Stochastic Newton equation in strong potential limit

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Abstract

We consider a type of stochastic Newton equations, with single-well potential functions, and study the limiting behaviors of their solution processes when the coefficients of the potentials diverge to infinity. We prove that for dimension 1, the stochastic solution processes converge. The explicit descriptions of the limiting processes are also given. Especially, the limiting processes are deterministic for special initial conditions.

Keywords: stochastic Newton equation, diffusion, potential, convergence

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1 Introduction

We consider the motion of a particle with its position X_t^λ and velocity V_t^λ at time t given by the following stochastic differential equation:

$$\begin{cases} dX_t^\lambda = V_t^\lambda dt \\ dV_t^\lambda = -bV_t^\lambda dt - \lambda \nabla g(X_t^\lambda)dt + \sigma(X_t^\lambda)dB_t, \\ (X_0^\lambda, V_0^\lambda) = (X_0, V_0). \end{cases} \tag{1.1}$$

Considering the Hamiltonian $H(x, v) = \frac{1}{2}|v|^2 + \lambda g(x)$, (1.1) can be seen as a randomized and damped Hamiltonian system. $b > 0$ is the damping parameter, B is a one dimensional Brownian motion, and we assume that $\sigma \in C^\infty(\mathbf{R}, \mathbf{R})$ is bounded and positive uniformly. $\lambda \geq 1$ is a parameter (later on the limit $\lambda \to \infty$ will be taken), and the function g is a potential. In this paper, we consider the case where g is a single-well function (see below for details).

We are interested in the behavior of the particle described by (1.1) when $\lambda \to \infty$. This type of problem – add some perturbation given by Brownian motion to a Newton mechanical system, and consider the limit of the solution – has been studied by many authors, for example, Kesten-Papanicolaou [4] considered the limit of the distribution of the solution, when the force converges to 0, and Albeverio-Smii [1] considered the asymptotic expansion of the solution. However, in the literature, to the best knowledge of the author, there are not so many papers concerning with our problem of taking the potential to infinity. (Some references in this line will be given later in this section). Same as in the relation between [7] and [8], this problem, which is interesting in itself, is also closely related to the problem of "mechanical models of Brownian motions" with negative resulting-potentials between the massive particles and with the massive particles evolving

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classically (instead of relativistically, as in [8]). The limit $\lambda \to \infty$ corresponds to the fact that the mass of the environmental light particles converges to 0 in that model (see [6, (3.30)] and [8, (3.4)] for details).

In this paper, we consider the simplest case where the system has dimension $d = 1$, so $X_t^\lambda, V_t^\lambda \in \mathbf{R}$. Also, we concentrate ourselves to the case where $g \in C_0^\infty(\mathbf{R}; \mathbf{R})$ is a single well potential. Precisely, we assume that g satisfies the following conditions (see Figure 1 below).

(A1) There exist constants $r_3 > r_2 > r_1 > 0$ such that $g(x) = 0$ if $x \geq r_3$ or $x = r_1$, $g'(r_1) < 0$ and $g(x) > 0$ if $x < r_1$. Also, $g(x)$ is strictly monotone decreasing in $x \in (r_1, r_2)$ and is strictly monotone increasing in $x \in (r_2, r_3)$.

Also, we assume the following.

(A2) There exists a constant $a_0 > 0$ such that g' is monotone decreasing in $(r_3 - a_0, r_3)$ and $C_1 := \inf_{y \in [r_3 - a_0, r_3)} \frac{g'(y)}{|g(y)|} > 0$.

The single-well assumption of (A1) is necessary for our function $a(\cdot)$ (see (1.3) below for its definition) to be well-defined, which is necessary for the statement of our result. We also remark that by (A1), r_2 is the unique minimum point of g. The growth condition (A2) is a technical condition, and is used in this paper to estimate the sojourn time of the particle at its right-end when it is near to r_3 (see Remark 5 below). By considering the balance between its time spent for and its energy loss during each round-trip, this ensures that the particle stays in (r_1, r_3) as soon as it enters this interval (see the paragraph before (1.2) and Proposition 3.7 for details).

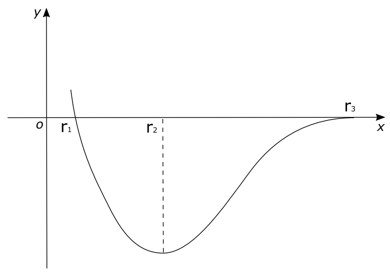

Figure 1: A sketch of the function g

The same question for the case with $d \geq 2$ (with g spherical symmetric) will be studied in a forthcoming paper, with the help of the results of this paper. Heuristically, as $\lambda \to \infty$, the potential force $-\lambda \nabla g$, which accelerates the particle, becomes stronger and stronger, so it is not so strange to expect that randomness vanishes in the limit. Our result of this paper shows that this is the case for dimension 1 under our assumption, and gives a precise description of the limiting process (see Theorem 1.1 below). However, it seems that the situation is different for $d \geq 2$: although $|X_t|$, the distance of the particle from the origin, has the same limit behavior (*i.e.*, oscillates in a certain subinterval of (r_1, r_3)), the direction of the particle keeps random – the strong potential force restrains only the distribution of the absolute value $|X_t|$, and cannot stick the direction of the particle.

Kusuoka [5] considered a similar question in $\mathbf{R}^d$, in the case where g is positive in an inner neighborhood of $\partial(\mathrm{supp}(g))$, the boundary of its support, with the initial position $X_0 \in \overline{\mathrm{supp}(g)}^C$, and got a limiting process given as a diffusion process reflecting at the boundary of $\mathrm{supp}(g)$. The idea is that, in this case, as soon as the particle arrives the boundary of $\mathrm{supp}(g)$ and attempts to enter this region, when $\lambda \to \infty$, the potential force $-\lambda \nabla g$ is so strong that it pushes the particle back to the region $\mathrm{supp}(g)$ in an instant. We notice that we have a totally different situation in this paper: in this paper, g is negative in an inner neighborhood of $\partial(\mathrm{supp}(g))$, so as soon as the particle enters $\mathrm{supp}(g)$, the potential force $-\lambda \nabla g$ accelerates the particle in an instant.

Also, [7] considered a similar question in $\mathbf{R}^d$ in the case where g has a negative region as same as in this paper, but with the particle evolving relativistically, precisely, [7] considered a randomized and damped Hamiltonian system with $H(p,q) = \sqrt{1+p^2} + \lambda g(q)$, or equivalently, the position Q_t^λ and the momentum P_t^λ of the particle at time t are given by $dQ_t^\lambda = \frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}} dt, dP_t^\lambda = -b\frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}} dt - \lambda \nabla g(Q_t^\lambda)dt + \sigma(Q_t^\lambda)dB_t, (Q_0^\lambda, P_0^\lambda) = (q_0, p_0)$. As explained, since g is negative in an inner neighborhood of $\partial(\mathrm{supp}(g))$, when $\lambda \to \infty$, one gets that $|P_t^\lambda| \to \infty$ in this region. However, since we are interested in the limit of $V_t^\lambda = \frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}}$ instead of P_t^λ, and $|V_t^\lambda|$ is always dominated by 1, it is still possible to consider the limit of $\{(Q_t, V_t); t \geq 0\}$. Indeed, by introducing several new stochastic processes, [7] proved that the evolution of the particle converges to a stochastic process with two phases, called diffusion phase and uniform motion phase (corresponding to $\{x : g(x) < 0\}$ and $(\mathrm{supp}(g))^C$, respectively). This is, again, not the case for our present model, since in our model, the velocity V_t^λ itself diverges to ∞.

In this paper, we consider the behavior of the particle described by (1.1) with $X_0 > r_3$ when $\lambda \to \infty$. Notice that until the first time τ_0 that the particle hits $X_t < r_3$, we have that $g = 0$, which means that the behavior of the particle does not depend on λ. Therefore, it is trivial that the distribution of $\left\{(X_{t\wedge\tau_0}^\lambda, V_{t\wedge\tau_0}^\lambda); t \geq 0\right\}$ is equal to, hence certainly converges to, the distribution of the τ_0-stopped diffusion process given by

$$\begin{cases} dX_t = V_t dt \\ dV_t = -bV_t dt + \sigma(X_t)dB_t, \\ (X_0, V_0) = (X_0, V_0). \end{cases}$$

So we could concentrate ourselves to the behavior of the particle after τ_0. *i.e.*, by time-shifting, we assume from now on that

$$X_0 = r_3, \qquad V_0 < 0.$$

In our present model, as claimed, the potential force $-\lambda \nabla g$ is an attractive force right after the particle enters $X_t < r_3$, which means that when $\lambda \to \infty$, V_t becomes infinity in an instant. So it is meaningless to consider the limit behavior of X_t^λ itself when $\lambda \to \infty$. Instead, we consider the limit behavior of $\left\{Y_t^{f,\lambda}; t \geq 0\right\}$ given by

$$Y_t^{f,\lambda} := \int_0^t f(X_s^\lambda)ds$$

for any $f \in C_b(\mathbf{R})$.

Choose and fix any $f \in C_b(\mathbf{R})$. Since f is bounded, it is easy to see (see Corollary 2.4) that $\left\{ \text{the distribution of } \left\{Y_t^{f,\lambda}; t \geq 0\right\}; \lambda \geq 1\right\}$ is tight in the meaning described below. However, it is not so easy to give the precise expression of the limiting process. We answer this problem in this paper.

Before formulating our result of this paper, let us first explain the situation heuristically. As claimed, for large λ, $|V_t^\lambda|$ is also very large as soon as the particle enters the range (r_1, r_3). So by the virtue of the damping force, the total energy (H_t^λ defined by (1.2) below) becomes negative. Therefore, in the limit $\lambda \to \infty$, we have with probability 1 that the particle could not leave from $[r_1, r_3]$, the closure of the domain where g is negative. Indeed, we will prove that it could not even reach the boundary r_1 and r_3 (see Proposition 3.7). On the other hand, the total energy of the particle also suggests that the limiting process does not stop at r_2, the unique minimum point of the potential function, either, in any finite time (see Remark 2 and Lemma 2.16 (1)). So in the limit, the particle keeps on oscillating in a certain range that is contained in (r_1, r_3), with its velocity very large. Therefore, the problem is to find the range of the particle in the limiting process around any given time. As we prove in this paper, this "range of the particle" depends on time.

In order to formulate our result, let us first define several notations. First, for any $\lambda > 0$, let

$$\begin{aligned} H_t^\lambda &:= \frac{1}{2}|V_t^\lambda|^2 + \lambda g(X_t^\lambda), \\ J_t^\lambda &:= \lambda^{-1} H_t^\lambda. \end{aligned} \tag{1.2}$$

As explained, the limit of J_t^λ as $\lambda \to \infty$ plays an important role as an index of "the range of the particle" at time t.

By assumption (A1), both $g\Big|_{[r_1,r_2)}$ and $g\Big|_{(r_2,r_3]}$ are strictly monotone with the same range $(g(r_2), 0]$, so both of them have inverse functions on $(g(r_2), 0]$. Write their inverse functions as $g^{-1,1}$ and $g^{-1,2}$, respectively. For any $f \in C_b(\mathbf{R})$, let

$$S_f(j) := S_f^g(j) := \sqrt{2}\int_{g^{-1,1}(j)}^{g^{-1,2}(j)} \frac{f(y)}{\sqrt{j - g(y)}} dy,$$

and let

$$A^g f(j) := \frac{S_f(j)}{S_1(j)}, \qquad j \in (g(r_2), 0).$$

Remark 1 *The intuitive meanings of $S_f(j)$ and $A^g f(j)$ are as follows: consider the case where $b = 0 = \sigma$ (i.e., the Hamiltonian system without randomizing or damping), then $\lambda^{-1/2} S_f(j)$ represents the line integral of the function f along the orbit $l_j : H(x,v) = \lambda j$ with respect to the Liouville measure of the system, and $\lambda^{-1/2} S_1(j)$ is the time period of the same orbit l_j. So $A^g f(j)$ is nothing but the mean value of f along the same orbit l_j.*

Let

$$a(j) = 2\sqrt{2}\int_{g^{-1,1}(j)}^{g^{-1,2}(j)} \sqrt{j - g(y)} dy, \qquad j \in (g(r_2), 0]. \tag{1.3}$$

Then $a(\cdot)$ is continuous on $j \in (g(r_2), 0]$, and

$$a'(j) = \sqrt{2}\int_{g^{-1,1}(j)}^{g^{-1,2}(j)} \frac{1}{\sqrt{j - g(y)}} dy = S_1(j) > 0, \quad j \in (g(r_2), 0).$$

So $a(\cdot)$ is bijective on $(g(r_2), 0]$. Write the inverse function of a as a^{-1}. Let

$$j_t = a^{-1}\Big(2\sqrt{2}e^{-bt}\int_{r_1}^{r_3}\sqrt{-g(y)}dy\Big), \tag{1.4}$$

and let

$$y_t^f = \int_0^t A^g f(j_s)ds, \qquad t \geq 0.$$

Finally, let $W = C([0,\infty); \mathbf{R}^2)$, and let

$$dist(w_1, w_2) = \sum_{n=1}^{\infty} 2^{-n}\Big(1 \wedge \Big[\max_{t\in[0,n]}|w_1(t) - w_2(t)|\Big]\Big), \qquad w_1, w_2 \in W.$$

Our main result of this paper is the following.

THEOREM 1.1 *Under the above assumptions, for any $f \in C_b^1(\mathbf{R})$, we have that when $\lambda \to \infty$, $\Big\{(J_t^\lambda, Y_t^{f,\lambda}); t \in [0,\infty)\Big\}$ converge to $\{(j_t, y_t^f); t \in [0,\infty)\}$ in probability in $(W, dist)$.*

Remark 2 *We remark the following observations with respect to the behavior of j_t:*

1. *For any $t \in (0,\infty)$, we have by the definition of j_t that $j_t \in (g(r_2), 0)$, so Theorem 1.1 implies that in the limit $\lambda \to \infty$, the particle keeps on oscillating in any finite time, with its range around time t given by (1.4).*
2. *We have that $\lim_{t\to\infty} j_t = a^{-1}(0) = g(r_2)$, the minimum value of g, so Theorem 1.1 implies that in the limit $t \to \infty$, the particle of the limiting process seems to concentrate around the minimum point of the potential – the damping is the stronger.*

Finally, let us notice several observations with respect to $a(\cdot)$ and j_t. First, by a simply calculation, we get that

$$a(0) = 2\sqrt{2}\int_{r_1}^{r_3}\sqrt{-g(y)}dy, \tag{1.5}$$

$$j - A^g g(j) = \frac{\int_{g^{-1,1}(j)}^{g^{-1,2}(j)}\sqrt{j - g(y)}dy}{\int_{g^{-1,1}(j)}^{g^{-1,2}(j)}\frac{1}{\sqrt{j-g(y)}}dy} = \frac{1}{2}\frac{a(j)}{a'(j)}, \quad j \in (g(r_2), 0). \tag{1.6}$$

In Remark 3, we use these to get an ordinary differential equation for j_t.

Remark 3 *Consider the following ordinary differential equation with respect to $\{j_t; t \geq 0\}$:*

$$\begin{cases} dj_t = -2b\Big(j_t - A^g g(j_t)\Big)dt, \\ j_t < 0 \qquad \text{for all } t > 0, \\ j_0 = 0. \end{cases} \tag{1.7}$$

We notice that j_t *defined by (1.4) is the unique solution of (1.7). Indeed, as long as* $j_t < 0$, *we have by (1.6) that* $\frac{d}{dt}j_t = -2b\Big(j_t - A^g g(j_t)\Big) \Leftrightarrow \frac{d}{dt}a(j_t) = -ba(j_t)$. *Since* $a(\cdot)$ *is bijective, this combined with (1.5) implies that*

$$\begin{aligned}&\{j_t; t \geq 0\} \text{ satisfies (1.7)}\\ &\Leftrightarrow \{a(j_t); t \geq 0\} \text{ satisfies } \frac{d}{dt}a(j_t) = -ba(j_t), \quad a(j_0) = 2\sqrt{2}\int_{r_1}^{r_3} \sqrt{-g(y)}dy. \end{aligned} \tag{1.8}$$

It is trivial that (1.8) has a unique solution $a(j_t) = 2\sqrt{2}e^{-bt}\int_{r_1}^{r_3}\sqrt{-g(y)}dy, t \geq 0$. *Since* $a(\cdot)$ *is bijective on* $(g(r_2), 0]$, *we get our assertion.*

Remark 4 *Intuitive meaning of (1.7): Same as in Remark 1, if one consider the Hamiltonian system without randomizing or damping, then* $\lambda^{1/2}a(j)$ *represents the line integral of* $|v|^2$ *along the orbit* $l_j : H(x, v) = \lambda j$ *with respect to Liouville measure of the system, and* $\lambda^{-1/2}a'(j)$ *is the period of the same orbit, so by (1.6),* $2\lambda(j - A^g g(j))$ *is the mean value of* $|v|^2$ *along the same orbit. For our model (i.e., with randomizing and damping), although this does not hold rigidly, is still a good approximation. Since* $-b\lambda^{-1}|V_t^\lambda|^2$ *is the strongest term in the "derivative" of* J_t^λ *(see, for example, (2.6)), we get heuristically that* $-2b(J_t - A^g g(J_t))$ *is a good approximation of the "derivative" of* J_t^λ.

Remark 5 *As noticed,* $a'(j)$ *is not well-defined at* $j = 0$, *due to the divergence of*

$$I(j) := \int_{r_3 - a_0}^{g^{-1,2}(j)} \frac{1}{\sqrt{j - g(x)}}dx$$

as $j \to 0-$ *(the remainder* $a'(j) - I(j)$ *is bounded as* $j \to 0-$*). However, assumption (A2) implies that there exists a constant* C *such that* $I(j) \leq C|j|^{-1/2}$ *as* $j \to 0-$. *Indeed, for any* $j \in (g(r_3 - a_0), 0)$, *we have by (A2) that* $j - g(x) \geq g'(g^{-1,2}(j))(g^{-1,2}(j) - x) \geq C_1|g(g^{-1,2}(j))|(g^{-1,2}(j) - x) = C_1|j|(g^{-1,2}(j) - x)$ *for any* $x \in (r_3 - a_0, g^{-1,2}(j))$, *so*

$$I(j) \leq \int_{r_3 - a_0}^{g^{-1,2}(j)} \frac{1}{\sqrt{C_1|j|(g^{-1,2}(j) - x)}}dx = \frac{1}{\sqrt{C_1|j|}}2\sqrt{g^{-1,2}(j) - (r_3 - a_0)} \leq 2\sqrt{\frac{a_0}{C_1}}|j|^{-1/2}.$$

This computation is also used, for example, in the proof of Proposition 2.7 (see Section 5 for the proof).

Remark 6 *Our convergence in probability of Theorem 1.1 is equivalent to the convergence in distribution by Lemma 2.2 below. In this paper, we prove this convergence in distribution by proving that the considered family of distributions is tight, and each of its cluster points as* $\lambda \to \infty$ *satisfies (1.7). We want to emphasize that as* $a'(j)$ *is not well-defined at* $j = 0$, *the condition* $j_t < 0$ *for all* $t > 0$ *in (1.7) could not be omitted.*

We prove Theorem 1.1 in the rest of this paper. The basic idea is as follows. We first consider the deterministic case, especially we count the number of times that it oscillates during any time interval, by estimating the sojourn time of the particle near its right-end during each oscillation (see Subsection 2.3). We then convert our problem for the random case to the deterministic case, by loosing the initial velocity condition a little bit for each oscillation (see, *e.g.*, Lemma 2.15). We do so by introducing sets $F_1^\lambda \sim F_{11,c}^\lambda$, the probabilities of which are proven to be asymptotically 1 (see Subsection 2.5). Well-known properties of Brownian motion, such as the law of the iterated logarithm and Levy's modulus continuity, are used.

The rest of this paper is organized as follows. In Section 2, we give some preparations: we prove a useful estimate of the velocity V_t^λ in Subsection 2.1, which ensures that $|V_t^\lambda|$ could not be too large; in Subsection 2.2, we notice that it suffices to prove the convergence in distribution, and prove the tightness of the considered family of distributions; we summarize several useful results for the deterministic case in Subsection 2.3, with the proofs given in Section 5; in Subsection 2.4, we recall some necessary properties of Brownian motion; and in Subsection 2.5, we define the mentioned sets $F_1^\lambda \sim F_{11,c}^\lambda$ and prove that they do have asymptotically full probabilities. So the conditions in the definitions of these sets could be used freely, when we consider the limit $\lambda \to \infty$ in Sections 3 and 4. We prove Theorem 1.1 by proving that all cluster points when $\lambda \to \infty$ satisfy (1.7). This is done in Sections 3 and 4: We prove in Section 3 that under any cluster probability, J_t is strictly negative for any $t > 0$, so the particle could never hit the boundary $\{r_1, r_3\}$ except when $t = 0$; and in Section 4, we prove that under any cluster probability, the process satisfies the claimed ordinary differential equation.

2 Preparations

We make several preparations in this section. First notice that under our assumption, there exist constants $a_1, a_2 > 0$ such that

$$g'(x) < -a_2 \text{ if } x \in [r_1, r_1 + a_1). \tag{2.1}$$

Also, by re-choosing $a_0 > 0$ if necessary, we can assume without loss of generality that $r_1 + a_1 < r_3 - a_0$ and

$$g(x) < \frac{1}{3} g(r_3 - a_0) \text{ if } x \in (r_1 + a_1, r_3 - a_0). \tag{2.2}$$

Notice that in order to prove Theorem 1.1, a result with respect to $t \in [0, \infty)$, it suffices to prove our assertion with respect to $t \in [0, T]$ for any $T > 0$. Choose any $T > 0$ and fix it from now on.

2.1 An essential estimate

We have the following estimation with respect to V_t^λ. This estimation plays an essential role in this paper.

Lemma 2.1 *There exists a constant $C_2 > 0$ such that*

$$E\Big[\sup_{t \in [0,T]} |V_t^\lambda|^4\Big]^{1/4} \le C_2 \lambda^{\frac{1}{2}}, \qquad \lambda \ge 1.$$

Proof. We have by the definition of H_t^λ and Ito's formula that

$$dH_t^\lambda = -b|V_t^\lambda|^2 dt + V_t^\lambda \sigma(X_t^\lambda) dB_t + \frac{1}{2}\sigma(X_t^\lambda)^2 dt,$$

so for any $t \in [0,T]$, we have that

$$\begin{aligned} |V_t^\lambda|^2 &= 2H_t^\lambda - 2\lambda g(X_t^\lambda) \\ &= 2H_0 - 2b\int_0^t |V_s^\lambda|^2 ds + 2\int_0^t V_s^\lambda \sigma(X_s^\lambda) dB_s + \int_0^t \sigma(X_s^\lambda)^2 ds - 2\lambda g(X_t^\lambda) \\ &\le 2H_0 + 2\int_0^t V_s^\lambda \sigma(X_s^\lambda) dB_s + T\|\sigma\|_\infty^2 + 2\lambda \|g\|_\infty. \end{aligned}$$

Here $\|g\|_\infty := \max_{x\in\mathbf{R}} |g(x)|$. Therefore, by Doob's inequality, we get that

$$\begin{aligned} &E\Big[\sup_{t\in[0,T]} |V_t^\lambda|^4\Big] \\ &\le 2\Big(2H_0 + T\|\sigma\|_\infty^2 + 2\lambda\|g\|_\infty\Big)^2 + 2E\Big[\sup_{t\in[0,T]}\Big(2\int_0^t V_s^\lambda \sigma(X_s^\lambda) dB_s\Big)^2\Big] \\ &\le 2\Big(2H_0 + T\|\sigma\|_\infty^2 + 2\lambda\|g\|_\infty\Big)^2 + 32E\Big[\Big(\int_0^T V_s^\lambda \sigma(X_s^\lambda) dB_s\Big)^2\Big] \\ &\le 2\Big(2H_0 + T\|\sigma\|_\infty^2 + 2\lambda\|g\|_\infty\Big)^2 + 32\|\sigma\|_\infty^2 T E\Big[\sup_{t\in[0,T]} |V_t^\lambda|^4\Big]^{1/2}. \end{aligned}$$

In general, for any $c_1, c_2 \in \mathbf{R}^+$, we have that

$$x^2 \le c_1 + c_2 x \Rightarrow x \le \frac{c_2 + \sqrt{c_2^2 + 4c_1}}{2} \le \frac{c_2 + c_2 + \sqrt{4c_1}}{2} = c_2 + \sqrt{c_1}.$$

Therefore, we get that

$$E\Big[\sup_{t\in[0,T]} |V_t^\lambda|^4\Big]^{1/2} \le 32\|\sigma\|_\infty^2 T + \sqrt{2}\Big(2H_0 + T\|\sigma\|_\infty^2\Big) + 2\sqrt{2}\|g\|_\infty \lambda$$

for any $\lambda \ge 1$, which implies our assertion. ∎

2.2 The tightness

Choose and fix any $f \in C_b(\mathbf{R})$. First recall that our expected limit $\{(j_t, y_t^f)\}_{t\in[0,T]}$ is deterministic. As well-known (see Lemma 2.2 below), when the expected limit is deterministic, the convergence in probability is equivalent to the convergence in distribution.

Lemma 2.2 *Let (S,ρ) be any complete separable metric space. Let X_n, $n \in \mathbf{N}$ be a sequence of S-valued random variables and let $x_0 \in S$. Then $X_n \to x_0$ in probability $\Leftrightarrow$ $X_n \to x_0$ in distribution.*

Proof. The "$\Rightarrow$" part is trivial. We prove the "$\Leftarrow$" part. For any $\varepsilon > 0$, let $h(y) = \Big(1 - \frac{\rho(y,x_0)}{\varepsilon}\Big)^+$. Then h is bounded and continuous, and $1_{\{y\in S: \rho(y,x_0)\ge\varepsilon\}} \le 1 - h$. Therefore, $P(\{\rho(X_n,x_0) \ge \varepsilon\}) \le E[1 - h(X_n)] \to 1 - h(x_0) = 0$. ∎

Let P_λ denote the distribution of $\{(J_t^\lambda, Y_t^{f,\lambda})\}_{t\in[0,T]}$. By Lemma 2.2, in order to prove Theorem 1.1, it suffices to prove that P_λ converges to $\delta_{\{(j_t,y_t^f)\}_{t\in[0,T]}}$ as $\lambda \to \infty$. We do so by prove that $\{P_\lambda; \lambda \geq 1\}$ is tight, and that any of its cluster points has probability one on $\{(j_t, y_t^f)\}_{t\in[0,T]}$. We prove the tightness of $\{P_\lambda; \lambda \geq 1\}$ in this subsection.

First, as an easy result of Billingsley [2, Theorem 7.3] (or see Karatzas-Shreve [3, Theorem 1.4.10]), we have the following general result with respect to the tightness of probability measures on $C([0,T]; \mathbf{R})$:

Lemma 2.3 *Let Z_t^λ be any stochastic process given by*

$$dZ_t^\lambda = \sigma^{Z,\lambda}(t)dB_t + b^{Z,\lambda}(t)dt.$$

If Z_0^λ is bounded for $\lambda \geq 1$, and

$$\sup_{\lambda\geq 1}\Big(\sup_{t\in[0,T]} E\Big[|b^{Z,\lambda}(t)|^2\Big] + E\Big[\sup_{t\in[0,T]}|\sigma^{Z,\lambda}(t)|^2\Big]\Big) < \infty, \tag{2.3}$$

then we have that $\Big\{$ the distribution of $\big\{Z_t^\lambda\big\}_{t\in[0,T]}; \lambda \geq 1\Big\}$ is tight in $\wp(C([0,T], \mathbf{R}))$.

Proof. We prove this lemma by proving that all conditions of [2, Theorem 7.3] are satisfied.

Since Z_0^λ is bounded for $\lambda \geq 1$, it is trivial that the first condition of [2, Billingsley, Theorem 7.3] is satisfied. We prove that the second condition there is also satisfied, in words, for any $\varepsilon > 0$, we prove that

$$\lim_{\delta\to 0} P(\sup_{s,t\in[0,T],|s-t|\leq\delta} |Z_t^\lambda - Z_s^\lambda| > \varepsilon) = 0.$$

It suffices to prove that

$$\lim_{\delta\to 0} P\Big(\sup_{s,t\in[0,T],|s-t|\leq\delta} \Big|\int_s^t \sigma^{Z,\lambda}(u)dB_u\Big| > \varepsilon/2\Big) = 0, \tag{2.4}$$

$$\lim_{\delta\to 0} P\Big(\sup_{s,t\in[0,T],|s-t|\leq\delta} \Big|\int_s^t b^{Z,\lambda}(u)du\Big| > \varepsilon/2\Big) = 0. \tag{2.5}$$

For (2.5), we have by Markov's inequality and Schwartz's inequality that

$$\begin{aligned}
& P\Big(\sup_{s,t\in[0,T],|s-t|\leq\delta} \Big|\int_s^t b^{Z,\lambda}(u)du\Big| > \varepsilon/2\Big) \\
\leq\ & (\varepsilon/2)^{-1} E\Big[\sup_{s,t\in[0,T],|s-t|\leq\delta} \Big|\int_s^t b^{Z,\lambda}(u)du\Big|\Big] \\
\leq\ & (\varepsilon/2)^{-1} E\Big[\sup_{s,t\in[0,T],|s-t|\leq\delta} |t-s|^{1/2}\Big|\int_s^t |b^{Z,\lambda}(u)|^2 du\Big|^{1/2}\Big] \\
\leq\ & (\varepsilon/2)^{-1}\delta^{1/2} E\Big[\int_0^T |b^{Z,\lambda}(u)|^2 du\Big]^{1/2} \\
\leq\ & (\varepsilon/2)^{-1}\delta^{1/2} T^{1/2} \sup_{t\in[0,T]} E\Big[|b^{Z,\lambda}(t)|^2\Big].
\end{aligned}$$

So we get (2.5).

We next prove (2.4). Since $\{\int_0^t \sigma^{Z,\lambda}(u)dB_u; t \geq 0\}$ is a continuous martingale, we have (see, for example, [3, Theorem 3.4.6 and Problem 3.4.7]) that there exists a Brownian motion $\{W(t); t \geq 0\}$ such that $\int_0^t \sigma^{Z,\lambda}(u)dB_u = W(\int_0^t |\sigma^{Z,\lambda}(u)|^2 du)$. So for any $\overline{T}, \eta > 0$, we have by Markov's inequality that

$$\begin{aligned} & P\Big(\sup_{s,t\in[0,T],|s-t|\leq\delta} |\int_s^t \sigma^{Z,\lambda}(u)dB_u| > \varepsilon/2\Big) \\ \leq \ & P\Big(\int_0^T |\sigma^{Z,\lambda}(u)|^2 du > \overline{T}\Big) + P\Big(\sup_{s,t\in[0,T],|s-t|\leq\delta} \int_s^t |\sigma^{Z,\lambda}(u)|^2 du > \eta\Big) \\ & \quad + P\Big(\sup_{0\leq\overline{s}\leq\overline{t}\leq\overline{T},|\overline{s}-\overline{t}|\leq\eta} |W(\overline{s}) - W(\overline{t})| > \varepsilon/2\Big) \\ \leq \ & \overline{T}^{-1} T \sup_{u\in[0,T]} E[|\sigma^{Z,\lambda}(u)|^2] + \eta^{-2} E\Big[\delta \sup_{u\in[0,T]} |\sigma^{Z,\lambda}(u)|^2\Big] \\ & \quad + P\Big(\sup_{0\leq\overline{s}\leq\overline{t}\leq\overline{T},|\overline{s}-\overline{t}|\leq\eta} |W(\overline{s}) - W(\overline{t})| > \varepsilon/2\Big). \end{aligned}$$

By the Lévy's modulus of continuity of Brownian motion (see, for example, [9, page 30, Theorem 2.7], which is also quoted in this paper as Lemma 2.9 (3) below), the last term on the right hand side above converges to 0 as $\eta \to 0$ for any $\overline{T} > 0$. Therefore, taking first $\overline{T} \to \infty$, then $\eta \to 0$ and finally $\delta \to 0$, we get (2.4).

This completes the proof of our assertion. ∎

As a direct consequence of Lemmas 2.1 and 2.3, we get the following.

COROLLARY 2.4 *We have that $\{P_\lambda; \lambda \geq 1\}$ is tight in $\wp(C([0,T];\mathbf{R}) \times C([0,T];\mathbf{R}))$.*

Proof. By definition and Ito's formula, we have that

$$\begin{aligned} dY_t^{f,\lambda} &= f(X_t^\lambda)dt, \\ dJ_t^\lambda &= \lambda^{-1} V_t^\lambda \sigma(X_t^\lambda)dB_t - b\lambda^{-1}|V_t^\lambda|^2 dt + \frac{1}{2}\lambda^{-1}\sigma(X_t^\lambda)^2 dt. \end{aligned} \tag{2.6}$$

This combined with Lemmas 2.1 and 2.3 implies our assertion. ∎

Let P_∞ be any cluster point of $\{P_\lambda\}$ as $\lambda \to \infty$. So there exists a sequence $\lambda_n \to \infty$ ($n \to \infty$) such that $P_{\lambda_n} \to P_\infty$. We prove in Sections 3 and 4 that P_∞ has probability 1 on $\{(j_t, y_t^f)\}_{t\geq 0}$.

2.3 Several facts for deterministic case

We present several facts with respect to the deterministic case in this subsection. The proofs are given in Appendix.

For any $(x, v) \in [r_1, r_3] \times \mathbf{R}$ and $\lambda \geq 1$, let $(x_t^\lambda(x, v), v_t^\lambda(x, v))$ be the solution of

$$\begin{cases} dx_t = v_t dt, \\ dv_t = -bv_t dt - \lambda g'(x_t)dt, \\ (x_0(x, v), v_0(x, v)) = (x, v), \end{cases} \tag{2.7}$$

let

$$h_t^\lambda(x, v) := \frac{1}{2}|v_t^\lambda(x, v)|^2 + \lambda g(x_t^\lambda(x, v)), \qquad j_t^\lambda(x, v) := \lambda^{-1} h_t^\lambda(x, v),$$

and let

$$t_1^\lambda(x,v) = \inf\Big\{t > 0; x_t^\lambda(x,v) = r_3 - a_0\Big\}. \tag{2.8}$$

Also, we sometimes write $x_t^\lambda(x,v)$, $v_t^\lambda(x,v)$, $h_t^\lambda(x,v)$ as x_t, v_t, h_t, respectively, when there is no risk of confusion.

First notice that $dh_t^\lambda(x,v) = -b|v_t^\lambda(x,v)|^2dt$ by definition. So we get the following result, which is almost trivial, but will be used several times in this paper:

Lemma 2.5 *$h_t^\lambda(x,v)$ and $j_t^\lambda(x,v)$ are monotone non-increasing with respect to t.*

The following lemma claims that, after departing from r_3, the particle hits $r_3 - a_0$ in a very short time, and if λ is large enough such that the damping is strong enough, the particle will never leave the domain (r_1, r_3).

Lemma 2.6 *Assume that $x_0 = r_3$ and $v_0 < 0$. Then we have the followings:*

1. $\lim_{\lambda\to\infty} t_1^\lambda(x_0, v_0) = 0$,
2. *There exists a $\lambda_0 \geq 1$ such that for any $\lambda \geq \lambda_0$, we have that $x_t^\lambda(x_0, v_0) \in (r_1, r_3)$ for all $t > 0$.*

Our next result is with respect to the k-th sojourn time of the particle near its right-end.

PROPOSITION 2.7 *For any $c > 0$, there exist constants $C_3(c) > 0$ and $\lambda_1(c) \geq 1$ such that for any $k \in \mathbf{N}$,*

$$\begin{aligned} &\lambda \geq \lambda_1(c), x_0 = r_3 - a_0, v_0 > 0, \frac{1}{2}|v_0|^2 + \lambda g(x_0) \leq -c\lambda^{1/2}k \\ \Rightarrow\ & t_1^\lambda(x_0, v_0) \leq C_3(c)\lambda^{-1/4}k^{-1/2}. \end{aligned}$$

Remark 7 *Roughly speaking, each oscillation decreases the total energy with an order of $\lambda^{1/2}$ (see Lemma 3.3 for the random case. The calculation for the deterministic case is even easier). So for any $k \in \mathbf{N}$, when considering the k-th round trip with initial energy of order 1, the total energy is bounded above by $-c\lambda^{1/2}k$ with some proper constant c. This is the meaning of the condition $\frac{1}{2}|v_0|^2 + \lambda g(x_0) \leq -c\lambda^{1/2}k$ in Proposition 2.7. Also, by a trivial time shift, in Proposition 2.7, we wrote the time of the k-th left-to-right-crossing of the particle over the point $r_3 - a_0$ as 0.*

Let S_f and $A^g f$ be as defined in Section 1. Notice that S_f is continuous, and that $S_f(j) \neq 0$ for any $j \in (g(r_2), 0)$. So $A^g f$ is also continuous.

For any $\delta_1, \delta_2 \in (0, -g(r_2))$ satisfying $\delta_1 + \delta_2 \leq -g(r_2)$, let

$$b_{1,\delta_1,\delta_2,f}^\lambda := \sup_{(x,v):\frac{1}{2}\lambda^{-1}|v|^2+g(x)\in(g(r_2)+\delta_1,-\delta_2)} \Big|\lambda^{1/2}\int_0^{S_1(\frac{1}{2}\lambda^{-1}|v|^2+g(x))\lambda^{-1/2}} f(x_u^\lambda(x,v))du - S_f\Big(\frac{1}{2}\lambda^{-1}|v|^2+g(x)\Big)\Big|. \tag{2.9}$$

Then we have the following:

PROPOSITION 2.8 *For any $\delta_1, \delta_2 \in (0, -g(r_2))$ satisfying $\delta_1 + \delta_2 \le -g(r_2)$, we have that*

$$\lim_{\lambda\to\infty} b^{\lambda}_{1,\delta_1,\delta_2,f} = 0.$$

Heuristic meaning of Proposition 2.8: as noticed in Remark 1, $\lambda^{-1/2}S_f(j)$ is the line integral of the function f along the orbit l_j for the case where $b = 0 = \sigma$. Proposition 2.8 claims that the approximation error of this approximation is of order $o(\lambda^{-1/2})$, locally uniformly with respect to the initial energy.

2.4 Several basic properties of Brownian motion

In this subsection, we prepare several basic properties of Brownian motion that will be used later.

Lemma 2.9 *Let $\left\{\overline{B}_t\right\}_{t\ge 0}$ be any standard Brownian motion. Then we have the followings:*

1. $\lim_{a\to\infty} P\Big(\Big\{\inf_{u\ge 0}(\varepsilon u + \overline{B}_t) < -a\Big\}\Big) = 0$ *for any* $\varepsilon > 0$,
2. $\lim_{a\to\infty} P\Big(\Big\{\overline{B}_s - \varepsilon s \ge 0$ *for some* $s \ge a\Big\}\Big) = 0$ *for any* $\varepsilon > 0$,
3. $P\Big(\limsup_{\varepsilon\to 0}\Big\{\sup_{0\le s\le t+s\le T, t\le\varepsilon} \frac{|\overline{B}_{t+s}-\overline{B}_s|}{\sqrt{2\varepsilon\log\frac{1}{\varepsilon}}}\Big\} = 1\Big) = 1.$

Proof. For the first assertion, notice that for any $\varepsilon > 0$, we have that

$$\begin{aligned}\liminf_{s\to\infty}\frac{\overline{B}_s}{\sqrt{2s\log\log s}} = -1 \quad &\Rightarrow \quad \liminf_{s\to\infty}(\varepsilon s + \overline{B}_s) = \infty \\ &\Rightarrow \quad \inf_{s\ge 0}(\varepsilon s + \overline{B}_s) > -\infty.\end{aligned}$$

So

$$\begin{aligned}&\lim_{a\to\infty} P\Big(\Big\{\inf_{u\ge 0}(\varepsilon u + \overline{B}_t) < -a\Big\}\Big) \\ = \;& P\Big(\bigcap_{a\in\mathbf{N}}\Big\{\inf_{u\ge 0}(\varepsilon u + \overline{B}_t) < -a\Big\}\Big) = P\Big(\Big\{\inf_{u\ge 0}(\varepsilon u + \overline{B}_t) = -\infty\Big\}\Big) \\ \le \;& P\Big(\Big\{\liminf_{s\to\infty}\frac{\overline{B}_s}{\sqrt{2s\log\log s}} = -1\Big\}^C\Big) \\ = \;& 0,\end{aligned}$$

where when passing to the last line, we used the law of the iterated logarithm (see, *e.g.*, Revue-Yor [9, page 58]).

We next prove the second assertion. Notice that for any $a > 0$, we have that

$$\begin{aligned}&\text{there exists a } s \ge a \text{ such that } \overline{B}_s - \varepsilon s \ge 0 \\ \Rightarrow \quad &\text{there exists a } s \ge a \text{ such that } \frac{\overline{B}_s}{\sqrt{2s\log\log s}} \ge \frac{\varepsilon s}{\sqrt{2s\log\log s}} \\ \Rightarrow \quad &\sup_{s\ge a}\frac{\overline{B}_s}{\sqrt{2s\log\log s}} \ge \inf_{s\ge a}\frac{\varepsilon s}{\sqrt{2s\log\log s}}\end{aligned}$$

Also, since $\frac{\varepsilon s}{\sqrt{2s\log\log s}} \to \infty$ as $s \to \infty$, there exists a constant $A > 0$ such that $\inf_{s\geq a} \frac{\varepsilon s}{\sqrt{2s\log\log s}} \geq 2$ for any $a \geq A$. So

$$\begin{aligned}
&\bigcap_{a\geq A} \Big\{\exists s \geq a \text{ such that } \overline{B}_s - \varepsilon s \geq 0\Big\} \\
\subset\ &\bigcap_{a\geq A} \Big\{\sup_{s\geq a} \frac{\overline{B}_s}{\sqrt{2s\log\log s}} \geq 2\Big\} \subset \Big\{\limsup_{s\to\infty} \frac{\overline{B}_s}{\sqrt{2s\log\log s}} \geq 2\Big\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{a\to\infty} P\Big(\Big\{\overline{B}_s - \varepsilon s \geq 0 \text{ for some } s \geq a\Big\}\Big) \\
= \ & P\Big(\bigcap_{a\geq A} \Big\{\overline{B}_s - \varepsilon s \geq 0 \text{ for some } s \geq a\Big\}\Big) \\
\leq \ & P\Big(\Big\{\limsup_{s\to\infty} \frac{\overline{B}_s}{\sqrt{2s\log\log s}} \geq 2\Big\}\Big) \\
= \ & 0,
\end{aligned}$$

where when passing to the last line, again, we used the law of the iterated logarithm.

The last assertion is nothing but Lévy's modulus of continuity (see, for example, [9, page 30, Theorem 2.7]). ∎

2.5 Several useful sets

In this subsection, we define several useful sets (see $F_1^\lambda \sim F_{11,c}^\lambda$ below), and prove that they are asymptotically of full probabilities. These sets will be used in Sections 3 and 4. Precisely, since we are interested in the limit distribution as $\lambda \to \infty$, we can always ignore those sets with asymptotically null measures. So by assuming that we are on these (asymptotically-full-probability) sets, we can assume freely that the conditions in the definitions of these sets hold.

Let us first define several constants. Choose $x_0 \in (r_3 - a_0, r_3)$ such that

$$g(x) < \frac{1}{3} g(r_3 - a_0) \text{ for any } x \in (r_3 - a_0, x_0), \tag{2.10}$$

and let $C_4 := \Big(-\frac{1}{6}g(r_3-a_0)\Big)^{1/2}(x_0-(r_3-a_0))$. Also, choose and fix a constant $T_0 \in (0,T)$ such that $-g(r_2)\Big(1 - e^{-2(b+1)T_0}\Big) < -\frac{1}{8}g(r_3 - a_0)$, and choose and fix a constant $\varepsilon_0 > 0$ such that $-b + \varepsilon_0\|\sigma\|_\infty^2 < 0$. Let

$$C_5 := \frac{b - \varepsilon_0\|\sigma\|_\infty^2}{\sqrt{6}} |g(r_3 - a_0)|^{1/2}(r_3 - a_0 - r_1 - a_1), \tag{2.11}$$

$$C_6 := \frac{C_5}{4}\Big(-2g(r_2)\Big)^{-1/2}, \tag{2.12}$$

and let $C_7 = C_3(\frac{C_5}{2})$, where $C_3(\cdot)$ is given by Proposition 2.7.

Also, we define the following notations.

$$\begin{aligned}
&\tau_0^\lambda = 0, \\
&\tau_k^\lambda = \inf\{t > \tau_{k-1}^\lambda; X_t^\lambda = r_3 - a_0\}, \qquad k \geq 1.
\end{aligned} \tag{2.13}$$

Now we are ready to give the definitions of our announced sets $F_1^\lambda \sim F_{11,c}^\lambda$. Let

$$\begin{aligned}
F_1^\lambda &:= \Big\{ \sup_{t\in[0,T]} \Big| \int_0^t V_s^\lambda \sigma(X_s^\lambda) dB_s \Big| \leq \lambda^{3/4} \Big\}, \\
F_2^\lambda &:= \Big\{ \sup_{0\leq s\leq s+t\leq T} \Big| \int_s^{s+t} e^{2bu} V_u^\lambda \sigma(X_u^\lambda) dB_u \Big| \leq \lambda^{3/4} \Big\}, \\
F_3^\lambda &:= \Big\{ \sup_{t\in[0,T]} \Big| \int_0^t \sigma(X_s^\lambda) dB_s \Big| \leq \lambda^{1/2} \Big\}, \\
F_4^\lambda &:= \Big\{ \sup_{s\in[0,t_1^\lambda(X_0,\frac{V_0}{2})]} \int_0^s e^{bu} \sigma(X_u^\lambda) dB_u < -\frac{V_0}{4} \Big\}, \\
F_5^\lambda &:= \Big\{ \inf_{t\in[0,T]} \Big(\int_0^t e^{2(b+1)s} |V_s^\lambda|^2 ds + \int_0^t e^{2(b+1)s} V_s^\lambda \sigma(X_s^\lambda) dB_s \Big) \geq -\lambda^{\frac12} \Big\}, \\
F_6^\lambda &:= \Big\{ \sup_{0\leq s\leq s+t\leq T, t\leq C_7\lambda^{-1/4}} \Big| \int_s^{s+t} e^{bu} \sigma(X_u^\lambda) dB_u \Big| \leq C_6 \Big\}, \\
F_{7,t_0}^\lambda &:= \{\tau_1^\lambda \leq t_0\}, \\
F_8^\lambda &:= \Big\{ \int_0^{\tau_1^\lambda} |V_s^\lambda|^2 ds \geq C_4 \lambda^{\frac12} \Big\}, \\
F_9^\lambda &:= \{ \int_0^t V_s^\lambda \sigma(X_s^\lambda) dB_s - \varepsilon_0 \int_0^t |V_s^\lambda|^2 \sigma(X_s^\lambda)^2 ds \leq 0 \text{ for all } t \geq \tau_1^\lambda \}, \\
F_{10}^\lambda &:= \Big\{ H_t^\lambda < 0 \text{ for all } t \in [\tau_1^\lambda, T] \Big\}. \\
F_{11,c}^\lambda &:= \Big\{ \sup_{0\leq s\leq s+t\leq T, t\leq c\lambda^{-1/2}} \Big(|X_{s+t}^\lambda - x_t^\lambda(X_s^\lambda, V_s^\lambda)| + \lambda^{-\frac12} |V_{s+t}^\lambda - v_t^\lambda(X_s^\lambda, V_s^\lambda)| \Big) < \lambda^{-1/4} \Big\}.
\end{aligned}$$

Here t_0, $c > 0$ are arbitrary constants. Finally, let $\overline{F}_{t_0,c}^\lambda$ be the intersection of these sets, *i.e.*, $\overline{F}_{t_0,c}^\lambda := F_1^\lambda \cap \cdots \cap F_{11,c}^\lambda$.

We prove that all these sets have probability 1 asymptotically as $\lambda \to \infty$. See the explanations before each Lemma below for their meanings and heuristic reasons. Our main result of this section is the following.

PROPOSITION 2.10 *For any $t_0, c > 0$, we have that*

$$\lim_{\lambda\to\infty} P(\overline{F}_{t_0,c}^\lambda) = 1.$$

We prove Proposition 2.10 in the rest of this subsection, by proving that each of the sets $F_1^\lambda \sim F_{11,c}^\lambda$ has probability 1 asymptotically as $\lambda \to \infty$.

We first deal with $F_1^\lambda \sim F_3^\lambda$. Since V_t^λ is of order at most $\lambda^{1/2}$ by Lemma 2.1 and σ is bounded, it is heuristically clear that the three integrals in the definitions of $F_1^\lambda \sim F_3^\lambda$ are of orders at most $\lambda^{1/2}$, $\lambda^{1/2}$ and 1, respectively. So it is not strange that these three sets are asymptotically of probability 1. These sets are used to prove that the diffusion terms in the corresponding expressions could not be too large (see, for example, the proofs of Lemma 3.1 and Proposition 4.7 for the usage of F_1^λ, Lemma 4.3 for F_2^λ and (3.2) in the proof of Lemma 3.4 for F_3^λ).

Lemma 2.11 *For $j \in \{1,2,3\}$, we have that* $\lim_{\lambda\to\infty} P(F_j^\lambda) = 1$.

Proof. By Doob's inequality and Lemma 2.1, we have that

$$\begin{aligned} & E\Big[\sup_{t\in[0,T]}\Big|\int_0^t V_s^\lambda \sigma(X_s^\lambda)dB_s\Big|^2\Big] \\ \leq\ & 4E\Big[\Big|\int_0^T V_s^\lambda \sigma(X_s^\lambda)dB_s\Big|^2\Big] = 4\int_0^T E\Big[|V_s^\lambda|^2\sigma(X_s^\lambda)^2\Big]ds \\ \leq\ & 4T\|\sigma\|_\infty^2 E\Big[\sup_{s\in[0,T]}|V_s^\lambda|^2\Big] \leq 4T\|\sigma\|_\infty^2 C_2^2\lambda. \end{aligned}$$

Therefore, by Chebyshev's inequality,

$$\begin{aligned} P\Big((F_1^\lambda)^C\Big) &= P\Big(\sup_{t\in[0,T]}\Big|\int_0^t V_s^\lambda \sigma(X_s^\lambda)dB_s\Big| > \lambda^{3/4}\Big) \\ &\leq \Big(\lambda^{3/4}\Big)^{-2} E\Big[\sup_{t\in[0,T]}\Big|\int_0^t V_s^\lambda \sigma(X_s^\lambda)dB_s\Big|^2\Big] \\ &\leq \lambda^{-1/2}4T\|\sigma\|_\infty^2 C_2^2. \end{aligned}$$

This implies our assertion with $j = 1$.

The proof of the assertion with $j = 2$ is almost the same. By Doob's inequality and Lemma 2.1, we have that

$$\begin{aligned} & E\Big[\sup_{0\leq s\leq s+t\leq[0,T]}\Big|\int_s^{t+s} e^{2bu}V_u^\lambda \sigma(X_u^\lambda)dB_u\Big|^2\Big] \\ \leq\ & 4E\Big[\sup_{r\in[0,T]}\Big|\int_0^r e^{2bu}V_u^\lambda \sigma(X_u^\lambda)dB_u\Big|^2\Big] \leq 16E\Big[\Big|\int_0^T e^{2bu}V_u^\lambda \sigma(X_u^\lambda)dB_u\Big|^2\Big] \\ \leq\ & 16e^{4bT}\|\sigma\|_\infty^2 TE\Big[\sup_{t\in[0,T]}|V_u^\lambda|^2\Big] \leq 16e^{4bT}\|\sigma\|_\infty^2 TC_2^2\lambda, \end{aligned}$$

so by Chebyshev's inequality,

$$P\Big((F_2^\lambda)^C\Big) \leq \lambda^{-1/2}16e^{4bT}\|\sigma\|_\infty^2 TC_2^2.$$

This implies our assertion with $j = 2$.

The proof of the assertion with $j = 3$ is even more simple. Indeed, by Doob's inequality, we have that

$$E\Big[\sup_{t\in[0,T]}\Big|\int_0^t \sigma(X_s^\lambda)dB_s\Big|^2\Big] \leq 4E\Big[\Big|\int_0^T \sigma(X_s^\lambda)dB_s\Big|^2\Big] \leq 4T\|\sigma\|_\infty^2,$$

so by Chebyshev's inequality,

$$P\Big((F_3^\lambda)^C\Big) \le \lambda^{-1} E\Big[\sup_{t\in[0,T]} \Big|\int_0^t \sigma(X_s^\lambda)dB_s\Big|^2\Big] \le 4T\|\sigma\|_\infty^2 \lambda^{-1},$$

which converges to 0 as $\lambda \to \infty$. ∎

We next prove that F_4^λ is asymptotically of probability 1. This is heuristically clear since the integrand is bounded and the length of the time period converges to 0 by Lemma 2.6 (1). This set is used to prove that F_{7,t_0}^λ and F_8^λ are asymptotically of probability 1.

Lemma 2.12 *We have that* $\lim_{\lambda\to\infty} P(F_4^\lambda) = 1$.

Proof. By Lemma 2.6, we have that $\lim_{\lambda\to\infty} t_1^\lambda(X_0, V_0/2) = 0$. Therefore, by Chebyshev's inequality and Doob's inequality, we get that

$$\begin{aligned} P\Big((F_4^\lambda)^C\Big) &= P\Big(\Big\{\sup_{s\in[0,t_1^\lambda(X_0,V_0/2)]} \int_0^s e^{bu}\sigma(X_u^\lambda)dB_u \ge -\frac{V_0}{4}\Big\}\Big) \\ &\le \Big(-\frac{V_0}{4}\Big)^2 4E\Big[\Big|\int_0^{t_1^\lambda(X_0,V_0/2)} e^{bu}\sigma(X_u^\lambda)dB_u\Big|^2\Big] \\ &\le \Big(-\frac{V_0}{4}\Big)^2 4e^{2bT}\|\sigma\|_\infty^2 t_1^\lambda(X_0,V_0/2) \\ &\to 0, \qquad \lambda\to\infty. \end{aligned}$$

∎

We next deal with F_5^λ and F_6^λ. They are also used to control the diffusion terms in the corresponding expressions (see, for example, the proof of Lemma 2.16 for the usage of F_5^λ and the proof of Claim 1 in the proof of Lemma 3.4 for F_6^λ). The basic idea is the well-known fact that a continuous martingale can be expressed as a time-changed Brownian motion.

Lemma 2.13 *We have that* $\lim_{\lambda\to\infty} P(F_5^\lambda) = 1$.

Proof. For any $\lambda \ge 1$, let $M_1^\lambda(t) := \int_0^t e^{2(b+1)s} V_s^\lambda \sigma(X_s^\lambda)dB_s$. Then $\{M_1^\lambda(t)\}_{t\ge 0}$ is a continuous martingale. So same as in the proof of Lemma 2.3, there exists a Brownian motion $\{W_1^\lambda(t)\}_{t\ge0}$ such that $M_1^\lambda(t) = W_1^\lambda\Big(\langle M_1^\lambda, M_1^\lambda\rangle_t\Big)$.

Also, let $C_8 := \|\sigma\|_\infty^2 e^{2(b+1)T}$. Then we have that

$$\langle M_1^\lambda, M_1^\lambda\rangle_t = \int_0^t e^{4(b+1)s}|V_s^\lambda|^2\sigma(X_s^\lambda)^2 ds \le C_8 \int_0^t e^{2(b+1)s}|V_s^\lambda|^2 ds.$$

So

$$\begin{aligned} &\inf_{t\in[0,T]} \Big(\int_0^t e^{2(b+1)s}|V_s^\lambda|^2 ds + \int_0^t e^{2(b+1)s} V_s^\lambda\sigma(X_s^\lambda)dB_s\Big) \\ \ge\ &\inf_{t\in[0,T]} \Big\{\frac{1}{C_8}\langle M_1^\lambda, M_1^\lambda\rangle_t + W_1^\lambda\Big(\langle M_1^\lambda, M_1^\lambda\rangle_t\Big)\Big\} \\ \ge\ &\inf_{s\ge0}\Big\{\frac{1}{C_8}s + W_1^\lambda(s)\Big\}. \end{aligned}$$

Therefore, by Lemma 2.9 (1), we get that

$$P\Big((F_5^\lambda)^C\Big) \le P\Big(\inf_{s\ge 0}\Big\{\frac{1}{C_8}s + W_1^\lambda(s)\Big\} < -\lambda^{1/2}\Big) \to 0, \qquad \lambda\to\infty.$$

∎

Lemma 2.14 *We have that* $\lim_{\lambda\to\infty} P(F_6^\lambda) = 1$.

Proof. For any $\lambda \ge 1$, let $M_2^\lambda(t) := \int_0^t e^{bu}\sigma(X_u^\lambda)dB_u$. Then M_2^λ is a continuous martingale, so there exists a Brownian motion $\{W_2^\lambda(\cdot)\}$ such that $M_2^\lambda(t) = W_2^\lambda\Big(\langle M_2^\lambda, M_2^\lambda\rangle_t\Big)$. Notice that

$$\begin{aligned}0\le s\le t+s\le T &\;\Rightarrow\; 0\le \langle M_2^\lambda, M_2^\lambda\rangle_s \le \langle M_2^\lambda, M_2^\lambda\rangle_{t+s} \le e^{2bT}\|\sigma\|_\infty^2 T,\\ t\le C_7\lambda^{-1/4} &\;\Rightarrow\; \langle M_2^\lambda, M_2^\lambda\rangle_{t+s} - \langle M_2^\lambda, M_2^\lambda\rangle_s \le e^{2bT}\|\sigma\|_\infty^2 C_7\lambda^{-1/4}.\end{aligned}$$

Therefore, with $T_1 := e^{2bT}\|\sigma\|_\infty^2 T$ and $C_9 := e^{2bT}\|\sigma\|_\infty^2 C_7$, we have that

$$\begin{aligned}&\sup_{0\le s\le s+t\le T, t\le C_7\lambda^{-1/4}} \Big|\int_s^{s+t} e^{bu}\sigma(X_u^\lambda)dB_u\Big|\\ =\;& \sup_{0\le s\le s+t\le T, t\le C_7\lambda^{-1/4}} \Big|W_2^\lambda\Big(\langle M_2^\lambda, M_2^\lambda\rangle_{t+s}\Big) - W_2^\lambda\Big(\langle M_2^\lambda, M_2^\lambda\rangle_s\Big)\Big|\\ \le\;& \sup_{0\le \bar s\le \bar t+\bar s\le T_1, \bar t\le C_9\lambda^{-1/4}} \Big|W_2^\lambda(\bar t+\bar s) - W_2^\lambda(\bar s)\Big|.\end{aligned}$$

So

$$P\Big((F_6^\lambda)^C\Big) \le P\Big(\sup_{0\le s\le s+t\le T_1, t\le C_9\lambda^{-1/4}} |\overline{B}_{t+s} - \overline{B}_s| > C_6\Big).$$

By Lemma 2.9 (3), the right hand side above converges to 0 as $\lambda\to\infty$. ∎

We next prove that the particle hits $r_3 - a_0$ very quickly for λ large enough (see Lemma 2.15 below), which is heuristically clear since the potential attractive force is extremely strong.

Lemma 2.15

1. *We have on the set* F_4^λ *that* $\tau_1^\lambda \le t_1^\lambda(X_0, V_0/2)$ *and that* $X_t^\lambda \in (r_3 - a_0, r_3)$ *for any* $t\in(0,\tau_1^\lambda)$,
2. *for any* $t_0 > 0$*, there exists a* $\lambda_2(t_0)\ge 1$ *such that for any* $\lambda\ge\lambda_2(t_0)$*, we have that* $F_4^\lambda \subset F_{7,t_0}^\lambda$,
3. *for any* $t_0>0$*, we have that* $\lim_{\lambda\to 0} P(F_{7,t_0}^\lambda) = 1$.

Proof. We have by definition that $d(e^{bt}V_t^\lambda) = -e^{bt}\lambda g'(X_t^\lambda)dt + e^{bt}\sigma(X_t^\lambda)dB_t$, hence

$$e^{bt}V_t^\lambda = V_0 - \lambda\int_0^t e^{bu}g'(X_u^\lambda)du + \int_0^t e^{bu}\sigma(X_u^\lambda)dB_u. \tag{2.14}$$

Similarly,

$$e^{bt}v_t^\lambda(X_0, V_0/2) = V_0/2 - \lambda\int_0^t e^{bu}g'(x_u^\lambda(X_0, V_0/2))du. \tag{2.15}$$

These two equations will be used later.

Let $\sigma_0^\lambda := \inf\Big\{t > 0; X_t^\lambda \geq x_t^\lambda(X_0, V_0/2)\Big\}$. We first notice that

$$g'(X_u^\lambda) \geq g'(x_u^\lambda(X_0, V_0/2)), \qquad \text{for any } u \leq \sigma_0^\lambda \wedge \tau_1^\lambda \wedge t_1^\lambda(X_0, V_0/2). \tag{2.16}$$

Indeed, $u \leq \sigma_0^\lambda$ implies that $X_u^\lambda < x_u^\lambda(X_0, V_0/2)$, $u \leq \tau_1^\lambda$ implies that $X_u^\lambda > r_3 - a_0$, and $u \leq t_1^\lambda(X_0, V_0/2)$ implies that $x_u^\lambda(X_0, V_0/2) > r_3 - a_0$. On the other hand, we have by assumptions (A1) and (A2) that $g'(\cdot)$ is monotone non-increasing on $(r_3 - a_0, \infty)$. These imply (2.16).

Assume that $\omega \in F_4^\lambda$. Then by (2.16), for any $s \leq \sigma_0^\lambda \wedge \tau_1^\lambda \wedge t_1^\lambda(X_0, V_0/2)$, we have by (2.14) and (2.15) that

$$\begin{aligned} e^{bs} V_s^\lambda &= V_0 - \lambda \int_0^s e^{bu} g'(X_u^\lambda) du + \int_0^s e^{bu} \sigma(X_u^\lambda) dB_u \\ &\leq V_0 - \lambda \int_0^s e^{bu} g'(x_u^\lambda(X_0, V_0/2)) du - V_0/4 \\ &= e^{bs} v_s^\lambda(X_0, V_0/2) + V_0/4, \end{aligned}$$

therefore,

$$V_s^\lambda \leq v_s^\lambda(X_0, V_0/2) + e^{-bs} V_0/4, \qquad \text{for any } s \leq \sigma_0^\lambda \wedge \tau_1^\lambda \wedge t_1^\lambda(X_0, V_0/2),$$

hence

$$X_s^\lambda \leq x_s^\lambda(X_0, V_0/2) + \int_0^s e^{-bu} \frac{V_0}{4} du, \qquad \text{for any } s \in (0, \sigma_0^\lambda \wedge \tau_1^\lambda \wedge t_1^\lambda(X_0, V_0/2)].$$

Therefore, we get that $\sigma_0^\lambda \wedge \tau_1^\lambda \wedge t_1^\lambda(X_0, V_0/2) < \sigma_0^\lambda$ on F_4^λ, hence

$$\tau_1^\lambda \wedge t_1^\lambda(X_0, V_0/2) < \sigma_0^\lambda, \qquad \text{on } F_4^\lambda. \tag{2.17}$$

Now, suppose that $\tau_1^\lambda > t_1^\lambda(X_0, V_0/2)$. Notice that $X_\cdot^\lambda \geq r_3 - a_0$ until τ_1^λ, so our assumption implies that $X^\lambda_{t_1^\lambda(X_0,V_0/2)} \geq r_3 - a_0 = x^\lambda_{t_1^\lambda(X_0,V_0/2)}$, therefore, $t_1^\lambda(X_0, V_0/2) \geq \sigma_0^\lambda$, hence $\tau_1^\lambda \wedge t_1^\lambda(X_0, V_0/2) = t_1^\lambda(X_0, V_0/2) \geq \sigma_0^\lambda$. This contradicts (2.17). Therefore, we have that $\tau_1^\lambda \leq t_1^\lambda(X_0, V_0/2)$ on F_4^λ. This completes the proof of our first assertion.

Notice that by Lemma 2.6, we have that $\lim_{\lambda\to\infty} t_1^\lambda(X_0, V_0/2) = 0$. This combined with the first assertion implies our second assertion.

The third assertion is a direct consequence of the second assertion and Lemma 2.12. ∎

We define two more notations. Let

$$\begin{aligned} &\widetilde{g} := g - g(r_2), \\ &\widetilde{H_t^\lambda} := H_t^\lambda - \lambda g(r_2) = \frac{1}{2}|V_t^\lambda|^2 + \lambda \widetilde{g}(X_t^\lambda). \end{aligned} \tag{2.18}$$

Lemma 2.16 *1. The followings hold on the set F_5^λ for any $t \in [0, T]$:*

(a)

$$H_t^\lambda \geq e^{-2(b+1)t}(H_0 - \lambda^{1/2}) + g(r_2)\lambda(1 - e^{-2(b+1)t}), \tag{2.19}$$

(b)

$$|V_t^\lambda|^2 \geq -2\lambda^{1/2} - 2\lambda\Big(- g(r_2)[1-e^{-2(b+1)t}] + g(X_t^\lambda)\Big). \tag{2.20}$$

2. *For any $t \in (0, T_0]$ and any $\lambda \geq \lambda_3 := \Big(- \frac{1}{8} g(r_3 - a_0)\Big)^{-2} \vee 1$, we have that the followings hold on the set F_5^λ:*

   (a)

$$H_t^\lambda \geq \frac{1}{4} g(r_3 - a_0)\lambda, \tag{2.21}$$

   (b) if $X_t^\lambda \in (r_1 + a_1, x_0)$ in addition, then

$$|V_t^\lambda|^2 \geq -\frac{1}{6} g(r_3 - a_0)\lambda. \tag{2.22}$$

Proof. Assume that $\omega \in F_5^\lambda$. We first prove the first assertion. By definition, we have that

$$d\Big(e^{2(b+1)t}\widetilde{H_t^\lambda}\Big) = e^{2(b+1)t}\Big(|V_t^\lambda|^2 dt + V_t^\lambda \sigma(X_t^\lambda) dB_t + 2(b+1)\lambda \widetilde{g}(X_t^\lambda)dt + \frac{1}{2}\sigma(X_t^\lambda)^2 dt\Big).$$

Since $\omega \in F_5^\lambda$ and $\widetilde{g} \geq 0$, this implies that

$$\begin{aligned} e^{2(b+1)t}\widetilde{H_t^\lambda} &\geq \widetilde{H_0^\lambda} + \int_0^t e^{2(b+1)s}|V_s^\lambda|^2 ds + \int_0^t e^{2(b+1)s} V_s^\lambda \sigma(X_s^\lambda) dB_s \\ &\geq \widetilde{H_0^\lambda} - \lambda^{1/2}. \end{aligned}$$

This implies (2.19). (2.20) is a direct consequence of (2.19) since $H_0 > 0$ and $|V_t^\lambda|^2 = 2H_t^\lambda - 2\lambda g(X_t^\lambda)$.

For the second assertion, notice that $e^{-2(b+1)t} \in (0,1]$; $\lambda \geq \Big(- \frac{1}{8} g(r_3 - a_0)\Big)^{-2}$ implies $-\lambda^{-1/2} \geq \frac{1}{8} g(r_3 - a_0)\lambda$; and $t \in (0, T_0]$ implies $-g(r_2)[1 - e^{-2(b+1)t}] \leq -\frac{1}{8} g(r_3 - a_0)$. Substituting these into (2.19), we get that

$$\begin{aligned} H_t^\lambda &\geq e^{-2(b+1)t}(H_0 - \lambda^{1/2}) + g(r_2)\lambda(1 - e^{-2(b+1)t}) \\ &\geq \frac{1}{8} g(r_3 - a_0)\lambda + \frac{1}{8} g(r_3 - a_0)\lambda = \frac{1}{4} g(r_3 - a_0)\lambda. \end{aligned}$$

So (2.21) holds under the present setting. Finally, if $X_t^\lambda \in (r_1 + a_1, x_0)$, then $g(X_t^\lambda) < \frac{1}{3} g(r_3 - a_0)$ by (2.2) and (2.10). Combining this with (2.21), we get that

$$\begin{aligned} |V_t^\lambda|^2 &= 2H_t^\lambda - 2g(X_t^\lambda)\lambda \\ &\geq \frac{1}{2} g(r_3 - a_0)\lambda - 2 \cdot \frac{1}{3} g(r_3 - a_0)\lambda = -\frac{1}{6} g(r_3 - a_0)\lambda. \end{aligned}$$

∎

The next lemma claims that the energy loss until τ_1^λ is already large enough. This is heuristically clear since $|V_t^\lambda|$ is of order $\lambda^{1/2}$ around $r_3 - a_0$. This helps us to prove (see the proof of Lemma 2.20) that with asymptotically full probability, the particle could never leave the interval $[r_1, r_3]$ after τ_1^λ.

Lemma 2.17 *1. There exists a $\lambda_4 \geq 1$ such that for any $\lambda \geq \lambda_4$, we have that*

$$F_5^\lambda \cap F_4^\lambda \subset F_8^\lambda \cap F_{7,T_0}^\lambda,$$

2. we have that $\lim_{\lambda\to\infty} P(F_8^\lambda) = 1$.

Proof. By Lemma 2.15 (2), for any $\lambda \geq \lambda_2(T_0)$, we have that $F_4^\lambda \subset F_{7,T_0}^\lambda = \{\tau_1^\lambda \leq T_0\}$, hence we have that

$$t \leq \tau_1^\lambda \Rightarrow t \leq T_0, \qquad \text{on the set } F_4^\lambda. \tag{2.23}$$

Let $\lambda_4 := \lambda_2(T_0) \vee \Big(-\frac{1}{8}g(r_3 - a_0)\Big)^{-2}$. Then by (2.23) and Lemma 2.16 (2b), we get that for any $\lambda \geq \lambda_4$, we have on the set $F_5^\lambda \cap F_4^\lambda$ that if $t \leq \tau_1^\lambda$ and $X_t^\lambda \in (r_3 - a_0, x_0)$, then $|V_t^\lambda|^2 \geq -\frac{1}{6}g(r_3 - a_0)\lambda$.

On the other hand, since $\{X_t^\lambda\}_t$ is continuous and $X_0^\lambda = r_3$, we have that until it arrives $r_3 - a_0$ at time τ_1^λ, it must passes through x_0 at least once. Let $\eta_{x_0} := \sup\{t < \tau_1^\lambda; X_t^\lambda \geq x_0\}$. Then $t \in (\eta_{x_0}, \tau_1^\lambda) \Rightarrow X_t^\lambda \in (r_3 - a_0, x_0)$.

Therefore, for any $\lambda \geq \lambda_4$, we have on the set $F_5^\lambda \cap F_4^\lambda$ that

$$\begin{aligned}\int_0^{\tau_1^\lambda} |V_s^\lambda|^2 ds &\geq \int_{\eta_{x_0}}^{\tau_1^\lambda} |V_s^\lambda|^2 ds \geq \int_{r_3-a_0}^{x_0} \Big(-\frac{1}{6}g(r_3 - a_0)\lambda\Big)^{1/2} dx \\ &= \Big(-\frac{1}{6}g(r_3 - a_0)\Big)^{1/2} (x_0 - (r_3 - a_0))\lambda^{1/2} = C_4\lambda^{1/2}.\end{aligned}$$

So $F_5^\lambda \cap F_4^\lambda \subset F_8^\lambda$. This implies our first assertion.

The second assertion is now easy by Lemmas 2.12 and 2.13. ∎

For any $\lambda \geq 1$, let

$$M_3^\lambda(t) := \int_0^t V_s^\lambda \sigma(X_s^\lambda) dB_s, \qquad t \geq 0.$$

This notation is used in the proofs of Lemmas 2.18 and 2.19.

F_9^λ is useful for the discussion with respect to the diffusion term of H_t^λ after the first hit time of the particle to $r_3 - a_0$. Precisely, on F_9^λ, the diffusion term is dominated by a small part of the drift part (see Lemma 2.19 and the proof of Lemma 2.20). We prove in the following that it is also asymptotically of probability 1. The basic idea is that, by restricting on the set F_8^λ, the quadratic variation of its diffusion term is large enough after τ_1^λ. So by re-expressing the diffusion term as a time-changed Brownian motion, we get our assertion as a direct result of Brownian motion's property.

Lemma 2.18 *We have that* $\lim_{\lambda\to\infty} P(F_9^\lambda) = 1$.

Proof. For any $\lambda \geq 1$, notice that M_3^λ is a continuous martingale. So there exists a Brownian motion $W_3^\lambda(\cdot)$ such that $M_3^\lambda(t) = W_3^\lambda\Big(\langle M_3^\lambda, M_3^\lambda\rangle_t\Big), t \geq 0$. On the other hand, let $C_{10} > 0$ be a constant such that $\sigma^2 \geq C_{10}$. Then on the set F_8^λ, we have that

$$\langle M_3^\lambda, M_3^\lambda\rangle_{\tau_1^\lambda} = \int_0^{\tau_1^\lambda} |V_s^\lambda|^2 \sigma(X_s^\lambda)^2 ds \geq C_{10} \int_0^{\tau_1^\lambda} |V_s^\lambda|^2 ds \geq C_{10}C_4\lambda^{1/2},$$

hence

$$\sup_{t\geq\tau_1^\lambda}\Big(M_3^\lambda(t)-\varepsilon_0\langle M_3^\lambda,M_3^\lambda\rangle_t\Big)\leq\sup_{s\geq C_{10}C_4\lambda^{1/2}}\Big(W_3^\lambda(s)-\varepsilon_0 s\Big).$$

Therefore,

$$P\Big((F_9^\lambda)^C\cap F_8^\lambda\Big)\leq P\Big(\Big\{\sup_{s\geq C_{10}C_4\lambda^{1/2}}\Big(W_3^\lambda(s)-\varepsilon_0 s\Big)\geq 0\Big\}\Big),$$

which, by Lemma 2.9 (2), converges to 0 as $\lambda\to\infty$. This combined with Lemma 2.17 (2) completes the proof of our assertion. ∎

Lemma 2.19 *We have that the following holds on the set* F_9^λ*:*

$$H_t^\lambda\leq\Big(-b+\varepsilon_0\|\sigma\|_\infty^2\Big)\int_0^t|V_s^\lambda|^2ds+H_0+\frac{T}{2}\|\sigma\|_\infty^2,\qquad \textit{for any } t\in[\tau_1^\lambda,T].$$

Proof. By the definition of $M_3^\lambda(\cdot)$, we have that

$$\langle M_3^\lambda,M_3^\lambda\rangle_t=\int_0^t|V_s^\lambda|^2\sigma(X_s^\lambda)^2ds\leq\|\sigma\|_\infty^2\int_0^t|V_s^\lambda|^2ds.$$

So on the set F_9^λ, we have for any $t\in[\tau_1^\lambda,T]$ that

$$\begin{aligned}H_t^\lambda &= H_0-b\int_0^t|V_s^\lambda|^2ds+M_3^\lambda(t)+\frac{1}{2}\int_0^t\sigma(X_s^\lambda)^2ds\\ &\leq M_3^\lambda(t)-\varepsilon_0\langle M_3^\lambda,M_3^\lambda\rangle_t+(-b+\varepsilon_0\|\sigma\|_\infty^2)\int_0^t|V_s^\lambda|^2ds+H_0+\frac{T}{2}\|\sigma\|_\infty^2\\ &\leq (-b+\varepsilon_0\|\sigma\|_\infty^2)\int_0^t|V_s^\lambda|^2ds+H_0+\frac{T}{2}\|\sigma\|_\infty^2. \qquad (2.24)\end{aligned}$$

∎

With the help of F_8^λ and F_9^λ, we prove that $\lim_{\lambda\to\infty}P(F_{10}^\lambda)=1$. This is heuristically clear since after the first hitting time to r_3-a_0, we have on $F_8^\lambda\cap F_9^\lambda$ that the drift term of the energy loss is large enough, and the random part is much weaker compared with the drift term.

Lemma 2.20 *We have that* $\lim_{\lambda\to\infty}P(F_{10}^\lambda)=1$.

Proof. Choose $\lambda_5\geq 1$ large enough such that $\Big(-b+\varepsilon_0\|\sigma\|_\infty^2\Big)C_4\lambda^{1/2}+H_0+T\|\sigma\|_\infty^2<0$ for all $\lambda\geq\lambda_5$. Notice that if $t\geq\tau_1^\lambda$, then on the set F_8^λ, we have that $\int_0^t|V_s^\lambda|^2ds\geq\int_0^{\tau_1^\lambda}|V_s^\lambda|^2ds\geq C_4\lambda^{1/2}$. So by Lemma 2.19, for any $\lambda\geq\lambda_5$, we have on the set $F_9^\lambda\cap F_8^\lambda$ that $H_t^\lambda<0$ for any $t\in[\tau_1^\lambda,T]$. So

$$\lambda\geq\lambda_5\Rightarrow F_9^\lambda\cap F_8^\lambda\subset F_{10}^\lambda.$$

This combined with Lemmas 2.17 (2) and 2.18 implies our assertion. ∎

Finally, we prove that when $\lambda\to\infty$, $P(F_{11,c}^\lambda)\to 1$. In other words, we prove that asymptotically, the solution (position and velocity) of the stochastic differential equation (1.1) could be approximated by that of the ordinary differential equation (2.7) with the same initial condition, for time short enough. This fact is heuristically almost trivial, since the involved processes are continuous. This is used in the proof of Lemma 4.2.

Lemma 2.21 *1. There exists a constant $C_{11} > 0$ such that for any $\lambda \geq 1$ and any $t_0 \in [0,T]$, the following holds:*

$$E\Big[\sup_{0\leq s\leq s+t\leq T, t\leq t_0}\Big(|X^\lambda_{s+t} - x^\lambda_t(X^\lambda_s, V^\lambda_s)| + \lambda^{-1/2}|V^\lambda_{s+t} - v^\lambda_t(X^\lambda_s, V^\lambda_s)|\Big)^2\Big]^{1/2}$$
$$\leq \lambda^{-1/2}\|\sigma\|_\infty 4\sqrt{T}\exp\Big(C_{11}\lambda^{1/2}t_0\Big).$$

2. For any $c > 0$, we have that $\lim_{\lambda\to\infty} P\Big(F^\lambda_{11,c}\Big) = 1$.

Proof. The second assertion is a direct consequence of the first assertion by Chebyshev's inequality. We prove the first assertion in the following.

We have by definition that

$$\begin{aligned}
&|X^\lambda_{s+t} - x^\lambda_t(X^\lambda_s, V^\lambda_s)| + \lambda^{-1/2}|V^\lambda_{s+t} - v^\lambda_t(X^\lambda_s, V^\lambda_s)| \\
\leq\ & \int_0^t \Big|V^\lambda_{s+u} - v^\lambda_u(X^\lambda_s, V^\lambda_s)\Big|du + b\int_0^t \lambda^{-1/2}\Big|V^\lambda_{s+u} - v^\lambda_u(X^\lambda_s, V^\lambda_s)\Big|du \\
& \quad + \lambda^{1/2}\|g''\|_\infty \int_0^t \Big|X^\lambda_{s+u} - x^\lambda_u(X^\lambda_s, V^\lambda_s)\Big|du + \lambda^{-1/2}\Big|\int_s^{s+t}\sigma(X^\lambda_u)dB_u\Big| \\
\leq\ & \Big(\lambda^{1/2}(\|g''\|_\infty + 1) + b\Big)\int_0^t \Big\{|X^\lambda_{s+u} - x^\lambda_u(X^\lambda_s, V^\lambda_s)| \\
& \quad + \lambda^{-1/2}|V^\lambda_{s+u} - v^\lambda_u(X^\lambda_s, V^\lambda_s)|\Big\}du + \lambda^{-1/2}\Big|\int_s^{s+t}\sigma(X^\lambda_u)dB_u\Big|.
\end{aligned}$$

Let $C_{11} := \|g''\|_\infty + 1 + b$, and let

$$r(u) := |X^\lambda_{s+u} - x^\lambda_u(X^\lambda_s, V^\lambda_s)| + \lambda^{-1/2}|V^\lambda_{s+u} - v^\lambda_u(X^\lambda_s, V^\lambda_s)|.$$

Then the calculation above implies that

$$r(t) \leq \lambda^{1/2}C_{11}\int_0^t r(u)du + \lambda^{-1/2}\Big|\int_s^{s+t}\sigma(X^\lambda_u)dB_u\Big|, \qquad \text{for all } t \geq 0.$$

So by Gronwall's Lemma, we get that

$$r(t) \leq \lambda^{-1/2}\Big|\int_s^{s+t}\sigma(X^\lambda_u)dB_u\Big|\exp\Big(\lambda^{1/2}C_{11}t\Big), \qquad \text{for all } t \geq 0.$$

We have by Doob's inequality that

$$\begin{aligned}
& E\Big[\sup_{0\leq s\leq s+t\leq T, t\leq t_0}\Big|\int_s^{s+t}\sigma(X^\lambda_u)dB_u\Big|^2\Big]^{1/2} \\
\leq\ & 2E\Big[\sup_{0\leq t\leq T}\Big|\int_0^t \sigma(X^\lambda_u)dB_u\Big|^2\Big]^{1/2} \leq 4E\Big[\Big|\int_0^T \sigma(X^\lambda_u)dB_u\Big|^2\Big]^{1/2} \\
\leq\ & 4\sqrt{T}\|\sigma\|_\infty.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& E\Big[\sup_{0\le s\le s+t\le T, t\le t_0}\Big(|X^\lambda_{s+t}-x^\lambda_t(X^\lambda_s,V^\lambda_s)|+\lambda^{-1/2}|V^\lambda_{s+t}-v^\lambda_t(X^\lambda_s,V^\lambda_s)|\Big)^2\Big]^{1/2} \\
\le\ & \lambda^{-1/2}\exp\Big(C_{11}\lambda^{1/2}t_0\Big)E\Big[\sup_{0\le s\le s+t\le T, t\le t_0}\Big|\int_s^{s+t}\sigma(X^\lambda_u)dB_u\Big|^2\Big]^{1/2} \\
\le\ & \lambda^{-1/2}\exp\Big(C_{11}\lambda^{1/2}t_0\Big)4\sqrt{T}\|\sigma\|_\infty.
\end{aligned}
$$

∎

We close this section by emphasizing again that for any $t_0, c > 0$, by Proposition 2.10, $\overline{F}^\lambda_{t_0,c}$ has probability 1 asymptotically, so all the conditions in the definitions of the sets $F^\lambda_1 \sim F^\lambda_{11,c}$ hold "asymptotically".

3 $\{J_t\}_{t>0}$ is non-increasing and negative under P_∞

We prove in this section that after taking $\lambda \to \infty$, the particle stays in the domain (r_1, r_3) for $t > 0$, with its oscillating range non-increasing.

We first have the following.

Lemma 3.1 *1. For any $\lambda \ge 1$, we have on the set F^λ_1 that*

$$
H^\lambda_t \le H^\lambda_s + 2\lambda^{3/4} + \frac{t-s}{2}\|\sigma\|^2_\infty, \qquad \textit{if } 0 \le s < t \le T,
$$

2. for any $\delta > 0$, we have that

$$
\lim_{\lambda\to\infty} P\Big(J^\lambda_t \le J^\lambda_s + \delta \textit{ for any } 0 \le s < t \le T\Big) = 1.
$$

Proof. For any $0 \le s < t \le T$, we have that

$$
\begin{aligned}
& H^\lambda_t - H^\lambda_s \\
=\ & \int_s^t V^\lambda_u \sigma(X^\lambda_u)dB_u - b\int_s^t |V^\lambda_u|^2 du + \frac{1}{2}\int_s^t \sigma(X^\lambda_u)^2 du \\
\le\ & \Big|\int_0^t V^\lambda_u\sigma(X^\lambda_u)dB_u\Big| + \Big|\int_0^s V^\lambda_u\sigma(X^\lambda_u)dB_u\Big| + \frac{t-s}{2}\|\sigma\|^2_\infty.
\end{aligned}
$$

This combined with the definition of F^λ_1 gives us our first assertion.

For any $\delta > 0$, let $\lambda_6(\delta) := \Big(T\|\sigma\|^2_\infty\delta^{-1}\Big) \vee (4\delta^{-1})^4 \vee 1$. Then for any $\lambda \ge \lambda_6(\delta)$, we have that $2\lambda^{3/4} + \frac{T}{2}\|\sigma\|^2_\infty \le \lambda\delta$, hence by our first assertion, we have for any $0 \le s < t \le T$ that $H^\lambda_t \le H^\lambda_s + \delta\lambda$ on the set F^λ_1. So

$$
P\Big(J^\lambda_t \le J^\lambda_s + \delta \text{ for any } 0 \le s < t \le T\Big) \ge P\Big(F^\lambda_1\Big), \qquad \lambda \ge \lambda_6(\delta).
$$

This combined with Lemma 2.11 implies our second assertion. ∎

The following is our first main result of this section:

PROPOSITION 3.2 *We have that*

$$P_\infty\Big(J.\ \textit{is monotone non-increasing and continuous}\Big) = 1.$$

Proof. Since $\{w \in C([0,T];\mathbf{R}) : \exists s,t \in [0,T], s.t.\ s < t \text{ and } w(s) < w(t) - \delta\}$ is open in $C([0,T];\mathbf{R})$ for any $\delta > 0$, Lemma 3.1 (2) implies that $J.$ under P_∞ is monotone non-increasing. The continuity is trivial by the definition of P_∞. ∎

As noticed in Remark 6, we need to confirm that J_t $(t > 0)$ of the limiting process is strictly negative with probability 1. We prove this in the rest of this section (see Proposition 3.7).

To simplify notations, from now on, when there are more than one λ in one notation, we write λ only once and omit the others. So we write $H^\lambda_{\tau^\lambda_{2k}}$ as $H^\lambda_{\tau_{2k}}$, and so on.

Lemma 3.3 *There exists a constant $\lambda_7 \geq 1$ such that for any $\lambda \geq \lambda_7$, we have that the following holds on the set $F^\lambda_5 \cap F^\lambda_9$:*

$$k \geq 1, \tau^\lambda_{2k} \leq T_0 \Rightarrow H^\lambda_{\tau_{2k}} \leq -C_5\lambda^{1/2}k.$$

Here C_5 is as defined in (2.11), and τ^λ_k is as defined in (2.13).

Proof. By Lemma 2.16 (2b), for any $\lambda \geq \lambda_3$ and $t \in (0, T_0]$, we have on the set F^λ_5 that

$$X^\lambda_t \in [r_1 + a_1, r_3 - a_0] \Rightarrow |V^\lambda_t|^2 \geq -\frac{1}{6}g(r_3 - a_0)\lambda > 0.$$

In particular, $|V^\lambda_t| \neq 0$ in this domain. Since V^λ_t is continuous, this means that the particle could not stop or turn back between $r_1 + a_1$ and $r_3 - a_0$. In other words, during the period $[\tau^\lambda_{2k-1}, \tau^\lambda_{2k}]$, the particle passes through $[r_1 + a_1, r_3 - a_0]$ exactly twice. Therefore,

$$\begin{aligned}\int_{\tau^\lambda_{2k-1}}^{\tau^\lambda_{2k}} |V^\lambda_s|^2 ds &\geq 2\int_{r_1+a_1}^{r_3-a_0} \sqrt{-\frac{1}{6}g(r_3 - a_0)\lambda}\, dx \\ &= \sqrt{\frac{2}{3}}|g(r_3 - a_0)|^{1/2}(r_3 - a_0 - r_1 - a_1)\lambda^{1/2}, \qquad \text{if } \tau^\lambda_{2k} \leq T_0,\end{aligned}$$

hence

$$\int_0^{\tau^\lambda_{2k}} |V^\lambda_t|^2 ds \geq \sqrt{\frac{2}{3}}|g(r_3 - a_0)|^{1/2}(r_3 - a_0 - r_1 - a_1)\lambda^{1/2}k, \qquad \text{if } \tau^\lambda_{2k} \leq T_0.$$

Since $\tau^\lambda_{2k} \geq \tau^\lambda_1$, this combinded with Lemma 2.19 implies that on the set $F^\lambda_5 \cap F^\lambda_9$, if $\tau^\lambda_{2k} \leq T_0$, then

$$\begin{aligned}H^\lambda_{\tau_{2k}} &\leq (-b + \varepsilon_0\|\sigma\|^2_\infty)\int_0^{\tau^\lambda_{2k}} |V^\lambda_s|^2 ds + H_0 + \frac{T}{2}\|\sigma\|^2_\infty \\ &\leq (-b + \varepsilon_0\|\sigma\|^2_\infty)\sqrt{\frac{2}{3}}|g(r_3 - a_0)|^{1/2}(r_3 - a_0 - r_1 - a_1)\lambda^{1/2}k + H_0 + \frac{T}{2}\|\sigma\|^2_\infty.\end{aligned}$$

So we get our assertion with $\lambda_7 := \lambda_3 \vee \Big((H_0 + T\|\sigma\|^2_\infty/2)C_5^{-1}\Big)^2$. ∎

Lemma 3.4 *There exist constants $C_{12} > 0$ and $\lambda_8 \geq 1$ such that for any $\lambda \geq \lambda_8$, the followings hold:*

1. *on the set $F_5^\lambda \cap F_{10}^\lambda \cap F_3^\lambda$, we have that*

$$k \geq 1, \tau_{2k}^\lambda \leq T_0 \Rightarrow \tau_{2k}^\lambda - \tau_{2k-1}^\lambda \leq C_{12}\lambda^{-1/2},$$

2. *on the set $F_5^\lambda \cap F_9^\lambda \cap F_{10}^\lambda \cap F_6^\lambda$, we have that*

$$k \geq 1, \tau_{2k}^\lambda \leq T_0, \tau_{2k+1}^\lambda \leq T \Rightarrow \tau_{2k+1}^\lambda - \tau_{2k}^\lambda \leq C_{12}\lambda^{-1/4}k^{-1/2}.$$

Proof. Let us first prove the first assertion. By Lemma 2.16 (2b), for any $\lambda \geq \lambda_3$, we have on the set F_5^λ that

$$t \in (0, T_0], X_t^\lambda \in [r_1 + a_1, r_3 - a_0] \Rightarrow |V_t^\lambda|^2 \geq -\frac{1}{6}g(r_3 - a_0)\lambda > 0.$$

In particular, since V_t^λ is continuous, we get that the particle could not stop or turn back in this domain. So if $\tau_{2k}^\lambda \leq T_0$, then during the period $[\tau_{2k-1}^\lambda, \tau_{2k}^\lambda]$, the particle passes through $r_1 + a_1$ exactly twice. Write them as ξ_1, ξ_2. Then

$$\begin{aligned}
&(\xi_1 - \tau_{2k-1}^\lambda) + (\tau_{2k}^\lambda - \xi_2) \\
\leq\ & 2\int_{r_1+a_1}^{r_3-a_0} \frac{1}{\sqrt{-\frac{1}{6}g(r_3 - a_0)\lambda}}dx \\
=\ & 2\Big(-\frac{1}{6}g(r_3 - a_0)\Big)^{-1/2}(r_3 - a_0 - r_1 - a_1)\lambda^{-1/2}, \qquad \text{if } \tau_{2k}^\lambda \leq T_0.
\end{aligned} \tag{3.1}$$

For the period $[\xi_1, \xi_2]$, notice that on the set F_{10}^λ, we have for any $t \in [\xi_1, \xi_2] \subset [\tau_1^\lambda, T]$ that $H_t^\lambda < 0$, hence $|V_t^\lambda|^2 = 2H_t^\lambda - 2\lambda g(X_t^\lambda) < -2\lambda g(r_2)$. Also, by (2.1), we have that $x \in [r_1, r_1 + a_1] \Rightarrow g'(x) < -a_2$. So if $\lambda \geq -8b^2a_2^{-2}g(r_2)$ in addition, then we have that $-bV_s^\lambda - \lambda g'(X_s^\lambda) \geq -b\sqrt{-2g(r_2)}\lambda^{1/2} + \lambda a_2 \geq \frac{1}{2}\lambda a_2$. Therefore, on the set $F_{10}^\lambda \cap F_3^\lambda$, we have that

$$\begin{aligned}
&2\sqrt{-2g(r_2)}\lambda^{1/2} \\
\geq\ & V_{\xi_2}^\lambda - V_{\xi_1}^\lambda = \int_{\xi_1}^{\xi_2}\Big(-bV_s^\lambda - \lambda g'(X_s^\lambda)\Big)ds + \int_{\xi_1}^{\xi_2}\sigma(X_s^\lambda)dB_s \\
\geq\ & \frac{1}{2}\lambda a_2(\xi_2 - \xi_1) - 2\sup_{t\in[0,T]}\Big|\int_0^t \sigma(X_s^\lambda)dB_s\Big| \\
\geq\ & \frac{1}{2}\lambda a_2(\xi_2 - \xi_1) - 2\lambda^{1/2},
\end{aligned} \tag{3.2}$$

hence

$$\xi_2 - \xi_1 \leq \frac{2(\sqrt{-2g(r_2)} + 1)\lambda^{1/2}}{\frac{1}{2}\lambda a_2} = \frac{4(\sqrt{-2g(r_2)} + 1)}{a_2}\lambda^{-1/2}.$$

This combined with (3.1) implies our first assertion.

We next prove the second assertion. Let C_5 and C_6 be as in (2.11) and (2.12). For any $\lambda \geq \Big(C_5^{-1}2C_6^2\Big)^2 \vee 1$, we have that $C_6\sqrt{-2g(r_2)}\lambda^{1/2} + \frac{1}{2}C_6^2 \leq \frac{C_5}{2}\lambda^{1/2}$, hence on the set F_{10}^λ, we have that $C_6 V_{\tau_{2k}}^\lambda + \frac{1}{2}C_6^2 \leq C_6\sqrt{-2g(r_2)}\lambda^{1/2} + \frac{1}{2}C_6^2 \leq \frac{C_5}{2}\lambda^{1/2}$. So for any $\lambda \geq \lambda_7 \vee \left(C_5^{-1}2C_6^2\right)^2 \vee \lambda_1(\frac{C_5}{2})$, by Lemma 3.3, we have on the set $F_5^\lambda \cap F_9^\lambda \cap F_{10}^\lambda$ that if $k \geq 1$ and $\tau_{2k}^\lambda \leq T_0$, then

$$\begin{aligned} \lambda g(X_{\tau_{2k}}^\lambda) + \frac{1}{2}(V_{\tau_{2k}}^\lambda + C_6)^2 &= H_{\tau_{2k}}^\lambda + C_6 V_{\tau_{2k}}^\lambda + \frac{1}{2}C_6^2 \\ \leq \quad -C_5\lambda^{1/2}k + \frac{C_5}{2}\lambda^{1/2} &\leq -\frac{C_5}{2}\lambda^{1/2}k. \end{aligned}$$

Also, $X_{\tau_{2k}}^\lambda = r_3 - a_0$ and $V_{\tau_{2k}}^\lambda + C_6 > 0$. Therefore, by Proposition 2.7 and the definition of C_7, we get that

$$t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) \leq C_7\lambda^{-1/4}k^{-1/2}. \tag{3.3}$$

Notice that we did not use the condition $\tau_{2k+1}^\lambda \leq T$ to get (3.3). This result will also be used in the proof of Lemma 3.5 later.

So in order to prove our second assertion, it suffices to prove that

$$\tau_{2k+1}^\lambda - \tau_{2k}^\lambda \leq t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6). \tag{3.4}$$

The proof of (3.4) is similar to that of Lemma 2.15 (1). Let

$$\sigma_k := \inf\left\{t > 0; X_{\tau_{2k}+t}^\lambda \geq x_t^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6)\right\}.$$

Claim 1. For any $\lambda \geq \lambda_7 \vee \left(C_5^{-1}2C_6^2\right)^2$, we have that the following holds on the set $F_5^\lambda \cap F_9^\lambda \cap F_{10}^\lambda \cap F_6^\lambda$:

$$k \geq 1, \tau_{2k}^\lambda \leq T_0, \tau_{2k+1}^\lambda \leq T \Rightarrow \sigma_k \wedge \left(\tau_{2k+1}^\lambda - \tau_{2k}^\lambda\right) \wedge t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) < \sigma_k.$$

Proof of Claim 1. Choose and fix any $t \in (0, \sigma_k \wedge \left(\tau_{2k+1}^\lambda - \tau_{2k}^\lambda\right) \wedge t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6)]$. By (3.3), we have that $t \leq C_7\lambda^{-1/4}$, so on the set F_6^λ, we get that

$$\int_{\tau_{2k}^\lambda}^{t+\tau_{2k}^\lambda} e^{b(s-\tau_{2k}^\lambda)}\sigma(X_s^\lambda)dB_s = e^{-b\tau_{2k}^\lambda}\int_{\tau_{2k}^\lambda}^{t+\tau_{2k}^\lambda} e^{bs}\sigma(X_s^\lambda)dB_s \leq e^{-b\tau_{2k}^\lambda}C_6 \leq C_6. \tag{3.5}$$

Also, for any $s \in (0,t)$, since $\tau_{2k}^\lambda < s+\tau_{2k}^\lambda < t+\tau_{2k}^\lambda \leq \tau_{2k+1}^\lambda$, we have that $X_{s+\tau_{2k}}^\lambda > r_3 - a_0$; since $s < t \leq t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6)$, we have that $x_s^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) \in (r_3 - a_0, r_3)$; finally, since $s < t \leq \sigma_k$, we have that $X_{s+\tau_{2k}}^\lambda < x_s^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6)$. Therefore, by assumption (A2), $g'(X_{s+\tau_{2k}}^\lambda) > g'(x_s^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6))$. So

$$\int_{\tau_{2k}^\lambda}^{t+\tau_{2k}^\lambda} e^{b(s-\tau_{2k}^\lambda)}g'(X_s^\lambda)ds = \int_0^t e^{bs}g'(X_{s+\tau_{2k}}^\lambda)ds > \int_0^t e^{bs}g'(x_s^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6))ds.$$

Combining this with (3.5), we get that

$$\begin{aligned} e^{bt}V_{t+\tau_{2k}}^\lambda &= V_{\tau_{2k}}^\lambda - \lambda\int_{\tau_{2k}^\lambda}^{t+\tau_{2k}^\lambda} e^{b(s-\tau_{2k}^\lambda)}g'(X_s^\lambda)ds + \int_{\tau_{2k}^\lambda}^{t+\tau_{2k}^\lambda} e^{b(s-\tau_{2k}^\lambda)}\sigma(X_s^\lambda)dB_s \\ &< V_{\tau_{2k}}^\lambda - \lambda\int_0^t e^{bs}g'(x_s^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6))ds + C_6 \\ &= e^{bt}v_t^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6). \end{aligned}$$

So $V^\lambda_{t+\tau_{2k}} < v^\lambda_t(X^\lambda_{\tau_{2k}}, V^\lambda_{\tau_{2k}}+C_6)$. This is true for any $t \in (0, \sigma_k \wedge \Big(\tau^\lambda_{2k+1} - \tau^\lambda_{2k}\Big) \wedge t^\lambda_1(X^\lambda_{\tau_{2k}}, V^\lambda_{\tau_{2k}} + C_6)]$. Therefore, $X^\lambda_{t+\tau_{2k}} < x^\lambda_t(X^\lambda_{\tau_{2k}}, V^\lambda_{\tau_{2k}} + C_6)$. So $t < \sigma_k$. This completes the proof of Claim 1. ∎

Assume that $\omega \in F^\lambda_5 \cap F^\lambda_9 \cap F^\lambda_{10} \cap F^\lambda_6$, and that $k \geq 1$, $\tau^\lambda_{2k} \leq T_0$, $\tau^\lambda_{2k+1} \leq T$. Then by Claim 1, we get that

$$\Big(\tau^\lambda_{2k+1} - \tau^\lambda_{2k}\Big) \wedge t^\lambda_1(X^\lambda_{\tau_{2k}}, V^\lambda_{\tau_{2k}} + C_6) < \sigma_k. \tag{3.6}$$

Assume that (3.4) were not the case. Then since $X^\lambda_{\cdot+\tau_{2k}} > r_3 - a_0$ until $\tau^\lambda_{2k+1} - \tau^\lambda_{2k}$, we get that $X^\lambda_{\tau_{2k}+t_1(X_{\tau_{2k}}, V_{\tau_{2k}}+C_6)} > r_3 - a_0 = x^\lambda_{t_1(X_{\tau_{2k}}, V_{\tau_{2k}}+C_6)}(X^\lambda_{\tau_{2k}}, V^\lambda_{\tau_{2k}} + C_6)$, hence $t^\lambda_1(X^\lambda_{\tau_{2k}}, V^\lambda_{\tau_{2k}}+C_6) \geq \sigma_k$. This combined with our assumption that $\tau^\lambda_{2k+1} - \tau^\lambda_{2k} > t^\lambda_1(X^\lambda_{\tau_{2k}}, V^\lambda_{\tau_{2k}} + C_6)$ contradicts (3.6). Therefore, (3.4) holds. This combined with (3.3) implies our second assertion. ∎

We prepare one more estimate before going further.

Lemma 3.5 *There exists a $\lambda_9 \geq 1$ such that the followings hold for any $\lambda \geq \lambda_9$:*

1. *we have on the set $F^\lambda_3 \cap F^\lambda_5 \cap F^\lambda_{10}$ that*

$$k \geq 1, \tau^\lambda_{2k-1} \leq \frac{3}{4} T_0 \Rightarrow \tau^\lambda_{2k} \leq T_0,$$

2. *we have on the set $F^\lambda_5 \cap F^\lambda_6 \cap F^\lambda_9 \cap F^\lambda_{10}$ that*

$$k \geq 1, \tau^\lambda_{2k} \leq \frac{T_0}{2} \Rightarrow \tau^\lambda_{2k+1} \leq \frac{3}{4} T_0,$$

3. *we have on the set $F^\lambda_3 \cap F^\lambda_5 \cap F^\lambda_6 \cap F^\lambda_9 \cap F^\lambda_{10}$ that*

$$k \geq 1, \tau^\lambda_{2k} \leq \frac{T_0}{2} \Rightarrow \tau^\lambda_{2(k+1)} \leq T_0.$$

Proof. (3) is obviously a direct consequence of the first two assertions. We prove (1) and (2). The calculation is similar to that of Lemma 3.4.

We prove the first assertion first. Recall that $X^\lambda_{\tau_{2k-1}} = r_3 - a_0$ and $V^\lambda_{\tau_{2k-1}} < 0$. Suppose that we are on the given set, and that $\lambda \geq \lambda_3 \vee \frac{6(12(r_3-a_0-r_1-a_1))^2}{|g(r_3-a_0)|T_0^2} \vee \Big(-8b^2 a_2^{-2} g(r_2)\Big) \vee \Big(\frac{48(\sqrt{-2g(r_2)}+1)}{T_0 a_2}\Big)^2$. As same as in the proof of Lemma 3.4, let

$$\xi_1 := \inf\{t > \tau^\lambda_{2k-1}; X^\lambda_t = r_1 + a_1\}, \quad \xi_2 := \inf\{t > \xi_1, X^\lambda_t = r_1 + a_1\}.$$

It suffice to prove (3.7) $\sim$ (3.9) below:

$$\xi_1 - \tau^\lambda_{2k-1} \leq \frac{T_0}{12}, \tag{3.7}$$

$$\xi_2 - \xi_1 \leq \frac{T_0}{12}, \tag{3.8}$$

$$\tau^\lambda_{2k} - \xi_2 \leq \frac{T_0}{12}. \tag{3.9}$$

We first prove that (3.7) holds. Suppose not. Then for any $s \in (\tau_{2k-1}^\lambda, \tau_{2k-1}^\lambda + \frac{T_0}{12})$, we have that $X_s \in (r_1 + a_1, r_3 - a_0)$. Also, since $\tau_{2k-1}^\lambda \le \frac{3}{4}T_0$ by assumption, we have that $s \in (0, T_0]$. So by Lemma 2.16 (2b), we have that $|V_s^\lambda|^2 \ge -\frac{1}{6}g(r_3 - a_0)\lambda > 0$, in particular, since $V_\cdot^\lambda$ is continuous, we get that $\{V_s^\lambda; s \in (\tau_{2k-1}^\lambda, \tau_{2k-1}^\lambda + \frac{T_0}{12})\}$ stays negative. So

$$r_1 + a_1 < X_{\tau_{2k-1}+\frac{T_0}{12}}^\lambda = X_{\tau_{2k-1}}^\lambda + \int_{\tau_{2k-1}^\lambda}^{\tau_{2k-1}^\lambda+\frac{T_0}{12}} V_s^\lambda ds \le r_3 - a_0 - \sqrt{-\frac{1}{6}g(r_3 - a_0)\lambda} \cdot \frac{T_0}{12}.$$

This contradicts the fact that $\lambda \ge \frac{6(12(r_3-a_0-r_1-a_1))^2}{|g(r_3-a_0)|T_0^2}$. Therefore, (3.7) holds.

We next prove (3.8). Suppose it were not the case. Then for any $s \in (\xi_1, \xi_1 + \frac{T_0}{12})$, we have that $s \in (\xi_1, \xi_2)$, so $X_s^\lambda \in (r_1, r_1 + a_1)$, hence $g'(X_s^\lambda) < -a_2$ by (2.1). Also, since $s \ge \xi_1 \ge \tau_1^\lambda$, we have that $H_s^\lambda < 0$, hence $|V_s^\lambda| < \sqrt{-2\lambda g(r_2)}$. Since $\lambda \ge -8b^2 a_2^{-2} g(r_2)$, these imply that $-bV_s^\lambda - \lambda g'(X_s^\lambda) \ge -b\sqrt{-2g(r_2)}\lambda^{1/2} + \lambda a_2 \ge \frac{1}{2}\lambda a_2$. Therefore,

$$\begin{aligned} 2\sqrt{-2g(r_2)}\lambda^{1/2} \;&>\; V_{\xi_1+\frac{T_0}{12}}^\lambda - V_{\xi_1}^\lambda = \int_{\xi_1}^{\xi_1+\frac{T_0}{12}} \Big(- bV_s^\lambda - \lambda g'(X_s^\lambda)\Big) ds + \int_{\xi_1}^{\xi_1+\frac{T_0}{12}} \sigma(X_s^\lambda) dB_s \\ &\ge\; \frac{1}{2}\lambda a_2 \cdot \frac{T_0}{12} - 2\lambda^{1/2}, \end{aligned}$$

hence $2(\sqrt{-2g(r_2)}+1) > \frac{T_0 a_2}{24}\lambda^{1/2}$. This contradicts the assumption that $\lambda \ge \Big(\frac{48(\sqrt{-2g(r_2)}+1)}{T_0 a_2}\Big)^2$. Therefore, (3.8) also holds.

Finally, we prove (3.9) by a similar method as that of (3.7). Suppose that $\tau_{2k}^\lambda - \xi_2 > \frac{T_0}{12}$. Then for any $s \in (\xi_2, \xi_2 + \frac{T_0}{12})$, we have that $X_s^\lambda \in (r_1 + a_1, r_3 - a_0)$. Also, with the help of (3.7) and (3.8), we have that $s \in (0, T_0]$. So by Lemma 2.16 (2b), we have that $|V_s^\lambda|^2 \ge -\frac{1}{6}g(r_3 - a_0)\lambda > 0$, in particular, since $V_\cdot^\lambda$ is continuous, we get that $\{V_s^\lambda; s \in (\xi_2, \xi_2 + \frac{T_0}{12})\}$ stays positive. So

$$r_3 - a_0 > X_{\xi_2+\frac{T_0}{12}}^\lambda = X_{\xi_2}^\lambda + \int_{\xi_2}^{\xi_2+\frac{T_0}{12}} V_s^\lambda ds \ge r_1 + a_1 + \sqrt{-\frac{1}{6}g(r_3 - a_0)\lambda} \cdot \frac{T_0}{12}.$$

This contradicts the fact that $\lambda \ge \frac{6(12(r_3-a_0-r_1-a_1))^2}{|g(r_3-a_0)|T_0^2}$. Therefore, (3.9) holds.

We next prove the second assertion of our lemma. Restrict ourselves on the given set. It suffices to prove that

$$\tau_{2k+1}^\lambda - \tau_{2k}^\lambda \le \frac{T_0}{4}. \tag{3.10}$$

Suppose not. Recall that $X_{\tau_{2k}}^\lambda = r_3 - a_0$ and $V_{\tau_{2k}}^\lambda > 0$. So for any $s \in (\tau_{2k}^\lambda, \tau_{2k}^\lambda + \frac{T_0}{4})$, we have that $X_s^\lambda \in (r_3 - a_0, r_3)$. As in the proof of Lemma 3.4, let $\sigma_k := \inf\Big\{t > 0; X_{\tau_{2k}+t}^\lambda \ge x_t^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6)\Big\}$. Then by exactly the same method as we used to prove Claim 1 in the proof of Lemma 3.4, we have that

$$\frac{T_0}{4} \wedge t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) \wedge (\tau_{2k+1}^\lambda - \tau_{2k}^\lambda) < \sigma_k. \tag{3.11}$$

On the other hand, as claimed in the proof of Lemma 3.4, we have that (3.3) holds in our present setting, too. So if $\lambda \geq (4C_7/T_0)^4$, then

$$t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) \leq C_7 \lambda^{-1/4} k^{-1/2} \leq C_7 \lambda^{-1/4} \leq T_0/4. \tag{3.12}$$

This combined with (3.11) implies that

$$t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) \wedge (\tau_{2k+1}^\lambda - \tau_{2k}^\lambda) < \sigma_k. \tag{3.13}$$

The rest is, again, the same as in the proof of Lemma 3.4. Precisely, if we assume that $\tau_{2k+1}^\lambda - \tau_{2k}^\lambda > t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6)$, then since $X_{\cdot+\tau_{2k}}^\lambda > r_3 - a_0$ until $\tau_{2k+1}^\lambda - \tau_{2k}^\lambda$, we get that $X_{\tau_{2k}+t_1(X_{\tau_{2k}}, V_{\tau_{2k}}+C_6)}^\lambda > r_3 - a_0 = x_{t_1(X_{\tau_{2k}}, V_{\tau_{2k}}+C_6)}^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6)$, hence $t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) \geq \sigma_k$. So $t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) \wedge (\tau_{2k+1}^\lambda - \tau_{2k}^\lambda) = t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6) \geq \sigma_k$. This contradicts (3.13). So $\tau_{2k+1}^\lambda - \tau_{2k}^\lambda \leq t_1^\lambda(X_{\tau_{2k}}^\lambda, V_{\tau_{2k}}^\lambda + C_6)$. This combined with (3.12) implies (3.10). ∎

Lemma 3.6 *For any $t \in (0, T_0/2]$, there exist constants $C_{13}(t) \in (0, -g(r_2))$ and $\lambda_{10}(t) \geq 1$ such that for any $\lambda \geq \lambda_{10}(t)$, we have on the set $F_1^\lambda \cap F_3^\lambda \cap F_5^\lambda \cap F_6^\lambda \cap F_{7,t/4}^\lambda \cap F_9^\lambda \cap F_{10}^\lambda$ that*

$$H_s^\lambda \leq -C_{13}(t)\lambda, \qquad \textit{for any } s \in [t, T].$$

Proof. Restrict ourselves on the given set. Let $K_t^\lambda := \inf\{k \in \mathbf{N}; \tau_{2k}^\lambda \geq t\}$. Since $\omega \in F_{7,t/4}^\lambda$, we have that $\tau_1^\lambda \leq t/4 \leq \frac{3}{4}T_0$, so by Lemma 3.5 (1), we get that $\tau_2^\lambda \leq T_0$. Therefore, by Lemma 3.4 (1), we get that $\tau_2^\lambda - \tau_1^\lambda \leq C_{12}\lambda^{-1/2}$. So if $\lambda \geq \left(\frac{4C_{12}}{t}\right)^2$, then $\tau_2^\lambda - \tau_1^\lambda \leq \frac{t}{4}$, hence $\tau_2^\lambda \leq \frac{t}{4} + \frac{t}{4} = \frac{t}{2}$. Therefore, we get by the definition of K_t^λ that $\sum_{k=2}^{K_t^\lambda} \left(\tau_{2k}^\lambda - \tau_{2(k-1)}^\lambda\right) = \tau_{2K_t}^\lambda - \tau_2^\lambda \geq t - \frac{t}{2} = \frac{t}{2}$.

On the other hand, we have that $\tau_{2(K_t-1)}^\lambda < t \leq T_0/2$, so by Lemma 3.5 (3), we have that $\tau_{2K_t}^\lambda \leq T_0$, hence by Lemma 3.4 (1) (2), we get that $\tau_{2k}^\lambda - \tau_{2(k-1)}^\lambda \leq C_{12}(\lambda^{-1/2} + \lambda^{-1/4}k^{-1/2})$ for any $k \in \{2, \cdots, K_t^\lambda\}$.

Combining the above, we get that if $\lambda \geq \left(\frac{4C_{12}}{t}\right)^2$, then

$$\frac{t}{2} \leq \sum_{k=2}^{K_t^\lambda} \left(\tau_{2k}^\lambda - \tau_{2(k-1)}^\lambda\right) \leq \sum_{k=2}^{K_t^\lambda} C_{12}(\lambda^{-1/2} + \lambda^{-1/4}k^{-1/2}). \tag{3.14}$$

Solving (3.14), we get that there exists a $C_{14}(t) > 0$ such that

$$K_t^\lambda \geq C_{14}(t)\lambda^{1/2}.$$

Indeed, first notice that in general, we have $\sum_{k=2}^m k^{-1/2} \leq 2\sqrt{m}$ for any $m \geq 2$, so (3.14) implies that $\frac{t}{2C_{12}} \leq \lambda^{-1/2}K_t^\lambda + 2\lambda^{-1/4}\sqrt{K_t^\lambda}$. This combined with $\lambda^{-1/4}\sqrt{K_t^\lambda} \geq 0$ implies that $\lambda^{-1/4}\sqrt{K_t^\lambda} \geq \sqrt{\frac{t}{2C_{12}} + 1} - 1$.

So with $C_{13}(t) := \frac{1}{2}C_5 C_{14}(t)$, we have by Lemma 3.3 that

$$H_{\tau_{2(K_t-1)}}^\lambda \leq -C_5 \lambda^{1/2} C_{14}(t) \lambda^{1/2} = -2C_{13}(t)\lambda + C_5 \lambda^{1/2}.$$

We have $t \geq \tau^\lambda_{2(K_t-1)}$. So if λ is large enough such that $2\lambda^{3/4} + \frac{T}{2}\|\sigma\|^2_\infty + C_5\lambda^{1/2} \leq C_{13}(t)\lambda$ in addition, then by Lemma 3.1 (1) and $\omega \in F^\lambda_1$, we have that the following holds for any $s \in [t, T]$:

$$\begin{aligned} H^\lambda_s &\leq H^\lambda_{\tau_{2(K_t-1)}} + 2\lambda^{3/4} + \frac{T}{2}\|\sigma\|^2_\infty \\ &\leq -2C_{13}(t)\lambda + C_{13}(t)\lambda = -C_{13}(t)\lambda. \end{aligned}$$

∎

Our second main result of this section is the following.

PROPOSITION 3.7 *We have that*

$$P_\infty\Big(J_t < 0 \text{ for any } t \in (0, T]\Big) = 1.$$

Proof. Choose any $t \in (0, T_0/2]$ and fix it for a while. By Lemma 3.6 we have that

$$P_\lambda(J_t > -C_{13}(t)) \leq P\Big(\Big(F^\lambda_1 \cap F^\lambda_3 \cap F^\lambda_5 \cap F^\lambda_6 \cap F^\lambda_{7,t/4} \cap F^\lambda_9 \cap F^\lambda_{10}\Big)^C\Big)$$

for any $\lambda \geq \lambda_{10}(t)$. So by Proposition 2.10, $\lim_{\lambda\to\infty} P_\lambda(J_t > -C_{13}(t)) = 0$. Since $\{J \in C([0, T]; \mathbf{R}); J_t > -C_{13}(t)\}$ is open in $C([0, T]; \mathbf{R})$, this implies that $P_\infty(J_t > -C_{13}(t)) = 0$. In particular, $P_\infty(J_t < 0) = 1$ for any $t \in (0, T_0/2]$. Since by Proposition 3.2, $J_\cdot$ is monotone non-increasing and continuous under P_∞, this implies our assertion. ∎

4 Proof of the main theorem

We give the proof of Theorem 1.1 in this section.

We first estimate the corresponding expression before taking limit $\lambda \to \infty$ (see Lemma 4.5 for the result). Let us first make several preparations.

Lemma 4.1 *1. For any $\lambda \geq (-\frac{1}{2}g(r_2))^{-2}$ and $t \in (0, T]$, we have on the set $F^\lambda_5 \cap F^\lambda_{7,t}$ that $J^\lambda_t \geq (1 - \frac{1}{2}e^{-2(b+1)T})g(r_2)$ and $S_1(J^\lambda_t) \geq C_{15}$ with $C_{15} := \inf\Big\{S_1(j); j \in [(1 - \frac{1}{2}e^{-2(b+1)T})g(r_2), 0)\Big\}$,*

2. if $\lambda \geq \lambda_{10}(t)$ in addition, then we have on the set $F^\lambda_1 \cap F^\lambda_3 \cap F^\lambda_5 \cap F^\lambda_6 \cap F^\lambda_{7,t/4} \cap F^\lambda_9 \cap F^\lambda_{10}$ that $J^\lambda_s \in [(1 - \frac{1}{2}e^{-2(b+1)T})g(r_2), -C_{13}(\frac{T_0}{2} \wedge t)]$ and

$$S_1(J^\lambda_s) \in [C_{15}, C_{16}(t)], \qquad \text{for any } s \in [t, T],$$

with $C_{16}(t) := \sup_{j \in [(1-\frac{1}{2}e^{-2(b+1)T})g(r_2), -C_{13}(\frac{T_0}{2}\wedge t)]} S_1(j)$. Here $\lambda_{10}(t)$ and $C_{13}(t)$ are given by Lemma 3.6.

Proof. Since $\lambda \geq (-\frac{1}{2}g(r_2))^{-2}$, we get from Lemma 2.16 (1a) that $J^\lambda_t \geq (1 - \frac{1}{2}e^{-2(b+1)T})g(r_2)$. Our assertions are now trivial by Lemma 3.6. ∎

For any $t \in (0, T]$, let $C_{17}(t) := C_{13}(\frac{T_0}{2} \wedge t)$.

Lemma 4.2 *For any $t_0, t_1 \in (0,T)$ with $t_0 < t_1$, there exists a constant $\lambda_{11}(t_0,t_1) \geq 1$ such that for any $\lambda \geq \lambda_{11}(t_0,t_1)$, the following holds on the set $\overline{F}^{\lambda}_{t_0/4, C_{16}(t_0)}$ for any $t \in [t_0, t_1]$:*

$$\begin{aligned} &\Big|\lambda^{1/2}\int_t^{t+S_1(J_t^\lambda)\lambda^{-1/2}} f(X_u^\lambda)du - S_f(J_t^\lambda)\Big| \\ \leq\ & b^{\lambda}_{1,-\frac{1}{2}g(r_2)e^{-2(b+1)T},C_{17}(t_0),f} + \|f'\|_\infty C_{16}(t_0)\lambda^{-1/4}. \end{aligned}$$

Here $b^\lambda_{1,*,*,f}$ is as defined in (2.9).

Proof. Assume that $\lambda \geq (-\frac{1}{2}g(r_2))^{-2} \vee \lambda_{10}(t_0)$ and $\omega \in \overline{F}^\lambda_{t_0/4, C_{16}(t_0)}$. Then for any $t \in [t_0, t_1]$, we have by Lemma 4.1 (2) that $S_1(J_t^\lambda) \leq C_{16}(t_0)$. Assume $\lambda \geq \left(\frac{C_{16}(t_0)}{T-t_1}\right)^2$ in addition. Then $C_{16}(t_0)\lambda^{-1/2} \leq T - t_1$, so for any $t \in [t_0, t_1]$, we have that $t + S_1(J_t^\lambda)\lambda^{-1/2} \leq t_1 + C_{16}(t_0)\lambda^{-1/2} \leq T$. Therefore, by the definition of $F^\lambda_{11, C_{16}(t_0)}$, we have that

$$\begin{aligned} &\Big|\lambda^{1/2}\int_t^{t+S_1(J_t^\lambda)\lambda^{-1/2}} f(X_u^\lambda)du - \lambda^{1/2}\int_0^{S_1(J_t^\lambda)\lambda^{-1/2}} f(x_u^\lambda(X_t^\lambda, V_t^\lambda))du\Big| \\ \leq\ & \lambda^{1/2}\|f'\|_\infty \int_0^{S_1(J_t^\lambda)\lambda^{-1/2}} \Big|X^\lambda_{u+t} - x_u^\lambda(X_t^\lambda, V_t^\lambda)\Big| du \\ \leq\ & \lambda^{1/2}\|f'\|_\infty C_{16}(t_0)\lambda^{-1/2}\lambda^{-1/4} = \|f'\|_\infty C_{16}(t_0)\lambda^{-1/4}. \end{aligned}$$

This combined with the definition of $b^\lambda_{1,*,*}$ implies our assertion. ∎

Lemma 4.3 *For any $\lambda \geq 1$, we have on the set F_2^λ that*

$$J^\lambda_{t+s} \geq e^{-2bt}J_s^\lambda + (1-e^{-2bt})g(r_2) - \lambda^{-1/4}$$

as long as $0 \leq s \leq s+t \leq T$.

Proof. Let $\widetilde{H_t^\lambda}$ be as defined in (2.18). Then we get that $e^{2b(t+s)}\widetilde{H^\lambda_{t+s}} \geq e^{2bs}\widetilde{H^\lambda_s} + \int_s^{t+s} e^{2bu}V_u^\lambda \sigma(X_u^\lambda)dB_u$. So on the set F_2^λ, we have that $e^{2b(t+s)}\widetilde{H^\lambda_{t+s}} \geq e^{2bs}\widetilde{H^\lambda_s} - \lambda^{3/4}$, hence $H^\lambda_{t+s} \geq e^{-2bt}H_s^\lambda + (1-e^{-2bt})g(r_2)\lambda - \lambda^{3/4}$. This implies our assertion. ∎

We define two more notations. For any $s \in (0,T]$, let

$$b^\lambda_{2,s,f} := \frac{1}{C_{15}}\Big(b^\lambda_{1,-\frac{1}{2}g(r_2)e^{-2(b+1)T},C_{17}(s),f} + \|f'\|_\infty C_{16}(s)\lambda^{-1/4}\Big).$$

Then by Proposition 2.8, it is trivial that

$$\lim_{\lambda\to\infty} b^\lambda_{2,s,f} = 0. \tag{4.1}$$

Also, for any $c \in (0, -g(r_2))$ and any $\varepsilon > 0$, let

$$b_{3,c,\varepsilon,f} := \sup_{x,y\in((1-\frac{1}{2}e^{-2(b+1)T})g(r_2),-c],|x-y|\leq\varepsilon} \Big|A^g f(x) - A^g f(y)\Big|.$$

Then by the continuity of A^g claimed before, we have that

$$\lim_{\varepsilon\to 0} b_{3,c,\varepsilon,f} = 0. \tag{4.2}$$

Lemma 4.4 *For any $s_1 \in (0, T]$ and any $\theta \in (0, -g(r_2))$, there exist constants $t_0(\theta) \in (0, T]$ and $\lambda_{12}(\theta, s_1) \geq 1$ such that for any $\lambda \geq \lambda_{12}(\theta, s_1)$, we have on the set $\overline{F}^{\lambda}_{s_1/4, C_{16}(s_1)}$ that the following holds for any $s \in [s_1, T]$ and $t \in (0, t_0(\theta)]$ satisfying $s + t \leq T$:*

$$\left| A^g f(J^{\lambda}_{t+s}) - A^g f(J^{\lambda}_s) \right| \leq b_{3, C_{17}(s_1), 2\theta, f}.$$

Proof. Assume that $\omega \in \overline{F}^{\lambda}_{s_1/4, C_{16}(s_1)}$. Then for any $s \geq s_1$, we have that $H^{\lambda}_s < 0$, hence $(e^{-2bt} - 1)H^{\lambda}_s > 0$. Therefore, by Lemma 4.3, we have that

$$J^{\lambda}_{t+s} - J^{\lambda}_s \geq (1 - e^{-2bt})g(r_2) - \lambda^{-1/4}, \qquad \text{if } s_1 \leq s \leq s + t \leq T. \tag{4.3}$$

Let $t_0(\theta) := \frac{1}{2b} \left| \log(1 + \frac{\theta}{g(r_2)}) \right| \wedge \left(2\theta \|\sigma\|_{\infty}^{-2} \right) \wedge T$. Then $t_0(\theta) > 0$, and we have that

$$\begin{aligned} t < t_0(\theta) \quad &\Rightarrow \quad t < \frac{1}{2b} \left| \log(1 + \frac{\theta}{g(r_2)}) \right| \\ &\Rightarrow \quad (1 - e^{-2bt}) g(r_2) > -\theta. \end{aligned}$$

So if $\lambda \geq (16\theta^{-4}) \vee 1$, then (4.3) implies that $J^{\lambda}_{t+s} - J^{\lambda}_s \geq -2\theta$ for any $s \in [s_1, T]$ and $t \in (0, t_0(\theta)]$ satisfying $s + t \leq T$.

On the other hand, we have by Lemma 3.1 (1) that $J^{\lambda}_{t+s} - J^{\lambda}_s \leq 2\lambda^{-1/4} + \frac{t}{2} \|\sigma\|^2_{\infty} \lambda^{-1} \leq \theta + \theta = 2\theta$.

In conclusion, with $\lambda_{12}(\theta, s_1) := 1 \vee (16\theta^{-4}) \vee \lambda_{10}(s_1) \vee (-\frac{1}{2} g(r_2))^{-2}$, we have for any $\lambda \geq \lambda_{12}(\theta, s_1)$ that $\left| J^{\lambda}_{t+s} - J^{\lambda}_s \right| \leq 2\theta$ if $s \in [s_1, T]$, $t \in (0, t_0(\theta)]$ and $s + t \leq T$.

Also, since $s, s + t \in [s_1, T]$, we have by Lemma 4.1 (2) that

$$J^{\lambda}_{t+s}, J^{\lambda}_s \in [(1 - \frac{1}{2} e^{-2(b+1)T}) g(r_2), -C_{17}(s_1)].$$

Combining the above, we get our assertion by the definition of $b_{3, C_{17}(s_1), 2\theta, f}$. ∎

Lemma 4.5 *For any $s_1 \in (0, T/3]$ and any $\theta \in (0, -g(r_2))$, there exist constants $t_1(\theta, s_1) > 0$ and $\lambda_{13}(\theta, s_1) \geq 1$ such that for any $\lambda \geq \lambda_{13}(\theta, s_1)$, we have on the set $\overline{F}^{\lambda}_{s_1/4, C_{16}(s_1)}$ that the following holds for any $s \in [s_1, T - s_1]$ and any $t \in (0, t_1(\theta, s_1)]$:*

$$\left| \frac{1}{t} \int_s^{s+t} f(X^{\lambda}_u) du - A^g f(J^{\lambda}_s) \right| \leq b^{\lambda}_{2, s_1, f} + b_{3, C_{17}(s_1), 2\theta, f} + 2\|f\|_{\infty} \frac{C_{16}(s_1)}{t} \lambda^{-1/2}.$$

Proof. Let $t_1(\theta, s_1) := \frac{s_0}{2} \wedge t_0(\theta)$. Choose and fix any $s \in [s_1, T - s_1]$ and $t \in (0, t_1(\theta, s_1)]$, and define

$$\begin{aligned} &t^{\lambda}_0 := s, \\ &t^{\lambda}_j := t^{\lambda}_{j-1} + S_1(J^{\lambda}_{t_{j-1}}) \lambda^{-1/2}, \\ &K := K_{s,t,\lambda} := \inf\{k \in \mathbf{N}; t^{\lambda}_k \geq s + t\} - 1. \end{aligned}$$

Then

$$\begin{aligned} &\left| \frac{1}{t} \int_s^{s+t} f(X^{\lambda}_u) du - A^g f(J^{\lambda}_s) \right| \\ \leq \; &\left| \frac{1}{t^{\lambda}_K - s} \int_s^{t^{\lambda}_K} f(X^{\lambda}_u) du - A^g f(J^{\lambda}_s) \right| + \left| \frac{1}{t} \int_s^{s+t} f(X^{\lambda}_u) du - \frac{1}{t^{\lambda}_K - s} \int_s^{t^{\lambda}_K} f(X^{\lambda}_u) du \right|. \end{aligned}$$

We first deal with the first term on the right hand side above. By Lemmas 4.2, 4.1 (1) and 4.4, for any $\lambda \geq \lambda_{11}(s_1, T-\frac{s_1}{2}) \vee H_0^2 \vee \lambda_{12}(\theta, s_1) \vee (-\frac{1}{2}g(r_2))^{-2}$, on the set $\overline{F}^{\lambda}_{s_1/4, C_{16}(s_1)}$, we have for any $u \in [s, s+t] \subset [s_1, T-\frac{s_1}{2}]$ that

$$\begin{aligned}
&\Big|\frac{\lambda^{1/2}}{S_1(J_u^\lambda)}\int_u^{u+S_1(J_u^\lambda)\lambda^{-1/2}} f(X_r^\lambda)dr - A^g f(J_u^\lambda)\Big| \\
\leq\ & \frac{1}{S_1(J_u^\lambda)}\Big(b^\lambda_{1,-\frac{1}{2}g(r_2)e^{-2(b+1)T}, C_{17}(s_1), f} + \|f'\|_\infty C_{16}(s_1)\lambda^{-1/4}\Big) \\
\leq\ & \frac{1}{C_{15}}\Big(b^\lambda_{1,-\frac{1}{2}g(r_2)e^{-2(b+1)T}, C_{17}(s_1), f} + \|f'\|_\infty C_{16}(s_1)\lambda^{-1/4}\Big) = b^\lambda_{2,s_1,f},
\end{aligned}$$

and $\Big|A^g f(J_u^\lambda) - A^g f(J_s^\lambda)\Big| \leq b_{3,C_{17}(s_1),2\theta,f}$. Therefore,

$$\begin{aligned}
&\Big|\frac{\lambda^{1/2}}{S_1(J_u^\lambda)}\int_u^{u+S_1(J_u^\lambda)\lambda^{-1/2}} f(X_r^\lambda)dr - A^g f(J_s^\lambda)\Big| \\
\leq\ & \Big|\frac{1}{S_f(J_u^\lambda)}\int_u^{u+S_f(J_u^\lambda)\lambda^{-1/2}} f(X_r^\lambda)dr - A^g f(J_u^\lambda)\Big| + \Big|A^g f(J_u^\lambda) - A^g f(J_s^\lambda)\Big| \\
\leq\ & b^\lambda_{2,s_1,f} + b_{3,C_{17}(s_1),2\theta,f}.
\end{aligned} \tag{4.4}$$

Notice that in general, if $|x_i - c| \leq b$ for any $i \in \{1, \cdots, n\}$, then for any $a_1, \cdots, a_n > 0$, we have that $\Big|\frac{a_1x_1+\cdots+a_nx_n}{a_1+\cdots+a_n} - c\Big| \leq b$. We have by definition that $t_K^\lambda - s = \sum_{j=1}^K S_1(J^\lambda_{t_{j-1}})\lambda^{-1/2}$. Therefore, (4.4) implies that

$$\Big|\frac{1}{t_K^\lambda - s}\int_s^{t_K^\lambda} f(X_u^\lambda)du - A^g f(J_s^\lambda)\Big| \leq b^\lambda_{2,s_1,f} + b_{3,C_{17}(s_1),2\theta,f}.$$

Now, it suffices to prove that

$$\Big|\frac{1}{t}\int_s^{s+t} f(X_u^\lambda)du - \frac{1}{t_K^\lambda - s}\int_s^{t_K^\lambda} f(X_u^\lambda)du\Big| \leq 2\|f\|_\infty \frac{C_{16}(s_1)}{t}\lambda^{-1/2}.$$

Notice that in general, for any $A, B, C, D \in \mathbf{R}$, we have that

$$\Big|\frac{A+B}{C+D} - \frac{A}{C}\Big| \leq \Big|\frac{B}{C+D}\Big| + \Big|\frac{A}{C}\Big| \cdot \Big|\frac{D}{C+D}\Big|.$$

Also notice that $\Big|\int_{t_K^\lambda}^{s+t} f(X_u^\lambda)du\Big| \leq (s+t-t_K^\lambda)\|f\|_\infty$ and $\Big|\frac{\int_s^{t_K^\lambda} f(X_u^\lambda)du}{t_K^\lambda - s}\Big| \leq \|f\|_\infty$. Therefore,

$$\begin{aligned}
&\Big|\frac{1}{t}\int_s^{s+t} f(X_u^\lambda)du - \frac{1}{t_K^\lambda - s}\int_s^{t_K^\lambda} f(X_u^\lambda)du\Big| \\
\leq\ & \Big|\frac{1}{t}\int_{t_K^\lambda}^{s+t} f(X_u^\lambda)du\Big| + \Big|\frac{\int_s^{t_K^\lambda} f(X_u^\lambda)du}{t_K^\lambda - s}\Big| \cdot \Big|\frac{t+s-t_K^\lambda}{t}\Big| \\
\leq\ & 2\|f\|_\infty \frac{s+t-t_K^\lambda}{t}.
\end{aligned}$$

If $\lambda \geq \lambda_{10}(s_1)$ in addition, then by Lemma 4.1 (2), we have on $\overline{F}^{\lambda}_{s_1/4,C_{16}(s_1)}$ that $s+t-t^{\lambda}_K \leq \lambda^{-1/2} S_1(J^{\lambda}_{t_K}) \leq \lambda^{-1/2} C_{16}(s_1)$. This completes the proof of our assertion. ∎

Now we are ready to prove Theorem 1.1. By Proposition 3.7, it is obviously a direct consequence of Propositions 4.6 and 4.7 given below.

PROPOSITION 4.6 *1. For any $s_0 \in (0, \frac{T}{3}]$, we have that the following holds P_∞-almost surely:*

$$\limsup_{t \to 0} \sup_{s \in [s_0, T-s_0]} \left| \frac{1}{t}\left(Y^f_{s+t} - Y^f_s\right) - A^g f(J_s) \right| = 0.$$

2. For any $s > 0$, we have P_∞-almost surely that

$$\frac{d}{ds} Y^f_s = A^g f(J_s).$$

Proof. The second assertion is trivial by the first assertion. We prove the first one in the following.

Choose and fix any $\theta > 0$ and $t_2 \in (0, t_1(\theta, s_0))$. For any $\varepsilon > 0$, since

$$\lim_{\lambda \to \infty} \left(b^{\lambda}_{2,s_0,f} + 2\|f\|_\infty \frac{C_{16}(s_0)}{t_2} \lambda^{-1/2} \right) = 0,$$

we have that there exists a $\lambda_{14}(\varepsilon, \theta, s_0, t_2) \geq \lambda_{13}(\theta, s_0)$ such that for any $\lambda \geq \lambda_{14}(\varepsilon, \theta, s_0, t_2)$ and any $t \geq t_2$, we have that $b^{\lambda}_{2,s_0,f} + 2\|f\|_\infty \frac{C_{16}(s_0)}{t_2} \lambda^{-1/2} < \varepsilon$, hence by Lemma 4.5 and Proposition 2.10, we get that

$$\begin{aligned} \lim_{\lambda \to \infty} P_\lambda \Big(&\exists (s,t) \in (s_0, T-s_0) \times (t_2, t_1(\theta, s_0)) s.t. \\ &J_s < 0 \text{ and } \left| \frac{1}{t}(Y^f_{s+t} - Y^f_s) - A^g f(J_s) \right| > b_{3,C_{17}(s_0),2\theta,f} + \varepsilon \Big) = 0. \end{aligned}$$

Since $\{(Y,J) \in C([0,T]; \mathbf{R}) \times C([0,T]; \mathbf{R}) \Big| J_s < 0 \text{ and } \left| \frac{1}{t}(Y_{t+s} - Y_s) - A^g f(J_s) \right| > \delta$ for some $(s,t) \in (s_1, s_2) \times (t_1, t_2)\}$ is open in $C([0,T]; \mathbf{R}) \times C([0,T]; \mathbf{R})$, this implies that

$$\begin{aligned} P_\infty \Big(&\exists (s,t) \in (s_0, T-s_0) \times (t_2, t_1(\theta, s_0)) s.t. \\ &J_s < 0 \text{ and } \left| \frac{1}{t}(Y^f_{s+t} - Y^f_s) - A^g f(J_s) \right| > b_{3,C_{17}(s_0),2\theta,f} + \varepsilon \Big) = 0. \end{aligned}$$

So by Proposition 3.7, we get that

$$P_\infty \Big(\exists (s,t) \in (s_0, T-s_0) \times (t_2, t_1(\theta, s_0)) s.t. \left| \frac{1}{t}(Y^f_{s+t} - Y^f_s) - A^g f(J_s) \right| > b_{3,C_{17}(s_0),2\theta,f} + \varepsilon \Big) = 0.$$

This is true for any $\varepsilon > 0$, any $\theta > 0$ and any $t_2 > 0$.

Taking $\varepsilon \to 0+$, we get that

$$P_\infty \Big(\sup_{(s,t) \in (s_0, T-s_0) \times (t_2, t_1(\theta, s_0))} \left| \frac{1}{t}(Y^f_{s+t} - Y^f_s) - A^g f(J_s) \right| \leq b_{3,C_{17}(s_0),2\theta,f} \Big) = 1. \tag{4.5}$$

Now take $t_2 \to 0$, hence we get that

$$P_\infty\Big(\sup_{(s,t)\in(s_0,T-s_0)\times(0,t_1(\theta,s_0))}\Big|\frac{1}{t}(Y^f_{s+t}-Y^f_s)-A^g f(J_s)\Big| \le b_{3,C_{17}(s_0),2\theta,f}\Big)=1.$$

Finally take $\theta \to 0$, so by (4.2), we get that

$$P_\infty\Big(\limsup_{t\to 0+}\sup_{s\in(s_0,T-s_0)}\Big|\frac{1}{t}(Y^f_{s+t}-Y^f_s)-A^g f(J_s)\Big|=0\Big)=1.$$

The assertion for $t \to 0-$ is proved in the same way. Indeed, by exactly the same method as we used up to now, we get the following modification of (4.5):

$$P_\infty\Big(\exists(s,t)\in(s_0,T-s_0)\times(t_2,t_1(\theta,s_0))\, s.t. \Big|\frac{1}{t}(Y^f_{s+t}-Y^f_s)-A^g f(J_{t+s})\Big| > b_{3,C_{17}(s_0),2\theta,f}+\varepsilon\Big)=0$$

for any $\varepsilon > 0$, any $\theta > 0$ and any $t_2 > 0$. Without loss of generality, assume that $t_1(\theta, s_0) \le s_0$. Write $\widetilde{s} := s + t$. Therefore,

$$P_\infty\Big(\sup_{(\widetilde{s},t)\in(2s_0,T-s_0)\times(t_2,t_1(\theta,s_0))}\Big|\frac{1}{t}(Y^f_{\widetilde{s}}-Y^f_{\widetilde{s}-t})-A^g f(J_{\widetilde{s}})\Big| \le b_{3,C_{17}(s_1),2\theta,f}\Big)=1.$$

Now we get our assertion for $t \to 0-$ by the same method as that for $t \to 0+$. ∎

PROPOSITION 4.7 *For any $s \in (0, T)$, we have P_∞-almost surely that*

$$\frac{d}{ds}J_s = -2b(J_s - A^g g(J_s)).$$

Proof. Fix any $s_0 \in (0, T)$ and let $s \in (s_0, T - s_0)$. Since

$$dJ^\lambda_t = -2bJ^\lambda_t dt + 2bg(X^\lambda_t)dt + \lambda^{-1}V^\lambda_t\sigma(X^\lambda_t)dB_t + \frac{1}{2\lambda}\sigma(X^\lambda_t)^2 dt,$$

we have that

$$J^\lambda_{t+s}-J^\lambda_s = -2b\int_s^{t+s}J^\lambda_u du+2b\int_s^{t+s}g(X^\lambda_u)du+\lambda^{-1}\int_s^{t+s}V^\lambda_u\sigma(X^\lambda_u)dB_u+\frac{1}{2\lambda}\int_s^{t+s}\sigma(X^\lambda_u)^2du.$$

Therefore,

$$\begin{aligned}
&\Big|\frac{1}{t}(J^\lambda_{t+s}-J^\lambda_s)+2b(J_s-A^g g(J_s))\Big| \\
\le\ & 2b\frac{1}{t}\int_s^{t+s}|J^\lambda_u-J^\lambda_s|du+2b\Big|\frac{1}{t}\int_s^{t+s}g(X^\lambda_u)du-A^g g(J^\lambda_s)\Big| \\
&+\lambda^{-1}\Big|\frac{1}{t}\int_s^{t+s}V^\lambda_u\sigma(X^\lambda_u)dB_u\Big|+\frac{1}{2\lambda}\Big|\frac{1}{t}\int_s^{t+s}\sigma(X^\lambda_u)^2du\Big|. \qquad (4.6)
\end{aligned}$$

We have on the set F^λ_1 that

$$\lambda^{-1}\Big|\frac{1}{t}\int_s^{t+s}V^\lambda_u\sigma(X^\lambda_u)dB_u\Big| \le \lambda^{-1}\frac{1}{t}2\lambda^{3/4}=\frac{2}{t}\lambda^{-1/4}. \qquad (4.7)$$

Also, we have that

$$(2\lambda)^{-1}\Big|\frac{1}{t}\int_s^{t+s}\sigma(X_u^\lambda)^2du\Big| \le (2\lambda)^{-1}\|\sigma\|_\infty^2. \qquad (4.8)$$

We next deal with the first term on the right hand side of (4.6). For any $\varepsilon > 0$, there exists a $t_3(\varepsilon) \in (0, s_0)$ such that $-g(r_2)(1-e^{-2bt}) < \frac{\varepsilon}{4b}$ for any $t \le t_3(\varepsilon)$. Notice that $s+t_3(\varepsilon) \le T$. Also, $J_s^\lambda < 0$ on $F_{7,s_0/4}^\lambda \cap F_{10}^\lambda$. Therefore, by Lemma 4.3, for any $\lambda \ge (4b/\varepsilon)^4$, we have on the set $F_2^\lambda \cap F_{7,s_0/4}^\lambda \cap F_{10}^\lambda$ that the following holds for any $u \in (s, s+t_3(\varepsilon))$:

$$\begin{aligned} J_u^\lambda - J_s^\lambda &\ge -(1-e^{-2b(u-s)})(J_s^\lambda - g(r_2)) - \lambda^{-1/4} \\ &\ge (1-e^{-2b(u-s)})g(r_2) - \lambda^{-1/4} \ge -\frac{\varepsilon}{4b} - \frac{\varepsilon}{4b} = -\frac{\varepsilon}{2b}. \end{aligned}$$

On the other hand, choose $\lambda_{15}(\varepsilon) \ge 1$ such that $\lambda \ge \lambda_{15}(\varepsilon) \Rightarrow 2\lambda^{-1/4}+(2\lambda)^{-1}\|\sigma\|_\infty^2 T \le \frac{\varepsilon}{2b}$, then for any $\lambda \ge \lambda_{15}(\varepsilon)$, we have on the set F_1^λ that

$$\begin{aligned} & J_u^\lambda - J_s^\lambda \\ = & -b\int_s^u \lambda^{-1}|V_r^\lambda|^2dr + \int_s^u \lambda^{-1}V_r^\lambda\sigma(X_r^\lambda)dB_r + \int_s^u (2\lambda)^{-1}\sigma(X_r^\lambda)^2dr \\ \le & \ 0 + 2\lambda^{-1/4} + (2\lambda)^{-1}\|\sigma\|_\infty^2 T \le \frac{\varepsilon}{2b}. \end{aligned}$$

Let $\lambda_{16}(\varepsilon) := (4b/\varepsilon)^4 \vee \lambda_{15}(\varepsilon)$. Then our calculation implies that, for any $\lambda \ge \lambda_{16}(\varepsilon)$, we have on the set $F_1^\lambda \cap F_2^\lambda \cap F_{7,s_0/4}^\lambda \cap F_{10}^\lambda$ that

$$|J_u^\lambda - J_s^\lambda| \le \frac{\varepsilon}{2b}. \qquad (4.9)$$

Combining (4.6), (4.7), (4.8) and (4.9) with Lemma 4.5, we get for any $\lambda \ge \lambda_{16}(\varepsilon)$, any $t_0 > 0$ and any $t \in (t_0, t_3(\varepsilon) \wedge t_1(\theta, s_0))$ that the following holds on the set $\overline{F}^\lambda_{s_1/4, C_{16}(s_0)}$:

$$\begin{aligned} & \Big|\frac{1}{t}(J_{t+s}^\lambda - J_s^\lambda) + 2b(J_s - A^g g(J_s))\Big| \\ \le & \ 2b\cdot\frac{\varepsilon}{2b} + 2b\Big(b^\lambda_{2,s_0,g} + b_{3,C_{17}(s_0),2\varepsilon,g} + 2\|g\|_\infty\frac{C_{16}(s_0)}{t_0}\lambda^{-1/2}\Big) + \frac{2}{t_0}\lambda^{-1/4} + (2\lambda)^{-1}\|\sigma\|_\infty^2. \end{aligned}$$

Combining this with (4.1), we get the following: for any $\overline{\varepsilon} > 0$ and any $t_0 > 0$, there exists a $\lambda_{17}(\varepsilon, s_0, \overline{\varepsilon}, t_0) \ge 1$ such that for any $\lambda \ge \lambda_{17}(\varepsilon, s_0, \overline{\varepsilon}, t_0)$, we have that

$$\begin{aligned} & P_\lambda\Big(\exists(s,t) \in (s_0, T-s_0)\times(t_0, t_3(\varepsilon)\wedge t_1(\theta,s_0)) s.t. \\ & \qquad J_s < 0 \text{ and } \Big|\frac{1}{t}(J_{s+t} - J_s) + 2b(J_s - A^g g(J_s))\Big| > \varepsilon + 2bb_{3,C_{17}(s_0),2\varepsilon,f} + \overline{\varepsilon}\Big) \\ \le & \ P\Big((\overline{F}^\lambda_{s_1/4,C_{16}(s_0)})^C\Big). \end{aligned}$$

The right hand side above converges to 0 as $\lambda \to \infty$ by Proposition 2.10. Since $\{J \in C([0,T];\mathbf{R})\Big|J_s < 0 \text{ and } \Big|\frac{1}{t}(J_{t+s} - J_s) + 2b(J_s - A^g g(J_s))\Big| > \delta$ for some $(s,t) \in (s_1, s_2) \times (t_1, t_2)\}$ is open in $C([0,T];\mathbf{R})$, this implies that

$$\begin{aligned} & P_\infty\Big(\exists(s,t) \in (s_0, T-s_0)\times(t_0, t_3(\varepsilon)\wedge t_1(\theta,s_0)) s.t. \\ & \qquad J_s < 0 \text{ and } \Big|\frac{1}{t}(J_{s+t} - J_s) + 2b(J_s - A^g g(J_s))\Big| > \varepsilon + 2bb_{3,C_{17}(s_0),2\varepsilon,f} + \overline{\varepsilon}\Big) = 0. \end{aligned}$$

Taking $\bar{\varepsilon} \to 0$, $t_0 \to 0$ and $\varepsilon \to 0$ in turn, we get that

$$P_\infty\Big(\limsup_{t\to 0+} \sup_{s\in(s_0,T-s_0)} \Big|\frac{1}{t}(J_{s+t}-J_s)+2b(J_s-A^g g(J_s)\Big| = 0\Big) = 1.$$

The assertion for $t \to 0-$ is proved in the same way. ∎

5 Appendix

This appendix provides the proofs of our results respect to the deterministic case presented in Subsection 2.3. We omit the superscript λ when there is no risk of confusion.

Proof of Lemma 2.6. Choose any $\varepsilon \in (0, a_0 \vee (\frac{-v_0}{b}))$, and let $\xi_\varepsilon := \inf\{t > 0; x_t = r_3 - \varepsilon\}$. Then for any $t \in [0, \xi_\varepsilon]$, we have that $\frac{d}{dt}\Big(e^{bt}v_t\Big) = -\lambda e^{bt} g'(x_t) \le 0$. So

$$v_t \le e^{-bt} v_0, \qquad \text{for any } t \in [0, \xi_\varepsilon].$$

Therefore, for any $t \in [0, \xi_\varepsilon]$, we have that

$$x_t = x_0 + \int_0^t v_s ds \le x_0 + \int_0^t e^{-bs} v_0 ds = x_0 - \frac{v_0}{b}(e^{-bt}-1).$$

In particular, $r_3 - \varepsilon = x_{\xi_\varepsilon} \le r_3 - \frac{v_0}{b}(e^{-b\xi_\varepsilon} - 1)$. Solving this, we get that

$$\xi_\varepsilon \le -b^{-1} \log\Big(1 + (b\varepsilon)/v_0\Big). \tag{5.1}$$

We next estimate $t_1^\lambda(x_0, v_0) - \xi_\varepsilon$. First notice that by assumption (A2), we have that

$$t \in [\xi_\varepsilon, t_1^\lambda(x_0, v_0)] \Rightarrow x_t \in [r_3 - a_0, r_3 - \varepsilon] \Rightarrow g'(x_t) \ge g'(r_3 - \varepsilon).$$

Also, for any $t \ge 0$, we have by Lemma 2.5 that $\frac{1}{2}|v_t|^2 + \lambda g(x_t) = h_t \le h_0$, hence $|v_t| \le \sqrt{2(h_0 - \lambda g(x_t))} \le \sqrt{2(h_0 - \lambda g(r_2))}$. Therefore, for any $\lambda \ge 1$ large enough such that $b\sqrt{2(h_0 - \lambda g(r_2))} \le \frac{\lambda}{2} g'(r_3 - \varepsilon)$, we have for any $t \in [\xi_\varepsilon, t_1^\lambda(x_0, v_0)]$ that

$$\frac{d}{dt} v_t = -\lambda g'(x_t) - b v_t \le -\lambda g'(r_3 - \varepsilon) + b\sqrt{2(h_0 - \lambda g(r_2))} \le -\frac{\lambda}{2} g'(r_3 - \varepsilon).$$

Since $v_{\xi_\varepsilon} \le 0$, this implies that

$$v_t \le -\frac{\lambda}{2} g'(r_3 - \varepsilon)(t - \xi_\varepsilon), \qquad \text{for any } t \in [\xi_\varepsilon, t_1^\lambda(x_0, v_0)].$$

Therefore, we get that

$$\begin{aligned} r_3 - a_0 &= x_{\xi_\varepsilon} + \int_{\xi_\varepsilon}^{t_1^\lambda(x_0,v_0)} v_s ds \\ &\le r_3 - \varepsilon - \frac{\lambda}{2} g'(r_3 - \varepsilon) \int_{\xi_\varepsilon}^{t_1^\lambda(x_0,v_0)} (s - \xi_\varepsilon) ds \\ &= r_3 - \varepsilon - \frac{\lambda}{2} g'(r_3 - \varepsilon) \cdot \frac{1}{2}\Big(t_1^\lambda(x_0, v_0) - \xi_\varepsilon\Big)^2. \end{aligned}$$

Solving this, we get that $t_1^\lambda(x_0, v_0) - \xi_\varepsilon \le \sqrt{\frac{4(a_0-\varepsilon)}{\lambda g'(r_3-\varepsilon)}}$. Combining this with (5.1), by taking first $\varepsilon > 0$ small enough and then $\lambda \ge 1$ large enough, we get our first assertion.

The second assertion is proved by the same method as the random case (see the proofs of Lemmas 2.16 and 2.17), and we omit the proof. ∎

Proof of Proposition 2.7. Since $h_0 < 0$ by assumption, we get by Lemma 2.5 that $h_t < 0$ for any $t \ge 0$, hence the particle stays in the domain $x_t \in (r_1, r_3)$. Let $\xi_1 := \inf\Big\{t > 0; v_t = 0\Big\}$. We remark that $x_{\xi_1} < r_3$.

Let us first make a preparation. Notice that since $x_{\xi_1} \in (r_3 - a_0, r_3)$, we have by assumption (A2) that $g'(x_{\xi_1}) \ge C_1|g(x_{\xi_1})|$. Also, by assumption, we have that $|g(x_{\xi_1})| = \lambda^{-1}|h_{\xi_1}| \ge \lambda^{-1}|h_0| \ge c\lambda^{-1/2}k$. Therefore, $g'(x_{\xi_1}) \ge C_1 c\lambda^{-1/2}k$. On the other hand, by (A2) and the mean value theorem, we have for any $y \in (r_3 - a_0, x_{\xi_1})$ that $g(x_{\xi_1}) - g(y) \ge g'(x_{\xi_1})(x_{\xi_1} - y)$, so

$$\begin{aligned}\int_{r_3-a_0}^{x_{\xi_1}} \frac{1}{\sqrt{g(x_{\xi_1}) - g(y)}}dy &\le \frac{1}{\sqrt{g'(x_{\xi_1})}}\int_{r_3-a_0}^{x_{\xi_1}} \frac{1}{\sqrt{x_{\xi_1} - y}}dy \\ &= \frac{1}{\sqrt{g'(x_{\xi_1})}} \cdot 2\sqrt{x_{\xi_1} - (r_3 - a_0)} \le \frac{2\sqrt{a_0}}{\sqrt{g'(x_{\xi_1})}} \\ &\le 2\sqrt{a_0/(C_1c)}\lambda^{1/4}k^{-1/2}. \qquad (5.2)\end{aligned}$$

Now we are ready to estimate ξ_1. For any $s \in (0, \xi_1)$, we have that $h_s \ge h_{\xi_1} = \lambda g(x_{\xi_1})$, so $|v_s| = \sqrt{2}\sqrt{h_s - \lambda g(x_s)} \ge \sqrt{2}\sqrt{\lambda g(x_{\xi_1}) - \lambda g(x_s)}$. Combining this with (5.2), we get that

$$\begin{aligned}\xi_1 &= \int_0^{\xi_1} ds \le \int_0^{\xi_1} \frac{|v_s|}{\sqrt{2}\sqrt{\lambda g(x_{\xi_1}) - \lambda g(x_s)}}ds \\ &= \int_{r_3-a_0}^{x_{\xi_1}} \frac{1}{\sqrt{2}\lambda^{1/2}\sqrt{g(x_{\xi_1}) - g(y)}}dy \le \sqrt{2a_0/(C_1c)}\lambda^{-1/4}k^{-1/2}. \qquad (5.3)\end{aligned}$$

We next estimate $t_1^\lambda(x_0, v_0) - \xi_1$. Let $\lambda_1(c) := \Big(64b^4C_1^{-4}c^{-2}\Big) \vee 1$.

Claim 2. Suppose that $\lambda \ge \lambda_1(c)$, $r \ge 0$, $h_r \le -c\lambda^{1/2}$, $v_r \le 0$, $t > r$ and $x_u \in [r_3 - a_0, r_3)$ for any $u \in (r, t)$, then $|v_u|^2 + \lambda g(x_u)$ is monotone non-decreasing with respect to $u \in (r, t)$.

Proof of Claim 2. For any $u \in (r, t)$, we have by assumption and Lemma 2.5 that

$$\lambda|g(x_u)| = -\lambda g(x_u) = -h_u + \frac{1}{2}|v_u|^2 \ge -h_u \ge -h_r \ge c\lambda^{1/2}.$$

Also, since $x_u \in [r_3 - a_0, r_3)$, we have by (A2) that $\frac{|g(x_u)|}{|g'(x_u)|} \le C_1^{-1}$. So

$$\begin{aligned}\frac{|v_u|}{\lambda|g'(x_u)|} &\le \frac{\sqrt{2\lambda|g(x_u)|}}{\lambda|g'(x_u)|} = \sqrt{2} \cdot \frac{|g(x_u)|}{|g'(x_u)|} \cdot \frac{1}{\sqrt{\lambda|g(x_u)|}} \\ &\le \sqrt{2}C_1^{-1}(c\lambda^{1/2})^{-1/2}.\end{aligned}$$

Therefore, $2bv_u + \lambda g'(x_u) \ge \lambda g'(x_u)\Big(1 - 2b\sqrt{2}C_1^{-1}c^{-1/2}\lambda^{-1/4}\Big)$. Notice that $\lambda \ge \lambda_1(c)$ implies that $1 - 2b\sqrt{2}C_1^{-1}c^{-1/2}\lambda^{-1/4} \ge 0$. So for any $u \in (r, t)$, since $g'(x_u) > 0$ and

$v_u < 0$ by assumption, we get that $\frac{d}{du}\Big(|v_u|^2 + \lambda g(x_u)\Big) = -\Big(2bv_u + \lambda g'(x_u)\Big)v_u \geq 0$. So $|v_u|^2 + \lambda g(x_u)$ is monotone non-decreasing with respect to $u \in (r, t)$. ∎

Let us come back to the proof of Proposition 2.7. We have that all the assumptions of Claim 2 are satisfied with $r = \xi_1$, and $v_r = 0$ in further. Therefore, by Claim 2, for any $u \in (r, t_1^\lambda(x_0, v_0))$, we have that $|v_u|^2 + \lambda g(x_u) \geq \lambda g(x_{\xi_1})$, hence $|v_u| \geq \lambda^{1/2}\sqrt{g(x_{\xi_1}) - g(x_u)}$. Combining this with (5.2), we get that

$$\begin{aligned} t_1^\lambda(x_0, v_0) - \xi_1 &= \int_{\xi_1}^{t_1^\lambda(x_0,v_0)} du \leq \int_{\xi_1}^{t_1^\lambda(x_0,v_0)} \frac{|v_u|}{\lambda^{1/2}\sqrt{g(x_{\xi_1}) - g(x_u)}} du \\ &= \lambda^{-1/2} \int_{r_3 - a_0}^{x_{\xi_1}} \frac{1}{\sqrt{g(x_{\xi_1}) - g(y)}} dy \\ &\leq 2\sqrt{a_0/(C_1 c)}\lambda^{-1/4}k^{-1/2}. \end{aligned}$$

This combined with (5.3) implies our assertion with $C_3(c) := (2 + \sqrt{2})\sqrt{a_0/(C_1 c)}$. ∎

Lemma 5.1 *Assume that $j_0 < 0$ and let $\eta > 0$ be any constant. Also, assume that either $g(x_0) = j_0 - \eta$ and $v_0 \cdot (x_0 - r_2) < 0$ or $g(x_0) < j_0 - \eta$. Let $\xi := \inf\{t > 0; g(x_t) = j_0 - \eta\}$. Then for any $\lambda \geq (-4bg(r_2)(r_3 - r_1))^2\eta^{-3}$, we have the followings:*

1. *$\xi \leq (r_3 - r_1)\eta^{-1/2}\lambda^{-1/2}$,*
2. *$|j_u - j_0| \leq -2bg(r_2)(r_3 - r_1)\eta^{-1/2}\lambda^{-1/2}$ for any $u \in (0, \xi)$,*
3. *$j_u - g(x_u) \geq \eta/2$ for any $u \in (0, \xi)$.*

Proof of Lemma 5.1. First, since j_t is monotone non-increasing with respect to t by Lemma 2.5, we have that $j_t \leq j_0 < 0$, hence $\{x_t\}$ stays in (r_1, r_3). Also, since $\frac{d}{dt}j_t = -b\lambda^{-1}|v_t|^2 \geq 2bg(r_2)$, we have that

$$j_t - j_0 \geq 2bg(r_2)t, \qquad \text{for any } t \geq 0. \tag{5.4}$$

For any $u \in (0, \xi)$, we have that $g(x_u) \leq j_0 - \eta$, hence by (5.4), we have that

$$\frac{1}{2}\lambda^{-1}|v_u|^2 = j_u - g(x_u) \geq j_u - (j_0 - \eta) \geq 2bg(r_2)u + \eta, \qquad \text{for any } u \in (0, \xi). \tag{5.5}$$

Now, choose and fix any $s_0 \leq \eta(-4bg(r_2))^{-1}$ for a while. Then for any $u \in (0, s_0 \wedge \xi)$, we have by (5.5) that $\frac{1}{2}\lambda^{-1}|v_u|^2 \geq 2bg(r_2)u + \eta \geq \eta/2$, hence $|v_u| \geq \eta^{1/2}\lambda^{1/2}$ for any $u \in (0, s_0 \wedge \xi)$. In particular, since $v_\cdot$ is continuous, this implies that v_u has the same sign for all $u \in (0, s_0 \wedge \xi)$.

Therefore, if $\xi \geq s_0$, then

$$r_3 - r_1 \geq |x_{s_0} - x_0| = \int_0^{s_0} |v_u| du \geq \eta^{1/2}\lambda^{1/2}s_0,$$

hence $s_0 \leq (r_3 - r_1)\eta^{-1/2}\lambda^{-1/2}$. In conclusion, we have proved that

$$s_0 \leq \eta(-4bg(r_2))^{-1}, \; s_0 \leq \xi \quad \Rightarrow \quad s_0 \leq (r_3 - r_1)\eta^{-1/2}\lambda^{-1/2}.$$

So $\eta(-4bg(r_2))^{-1} \wedge \xi \leq (r_3 - r_1)\eta^{-1/2}\lambda^{-1/2}$. Since

$$\eta(-4bg(r_2))^{-1} \geq (r_3 - r_1)\eta^{-1/2}\lambda^{-1/2} \tag{5.6}$$

by assumption, we get that $\xi \leq (r_3 - r_1)\eta^{-1/2}\lambda^{-1/2}$. This completes the proof of our first assertion.

The second assertion is now easy by (5.4) and Lemma 2.5. The third assertion is nothing but a combination of the first assertion, (5.5) and (5.6). ∎

Proof of Proposition 2.8. Let

$$\begin{aligned} C_{18}(\delta_1, \delta_2) &:= \sup_{j \in [g(r_2) + \frac{1}{4}\delta_1, -\delta_2]} S_1(j), \\ C_{19}(\delta_1, \delta_2) &:= \inf_{y \in [g^{-1,2}(g(r_2) + \frac{\delta_1}{4}), g^{-1,2}(-\delta_2)]} g'(y), \\ C_{20}(\delta_1, \delta_2) &:= \inf_{y \in [g^{-1,1}(-\delta_2), g^{-1,1}(g(r_2) + \frac{\delta_1}{4})]} |g'(y)|. \end{aligned}$$

Then $C_{18}(\delta_1, \delta_2), C_{19}(\delta_1, \delta_2), C_{20}(\delta_1, \delta_2) \in (0, \infty)$.

Choose any $j \in (g(r_2) + \delta_1, -\delta_2)$ and assume that $j_0 = \frac{1}{2}\lambda^{-1}|v_0|^2 + g(x_0) = j$. Since j_t is monotone non-increasing with respect to t by Lemma 2.5, we have that $\frac{1}{2}\lambda^{-1}|v_t|^2 + g(x_t) = j_t \leq j_0 = j < 0$, hence $\frac{1}{2}\lambda^{-1}|v_t|^2 \leq -g(r_2)$. Therefore, $\frac{d}{dt}j_t \geq 2bg(r_2)$, so

$$t \in (0, 2S_1(j)\lambda^{-1/2}) \Rightarrow j_t \in (j + 4bg(r_2)C_{18}(\delta_1, \delta_2)\lambda^{-1/2}, j). \tag{5.7}$$

In particular, if $\lambda \geq \Big(-8bg(r_2)C_{18}(\delta_1, \delta_2)\delta_1^{-1}\Big)^2$, then we have that $j_t \in (g(r_2) + \frac{1}{2}\delta_1, -\delta_2)$ as long as $t \in (0, 2S_1(j)\lambda^{-1/2})$.

Choose and fix any $j \in (g(r_2) + \delta_1, -\delta_2)$ and any $\eta \in (0, \frac{\delta_1}{4})$ for a while. Divide the period $(0, S_1(j)\lambda^{-1/2})$ into the periods that x_t stays in $(g^{-1,1}(j), g^{-1,1}(j - \eta))$, $(g^{-1,1}(j - \eta), g^{-1,2}(j - \eta))$ and $(g^{-1,2}(j - \eta), g^{-1,2}(j))$, respectively. In the following, we prove our assertion by considering each of these sojourn times. We shift the time such that each period starts from time 0 (hence (x_0, v_0) is different from the one up to now).

By first taking $\lambda \to \infty$ with η fixed, then taking $\eta \to 0$, our assertion is a direct consequence of Claim 3 given below. (For the sake of simplicity, we write $(x_u^\lambda(x_0, v_0), v_u^\lambda(x_0, v_0))$ as (x_t, v_t)). Precisely, assertions (1) $\sim$ (4) of Claim 3 estimate the sojourn time of the particle in $A := \{y : g(y) > j - \eta\}$, which combined with the boundedness of f gives us an estimate of the integral of $f(x_u(x, v))$ on $[0, S_1(\frac{1}{2}\lambda^{-1}|v|^2 + g(x))] \cap \{u : x_u(x, v) \in A\}$; (5) deals with the corresponding term for the case with $b = 0 = \sigma$; and (6) estimates the difference of these two integrals on $\{u : x_u(x, v) \in A^C\}$.

Claim 3. Assume that $j_0 \in [j - \frac{\eta}{2}, j]$. Also, assume that $\lambda \geq \Big(-8bg(r_2)C_{18}(\delta_1, \delta_2)\delta_1^{-1}\Big)^2 \vee \Big(-32b^2g(r_2)C_{19}(\delta_1, \delta_2)^{-2}\Big) \vee \Big(-32b^2g(r_2)C_{20}(\delta_1, \delta_2)^{-2}\Big) \vee \Big(-4bg(r_2)(r_3 - r_1)\Big)^2\eta^{-3}$. Let $\xi_1 := \inf\{t > 0; v_t = 0\}$ and $\xi_2 := \inf\{t > 0; g(x_t) = j - \eta\}$. Then we have the following:

(1) Assume that $v_0 > 0$, $x_0 \in (r_2, r_3)$ and $g(x_0) \geq j - \eta$. Then

$$\xi_1 \leq \sqrt{2}C_{19}(\delta_1, \delta_2)^{-1/2}\lambda^{-1/2}\sqrt{g^{-1,2}(j) - g^{-1,2}(j - \eta)}.$$

(2) Assume that $v_0 = 0$ and $x_0 \in (r_2, r_3)$. Then

$$\xi_2 \le 2C_{19}(\delta_1, \delta_2)^{-1/2}\lambda^{-1/2}\sqrt{g^{-1,2}(j) - g^{-1,2}(j-\eta)}.$$

(3) Assume that $v_0 < 0$, $x_0 \in (r_1, r_2)$ and $g(x_0) \ge j - \eta$. Then

$$\xi_1 \le \sqrt{2}C_{20}(\delta_1, \delta_2)^{-1/2}\lambda^{-1/2}\sqrt{g^{-1,1}(j-\eta) - g^{-1,1}(j)}.$$

(4) Assume that $v_0 = 0$ and $x_0 \in (r_1, r_2)$. Then

$$\xi_2 \le 2C_{20}(\delta_1, \delta_2)^{-1/2}\lambda^{-1/2}\sqrt{g^{-1,1}(j-\eta) - g^{-1,1}(j)}.$$

(5) We have that

$$\begin{aligned} &\int_{g^{-1,1}(j)}^{g^{-1,1}(j-\eta)} \frac{|f(y)|}{\sqrt{j - g(y)}}dy + \int_{g^{-1,2}(j-\eta)}^{g^{-1,2}(j)} \frac{|f(y)|}{\sqrt{j - g(y)}}dy \\ \le\ & 2\|f\|_\infty\Big(C_{19}(\delta_1, \delta_2)^{-1/2}\sqrt{g^{-1,2}(j) - g^{-1,2}(j-\eta)} \\ &\quad + C_{20}(\delta_1, \delta_2)^{-1/2}\sqrt{g^{-1,1}(j-\eta) - g^{-1,1}(j)}\Big). \end{aligned}$$

(6) For any $x_0, z \in [g^{-1,1}(j-\eta), g^{-1,2}(j-\eta)]$, let $\xi_z = \inf\{t > 0; x_t = z\}$, and let $|x_0, z|$ denote the interval $[x_0, z]$ if $x_0 < z$, or the interval $[z, x_0]$ if $z < x_0$. Then

$$\begin{aligned} &\sup_{\substack{x_0, z \in [g^{-1,1}(j-\eta), g^{-1,2}(j-\eta)],\\ v_0\cdot(z-x_0)>0}} \Big|\lambda^{1/2}\int_0^{\xi_z} f(x_u)du - \int_{|x_0,z|} \frac{f(y)}{\sqrt{2}\sqrt{j - g(y)}}dy\Big| \\ \le\ & \eta^{-3/2}\|f\|_\infty(r_3 - r_1)\Big(- 2bg(r_2)(r_3 - r_1)\eta^{-1/2}\lambda^{-1/2} + |j_0 - j|\Big). \end{aligned}$$

Proof of Claim 3. Since $\lambda \ge \Big(- 8bg(r_2)C_{18}(\delta_1, \delta_2)\delta_1^{-1}\Big)^2$, by the same method as in the beginning of the proof of this proposition, we have that $u \in (0, 2S_1(j)\lambda^{-1/2}) \Rightarrow j_u \in (g(r_2) + \frac{1}{4}\delta_1, -\delta_2)$. Indeed, we have that $j_0 \ge j - \eta/2 \ge j - \delta_1/8$, so for any $t \in (0, 2S_1(j)\lambda^{-1/2})$, we have by (5.7) that $j_t \ge j - \delta_1/8 + 4bg(r_2)C_{18}(\delta_1, \delta_2)\lambda^{-1/2} \ge j - \frac{5}{8}\delta_1$. This combined with $j \ge g(r_2) + \delta_1$ implies that $j_t \ge g(r_2) + \frac{1}{4}\delta_1$.

The proofs of (1) and (2) given in the following are almost the same as that of Proposition 2.7.

(1) Since j_t is monotone non-increasing by Lemma 2.5, we have that $g(x_{\xi_1}) = j_{\xi_1} \le j_0 \le j$, hence $x_{\xi_1} \le g^{-1,2}(j)$. Also, for any $u \in (0, \xi_1)$, we have that $\frac{1}{2}\lambda^{-1}|v_u|^2 + g(x_u) = j_u \ge j_{\xi_1} = g(x_{\xi_1})$, hence $|v_u| \ge \sqrt{2\lambda}\sqrt{g(x_{\xi_1}) - g(x_u)}$. Moreover, we have by the mean-value theorem that $g(x_{\xi_1}) - g(y) \ge C_{19}(\delta_1, \delta_2)(x_{\xi_1} - y)$ for any $y \in [g^{-1,2}(g(r_2) + \frac{\delta_1}{4}), x_{\xi_1}]$.

Therefore,

$$\begin{aligned}
\xi_1 &= \int_0^{\xi_1} du \le \int_0^{\xi_1} \frac{v_u}{\sqrt{2\lambda}\sqrt{g(x_{\xi_1}) - g(x_u)}} du \\
&\le \int_{g^{-1,2}(j-\eta)}^{x_{\xi_1}} \frac{1}{\sqrt{2\lambda}\sqrt{g(x_{\xi_1}) - g(y)}} dy \\
&\le \frac{1}{\sqrt{2\lambda C_{19}(\delta_1,\delta_2)}} \int_{g^{-1,2}(j-\eta)}^{x_{\xi_1}} \frac{1}{\sqrt{x_{\xi_1} - y}} dy \\
&= \frac{1}{\sqrt{2\lambda C_{19}(\delta_1,\delta_2)}} 2\sqrt{x_{\xi_1} - g^{-1,2}(j-\eta)} \\
&\le \sqrt{2} C_{19}(\delta_1,\delta_2)^{-1/2} \lambda^{-1/2} \sqrt{g^{-1,2}(j) - g^{-1,2}(j-\eta)}.
\end{aligned}$$

(2) We have that $\lambda \ge -32b^2 g(r_2) C_{19}(\delta_1,\delta_2)^{-2}$. So for any $u \in (0,\xi_2)$, $2bv_u + \lambda g'(x_u)$ has the same sign as $g'(x_u)$. Indeed, we have that $\left|\frac{2bv_u\lambda^{-1/2}}{g'(x_u)}\right| \le 2b\sqrt{-2g(r_2)} C_{19}(\delta_1,\delta_2)^{-1}$, hence $\frac{2bv_u + \lambda g'(x_u)}{\lambda g'(x_u)} = 1 + \frac{2bv_u\lambda^{-1/2}}{g'(x_u)}\lambda^{-1/2} \ge 1 - 2b\sqrt{-2g(r_2)} C_{19}(\delta_1,\delta_2)^{-1}\lambda^{-1/2} \ge \frac{1}{2}$.

So in the present case, we have for any $u \in (0,\xi_2)$ that $2bv_u + \lambda g'(x_u) > 0$. Also, v_u is negative in the present case for $u \in (0,\xi_2)$. So

$$\frac{d}{du}\Big(|v_u|^2 + \lambda g(x_u)\Big) = -v_u\Big(2bv_u + \lambda g'(x_u)\Big) > 0, \qquad u \in (0,\xi_2).$$

Therefore, $|v_u|^2 + \lambda g(x_u)$ is monotone non-decreasing with respect to $u \in (0,\xi_2)$. So for any $u \in (0,\xi_2)$, we have that $|v_u|^2 + \lambda g(x_u) \ge \lambda g(x_0)$, hence $|v_u| \ge \lambda^{1/2}\sqrt{g(x_0) - g(x_u)}$. Also, we have that $x_0 \le g^{-1,2}(j)$, and that $g(x_0) - g(y) \ge C_{19}(\delta_1,\delta_2)(x_0 - y)$ for any $y \in (g^{-1,2}(j-\eta), x_0)$. Therefore,

$$\begin{aligned}
\xi_2 &= \int_0^{\xi_2} du \le \int_0^{\xi_2} \frac{|v_u|}{\lambda^{1/2}\sqrt{g(x_0) - g(x_u)}} du \\
&\le \int_{g^{-1,2}(j-\eta)}^{x_0} \frac{1}{\lambda^{1/2}\sqrt{g(x_0) - g(y)}} dy \\
&\le \lambda^{-1/2} \frac{1}{\sqrt{C_{19}(\delta_1,\delta_2)}} \int_{g^{-1,2}(j-\eta)}^{x_0} \frac{1}{\sqrt{x_0 - y}} dy \\
&= \lambda^{-1/2} C_{19}(\delta_1,\delta_2)^{-1/2} 2\sqrt{x_0 - g^{-1,2}(j-\eta)} \\
&\le \lambda^{-1/2} C_{19}(\delta_1,\delta_2)^{-1/2} 2\sqrt{g^{-1,2}(j) - g^{-1,2}(j-\eta)}.
\end{aligned}$$

(3) is proved in exactly the same way as that for (1), and (4) is proved in exactly the same way as that for (3). (5) is proved similarly, and we omit the proof here.

(6)Assume that $v_0 > 0$ and $z > x_0$. First notice that

$$\int_{[x_0,z]} \frac{f(y)}{\sqrt{2}\sqrt{j - g(y)}} dy = \int_0^{\xi_z} \frac{f(x_u)}{\sqrt{2}\sqrt{j - g(x_u)}} v_u du.$$

Also, since $j_u \le j$, we have that

$$\begin{aligned} & \Big|(j_u - g(x_u))^{-1/2} - (j - g(x_u))^{-1/2}\Big| \\ \le\ & \frac{1}{2}(j_u - g(x_u))^{-3/2}|j_u - j| \\ \le\ & \frac{1}{2}(j_u - g(x_u))^{-3/2}\Big(|j_u - j_0| + |j_0 - j|\Big). \end{aligned}$$

Since $\lambda \ge (-4bg(r_2)(r_3 - r_1))^2\eta^{-3}$, this combined with Lemma 5.1 implies that for any $u \in [0, \xi_z]$, we have that

$$\Big|(j_u - g(x_u))^{-1/2} - (j - g(x_u))^{-1/2}\Big| \le \frac{1}{2}(\eta/2)^{-3/2}\Big(-2bg(r_2)(r_3 - r_1)\eta^{-1/2}\lambda^{-1/2} + |j_0 - j|\Big).$$

Therefore,

$$\begin{aligned} & \Big|\lambda^{1/2}\int_0^{\xi_z} f(x_u)du - \int_{|x_0,z|}\frac{f(y)}{\sqrt{2}\sqrt{j - g(y)}}dy\Big| \\ =\ & \Big|\int_0^{\xi_z} v_u f(x_u)\frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{j_u - g(x_u)}} - \frac{1}{\sqrt{j - g(x_u)}}\Big)du\Big| \\ \le\ & \frac{1}{\sqrt{2}}\int_0^{\xi_z} |v_u||f(x_u)|\Big|\frac{1}{\sqrt{j_u - g(x_u)}} - \frac{1}{\sqrt{j - g(x_u)}}\Big|du \\ \le\ & \eta^{-3/2}\Big(-2bg(r_2)(r_3 - r_1)\eta^{-1/2}\lambda^{-1/2} + |j_0 - j|\Big)\int_0^{\xi_z} |v_u||f(x_u)|du. \end{aligned}$$

Since $v_0 \cdot (z - x_0) > 0$, we have that $\int_0^{\xi_z} |v_u||f(x_u)|du = \int_{|x_0,z|} |f(y)|dy \le \|f\|_\infty (r_3 - r_1)$, this completes the proof of our assertion. ∎

As explained, this completes the proof of Proposition 2.8. ∎

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