# Hausdorff dimension of asymptotic self-similar sets

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# Abstract

In the present dissertation, we introduce the notion of almost similarity maps extending that of similarity maps in order to construct asymptotic self-similar sets on curved metric spaces, and determine the Hausdorff dimensions of such asymptotic self-similar sets.

Let $X$ be a complete doubling metric space. Let $\bar{U} \supset \bar{V}$ be bounded domains in $X$ homeomorphic to each other, where $\bar{U}$ and $\bar{V}$ denote the closures of the open subsets $U$ and $V$. Fix constants $0 < \lambda < 1$, $0 < \nu < 1$ and a continuous monotone non-decreasing function $\varphi : (0, \infty) \to [0, \infty)$ with $\lim_{x \to +0} \varphi(x) = 0$. We call a homeomorphism $f : \bar{U} \to \bar{V}$ a $(\lambda, \varphi(|\bar{U}|), \nu)$-*almost similarity map* if for every $x, y \in \bar{U}$

$$\left| \frac{|f(x), f(y)|}{|x, y|} - \lambda \right| \leq \lambda\varphi(|U|),$$
$$|V| \leq \nu|U|.$$

Where $|\bar{U}|$ is the diameter of $\bar{U}$. Then the set $\bar{V}$ is called a $(\lambda, \varphi(|\bar{U}|), \nu)$-almost similar set of $\bar{U}$.

For a fixed positive integer $k$, we denote by $\mathcal{I} = \mathcal{I}^k$ the set of all ordered multi-indices $I = i_1 \cdots i_n$ with $n \geq 1$, $1 \leq i_j \leq k$ for every $1 \leq j \leq n$. We set $|I| = |i_1 \cdots i_n| = n$ and call it the length of $I$. Let $\mathcal{I}_n$ denote the set of all $I \in \mathcal{I}$ of length $n$.

An asymptotic self-similar set is defined under the following hypothesis: For $0 < \nu < 1$ and $a > 0$, let $\varphi : (0, \infty) \to [0, \infty)$ be a continuous function with $\lim\limits_{x \to 0} \varphi(x) = 0$ satisfying conditions (1) and (2) in introduction.

**Definition 1.** ([51]) Suppose that *ratio coefficients* $0 < \lambda_i < 1$, $(i = 1, \ldots, k)$ together with a non-empty open subset $V \subset X$ are given for which we have

(1) for each $1 \leq i \leq k$, a $(\lambda_i, \varphi(|\bar{V}|), \nu)$-almost similarity map

$$f_i : \bar{V} \to \bar{V}_i \subset \bar{V},$$

is given in such a way that $V_i \cap V_j = \emptyset$ for every $1 \leq i \neq j \leq k$, where $V_i := f_i(V)$;

(2) for each $1 \leq i, j \leq k$, a $(\lambda_j, \varphi(|\bar{V}_i|), \nu)$-almost similarity map

$$f_{ij} : \bar{V}_i \to \bar{V}_{ij} \subset \bar{V}_i,$$

is given in such a way that $V_{ij} \cap V_{ij'} = \emptyset$ for every $1 \leq j \neq j' \leq k$, where $V_{ij} := f_{ij}(V_i)$;

(3) for each $I' \in \mathcal{I}_{n-1}$ and $1 \le i_n \le k$ with $I := I'i_n$, a $(\lambda_{i_n}, \varphi(|\bar{V}_{I'}|), \nu)$-almost similarity map
$$f_I : \bar{V}_{I'} \to \bar{V}_I \subset \bar{V}_{I'},$$
is defined in such a way that $V_{I'i} \cap V_{I'j} = \emptyset$ for every $1 \le i \ne j \le k$, where $V_I := f_I(V_{I'})$.

We call $\{(\bar{V}_I, f_I)\}_{I\in\mathcal{I}}$ an $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-*asymptotic similarity system.* Then the set $K$ defined as
$$K = \bigcap_{n=1}^{\infty} \left( \bigcup_{I\in\mathcal{I}_n} \bar{V}_I \right)$$
is called an *asymptotic self-similar set* in $X$.

Our main results in the present dissertation are stated as follows.

**Theorem 2.** ([51]) *Let $X$ be a complete doubling metric space and let $K$ be the asymptotic self-similar set associated with a $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I\in\mathcal{I}}$. Then the Hausdorff and the box dimensions of the asymptotic self-similar set $K$ are given as*
$$\dim_H K = \dim_B K = s,$$
*where $s$ is a unique number satisfying* $\displaystyle\sum_{i=1}^{k} \lambda_i^s = 1.$

**Theorem 3.** ([51]) *If a geodesic triangle domain $\Delta$ in a convex domain on a surface is asymptotically non-degenerate, then*

*(1) for some $0 < \nu < 1$ there exists a $(\{1/2, 1/2, 1/2\}, \varphi, \nu)$-asymptotic similarity system $\{(\Delta_I, f_I)\}_{I\in\mathcal{I}^3}$ associated with $\Delta$, where $\varphi(x) = cx^2$ for some constant $c > 0$;*

*(2) the Hausdorff and box dimensions of the Sierpinski gasket $K_\Delta$ associated with $\Delta$ are given by*
$$\dim_H K_\Delta = \dim_B K_\Delta = \frac{\log 3}{\log 2}.$$

**Corollary 4.** ([51]) *A geodesic triangle domain $\Delta$ in a convex domain on a surface is asymptotically non-degenerate if and only if for some $0 < \nu < 1$ there exists a $(\{1/2, 1/2, 1/2\}, \varphi, \nu)$-asymptotic similarity system $\{(\Delta_I, f_I)\}_{I\in\mathcal{I}^3}$ associated with $\Delta$, where $\varphi(x) = cx^2$ for some constant $c > 0$.*

# Contents

# Chapter 0

# Introduction and Main results

A fractal is a set whose Hausdorff dimension is not an integer. A bijective map $f : \mathbb{R}^d \to \mathbb{R}^d$ is called a *contracting similarity map* if there exists a real number $0 < \lambda < 1$, such that $d(f(x), f(y)) = \lambda d(x, y)$ for every $x,\ y \in \mathbb{R}^d$. The notion of self-similar sets or general Cantor sets have played significant roles in fractal geometry. These sets are usually defined by means of iterated function systems $\{f_1, \cdots, f_k\}$ consisting of contracting similarity maps on a complete metric space as the unique nonempty compact set $K$, called an attractor or an invariant set, satisfying $K = \bigcup_{i=1}^{n} f_i(K)$. Moran constructed general cantor sets in $\mathbb{R}^d$ by using the notion of *contracting similarity maps*, and determined the Hausdorff dimension of them as the similarity dimension (see [41], for instance). Hutchinson [21] (cf. Kigami [27], Schief [45]) introduced the notion of the open set condition and determined the Hausdorff dimension of self-similar sets in Euclidean space $\mathbb{R}^d$ satisfying the open set condition. Balogh and Rohner extended Hutchinson's result to doubling metric spaces ([7]). However, it is difficult to construct a contracting similarity map in general metric spaces. Actually, similarity maps do not always exist on curved metric spaces. To overcome this difficulty, in the present dissertation we introduce the notion of $(\lambda, \varphi, \nu)$-*almost similarity maps* extending that of $\lambda$-similarity maps, that is defined as follows: Let $\bar{U} \supset \bar{V}$ be bounded domains in a metric space $X$ homeomorphic to each other, where $\bar{U}$ and $\bar{V}$ denote the closures of the open subsets $U$ and $V$. Fix constants $0 < \lambda < 1$, $0 < \nu < 1$ and a continuous monotone non-decreasing function $\varphi : (0, \infty) \to [0, \infty)$ with $\lim_{x \to +0} \varphi(x) = 0$. We call a homeomorphism $f : \bar{U} \to \bar{V}$ a $(\lambda, \varphi(|\bar{U}|), \nu)$-*almost similarity map* if for every $x, y \in \bar{U}$,

$$\left| \frac{|f(x), f(y)|}{|x, y|} - \lambda \right| \le \lambda \varphi(|U|),$$
$$|V| \le \nu |U|.$$

Where $|U|$ is the diameter of $U$. Then the set $\bar{V}$ is called a $(\lambda, \varphi(|\bar{U}|), \nu)$-almost similar set of $\bar{U}$.

In the present dissertation, we extend both Balogh and Rohner 's result.

A metric space $X$ is said to be *doubling* if there exists a number $C$ such that for any $x \in X$ and any $r > 0$, there exist $\{x_i\}_{i=1}^C \subset X$ such that

$$B(x, r) \subset \bigcup_{i=1}^{C} B(x_i, r/2)$$

Note that $C$, called the *doubling constant* of $X$, does not dependent on the choices of $x$ or $r$.

In the present dissertation, we investigate asymptotic self-similar sets on the doubling metric spaces, that are defined by using the notion of $(\lambda, \varphi, \nu)$-*almost similarity maps*, and as applications, we construct asymptotic self-similar sets on Riemannian manifolds and surfaces.

In recent years, geomeric analysis on doubling metric measure spaces has been very active (see for instance Assouad [1], Gromov[20], Heinonen [23], Villani[48]), and therefore it is quite natural to study self-similarity sets in such doubling metric spaces.

In the present dissertation, all spaces are assumed to be proper complete metric spaces. We first introduce the notion of $(\lambda, c, \nu)$-*similarity maps* to define asymptotically generalized Cantor sets, and determine the Hausdorff dimension of such an asymptotically generalized Cantor set. Let $X$ be a metric space. Let $\bar{U} \supset \bar{V}$ be bounded domains in $X$ homeomorphic to each other, where $\bar{U}$ and $\bar{V}$ denote the closures of the open subsets $U$ and $V$. Fix constants $0 < \lambda < 1$, $c > 0$, and $0 < \nu < 1$.

**Definition 1.** ([50]) We call a homeomorphism $f : \bar{U} \to \bar{V}$ is a $(\lambda, c, \nu)$-*similarity map* if for every $x, y \in \bar{U}$,

$$\left| \frac{d(f(x), f(y))}{d(x, y)} - \lambda \right| \leq \lambda c |U|$$
$$|V| \leq \nu |U|$$

Then the set $\bar{V}$ is called a $(\lambda, c, \nu)$-similar set of the set $\bar{U}$.

Using this notation, we can define an *asymptotically generalized Cantor set* in $X$ as follows:

**Definition 2.** ([50]) Suppose that *ratio coefficients* $0 < \lambda_i < 1$, $(i = 1, \dots, k)$ and constants $c > 0$, $0 < \nu < 1$ are given for which we have

(1) Consider $k$ subsets $\Delta_1, \cdots, \Delta_k$ of $X$, each of which is bounded and closed, satisfying $\overline{(\Delta_i)^0} = \Delta_i$, $\Delta_i \bigcap \Delta_j = \emptyset$ $(1 \leq i \neq j \leq k)$, where $\Delta^0$ and $\bar{\Delta}$ denote the interior and the closure of $\Delta$ respectively. These sets are called *basic sets*.

(2) For any $1 \leq i, j \leq k$, let $\Delta_{ij}$ be $(\lambda_j, c, \nu)$-similar sets of $\Delta_i$ such that $\Delta_{ij} \bigcap \Delta_{ij'} = \emptyset$ $(j \neq j')$ $(1 \leq j, j' \leq k)$.

(3) For any $n \geq 2$ and $\omega_1, \cdots, \omega_n \in \{1, 2, \cdots, k\}$, construct $(\lambda_{\omega_n}, c, \nu)$-similar sets $\Delta_{\omega_1 \cdots \omega_n}$ of $\Delta_{\omega_1 \cdots \omega_{n-1}}$ such that $\Delta_{\omega_1 \cdots \omega_n} \bigcap \Delta_{\omega_1 \cdots \omega'_n} = \emptyset$ $(1 \leq \omega_n \neq \omega'_n \leq k)$.

Then the set $C$ defined as

$$C := \bigcap_{n=1}^{\infty} \left( \bigcup_{\omega_1, \ldots, \omega_n = 1}^{k} \Delta_{\omega_1 \cdots \omega_n} \right)$$

is called an *asymptotically generalized Cantor set* in $X$.

**Theorem 3.** ([50]) *Let $X$ be a complete doubling metric space. Let $C$ be an asymptotically generalized Cantor set in $X$ with ratio coefficients $\lambda_1, \ldots, \lambda_k$ defined above. Then the Hausdorff dimension of $C$ is equal to the similarity dimension. Namely it is equal to $t$ such that $\sum_{i=1}^{k} \lambda_i^t = 1$.*

For a fixed positive integer $k$, we denote by $\mathcal{I} = \mathcal{I}^k$ the set of all ordered multi-indices $I = i_1 \cdots i_n$ with $n \geq 1$, $1 \leq i_j \leq k$ for every $1 \leq j \leq n$. We set $|I| = |i_1 \cdots i_n| = n$ and call it the length of $I$. Let $\mathcal{I}_n$ denote the set of all $I \in \mathcal{I}$ of length $n$.

As an application of Theorem 3 , we consider an asymptotically generalized Cantor set on a complete Riemannian manifold, which is constructed as follows. Fix constants $0 < \lambda < 1$. Let $B(r)$ be a closed ball of radius $r$ on a Riemannian manifold $M$, $k$ be an integer. First, we take $k$ disjoint closed balls $B_i(\lambda r)$ of radius $\lambda r$ in $B(r)$ $(1 \leq i \leq k)$. Next, we also take $k$ disjoint closed balls $B_{ij}(\lambda^2 r)$ of radius $\lambda^2 r$ in each $B_i(\lambda r)$ $(1 \leq j \leq k)$. Repeating this procedure for each $B_{ij}(\lambda^2 r)$ infinitely many times, we obtain a family of disjoint closed balls $\{B_I(\lambda^{|I|} r)\}_{I \in \mathcal{I}^k}$. The asymptotically generalized Cantor set $K_B$ on $M$ associated with $\{B_I(\lambda^{|I|} r)\}_{I \in \mathcal{I}^k}$ is defined as

$$K_B := \bigcap_{n=1}^{\infty} \left( \bigcup_{I \in \mathcal{I}_n} B_I(\lambda^{|I|} r) \right)$$

If $r$ is sufficiently small, then the Hausdorff dimension of $K_B$ is given by

$$\dim_H K_B = \frac{\log k}{-\log \lambda}.$$

Next, we introduce the notion of $(\lambda, \varphi, \nu)$-almost similarity maps in doubling metric spaces to define asymptotic self-similar sets satisfying the generalized open set condition, and determine their Hausdorff dimensions.

Fix a continuous monotone non-decreasing function $\varphi : (0, \infty) \to [0, \infty)$ with $\lim_{x \to +0} \varphi(x) = 0$.

In this dessertation, we assume the following conditions for $\varphi$ :

(1) $\varphi : (0, \infty) \to [0, \infty)$ is non-decreasing with $\lim_{x \to +0} \varphi(x) = 0$;

(2) $\displaystyle\int_1^{\infty} \varphi(a\nu^x)\, dx < \infty$ for some constants $a > 0$ and $0 < \nu < 1$.

Note that the second condition (2) above does not depend on the choice of $a > 0$ and $0 < \nu < 1$, and that for any $\alpha > 0$ and any positive integer $n$, the following functions satisfy the above conditions:

$$\varphi(y) = y^{\alpha}, \quad \varphi(y) = -(\log y)^{-1-\frac{2}{2n+1}}.$$

An asymptotic self-similar set in $X$ is defined under the following hypothesis: For $0 < \nu < 1$ and $a > 0$, let $\varphi : (0, \infty) \to (0, \infty)$ be a continuous function satisfying the above conditions (1) and (2).

**Definition 4.** ([51]) Suppose that *ratio coefficients* $0 < \lambda_i < 1$, $(i = 1, \ldots, k)$ together with a non-empty open subset $V \subset X$ are given for which we have

(1) for each $1 \leq i \leq k$, a $(\lambda_i, \varphi(|\bar{V}|), \nu)$-almost similarity map

$$f_i : \bar{V} \to \bar{V}_i \subset \bar{V},$$

is given in such a way that $V_i \cap V_j = \emptyset$ for every $1 \leq i \neq j \leq k$, where $V_i := f_i(V)$ are open subsets.

(2) for each $1 \leq i, j \leq k$, a $(\lambda_j, \varphi(|\bar{V}_i|), \nu)$-almost similarity map

$$f_{ij} : \bar{V}_i \to \bar{V}_{ij} \subset \bar{V}_i,$$

is given in such a way that $V_{ij} \cap V_{ij'} = \emptyset$ for every $1 \leq j \neq j' \leq k$, where $V_{ij} := f_{ij}(V_i)$ are open subsets.

(3) for each $I' \in \mathcal{I}_{n-1}$ and $1 \leq i_n \leq k$ with $I := I' i_n$, a $(\lambda_{i_n}, \varphi(|\bar{V}_{I'}|), \nu)$-almost similarity map

$$f_I : \bar{V}_{I'} \to \bar{V}_I \subset \bar{V}_{I'},$$

is defined in such a way that $V_{I'i} \cap V_{I'j} = \emptyset$ for every $1 \leq i \neq j \leq k$, where $V_I := f_I(V_{I'})$ are open subsets.

We call $\{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}}$ an $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-*asymptotic similarity system*. Then the set $K$ defined as

$$K = \bigcap_{n=1}^{\infty} \left( \bigcup_{I \in \mathcal{I}_n} \bar{V}_I \right),$$

is called an *asymptotic self-similar set* in $X$.

In some sense, asymptotic self-similar sets are generalization of asymptotically generalized cantor sets satisfying open set condition.

Let us consider the case of iterated function system $\{f_1, \ldots, f_k\}$ of contracting similarity maps with open set condition

(1) $V \supset f_1(V) \cup \cdots \cup f_k(V)$;

(2) $f_i(V) \cap f_j(V) = \emptyset$ for every $i \neq j$;

for some non-empty open set $V \subset X$. For $I = i_1 \cdots i_n$, let $V_I := f_{i_n} \circ \cdots \circ f_{i_1}(V)$ and let $f_I := f_{i_n} : \bar{V}_{I'} \to \bar{V}_I$ where $I' = i_1 \cdots i_{n-1}$. Then this gives a $(\{\lambda_i\}_{i=1}^k, \varphi = 0, \lambda_{\max})$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}}$, where $\lambda_{\max} = \max \lambda_i$. Thus the notion of $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system is an extension of iterated function system of contracting similarity maps with open set condition.

**Theorem 5.** ([51]) *Let $X$ be a complete doubling metric space and let $K$ be the asymptotic self-similar set associated with a $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}}$. Then the Hausdorff and the box dimensions of $K$ are given as*

$$\dim_H K = \dim_B K = s,$$

*where $s$ is a unique number satisfying $\displaystyle\sum_{i=1}^k \lambda_i^s = 1$.*

In [7], Balogh and Rohner suggested a problem: What happens if an iterated function system $\{f_1, \ldots, f_k\}$ of contracting similarity maps is replaced by one of contracting *asymptotically similarity maps* in the sense that for all $I = i_i \cdots i_n \in \mathcal{I}$

$$c_1 \lambda_I \leq \frac{|f_I(x), f_I(y)|}{|x, y|} \leq c_2 \lambda_I,$$

where $f_I = f_{i_n} \circ \cdots \circ f_{i_1}$, $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_n}$ and $c_1$, $c_2$ are uniform positive constants. Our $(\{\lambda_i\}_{i=1}^k, \varphi, \nu)$-asymptotic similarity system $\{(\bar{V}_I, f_I)\}_{I \in \mathcal{I}}$ is closely related with the above iterated function system of contracting asymptotically similarity maps under the open set condition (see Lemma 3.3.2 ). Thus Theorem 5 can be thought of as a partial answer to the question raised by Balogh and Rohner in a more general situation than an iterated function system.

As an application of Theorem 5, we consider a Sierpinski gasket on a complete surface $M$, which is naturally defined in a geometric way as follows. Let $\Delta$ be a closed domain bounded by a geodesic triangle. By joining the midpoints of the edges of $\Delta$, we divide $\Delta$ into four triangles, and remove the center triangle to get three triangles $\Delta_1$, $\Delta_2$ and $\Delta_3$. Repeating this procedure for each $\Delta_i$ infinitely many times, we obtain a system of geodesic triangles $\{\Delta_I\}_{I \in \mathcal{I}^3}$. The *generalized Sierpinski gasket $K_\Delta$* on $M$ associated with $\Delta$ is defined as

$$K_\Delta = \bigcap_{n=1}^{\infty} \left( \bigcup_{I \in \mathcal{I}_n} \Delta_I \right),$$

A geodesic triangle region $\Delta$ is called *$\delta$-non-degenerate* if each angle $\tilde{\alpha}$ of a comparison triangle $\tilde{\Delta}$ of $\Delta$ in $\mathbb{R}^2$ satisfies $\delta < \tilde{\alpha} < \pi - \delta$, where a comparison triangle means that $\tilde{\Delta}$ has the same side-length as $\Delta$. We say that $\Delta$ is *asymptotically non-degenerate* if all the divided small triangles $\Delta_I$ are $\delta$-non-degenerate for some constant $\delta > 0$. For example, every geodesic triangle region $\Delta$ of perimeter less than $2\pi$ on a unit sphere is asymptotically non-degenerate (see Example 4.2.3). We show that a small geodesic triangle region on a surface is asymptotically non-degenerate (see Lemma 4.2.9).

**Theorem 6.** ([51]) *If a geodesic triangle domain $\Delta$ in a convex domain on a surface is asymptotically non-degenerate, then*

*(1) for some $0 < \nu < 1$ there exists a $(\{1/2, 1/2, 1/2\}, \varphi, \nu)$-asymptotic similarity system $\{(\Delta_I, f_I)\}_{I \in \mathcal{I}^3}$ associated with $\Delta$, where $\varphi(x) = cx^2$ for some constant $c > 0$;*

*(2) the Hausdorff and box dimensions of the Sierpinski gasket $K_\Delta$ associated with $\Delta$ are given by*

$$\dim_H K_\Delta = \dim_B K_\Delta = \frac{\log 3}{\log 2}.$$

The following result gives a condition for $\Delta$ to be asymptotically non-degenerate.

**Corollary 7.** ([51]) *A geodesic triangle domain $\Delta$ in a convex domain on a surface is asymptotically non-degenerate if and only if for some $0 < \nu < 1$ there exists a $(\{1/2, 1/2, 1/2\}, \varphi, \nu)$-asymptotic similarity system $\{(\Delta_I, f_I)\}_{I \in \mathcal{I}^3}$ associated with $\Delta$, where $\varphi(x) = cx^2$ for some constant $c > 0$.*

In the present dissertation, we also investigate self-similar sets by using $\lambda$-similarity maps in some trees. Indeed, this is also the special cases of asymptotic self-similar sets. In these cases, $\varphi = 0$, and $\nu = \frac{\lambda}{2}$.

Let us consider a tree $X_0$ with vertices $\{\ P_0,\ P_{w_1 \cdots w_k} \mid w_i \in \{1, 2\},\ \ i = 1, 2, \cdots, k,\ k \geq 1,\ \}$ defined as follows. First we fix a constant $0 < \lambda < 1$. We begin with the two edges $[P_0 P_1], [P_0 P_2]$ of length $\frac{\lambda}{2}$. For each $w \in \{1, 2\}$, the edge $[P_0 P_w]$ branches at $P_w$ into two edges $[P_w P_{w1}], [P_w P_{w2}]$ of length $(\frac{\lambda}{2})^2$. In general, for $w_1, \cdots, w_k \in \{1, 2\}$, the edge $[P_{w_1 \cdots w_{k-1}} P_{w_1 w_2 \cdots w_k}]$ branches at $P_{w_1 w_2 \cdots w_k}$ into two edges $[P_{w_1 w_2 \cdots w_k} P_{w_1 w_2 \cdots w_k 1}]$, $[P_{w_1 w_2 \cdots w_k} P_{w_1 w_2 \cdots w_k 2}]$ of length $(\frac{\lambda}{2})^{k+1}$. In this way, we construct the infinite tree $X_0$. The distance on $X_0$ is naturally defined by using the length of edges. Let X be the completion of $X_0$, and let $\partial X$ be the set of points of X at which some shortest path cannot extend anymore. We define

$$C := \partial X = X \backslash X_0.$$

Obviously, $X$ is a CAT(0) space. Next, we consider $C$ as a self-similar set ([49], [7]). Let $X_{w_1 w_2 \cdots w_k}$ be the union of shortest paths from $P_{w_1 w_2 \cdots w_k}$ to $C$. For each $w \in \{1, 2\}$, we define the map $f_w : X \to X_w$ by

$$f_w(P_0) = P_w,$$
$$f_w(P_{w_1 w_2 \cdots w_k}) = P_{w w_1 w_2 \cdots w_k}.$$

It is a $\frac{\lambda}{2}$-similarity map. From the iterated function system $\{f_1, f_2\}$, we have a self-similar set $C$. In a terminology of asymptotic self-similar sets, we proceed as follows.

In general, for $w_1, w_2, \cdots, w_k \in \{1, 2\}$, we define the map $f_{w_1 w_2 \cdots w_k} : X_{w_1 w_2 \cdots w_{k-1}} \to X_{w_1 w_2 \cdots w_k}$ by

$$f_{w_1 w_2 \cdots w_k} = f_{w_k}|_{X_{w_1 w_2 \cdots w_{k-1}}}.$$

It is also a $\frac{\lambda}{2}$-similarity map. We define C as the limiting set

$$C := \bigcap_{k=1}^{\infty} ( \bigcup_{w_1,\ldots,w_k=1}^{2} X_{w_1 \cdots w_k} ).$$

Let $X$ be a metric space. We denote by $Isom(X)$ the set of all isometries on $X$. We say that $A \subset X$ is *homogeneous* in $X$ if for any $x, y \in A$ there is a isometry $g \in Isom(X)$ such that $g(x) = y$ and $g(A) = A$.

We have the following results.

**Proposition 8.** *For any constant $s$ with $0 < s < 1$, there exists some 1-dimensional CAT(0) space $X$ such that*

(*1*) $\dim_H(\partial X) = s$.

(*2*) *$\partial X$ is a self-similar set.*

(*3*) *$\partial X$ is homogeneous in $X$.*

(*4*) *$L(X) < \infty$, where $L$ denote the length of $X$.*

**Proposition 9.** *For $s = 1$, there exists some 1-dimensional CAT(0) space $X$ such that*

(*1*) *$dim_H(\partial X) = 1$.*

(*2*) *$\partial X$ is a self-similar set.*

(*2*) *$\partial X$ is homogeneous in $X$.*

**Corollary 10.** *For any given natural number $n$, there exists some $n$-dimensional $CAT(0)$ space $Y$ such that*

(*1*) *$dim_H \partial Y = n$, and*

(*2*) *$\partial Y$ is homogeneous in $Y$.*

## 0.1 Organization of the paper

The organization of this dissertation is as follows.

This dissertation consists of six chapters.

In chapter 1, we give several basic definitions on metric spaces, self-similar sets, Hausdorff and box dimensions, self-similar measures and CAT(0)-spaces. We also recall some results in Riemannian geometry.

In chapter 2, using the properties of doubling metric spaces, we prove Theorem 3.

In chapter 3, using the properties of the generalized open set condition and Borel probability measures, we give the proof of Theorem 5.

In chapter 4, we give several examples of asymptotic self-similar sets on the curved spaces by using asymptotic similarity maps, and determine their Hausdorff dimensions.

In chapter 5, we give self-simialr sets by using similarity maps in some tree.

# Chapter 1

# Preliminaries

In this chapter, we give several basic definitions and results on metric spaces and complete Riemannian manifolds. We will mainly present the definitions of, Hausdorff and box dimensions, CAT(0)-spaces, Rauch comparison theorem, Bishop-Gromov comparison theorem and self-similar sets.

## 1.1 Definitions and Notations

In this section, we give some definitions and notations.

Let $X$ be a metric space, $x, y \in X$, $A, B \subset X$. The distance between $x$ and $y$ is denoted by $|x, y|$. We denote the diameter of $A$ by $|A| = \sup\{\ |x, y| \mid x,\ y \in A\ \}$, and the distance between $x$ and $A$ by $|x, A| = \inf\{\ |x, y| \mid y \in A\ \}$, and the distance between $A$ and $B$ by $|A, B| = \inf\{\ |x, y| \mid x \in A,\ y \in B\ \}$. The interior and the closure of $A$ in $X$ is denoted by $A^0$ and $\bar{A}$, respectively. For each $r > 0$, the closed metric ball with radius $r$ and center $x$ is denoted by $B(x, r) = \{\ y \in X \mid |y, x| \leq r\ \}$, and the $r$-neighborhood of $A$ by $U_r(A) = \{\ y \in X \mid |y, A| < r\ \}$. *The Hausdorff distance* between $A$ and $B$, denoted by $d_H(A, B)$, is defined as

$$d_H(A, B) = \inf\{\ r > 0 \mid A \subset U_r(B) \ \mathit{and}\ B \subset U_r(A)\ \}.$$

An $\epsilon$*-cover* $\{U_i\}$ of $A$ is a finite or countable collection of sets $U_i$ covering $A$ with $|U_i| \leq \epsilon$. Let $\alpha$ be a nonnegative real number . The $\alpha$*-dimensional Hausdorff measure* of $A$, denoted by $\mathcal{H}^\alpha(A)$, is defined by the formula

$$\mathcal{H}^\alpha(A) := \lim_{\epsilon \to 0} \inf \Big\{ \sum_{i=1}^{\infty} |U_i|^\alpha \ \Big|\ \{U_i\} :\ \epsilon\text{-cover of } A \Big\},$$

and *The Hausdorff dimension* of $A$, denoted by $\dim_H A$, is defined as

$$\dim_H A := \sup\{\alpha \geq 0 | \mathcal{H}^\alpha(A) = \infty\} = \inf\{\alpha \geq 0 | \mathcal{H}^\alpha(A) = 0\}.$$

Let $N_\epsilon(A)$ denote the minimal number of subsets of diameter $\leq \epsilon$ needed to cover $A$. The *lower box dimension* and the *upper box dimension* of $A$ are defined respectively as

$$\underline{\dim}_B A = \underline{\lim}_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon},$$
$$\overline{\dim}_B A = \overline{\lim}_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}.$$

When both the lower and the upper box dimensions are equal, the common value

$$\dim_B A = \lim_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}$$

is called the *box dimension* of $A$.

Also, the lower and the upper box dimensions can be defined as follows: Let $D{=}D(A, \epsilon)$ be the collection of all countable open covers $\mathcal{U}$ of $A$ such that $|U| = \epsilon$ for every $U \in \mathcal{U}$. We define $\underline{r}(A, \alpha)$ and $\overline{r}(A, \alpha)$ respectively by

$$\underline{r}(A, \alpha) = \underline{\lim}_{\epsilon \to 0} \inf_D \sum_i |U|^\alpha,$$
$$\overline{r}(A, \alpha) = \overline{\lim}_{\epsilon \to 0} \inf_D \sum_i |U|^\alpha,$$

then the *lower box dimension* and the *upper box dimension* of $A$ are defined respectively as

$$\begin{aligned}
\underline{\dim}_B A &= \inf\{\alpha > 0 \mid \underline{r}(A, \alpha) = 0\} \\
&= \sup\{\alpha > 0 \mid \underline{r}(A, \alpha) = \infty\}, \\
\overline{\dim}_B A &= \inf\{\alpha > 0 \mid \overline{r}(A, \alpha) = 0\} \\
&= \sup\{\alpha > 0 \mid \overline{r}(A, \alpha) = \infty\}.
\end{aligned}$$

The following is an immediate consequence.

**Lemma 1.1.1.** (cf. [41])

$$\dim_H A \leq \underline{\dim}_B A \leq \overline{\dim}_B A.$$

*Proof.* By the definitions of the Hausdorff dimension and the Box dimension, we have

$$\mathcal{H}^\alpha(A) \leq \underline{r}(A, \alpha) \leq \overline{r}(A, \alpha),$$

from which the conclusion follows immediately. □

For a metric space $X$ and $0 < \lambda < 1$, a map $f : X \to X$ is called a $\lambda$-*contracting similarity map* if $|f(x), f(y)| \leq \lambda |x, y|$ holds for every $x,\ y \in X$.

Let $\mathcal{M}(X)$ be the set of all Borel probability measures on $X$. Define the metric on $\mathcal{M}(X)$ by

$$d_{\mathcal{M}}(\mu_1, \mu_2) = \sup_\phi \left\{ \left| \int_X \phi \, d\mu_1 - \int_X \phi \, d\mu_2 \right| \right\},$$

where $\phi : X \to \mathbb{R}$ runs over all Lipschitz function with Lipschitz constant $L(\phi) \leq 1$. By Riesz's representation formula, we have the following.

**Lemma 1.1.2.** (cf. [25]) *$\mathcal{M}(X)$ is a complete metric space.*

In the proof of the lemma 1.1.4, the following lemma will be used.

**Lemma 1.1.3.** (cf. [4]) *Let $X$ be a compact metric space, and let $A$, $B$, $C$, and $D$ be subsets in $X$. Then*

$$d_H(A \cup B,\ C \cup D) \leq \max\big\{d_H(A, C), d_H(B, D)\big\}.$$

**Lemma 1.1.4.** (cf. [25]) *Let $\{f_i\}_{i=1}^m$ be a family of contracting similarity maps in a complete metric space $X$:*

$$\frac{|f_i(x), f_i(y)|}{|x, y|} \leq \lambda_i < 1,$$

*for every $1 \leq i \leq k$. Then*

*(1) there exists a compact subset $K$ of $X$ such that $K = f_1(K) \cup \cdots \cup f_m(K)$;*

*(2) for any positive numbers $q_i$, $i = 1, \cdots, m$, with $\sum_{i=1}^m q_i = 1$, there exists a unique Borel probability measure $\mu_0$ with support $K$ such that*

$$\mu_0(A) = q_1\mu_0(f_1^{-1}(A)) + \cdots + q_m\mu_0(f_m^{-1}(A))$$

*for every measurable subset $A \subset X$. In other words,*

$$\mu_0 = \sum_{i=1}^m q_i (f_i)_*(\mu_0),$$

*where $(f_i)_*(\mu_0)$ is the push-forward measure of $\mu_0$ by $f_i$.*

The measure $\mu_0$ is called a *self-similar measure.*

*Proof of Lemma 1.1.4.* (1). Let $C(X)$ denote the set of nonempty compact subsets of $X$ equipped with the Hausdorff distance $d_H$. Then we see that the $C(X)$ is a complete metric space.

For $A \in C(X)$, let $f(A)$ denote $f(A) = \{f(x) : x \in A\}$. If $f$ is a $\lambda$-contracting similarity map on $X$, then $f$ is also an $\lambda$-contracting similarity map on $C(X)$.

For $\lambda_i$-contracting similarity map $f_i$, $i = 1, \cdots, m$, we define the map $F : C(X) \to C(X)$ by

$$F(A) = f_1(A) \cup \cdots \cup f_m(A).$$

Since

$$d_H(A \cup B,\ C \cup D) \leq \max\big\{d_H(A, C), d_H(B, D)\big\},$$

the map $F$ is a $\max\{\lambda_1, \cdots, \lambda_m\}$-contracting map. Since $A$, $F(A)$, $F^2(A)$, $\cdots$ is a Cauchy sequence in $C(X)$, for any $A \in C(X)$, and it converges to a set $K \in C(X)$ with $F(K) = K$, that is

$$K = F(K) = f_1(K) \cup \cdots \cup f_m(K).$$

Such a set $K$ is unique because

$$d_H(F(A), F(B)) \leq \max\{r_1, \cdots, r_m\} d_H(A, B),$$

for any $A,\ B \in C(X)$.

(2). Let $M(X)$ and $M(K)$ denote the sets of Borel probability measures with supports on $X$ and $K$, respectively. Then $M(X)$ is a complete metric space.

Define $F_*(p_1, \cdots, p_m) : M(X) \to M(X)$ by

$$(F_*(p_1, \cdots, p_m)\mu)(A) = p_1\mu\big(f_1^{-1}(A)\big) + \cdots + p_m\mu\big(f_m^{-1}(A)\big),$$

for any $\mu \in M(X)$.

Then $F_*(p_1, \cdots, p_m)$ is a contracting map. Also, by the Riesz representation theorem, we see that $M(K)$ is a complete metric space, and for any $\mu \in M(K)$, we have $F_*(p_1, \cdots, p_m)\mu \in M(K)$, and $F_*(p_1, \cdots, p_m) : M(K) \to M(K)$ is a contracting map. Therefore there exists a unique $\mu_0 \in M(K)$, such that $F_*(p_1, \cdots, p_m)\mu_0 = \mu_0$. $\square$

To prove Theorems 3 and 5, we need the following.

**Lemma 1.1.5.** (cf. [7]) *Let $X$ be a doubling metric space with doubling constant $C$. For any $0 < \delta < 1$, there exists a constant $C_0 = C_0(C, \delta)$ such that the number of mutually disjoint balls $B(x_i, \delta r)$ in a ball $B(x, r)$ of $X$ is bounded by $C_0(C, \delta)$.*

## 1.2 CAT(0)-spaces

In this section, we give some definition in the geodesic metric spaces. We will mainly review the definitions of, *geodesic, geodesic metric spaces, CAT(0)-spaces.* In this section, we mainly refer to [4], [3].

Let $X$ be a metric space, $x, y \in X$. Let $\sigma : [0, l] \subset \mathbb{R} \to X$ be a map which satisfies $\sigma(0) = x$, $\sigma(l) = y$. $\sigma$ is called a *geodesic path* joining $x$ to $y$ if $|\sigma(t), \sigma(s)| = |t - s|$ for every $t, s \in [0, l]$. Then we say the image $\gamma$ of $\sigma$ a *geodesic segment* with endpoint $x$ and $y$. A geodesic segment joining $x$ and $y$ is denoted by $[x, y]$. Three points $x, y, z \in X$, and three geodesic segments $[x, y], [y.z], [z, x]$ joining them is called a *geodesic triangle*, which is denoted by $\Delta([x, y], [y, z], [z, x])$ or $\Delta(x, y, z)$. For a point $q \in X$, $q \in \Delta$ means that $q$ lies in the union of $[x, y], [y, z]$ and $[z, x]$.

A triangle $\tilde{\Delta} = \Delta(\tilde{x}, \tilde{y}, \tilde{z})$ of an Euclidean space $\mathbb{R}^n$ is a *comparison triangle* of $\Delta(x, y, z)$ if $|\tilde{x}, \tilde{y}| = |x, y|$, $|\tilde{y}, \tilde{z}| = |y, z|$, $|\tilde{z}, \tilde{x}| = |z, x|$. A point $\tilde{q} \in [\tilde{x}, \tilde{y}]$ is a *comparison point* of $q \in [x, y]$ if $|x, q| = |\tilde{x}, \tilde{q}|$.

A map $\sigma : [0, \infty) \to X$ is called a *geodesic ray* if $|\sigma(t), \sigma(s)| = |t - s|$ for every $t, s \in [0, \infty)$. A map $\sigma : \mathbb{R} \to X$ is called a *geodesic line* if $|\sigma(t), \sigma(s)| = |t - s|$ for every $t, s \in \mathbb{R}$. A metric space $X$ is called *geodesic metric space* if any two points in $X$ are joined by a geodesic.

A subset $A$ of $X$ is *convex* if any $x, y \in A$ can be joined by a unique geodesic $\gamma$ of $X$ such that the image of $\gamma$ is included in $A$.

Let $X$ be a geodesic metric space, $\Delta \subset X$ a geodesic triangle, and $\tilde{\Delta}$ a comparison triangle of $\Delta$ in $\mathbb{R}^2$. For any $x \in \Delta$, $\tilde{x}$ denotes a comparison point on $\tilde{\Delta}$

**Definition 1.2.1.** (cf. [3]) We say that $X$ is a *CAT(0)-space* if any $\Delta$ satisfies

$$|x, y| \leq |\tilde{x}, \tilde{y}|.$$

for any $x, y \in \Delta$ and comparison points $\tilde{x}, \tilde{y} \in \tilde{\Delta}$ respectively.

For example, in geodesic metric spaces, a tree is a CAT(0)-space. Indeed, any comparison triangle of trees is degenerate.

The following is a basic property of CAT(0)-spaces. This follows from the definitions immediately. This will be used in Chapter 5.

**Proposition 1.2.2.** (cf. [3]). *The product of CAT(0)-spaces is a CAT(0)-space.*

## 1.3 Rauch Comparison Theorem

The Rauch comparison theorem is very important tool when we determine the Hausdorff dimension of a generalized Sierpinski gasket constructed on surfaces. In this section, we mainly refer to [9]

Let $M$ be a smooth finite-dimensional manifold, $p \in M$. $T_pM$ denote the tangent space of $M$ at $p$, and $TM$ denote the tangent bundle of $M$. For $\upsilon, \nu \in T_pM$, by $\langle \upsilon, \nu \rangle$ we donote the riemannian scalar product between $\upsilon$ and $\nu$.

Let $\gamma_\upsilon : [0, b] \to M$ be a geodesic of $M$ satisfying $\gamma_\upsilon(0) = p$, $\dot{\gamma}_\upsilon(0) = \upsilon$, and $1 \in [0, b]$.

The *exponential map* $\exp_p : T_pM \to M$ is defined by

$$\exp_p(\upsilon) = \gamma_\upsilon(1)$$

for any $\upsilon \in T_pM$.

The following Gauss lemma is also very useful when we determine the Hausdorff dimension of the generalized Sierpinski gasket on a surface, and will be used in Chapter 4.

**Lemma 1.3.1.** (cf. [9]) *Given a point $p \in M$. Suppose $\upsilon, \nu \in T_pM$. Then we have.*

$$d(\exp_p)_\upsilon \upsilon = \gamma'(1), \qquad \langle \gamma'_\upsilon(1), d(\exp_p)_\upsilon \nu \rangle = \langle \upsilon, \nu \rangle.$$

The following theorem is very important for considering asymptotic self-similar sets on curved spaces, and will be used in Chapter 4.

**Theorem 1.3.2.** (cf. [9]) *Let $M$, $N$ be riemannian manifolds such that* $\dim N \geq \dim M$, *and let $\gamma_1$, $\gamma_2 : [0, l] \to M, N$ be normal geodesics, and put $\gamma_1' = T_1$, $\gamma_2' = T_2$. Given $t \in [0, l]$, and two tangent vectors $X_1, X_2$ such that $X_1 \in M_{\gamma_1(t)}$, $X_2 \in N_{\gamma_2(t)}$. Suppose that the sectional curvatures $\sigma_1, \sigma_2$ spanned by $T_1, X_1$ and $T_2, X_2$ satisfy $K(\sigma_2) \geq K(\sigma_1)$.*

*Assume further that for no $t \in [0,l]$ is $\gamma_2(t)$ conjugate to $\gamma_2(0)$ along $\gamma_1$. Let $V_1, V_2$ be Jacobi fields along $\gamma_1, \gamma_2$ such that $V_1(0), V_2(0)$ are tangent to $\gamma_1,\ \gamma_2$ and*

$$\| V_1(0) \| = \| V_2(0) \|,\ \langle T_1, V_1'(0)\rangle = \langle T_2, V_2'(0)\rangle,\ \| V_1'(0) \| = \| V_2'(0) \| .$$

*Then we have*

$$\| V_1(t) \| \geq \| V_2(t) \|$$

*for every $t \in [0,l]$.*

## 1.4 Bishop-Gromov Comparison Theorem

The Bishop-Gromov Comparison Theorem is very important tool when we consider constructions of asymptotic self-similar sets on Riemannian manifolds, and will be used in Chapter 4. In this section, we mainly refer to [4].

Fix a constant $\kappa \in \mathbb{R}$. For an integer $n \geq 2$, we denote by $M_\kappa^n$ the $n$-dimensional space form of curvature $\kappa$, where a *space form* is a simply connected complete space whose curvature is constant $\kappa$. Spheres, Euclidean spaces and hyperbolic spaces are space forms. For a fixed positive integer $n$, $V_\kappa^n(r)$ denotes the volume of a $r$-ball in $M_\kappa^n$.

Let $X$ be a metric space, $x, y \in X$. For any $\epsilon > 0$, $x, y$ are *$\epsilon$-close* if $|x,y| \leq \epsilon$. $X$ is called a *intrinsic metric space* if for any $\epsilon > 0$, $x, y \in X$, there is a finite sequence $\{x_1, x_2, \cdots, x_k\}$ such that every two neighboring points in this sequence are $\epsilon$-close, and $\sum_{i=1}^{k-1} |x_i, x_{i+1}| < |x_1, x_k| + \epsilon$.

Let $X$ be a metric space. For $x, y, z \in X$, we denote by $\tilde{\angle} yxz$ the angle at the vertex $\tilde{x}$ of the comparison triangle $\tilde{\triangle} yxz$ in $M_\kappa^n$ of a triangle $\triangle yxz$ in $X$, where we set $\tilde{\triangle} yxz := \triangle \tilde{y}\tilde{x}\tilde{z}$.

**Definition 1.4.1.** We say that $X$ is a *Alexandrov space of curvature* $\geq \kappa$ if $X$ is a locally complete intrinsic metric space such that for any point $x \in X$, there is a neighborhood $U_x$ such that

$$\tilde{\angle} bac + \tilde{\angle} cad + \tilde{\angle} dab \leq 2\pi$$

for any point $a \in U_x$ and any $b, c, d \in U_x - \{a\}$.

The following theorem will be used in the proof of Example 4.1.2 in Chapter 4.

**Theorem 1.4.2.** (cf. [4]) *If $X$ is an $n$-dimensional complete Alexandrov space of curvature $\geq \kappa$, then we have*

$$\frac{\mu_n\big(B_R(p)\big)}{V_\kappa^n(R)} \leq \frac{\mu_n\big(B_r(p)\big)}{V_\kappa^n(r)}$$

*for any $p \in X$ and $R \geq r > 0$.*

## 1.5 Self-similar sets

In this section, we give a definition of a self-similar set. We mainly refer to [21],[7]. Let $X$ be a complete metric space.

**Definition 1.5.1.** We say that the compact set $K \subset X$ is a *self-similar set* if there is a finite family $\mathcal{F} = \{f_1, \cdots, f_k\}$ of contracting similarity maps on $X$ such that

$$K = \bigcup_{i=1}^{k} f_i(K).$$

The following is called the *open set condition* introduced by Hutchinson.

Let $\mathcal{F} = \{f_1, \cdots, f_k\}$ is a family of contracting similarity maps. We say that $\mathcal{F}$ satisfies the *open set condition* if there is a non-empty open set $A \subset X$ such that

(1) $A \supset f_1(A) \cup \cdots \cup f_k(A)$;

(2) $f_i(A) \cap f_j(A) = \emptyset$ for every $i \neq j$.

Hutchinson proved the following theorem for self-similar sets satisfying the open set conditions in Euclidean spaces. Balogh and Roner proved this theorem for self-similar sets satisfying the open set conditions in the doubling metric spaces.

**Theorem 1.5.2.** (cf. [7]) *Let $X$ be a metric space and let $K$ be the self-similar set with respect to $(\{\lambda_i\}_{i=1}^k)$- contracting similarity maps $f_i$. Then the Hausdorff dimension of the self-similar set $K$ is given as*

$$\dim_H K = s,$$

*where $s$ is a unique number satisfying* $\sum_{i=1}^{k} \lambda_i^s = 1$.

# Chapter 2

# Proof of Theorem 3

In this chapter, we prove Theorem 3. In the section 2.1, we show $\dim_H C \le t$. In the section 2.2, we show $\dim_H C \ge t$. We will use the properties of the doubling metric spaces in the proof of $\dim_H C \ge t$.

## 2.1 Proof of $\dim_H C \le t$

Let $n$ the *depth* of the basic set $\Delta_{\omega_1 \cdots \omega_n}$ of $C$.

**Lemma 2.1.1.** $\dim_H C \le t$.

*Proof.* Let $c$ be the constant in the definition of a $(\lambda, c, \nu)$-similarity map in Introduction. By the construction of $C$, we have

$$|\Delta_{\omega_1 \cdots \omega_n}| \le |\Delta_{\omega_1 \cdots \omega_{n-1}}| \nu.$$

Obviously there exists a number $n_0 (n_0 \gg 1)$ such that

$$c|\Delta_{\omega_1 \cdots \omega_{n_0}}| < 1,$$

For any $\varepsilon > 0$, let $n$ be sufficiently large $(n > n_0)$ such that

$$\mathcal{U} = \{ \ \Delta_{\omega_1 \cdots \omega_n} \mid 1 \le \omega_j \le k, 1 \le j \le n \ \}$$

is an $\varepsilon$-cover of $C$. By the definition of $(\lambda, c, \nu)$-similarity map $f : \Delta_{\omega_1 \cdots \omega_{n-1}} \to \Delta_{\omega_1 \cdots \omega_n}$, we have

$$|\Delta_{\omega_1 \cdots \omega_n}| \le \lambda_{\omega_n} (1 + c|\Delta_{\omega_1 \cdots \omega_{n-1}}|) |\Delta_{\omega_1 \cdots \omega_{n-1}}|.$$

Let $n = n_0 + m$, then

$$c|\Delta_{\omega_1 \cdots \omega_{n-1}}| \le c|\Delta_{\omega_1 \cdots \omega_{n_0}}| \nu^{m-1} \le \nu^{m-1}.$$

Thus we see

$$\begin{aligned}
m(C,t,\varepsilon) &\leq \sum_{(\omega_1,\cdots,\omega_n)} |\Delta_{\omega_1\cdots\omega_n}|^t \\
&= \sum_{(\omega_1,\cdots,\omega_{n-1})} (\ |\Delta_{\omega_1\cdots\omega_{n-1}1}|^t + \cdots + |\Delta_{\omega_1\cdots\omega_{n-1}k}|^t\ ) \\
&\leq \sum_{(\omega_1,\cdots,\omega_{n-1})} (1+c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t |\Delta_{\omega_1\cdots\omega_{n-1}}|^t (\lambda_1^t + \cdots + \lambda_k^t) \\
&\leq \sum_{(\omega_1,\cdots,\omega_{n-1})} (1+\nu^{m-1})^t |\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&= (1+\nu^{m-1})^t \sum_{(\omega_1,\cdots,\omega_{n-1})} |\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&\leq \cdots < (1+\nu^{m-1})^t \cdots (1+\nu)^t 2^t \sum_{\omega_1,\cdots,\omega_{n_0}} |\Delta_{\omega_1\cdots\omega_{n_0}}|^t.
\end{aligned}$$

Here when $m \to \infty$ the sequence $a_m = (1+\nu^{m-1})^t \cdots (1+\nu)^t 2^t$ converges. Hence $m(C,t) \leq K_0$ for some constant $K_0$, and therefore $\dim_H C \leq t$. □

## 2.2 Proof of $\dim_H C \geq t$

To prove $\dim_H C \geq t$, we first show Lemmas 2.2.1 and 2.2.5.

**Lemma 2.2.1.** *There exists a constant $K_0$, chosen independently of any cover, such that if $\mathcal{U}$=$\{U_i\}$ is any cover of $C$ such that each $U_i$ is a basic set, then*

$$\sum_i |U_i|^t \geq K_0 > 0.$$

Let $\mathcal{U}$ be a cover of $C$. $\mathcal{U}$ is called *minimal* if no proper subcollection of $\mathcal{U}$ covers $C$.

*Proof of Lemma 2.2.1.* Let $\mathcal{U}$=$\{U_i\}$ be any cover of $C$ by basic sets. Because $C$ is compact, it suffices to eatablish

$$\sum_i |U_i|^t \geq K > 0$$

for $\mathcal{U}$ is finite and minimal.

Let $n$ be the maximum of the depths of all basic sets in $\mathcal{U}$, and let $\Delta_{\omega_1\cdots\omega_n}$ be a basic set of maximal depth in $\mathcal{U}$. Since $\mathcal{U}$ is minimal, it does not contain the basic set $\Delta_{\omega_1\cdots\omega_{n-1}}$. It follows that each of the basic set $\Delta_{\omega_1\cdots\omega_{n-1}j}$ for $j=1,\cdots,k$ is contained in $\mathcal{U}$.

Thus the sum $\sum_i |U_i|^t$ contains the partial sum

$$|\Delta_{\omega_1\cdots\omega_{n-1}1}|^t + \cdots + |\Delta_{\omega_1\cdots\omega_{n-1}k}|^t.$$

By the definition of $(\lambda, c, \nu)$-similarity map and $t$, we see

$$\begin{aligned}
|\Delta_{\omega_1\cdots\omega_{n-1}1}|^t + \cdots + |\Delta_{\omega_1\cdots\omega_{n-1}k}|^t \\
&\geq \lambda_1^t(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t|\Delta_{\omega_1\cdots\omega_{n-1}}|^t + \\
&\cdots + \lambda_k^t(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t|\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&= (\lambda_1^t + \cdots + \lambda_k^t)(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t|\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&= (1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t|\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&\geq (1 - \nu^{m-1})^t|\Delta_{\omega_1\cdots\omega_{n-1}}|^t.
\end{aligned}$$

We replace $\{\Delta_{\omega_1\cdots\omega_{n-1}j}\}_{j=1}^k$ by $\Delta_{\omega_1\cdots\omega_{n-1}}$. In this way we replace all the basic sets in $\mathcal{U}$ of depth $n$ by the corresponding sets of depth $n-1$, to obtain a new covering $\mathcal{U}'$ by basic sets. We may assume that $\mathcal{U}'$ is minimal. Then we can repeat the previous argument, and obtain

$$\sum_i |U_i|^t \geq (1 - \nu^{m-1})^t \cdots (1 - \nu)^t (1 - c|\Delta_{\omega_1\cdots\omega_{n_0}}|)^t|\Delta_{\omega_1\cdots\omega_{n_0}}|^t.$$

But in the last expression, $a_m = (1 - \nu^{m-1})^t \cdots (1 - \nu)^t$ converges to a positive number and $(1 - c|\Delta_{\omega_1\cdots\omega_{n_0}}|)^t|\Delta_{\omega_1\cdots\omega_{n_0}}|^t$ is uniformly bounded from below. Therefore

$$\sum_i |U_i|^t \geq K_0 > 0$$

for a uniform positive number $K_0$. □

To show Lemma 2.2.5, we first show the following Lemmas 2.2.2, 2.2.3, and Lemma 2.2.4.

**Lemma 2.2.2.** *Let* $\lambda_{\min}$=min$\{\lambda_1, \cdots, \lambda_k\}$. *For each* $r > 0$, *set*

$$V(r) = \{\ \Delta_{\omega_1\cdots\omega_n} \mid r\lambda_{\min} \leq |\Delta_{\omega_1\cdots\omega_n}| \leq \frac{r}{\lambda_{\min}}\ \}, \tag{2.2.1}$$

*and given* $x \in X$, *define*

$$V_x(r) = \{\ V \in V(r) \mid x \in V\ \}.$$

*Let* $N$ *be the number of elements of* $V_x(r)$. *Then* $N \leq M$, *where* $M$ *is independent of* $x$ *and* $r$.

*Proof.* First we consider the case $x \in C$. We can write given $x \in C$ as

$$\{x\} = \bigcap_{n\geq 1} \Delta_{\omega_1\cdots\omega_n}.$$

For the infinite sequence $\omega_1,\ \omega_2,\ \cdots,\ \omega_n, \cdots$, define the set $E$ as

$$E = \{\ n \mid r\lambda_{\min} \leq |\Delta_{\omega_1\cdots\omega_n}| \leq \frac{r}{\lambda_{\min}}\ \}. \tag{2.2.2}$$

Then the number of elements of $E$ is equal to the number $N$ of elements of $V_x(r)$. Now let $n' = \min E,\ n'' = \max E$, and let $n'' = n' + m, n'' \geq n_0,\ n' \geq n_0$. Beacause

$$|\Delta_{\omega_1\cdots\omega_{n''}}| = |\Delta_{\omega_1\cdots\omega_{n'+m}}| \leq |\Delta_{\omega_1\cdots\omega_{n'}}|\nu^m,$$

by the definition of $n'$ , $n''$, we have

$$r\lambda_{\min} \leq |\Delta_{\omega_1\cdots\omega_{n''}}| \leq |\Delta_{\omega_1\cdots\omega_{n'}}|\nu^m \leq \frac{r}{\lambda_{\min}}\nu^m.$$

Therefore,

$$r\lambda_{\min} \leq \frac{r}{\lambda_{\min}}\nu^m.$$

Hence,

$$m \leq 2\frac{\log \lambda_{\min}}{\log \nu} = M.$$

Next, we consider the general case $x \in X$. For any $x{\in}X$, define $E$ as

$$E = \{\ \Delta_{\omega_1\cdots\omega_n}\ |\ x \in \Delta_{\omega_1\cdots\omega_n}\ \}.$$

If $n = 1$, there exits unique $\omega_1$ such that $x \in \Delta_{\omega_1}$; if $n = 2$, there exits unique $\omega_2$ such that $x \in \Delta_{\omega_1\omega_2}$; similarly there exits unique $\omega_n$ such that $x \in \Delta_{\omega_1\cdots\omega_n}$. If $E$ is an infinite set, then $x \in C$. Because there exits unique infinite sequence $\omega_1,\ \omega_2,\ \cdots,\ \omega_n, \cdots$ such that

$$x \in \Delta_{\omega_1\cdots\omega_n},$$

and

$$\{x\} = \bigcap_{n\geq 1} \Delta_{\omega_1\cdots\omega_n}$$

for any $n(n \geq 1)$. Therefore, $x{\in}C$.

If $E$ is a finite set, namely,

$$E = \{\ \Delta_{\omega_1}, \Delta_{\omega_1\omega_2}, \cdots, \Delta_{\omega_1\cdots\omega_n}\ \},$$

then we have

$$V_x(r) = \{\ \Delta_{\omega_1\cdots\omega_{n_0}}, \cdots, \Delta_{\omega_1\cdots\omega_{n_0+m}}\ \}$$

for suitable $n_0$ and $m$. Thus by an argument similar to Lemma 2.2.2, the number of elements of $V_x(r)$ is bounded above by a constant $M$ (which is independent of $x$ and $r$). $\square$

**Lemma 2.2.3.** *If* $b_{\omega_1\cdots\omega_n} = \max\{\ r\ |\ B(x,r) \subset \Delta_{\omega_1\cdots\omega_n}\ \}$, *then*

$$b_{\omega_1\cdots\omega_n} \geq \lambda_{\omega_n} b_{\omega_1\cdots\omega_n-1}(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)$$

.

*Proof.* Let $x$ be the center point of a largest ball included in $\Delta_{\omega_1\cdots\omega_{n-1}}$. By the definition of $(\lambda_{\omega_n}, c, \nu)$-similarity map $f : \Delta_{\omega_1\cdots\omega_{n-1}} \to \Delta_{\omega_1\cdots\omega_n}$, we have

$$B\big(f(x), \lambda_{\omega_n} b_{\omega_1\cdots\omega_{n-1}}(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)\big) \subset f\big(B(x, b_{\omega_1\cdots\omega_{n-1}})\big)$$

Thus we see

$$B\big(f(x), \lambda_{\omega_n} b_{\omega_1\cdots\omega_{n-1}}(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)\big) \subset \Delta_{\omega_1\cdots\omega_n}.$$

Therefore we obtain

$$b_{\omega_1\cdots\omega_n} \geq \lambda_{\omega_n} b_{\omega_1\cdots\omega_n-1}(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|).$$

□

**Lemma 2.2.4.** *If* $b_{\omega_1\cdots\omega_n} = \max\{\ r \mid B(x,r) \subset \Delta_{\omega_1\cdots\omega_n}\ \}$, *then there exists a constant* $k_0$ *such that*

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \leq k_0, \tag{2.2.3}$$

*for any* $n$ *and any* $\omega_1, \omega_2, \cdots, \omega_n$.

*Proof.* By the definition of $(\lambda_{\omega_n}, c, \nu)$-similarity map $f : \Delta_{\omega_1\cdots\omega_{n-1}} \to \Delta_{\omega_1\cdots\omega_n}$, we have

$$|\Delta_{\omega_1\cdots\omega_n}| \leq \lambda_{\omega_n}(1 + c|\Delta_{\omega_1\cdots\omega_{n-1}}|)|\Delta_{\omega_1\cdots\omega_{n-1}}|\ ,$$

Therefore we obtain

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \leq \frac{(1 + c|\Delta_{\omega_1\cdots\omega_{n-1}}|)|\Delta_{\omega_1\cdots\omega_{n-1}}|}{(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)b_{\omega_1\cdots\omega_{n-1}}}.$$

There exists $n_0$ such that for any $n \geq n_0$

$$\frac{1 + c|\Delta_{\omega_1\cdots\omega_{n-1}}|}{1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|} \leq 1 + 3c|\Delta_{\omega_1\cdots\omega_{n-1}}|.$$

Thus we have

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \leq \frac{|\Delta_{\omega_1\cdots\omega_{n-1}}|}{b_{\omega_1\cdots\omega_{n-1}}}(1 + 3c|\Delta_{\omega_1\cdots\omega_{n-1}}|).$$

By the construction of $C$, we have

$$|\Delta_{\omega_1\cdots\omega_n}| \leq |\Delta_{\omega_1\cdots\omega_{n-1}}|\nu.$$

Hence, there exists $n_1 \geq n_0$ such that

$$3c|\Delta_{\omega_1\cdots\omega_{n_1}}| < 1.$$

Now let $n = n_1 + m$, then we get

$$|\Delta_{\omega_1\cdots\omega_n}| \leq |\Delta_{\omega_1\cdots\omega_{n_1}}|\nu^m.$$

Therefore we obtain

$$3c|\Delta_{\omega_1\cdots\omega_{n-1}}| \le 3c|\Delta_{\omega_1\cdots\omega_{n_1}}|\nu^{m-1} \le \nu^{m-1},$$

and hence

$$1 + 3c|\Delta_{\omega_1\cdots\omega_{n-1}}| \le 1 + \nu^{m-1}.$$

Thus we have

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \le \frac{|\Delta_{\omega_1\cdots\omega_{n-1}}|}{b_{\omega_1\cdots\omega_{n-1}}}(1+\nu^{m-1}).$$

Therefore we obtain

$$\begin{aligned}\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} &\le \frac{|\Delta_{\omega_1\cdots\omega_{n-1}}|}{b_{\omega_1\cdots\omega_{n-1}}}(1+\nu^{m-1})\\ &\le \frac{|\Delta_{\omega_1\cdots\omega_{n-2}}|}{b_{\omega_1\cdots\omega_{n-2}}}(1+\nu^{m-2})(1+\nu^{m-1})\\ &\le \cdots \le \frac{|\Delta_{\omega_1\cdots\omega_{n_1}}|}{b_{\omega_1\cdots\omega_{n_1}}}2(1+\nu)\cdots(1+\nu^{m-2})(1+\nu^{m-1}).\end{aligned}$$

Here when $m \to \infty$ the sequence $a_m = 2(1+\nu)\cdots(1+\nu^{m-1})$ converges. Thus there exists a constant $k_1$ such that

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \le k_1,$$

for any $n \ge n_1$. Let

$$k_2 = \max\left\{\frac{|\Delta_{\omega_1}|}{b_{\omega_1}}, \frac{|\Delta_{\omega_1\omega_2}|}{b_{\omega_1\omega_2}}, \cdots, \frac{|\Delta_{\omega_1\cdots\omega_{n_1}}|}{b_{\omega_1\cdots\omega_{n_1}}}\right\},$$

and let $k_0 = \max\{k_1,\ k_2\}$. Then we have

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \le k_0, \tag{2.2.4}$$

for any $n$ and any $\omega_1, \omega_2, \cdots, \omega_n$. $\square$

Next, we shall prove the follwing lemma.

**Lemma 2.2.5.** *Let $U$ be a bounded subset of $X$, and write $r = |U|$. Then $U$ intersects at most $M' = C(\delta)M$ elements of $V(r)$, where $M$ is the constant given in Lemma* 2.2.2 *and* $\delta = \frac{\lambda_{\min}^2}{2k_0 + 2k_0\lambda_{\min} + \lambda_{\min}^2}$.

*Proof.* Fix an arbitrary point $x_0 \in U$, and consider the ball

$$B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r\right) \subset X.$$

Then we have

$$U \subset B\left(x_0, \Big(1 + \frac{1}{\lambda_{\min}}\Big) r\right),$$

and choose maximal points $\{x_i\}_{i=1}^N \subset B\left(x_0, \Big(1 + \frac{1}{\lambda_{\min}}\Big) r\right)$ such that

$$d(x_i, x_j) \geq \frac{r\lambda_{\min}}{k_0}$$

for any $i \neq j$, where $k_0$ is a constant defined in Lemma 2.2.4.

Next, we show

**Sublemma 2.2.6.**

$$N \leq C_0(C, \delta),$$

*where* $\delta = \frac{\lambda_{\min}^2}{2k_0 + 2k_0\lambda_{\min} + \lambda_{\min}^2}$*, and* $C_0(C, \delta)$ *is the constant given Lemma 1.1.5.*

*Proof.* We consider the ball $B\Big(x_i, \frac{r\lambda_{\min}}{2k_0}\Big)$, and the ball $B\left(x_0, \Big(1 + \frac{1}{\lambda_{\min}}\Big) r + \frac{r\lambda_{\min}}{2k_0}\right)$. Then we have

$$\bigcup_{i=1}^N B\Big(x_i, \frac{r\lambda_{\min}}{2k_0}\Big) \subset B\left(x_0, \Big(1 + \frac{1}{\lambda_{\min}}\Big) r + \frac{r\lambda_{\min}}{2k_0}\right).$$

Since

$$B\Big(x_i, \frac{r\lambda_{\min}}{2k_0}\Big) \bigcap B\Big(x_j, \frac{r\lambda_{\min}}{2k_0}\Big) = \emptyset \quad (i \neq j).$$

Thus, by lemma 1.1.5 we have

$$N \leq C_0(C, \delta),$$

where $\delta = \frac{\lambda_{\min}^2}{2k_0 + 2k_0\lambda_{\min} + \lambda_{\min}^2}$.

Next we are going to show

**Sublemma 2.2.7.** *If* $V \in V(r)$ *intersects* $U$*, then it must contain one of* $\{x_i\}$.

*Proof.* We take a point $y$ such that

$$y \in V \bigcap U.$$

Let $x$ be the center point of a largest ball included in $V$. Then we have

$$\begin{aligned} d(x, x_0) &\leq d(x, y) + d(y, x_0) \\ &\leq |V| + |U| \leq \Big(\frac{1}{\lambda_{\min}} + 1\Big) r. \end{aligned}$$

Therefore, we obtain

$$x \in B\left(x_0, \Big(1 + \frac{1}{\lambda_{\min}}\Big) r\right).$$

Furthermore, we have

$$B\left(x_0, \Big(1 + \frac{1}{\lambda_{\min}}\Big) r\right) \subset \bigcup_{i=1}^{N} B\Big(x_i, \frac{r\lambda_{\min}}{k_0}\Big).$$

Thus there exists a point $x_i$ $(i = 1, 2, \cdots, N)$ such that

$$x \in B\left(x_i, \frac{r\lambda_{\min}}{k_0}\right).$$

Hence we see

$$x_i \in B\Big(x, \frac{r\lambda_{\min}}{k_0}\Big).$$

By Lemma 2.2.4, we have

$$\frac{|V|}{b(V)} \leq k_0.$$

Therefore, we obtain

$$b(V) \geq \frac{|V|}{k_0} \geq \frac{r\lambda_{\min}}{k_0}.$$

Then we see

$$V \supset B\Big(x, b(V)\Big) \supset B\Big(x, \frac{r\lambda_{\min}}{k_0}\Big).$$

Hence, $x_i \in V$. Because each of $\{x_i\}$ is contained in at most $M$ such sets $V$, it follows that the total number of elements $V$ of $V(r)$ which intersect $U$ is bounded above by $M' = C_0(C, \delta)M$. □

Now, we can show the following.

**Lemma 2.2.8.** $\dim_H C \geq t$

*Proof.* Let $\mathcal{U} = \{U_i\}$ be any $\varepsilon$-cover of $C$. For each $U_i$, write $r_i = |U_i|$, and let $U_{i,1}, \cdots, U_{i,m(i)}$ be the basic sets in $V(r_i)$ which intersect $U_i$. By Lemma 2.2.5, we have

$$m(i) \leq M'.$$

Furthermore, from 2.2.1, we see

$$|U_{i,j}| \leq \frac{|U_i|}{\lambda_{\min}},$$

and

$$\sum_{j=1}^{m(i)} |U_{i,j}|^t \leq m(i) \frac{|U_i|^t}{{\lambda_{\min}}^t} \leq \frac{M'}{{\lambda_{\min}}^t} |U_i|^t.$$

Then we have

$$|U_i|^t \geq \frac{{\lambda_{\min}}^t}{M'} \sum_{j=1}^{m(i)} |U_{i,j}|^t.$$

Summing over all the elements of $\mathcal{U}$ yields

$$\sum_i |U_i|^t \geq \frac{\lambda_{\min}}{M'}^t \sum_i \sum_{j=1}^{m(i)} |U_{i,j}|^t.$$

Since $\{U_{i,j}\}$ is a cover of $C$ by basic sets, we may apply Lemma 2.2.1 to obtain

$$\sum_i |U_i|^t \geq \frac{\lambda_{\min}}{M'}^t K_0 > 0,$$

where $K_0$ is the constant in Lemma 2.2.1. Hence we obtain $\dim_H C \geq t$ $\square$

This completes the proof of Theorem 3.

# Chapter 3

# Proof of Theorem 5

In this chapter, we give the proof of Theorem 5. We will use properties of some Borel probability measure which is determined by our generalized open set condition, to prove Theorem 5. We will show that $\dim_H K \leq s$, $\dim_H K \geq s$, and $\overline{\dim}_B K \leq s$ in this chapter.

## 3.1 Preliminaries

**Definition 3.1.1.** We call a set $\mathcal{S}$ consisting of $I \in \mathcal{I}$ a *simple family* if the following conditions are satisfied:

(1) $\displaystyle\bigcup_{I \in \mathcal{S}} \bar{W}_I \supset K$;

(2) If $I = i_1 \cdots i_{m-1} i_m \in \mathcal{S}$, then both $I_0 = i_1 \cdots i_{m-1}$ and $I_1 = i_1 \cdots i_{m-1} i_m i$ do not belong to $\mathcal{S}$ for all $1 \leq i \leq k$.

**Lemma 3.1.2.** *Let $\varphi : (0, \infty) \to [0, \infty)$ be a continuous function satisfying the conditions* (1), (2) *in Introduction. Then*

$$\prod_{i=0}^{\infty} (1 + \varphi(\nu^i |V|) < \infty, \quad \prod_{i=0}^{\infty} (1 - \varphi(\nu^i |V|) > 0.$$

*Proof.* By the condition on $\varphi$, we have

$$\sum_{i=0}^{\infty} \log(1 + \varphi(\nu^i |V|)) \leq \sum_{i=0}^{\infty} \varphi(\nu^i |V|) < \infty.$$

Similarly we have

$$\sum_{i=0}^{\infty} \log(1 - \varphi(\nu^i |V|)) \geq -2 \sum_{i=0}^{\infty} \varphi(\nu^i |V|) > -\infty.$$

These complete the proof. □

**Lemma 3.1.3.** *Let $X$ be as in Theorem 5, and let $\mathcal{V} = \{V_i\}$ be a collection of disjoint open sets of $X$ such that each $V_i$ contains a closed ball of radius $c_1\rho$ and is included in a closed ball of radius $c_2\rho$ for some positive constants $c_1 < c_2$ and $\rho$. Then every closed $\rho$-ball $B(x,\rho)$in $X$ intersects at most $C(\delta)$ elements of $\bar{\mathcal{V}} = \{\bar{V}_i\}$, where $\delta = \frac{c_1}{c_1+4c_2+2}$ and $C(\delta)$ is a constant given in Lemma 1.1.5*

*Proof.* Take $x_1^i, x_2^i \in X$ satisfying $B(x_1^i, c_1\rho) \subset V_i \subset B(x_2^i, c_2\rho)$. Let $\bar{V}_1, \cdots, \bar{V}_N$ intersect $B(x,\rho)$.
Taking any point $z \in \bar{V}_i \cap B(x,\rho)$, we have

$$|x_1^i, x| \le |x_1^i, z| + |z, x| \le (2c_2+1)\rho.$$

Furthermore, for any $y \in B(x_1^i, c_1\rho)$, we have

$$|y, x| \le |y, x_1^i| + |x_1^i, x| \le (c_1 + 2c_2 + 1)\rho.$$

Thus we get

$$\bigcup_{i=1}^{N} B(x_1^i, c_1\rho) \subset B\big(x, (c_1+2c_2+1)\rho\big).$$

Since $B(x_1^i, c_1\rho)$ are mutually disjoint, from Lemma 1.1.5 we obtain the conclusion of the lemma. This completes the proof. □

We will use the following setting in the proof of Theorem 5.
For each $I = i_1 \cdots i_n \in \mathcal{I}$, we set

$$g_I := f_I \circ \cdots \circ f_{i_1 i_2} \circ f_{i_1} : \bar{V} \to \bar{V},$$

and $\bar{V}_I := g_I(\bar{V}) \subset \bar{V}$. Note that

$$|V_I| \le \nu^{|I|}|V|.$$

Let $s$ be a unique solution of $\displaystyle\sum_{i=1}^{k} \lambda_i^s = 1$

## 3.2 Proof of $\dim_H K \le s$

In this section, we show $\dim_H K \le s$.

**Lemma 3.2.1.** $\dim_H K \le s$

*Proof.* By the construction of $K$, we have $|V_{i_1\cdots i_n}| \le |V_{i_1\cdots i_{n-1}}|\nu$. For any $\epsilon > 0$ take a sufficiently large $n$ such that $\mathcal{U}_n := \{\ V_I \mid I \in \mathcal{I}_n\}$ is an $\epsilon$-cover of $K$. From the definition of $(\lambda_{i_n}, \varphi, \nu)$-almost similarity map $f_I : V_{I'} \to V_I$, $I = I'i_n$, we have

$$|V_I| \le \lambda_{i_n}(1 + \varphi(|V_{I'}|)|V_{I'}|.$$

It follows from Lemma 3.1.2 that

$$\begin{aligned}
\mathcal{H}^s_\epsilon(K) &\leq \sum_{I\in\mathcal{I}_n} |V_I|^s \\
&= \sum_{I'\in\mathcal{I}_{n-1}} (\ |V_{I'1}|^s + \cdots + |V_{I'k}|^s\ ) \\
&\leq \sum_{I'\in\mathcal{I}_{n-1}} (1+\varphi(|V_{I'}|))^s |V_{I'}|^s (\lambda_1^s + \cdots + \lambda_k^s) \\
&\leq (1+\varphi(\nu^{n-1}|V|))^s \sum_{I'\in\mathcal{I}_{n-1}} |V_{I'}|^s \\
&\leq \cdots < \prod_{i=0}^{\infty} (1+\varphi(\nu^i |V|))^s |V| < C|V|,
\end{aligned}$$

where $C$ is a constant, and therefore $\dim_H K \leq s$.

$\square$

## 3.3 Proof of $\dim_H K \geq s$

In this section, we show $\dim_H K \geq s$. Indeed, we will show $\dim_H K_{I_0} \geq s$ for certain subset $K_{I_0} \subset K$.

**Lemma 3.3.1.** *$\dim_H K \geq s$.*

We set

$$\bar{V}^n := \bigcup_{I\in\mathcal{I}_n} \bar{V}_I.$$

Note that

$$K = \bigcap_{n=1}^{\infty} \bar{V}^n.$$

For a large $n_0$, fix an abitrary $I_0 = i_1 \cdots i_{n_0} \in \mathcal{I}_{n_0}$, and consider $\bar{V}_{I_0} = g_{I_0}(\bar{V}) = f_{I_0} \circ \cdots f_{i_1 i_2} \circ f_{i_1}(\bar{V})$. It suffices to prove that $\dim_H K_{I_0} \geq s$ for $K_{I_0} := K \cap V_{I_0}$. Therefore we start with $W := V_{I_0}$ instead of $V$.

For every $1 \leq i \leq k$, put

$$h_i := f_{I_0 i} : \bar{W} \to \bar{W}_i = h_i(\bar{W}) \subset \bar{W},$$

and recall from the definition

$$\left| \frac{|h_i(x), h_i(y)|}{|x,y|} - \lambda_i \right| < o(n_0),$$

where $o(n_0) = \lambda_i \varphi(\nu^{n_0}|V|)$ and therefore $\lim_{n_0\to\infty} o(n_0) = 0$. For $J = j_1 \cdots j_m$, define $g_J : \bar{W} \to \bar{W}_J$ by

$$g_J := h_J \circ \cdots \circ h_{j_1 j_2} \circ h_{j_1},$$

where we use the notation

$$h_{j_1 \cdots j_\ell} := f_{I j_1 \cdots j_\ell} : \bar{W}_{j_1 \cdots j_{\ell-1}} \to \bar{W}_{j_1 \cdots j_\ell},$$

as before.

**Lemma 3.3.2.** *For every $x, y \in \bar{W}$, we have*

$$\left| \frac{|g_J(x), g_J(y)|}{|x, y|} - \lambda_J \right| < o(n_0)\lambda_J,$$

*where $\lambda_J = \lambda_{j_1} \cdots \lambda_{j_m}$.*

*Proof.* Put $J_\ell := j_1 \cdots j_\ell$ for each $1 \le \ell \le m$. From Lemma 3.1.2, we obtain

$$\begin{aligned} \frac{|g_J(x), g_J(y)|}{|x, y|} &= \frac{|g_{J_m}(x), g_{J_m}(y)|}{|g_{J_{m-1}}(x), g_{J_{m-1}}(y)|} \cdots \frac{|g_{J_2}(x), g_{J_2}(y)|}{|g_{J_1}(x), g_{J_1}(y)|} \frac{|g_{J_1}(x), g_{J_1}(y)|}{|x, y|} \\ &\le \lambda_J \prod_{\ell=0}^{\infty} (1 + \varphi(\nu^{n_0+\ell}|V|)) \\ &= \lambda_J(1 + o(n_0)). \end{aligned}$$

An estimate from below is similar, and hence omitted. □

For a small $\epsilon > 0$ compared with $|W|$, let $\{U_i\}$ be any $\epsilon$-covering of $K := K_{I_0}$. Replacing $U_i$ by balls $B_i$ of radius less than $|U_i|$, we have a covering $\{B_i\}$ of $K$. Thus

$$\sum |U_i|^s \ge 2^{-s} \sum |B_i|.$$

Fix $B_i$ and take $c_1 > 0$ and $c_2 > 0$ such that $W$ contains a ball of radius $c_1|W|$ and is contained in a ball of radius $c_2|W|$.

**Assertion 3.3.3.** *For each $i$, there is a simple family $\mathcal{S} = \mathcal{S}_i$ consisting of $J$ satisfying that $\bar{W}_J$ is contained in a ball of radius $c_2|B_i|$ and contains a ball of radius $\tilde{\lambda}_{min} c_1 c_2 |B_i|$ for some uniform constant $0 < \tilde{\lambda}_{min} \le \lambda_{\min}$.*

*Proof.* We denote by $\mathcal{I}_\infty$ the set of all infinite sequences $J_\infty = j_1 j_2 \cdots$ with $1 \le j_\ell \le k$ for all $\ell \ge 1$. For each $J_\infty = j_1 j_2 \cdots \in \mathcal{I}_\infty$, there is a unique $m$ such that $|W_{j_1 \cdots j_{m-1}}| > c_2|B_i|$ and $|W_{j_1 \cdots j_m}| \le c_2|B_i|$. Set $J := j_1 \cdots j_m$. Obviously, $W_J$ is contained in a a ball of rdius $c_2|B_i|$. Since $W$ contains a ball of radius $c_1|W|$ and since $W_J$ is open, $W_J$ contains a ball of radius $(1 - o(n_0))\lambda_J c_1 |W|$. From the choice of $J$,

$$(1 - o(n_0))\lambda_J c_1 |W| \ge (1 - o(n_0))^2 \lambda_{j_m} c_1 c_2 |B_i|.$$

This completes the proof. □

**Assertion 3.3.4.** *There is a measure $\mu = \mu_{\mathcal{S}}$ such that*

$$\mu = \sum_{I \in \mathcal{S}} \lambda_I^s (g_I)_*(\mu),$$

*where $\lambda_I^s = (\lambda_I)^s$.*

*Proof.* Define $F : \mathcal{M}(\bar{W}) \to \mathcal{M}(\bar{W})$ by

$$F(\sigma) = \sum \lambda_I^s (g_I)_*(\sigma).$$

It is straightforward to see that $F$ is contracting. Then the conclusion follows from the contraction mapping theorem. $\square$

For any $\bar{W}_J \in \mathcal{S}$,

$$c_2^s |B_i|^s \geq |W_J|^s \geq |K_J|^s \geq (1 - o(n_0))\lambda_J^s |K|^s. \tag{3.3.1}$$

By Lemma 3.1.3, the number of $\bar{W}_J \in \mathcal{S}$ meeting $B_i$ is uniformly bounded by some constant $C = C(\delta)$, where $\delta = \delta(c_1, c_2, \tilde{\lambda}_{\min})$. Then

$$\mu(B_i) = \sum_{I \in \mathcal{S}} \lambda_I^s \mu_I(B_i) = \sum_{I \in \mathcal{S}} \lambda_I^s \mu_I(B_i \cap \bar{W}_I) \tag{3.3.2}$$

$$\leq C(\delta) \max_{I \in \mathcal{S}, \bar{W}_I \cap B_i \neq \phi} \lambda_I^s. \tag{3.3.3}$$

It follows from (3.3.1) and (3.3.3) that

$$c_2^s |B_i|^s \geq (1 - o(n_0)) C(\delta)^{-1} |K|^s \mu(B_i).$$

Since

$$\sum_{|J|=m} \lambda_J^s = 1,$$

for each $m \geq 1$, by the same reason as Lemma 1.1.4, we have a unique measure $\mu_m$ such that

$$\mu_m = \sum_{|J|=m} \lambda_J^s (g_J)_*(\mu_m).$$

**Assertion 3.3.5.** *For $m > \max_{I \in \mathcal{S}} |I|$, we have $\mu = \mu_m$.*

*Proof.* For each $J$ with $|J| = m$, there are unique $I \in \mathcal{S}$ and $J_\alpha$ such that $J = IJ_\alpha$. Let $A_I$ be the set of all the indices $\alpha$ with $J = IJ_\alpha$ for some $J \in \mathcal{I}_m$ We now write as

$$\mu_m = \sum_{I \in \mathcal{S}, \alpha \in A_I} \lambda_{IJ_\alpha}^s (g_{IJ_\alpha})_*(\mu_m).$$

By iterating $\ell$-times, we have

$$\begin{aligned} \mu_m &= \sum_{J_1, \ldots, J_\ell \in \mathcal{I}_m} \lambda_{J_1}^s \cdots \lambda_{J_\ell}^s (g_{J_1} \circ \cdots \circ g_\ell)_*(\mu_m) \\ &= \sum_{I_i \in \mathcal{S}, \alpha_i \in A_{I_i}} \lambda_{I_1 J_{\alpha_1}}^s \cdots \lambda_{I_\ell J_{\alpha_\ell}}^s (g_{J_1} \circ \cdots \circ g_{J_\ell})_*(\mu_m) \end{aligned}$$

and similarly, together with

$$\sum_{\alpha \in A_I} \lambda^s_{J_\alpha} = 1,$$

we obtain

$$\begin{aligned}
\mu &= \sum_{I \in \mathcal{S}} \lambda^s_I (g_I)_*(\mu) \\
&= \sum_{I_1, \dots, I_\ell \in \mathcal{S}} \lambda^s_{I_1 \cdots I_\ell} (g_{I_1} \circ \cdots \circ g_{I_\ell})_*(\mu) \\
&= \sum_{I_i \in \mathcal{S}, \alpha_i \in A_{I_i}} \lambda^s_{I_1 J_{\alpha_1}} \cdots \lambda^s_{I_\ell J_{\alpha_\ell}} (g_{I_1} \circ \cdots \circ g_{I_\ell})_*(\mu)
\end{aligned}$$

It follows that

$$\begin{aligned}
d_{\mathcal{M}}(\mu, \mu_m) \leq & \sum_{I_i \in \mathcal{S}, \alpha_i \in A_{I_i}} \lambda^s_{I_1 J_{\alpha_1}} \cdots \lambda^s_{I_\ell J_{\alpha_\ell}} \\
& \sup_{L(\phi) \leq 1} \left| \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu - \int \phi \circ g_{J_\ell} \circ \cdots \circ g_{J_1} \, d\mu_m \right|
\end{aligned}$$

Here,

$$\begin{aligned}
| \int \phi \circ g_{I_\ell} & \circ \cdots \circ g_{I_1} \, d\mu - \int \phi \circ g_{J_\ell} \circ \cdots \circ g_{J_1} \, d\mu_m | \\
& \leq \left| \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu - \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu_m \right| \\
& + \left| \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu_m - \int \phi \circ g_{J_\ell} \circ \cdots \circ g_{J_1} \, d\mu_m \right|.
\end{aligned}$$

For a constant $\tilde{\lambda}$ with $\lambda_{\max} < \tilde{\lambda} < 1$, choose a large $n_0$ such that $(1 + o(n_0))\lambda_{\max} < \tilde{\lambda} < 1$ for some uniform constant $\tilde{\lambda} < 1$. Then the Lipschitz constant of $g_{I_\ell} \circ \cdots \circ g_{I_1}$ satisfies

$$L(g_{I_\ell} \circ \cdots \circ g_{I_1}) \leq (1 + o(n_0))^\ell \lambda_{I_\ell} \cdots \lambda_{I_1} < \tilde{\lambda}_{I_1 \cdots I_\ell}$$

Therefore we obtain

$$\begin{aligned}
| \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu & - \int \phi \circ g_{I_\ell} \circ \cdots \circ g_{I_1} \, d\mu_m | \\
& \leq \tilde{\lambda}_{I_1 \cdots I_\ell}.
\end{aligned}$$

On the other hand, from the inclusion

$$g_{I_\ell} \circ \cdots \circ g_{I_1}(\bar{W}) \supset g_{J_\ell} \circ \cdots \circ g_{J_1}(\bar{W}),$$

we have

$$\begin{aligned}\sup_{x\in\bar{W}} |\phi\circ g_{I_\ell}\circ\cdots\circ g_{I_1}(x) - \phi\circ g_{J_\ell}\circ\cdots\circ g_{J_1}(x)| \\ &\le |g_{I_\ell}\circ\cdots\circ g_{I_1}(\bar{W})| \\ &\le (1+o(n_0))^\ell \lambda_{I_\ell}\cdots\lambda_{I_1} < \tilde{\lambda}_{I_1\cdots I_\ell}\end{aligned}$$

Thus letting $n = \min_{I\in\mathcal{S}} |I|$, we have

$$\begin{aligned} d_{\mathcal{M}}(\mu,\mu_m) &\le \sum_{I_1,\ldots,I_\ell,\alpha_1,\ldots,\alpha_\ell} \lambda^s_{I_1 J_{\alpha_1}}\cdots\lambda^s_{I_\ell J_{\alpha_\ell}} \tilde{\lambda}_{I_1\cdots I_\ell}(d_{\mathcal{M}}(\mu,\mu_m)+1) \\ &\le \tilde{\lambda}^{n\ell} \sum_{I_1,\ldots,I_\ell,\alpha_1,\ldots,\alpha_\ell} \lambda^s_{I_1 J_{\alpha_1}}\cdots\lambda^s_{I_\ell J_{\alpha_\ell}}(d_{\mathcal{M}}(\mu,\mu_m)+1) \\ &= \tilde{\lambda}^{n\ell}\sum_{I_1,\ldots,I_\ell\in\mathcal{S}} \lambda^s_{I_1}\cdots\lambda^s_{I_\ell}(d_{\mathcal{M}}(\mu,\mu_m)+1) \\ &= \tilde{\lambda}^{n\ell}(d_{\mathcal{M}}(\mu,\mu_m)+1), \end{aligned}$$

which yields

$$d_{\mathcal{M}}(\mu,\mu_m) < \frac{1}{1-\tilde{\lambda}^{n\ell}}\tilde{\lambda}^{n\ell}.$$

Letting $\ell\to\infty$, we conclude that $\mu = \mu_m$. $\square$

From the last assertion, we have

$$\operatorname{supp}(\mu) \subset \bigcap_{m=1}^{\infty}\left(\bigcup_{|J|=m} g_J(\bar{W})\right) = K.$$

It follows that

$$\begin{aligned}\sum 2^{-s}|B_i|^s &\ge (1-o(n_0))c_2^{-s}C(\delta)^{-1}|K|\sum\mu(B_i) \\ &\ge (1-o(n_0))c_2^{-s}C(\delta)^{-1}|K|.\end{aligned}$$

This shows that $\dim_H K \ge s$. We have completed the proof of lemma 3.3.1. $\square$

These complete the proof of $\dim_H K = s$ in Theorem 5.

## 3.4 Proof of $\overline{\dim}_B K \le s$

In this section, we show $\overline{\dim}_B K \le s$. The notation of Section 3.3 will be used in this section.

**Lemma 3.4.1.** $\overline{\dim}_B K \le s$.

*Proof.* For every $\epsilon > 0$ and $J = j_1 j_2 \cdots \in \mathcal{I}_\infty$, take a minimal $m$ satisfying $|W_{J'}| \le \epsilon$ for $J' := J_m = j_1 \cdots j_m$. Note that

$$\epsilon \lambda_{\min}/2 \le |W_{J'}|. \tag{3.4.4}$$

Thus we have a simple family $\mathcal{S} = \{J' \mid J \in \mathcal{I}_\infty\}$. Note also that

$$\sum_{J' \in \mathcal{S}} \lambda_{J'}^s = 1. \tag{3.4.5}$$

By Lemma 3.3.2, we have

$$\left| \frac{|W_{J'}|}{|W|} - \lambda_{\lambda'_J} \right| < \lambda_{J'} o(n_0). \tag{3.4.6}$$

It follows from (3.4.4) and (3.4.6) that

$$(\epsilon \lambda_{\min}/2)^s \le 2^s \lambda_{J'}^s |W|^s.$$

Using (3.4.5), we obtain

$$\sum_{J' \in \mathcal{S}} (\epsilon \lambda_{\min}/2)^s \le 2^s |W|^s,$$

which yields that

$$N_\epsilon(K) \le 2^s |W|^s (\epsilon \lambda_{\min}/2)^{-s}.$$

This shows that $\overline{\dim}_B K \le s$. □

It follows from Lemmas 3.3.1, 3.4.1 and 1.1.1 that $\dim_H K = \dim_B K = s$. This completes the proof of Theorem 5.

# Chapter 4

# Examples of asymptotic self-similar sets

In this chapter, we give several examples of asymptotic self-similar sets on the curved spaces by using asymptotic similarity maps, which is introduced in Introduction. In Section 4.1, we will use $(\lambda, c, \nu)$-almost similarity maps. In Section 4.2, we will use $(\lambda, \varphi, \nu)$-almost similarity maps.

## 4.1 Asymptotic self-similar sets in Riemannian manifolds

In this section, we construct an asymptotically generalized Cantor set in a complete Riemannian manifold by using notion of $(\lambda, c, \nu)$-almost similarity maps.

Let $M$ be a complete Riemannian manifold. For a point $p \in M$, let $B(0,r) = \{v \in T_pM \mid \|v\| \leq r\}$. If $r$ is sufficiently small, then the exponential map $exp_p : B(0,r) \to M$ is a diffeomorphism onto $B(p,r) = \{q \in M | d(p,q) \leq r\}$. For any $v \in B(0,r)$, let $\gamma_v$ be a geodesic such that $\gamma_v(0) = p, \dot{\gamma}_v(0) = v$. Then by definition, $exp_p(v) = \gamma_v(1)$.

Let $K_M$ be the sectional curvature of $M$. Take a positive number $\Lambda$ such that $-\Lambda^2 \leq K_M \leq \Lambda^2$ on $B(p,r)$. By Rauch Comparison Theorem(cf.[9]), for any $u,\ v \in B(0,r)$,

$$\frac{\sin \Lambda r}{\Lambda r} \leq \frac{d(exp_p(u), exp_p(v))}{\|u - v\|} \leq \frac{\sinh \Lambda r}{\Lambda r}.$$

**Proposition 4.1.1.** For a constant $\lambda$ with $0 < \lambda < 1$, let $p_1 \in B(p,r) \subset M$ with $d(p_1, p) \leq (1-\lambda)r$. Let $\tilde{f}_1 : T_{p_1}M \to T_{p_1}M$ be the $\lambda$-similarity map given by $v \mapsto \lambda v$. Let $I_0 : T_pM \to T_{p_1}M$ be a linear isometry. Let $A_0 := B(p,r),\ \tilde{A}_0 := exp_p^{-1}(A_0) = B(0_p, r) \subset T_pM,\ \tilde{A}_1 := I_0(\tilde{A}_0) = B(0_{p_1}, r) \subset T_{p_1}M,\ \tilde{B}_1 := \tilde{f}_1(\tilde{A}_1) = B(0_{p_1}, \lambda r),\ A_1 := exp_{p_1}(\tilde{B}_1) = B(p_1, \lambda r)$. Then $f_0 := exp_{p_1} \circ \tilde{f}_1 \circ I_0 \circ exp_p^{-1} : A_0 \to A_1$ is a $(\lambda, c, \nu)$-similarity map, where $c = \frac{\Lambda^2}{16}(\lambda^2 + 1)$.

*Proof.* For any $x,\ y \in A_0$, by Rauch Comparison Theorem ([9]), we have

$$\frac{d(e^{-1}(x), e^{-1}(y))}{d(x,y)} \le \frac{\Lambda r}{\sin \Lambda r}, \quad \frac{d(f_0(x), f_0(y))}{d(\tilde{f}_1(I_0(e^{-1}(x))), \tilde{f}_1(I_0(e^{-1}(y))))} \le \frac{\sinh \Lambda\lambda r}{\Lambda\lambda r},$$

and therefore

$$\frac{d(f_0(x), f_0(y))}{d(x,y)} \le \lambda \frac{\Lambda r}{\sin \Lambda r} \frac{\sinh \Lambda\lambda r}{\Lambda\lambda r},$$

where $e^{-1} = exp_p^{-1}$.

When $r \ll 1$, by Taylor expansion we get

$$\frac{\Lambda r}{\sin \Lambda r} \le 1 + \frac{1}{7}\Lambda^2 r^2, \quad \frac{\sinh \Lambda\lambda r}{\Lambda\lambda r} \le 1 + \frac{1}{7}\Lambda^2\lambda^2 r^2.$$

Thus, we have

$$\begin{aligned}
\frac{d(f_0(x), f_0(y))}{d(x,y)} &\le \lambda \frac{\Lambda r}{\sin \Lambda r} \frac{\sinh \Lambda\lambda r}{\Lambda\lambda r} \\
&\le \lambda(1 + \frac{1}{7}\Lambda^2 r^2)(1 + \frac{1}{7}\Lambda^2\lambda^2 r^2) \\
&\le \lambda(1 + \frac{1}{7}\Lambda^2\lambda^2 r^2 + \frac{1}{7}\Lambda^2 r^2 + \frac{1}{49}\Lambda^4\lambda^2 r^4) \\
&\le \lambda(1 + \frac{1}{8}\Lambda^2\lambda^2 r^2 + \frac{1}{8}\Lambda^2 r^2) \qquad\qquad (r \ll 1) \\
&\le \lambda + \lambda(\frac{1}{8}\Lambda^2\lambda^2 r^2 + \frac{1}{8}\Lambda^2 r^2).
\end{aligned}$$

Furthermore, since $|A_0| = 2r$, we obtain

$$\begin{aligned}
\frac{d(f_0(x), f_0(y))}{d(x,y)} - \lambda &\le \frac{\lambda\Lambda^2}{8}(\lambda^2 + 1) r^2 \qquad\qquad (r \ll 1) \\
&\le \frac{\lambda\Lambda^2}{16}(\lambda^2 + 1) 2r \\
&= \frac{\lambda\Lambda^2}{16}(\lambda^2 + 1)|A_0|.
\end{aligned}$$

Similarly, we have $\frac{d(f_0(x), f_0(y))}{d(x,y)} - \lambda \ge -\frac{\lambda\Lambda^2}{16}(\lambda^2+1)|A_0|$. Letting $c = \frac{\Lambda^2}{16}(\lambda^2+1)$, we obtain $|\frac{d(f_0(x), f_0(y))}{d(x,y)} - \lambda| \le \lambda c|A_0|$, and hence $f_0$ is a $(\lambda, c, \nu)$-similarity map. □

Using proposition as above, we can obtain the following.

**Example 4.1.2.** For $0 < \lambda \le \frac{1}{2}$, let $k_0$ be a maximal number of disjoint closed balls of radius $\lambda$ which is contained in the unit ball of $\mathbb{R}^n$. Let $M$ be an $n$-dimensional complete Riemannian manifold of Ricci curvature $\ge (n-1)\kappa$ and $p \in M$ for a constant $\kappa$. If $r$ is sufficiently small, then $B(p, r)$ is almost isometric to $B(0, r) \subset T_pM$. Let $1 < k \le k_0$ and $r_1 = \lambda r$. Then we can take $k$ disjoint balls $\{B(p_i, r_1)\}_{i=1}^k$ in $B(p, r)$. By Proposition 4.1.1,

$B(p_i, r_1)$ is a $(\lambda, c, \nu)$-similar set of $B(p, r)$ for some uniform constant $c$. let $r_2 = \lambda r_1$, then we can take $k$ disjoint balls $\{B(p_{ij}, r_2)\}_{j=1}^k$ in each ball $B(p_i, r_1)$, and $B(p_{ij}, r_2)$ is a$(\lambda, c, \nu)$-similar set of $B(p_i, r_1)$. Repeating this procedure, we can construct basic sets $B(p_{i_1 \cdots i_n}, r_n)$ $(r_n = \lambda^n r,\ i_1, \cdots, i_n = 1, 2, \cdots k)$, and we can define an asymptotic self-similar set $C$ in $M$ as

$$C := \bigcap_{n=1}^{\infty} \Big( \bigcup_{i_1, \ldots, i_n = 1}^{k} B(p_{i_1 \cdots i_n}, r_n) \Big).$$

Let $\mu$ be the Riemannian measure of $M$. We denote by $V_\kappa^n(r)$ the volume of a $r$-ball in the $n$-dimensional space form $M_\kappa^n$ of constant curvature $\kappa$. By Theorem 1.4.2, we have

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, \delta r))} \leq \frac{V_\kappa^n(r)}{V_\kappa^n(\delta r)} = \frac{\int_0^r \sinh \sqrt{|\kappa|} t dt}{\int_0^{\delta r} \sinh \sqrt{|\kappa|} t dt} \leq C_{n,\kappa}(\delta),$$

for any $x_0 \in M$ and $0 < \delta, r < 1$, where $C_{n,\kappa}(\delta)$ is a positive constant depending only on $n, \kappa$ and $\delta$.

Hence by Theorem 3, we have $\dim_H C = -\frac{\log k}{\log \lambda}$.

## 4.2 Sierpinski gaskets on surfaces

In this section, we determine the Hausdorff dimensions of the generalized Sierpinski gaskets, which is constructed on the convex domains of surfaces.

Let $D$ be a domain in a complete surface $M$. We assume that $D$ is convex in the sense that for every $p \in D$, the distance function $d_p(\cdot) = d(p, \cdot)$ from $p$ is convex in $D$. For simplicity, we assume that the absolute value of the Gaussian curvature of $M$ is at most 1 on $D$. Let $\Delta$ be a domain in $D$ bounded by a geodesic triangle. We call $\Delta$ a *geodesic triangle region.*

**Definition 4.2.1.** We say that $\Delta$ is *$\delta$-non-degenerate* if each angle $\tilde{\alpha}$ of a comparison triangle $\tilde{\Delta}$ of $\Delta$ in $\mathbb{R}^2$ satisfies $\delta < \tilde{\alpha} < \pi - \delta$, where a comparison triangle means that $\tilde{\Delta}$ has the same side-length as $\Delta$.

Let $\{\Delta_I\}_{I \in \mathcal{I}^3}$ be the system of geodesic triangles obtained by dividing $\Delta$ into smaller triangles $\Delta_I$ consecutively, as stated in Introduction.

**Definition 4.2.2.** We say that the system $\{\Delta_I\}_{I \in \mathcal{I}^3}$ is *non-degenerate* if there is a $\delta > 0$ such that $\Delta_I$ is $\delta$-non-degenerate for every $I \in \mathcal{I}^3$. In this case, we also say that $\Delta$ is *asymptotically non-degenerate.*

**Example 4.2.3.** Let $\mathbb{S}^2$ denote the unit sphere around the origin in $\mathbb{R}^3$, and let $\Delta$ be a geodesic triangle domain on $\mathbb{S}^2$ of perimeter less than $2\pi$. Joining the vertexes $p_1, p_2, p_3$ of $\Delta$ by shortest segments in $\mathbb{R}^3$, we have a geodesic triangle region $\tilde{\Delta}$ on the plane through $p_1, p_2, p_3$. By the projection along the rays from the origin of $\mathbb{R}^3$, we have a canonical map

$$\pi : \Delta \to \tilde{\Delta},$$

which is a bi-Lipschitz homeomorphism. From the canonical decomposition $\{\Delta_I\}_{I\in\mathcal{I}^3}$ of $\Delta$, setting $\tilde{\Delta}_I := \pi(\Delta_I)$, we have the canonical decomposition $\{\tilde{\Delta}_I\}_{I\in\mathcal{I}^3}$ of $\tilde{\Delta}$. Note that each $\tilde{\Delta}_I$ is $2^{-|I|}$-similar to $\tilde{\Delta}$ in the usual sense. Since $\Delta_I$ is bi-Lipschitz homeomorphic to $\tilde{\Delta}_I$,

$$\mathrm{Area}(\Delta_I) \geq L^{-2}\mathrm{Area}(\tilde{\Delta}_I),$$

where $L$ is the bi-Lipschitz constant of $\pi$. It follows that $\Delta$ is asymptotically non-degenerate. Now we have the formula (6) for the Sierpinsli gasket $K_\Delta$ associated with $\Delta$ by two reasons. One is by Theorem 6 and the other one is due to the well-known formula for $K_{\tilde{\Delta}}$.

Example 4.2.3 is the special case. For a geodesic triangle region on a general complete surface, there is no canonical map $\tilde{\Delta} \to \Delta$ as in Example 4.2.3. It seems impossible to reduce the problem to a triangle region in $\mathbb{R}^2$ in general.

The main purpose of this section is to prove the following result.

**Theorem 4.2.4.** *For every $\delta > 0$ there exists an $r > 0$ such that*

1. *every geodesic triangle region $\Delta$ on $D$ with $|\Delta| \leq r$ is asymptotically non-degenerate;*
2. *the Hausdorff and box dimensions of the Sierpinski gasket $K_\Delta$ associated with $\Delta$ are given by* (6).

If $\Delta$ be asymptotically non-degenerate as in Theorem 6, we can apply Theorem 4.2.4 to $\Delta_I$ for each $I \in \mathcal{I}^3$ with large enough $|I|$. Therefore Theorem 4.2.4 yields Theorem 6.

The following lemma is a consequence of law of cosine, and hence is omitted.

**Lemma 4.2.5.** *For any $\delta > 0$ there exists an $\epsilon > 0$ such that if a geodesic triangle $\Delta$ of side length $(a_1, a_2, a_3)$ is $\delta$-non-degenerate, and if the side length $(a_1', a_2', a_3')$ of a geodesic triangle $\Delta'$ satisfies*

$$(1-\epsilon)\frac{a_j}{a_i} < \frac{a_j'}{a_i'} < (1+\epsilon)\frac{a_j}{a_i}, \tag{4.2.1}$$

*for any $i \neq j$, then $\Delta'$ is $\delta/2$-non-degenerate.*

*Proof.* We may assume that $\Delta$ and $\Delta'$ are triangles in $\mathbb{R}^2$. Set $(a, b, c) := (a_1, a_2, a_3)$ and $(a', b', c') := (a_1', a_2', a_3')$ for simplicity. Rescaling $\Delta'$, we may assume that $c = c'$. It suffices to show that if $\Delta'$ has side-length $(a', b', c') = (a', b, c)$ satisfying (4.2.1), then the angles $\alpha$, $\beta$ (resp. $\alpha'$, $\beta'$) opposite to the edges of length $a$ and $b$ in $\Delta$ (resp $a'$ and $b$ in $\Delta'$) satisfy that $|\alpha' - \alpha| < \delta/4$ and $|\beta' - \beta| < \delta/4$ for a suitable $\epsilon = \epsilon(\delta) > 0$.

**Sublemma 4.2.6.** *If a geodesic triangle $\Delta$ of side lengths $(a_1, a_2, a_3)$ is $\delta$-non-degenerate, then there exists a constant $C(\delta)$ such that*

$$C(\delta)^{-1} < \frac{a_j}{a_i} < C(\delta),$$

*for every $1 \leq i, j \leq 3$.*

*Proof.* This is an immediate consequence of the law of sines. One can take $C(\delta) = 1/\sin\delta$. $\square$

By trigonometry, we have

$$\sin^2\alpha/2 = (a+c)(a+b)/bc,\ \sin^2\alpha'/2 = (a'+c)(a'+b)/bc.$$

It follows from the assumption and Sublemma 4.2.6 with $|a'-a| < \epsilon a$ that

$$|\sin^2\alpha'/2 - \sin^2\alpha/2| \le a(a+a'b+c)\epsilon/bc \le 5C(\delta)^2\epsilon. \tag{4.2.2}$$

Since $\sin\alpha'/2 + \sin\alpha/2 > \sin(\delta/2)$, we obtain

$$|\sin\alpha'/2 - \sin\alpha/2| \le 5C(\delta)^2\epsilon/\sin(\delta/2).$$

From $\alpha < \pi - 2\delta$, we have $\cos\frac{\alpha'+\alpha}{4} > \sin(\delta/4)$. It follows that

$$|\alpha'-\alpha| \le 8\left|\sin\frac{\alpha'-\alpha}{4}\right| < 5C(\delta)^2\epsilon/\sin^2(\delta/4). \tag{4.2.3}$$

Similarly we have

$$\begin{aligned}|\sin^2\beta'/2 - \sin^2\beta/2| &= |a-a'|b(b+c)/aa'c \le b(b+c)\epsilon/ca' \\ &\le \frac{\epsilon}{1-\epsilon}\frac{b(b+c)}{a} \le \frac{\epsilon}{1-\epsilon}2C(\delta)^2,\end{aligned}$$

which implies

$$|\beta'-\beta| < \frac{8\epsilon}{1-\epsilon}\left(\frac{C(\delta)}{\sin(\delta/2)}\right)^2. \tag{4.2.4}$$

Thus from (4.2.3), (4.2.4), we obtain $|\alpha'-\alpha| < \delta/4$ and $|\beta'-\beta| < \delta/4$ for a suitable $\epsilon \le \epsilon(\delta)$. This completes the proof. $\square$

Let $\Delta$ be a geodesic triangle region on $D$ bounded by a geodesic triangle $(\gamma_1,\gamma_2,\gamma_3)$ with vertices $p_1,p_2,p_3$. By the convexity of $D$, we have

$$|\Delta| = \max_{1\le i\le 3} a_i,$$

where we put $a_i := L(\gamma_i)$. Fix a vertex $p_1$ and let $\gamma_i$ be parametrized on $[0,1]$ in such a way that $\gamma_2(0) = \gamma_3(0) = p_1$. Let $\varphi : [0,1]\times[0,1] \to \Delta$ be a parametrization of $\Delta$ such that $t \to \varphi(t,s)$, $0 \le t \le 1$, is the geodesic, denoted by $\sigma_s$, from $\gamma_2(s)$ to $\gamma_3(s)$ for each $s \in [0,1]$. Namely $\varphi(t,s) = \sigma_s(t)$. We set

$$a_1(s) := L(\sigma_s).$$

Now define the map $f_1 : \Delta \to \Delta$ by

$$f_1(\varphi(t,s)) = \varphi(t,s/2).$$

Note that the image $\Delta_1$ of $f_1$ is the geodesic triangle region bounded by $(\gamma_2|_{[0,1/2]}, \gamma_3|_{[0,1/2]}, \sigma_{1/2})$ and that $\Delta_1$ has side-length $(a_1(1/2), a_2/2, a_3/2)$. We put

$$r := |\Delta|.$$

**Lemma 4.2.7.** *For any $s \in (0,1)$, we have*
$$1 - r^2 < \frac{a_1(s)}{sa_1} < 1 + r^2.$$
*In particular,* $|\Delta_1| \le \frac{1}{2}(1+r^2)|\Delta|$.

*Proof.* Let $\tilde{\gamma}_i(s) := \exp_{p_1}^{-1}(\gamma_i(s))$, $i = 2,3$. The Rauch comparison theorem (see [9]) implies
$$\frac{\sin r}{r} < \frac{a_1}{d(\tilde{\gamma}_2(1), \tilde{\gamma}_3(1))} < \frac{\sinh r}{r} \tag{4.2.5}$$
$$\frac{\sin r}{r} < \frac{a_1(s))}{d(\tilde{\gamma}_2(s), \tilde{\gamma}_3(s))} < \frac{\sinh r}{r}. \tag{4.2.6}$$
Since $d(\tilde{\gamma}_2(s), \tilde{\gamma}_3(s)) = sd(\tilde{\gamma}_2(1), \tilde{\gamma}_3(1)$, the conclusion follows. □

Let us denote by $(a_{1,1}, a_{1,2}, a_{1,3})$ the side length $(a_1(1/2), a_2/2, a_3/2)$ of $\Delta_1$. Lemma 4.2.7 implies that
$$(1-r^2)\frac{a_i}{a_j} < \frac{a_{1,i}}{a_{1,j}} < (1+r^2)\frac{a_i}{a_j}, \tag{4.2.7}$$
for every $1 \le i, j \le 3$.

In a similar way, we construct a map $f_{i_1} : \Delta \to \Delta_{i_1} \subset \Delta$ for each $1 \le i_1 \le 3$. Repeating this procedure for each $\Delta_i$ inductively, for each multi-index $I = i_1 \cdots i_{n-1} i_n$, we have a geodesic triangle region $\Delta_I$ and a map $f_I : \Delta_{I'} \to \Delta_I$, where $I' = i_1 \cdots i_{n-1}$. The side-length $(a_{I,1}, a_{I,2}, a_{I,3})$ of $\Delta_I$ is also suitably defined inductively. Take $r < 1$ and set
$$\nu := \frac{1}{2}(1 + r^2) < 1.$$

**Lemma 4.2.8.** *There exists an $L(r) > 1$ such that for every $I$ and $1 \le i, j \le 3$*
$$L(r)^{-1}\frac{a_i}{a_j} < \frac{a_{I,i}}{a_{I,j}} < L(r)\frac{a_i}{a_j}.$$

*Proof.* Repeating use of (4.2.7) and Lemma 4.2.7 applied to $s = 1/2$ implies that for each $I = i_1 \cdots i_m$,
$$\begin{aligned}(1-r_m^2)\cdots(1-r_1^2)(1-r^2)&\frac{a_i}{a_j}\\ &< \frac{a_{I,i}}{a_{I,j}} < (1+r_m^2)\cdots(1+r_1^2)(1+r^2)\frac{a_i}{a_j}\end{aligned}$$
for every $1 \le i, j \le 3$, where $r_k := |\Delta_{i_1 \cdots i_k}|$, $1 \le k \le m$. Since
$$r_k \le \frac{1}{2}(1 + r_{k-1}^2) r_{k-1} < \nu r_{k-1} < \cdots < \nu^k r.$$
it follows that
$$\Pi_{m=0}^{\infty}\left(1 - \nu^{2m} r^2\right)\frac{a_i}{a_j} < \frac{a_{I,i}}{a_{I,j}} < \Pi_{m=1}^{\infty}\left(1 + \nu^{2m} r^2\right)\frac{a_i}{a_j}. \tag{4.2.8}$$
This completes the proof. □

From (4.2.8), one can take $L(r)$ as

$$L(r) := e^{\frac{2r^2}{1-\nu^2}}.$$

For every $s \in (0,1]$ we denote by $\Delta(1:s)$ the geodesic triangle $(\gamma_2|_{[0,s]}, \gamma_3|_{[0,s]}, \sigma_s)$. Similarly, $\Delta(i:s)$ and $\Delta_I(i:s)$ are defined for every $1 \le i \le 3$ and every multi-index $I \in \mathcal{I}^3$.

Lemmas 4.2.5, 4.2.7 and 4.2.8 imply

**Lemma 4.2.9.** *For every $\delta > 0$, there exists a positive number $r$ such that if $\Delta$ is $\delta$-non-degenerate and the diameter $|\Delta|$ of $\Delta$ is less than $r$, then $\Delta_I$ as well as $\Delta_I(i:s)$ is $\delta/2$-non-degenerate for every multi-index $I$, $1 \le i \le 3$ and $s \in (0,1)$.*

By Lemma 4.2.9, we get the conclusion (1) of Theorem 4.2.4. In view of Theorem 5, to prove the conclusion (2) of Theorem 4.2.4, it suffices to prove the following.

**Theorem 4.2.10.** *There is a positive numbers $c = c(\delta)$ such that $\{(\Delta_I, f_I)\}_{I\in\mathcal{I}^3}$ gives a $(1/2, \varphi_c, \nu)$-asynptotic similarity system, where $\varphi_c(x) = cx^2$.*

*Proof.* In view of Lemma 4.2.9, it suffices to prove that the map $f := f_1 : \Delta \to \Delta_1 \subset \Delta$ is a $(1/2, \varphi_c, \nu)$-almost similarity map for a uniform positive constant $c = c(\delta)$. Note that $J_s(t) := \frac{\partial\varphi}{\partial s}(t,s)$ is a Jacobi field along $\sigma_s$. Set $T_s(t) := \frac{\partial\varphi}{\partial t}(t,s) = \dot{\sigma}_s(t)$. Observe that

$$df(T_s(t)) = T_{s/2}(t), \;\; df(J_s(t)) = \frac{1}{2}J_{s/2}(t). \tag{4.2.9}$$

Lemma 4.2.7 shows that

$$\left|\frac{L(\sigma_{s/2})}{L(\sigma_s)} - \frac{1}{2}\right| < 3r^2,$$

which implies that

$$\left|\frac{|df(T_s)|}{|T_s|} - \frac{1}{2}\right| < 3r^2. \tag{4.2.10}$$

Next we show

**Lemma 4.2.11.** *For every $s, u \in (0,1]$ and $t \in [0,1]$, we have*

$$\left|\frac{|J_u(t)|}{|J_s(t)|} - 1\right| < C(\delta)r^2. \tag{4.2.11}$$

From now on, we shall use the general symbols $C(\delta)$ or $c(\delta)$ to denote constants depending only on $\delta$ unless otherwise stated.

*Proof.* For any fixed $s$, take unique Jacobi fields $Y_1$ and $Y_2$ along $\sigma_s$ and the reverse geodesic $\sigma_s^-(t) := \sigma(1-t)$ respectively such that

$$Y_1(0) = 0,\, Y_1(1) = J_s(1),\;\; Y_2(1) = J_s(0),\, Y_2(0) = 0,$$

to have

$$J_s(t) = Y_1(t)) + Y_2(1-t).$$

We dente by $\mathbb{S}^2$ and $\mathbb{H}^2$ the sphere and the hyperbolic plane of constant curvature 1 and $-1$ respectively.

Recall that $\Delta$ is a $\delta$-non-degenerate geodesic triangle region of side lengths $(a_1, a_2, a_3)$ in $D$ whose diameter is denoted by $r$.

**Lemma 4.2.12.** *Let $\alpha_{i+}$ and $\alpha_{i-}$be the angles of comparison triangles $\Delta_+$ and $\Delta_-$ of $\Delta$ in $\mathbb{S}^2$ and $\mathbb{H}^2$ respectively at the vertices opposite to the edge of length $a_i$. Then we have*

$$|\alpha_{i+} - \alpha_{i-}| < C(\delta)r^2.$$

*Proof.* Put $(a,b,c) := (a_1,a_2,a_3)$, and let $\alpha_+$, $\alpha_-$ and $\alpha$ be the angles of comparison triangles of $\Delta$ in $\mathbb{S}^2$, $\mathbb{H}^2$ and $\mathbb{R}^2$ respectively at the vertices opposite to the edge of length $a$. By the laws of cosines, we have

$$\begin{aligned} \sin b \sin c \cos \alpha_+ &= \cos a - \cos b \cos c \\ \sinh b \sinh c \cos \alpha_- &= \cosh b \cosh c - \cosh a \\ 2bc \cos \alpha &= b^2 + c^2 - a^2, \end{aligned}$$

which imply

$$\begin{aligned} 2bc \cos \alpha_+ &= 2bc \cos \alpha + O(b^3c) + O(bc^3) + O(b^2c^2) + O(a^4) \\ 2bc \cos \alpha_- &= 2bc \cos \alpha + O(b^3c) + O(bc^3) + O(b^2c^2) + O(a^4). \end{aligned}$$

It follows from Sublemma4.2.6 that

$$\begin{aligned} |\cos \alpha_+ - \cos \alpha| &\le O(b^2) + O(c^2) + O(bc) + O(a^4/bc) \\ &\le C(\delta)r^2. \end{aligned}$$

Since $\delta < \alpha < \pi - \delta$, we obtain $|\alpha_+ - \alpha| \le C(\delta)r^2$. Similarly we get $|\alpha_- - \alpha| \le C(\delta)r^2$, and hence $|\alpha_+ - \alpha_-| \le C(\delta)r^2$. □

Let $\alpha_s$ and $\beta_s$ be the angle of the geodesic triangle $\Delta(1:s) = (\gamma_2|_{[0,s]}, \gamma_3|_{[0,s]}, \sigma_s)$ at $\gamma_2(s)$ and $\gamma_3(s)$ respectively.

**Lemma 4.2.13.**

$$|\alpha_s - \alpha_t| < c(\delta)r^2, \ |\beta_s - \beta_t| < c(\delta)r^2,$$

*for every $s, t \in (0,1]$.*

*Proof.* Let $\alpha_s^+$ , $\alpha_s^-$ , $\alpha_s^0$ denote the angles of comparison triangles in $\mathbb{S}^2$, $\mathbb{H}^2$, and $\mathbb{R}^2$ respectively at the vertices coresponding $\gamma_2(s)$. By Toponogov's theorem (cf. [9]), we have

$$\alpha_s^- \le \alpha_s, \ \alpha_s^0 \le \alpha_s^+. \tag{4.2.12}$$

By the law of cosines, we have

$$\cos\alpha_s^0 = \frac{a_2^2 + (a_1(s)/s)^2 - a_3^2}{2a_2(a_1(s)/s)}$$
$$\cos\alpha_t^0 = \frac{a_2^2 + (a_1(t)/t)^2 - a_3^2}{2a_2(a_1(t)/t)},$$

which imply with Lemma4.2.7

$$\begin{aligned}
&\cos\alpha_s^0 - \cos\alpha_t^0 \\
&\quad\le \frac{a_2^2 + a_1^2(1+r^2) - a_3^2}{2a_2a_1(1-r^2)} - \frac{a_2^2 + a_1^2(1-r^2) - a_3^2}{2a_2a_1(1+r^2)} \\
&\quad= \frac{r^2(2a_1^2 + a_2^2 - a_3^2)}{a_1a_2(1-r^2)(1+r^2)} \\
&\quad= \frac{r^2}{1-r^4}\left(\frac{2a_1}{a_2} + \frac{a_2}{a_1} - \frac{a_3^2}{a_1a_2}\right) \\
&\quad\le C(\delta)r^2.
\end{aligned}$$

Revercing the role of $s$ and $t$, we have

$$|\cos\alpha_s^0 - \cos\alpha_t^0| \le C(\delta)r^2.$$

By Lemma 4.2.9, we have $\delta/2 < (\alpha_s^0+\alpha_t^0)/2 < \pi-\delta/2$, which implies $\sin\frac{\alpha_s^0+\alpha_t^0}{2} > \sin(\delta/2)$. Therefore we conclude that

$$|\alpha_s^0 - \alpha_t^0| \le 4\left|\sin\left(\frac{\alpha_s^0 - \alpha_t^0}{2}\right)\right| \le C_1(\delta)r^2.$$

where $C_1(\delta) := \frac{2C(\delta)}{\sin(\delta/2)}$ Using (4.2.12) and Lemma 4.2.12, we see

$$\begin{aligned}
\alpha_s &\le \alpha_s^0 + C(\delta)r^2 \\
&\le \alpha_t^0 + C(\delta)r^2 + C_1(\delta)r^2 \\
&\le \alpha_t + 2C(\delta)r^2 + C_1(\delta)r^2.
\end{aligned}$$

Reversing the role of $s$ and $t$ completes the proof. □

Next we analyze the behavior of the norm of Jacobi field $J_s$. For a fixed $s \in (0,1]$, let $Y_i(t) = Y_i^N(t) + Y_i^T(t)$, $i = 1, 2$, be the orthogonal decompositions of $Y_i$ to the normal and tangential components to $\dot\sigma_s$. We can write $Y_i(t)$ and $Y_i(t)^N$ as

$$Y_1(t) = d\exp_{\gamma_2(s)}(t(V_1)_{t\dot\sigma_s(0)}),\ Y_2(t) = d\exp_{\gamma_3(s)}(t(V_2)_{t\dot\sigma_s^-(0)}), \tag{4.2.13}$$
$$Y_1^N(t) = d\exp_{\gamma_2(s)}(t(V_1^N)_{t\dot\sigma_s(0)}),\ Y_2^N(t) = d\exp_{\gamma_3(s)}(t(V_2^N)_{t\dot\sigma_s^-(0)}), \tag{4.2.14}$$

where $V_1$ and $V_2$ are some parallel vector fields on the tangent spaces satisfying

$$d\exp_{\gamma_2(s)}((V_1)_{\dot\sigma_s(0)}) = \dot\gamma_3(s),\ \ d\exp_{\gamma_3(s)}((V_2)_{\dot\sigma_s^-(0)}) = \dot\gamma_2(s).$$

The Rauch comparison theorem shows that

$$|Y_1^N(t)| \fallingdotseq t|V_1^N| \fallingdotseq t|\dot\gamma_3(t)^N|,\ \ |Y_2^N(1-t)| \fallingdotseq (1-t)|V_2^N| \fallingdotseq (1-t)|\dot\gamma_2(t)^N|.$$

Here and hereafter we use the symbol $a \fallingdotseq b$ whenever $\left|\frac{a}{b}-1\right| < C(\delta)r^2$. It follows from $\dim M = 2$ that

$$\begin{aligned}
|J_s^N(t)| &= |Y_1^N(t)| + |Y_2^N(1-t)| && (4.2.15)\\
&\fallingdotseq t|\dot\gamma_3(t)^N| + (1-t)|\dot\gamma_2(t)^N| && (4.2.16)\\
&= t\sin\beta_s a_3 + (1-t)\sin\alpha_s a_2, && (4.2.17)
\end{aligned}$$

where we recall $a_i = L(\gamma_i) = |\dot\gamma_i(t))|$. Similarly we have

$$|J_u^N(t)| \fallingdotseq t\sin\beta_u a_3 + (1-t)\sin\alpha_u a_2.$$

It follows from that

$$|J_s^N(t)| \fallingdotseq |J_u^N(t)|. \tag{4.2.18}$$

Next we show that

$$|J_s^T(t)| \fallingdotseq |J_u^T(t)|. \tag{4.2.19}$$

We use the expression (4.2.13) with Gauss's lemma to obtain

$$\begin{aligned}
\langle Y_1(t), T_s(t)\rangle &= ta_3|T_s|\cos\beta_s,\\
\langle Y_2(t), T_s(t)\rangle &= -(1-t)a_2|T_s|\cos\alpha_s.
\end{aligned}$$

Thus we get

$$|J_s^T(t)| = |ta_3\cos\beta_s - (1-t)a_2\cos\alpha_s|.$$

From an inequality for $|J_u^T(t)|$ similar to the above and Lemma 4.2.13, we have (4.2.19). Now (4.2.11) follows from (4.2.18), (4.2.19). Thus we have completed the proof of Lemma 4.2.11. □

The expression (4.2.13) also yields

$$|Y_1(t)| \fallingdotseq t|V_1| \fallingdotseq ta_3,\ \ |Y_2(1-t)| \fallingdotseq (1-t)|V_2| \fallingdotseq (1-t)a_2.$$

In particular we have

$$|J_s(t)| \le 2r. \tag{4.2.20}$$

Since $|J_s^N(t)| \ge c(\delta)r$ from (4.2.17), (4.2.20) implies that the angle $\theta_s(t) := \angle(J_s(t), T_s(t))$ has definite lower and upper bounds:

$$0 < c(\delta) \le \theta_s(t) \le \pi - c(\delta). \tag{4.2.21}$$

(4.2.9), (4.2.10), (4.2.11) and (4.2.21) yield that

$$\left| \frac{|df(v)|}{|v|} - \frac{1}{2} \right| < C(\delta) r^2,$$

for every tangent vector $v$. Thus we conclude that $f : \Delta \to \Delta_1$ is a $(1/2, \varphi_{C(\delta)}, \nu)$-almost similarity map, with $\varphi_{C(\delta)}(x) = C(\delta)x^2$. This completes the proof of Theorem 4.2.10. $\square$

Finally we show Corollary 7 in Introduction .

*Proof of Corollary 7.* In view of Theorem 5, it suffices to show that for a geodesic triangle region $\Delta$ on a convex domain of a complete surface, if the collection $\{(\Delta_I, f_I)\}_{I \in \mathcal{I}^3}$ gives a $(\{1/2, 1/2, 1/2\}, \varphi_C, \nu)$-asymptotic similarity system with $\varphi_C(x) = Cx^2$ and $0 < \nu < 1$, then $\Delta$ is asymptotically non-degenerate.

For a large $n_0$, fix an abitrary $I_0 = i_1 \cdots i_{n_0} \in \mathcal{I}_{n_0}$, and set

$$W := \Delta_{I_0} = g_{I_0}(\Delta) = f_{I_0} \circ \cdots f_{i_1 i_2} \circ f_{i_1}(\Delta).$$

For every $1 \le i \le k$, put

$$h_i := f_{I_0 i} : W \to W_i = h_i(W) \subset W,$$

and recall from the definition

$$\left| \frac{|h_i(x), h_i(y)|}{|x, y|} - \lambda_i \right| < o(n_0),$$

where $o(n_0) = \lambda_i \varphi(\nu^{n_0} |\Delta|)$ and therefore $\lim_{n_0 \to \infty} o(n_0) = 0$. For $J = j_1 \cdot \cdot j_m$, define $g_J : W \to W_J$ by

$$g_J := h_J \circ \cdots \circ h_{j_1 j_2} \circ h_{j_1},$$

where we use the notation

$$h_{j_1 \cdot \cdot j_\ell} := f_{I j_1 \cdot \cdot j_\ell} : W_{j_1 \cdot \cdot j_{\ell-1}} \to W_{j_1 \cdot \cdot j_\ell},$$

as before. By Lemma 3.3.2, we have

$$\left| \frac{|g_J(x), g_J(y)|}{|x, y|} - \lambda_J \right| < o(n_0) \lambda_J,$$

for every $x, y \in W$. We denote by $\mathrm{inrad}(W)$, the inradius of $W$, the largest $r > 0$ such that an $r$-ball is contained in $W$. It follows that

$$\frac{|W_J|}{\mathrm{inrad}(W_J)} \le \frac{1 + o(n_0)}{1 - o(n_0)} \frac{|W|}{\mathrm{inrad}(W)},$$

for every $J \in \mathcal{I}^3$. This implies that there exists a $\delta > 0$ such that $\Delta_I$ is $\delta$-nondegenerate for every $I \in \mathcal{I}^3$. $\square$

This completes the proof of Corollary 7.

# Chapter 5

# Self-similar sets as boundaries of trees

In this chapter, we consider the self-similar sets by using contracting similarity maps in some trees, and give several examples of CAT(0)-spaces.

Let $X$ be a metric space. Let $Isom(X)$ denote the set of all isometries on $X$. We say that $A \subset X$ is *homogeneous* in $X$ if for any $x, y \in A$ there is a isometry $g \in Isom(X)$ such that $g(x) = y$ and $g(A) = A$.

In this chapter, we prove the following.

**Proposition 5.0.14.** *For any constant $s$ with $0 < s < 1$, there exists some 1-dimensional CAT(0) space $X$ such that*

(*1*) $\dim_H(\partial X) = s$.

(*2*) *$\partial X$ is a self-similar set.*

(*3*) *$\partial X$ is homogeneous in $X$.*

(*4*) *$L(X) < \infty$, where $L$ denote the length of $X$.*

**Proposition 5.0.15.** *For $s = 1$, there exists some 1-dimensional CAT(0) space $X$ such that*

(*1*) $dim_H(\partial X) = 1$.

(*2*) *$\partial X$ is a self-similar set.*

(*2*) *$\partial X$ is homogeneous in $X$.*

## 5.1 Self-similar sets in trees

In this section, we first construct some tree, and define a self-similar set as following.

Let us consider a tree $X_0$ with vertices $\{ P_0,\ P_{w_1 \cdots w_k} \mid w_i \in \{1,2\},\ \ i = 1, 2, \cdots, k,\ k \geq 1,\ \}$ defined as follows. First we fix a constant $0 < \lambda < 1$. We begin with the two edges $[P_0P_1], [P_0P_2]$ of length $\frac{\lambda}{2}$. For each $w \in \{1,2\}$, the edge $[P_0P_w]$ branches at $P_w$ into two edges $[P_wP_{w1}], [P_wP_{w2}]$ of length $(\frac{\lambda}{2})^2$. In general, for $w_1, \cdots, w_k \in \{1,2\}$, the edge $[P_{w_1 \cdots w_{k-1}} P_{w_1w_2\cdots w_k}]$ branches at $P_{w_1w_2\cdots w_k}$ into two edges $[P_{w_1w_2\cdots w_k} P_{w_1w_2\cdots w_k1}]$, $[P_{w_1w_2\cdots w_k} P_{w_1w_2\cdots w_k2}]$ of length $(\frac{\lambda}{2})^{k+1}$. In this way, we construct the infinite tree $X_0$. Note that the distance on $X_0$ is naturally defined by using the length of edges. Let X be the completion of $X_0$, and let $\partial X$ be the set of points of X at which some shortest path cannot extend anymore. We define

$$C := \partial X = X \backslash X_0.$$

Obviously, $X$ is a CAT(0) space. Next, we consider $C$ as a self-similar set ([49], [7]) as follows. Let $X_{w_1w_2\cdots w_k}$ be the union of shortest paths from $P_{w_1w_2\cdots w_k}$ to $C$. For each $w \in \{1,2\}$, we define the map $f_w : X \to X_w$ by

$$f_w(P_0) = P_w,$$
$$f_w(P_{w_1w_2\cdots w_k}) = P_{ww_1w_2\cdots w_k}.$$

Then, it is a $\frac{\lambda}{2}$-similarity map. From the iterated function system $\{f_1, f_2\}$, we have a self-similar set $C$. In a terminology of asymptotic self-similar sets, we proceed as follows.

In general, for $w_1, w_2, \cdots, w_k \in \{1,2\}$, we define the map $f_{w_1w_2\cdots w_k} : X_{w_1w_2\cdots w_{k-1}} \to X_{w_1w_2\cdots w_k}$ by

$$f_{w_1w_2\cdots w_k} = f_{w_k}|_{X_{w_1w_2\cdots w_{k-1}}}.$$

More concretely,

$$f_{w_1w_2\cdots w_k}(P_{w_1w_2\cdots w_{k-1}}) = P_{w_1w_2\cdots w_k},$$
$$f_{w_1w_2\cdots w_k}(P_{w_1\cdots w_{k-1}v_k\cdots v_l}) = P_{w_1\cdots w_kv_k\cdots v_l}.$$

It is also a $\frac{\lambda}{2}$-similarity map. We define C as the limiting set

$$C := \bigcap_{k=1}^{\infty} ( \bigcup_{w_1,\ldots,w_k=1}^{2} X_{w_1\cdots w_k}). \tag{5.1.1}$$

Then $C$ is a self-similar set in the sense [7]. Furthermore, $C$ satisfies the following property.

**Lemma 5.1.1.** *$C$ is homogeneous in $X$.*

*Proof.* Let $\varphi_0 \in Isom(X)$ be the reflection of $X$ at $P_0$ such that for $\{w_1, w_1'\} = \{1,2\}$

$$\varphi_0(P_0) = P_0, \quad \varphi_0(P_{w_1}) = P_{w_1'},$$
$$\varphi_0(P_{w_1w_2\cdots w_k}) = P_{w_1'w_2\cdots w_k}$$

for any $w_2, \cdots, w_k \in \{1,2\}$. For $w_1, w_2, \cdots, w_k \in \{1,2\}$ we set

$$F_{w_1\cdots w_k} := f_{w_1\cdots w_k} \circ \cdots \circ f_{w_1w_2} \circ f_{w_1},$$

where $f_{w_1\cdots w_k}$ is the $\frac{\lambda}{2}$-similarity maps defined in the construction of $C$, $k = 1,2,3,\cdots$. Let

$$\varphi_{w_1\cdots w_k} := \begin{cases} F_{w_1\cdots w_k} \circ \varphi_0 \circ {F_{w_1\cdots w_k}}^{-1} & on\ X_{w_1\cdots w_k} \\ id & on\ X \backslash X_{w_1\cdots w_k}. \end{cases}$$

Then, $\varphi_{w_1\cdots w_k} \in Isom(X)$, and it is the reflection of $X$ at $P_{w_1\cdots w_k}$.
Namely, that satisfies

$$\begin{aligned} &\varphi_{w_1\cdots w_k}(P_{w_1\cdots w_k}) = P_{w_1\cdots w_k}, \\ &\varphi_{w_1\cdots w_k}(P_{w_1\cdots w_k w_{k+1}}) = P_{w_1\cdots w_k w'_{k+1}}, \\ &\varphi_{w_1\cdots w_k}(P_{w_1\cdots w_k w_{k+1}\cdots w_l}) = P_{w_1\cdots w_k {w_{k+1}}'\cdots w_l}, \end{aligned}$$

where $\{w_{k+1}, w'_{k+1}\} = \{1,2\}$.

For any $x, y \in C$, we take sequences $\{P_{w_1\cdots w_k}\}$ and $\{P_{v_1\cdots v_k}\}$ such that

$$\begin{aligned} x &= \lim_{k\to\infty} P_{w_1\cdots w_k}, \\ y &= \lim_{k\to\infty} P_{v_1\cdots v_k}. \end{aligned}$$

Put

$$I = \{\ i \in \mathbb{N} \mid w_i \neq v_i\ \} = \{\ i_1 < i_2 < \cdots < i_k \cdots\}.$$

Then, for any $i \in I$, we have $v_i = w'_i$ because $\{w_i, w'_i\} = \{1,2\}$. Also we have $v_i = w_i$ for any $i \in \mathbb{N} - I$. Then we can let

$$\begin{aligned} x &= \lim_{k\to\infty} P_{w_1\cdots w_{i_1-1}w_{i_1}\cdots w_{i_2}\cdots w_{i_k}}, \\ y &= \lim_{k\to\infty} P_{v_1\cdots v_{i_1-1}v_{i_1}\cdots v_{i_2}\cdots v_{i_k}} \\ &= \lim_{k\to\infty} P_{w_1\cdots w_{i_1-1}w'_{i_1}\cdots w'_{i_2}\cdots w'_{i_k}}. \end{aligned}$$

We set

$$y_k = \varphi_{w_1\cdots w'_{i_1}\cdots w'_{i_2}\cdots w'_{i_k-1}} \circ \cdots \circ \varphi_{w_1\cdots w'_{i_1}\cdots w_{i_2-1}} \circ \varphi_{w_1\cdots w_{i_1-1}}(x).$$

Then by the definition of $\varphi_{w_1\cdots w_k}$, we have

$$y = \lim_{k\to\infty} y_k \tag{5.1.2}$$

We set

$$g_k = \varphi_{w_1\cdots w'_{i_1}\cdots w'_{i_2}\cdots w'_{i_k-1}} \circ \cdots \circ \varphi_{w_1\cdots w'_{i_1}\cdots w_{i_2-1}} \circ \varphi_{w_1\cdots w_{i_1-1}},$$

and define $g \in Isom(X)$ by $g = \lim\limits_{k\to\infty} g_k$. Namely,

$$g(z) = \lim_{k\to\infty} g_k(z),$$

for any $z \in X$. Then (5.1.2) means $g(x) = y$. This completes the proof. □

## 5.2 Properties

In this section, we will prove that the tree $X$ is a doubling metric space. Note that $X$ is the tree constructed in Section 5.1.

Indeed, $X$ satisfies the following property.

**Lemma 5.2.1.** *Let $X$ be the tree defined as above, and let $\mu$ be the measure determined by length. Then, for any $x_0 \in X$, $r > 0$, and $0 < \delta < 1.$,*

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, \delta r))} \leq c_\lambda(\delta),$$

*where $c_\lambda(\delta) > 0$ is a constant depended only on $\delta$ and $\lambda$.*

*Proof.* Let $\partial B_r := \partial B(x_0, r)$, $\mu_r := \mu\big(B(x_0, r)\big)$, and $a_n := d(P_{w_1 \cdots w_{n-1}}, P_{w_1 \cdots w_n}) = (\frac{\lambda}{2})^n$. Then we have

$$\begin{aligned} \mu(X_{w_1 \cdots w_k}) &= 2\frac{a_{k+1}}{1-\lambda}, \\ d(P_{w_1 \cdots w_k}, C) &= \frac{a_{k+1}}{1-\frac{\lambda}{2}}. \end{aligned}$$

Note that

$$d(P_{k+1}, C) < d(P_k, P_{k+1}),$$

and there some integers $k, l, n$ satisfy $k < l < n$.

Let

$$\begin{aligned} &\partial B_r \cap (\ P_{w_1 \cdots w_{k-1}},\ \ P_{w_1 \cdots w_k}\ ] \neq \emptyset, && \alpha := d\big(\partial B_r,\ P_{w_1 \cdots w_k}\big), \\ &\partial B_{\delta r} \cap (\ P_{w_1 \cdots w_{l-1}},\ \ P_{w_1 \cdots w_l}\ ] \neq \emptyset, && \beta := d\big(\partial B_{\delta r},\ P_{w_1 \cdots w_l}\big). \end{aligned}$$

To prove the lemma, we consider the following three **Cases(1)(2)(3)**.

**Case (1).** The center $x_0$ of the balls is on the boundary of the tree ($x_0 \in C$).

Then we have

$$\begin{aligned} r &= \alpha + d(P_{w_1 \cdots w_k},\ x_0) = \alpha + \frac{a_{k+1}}{1-\frac{\lambda}{2}}, && \alpha \in [\ 0,\ a_k\ ), \\ \delta r &= \beta + d(P_{w_1 \cdots w_l},\ x_0) = \beta + \frac{a_{l+1}}{1-\frac{\lambda}{2}}, && \beta \in [\ 0,\ a_l\ ), \end{aligned}$$

and hence

$$\delta = \frac{\delta r}{r} = \frac{\beta + \frac{a_{l+1}}{1-\frac{\lambda}{2}}}{\alpha + \frac{a_{k+1}}{1-\frac{\lambda}{2}}} = \frac{\beta(1-\frac{\lambda}{2}) + a_{l+1}}{\alpha(1-\frac{\lambda}{2}) + a_{k+1}} < \frac{\beta + a_{l+1}}{\alpha(1-\lambda) + a_{k+1}},$$

which gives

$$\alpha(1-\lambda) + a_{k+1} < \frac{1}{\delta}(\beta + a_{l+1}). \tag{5.2.3}$$

Furthermore, we have

$$\begin{cases} \mu_r = 2\alpha + \frac{a_{k+1}}{1-\lambda}, & if\ \alpha \in [0, a_{k+1}], \\ \mu_r \le \alpha + 2\frac{a_{k+1}}{1-\lambda}, & if\ \alpha \in (a_{k+1}, a_k), \end{cases}$$

and so for any $0 \le \alpha < a_k$

$$\mu_r < 2\alpha + 2\frac{a_{k+1}}{1-\lambda}. \tag{5.2.4}$$

Similarly, we see that

$$\begin{cases} \mu_{\delta r} \ge 2\beta + \frac{a_{l+1}}{1-\lambda}, & if\ \beta \in [0, d(p_{w_1\cdots w_l}, C)], \\ \mu_{\delta r} = \beta + 2\frac{a_{l+1}}{1-\lambda}, & if\ \beta \in (d(p_{w_1\cdots w_l}, C), a_l), \end{cases}$$

and so for any $0 \le \beta < a_l$

$$\mu_{\delta r} > \beta + \frac{a_{l+1}}{1-\lambda}. \tag{5.2.5}$$

Finally, using the inequalities (5.2.3), (5.2.4), (5.2.5), we have

$$\begin{aligned} \frac{\mu_r}{\mu_{\delta r}} &< \frac{2\alpha + 2\frac{a_{k+1}}{1-\lambda}}{\beta + \frac{a_{l+1}}{1-\lambda}} = 2 \cdot \frac{\alpha(1-\lambda) + a_{k+1}}{\beta(1-\lambda) + a_{l+1}} \\ &< 2 \cdot \frac{\frac{1}{\delta}(\beta + a_{l+1})}{\beta(1-\lambda) + a_{l+1}} = 2 \cdot \frac{1}{\delta} \cdot \frac{\beta + a_{l+1}}{\beta(1-\lambda) + a_{l+1}} \\ &= 2 \cdot \frac{1}{\delta}\Big(\frac{\beta}{\beta(1-\lambda) + a_{l+1}} + \frac{a_{l+1}}{\beta(1-\lambda) + a_{l+1}}\Big) \\ &< 2 \cdot \frac{1}{\delta}\Big(\frac{1}{1-\lambda} + 1\Big) = 2 \cdot \frac{2-\lambda}{1-\lambda} \cdot \frac{1}{\delta}. \end{aligned}$$

Thus, we can take a constant $c_\lambda(\delta)$ as

$$c_\lambda(\delta) \ge 2 \cdot \frac{2-\lambda}{1-\lambda} \cdot \frac{1}{\delta}.$$

**Case (2).** The center $x_0$ of the balls coincides with some vertex $P_{w_1\cdots w_n}$ $(n < \infty)$ of the tree $(x_0 = P_{w_1\cdots w_n})$.

Then we have

$$r = \alpha + d(P_{w_1\cdots w_k},\ P_{w_1\cdots w_n}) = \alpha + \frac{a_{k+1} - a_{n+1}}{1-\frac{\lambda}{2}}, \qquad \alpha \in [\ 0,\ a_k\ \big),$$
$$\delta r = \beta + d(P_{w_1\cdots w_l},\ P_{w_1\cdots w_n}) = \beta + \frac{a_{l+1} - a_{n+1}}{1-\frac{\lambda}{2}}, \qquad \beta \in [\ 0,\ a_l\ \big),$$

and hence

$$\delta = \frac{\delta r}{r} = \frac{\beta + \frac{a_{l+1}-a_{n+1}}{1-\frac{\lambda}{2}}}{\alpha + \frac{a_{k+1}-a_{n+1}}{1-\frac{\lambda}{2}}} = \frac{\beta(1-\frac{\lambda}{2}) + a_{l+1} - a_{n+1}}{\alpha(1-\frac{\lambda}{2}) + a_{k+1} - a_{n+1}}$$
$$< \frac{\beta + a_{l+1} - a_{n+1}}{\alpha(1-\lambda) + a_{k+1} - a_{n+1}},$$

which gives

$$\alpha(1-\lambda) + a_{k+1} < \frac{1}{\delta}(\beta + a_{l+1}), \tag{5.2.6}$$

where we use the fact that $\delta < 1$.
Furthermore, since $d(x_0, C) < r$, we have

$$\begin{cases} \mu_r = 2\alpha + \frac{a_{k+1}}{1-\lambda}, & if\ \alpha \in [0, a_{k+1}], \\ \mu_r \le \alpha + 2\frac{a_{k+1}}{1-\lambda}, & if\ \alpha \in (a_{k+1}, a_k), \end{cases}$$

and so for any $0 \le \alpha < a_k$

$$\mu_r < 2\alpha + 2\frac{a_{k+1}}{1-\lambda}. \tag{5.2.7}$$

Similarly, we see that

$$\begin{cases} \mu_{\delta r} \ge 2\beta + \frac{a_{l+1}}{1-\lambda}, & if\ \beta \in [0, d(p_{w_1\cdots w_l}, C)], \\ \mu_{\delta r} = \beta + 2\frac{a_{l+1}}{1-\lambda}, & if\ \beta \in (d(p_{w_1\cdots w_l}, C), a_l), \end{cases}$$

and so for any $0 \le \beta < a_l$

$$\mu_{\delta r} > \beta + \frac{a_{l+1}}{1-\lambda}. \tag{5.2.8}$$

Finally, using the inequalities (5.2.6), (5.2.7), (5.2.8), we have

$$\begin{aligned}
\frac{\mu_r}{\mu_{\delta r}} &< \frac{2\alpha + 2\frac{a_{k+1}}{1-\lambda}}{\beta + \frac{a_{l+1}}{1-\lambda}} \\
&= 2 \cdot \frac{\alpha(1-\lambda) + a_{k+1}}{\beta(1-\lambda) + a_{l+1}} \\
&< 2 \cdot \frac{\frac{1}{\delta}(\beta + a_{l+1})}{\beta(1-\lambda) + a_{l+1}} \\
&= 2 \cdot \frac{1}{\delta} \cdot \frac{\beta + a_{l+1}}{\beta(1-\lambda) + a_{l+1}} \\
&= 2 \cdot \frac{1}{\delta} \Big( \frac{\beta}{\beta(1-\lambda) + a_{l+1}} + \frac{a_{l+1}}{\beta(1-\lambda) + a_{l+1}} \Big) \\
&< 2 \cdot \frac{1}{\delta} \Big( \frac{1}{1-\lambda} + 1 \Big) = 2 \cdot \frac{2-\lambda}{1-\lambda} \cdot \frac{1}{\delta}.
\end{aligned}$$

Thus, we can take a constant $c_\lambda(\delta)$ as

$$c_\lambda(\delta) \geq 2 \cdot \frac{2-\lambda}{1-\lambda} \cdot \frac{1}{\delta}.$$

**Case (3).** The center $x_0$ of the balls belongs to some open edge $(P_{w_1\cdots w_n},\ P_{w_1\cdots w_{n+1}})$ of the tree X.

In this case, we consider the three cases (**a**)(**b**)(**c**) bellow. Here we set $\alpha_0 := d(P_{w_1\cdots w_n},\ x_0)$.

(**a**). $n \geq l+1,\ n \geq 3$.

Then we have

$$r = \alpha + d(p_{w_1\cdots w_k}, p_{w_1\cdots w_n}) + \alpha_0 = \alpha + \frac{a_{k+1} - a_{n+1}}{1-\frac{\lambda}{2}} + \alpha_0, \qquad \alpha \in [\ 0,\ a_k\ ),$$

$$\delta r = \beta + d(p_{w_1\cdots w_l}, p_{w_1\cdots w_n}) + \alpha_0 = \beta + \frac{a_{l+1} - a_{n+1}}{1-\frac{\lambda}{2}} + \alpha_0, \qquad \beta \in [\ 0,\ a_l\ ),$$

and hence

$$\begin{aligned}
\delta = \frac{\delta r}{r} &= \frac{\beta + \frac{a_{l+1}-a_{n+1}}{1-\frac{\lambda}{2}} + \alpha_0}{\alpha + \frac{a_{k+1}-a_{n+1}}{1-\frac{\lambda}{2}} + \alpha_0} \\
&= \frac{\beta(1-\frac{\lambda}{2}) + a_{l+1} - a_{n+1} + \alpha_0(1-\frac{\lambda}{2})}{\alpha(1-\frac{\lambda}{2}) + a_{k+1} - a_{n+1} + \alpha_0(1-\frac{\lambda}{2})} \\
&< \frac{\beta + a_{l+1} - a_{n+1} + \alpha_0}{\alpha(1-\lambda) + a_{k+1} - a_{n+1}},
\end{aligned}$$

which gives

$$\alpha(1-\lambda)+a_{k+1}<\frac{1}{\delta}(\beta+a_{l+1}+\alpha_0). \tag{5.2.9}$$

Therefore, since $n \geq l+1$ and $d(x_0, C) < r$, we have

$$\begin{cases} \mu_r = 2\alpha + \frac{a_{k+1}}{1-\lambda}, & if\ \alpha \in [0, a_{k+1}], \\ \mu_r \leq \alpha + 2\frac{a_{k+1}}{1-\lambda}, & if\ \alpha \in (a_{k+1}, a_k), \end{cases}$$

and so for any $0 \leq \alpha < a_k$

$$\mu_r < 2\alpha + 2\frac{a_{k+1}}{1-\lambda}. \tag{5.2.10}$$

Similarly, we see that

$$\begin{cases} \mu_{\delta r} \geq 2\beta + \frac{a_{l+1}}{1-\lambda}, & if\ \beta \in [0, d(p_{w_1\cdots w_l}, C)], \\ \mu_{\delta r} = \beta + 2\frac{a_{l+1}}{1-\lambda}, & if\ \beta \in (d(p_{w_1\cdots w_l}, C), a_l), \end{cases}$$

and so for any $0 \leq \beta < a_l$

$$\mu_{\delta r} > \beta + \frac{a_{l+1}}{1-\lambda}. \tag{5.2.11}$$

Finally, using the inequalities (5.2.9), (5.2.10), (5.2.11), we have

$$\begin{aligned} \frac{\mu_r}{\mu_{\delta r}} &< \frac{2\alpha + 2\frac{a_{k+1}}{1-\lambda}}{\beta + \frac{a_{l+1}}{1-\lambda}} \\ &= 2\cdot\frac{\alpha(1-\lambda)+a_{k+1}}{\beta(1-\lambda)+a_{l+1}} \\ &< 2\cdot\frac{\frac{1}{\delta}(\beta+a_{l+1}+\alpha_0)}{\beta(1-\lambda)+a_{l+1}} \\ &= 2\cdot\frac{1}{\delta}\cdot\frac{\beta+a_{l+1}+\alpha_0}{\beta(1-\lambda)+a_{l+1}} \\ &= 2\cdot\frac{1}{\delta}\Big(\frac{\beta}{\beta(1-\lambda)+a_{l+1}}+\frac{a_{l+1}}{\beta(1-\lambda)+a_{l+1}}+\frac{\alpha_0}{\beta(1-\lambda)+a_{l+1}}\Big) \\ &< 2\cdot\frac{1}{\delta}\Big(\frac{1}{1-\lambda}+1+1\Big) = 2\cdot\frac{3-2\lambda}{1-\lambda}\cdot\frac{1}{\delta}, \end{aligned}$$

where we use the fact that $a_0 \leq a_{n+1}$ and hence $a_0 \leq a_{l+1}$.

Thus, we can take a constant $c_\lambda(\delta)$ as

$$c_\lambda(\delta) \geq 2\cdot\frac{3-2\lambda}{1-\lambda}\cdot\frac{1}{\delta}.$$

(**b**). Let

$$\partial B_r \cap (\ P_{w_1\cdots w_k}\quad P_{w_1\cdots w_{k+1}}\ ] \neq \emptyset, \qquad \alpha := d\big(\partial B_r,\ P_{w_1\cdots w_{k+1}}\big),$$
$$\partial B_{\delta r} \cap (\ P_{w_1\cdots w_l}\quad P_{w_1\cdots w_{l+1}}\ ] \neq \emptyset, \qquad \beta := d\big(\partial B_{\delta r},\ x_0\big).$$

$(b_1)$. $n = l,\ l \geq k+1$.

Then we have

$$\begin{aligned} r &= \alpha + d(p_{w_1\cdots w_{k+1}}, p_{w_1\cdots w_n}) + \alpha_0 = \alpha + \frac{a_{k+2} - a_{n+1}}{1 - \frac{\lambda}{2}} + \alpha_0, \\ \delta r &= \beta, \end{aligned}$$

where $\alpha \in [\ 0,\ a_{k+1}\ )$ and $\beta \in [\ 0,\ a_{l+1}\ )$,
and hence

$$\begin{aligned} \delta = \frac{\delta r}{r} &= \frac{\beta}{\alpha + \frac{a_{k+2} - a_{n+1}}{1 - \frac{\lambda}{2}} + \alpha_0} \\ &= \frac{\beta(1 - \frac{\lambda}{2})}{\alpha(1 - \frac{\lambda}{2}) + a_{k+2} - a_{n+1} + \alpha_0(1 - \frac{\lambda}{2})} \\ &< \frac{\beta}{\alpha(1 - \lambda) + a_{k+2} - a_{n+1}} \\ &\leq \frac{\beta}{\alpha(1 - \lambda) + a_{k+2} - r}, \end{aligned}$$

which gives

$$\alpha(1 - \lambda) + a_{k+2} \leq \frac{1}{\delta}\beta + r. \tag{5.2.12}$$

Furthermore, we have

$$\begin{cases} \mu_r = 2\alpha + \frac{a_{k+2}}{1-\lambda}, & if\ \alpha \in [0, a_{k+2}], \\ \mu_r \leq \alpha + 2\frac{a_{k+1}}{1-\lambda}, & if\ \alpha \in (a_{k+2}, a_{k+1}), \end{cases}$$

and so for any $0 \leq \alpha < a_{k+1}$

$$\mu_r < 2\alpha + 2\frac{a_{k+2}}{1 - \lambda}. \tag{5.2.13}$$

Obviously, we see that

$$\mu_{\delta r} > \beta. \tag{5.2.14}$$

Finally, using the inequalities (5.2.12), (5.2.13), (5.2.14), we have

$$\begin{aligned}\frac{\mu_r}{\mu_{\delta r}} &< \frac{2\alpha + 2\frac{a_{k+2}}{1-\lambda}}{\beta} \\ &= 2\frac{1}{1-\lambda}\frac{\alpha(1-\lambda)+a_{k+2}}{\beta} \\ &< 2\frac{1}{1-\lambda}\frac{r+\frac{1}{\delta}\beta}{\beta} \\ &= 2\frac{1}{1-\lambda}(\frac{r}{\delta r}+\frac{\frac{1}{\delta}\beta}{\beta}) \\ &= 4\cdot\frac{1}{1-\lambda}\cdot\frac{1}{\delta}.\end{aligned}$$

Thus, we can take a constant $c_\lambda(\delta)$ as

$$c_\lambda(\delta) \geq 4\cdot\frac{1}{1-\lambda}\cdot\frac{1}{\delta}.$$

$(b_2)$ $\ n=l,\ l=k+1.$

Then we have

$$\begin{aligned} r &= \alpha+\alpha_0 & \alpha &\in [\,0,\ a_{k+1}\,), \\ \delta r &= \beta, & \beta &\in [\,0,\ a_{l+1}\,), \end{aligned}$$

Therefore, we have

$$\mu_r < \frac{4r}{1-\lambda}. \tag{5.2.15}$$

Obviously, we see that

$$\mu_{\delta r} > \delta r. \tag{5.2.16}$$

Finally, using the inequalities (5.2.15), (5.2.16), we have

$$\frac{\mu_r}{\mu_{\delta r}} < 4\cdot\frac{1}{1-\lambda}\cdot\frac{1}{\delta}.$$

Thus, we can take a constant $c_\lambda(\delta)$ as

$$c_\lambda(\delta) \geq 4\cdot\frac{1}{1-\lambda}\cdot\frac{1}{\delta}.$$

(**c**). $x_0 \in (p_0, p_w)$, for some $w\in\{1,2\}$. This case we set $\beta_0 := d(x_0, P_1)$.

Let $r=\alpha$ and $\delta r=\beta$. Then we have

$$\delta=\frac{\delta r}{r}=\frac{\beta}{\alpha},$$

which gives

$$\frac{1}{\delta}=\frac{\alpha}{\beta}. \tag{5.2.17}$$

Furthermore, we have

$$\begin{cases} \mu_r \le \alpha+\beta_0+2\frac{a_2}{1-\lambda}, & if\ \alpha \ge a_2+\beta_0, \\ \mu_r < 3\alpha, & if\ \alpha < a_2+\beta_0, \end{cases}$$

and so for any $0<\alpha\le a_1$

$$\mu_r \le \frac{(3-\lambda)\alpha}{1-\lambda}. \tag{5.2.18}$$

Obviously, we see that

$$\mu_{\delta r} > \beta. \tag{5.2.19}$$

Finally, using the inequalities (5.2.17), (5.2.18), (5.2.19), we have

$$\begin{aligned} \frac{\mu_r}{\mu_{\delta r}} &< \frac{\frac{(3-\lambda)\alpha}{1-\lambda}}{\beta} \\ &= \frac{3-\lambda}{1-\lambda}\cdot\frac{\alpha}{\beta} \\ &= \frac{3-\lambda}{1-\lambda}\cdot\frac{1}{\delta} \end{aligned}$$

Thus, we can take a constant $c_\lambda(\delta)$ as

$$c_\lambda(\delta) \ge \frac{3-\lambda}{1-\lambda}\cdot\frac{1}{\delta}.$$

The proof of Lemma 5.2.1 is completed. □

By using Lemmas 5.2.1 and 1.1.5, we have the following immediately.

**Lemma 5.2.2.** *$X$ is a doubling metric space.*

## 5.3 Proofs of Propositions

In this section, we give the proofs of Propositions 5.0.14 and 5.0.15.

*Proof of Proposition 5.0.14.* For any $0<\lambda<1$, let $X_0$ be the tree constructed in Section 5.1, and $X$ the completion of $X_0$. From the construction of the boundary set $\partial X$

of $X$, $\partial X$ can be considered as a self-similar set in the sense of ([50]). Thus, by Main Theorem ([50]) and Lemma 5.2.1, we have

$$\dim_H(\partial X) = \frac{\log 2}{\log 2 - \log \lambda}.$$

We put $D(\lambda) = \frac{\log 2}{\log 2 - \log \lambda}$ for any $\lambda \in (0, 1]$. Clearly, $D(\lambda)$ is a continuous function on $(0, 1]$, is monotone increasing, and satisfies

$$\lim_{\lambda \to 0} D(\lambda) = 0, \quad D(1) = 1.$$

Hence, for any $s \in (0, 1)$, there exists a unique $\lambda \in (0, 1)$ satisfying

$$\frac{\log 2}{\log 2 - \log \lambda} = s.$$

By the construction of $X$, we have

$$L(X) = \frac{\lambda}{1 - \lambda} < \infty.$$

By the lemma 5.1.1, we see that $\partial X$ is homogeneous in $X$. The proof of Proposition 5.0.14 is completed. $\square$

Next, we prove Proposition 5.0.15.

*Proof of Proposition 5.0.15.* Let $X_\lambda$ be the tree constructed in the section 5.1, and $X$ the tree with $\lambda = 1$. Let $f : X_\lambda \to X$ be the natural bijection, such that

$$f(p_{w_1 \cdots w_n}) = p_{w_1 \cdots w_n}$$

for any $p_{w_1 \cdots w_n} \in X_\lambda$. Then, it follows $f$ is a expanding map. Namely,

$$d(f(x_\lambda), f(y_\lambda)) \geq d(x_\lambda, y_\lambda)$$

for any $x_\lambda, y_\lambda \in X_\lambda$. In particular, we see

$$d(f(x), f(y)) \geq d(x, y)$$

for any $x, y \in \partial X_\lambda$. It follows that $\dim_H(\partial X_\lambda)) \leq \dim_H(\partial X)$. Namely

$$\dim_H(\partial X) \geq \frac{\log 2}{\log 2 - \log \lambda}$$

for any $0 < \lambda < 1$. Thus we have $\dim_H \partial X \geq 1$.

The proof of $\dim_H \partial X \leq 1$ follows in the same way as the proof of Lemma 2.1.1. From the construction of the boundary set $\partial X$ of $X$, we see that $\partial X$ is a self-similar set in the sense of ([50]). By Lemma 5.1.1, it follows $\partial X$ is homogeneous in $X$. This completes the proof. $\square$

In general, we can also obtain the following.

**Corollary 5.3.1.** *For any given natural number $n$, there exists some $n$-dimensional $CAT(0)$ space $Y$ such that*

*(1)* $dim_H \partial Y = n$, *and*

*(2)* $\partial Y$ *is homogeneous in* $Y$.

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