problems $(CP1)$, $(CP2)$ and $(CP3)$.

$(CP1)$: $u^{(1)}(t,x)$ and $u_0^{(1)}(x) = u_1^{(1)}(x) = \{1 - \chi(|x| - 2)\}e^{-x^2/2}$,
$(CP2)$: $u^{(2)}(t,x)$ and $u_0^{(2)}(x) = u_1^{(2)}(x) = \chi(|x| - 2)e^{-x^2/2}$,
$(CP3)$: $u^{(3)}(t,x)$ and $u_0^{(3)}(x) = u_1^{(3)}(x) = \chi(|x| - 2)\{1 - \chi(|x| - 5)\}e^{-x^2/2}$.

We immediately see that superposition principle gives the solution of (1.7) $u(t,x) \equiv u^{(1)}(t,x) + u^{(2)}(t,x)$. By Remark 1.2 we obtain the solutions $u^{(1)}(t,x)$ and $u^{(3)}(t,x)$ of $(CP1)$ and $(CP3)$ such that the loss of derivatives does not occur. So, our concern is the loss of derivatives of $u^{(2)}(t,x)$. By uniqueness and finite propagation speed, we know that

$$u^{(2)}(t,x) \equiv \begin{cases} e^{t-x^2/2} \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad |x| \geq 2 + R + \sqrt{A_1}, \\ u^{(3)}(t,x) \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad |x| \leq 5 - \sqrt{A_1}, \end{cases}$$

noting that $u_0^{(2)}(x) \equiv u_0^{(3)}(x)$ for all $x \in [-5,5]$. Since $R = 1$ and $A_1 \leq 1$, $u^{(2)}(t,x)$ is determined for all $(t,x) \in [0,1] \times \mathbf{R}_x$. Thus, we find that $u^{(2)}(t,x)$ has a super-exponential decay such that the loss of derivatives does not occur. This also means (ii) for $u(t,x)$.

From the above example, the following questions arise when $a(t,x)$ has a polynomial decay and the initial data u_0, u_1 have a (sub- or super-)exponential decay:

Question 1: Would not the (second type of) loss of derivatives for the solution of the Cauchy problem (1.1) under (1.2) occur in general?
Question 2: Can we expect that the solution keeps to have a (sub- or super-) exponential decay same as the initial data?

We shall give answers for these questions in the next section.

2 Energy Estimate

We may suppose that $A := \sup_{(t,x)\in[0,T]\times\mathbf{R}_x} a(t,x) \leq 1$ by considering $u(t,x) = U(t, x/\sqrt{A})$ without loss of generality. Now we define the energy

$$E(t)^2 := \left\| e^{\rho(t)a(t,x)^{-\gamma}} \partial_t u \right\|_{L^2}^2 + \left\| a(t,x)^{1/2} e^{\rho(t)a(t,x)^{-\gamma}} \partial_x u \right\|_{L^2}^2 + \delta \left\| e^{\rho(t)a(t,x)^{-\gamma}} u \right\|_{L^2}^2, \tag{2.1}$$

where $\gamma > 0$ and $\rho(t)$ is a differentiable function such that $\rho'(t) < 0$.

Remark 2.1 *Under our condition (1.2), we remark that* $a(t,x)^{1/2} e^{\rho(t)a(t,x)^{-\gamma}} \geq c_\varepsilon e^{(\rho(t)-\varepsilon)a(t,x)^{-\gamma}}$. *Then, we get*

$$\left\| a(t,x)^{1/2} e^{\rho(t)a(t,x)^{-\gamma}} \partial_x u \right\|_{L^2}^2 \geq c_\varepsilon \left\| e^{(\rho(t)-\varepsilon)a(t,x)^{-\gamma}} \partial_x u \right\|_{L^2}^2, \tag{2.2}$$

even when $\lim_{|x|\to\infty} a(t,x) = 0$. *Consequently, this enables us to avoid a second type of loss of derivatives. In the below proof our task is to check carefully whether the use of the weight* $e^{\rho(t)a(t,x)^{-\gamma}}$ *is suitable for the energy computations.*

By differentiating $E(t)^2$ in t, we have

$$
\begin{aligned}
& \partial_t\{E(t)^2\} \\
= \; & \partial_t\Big\{\Big(e^{\rho a^{-\gamma}}u_t,\ e^{\rho a^{-\gamma}}u_t\Big) + \Big(ae^{\rho a^{-\gamma}}u_x,\ e^{\rho a^{-\gamma}}u_x\Big) + \delta\Big(e^{\rho a^{-\gamma}}u,\ e^{\rho a^{-\gamma}}u\Big)\Big\} \\
\leq \; & 2\rho'\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + 2\gamma|\rho|\Big\||a_t|^{1/2}a^{-(\gamma+1)/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 \\
& +2\Re\Big(e^{\rho a^{-\gamma}}u_{tt},\ e^{\rho a^{-\gamma}}u_t\Big) \\
& +2\rho'\Big\|a^{(1-\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + 2\gamma|\rho|\Big\||a_t|^{1/2}a^{-\gamma/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 \\
& +\Big\||a_t|^{1/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + 2\Re\Big(ae^{\rho a^{-\gamma}}u_{tx},\ e^{\rho a^{-\gamma}}u_x\Big) \\
& +2\delta\rho'\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u\Big\|^2 + 2\delta\gamma|\rho|\Big\||a_t|^{1/2}a^{-(\gamma+1)/2}e^{\rho a^{-\gamma}}u\Big\|^2 \\
& +2\delta\Re\Big(e^{\rho a^{-\gamma}}u_t,\ e^{\rho a^{-\gamma}}u\Big) \\
= \; & 2\rho'\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + 2\gamma|\rho|\Big\||a_t|^{1/2}a^{-(\gamma+1)/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 \\
& +2\Re\Big(e^{\rho a^{-\gamma}}au_{xx},\ e^{\rho a^{-\gamma}}u_t\Big) \\
& +2\rho'\Big\|a^{(1-\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + 2\gamma|\rho|\Big\||a_t|^{1/2}a^{-\gamma/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 \\
& +\Big\||a_t|^{1/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 - 2\Re\Big(u_t,\ \partial_x ae^{2\rho a^{-\gamma}}u_x\Big) \\
& +2\delta\rho'\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u\Big\|^2 + 2\delta\gamma|\rho|\Big\||a_t|^{1/2}a^{-(\gamma+1)/2}e^{\rho a^{-\gamma}}u\Big\|^2 \\
& +2\delta\Re\Big(e^{\rho a^{-\gamma}}u_t,\ e^{\rho a^{-\gamma}}u\Big) \\
= \; & 2\rho'\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + 2\rho'\Big\|a^{(1-\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + 2\delta\rho'\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u\Big\|^2 \\
& +2\gamma|\rho|\Big\||a_t|^{1/2}a^{-(\gamma+1)/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + 2\delta\gamma|\rho|\Big\||a_t|^{1/2}a^{-(\gamma+1)/2}e^{\rho a^{-\gamma}}u\Big\|^2 \\
& +2\gamma|\rho|\Big\||a_t|^{1/2}a^{-\gamma/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + \Big\||a_t|^{1/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 \\
& -2\Re\Big(u_t,\ a_xe^{2\rho a^{-\gamma}}u_x\Big) + 4\gamma\rho\Re\Big(u_t,\ a^{-\gamma}a_xe^{2\rho a^{-\gamma}}u_x\Big) + 2\delta\Re\Big(e^{\rho a^{-\gamma}}u_t,\ e^{\rho a^{-\gamma}}u\Big) \\
=: \; & I + II + III + IV + V + VI + VII + VIII + IX + X.
\end{aligned}
$$

By (1.5) we also get

$$
IV + V \;\leq\; 2M\gamma|\rho|\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + 2M\delta\gamma|\rho|\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u\Big\|^2,
$$

$$
\begin{aligned}
VI &\leq 2M\gamma|\rho|\Big\|a^{(1-\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2, \\
VII &\leq M\Big\|a^{1/2}e^{\rho a^{-\gamma}}u_x\Big\|^2.
\end{aligned}
$$

By (1.4) from the Glaeser inequality with respect to x, i.e., $|\partial_x a(t,x)| \leq Ka(t,x)^{1/2}$ for $(t,x) \in \mathbf{R}^2$, we get

$$
\begin{aligned}
VIII &= -2\Re\Big(K^{1/2}a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t,\ K^{-1/2}a^{\gamma/2}a_x e^{\rho a^{-\gamma}}u_x\Big) \\
&\leq K\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + K^{-1}\Big\|a^{\gamma/2}a_x e^{\rho a^{-\gamma}}u_x\Big\|^2 \\
&\leq K\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + K\Big\|a^{(1+\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2, \\
IX &= -4\gamma\rho\Re\Big(K^{1/2}a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t,\ K^{-1/2}a^{-\gamma/2}a_x e^{\rho a^{-\gamma}}u_x\Big) \\
&\leq 2K\gamma|\rho|\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + 2K^{-1}\gamma|\rho|\Big\|a^{-\gamma/2}a_x e^{\rho a^{-\gamma}}u_x\Big\|^2 \\
&\leq 2K\gamma|\rho|\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + 2K\gamma|\rho|\Big\|a^{(1-\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2, \\
X &= 2\delta\Re\Big(e^{\rho a^{-\gamma}}u_t,\ e^{\rho a^{-\gamma}}u\Big) \leq \delta\Big\|e^{\rho a^{-\gamma}}u_t\Big\|^2 + \delta\Big\|e^{\rho a^{-\gamma}}u\Big\|^2.
\end{aligned}
$$

Summing up these terms and taking $\delta = 1$, we have

$$
\begin{aligned}
&\partial_t\{E(t)^2\} \\
\leq\ & 2\Big\{\rho' + \gamma(M+K)|\rho|\Big\} \\
&\times\Big\{\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + \Big\|a^{(1-\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + \delta\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u\Big\|^2\Big\} \\
&+(M+K)\Big\{\Big\|a^{1/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + \Big\|a^{(1+\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2\Big\} \\
&+\delta\Big\{\Big\|e^{\rho a^{-\gamma}}u_t\Big\|^2 + \Big\|e^{\rho a^{-\gamma}}u\Big\|^2\Big\} \\
\leq\ & 2\Big\{\rho' + \gamma(M+K)|\rho| + \frac{1}{2}(M+K+1)\Big\} \\
&\times\Big\{\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2 + \Big\|a^{(1-\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + \Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u\Big\|^2\Big\}, \quad (2.3)
\end{aligned}
$$

here we used that $a^{1/2} \leq a^{(1-\gamma)/2}$, $a^{(1+\gamma)/2} \leq a^{(1-\gamma)/2}$ and $a^{-\gamma/2} \geq 1$ for $\gamma > 0$, thanks to $0 < a \leq 1$. Now we shall solve the ordinary differential equation

$$
\rho' + \gamma(M+K)|\rho| + \frac{1}{2}(M+K+1) = 0. \tag{2.4}
$$

When $\rho_0 > 0$, we shall put the one point $t = t_0 > 0$ such that

$$
\rho_0 = \frac{1}{2}\int_0^{t_0} e^{\gamma\int_0^s (M+K)d\tau}(M+K+1)ds
$$

and choose

$$\rho(t)=\begin{cases} e^{\gamma\int_0^t (M+K)d\tau}\Big\{\rho_0-\frac{1}{2}\int_0^t e^{-\gamma\int_0^s (M+K)d\tau}(M+K+1)ds\Big\} \\ \qquad\qquad\qquad\qquad \text{if}\ \ \rho_0\le 0\ \ \text{and}\ \ 0\le t\le T, \\ e^{-\gamma\int_0^t (M+K)d\tau}\Big\{\rho_0-\frac{1}{2}\int_0^t e^{\gamma\int_0^s (M+K)d\tau}(M+K+1)ds\Big\} \\ \qquad\qquad\qquad\qquad \text{if}\ \ \rho_0> 0\ \ \text{and}\ \ 0\le t\le t_0, \\ e^{\gamma\int_{t_0}^t (M+K)d\tau}\Big\{\rho(t_0)-\frac{1}{2}\int_{t_0}^t e^{-\gamma\int_{t_0}^s (M+K)d\tau}(M+K+1)ds\Big\} \\ \qquad\qquad\qquad\qquad \text{if}\ \ \rho_0> 0\ \ \text{and}\ \ t_0\le t\le T, \end{cases} \tag{2.5}$$

where we remark that $\rho(t_0)=0$. Then, we see that

$$E(t)^2\le E(0)^2.$$

Hence, it follows that for $\gamma>0$

$$\begin{aligned} &\left\|e^{\rho(t)a(t,x)^{-\gamma}}\partial_t u\right\|_{L^2}^2+\left\|a(t,x)^{1/2}e^{\rho(t)a(t,x)^{-\gamma}}\partial_x u\right\|_{L^2}^2+\left\|e^{\rho(t)a(t,x)^{-\gamma}}u\right\|_{L^2}^2 \\ &\le \left\|e^{\rho_0 a(0,x)^{-\gamma}}u_1\right\|_{L^2}^2+\left\|a(0,x)^{1/2}e^{\rho_0 a(0,x)^{-\gamma}}\partial_x u_0\right\|_{L^2}^2+\left\|e^{\rho_0 a(0,x)^{-\gamma}}u_0\right\|_{L^2}^2. \end{aligned} \tag{2.6}$$

Thus, noting (2.2), we can avoid the (second type of) loss of derivatives. This is an answer of the *Question 1* in §1. In particular, when there exist $c_T>0$, $C_T>0$ and $\alpha\ge 0$ such that

$$c_T\langle x\rangle^{-\alpha}\le a(t,x)\le C_T\langle x\rangle^{-\alpha}\ \ \text{for all}\ \ (t,x)\in[0,T]\times\mathbf{R}_x, \tag{2.7}$$

noting the signature of $\rho(t)$ and ρ_0, we have for $\gamma>0$

$$\begin{aligned} &\left\|e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}\langle x\rangle^{\alpha\gamma}}\partial_t u\right\|_{L^2}^2+c_T\left\|\langle x\rangle^{-\alpha/2}e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}\langle x\rangle^{\alpha\gamma}}\partial_x u\right\|_{L^2}^2 \\ &\qquad +\left\|e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}\langle x\rangle^{\alpha\gamma}}u\right\|_{L^2}^2 \\ &\le \left\|e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}\langle x\rangle^{\alpha\gamma}}u_1\right\|_{L^2}^2+C_T\left\|\langle x\rangle^{-\alpha/2}e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}\langle x\rangle^{\alpha\gamma}}\partial_x u_0\right\|_{L^2}^2 \\ &\qquad +\left\|e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}\langle x\rangle^{\alpha\gamma}}u_0\right\|_{L^2}^2. \end{aligned} \tag{2.8}$$

So, the solution u also has a (sub- or super-)exponential decay.

Remark 2.2 *In order to derive (2.8), we have not directly used the energy as* $E(t)^2:=\|e^{\rho(t)\langle x\rangle^\gamma}\partial_t u\|_{L^2}^2+\|a(t,x)^{1/2}e^{\rho(t)\langle x\rangle^\gamma}\partial_x u\|_{L^2}^2+\|e^{\rho(t)\langle x\rangle^\gamma}u\|_{L^2}^2$ *for the energy computations. This would make the energy computations complicated in our case. After getting the desired energy inequality with the energy (2.1), we have replaced* $a(t,x)^{-\gamma}$ *by the power of* $\langle x\rangle$ *in the exponent as the above.*

Thus, we conclude the following theorem:

Theorem 2.3 *Let ρ be defined by (2.5). Assume that $a(t,x)$ satisfies (1.2), (1.4) and (1.5). Then the Cauchy problem (1.1) has a unique solution u satisfying (2.6). In particular, if $a(t,x)$ has a polynomial decay as (2.7) and the initial data u_0, u_1 have (sub- or super-)exponential decay, the Cauchy problem (1.1) has a unique solution u having a (sub- or super-)exponential decay and satisfying (2.8).*

Remark 2.4 *It is not necessary that the decreasing function $\rho(t)$ is positive on $[0,T]$ for the given $T>0$. If we take a sufficiently large $\rho_0>0$ such that $\rho(t)>0$ for $t\in[0,T]$, the (second type of) loss of derivatives does not occur, since the weight functions in the 2nd terms in the left hand sides of (2.6) and (2.8) do not disappear.*

Remark 2.5 *In general, weakly hyperbolic equations are concerned with the study of the Levi condition ([3], [5], etc). If we consider the equation with the lower order terms, i.e.,*

$$\partial_t^2 u - a(t,x)\partial_x^2 u + b(t,x)\partial_x u + d(t,x)\partial_t u + c(t,x)u = 0,$$

we also need Levi conditions

$$|b(t,x)| \le M(t)a(t,x)^{1/2} \text{ with } M(t)\in L^1(0,T) \text{ for } (t,x)\in[0,T]\times\mathbf{R}_x, \quad (2.9)$$
$$|d(t,x)|+|c(t,x)| \le M(t)\in L^1(0,T) \quad \text{for } (t,x)\in[0,T]\times\mathbf{R}_x, \quad (2.10)$$

where $M(t)$ is arranged to be a positive function ($M(t)^{-1/2}$, $M(t)^{-1}$ exist). Indeed, we have the additional terms in the energy computations, which can be estimated as follows:

$$\begin{aligned}
-2\Re\Big(e^{\rho a^{-\gamma}}bu_x,\ e^{\rho a^{-\gamma}}u_t\Big) &= -2\Re\Big(M^{-1/2}a^{\gamma/2}be^{\rho a^{-\gamma}}u_x,\ M^{1/2}a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big)\\
&\le M^{-1}\Big\|a^{\gamma/2}be^{\rho a^{-\gamma}}u_x\Big\|^2 + M\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2\\
&\le M\Big\|a^{(1+\gamma)/2}e^{\rho a^{-\gamma}}u_x\Big\|^2 + M\Big\|a^{-\gamma/2}e^{\rho a^{-\gamma}}u_t\Big\|^2,\\
-2\Re\Big(e^{\rho a^{-\gamma}}du_t,\ e^{\rho a^{-\gamma}}u_t\Big) &\le 2M\Big\|e^{\rho a^{-\gamma}}u_t\Big\|^2,\\
-2\Re\Big(e^{\rho a^{-\gamma}}cu,\ e^{\rho a^{-\gamma}}u_t\Big) &\le M\Big\|e^{\rho a^{-\gamma}}u_x\Big\|^2 + M\Big\|e^{\rho a^{-\gamma}}u_t\Big\|^2. \qquad (2.11)
\end{aligned}$$

When $c(t,x)>0$, we may replace $\delta\|e^{\rho(t)a(t,x)^{-\gamma}}u\|_{L^2}^2$ by $\|c(t,x)^{1/2}e^{\rho(t)a(t,x)^{-\gamma}}u\|_{L^2}^2$ in the energy (2.1). Then the term (2.11) is canceled in the energy computations.

The energy (2.1) includes the parameter $\gamma > 0$. We can also consider the case corresponding to $\gamma = 0$ with the following energy instead of (2.1):

$$
\begin{aligned}
&E(t)^2 \\
&:= \left\| e^{\rho(t) \log a(t,x)^{-1}} \partial_t u \right\|_{L^2}^2 + \left\| a(t,x)^{1/2} e^{\rho(t) \log a(t,x)^{-1}} \partial_x u \right\|_{L^2}^2 + \delta \left\| e^{\rho(t) \log a(t,x)^{-1}} u \right\|_{L^2}^2 \\
&\Big(= \left\| a(t,x)^{-\rho(t)} \partial_t u \right\|_{L^2}^2 + \left\| a(t,x)^{1/2-\rho(t)} \partial_x u \right\|_{L^2}^2 + \delta \left\| a(t,x)^{-\rho(t)} u \right\|_{L^2}^2 \Big). \qquad (2.12)
\end{aligned}
$$

In the energy computations, we replace γ by 0 in the inside of the norms or the inner products (but replace γ by 1 in the outside of them). Similarly we have the estimate instead of (2.3)

$$
\begin{aligned}
\partial_t \{E(t)^2\} \quad \le \quad & 2\Big\{\rho' + (M+K)|\rho| + \frac{1}{2}(M+K+1)\Big\}\Big\{\left\| (\log a^{-1})^{1/2} e^{\rho \log a^{-1}} u_t \right\|^2 \\
& + \left\| (a \log a^{-1})^{1/2} e^{\rho \log a^{-1}} u_x \right\|^2 + \left\| (\log a^{-1})^{1/2} e^{\rho \log a^{-1}} u \right\|^2 \Big\}, \quad (2.13)
\end{aligned}
$$

and solve the ordinary differential equation

$$
\rho' + (M+K)|\rho| + \frac{1}{2}(M+K+1) = 0.
$$

When $\rho_0 > 0$, we shall put the one point $t = t_0 > 0$ such that

$$
\rho_0 = \frac{1}{2} \int_0^{t_0} e^{\int_0^s (M+K) d\tau} (M+K+1) ds
$$

and choose

$$
\rho(t) = \begin{cases}
e^{\int_0^t (M+K) ds} \Big\{ \rho_0 - \frac{1}{2} \int_0^t e^{-\int_0^s (M+K) d\tau} (M+K+1) ds \Big\} \\
\qquad \text{if} \quad \rho_0 \le 0 \quad \text{and} \quad 0 \le t \le T, \\
e^{-\int_0^{t_0} (M+K) ds} \Big\{ \rho_0 - \frac{1}{2} \int_0^t e^{\int_0^s (M+K) d\tau} (M+K+1) ds \Big\} \\
\qquad \text{if} \quad \rho_0 > 0 \quad \text{and} \quad 0 \le t \le t_0, \\
e^{\int_{t_0}^t (M+K) ds} \Big\{ \rho(t_0) - \frac{1}{2} \int_{t_0}^t e^{-\int_{t_0}^s (M+K) d\tau} (M+K+1) ds \Big\} \\
\qquad \text{if} \quad \rho_0 > 0 \quad \text{and} \quad t_0 \le t \le T,
\end{cases} \qquad (2.14)
$$

where we remark that $\rho(t_0) = 0$. Then, we see that $\partial_t \{E(t)^2\} \le 0$ and $E(t)^2 \le E(0)^2$. Hence, by it follows that

$$
\begin{aligned}
& \left\| a(t,x)^{-\rho(t)} \partial_t u \right\|_{L^2}^2 + \left\| a(t,x)^{1/2-\rho(t)} \partial_x u \right\|_{L^2}^2 + \left\| a(t,x)^{-\rho(t)} u \right\|_{L^2}^2 \\
& \le \left\| a(0,x)^{-\rho_0} u_1 \right\|_{L^2}^2 + \left\| a(0,x)^{1/2-\rho_0} \partial_x u_0 \right\|_{L^2}^2 + \left\| a(0,x)^{-\rho_0} u_0 \right\|_{L^2}^2, \qquad (2.15)
\end{aligned}
$$

and in particular under (2.7) with $c_T > 0$ and $C_T > 0$

$$
\begin{aligned}
&\Big\|\langle x\rangle^{\alpha\rho(t)}\partial_t u\Big\|_{L^2}^2 + c_T\Big\|\langle x\rangle^{\alpha\{\rho(t)-1/2\}}\partial_x u\Big\|_{L^2}^2 + \Big\|\langle x\rangle^{\alpha\rho(t)} u\Big\|_{L^2}^2 \\
&\le \Big\|\langle x\rangle^{\alpha\rho_0} u_1\Big\|_{L^2}^2 + C_T\Big\|\langle x\rangle^{\alpha\{\rho_0-1/2\}}\partial_x u_0\Big\|_{L^2}^2 + \Big\|\langle x\rangle^{\alpha\rho_0} u_0\Big\|_{L^2}^2. \qquad (2.16)
\end{aligned}
$$

Thus, we get the following:

Theorem 2.6 *Let ρ be defined by (2.14). Assume that $a(t,x)$ satisfies (1.2), (1.4) and (1.5). Then the Cauchy problem (1.1) has a unique solution u satisfying (2.15). In particular, if $a(t,x)$ has a polynomial decay as (2.7) and the initial data u_0, u_1 have polynomial decays, the Cauchy problem (1.1) has a unique solution u having a polynomial decay and satisfying (2.16).*

A result when the initial data having *less regularities* can be given, is better in the study of the wellposedness of Cauchy problems. So, one will also think that Theorem 2.6 is a better result than Theorem 2.3 in sense that the initial data having *less decays* can be given. Similarly as the technical term "finite loss of derivatives", we may call "finite loss of decays" for (2.16). The energy inequality (2.16) corresponding to $\gamma = 0$ with $\alpha = 2$ (and especially $\rho_0 = 1/2$) becomes

$$
\begin{aligned}
&\|\langle x\rangle^{2\rho(t)}\partial_t u\|_{L^2}^2 + c_T\|\langle x\rangle^{2\rho(t)-1}\partial_x u\|_{L^2}^2 + \|\langle x\rangle^{2\rho(t)} u\|_{L^2}^2 \\
&\qquad \le \|\langle x\rangle u_1\|_{L^2}^2 + C_T\|\partial_x u_0\|_{L^2}^2 + \|\langle x\rangle u_0\|_{L^2}^2. \qquad (2.17)
\end{aligned}
$$

But, if we consider the Cauchy problem (1.7), the energy inequality (2.17) does not show a positive answer for the *Question 2* in §1. On the other hand, the energy inequality (2.8) with $\gamma = 1$ and $\alpha = 2$ in Theorem 2.3 just becomes

$$
\begin{aligned}
&\Big\|e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}\langle x\rangle^2}\partial_t u\Big\|_{L^2}^2 + c_T\Big\|\langle x\rangle^{-1}e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}\langle x\rangle^2}\partial_x u\Big\|_{L^2}^2 \\
&\qquad + \Big\|e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}\langle x\rangle^2} u\Big\|_{L^2}^2 \\
&\le \Big\|e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}\langle x\rangle^2} u_1\Big\|_{L^2}^2 + C_T\Big\|\langle x\rangle^{-1}e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}\langle x\rangle^2}\partial_x u_0\Big\|_{L^2}^2 \\
&\qquad + \Big\|e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}\langle x\rangle^2} u_0\Big\|_{L^2}^2. \qquad (2.18)
\end{aligned}
$$

(2.18) means that the solution keeps to have a super-exponential decay same as the initial data (see the example in §1). This is an answer of the *Question 2*.

Remark 2.7 *If one considers the standard energy (1.6), that is (2.12) with* $\rho = 0$*, one would obtain the following instead of (2.13):*

$$\partial_t\{E(t)^2\} \le (M+K+1)\Big\{\|u_t\|^2 + \|a^{1/2}u_x\|^2 + \|u\|^2\Big\} = (M+K+1)E(t)^2,$$

and $E(t)^2 \le e^{\int_0^t (M+K+1)d\tau} E(0)^2$ *and*

$$\|\partial_t u\|^2 + \|a^{1/2}\partial_x u\|^2 + \|u\|^2 \le e^{\int_0^t (M+K+1)d\tau}\Big\{\|u_1\|^2 + \|a(0,x)^{1/2}\partial_x u_0\|^2 + \|u_0\|^2\Big\}.$$

Under the condition (1.2) the left hand side should be changed into $\|\partial_t u\|^2 + \|u\|^2$*. So, the second type of loss of derivatives occurs for the standard energy (1.6). From this inequality with the standard energy, we do not know any information about the loss of decay.*

Remark 2.8 *Thanks to the use of the loss of decays, we can avoid the second type of loss of derivatives for the decaying coefficient* $a(t,x)$*, but can not expect the time decay of solutions. [13] and [14] developed the theory of Fourier integral operators which help us to construct the parametrix, and gave weighted estimates for hyperbolic equations, where the decay in* x *is translated into the time decay of solutions.*

In the above, we have proved the global results with respect to x by the global condition (1.2). If we give compactly supported initial data, by the finite propagation property we may consider the problem locally with respect to x. Instead of (1.2), we shall assume

$$(A \ge) a(t,x) > 0 \quad \text{for all} \quad (t,x) \in [0,T] \times \mathbf{R}_x \backslash \{0\}, \tag{2.19}$$

which is regarded as a local condition around $x = 0$. Without changes of other conditions and the proof, we can also obtain the following:

Corollary 2.9 *Let* ρ *be defined by (2.5) (resp. (2.14)). Assume that* $a(t,x)$ *satisfies (1.4), (1.5) and (2.19). Then the Cauchy problem (1.1) has a unique solution* u *satisfying (2.6) (resp. (2.15)). In particular, if there exist* $c_T > 0$*,* $C_T > 0$ *and* $\alpha \ge 0$ *such that*

$$c_T|x|^\alpha \le a(t,x) \le C_T|x|^\alpha \quad \text{for all} \quad (t,x) \in [0,T] \times \mathbf{R}_x, \tag{2.20}$$

the Cauchy problem (1.1) has a unique solution u satisfying

$$
\begin{aligned}
&\left\|e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}|x|^{-\alpha\gamma}}\partial_t u\right\|_{L^2}^2 + c_T\left\||x|^{\alpha/2}e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}|x|^{-\alpha\gamma}}\partial_x u\right\|_{L^2}^2 \\
&\qquad\qquad + \left\|e^{\min\{\rho(t)c_T,\ \rho(t)C_T\}|x|^{-\alpha\gamma}}u\right\|_{L^2}^2 \\
&\le \left\|e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}|x|^{-\alpha\gamma}}u_1\right\|_{L^2}^2 + C_T\left\||x|^{\alpha/2}e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}|x|^{-\alpha\gamma}}\partial_x u_0\right\|_{L^2}^2 \\
&\qquad\qquad + \left\|e^{\max\{\rho_0 c_T,\ \rho_0 C_T\}|x|^{-\alpha\gamma}}u_0\right\|_{L^2}^2
\end{aligned}
$$

$$
\Big(\textit{resp.}\quad
\begin{aligned}
&\left\||x|^{-\alpha\rho(t)}\partial_t u\right\|_{L^2}^2 + c_T\left\||x|^{-\alpha\{\rho(t)-1/2\}}\partial_x u\right\|_{L^2}^2 + \left\||x|^{-\alpha\rho(t)}u\right\|_{L^2}^2 \\
&\le \left\||x|^{-\alpha\rho_0}u_1\right\|_{L^2}^2 + C_T\left\||x|^{-\alpha\{\rho_0-1/2\}}\partial_x u_0\right\|_{L^2}^2 + \left\||x|^{-\alpha\rho_0}u_0\right\|_{L^2}^2 \quad \Big).
\end{aligned}
$$

Remark 2.10 *In order that $a(t,x)$ belongs to C^2 with respect to x as (1.4), actually α should satisfy $\alpha \ge 2$.*

The last energy inequality corresponding to $\gamma = 0$ with $\alpha = 2$ (and especially $\rho_0 = 1/2$) becomes

$$
\begin{aligned}
&\||x|^{-2\rho(t)}\partial_t u\|_{L^2}^2 + c_T\||x|^{-2\rho(t)+1}\partial_x u\|_{L^2}^2 + \||x|^{-2\rho(t)}u\|_{L^2}^2 \\
&\qquad\qquad \le \||x|^{-1}u_1\|_{L^2}^2 + C_T\|\partial_x u_0\|_{L^2}^2 + \||x|^{-1}u_0\|_{L^2}^2.
\end{aligned}
$$

If the initial data u_0 and u_1 have degeneration of polynomial order at $x = 0$, the right hand side is finite. So, we find that the solution u must degenerate at $x = 0$ for the degenerating coefficient $a(t,x)$ (propagation speed 0 at $x = 0$).

Remark 2.11 *The case when the coefficient is just $a(t,x) \equiv x^2$, satisfies (2.20) with $c_T = C_T = 1$ and $\alpha = 2$, but not (2.19) with $A < \infty$. If we give compactly supported initial data, the finite propagation property allows the coefficient $a(t,x) \equiv x^2$ with some modifications to satisfy (2.19). Indeed, [1] shows the following representation formula of the exact solution u with $a(t,x) \equiv x^2$ and $u_0 \equiv 0$:*

$$
u(t,x) \;=\; t\int_{-1}^{1} \frac{u_1(xe^{yt})}{2e^{\frac{yt}{2}}} J_0\left(\frac{t\sqrt{1-y^2}}{2}\right) dy,
$$

where $J_0(x)$ is the Bessel function of order 0, i.e., $J_0(x) = \sum_{n=0}^{\infty}(-\frac{x^2}{4})^n/(n!)^2$. Hence, we really find that the solution u degenerates at $x = 0$ if $u_1(0) = 0$.

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