Representations of Algebraic Supergroups

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Abstract

This thesis consists of two parts. Part I gives a construction of algebraic supergroups over a commutative ring, by using the concept of Harish-Chandra pairs. Part II studies representations of quasireductive supergroups over an arbitrary field. Quasireductive supergroups $\mathbf{G}$ form a wide class of algebraic supergroups which includes all Chevalley supergroups of classical type. We give a systematic construction of their irreducible representations in arbitrary characteristic. When $\mathbf{G}$ has what we called a distinguished parabolic subsupergroup, we prove a super-analogue of the Kempf vanishing theorem for $\mathbf{G}$, and classify the irreducible representations of $\mathbf{G}$.

Contents

Part I

Algebraic Supergroups

Chapter 1

Introduction

We work over a non-zero commutative ring $\Bbbk$. The unadorned $\otimes$ is the tensor product over $\Bbbk$.

The word "super" is used as a synonym of "graded by $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$". Ordinary objects, such as Lie/Hopf algebras, which are defined in the tensor category of $\Bbbk$-modules, given the trivial symmetry $V \otimes W \to W \otimes V$; $v \otimes w \mapsto w \otimes v$, are generalized by their super-analogues, such as Lie/Hopf superalgebras, which are defined in the tensor category of $\mathbb{Z}_2$-graded $\Bbbk$-modules, given the super-symmetry

$$V \otimes W \longrightarrow W \otimes V; \quad v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v,$$

see (I.2.1.1) for the details. Our main concern are the super-analogues of affine/algebraic groups. By saying *affine groups* (resp., *algebraic groups*), we mean, following Jantzen [16], what are formally called affine group schemes (resp., affine algebraic group schemes), and we will use analogous simpler names for their super analogues.

An *algebraic supergroup* (over $\Bbbk$) is thus a representable group-valued functor $\mathbf{G}$ defined on the category of commutative superalgebras over $\Bbbk$, such that the commutative Hopf superalgebra $\mathcal{O}(\mathbf{G})$ representing $\mathbf{G}$ is finitely generated; see [5, Chapter 11], for example. Associated with such $\mathbf{G}$ are a Lie superalgebra, $\mathrm{Lie}(\mathbf{G})$, and an algebraic group, $\mathbf{G}_{\mathrm{ev}}$. The latter is the (necessarily, representable) group-valued functor obtained from $\mathbf{G}$ by restricting the domain to the category of commutative algebras.

Important examples of algebraic supergroups over the complex number field $\mathbb{C}$ are *Chevalley* $\mathbb{C}$-*supergroups*; they are the algebraic supergroups $\mathbf{G}$ over $\mathbb{C}$ such that $\mathrm{Lie}(\mathbf{G})$ is one of the complex simple Lie superalgebras, which were classified by Kac [17]. Just as Kostant [18] once did in the classical, non-super situation, Fioresi and Gavarini constructed natural $\mathbb{Z}$-forms of the Chevalley $\mathbb{C}$-supergroups; see [9, 11, 10]. Those $\mathbb{Z}$-forms, called *Chevalley* $\mathbb{Z}$-*supergroups*, are important, and would be useful especially to study Chevalley supergroups in positive characteristic. A motivation of this paper is to make part of Fioresi and Gavarini's construction simpler and more rigorous, and we realize it by using *Harish-Chandra pairs*, as will be explained below. Their construction is parallel to the classical one; it starts with (1) proving the existence of "Chevalley basis" for each complex simple Lie superalgebra $\mathfrak{g}$, and then turns to (2) constructing from the basis a natural $\mathbb{Z}$-form, called a *Kostant superalgebra*, of $\mathcal{U}(\mathfrak{g})$. Our construction, which will be given in Part II Section 3.3, uses results from these (1) and (2), but dispenses with the following procedures, which include to choose a faithful representation of $\mathfrak{g}$ on a finite-dimensional complex supervector space including an appropriate $\mathbb{Z}$-lattice; see Remarks 3.3.3 and 3.3.8.

In this and the following paragraphs, let us suppose that $\Bbbk$ is a field of characteristic not equal 2. Even in this case, algebraic supergroups have not been studied so long as Lie supergroups. Indeed, the latter has a longer history of study founded by Kostant [19], Koszul [20] and others in the 1970's. An important result from the study is the equivalence, shown by Kostant, between the category of Lie supergroups and the category of Harish-Chandra pairs; see [5, Section 7.4], [39]. The corresponding result for algebraic supergroups, that is, the equivalence

$$\mathsf{ASG} \approx \mathsf{HCP} \tag{I.1.0.1}$$

between the category ASG of algebraic supergroups and the category HCP of Harish-Chandra pairs, was only recently proved by Carmeli and Fioresi [6] when $\Bbbk = \mathbb{C}$, and then by Masuoka [24] for an arbitrary field of characteristic not equal to 2; see [24, 13] for applications of the result. As was done for Lie supergroups, Carmeli and Fioresi define a *Harish-Chandra pair* to be a pair $(G, \mathfrak{g})$ of an algebraic group G and a finite-dimensional Lie superalgebra $\mathfrak{g}$ which satisfy some conditions (see Definition 4.2.1), and proved that the equivalence (I.1.0.1) is given by $\mathbf{G} \mapsto (\mathbf{G}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{G}))$ (see the third paragraph above). In [24], the definition of Harish-Chandra pairs and the category equivalence are given by purely Hopf algebraic terms, but they will be easily seen to be essentially the same as those in [6] and in this part; see Remarks 4.2.3 and 4.5.6.

To prove the category equivalence, the articles [6] and [24] both use the following property of $\mathcal{O}(\mathbf{G})$, which was proved in [22] and will be re-produced as Theorem 3.1.3 below: given $\mathbf{G} \in \mathsf{ASG}$, the Hopf superalgebra $\mathcal{O}(\mathbf{G})$ is *split* in the sense that there exists a counit-preserving isomorphism

$$\mathcal{O}(\mathbf{G}) \simeq \mathcal{O}(\mathbf{G}_{\mathrm{ev}}) \otimes \wedge(W) \tag{I.1.0.2}$$

of left $\mathcal{O}(\mathbf{G}_{\mathrm{ev}})$-comodule superalgebras, where W is the odd component of the cotangent supervector space of $\mathbf{G}$ at $\bar{1}$, and $\wedge(W)$ is the exterior algebra on it. This basic property played a role in [28] as well; see also [25]. As another application of the property we will prove a representation-theoretic result, Part II, Corollary 2.4.10, which generalizes results which were proved in [3, 4, 34] for some special algebraic supergroups, see Part II.

Throughout in this part we mainly assume that $\Bbbk$ is a non-zero commutative ring which is 2-*torsion free*, or namely, is such that an element $a \in \Bbbk$ must be zero whenever $2a = 0$. We pose this assumption because it seems natural, in order to keep the super-symmetry (I.2.1.1) non-trivial. Theorem 4.5.1, proves the category equivalence (I.1.0.1) over such $\Bbbk$ as above. We pose some assumptions to objects in the relevant categories, which are necessarily satisfied if $\Bbbk$ is a field. Indeed, an algebraic supergroup $\mathbf{G}$ in ASG is required to satisfy, in particular, the condition that $\mathcal{O}(\mathbf{G})$ is split, while an object $(G, \mathfrak{g})$ in HCP is required to satisfy, in particular, the condition that $\mathfrak{g}$ is *admissible* (see Definition 2.5.2), and so, given an odd element $v \in \mathfrak{g}_{\bar{1}}$, the even component $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ must contain a unique element, $\frac{1}{2}[v, v]$, whose double equals $[v, v]$; see Section 4.3 and Definition 4.2.1 for the precise definitions of ASG and HCP, respectively. A novelty of our proof of the result is to construct a functor $\mathbf{G} : \mathsf{HCP} \to \mathsf{ASG}$, which will be proved an equivalence, as follows; given $(G, \mathfrak{g}) \in \mathsf{HCP}$, we realize the Hopf superalgebra $\mathcal{O}(\mathbf{G})$ corresponding to $\mathbf{G} = \mathbf{G}(G, \mathfrak{g})$ as a discrete Hopf super-subalgebra of some complete topological Hopf superalgebra, $\widehat{\mathcal{A}}$, that is simply constructed from the given pair. Indeed, this Hopf algebraic idea was used in [24], but our construction has been modified as to be applicable when $\Bbbk$ is a commutative ring. Based on the proved equivalence we will re-construct the Chevalley $\mathbb{Z}$-supergroups, by giving the corresponding Harish-Chandra pairs.

This part is organized as follows. In Chapter 2, we give necessary definitions and notations of super-objects. Chapter 3 is devoted to preliminaries on affine/algebraic supergroups and its Lie superalgebras. The category equivalence theorem, Theorem 4.5.1, is proved in Chapter 4, while the Chevalley $\mathbb{Z}$-supergroups will be re-constructed in Part II, Section 3.3. We prove in Corollary 4.1.3 that the universal envelope $\mathcal{U}(\mathfrak{g})$ of such a Lie superalgebra $\mathfrak{g}$ has the property which is dual to the splitting property (I.1.0.2); the corollary plays a role when we prove Theorem 4.5.1. After an earlier version of the paper [27] was submitted, the article [12] by Gavarini was in circulation. Theorem 4.3.14 of [12] essentially proves our category equivalence theorem in the generalized situation that $\Bbbk$ is an arbitrary commutative ring. A point is to use the additional structure, called 2-*operations*, on Lie superalgebras $\mathfrak{g}$, which generalizes the map $\mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$; $v \mapsto \frac{1}{2}[v, v]$ given on an admissible Lie superalgebra in our situation. Given a Harish-Chandra pair, Gavarini constructs an affine supergroup in a quite different method from ours, realizing it as a group valued functor. In Section 4.6, we will refine his category equivalence, using our construction and giving detailed arguments on 2-operations, in particular. This would not be meaningless because such detailed arguments are not be given in [12]; see Remark 4.6.10. Chapter 5 is devoted to give a simpler and more conceptual presentation of Gavarini's original construction ([12]). Section 5.3 starts with the subsection in which we re-prove Gavarini's category equivalence cited above, using our method of construction. This aims to supplement again Gavarini's original proof; see Remark 5.3.6. In Section 5.4, we suppose that $\Bbbk$ is a field of characteristic not equal to 2. As an application of our construction, given an algebraic supergroup $\mathbf{G}$ and its closed subsupergroup $\mathbf{H}$, we describe the normalizer $\mathcal{N}_{\mathbf{G}}(\mathbf{H})$ and the centralizer $\mathcal{Z}_{\mathbf{G}}(\mathbf{H})$ in terms of Harish-Chandra pairs; see Theorem 5.4.3.

Chapter 2

Preliminaries

Let $\Bbbk$ be a base commutative ring with 1. We assume that $\Bbbk$ is 2-*torsion free*, i.e., $2 : \Bbbk \to \Bbbk$; $c \mapsto 2c$ is injective. The unadorned $\otimes$ denotes the tensor product over $\Bbbk$. A module over $\Bbbk$ is said to be *finite* (resp. *flat, free*) if it is finitely generated (resp. flat, free) as $\Bbbk$-module.

2.1 Super-objects

Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the group of order 2. The group algebra $\Bbbk\mathbb{Z}_2$ of $\mathbb{Z}_2$ over $\Bbbk$ forms a Hopf algebra by the following coalgebra structures

$$\Delta(\epsilon) = \epsilon \otimes \epsilon, \quad \varepsilon(\epsilon) = 1,$$

where $\epsilon \in \mathbb{Z}_2$. Note that the antipode is given by $\mathcal{S}(\epsilon) = \epsilon$ for each $\epsilon \in \mathbb{Z}_2$. We let SMod denote the category of right $\Bbbk\mathbb{Z}_2$-comodules.

An object in SMod is called a *supermodule* over $\Bbbk$. In other words, a supermodule is a $\mathbb{Z}_2$-graded module over $\Bbbk$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a supermodule. For a homogeneous element $v \in V_\epsilon$ with $\epsilon \in \mathbb{Z}_2$, we denote its *parity* by $|v| := \epsilon$. Unless otherwise stated, an element of a supermodule is always assumed to be homogeneous. Let $\mathrm{Hom}_\Bbbk(V, W)$ be the set of all morphisms f from V to W in SMod. By definition, f satisfies $f(V_\epsilon) \subseteq W_\epsilon$ for each $\epsilon \in \mathbb{Z}_2$. An element of $\mathrm{Hom}_\Bbbk(V, W)$ is called a *parity preserving* $\Bbbk$-linear map.

We define a supermodule ΠV so that $(\Pi V)_\epsilon := V_{\epsilon+\bar{1}}$ for each $\epsilon \in \mathbb{Z}_2$. For a morphism $f : V \to W$ in SMod, we define a morphism $\Pi f : \Pi V \to \Pi W$ in SMod so that $\Pi f := f$. In this way, we get a functor

$$\Pi : \mathsf{SMod} \longrightarrow \mathsf{SMod},$$

called the *parity change functor*. For supermodules V, W, we define a supermodule $\underline{\mathrm{Hom}}_\Bbbk(V, W)$ by letting

$$\underline{\mathrm{Hom}}_\Bbbk(V, W)_{\bar{0}} := \mathrm{Hom}_\Bbbk(V, W), \qquad \underline{\mathrm{Hom}}_\Bbbk(V, W)_{\bar{1}} := \mathrm{Hom}_\Bbbk(V, \Pi W).$$

An element of $\underline{\mathrm{Hom}}_\Bbbk(V, W)$ is called a *homogeneous* $\Bbbk$-linear map. Given a supermodule V, we let V^* denote its $\Bbbk$-linear dual $\underline{\mathrm{Hom}}_\Bbbk(V, \Bbbk)$, called the *dual of* V. By definition, $(V^*)_\epsilon = (V_\epsilon)^*$ for $\epsilon \in \mathbb{Z}_2$.

For supermodules $V, W \in \mathsf{SMod}$, we define an object $V \otimes W$ in SMod as follows

$$(V \otimes W)_\epsilon := \bigoplus_{\substack{\epsilon', \epsilon'' \in \mathbb{Z}_2 \\ \epsilon' + \epsilon'' = \epsilon}} V_{\epsilon'} \otimes W_{\epsilon''},$$

where $\epsilon \in \mathbb{Z}_2$. Then the category SMod forms a tensor category with a unit object $\Bbbk$ which we regard as a purely even object $\Bbbk = \Bbbk \oplus 0$. The tensor category SMod is symmetric with respect to the *supersymmetry*

$$c_{V,W} : V \otimes W \longrightarrow W \otimes V; \quad v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v \tag{I.2.1.1}$$

for $V, W \in \mathsf{SMod}$ (see also [22, §2]). Here and in what follows, an element v in V is regarded as a homogeneous element of V. In this way, the category SMod forms a rigid symmetric tensor category.

A *superalgebra* (resp. *supercoalgebra, Hopf superalgebra* etc.) is defined to be an algebra (resp. coalgebra, Hopf algebra etc.) object in SMod. For example, a superalgebra R is nothing but a $\mathbb{Z}_2$-graded algebra. Note that the even component $R_{\bar{0}}$ of R is an ordinary algebra. The ordinary objects are regarded as *purely even* super-objects.

For a superalgebra R, a $\Bbbk$-subsupermodule I of R is called a *left (resp. right) super-ideal* if it satisfies $IR \subseteq R$ (resp. $RI \subseteq R$). A *two-sided super-ideal* is both a right super-ideal and a left super-ideal, as usual. By definition, a superalgebra R is a right $\Bbbk\mathbb{Z}_2$-comodule algebra. The comodule structure $\rho : R \to R \otimes \Bbbk\mathbb{Z}_2$ satisfies $a \in R_\epsilon$ if and only if $\rho(a) = a \otimes \epsilon$ for a fixed $\epsilon \in \mathbb{Z}_2$. A superalgebra R is said to be *commutative* if $ab = (-1)^{|a||b|} ba$ for all $a, b \in R$.

Example 2.1.1. Let $R := \Bbbk[T_1, \dots, T_n; \xi_1, \dots, \xi_m]$, where the $T_1, \dots, T_n$ are ordinary indeterminates and the $\xi_1, \dots, \xi_m$ are *odd indeterminates* which satisfy $\xi_i \xi_j = -\xi_j \xi_i$ for $1 \leqslant i, j \leqslant m$. Since $\Bbbk$ is 2-torsion free, we have $\xi_i^2 = 0$ for each $1 \leqslant i \leqslant m$. This R forms a commutative superalgebra. For $1 \leqslant r \leqslant m$, we denote $I_r := \{i_1, \dots, i_r \mid 1 \leqslant i_1 < \cdots < i_r \leqslant m\}$ and $\xi_{I_r} := \xi_{i_1} \cdots \xi_{i_r}$. Then we have

$$R_{\bar{0}} \text{ (resp., } R_{\bar{1}}) \;=\; \{ \sum_{r:\text{ even (resp., odd)}} f_{I_r} \xi_{I_r} \mid f_{I_r} \in \Bbbk[T_{i_1}, \dots, T_{i_r}],\ I_r \subseteq \{1, \dots, m\} \}.$$

Here, we treat the integer 0 as an even number. Let $\wedge(\xi_1, \dots, \xi_m)$ be the exterior algebra over $\Bbbk$ generated by $\xi_1, \dots, \xi_m$. This forms a commutative superalgebra in the obvious way. There is an isomorphism of superalgebras:

$$\Bbbk[T_1, \dots, T_n] \otimes \wedge(\xi_1, \dots, \xi_m) \xrightarrow{\simeq} R.$$

Note that, the non-zero elements $\xi_i \xi_j$ ($\in R_{\bar{0}}$) for $i \neq j$ are nilpotent.

Example 2.1.2. For $n, m \geqslant 0$, we let $A(m|n)$ denote the commutative superalgebra over $\Bbbk$ generated by $\{x_{ij}\}_{1\leqslant i,j\leqslant m+n}$ satisfying the relations

$$x_{ij}x_{k\ell} = \begin{cases} -x_{k\ell}x_{ij} & \text{for } i,k \leqslant m < j,\ell \text{ or } j,\ell \leqslant m < i,k, \\ x_{k\ell}x_{ij} & \text{otherwise.} \end{cases}$$

Moreover, $A(m|n)$ forms a commutative superbialgebra by letting

$$\Delta(x_{ij}) := \sum_{k=1}^{m+n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) := \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta.

Let C be a flat supercoalgebra (i.e., its underlying supermodule is flat). If D is a subsupermodule of C, then D is also flat and we have canonical injections $D^{\otimes n} \to C^{\otimes n}$ for each $n \geqslant 1$. Thus we say D is a *subsupercoalgebra* if $\Delta(D)$ is in $D \otimes D$, where Δ is the comultiplication of C. A supercoalgebra C is said to be *cocommutative* if $\Delta(c) = (-1)^{|c_1||c_2|} c_2 \otimes c_1$ for all $c \in C$.

Given a Hopf superalgebra A, we denote that comultiplication, the counit and the antipode of A by

$$\Delta : A \to A \otimes A, \quad \varepsilon : A \to \Bbbk, \quad \mathcal{S} : A \to A.$$

We use the "Heyneman-Sweedler notation" $\Delta(a) = a_1 \otimes a_2$ for $a \in A$, as usual.

A two-sided super-ideal I of A is called a *Hopf super-ideal* A if it satisfies $\Delta(I) \subseteq A \otimes I + I \otimes A$, $\varepsilon(I) = 0$ and $\mathcal{S}(I) \subseteq I$.

Example 2.1.3. For a supermodule V, we let $T^0(V) := \Bbbk$, $T^n(V) := V^{\otimes n}$ for each $n \geqslant 1$. Then $T(V) := \bigoplus_{n\geqslant 0} T^n(V)$ is $\mathbb{N}$-graded, and hence is $\mathbb{Z}_2$-graded. Explicitly, $T(V)_\epsilon := \bigoplus_{n\geqslant 0} T^{2n+\epsilon}(V)$ for $\epsilon \in \mathbb{Z}_2$. This uniquely forms a Hopf superalgebra by letting

$$\Delta(v) = v \otimes 1 + 1 \otimes v,$$

for each (homogeneous) $v \in V$. The antipode is given by $\mathcal{S}(v) = -v$ for $v \in V$. By definition, $T(V)$ is cocommutative. For an explicit description of the coproduct, see [24, Remark 3].

2.2 Supermodules

Let R be a superalgebra with the multiplication $m : R \otimes R \to R$ and the unit $u : \Bbbk \to R$. A *left R-supermodule* M is a supermodule endowed with a structure map $\alpha \in \mathrm{Hom}_{\Bbbk}(R \otimes M, M)$ satisfying the following familiar commutative diagrams in SMod

$$\begin{array}{ccc} R \otimes R \otimes M & \xrightarrow{m \otimes \mathrm{id}_M} & R \otimes M \\ {\scriptstyle \mathrm{id}_R \otimes \alpha}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \alpha} \\ R \otimes M & \xrightarrow[\alpha]{} & M, \end{array} \qquad \begin{array}{ccc} \Bbbk \otimes M & \xrightarrow{u \otimes \mathrm{id}_M} & R \otimes M \\ & \searrow^{\cong} \ \circlearrowleft & \downarrow{\scriptstyle \alpha} \\ & & M. \end{array}$$

Let $a.m := \alpha(a \otimes m)$ for each $a \in R$, $m \in M$. For two left R-supermodules M and N, we denote the set of all (parity preserving) left R-supermodule maps by $\mathrm{Hom}_R(M, N)$. Let $\underline{\mathrm{Hom}}_R(M, N)$ be the supermodule of the form $\mathrm{Hom}_R(M, N) \oplus \mathrm{Hom}_R(M, \Pi N)$, as before. We denote $\underline{\mathrm{Hom}}_R(M, M)$ by $\underline{\mathrm{End}}_R(M)$. Let ${}_R\mathsf{SMod}$ denote the category of left R-supermodules. Similarly, we define the notion of *right R-supermodules*. For a left R-supermodule M, we define

$$m.a := (-1)^{|a||m|} a.m \tag{I.2.2.1}$$

for $a \in R$, $m \in M$. By this structure, M forms a right A-supermodule.

For a left (resp. right) R-supermodule (M, α), the dual supermodule M^* is naturally regarded as a right (resp. left) R-supermodule by the following formula

$$\langle f.a, m\rangle := \langle f, a.m\rangle \quad (\text{resp. } \langle a.f, m\rangle := \langle f, m.a\rangle),$$

for $a \in R$, $f \in M^*$ and $m \in M$. Here $\langle\ ,\ \rangle \in \mathrm{Hom}_{\Bbbk}(M^* \otimes M, \Bbbk)$ is the canonical evaluation map.

The category of all left R-supermodules ${}_R\mathsf{SMod}$ is nothing but the category of left R- right $\Bbbk\mathbb{Z}_2$-Hopf modules ${}_R\mathsf{Mod}^{\Bbbk\mathbb{Z}_2}$. Since there is a unique Hopf algebra isomorphism $\Bbbk\mathbb{Z}_2 \cong (\Bbbk\mathbb{Z}_2)^*$, we have an equivalence

$${}_R\mathsf{SMod} \xrightarrow{\approx} {}_{R\rtimes\Bbbk\mathbb{Z}_2}\mathsf{Mod}\,; \quad M \longmapsto M. \tag{I.2.2.2}$$

For a R-supermodule M, the corresponding $R \rtimes \Bbbk\mathbb{Z}_2$-module structure on M is given by

$$(a \rtimes \epsilon).m := a(m_{\bar{0}} + (-1)^{\epsilon} m_{\bar{1}})$$

for $a \in R$, $\epsilon \in \mathbb{Z}_2$ and $m = m_{\bar{0}} + m_{\bar{1}}$ with $m_{\bar{0}} \in M_{\bar{0}}$ and $m_{\bar{1}} \in M_{\bar{1}}$.

A non-zero superalgebra is said to be *simple* if it has no non-trivial super-ideal.

Proposition 2.2.1. *Assume that $\Bbbk$ is a field. If R is a simple superalgebra with $R_{\bar{1}} \neq 0$, then $R_{\bar{0}}$ is Morita equivalent to $R \rtimes \Bbbk\mathbb{Z}_2$.*

Proof. Let $H := \Bbbk\mathbb{Z}_2$ and let $B := R^{\mathrm{co}H}$, for simplicity. Here $R^{\mathrm{co}H} := \{a \in R \mid \rho(a) = a \otimes \bar{0}\}$, where $\rho : R \to R \otimes \Bbbk\mathbb{Z}_2$ is the $\Bbbk\mathbb{Z}_2$-comodule structure map of R. Thus B coincides with $R_{\bar{0}}$. Since $R_{\bar{1}}R_{\bar{1}} \oplus R_{\bar{1}}$ is a non-zero super-ideal of R, we have $R_{\bar{1}}R_{\bar{1}} = R_{\bar{0}}$. Therefore, the $\mathbb{Z}_2$-grading of R is *strongly graded*. This is equivalent to saying that R/B is a *right H-Galois*, see [7]. Then by the theory of Hopf-Galois extension, the following is an equivalence

$${}_B\mathsf{Mod} \xrightarrow{\approx} {}_R\mathsf{Mod}^H\,; \quad V \longmapsto R \otimes_B V. \tag{I.2.2.3}$$

Combine the equivalence above with (I.2.2.2), the claim follows. □

2.3 Pairings

Let R be a superalgebra. Given R-supermodules M, N, we let $M \otimes_R N$ denote the quotient $\Bbbk$-supermodule of $M \otimes N$ defined by the relations

$$ma \otimes n = m \otimes an$$

for $a \in R$, $m \in M$, $n \in N$. This is naturally an R-supermodule. The category of R-supermodules ${}_R\mathsf{SMod}$ forms a symmetric tensor category, where the tensor product is the $\otimes_R$ just defined above, and the unit object is R. The symmetry is the one induced from the supersymmetry $c_{-,-}$ (see (I.2.1.1)), and it will be denoted by the same symbol.

A Hopf superalgebra over R is a Hopf-algebra object in ${}_R\mathsf{SMod}$. The structure maps of a Hopf superalgebra $\mathcal{A}$ over R will be denoted by

$$\Delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes_R \mathcal{A}, \quad \varepsilon_{\mathcal{A}} : \mathcal{A} \to R, \quad \mathcal{S}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}.$$

We use the notation $\Delta_{\mathcal{A}}(a) = a_1 \otimes_R a_2$ for $a \in \mathcal{A}$, as before.

A *pairing* between objects M and N in ${}_R\mathsf{SMod}$ is a morphism $M \otimes_R N \to R$ in ${}_R\mathsf{SMod}$, which will be often presented as

$$\langle\ ,\ \rangle : M \times N \to R, \qquad \langle m, n\rangle = \text{ the value of } m \otimes_R n.$$

The *tensor product* with another pairing $\langle\, ,\, \rangle : M' \otimes_R N' \to R$ is the pairing between $M \otimes_R M'$ and $N \otimes_R N'$ which is defined to be the composite

$$(M \otimes_R M') \otimes_R (N \otimes_R N') \overset{\mathrm{id}_M \otimes_R c_{M',N} \otimes_R \mathrm{id}_{N'}}{\longrightarrow} (M \otimes_R N) \otimes_R (M' \otimes_R N') \overset{\langle\, ,\rangle \otimes_R \langle\, ,\rangle}{\longrightarrow} R \otimes_R R \ \cong R.$$

Explicitly, it is defined by

$$\langle m \otimes_R m',\, n \otimes_R n'\rangle = (-1)^{|m'||n|} \langle m,\, n\rangle\, \langle m',\, n'\rangle, \tag{I.2.3.1}$$

where $m \in M$, $m' \in M'$, $n \in N$, $n' \in N'$.

Remark 2.3.1. If $R = \Bbbk$, then the sign $(-1)^{|m'||n|}$ above can be replaced by either $(-1)^{|m||n|}$, $(-1)^{|m'||n'|}$ or $(-1)^{|m||n'|}$.

Definition 2.3.2. Let $\mathcal{A}$, $\mathcal{B}$ be Hopf superalgebras over R. A pairing $\langle\, ,\, \rangle : \mathcal{A} \times \mathcal{B} \to R$ is called a *Hopf pairing*, if we have

$$\langle x,\, hk\rangle = \langle \Delta_{\mathcal{A}}(x),\, h \otimes_R k\rangle, \quad \langle xy,\, h\rangle = \langle x \otimes_R y,\, \Delta_{\mathcal{B}}(h)\rangle, \tag{I.2.3.2}$$

$$\langle x,\, 1\rangle = \varepsilon_{\mathcal{A}}(x), \quad \langle 1,\, h\rangle = \varepsilon_{\mathcal{B}}(h), \tag{I.2.3.3}$$

where $x, y \in \mathcal{A}$, $h, k \in \mathcal{B}$. On the right-hand sides of (I.2.3.2) appears the tensor product of two copies of the pairing.

One sees that the conditions imply

$$\langle \mathcal{S}_{\mathcal{A}}(x),\, h\rangle = \langle x,\, \mathcal{S}_{\mathcal{B}}(h)\rangle \tag{I.2.3.4}$$

for $x \in \mathcal{A}$, $h \in \mathcal{B}$.

Just as in the non-super situation, the set

$$\mathsf{Gpl}(\mathcal{A}) := \{g \in \mathcal{A}_{\bar{0}} \mid \Delta_{\mathcal{A}}(g) = g \otimes_R g,\ \varepsilon_{\mathcal{A}}(g) = 1\}$$

of all *even group-likes* in $\mathcal{A}$ forms a group under the multiplication of $\mathcal{A}$. On the other hand, the set

$$\mathsf{SAlg}_R(\mathcal{B}, R)$$

of all superalgebra maps $\mathcal{B} \to R$ over R is a group under the *convolution product* $*$. Namely, for $f, f' \in \mathsf{SAlg}_R(\mathcal{B}, R)$ we define $f * f' \in \mathsf{SAlg}_R(\mathcal{B}, R)$ as follows

$$f * f' : \mathcal{B} \xrightarrow{\Delta_{\mathcal{B}}} \mathcal{B} \otimes_R \mathcal{B} \xrightarrow{f \otimes_R f'} R \otimes_R R \ \cong R. \tag{I.2.3.5}$$

Lemma 2.3.3. *A Hopf pairing* $\langle\,,\,\rangle : \mathcal{A} \times \mathcal{B} \to R$ *induces the group map*

$$\mathsf{Gpl}(\mathcal{A}) \longrightarrow \mathsf{SAlg}_R(\mathcal{B}, R); \quad g \longmapsto \langle g,\, -\rangle.$$

Here is a typical example of Hopf pairings over $\Bbbk$.

Example 2.3.4 (cf. [24, Eq. (5)]). Let W be a finite and free module (over $\Bbbk$). We regard the exterior algebra $\wedge(W)$ on W as a Hopf superalgebra over $\Bbbk$ in which every element in W is an odd primitive. We have another such Hopf superalgebra $\wedge(W^*)$ over $\Bbbk$. A Hopf pairing $\langle\ ,\ \rangle$: $\wedge(W^*) \times \wedge(W) \to \Bbbk$ is defined by

$$\langle v_1 \wedge \cdots \wedge v_n,\ w_1 \wedge \cdots \wedge w_m\rangle := \delta_{n,m}\, (-1)^{n(n-1)/2} \det\big(v_i(w_j)\big)_{i,j}, \quad m, n \geq 0, \tag{I.2.3.6}$$

where $v_i \in W^*$, $w_i \in W$. Here $\delta_{n,m}$ is the Kronecker's delta.

Remark 2.3.5. In [24, 25, 29, 27], they use the simpler duality. Namely,

$$\langle v_1 \wedge \cdots \wedge v_n,\ w_1 \wedge \cdots \wedge w_m\rangle := \delta_{n,m} \det\big(v_i(w_j)\big)_{i,j}, \quad m, n \geq 0, \tag{I.2.3.7}$$

where $v_i \in W^*$, $w_i \in W$. Since this simpler duality (I.2.3.7) does not work well for Hopf superalgebras over R, in general, we have to use the duality (I.2.3.6). In Section 2.4, this circumstance is explained, and the difference caused by choices is described in terms of *cocycle deformations*.

2.4 Comparing dualities

In the situation above we suppose $R = \Bbbk$, and consider super-objects and pairings over $\Bbbk$.

Let $\mathcal{C}$ be a supercoalgebra over $\Bbbk$. Then we make the dual $\Bbbk$-supermodule $\mathcal{C}^*$ uniquely into a superalgebra so that the canonical pairing $\mathcal{C}^* \times \mathcal{C} \to \Bbbk$ satisfies the second equations of (I.2.3.2), (I.2.3.3). This is the same as saying that the canonical pairing $\mathcal{C} \times \mathcal{C}^* \to \Bbbk$ satisfies the first equations of (I.2.3.2), (I.2.3.3). The identity of $\mathcal{C}^*$ is the counit of $\mathcal{C}$, and the product is given by

$$(pq)(h) = (-1)^{|p||q|}\, p(h_1)\, q(h_2),$$

where $p, q \in \mathcal{C}^*$, $h \in \mathcal{C}$. We denote this superalgebra by $\mathcal{C}^{\bar{*}}$.

Similarly, if $\mathcal{A}$ is Hopf superalgebra over $\Bbbk$ which is finite projective, we make $\mathcal{A}^*$ uniquely into a Hopf superalgebra, so that $\mathcal{A}^* \times \mathcal{A} \to \Bbbk$ or $\mathcal{A} \times \mathcal{A}^* \to \Bbbk$ is a Hopf pairing. We also denote it by $\mathcal{A}^{\bar{*}}$. call the *dual Hopf superalgebra* of $\mathcal{A}$. Since the Hopf pairing given in Example 2.3.4 is non-degenerate, it follows that the Hopf superalgebras $\wedge(W)$ and $\wedge(W^*)$ are dual to each other.

Let

$$\langle\ ,\ \rangle : V \times W \longrightarrow \Bbbk, \qquad \langle\ ,\ \rangle : V' \times W' \longrightarrow \Bbbk$$

be pairings over $\Bbbk$. In the articles [24, 25, 27], the tensor product of pairings is supposed to be the ordinary one, just as in the non-super situation,

$$\langle v \otimes w,\ v' \otimes w' \rangle_{\mathrm{ord}} := \langle v,\, w\rangle\, \langle v',\, w'\rangle,$$

where $v \in V$, $w \in W$, $v' \in V'$, $w' \in W'$. This is justified, since it holds that $\langle\, ,\, \rangle_{\mathrm{ord}} \circ (c_{V,W} \otimes \mathrm{id}_{W' \otimes V'}) = \langle\, ,\, \rangle_{\mathrm{ord}} \circ (\mathrm{id}_{V \otimes W} \otimes c_{W',V'})$; see the proof of [24, Corollary 3].

On the other hand, over $R \in \mathsf{SAlg}_\Bbbk$ in general, this is *not* true any more. Therefore, we chose the definition as in (I.2.3.1), so that indeed, we have $\langle\, ,\, \rangle \circ (c_{M,N} \otimes_R \mathrm{id}_{N' \otimes_R M'}) = \langle\, ,\, \rangle \circ (\mathrm{id}_{M \otimes_R N} \otimes_R c_{N',M'})$, i.e.,

$$\begin{array}{ccc}
(M \otimes_R N) \otimes_R (N' \otimes_R M') & \xrightarrow{c_{M,N} \otimes_R \mathrm{id}_{N' \otimes_R M'}} & (N \otimes_R M) \otimes_R (N' \otimes_R M') \\
{\scriptstyle \mathrm{id}_{M \otimes_R N} \otimes_R c_{N',M'}} \big\downarrow & \circlearrowleft & \big\downarrow {\scriptstyle \langle , \rangle} \\
(M \otimes_R N) \otimes_R (M' \otimes_R N') & \xrightarrow[\langle , \rangle]{} & R.
\end{array}$$

Due to these different choices, the Hopf pairing given by (I.2.3.6) is different from the ordinary one given by (I.2.3.7) or [24, Eq. (5)]. Note also that, the dual (Hopf) superalgebras given above are different from those given in the cited articles. In the following, we are going to clarify this difference.

Let $\Bbbk^\times$ denote the multiplicative group of all units in $\Bbbk$, and regard it as a trivial module over the group $\mathbb{Z}_2$. Then the map $\sigma : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \Bbbk^\times$ defined by

$$\sigma(\epsilon, \eta) = (-1)^{\epsilon\eta}, \quad \epsilon, \eta \in \mathbb{Z}_2$$

satisfies

$$\sigma(\epsilon, \eta)\sigma(\delta, \epsilon + \eta) = \sigma(\delta + \epsilon, \eta)\sigma(\delta, \epsilon)$$

for $\delta, \epsilon, \eta \in \mathbb{Z}_2$. Thus σ is a 2-*cocycle*. Therefore, the identity functor

$$\mathsf{SMod} \longrightarrow \mathsf{SMod}; \quad V \longmapsto {}_\sigma V := V$$

together with the tensor structure

$$\begin{aligned} {}_\sigma V \otimes {}_\sigma W &\longrightarrow {}_\sigma(V \otimes W); \quad v \otimes w \longmapsto \sigma(|v|, |w|)\, v \otimes w, \\ \mathrm{id} : \Bbbk &\longrightarrow {}_\sigma\Bbbk \ (= \Bbbk) \end{aligned} \tag{I.2.4.1}$$

form a tensor equivalence.

Lemma 2.4.1. *The tensor functor ${}_\sigma(-)$ preserves the supersymmetry.*

Moreover, the functor ${}_\sigma(-)$ is an involution, since $\sigma(\epsilon, \eta)^2 = 1$ for $\epsilon, \eta \in \mathbb{Z}_2$. If $\Bbbk$ contains a square root $\sqrt{-1}$ of -1, then σ is the *coboundary* of the map

$$\nu : \mathbb{Z}_2 \to \Bbbk^\times, \qquad \nu(\bar{0}) := 1, \quad \nu(\bar{1}) := \sqrt{-1}.$$

Therefore,

$${}_\sigma V \longmapsto V; \quad v \longmapsto \nu(|v|)\, v$$

gives a natural isomorphism from the tensor equivalence ${}_\sigma(-)$ given by σ to the identity tensor functor.

It follows that if $\mathcal{A}$ is a super-object (e.g. a Hopf superalgebra) over $\Bbbk$, then ${}_\sigma\mathcal{A}$ is such an object, and ${}_\sigma({}_\sigma\mathcal{A})$ coincides with $\mathcal{A}$. This ${}_\sigma\mathcal{A}$ is called the (*cocycle*) *deformation* of $\mathcal{A}$ by σ; see [23, Section 1.1], for example. If $\sqrt{-1} \in \Bbbk$, then $\mathcal{A}$ and ${}_\sigma\mathcal{A}$ are naturally isomorphic.

Given two pairings over $\Bbbk$ as above, we have

$$\begin{aligned} \langle v \otimes w,\, v' \otimes w' \rangle &= \langle \sigma(|v|, |w|)\, v \otimes w,\, v' \otimes w' \rangle_{\mathrm{ord}} \\ &= \langle v \otimes w,\, \sigma(|v'|, |w'|)\, v' \otimes w' \rangle_{\mathrm{ord}}; \end{aligned}$$

see (I.2.4.1).

Therefore, the dual (Hopf) superalgebra $\mathcal{A}^{\bar{*}}$ given above coincides with the deformation ${}_\sigma(\mathcal{A}^*)$ of the one $\mathcal{A}^*$ treated in [24, 25, 27].

2.5 Lie Superalgebra

2.5.1 Admissible Lie superalgebras

A *Lie superalgebra* $\mathfrak{g}$ over $\Bbbk$ is a Lie algebra object in SMod. In other words, $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded module over $\Bbbk$ endowed with $[\ ,\ ] \in \mathrm{Hom}_\Bbbk(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$, called *super-bracket*, satisfying

(i) $[w, w] = 0$ for $w \in \mathfrak{g}_{\bar{0}}$,

(ii) $[[x, x], x] = 0$ for $x \in \mathfrak{g}_{\bar{1}}$,

(iii) $[\,,\,]\circ(\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}}+c_{\mathfrak{g},\mathfrak{g}})=0$, and

(iv) $[[\,,\,],\,]\circ(\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}}+c_{\mathfrak{g},\mathfrak{g}\otimes\mathfrak{g}}+c_{\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}})=0$,

where $c_{-,-}$ is the supersymmetry, see (I.2.1.1).

Remark 2.5.1. If $\mathfrak{g}_{\bar{1}}$ is 2-torsion free, then the condition (iv) restricted to $\mathfrak{g}_{\bar{1}}\otimes\mathfrak{g}_{\bar{1}}\otimes\mathfrak{g}_{\bar{1}}$ is automatically satisfied. Indeed, this follows by applying the condition (ii) to $x_1+x_2+x_3$ with $x_1,x_2,x_3\in\mathfrak{g}_{\bar{1}}$.

By definition, for a Lie superalgebra $\mathfrak{g}$, the even part $\mathfrak{g}_{\bar{0}}$ forms an ordinary Lie algebra. We treat a special class of Lie superalgebras. For a 2-torsion free module V, an element $v\in V$ is said to be 2-*divisible* if there exists $w\in V$ such that $v=2w$. Since such a w is uniquely determined, we will denote w by $\frac{1}{2}v$.

Definition 2.5.2 ([27, Definition 3.1])**.** Let $\mathfrak{g}$ be a Lie superalgebra over $\Bbbk$. The Lie superalgebra $\mathfrak{g}$ is called *admissible* if it satisfies the following conditions.

(A1) The even part $\mathfrak{g}_{\bar{0}}$ is flat

(A2) the odd part $\mathfrak{g}_{\bar{1}}$ is free, and

(A3) for all $x\in\mathfrak{g}_{\bar{1}}$, the element $[x,x]$ is 2-divisible.

Let $\mathfrak{g}$ be a Lie superalgebra. If $\mathfrak{g}_{\bar{1}}$ has a $\Bbbk$-free basis $\mathfrak{X}$ such that $[x,x]$ is 2-divisible for every $x\in\mathfrak{X}$, then the condition (A3) stated above is automatically satisfied.

Example 2.5.3. Let $\mathrm{Mat}_{r,s}(\Bbbk)$ denote the set of all r-by-s matrices with entries in $\Bbbk$. As in the ordinary case, $\mathfrak{gl}_r:=\mathrm{Mat}_{r,r}(\Bbbk)$ forms a Lie algebra.

(1) Set $\mathfrak{gl}(m|n):=\mathrm{Mat}_{m+n,m+n}(\Bbbk)$. This forms a superspace by the following $\mathbb{Z}_2$-grading.

$$\mathfrak{gl}(m|n)_{\bar{0}}:=\{\begin{pmatrix}A&0\\0&D\end{pmatrix}\mid A\in\mathrm{Mat}_{m,m}(\Bbbk),\ D\in\mathrm{Mat}_{n,n}(\Bbbk)\},$$

$$\mathfrak{gl}(m|n)_{\bar{1}}:=\{\begin{pmatrix}0&B\\C&0\end{pmatrix}\mid B\in\mathrm{Mat}_{m,n}(\Bbbk),\ C\in\mathrm{Mat}_{n,m}(\Bbbk)\}.$$

Moreover, $\mathfrak{gl}(m|n)$ becomes a Lie superalgebra by letting

$$[X,Y]:=XY-(-1)^{|X||Y|}YX,$$

for (homogeneous) elements $X,Y\in\mathfrak{gl}(m|n)$. The even part $\mathfrak{gl}(m|n)_{\bar{0}}$ of $\mathfrak{gl}(m|n)$ is isomorphic to $\mathfrak{gl}_m\oplus\mathfrak{gl}_n$ as Lie algebras.

(2) Let $\mathfrak{q}(n)$ be the Lie subsuperalgebra of $\mathfrak{gl}(n|n)$ consisting of matrices of the form

$$\begin{pmatrix}A&B\\B&A\end{pmatrix},\quad A,\,B\in\mathrm{Mat}_{n,n}(\Bbbk).$$

This is called a *queer Lie superalgebra.* Note that, the even part $\mathfrak{q}(n)_{\bar{0}}$ is isomorphic to $\mathfrak{gl}_n$ as Lie algebras.

One can easily check that the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{q}(n)$ stated above are admissible.

For an admissible Lie superalgebra $\mathfrak{g}$, we define the *universal enveloping superalgebra* $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ as the quotient Hopf superalgebra of $T(\mathfrak{g})$ by the Hopf super-ideal generated by the following homogeneous primitives

$$yz - (-1)^{|y||z|}zy - [y, z], \quad x^2 - \frac{1}{2}[x, x], \tag{I.2.5.1}$$

where $y, z \in \mathfrak{g}$ and $x \in \mathfrak{g}_{\bar{1}}$. This $\mathcal{U}(\mathfrak{g})$ is cocommutative. Note that if $2 \in \Bbbk^\times$, then the second element $x^2 - \frac{1}{2}[x, x]$ in (I.2.5.1) may be removed, since they are covered by the first one.

Let $\mathcal{U}(\mathfrak{g}_{\bar{0}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\bar{0}}$, as usual. Namely, this is the quotient cocommutative Hopf algebra of the tensor algebra $T(\mathfrak{g}_{\bar{0}})$ of $\mathfrak{g}_{\bar{0}}$ be the Hopf ideal generated by $yz - zy - [y, z]$ for $y, z \in \mathfrak{g}_{\bar{0}}$. Since $\mathfrak{g}_{\bar{0}}$ is flat $\Bbbk$-module, the canonical map $\mathfrak{g}_{\bar{0}} \to \mathcal{U}(\mathfrak{g}_{\bar{0}})$ is injective, see [14]. Through this injection, we may regard $\mathfrak{g}_{\bar{0}} \subset \mathcal{U}(\mathfrak{g}_{\bar{0}})$.

On the other hand, the inclusion $\mathfrak{g}_{\bar{0}} \subset \mathfrak{g}$ induces a Hopf superalgebra map

$$\mathcal{U}(\mathfrak{g}_{\bar{0}}) \longrightarrow \mathcal{U}(\mathfrak{g}).$$

2.5.2 2-Operations

In this subsection, we work over an arbitrary non-zero commutative ring $\Bbbk$.

Let $\mathfrak{g}$ be a Lie superalgebra.

Definition 2.5.4 ([12, Definition 2.2.1])**.** A 2-*operation* on $\mathfrak{g}$ is a map $(-)^{\langle 2\rangle} : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ such that

(i) $(cv)^{\langle 2\rangle} = c^2 v^{\langle 2\rangle}$,

(ii) $(v + w)^{\langle 2\rangle} = v^{\langle 2\rangle} + [v, w] + w^{\langle 2\rangle}$, and

(iii) $[v^{\langle 2\rangle}, z] = [v, [v, z]]$,

where $c \in \Bbbk$, $v, w \in \mathfrak{g}_{\bar{1}}$, $z \in \mathfrak{g}$.

This is related with the admissibility defined by Definition 2.5.2 as follows.

Lemma 2.5.5. *Assume that $\Bbbk$ is 2-torsion free. If $\mathfrak{g}$ is admissible, then*

$$v^{\langle 2\rangle} := \frac{1}{2}[v, v], \quad v \in \mathfrak{g}_{\bar{1}}$$

gives the unique 2-operation on $\mathfrak{g}$, and this is indeed the unique map $\mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ that satisfies (i), (ii) above.

Proof. The left and the right-hand sides of (i)–(iii) coincide since their doubles are seen to coincide. The uniqueness follows, since we see from (i), (ii) that $4v^{\langle 2\rangle} = (2v)^{\langle 2\rangle} = 2v^{\langle 2\rangle} + [v, v]$, and so $2v^{\langle 2\rangle} = [v, v]$. □

If $\Bbbk$ is 2-torsion free, an admissible Lie superalgebra is thus the same as a Lie superalgebra $\mathfrak{g}$ given a (unique) 2-operation, such that $\mathfrak{g}_{\bar{0}}$ is $\Bbbk$-flat and $\mathfrak{g}_{\bar{1}}$ is $\Bbbk$-free.

Let us return to the situation that $\Bbbk$ is arbitrary. Let $\mathfrak{g}$ be a Lie superalgebra given a 2-operation. One directly verifies the following.

Proposition 2.5.6. *Suppose that the odd component $\mathfrak{g}_{\bar{1}}$ is $\Bbbk$-free, and choose a totally ordered basis $\mathfrak{X}$ arbitrarily. Given a commutative algebra S, define a map*

$$(-)_S^{\langle 2\rangle} : \mathfrak{g}_{\bar{1}} \otimes S \longrightarrow \mathfrak{g}_{\bar{0}} \otimes S$$

by

$$\Big(\sum_{i=1}^{n} x_i \otimes c_i\Big)_S^{\langle 2\rangle} := \sum_{i=1}^{n} x_i^{\langle 2\rangle} \otimes c_i^2 + \sum_{i<j} [x_i,\, x_j] \otimes c_i c_j,$$

where $x_1 < \cdots < x_n$ in $\mathfrak{X}$, and $c_i \in S$. This definition is independent of choice of ordered bases, and the map gives a 2-operation on the S-Lie superalgebra $\mathfrak{g} \otimes S$. For arbitrary elements $v_i \in \mathfrak{g}_{\bar{1}}$, $c_i \in S$, $1 < i < m$, we have

$$\Big(\sum_{i=1}^{m} v_i \otimes c_i\Big)_S^{\langle 2\rangle} = \sum_{i=1}^{m} v_i^{\langle 2\rangle} \otimes c_i^2 + \sum_{i<j} [v_i,\, v_j] \otimes c_i c_j.$$

2.6 Supercomodules

Let C be a supercoalgebra. A *right C-supermodule V* is a supermodule endowed with a structure map $\rho \in \mathrm{Hom}_{\Bbbk}(V, V \otimes C)$ satisfying the following commutative diagrams in SMod.

$$\begin{array}{ccc} V \otimes C \otimes C & \xleftarrow{\mathrm{id}_V \otimes \Delta} & V \otimes C \\ {\scriptstyle \rho \otimes \mathrm{id}_C}\uparrow & \circlearrowleft & \uparrow{\scriptstyle \rho} \\ V \otimes C & \xleftarrow[\rho]{} & V, \end{array} \qquad \begin{array}{ccc} V \otimes \Bbbk & \xleftarrow{\mathrm{id}_V \otimes \varepsilon} & V \otimes C \\ & \nwarrow^{\cong} \quad \circlearrowleft & \uparrow{\scriptstyle \rho} \\ & & V. \end{array}$$

We let denote $\rho(v) = v_0 \otimes v_1$ for $v \in V$, as usual. Let $\mathrm{Hom}^C(V, W)$ denote the set of all (parity preserving) right C-supermodule maps from V to W and let $\underline{\mathrm{Hom}}^C(V, W) := \mathrm{Hom}^C(V, W) \oplus \mathrm{Hom}^C(V, \Pi W)$, as before. We denote $\underline{\mathrm{Hom}}^C(V, V)$ by $\underline{\mathrm{End}}^C(V)$. Let SMod^C denote the category of right C-supercomodules. Similarly, we define the notion of *left C-supercomodules*. For a left C-supercomodule V with its structure map $\rho : V \to C \otimes V$, we let denote $\rho(v) = v_{-1} \otimes v_0$ for $v \in V$.

Definition 2.6.1. Let C and C' be supercoalgebras. A *left C-, right C'-supercomodule* is a supermodule V satisfying the following conditions

(i) V is a left C-supercomodule with the structure map ρ and a right C'-supercomodule with the structure map ρ', and

(ii) $(\mathrm{id}_C \otimes \rho') \circ \rho = (\rho \otimes \mathrm{id}_{C'}) \circ \rho'$.

For a finite and flat left (resp. right) C-supercomodule V with its structure map ρ, we can naturally regard the dual superspace V^* as a right (resp. left) C-supercomodule. To explain this, we consider the following map

$$V^* \otimes V \xrightarrow{\mathrm{id} \otimes \rho} V^* \otimes C \otimes V \xrightarrow{c_{V^*, C} \otimes \mathrm{id}_V} C \otimes V^* \otimes V \xrightarrow{\mathrm{id} \otimes \langle\, ,\, \rangle} C$$

$$(\text{resp. } V^* \otimes V \xrightarrow{\mathrm{id}\otimes\rho} V^* \otimes V \otimes C \xrightarrow{\langle\,,\rangle\otimes\mathrm{id}} C).$$

Then we get an element of $\mathrm{Hom}_{\Bbbk}(V^* \otimes V, C) = \mathrm{Hom}_{\Bbbk}(V^*, \underline{\mathrm{Hom}}_{\Bbbk}(V, C))$. There is a canonical isomorphism

$$C \otimes V^* \cong \underline{\mathrm{Hom}}_{\Bbbk}(V, C)\,; \quad c \otimes f \longmapsto (v \mapsto c\langle f, v\rangle).$$

By using this isomorphism (resp. the composition of $c_{V^*,C}$ and this isomorphism), we get an element ρ^* in $\mathrm{Hom}_{\Bbbk}(V^*, V^* \otimes C)$ (resp. $\mathrm{Hom}_{\Bbbk}(V^*, C \otimes V^*)$). Explicitly,

$$\langle \rho^*(f), v\rangle = v_{-1}\langle f, v_0\rangle \quad (\text{resp. } \langle \rho^*(f), v\rangle = \langle f, v_0\rangle v_1)$$

for $c \in C$, $f \in V^*$ and $v \in V$. Now one can easily show that this ρ^* indeed define a right (resp. left) C-supercomodule structure on V^*.

Let $f : C \to D$ be a supercoalgebra map. For a right C-supercomodule V, we define

$$\rho|_D : V \xrightarrow{\rho} V \otimes C \xrightarrow{\mathrm{id}\otimes f} V \otimes D,$$

where ρ is the supercomodule structure map of V. One sees V becomes a right D-supercomodule with this new structure map $\rho|_D$. We denote this D-supercomodule by $\mathrm{res}^C_D(V)$. This $\mathrm{res}^C_D(-)$ gives a functor from the category of right C-supercomodules SMod^C to the category of right D-supercomodules SMod^D.

Let V be a right C-supercomodule, and W be a left C-supercomodule. We define the *cotensor product* of V and W by

$$V \,\square_C\, W := \mathrm{Ker}(V \otimes W \xrightarrow{\rho\otimes\mathrm{id}\,-\,\mathrm{id}\otimes\psi} V \otimes C \otimes W).$$

We regard $V \square_C W$ as a subsuperspace of $V \otimes W$. In this way, we have a left exact functor $(-)\square_C W$ form the category of right C-supercomodules SMod^C to the category of superspaces SMod.

We regard C as a left D-supercomodule by

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f\otimes\mathrm{id}_C} D \otimes C. \qquad \text{(I.2.6.1)}$$

For a right D-supercomodule V, we define a right C-supercomodule

$$\mathrm{ind}^C_D(V) := V \,\square_D\, C$$

whose structure map is given by $\mathrm{id}_V \otimes \Delta$. This $\mathrm{ind}^C_D(-)$ gives a left exact functor from the category of right D-supercomodules SMod^D to the category of right C-supercomodules SMod^C.

Lemma 2.6.2. *Let C be a supercoalgebra, and let V be a right C-supercomodule.*

(1) The following map is a natural isomorphism of right C-supercomodules.

$$V \,\square_C\, C \xrightarrow{\cong} V; \quad v \otimes c \longmapsto v\,\varepsilon(c),$$

where $\varepsilon : C \to \Bbbk$ is the counit of C.

(2) *Let C' be a supercoalgebra. For a left C- right C'-supercomodule W and a left C'-supercomodule X,*

$$V \,\square_C\, (W \,\square_{C'}\, X) \cong (V \,\square_C\, W) \,\square_{C'}\, X; \quad v \otimes (w \otimes x) \longmapsto (v \otimes w) \otimes x$$

is a natural isomorphism of superspaces

The following is a kind of dual result of Frobenius reciprocity.

Proposition 2.6.3 (Frobenius Reciprocity)**.** *Let V be a right D-supercomodule and let W be a right C-supercomodule. Then there exists an isomorphism of* SMod

$$\begin{array}{rcl} \underline{\mathrm{Hom}}^C(\mathrm{res}^D_C(V), W) & \xrightarrow{\cong} & \underline{\mathrm{Hom}}^D(V, \mathrm{ind}^D_C(W)); \\ \varphi & \mapsto & (\varphi \otimes \mathrm{id}_D) \circ \rho, \end{array}$$

where ρ is the supercomodule structure map of V.

Proof. For $\psi \in \underline{\mathrm{Hom}}^D(V, \mathrm{ind}^D_C(W))$, we define

$$V \xrightarrow{\psi} W \,\square_C\, D \overset{\mathrm{id}_W \otimes \varepsilon}{\longrightarrow} W,$$

where ε is the counit of D. One can easily check that this gives the inverse. Since f and the structure maps of V, W preserve the parity, we are done. □

In particular, we have an isomorphism

$$\mathrm{Hom}^C(\mathrm{res}^D_C(V), W) \xrightarrow{\cong} \mathrm{Hom}^D(V, \mathrm{ind}^D_C(W)) \tag{I.2.6.2}$$

of modules over $\Bbbk$. Since $\mathrm{ind}^D_C(-)$ is right adjoint to the restriction functor $\mathrm{res}^D_C(-)$, we have the following result.

Corollary 2.6.4. *The functor* $\mathrm{ind}^D_C(-)$ *preserves injective objects.*

Let A be a Hopf superalgebra. We regard $\Bbbk$ as a left A-supercomodule by

$$\Bbbk \to A \otimes \Bbbk; \quad 1 \mapsto 1_A \otimes 1,$$

where 1_A is the unit element of A. For a right A-supercomodule V, we define the *coinvariant subsuperspace* $V^{\mathrm{co}A}$ of V as follows

$$V^{\mathrm{co}A} := V \,\square_A\, \Bbbk.$$

Explicitly, $v \in V^{\mathrm{co}A}$ if and only if $\rho(v) = v \otimes 1_A$, where $\rho : V \to V \otimes A$ is the right A-supercomodule structure of V.

We can show a super-analogue of the tensor identity theorem ([16, Part I, 3.6]).

Proposition 2.6.5. *Let B a Hopf superalgebra with a Hopf superalgebra map $A \to B$. For $V \in$* SMod^A *and $W \in$* SMod^B*, we have an isomorphism*

$$V \otimes \mathrm{ind}^A_B(W) \xrightarrow{\cong} \mathrm{ind}^A_B(\mathrm{res}^A_B(V) \otimes W)$$

of right A-supercomodules.

Proof. For $v \otimes w \otimes a \in V \otimes \mathrm{ind}_B^A(W)$, the map $v \otimes w \otimes a \longmapsto (-1)^{|v_1||w|} v_0 \otimes w \otimes v_1 a$ gives the right A-supercomodule isomorphism. $\square$

Chapter 3

Supergroups

Let $\Bbbk$ be a non-zero base commutative ring.

3.1 Algebraic supergroups

An *affine supergroup scheme* (*supergroup*, for short) is a representable functor $\mathbf{G}$ from the category $\mathsf{SAlg}_{\Bbbk}$ of commutative superalgebras over $\Bbbk$ to the category Grp of groups. We denote the representing object of $\mathbf{G}$ by $\mathcal{O}(\mathbf{G})$. This $\mathcal{O}(\mathbf{G})$ forms a commutative Hopf superalgebra, by Yoneda's Lemma. A subsupergroup functor $\mathbf{K}$ of $\mathbf{G}$ is called a *closed subsupergroup* if there is a Hopf superalgebra surjection $\mathcal{O}(\mathbf{G}) \to \mathcal{O}(\mathbf{K})$. A supergroup $\mathbf{G}$ said to be *algebraic* (resp. *flat*) if $\mathcal{O}(\mathbf{G})$ is finitely generated as a superalgebra (resp. $\mathcal{O}(\mathbf{G})$ is $\Bbbk$-flat).

Conversely, for a commutative Hopf superalgebra A over $\Bbbk$, we get a supergroup $\mathrm{SSp}(A)$ as follows. As in (I.2.3.5), for a commutative superalgebra R, $\mathrm{SSp}(A)(R)$ is the set $\mathsf{SAlg}_{\Bbbk}(A, R)$ of all superalgebra maps from A to R over $\Bbbk$. For $f, f' \in \mathrm{SSp}(A)(R)$, the multiplication $*$ is given by

$$f * f' := m_R \circ (f \otimes f') \circ \Delta : A \longrightarrow R,$$

where $m_R : R \otimes R \to R$ is the multiplication on R and $\Delta : A \to A \otimes A$ is the comultiplication of A. The unit element of $\mathrm{SSp}(A)(R)$ is given by $u_R \circ \varepsilon : A \to R$, where $u_R : \Bbbk \to R$ is the unit of R and $\varepsilon : A \to \Bbbk$ is the counit of A. Finally, the $*$-inverse of the element $\varphi \in \mathrm{SSp}(A)(R)$ is given by $\varphi \circ \mathcal{S} : A \to R$, where $\mathcal{S} : A \to A$ is the antipode of A. As in ordinary case, one can easily show that $\mathrm{SSp}(A)$ forms a supergroup by these structure maps. In this way, one can see that affine supergroups correspond to commutative Hopf superalgebras.

For a commutative Hopf superalgebra A, we define

$$\overline{A} := A/(A_{\bar{1}}), \qquad W^A := (A^+/(A^+)^2)_{\bar{1}}, \tag{I.3.1.1}$$

where $(A_{\bar{1}})$ is the super-ideal of A generated by the odd part $A_{\bar{1}}$ of A and $A^+ := \operatorname{Ker} \varepsilon$. Note that $\overline{A}$ is an ordinary Hopf algebra. Let

$$A \longrightarrow \overline{A}; \quad a \longmapsto \overline{a} \tag{I.3.1.2}$$

be the quotient map. We regard A as a left $\overline{A}$-comodule superalgebra by

$$A \longrightarrow \overline{A} \otimes A; \quad a \longmapsto \overline{a_1} \otimes a_2,$$

where $\overline{a_1}$ denotes the canonical image of $a_1 \in A$ in $\overline{A}$.

Definition 3.1.1 ([27, Definition 2.1]). A commutative Hopf superalgebra A is said to be *split* if W^A is free and there exists an isomorphism $\psi : A \xrightarrow{\simeq} \overline{A} \otimes \wedge(W^A)$ of left $\overline{A}$-comodule superalgebras.

A split commutative Hopf superalgebra A is finitely generated if and only if $\overline{A}$ is finitely generated and W^A is $\Bbbk$-finite (free).

If such ψ exists, then we can re-chosen so that *counit-preserving* in the sense that

$$(\varepsilon_{\bar{A}} \otimes \varepsilon_{\wedge(W^A)}) \circ \psi = \varepsilon_A,$$

where ε_A (resp., $\varepsilon_{\bar{A}}$, $\varepsilon_{\wedge(W^A)}$) is the counit of A (resp., $\bar{A}$, $\wedge(W^A)$). Indeed, the map $a \mapsto \psi(a_1)\gamma \circ \mathcal{S}(a_2)$ gives the desired one, where $\gamma := (\varepsilon_{\bar{A}} \otimes \varepsilon_{\wedge(W^A)}) \circ \psi$.

Remark 3.1.2. We regard A as a right $\overline{A}$-comodule superalgebra by

$$A \longrightarrow A \otimes \overline{A}; \quad a \longmapsto a_1 \otimes \overline{a_2}. \tag{I.3.1.3}$$

The same condition as above is equivalent to the condition with the sides switched, that is, the condition that there exists a (counit-preserving) isomorphism $A \xrightarrow{\simeq} \wedge(W^A) \otimes \overline{A}$ of right $\overline{A}$-comodule superalgebras. Indeed, if ψ is a left- or right-sided isomorphism, then the composite $c \circ \psi \circ \mathcal{S}$, where $c = c_{\overline{A}, \wedge(W^A)}$ or $c = c_{\wedge(W^A), \overline{A}}$, gives an opposite-sided one.

Theorem 3.1.3 ([22, Theorem 4.5]). *If $\Bbbk$ is a field of characteristic $\neq 2$, then every commutative Hopf superalgebra is split.*

For a supergroup $\mathbf{G}$, we define its *even part* $\mathbf{G}_{\mathrm{ev}}$ as the restricted functor of $\mathbf{G}$ form category $\mathsf{Alg}_\Bbbk$ of commutative algebras over $\Bbbk$ to Grp. This $\mathbf{G}_{\mathrm{ev}}$ is an ordinary affine group (scheme) represented by the quotient Hopf algebra $\overline{\mathcal{O}(\mathbf{G})}$. If $\mathbf{G}$ is algebraic, then $\mathbf{G}_{\mathrm{ev}}$ is also algebraic. By definition, $W^{\mathcal{O}(\mathbf{G})}$ is the odd part of the cotangent space of $\mathbf{G}$ at the identity.

Example 3.1.4. Let M be a $\Bbbk$-supermodule. For a commutative superalgebra R over $\Bbbk$, we define

$$\mathbf{GL}(M)(R) := \underline{\mathrm{Aut}}_R(M \otimes R),$$

where $\underline{\mathrm{Aut}}_R(M \otimes R)$ is a subsuperspace of $\underline{\mathrm{End}}_R(M \otimes R)$ consisting of all invertible morphisms. By definition, $\mathbf{GL}(M)$ is a supergroup. If M is $\Bbbk$-free and finite rank such that $\mathrm{rank}\, M_{\bar{0}} = m$, $\mathrm{rank}\, M_{\bar{1}} = n$, then we can regard $\mathbf{GL}(M)$ as a matrix group as follows

$$\mathbf{GL}(M)(R) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \;\middle|\; \begin{array}{l} A \in \mathrm{GL}_m(R_{\bar{0}}),\ B \in \mathrm{Mat}_{m,n}(R_{\bar{1}}), \\ C \in \mathrm{Mat}_{n,m}(R_{\bar{1}}),\ D \in \mathrm{GL}_n(R_{\bar{0}}) \end{array} \right\}$$

where $\mathrm{GL}_m(-)$ is the ordinary general linear group (scheme) of degree m. In this case, we sometimes denote $\mathbf{GL}(M)$ by $\mathbf{GL}(m|n)$.

The supergroup $\mathbf{GL}(m|n)$ is an algebraic supergroup. Indeed, the Hopf superalgebra $\mathcal{O}(\mathbf{GL}(m|n))$ representing $\mathbf{G}(m|n)$ is given by the localization $A(m|n)_d$ of $A(m|n)$ at

$$d := \det(x_{ij})_{1\leqslant i,j\leqslant m} \det(x_{k\ell})_{m+1\leqslant k,\ell\leqslant m+n},$$

for the notation see Example 2.1.2. In particular, one sees that the antipode is given by

$$\mathcal{S}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} (A-BD^{-1}C)^{-1} & -A^{-1}B\mathcal{S}(D) \\ \hline -D^{-1}C\mathcal{S}(A) & (D-CA^{-1}B)^{-1} \end{array}\right),$$

where $A = (x_{ij})_{1\leqslant i,j\leqslant m}$, $B = (x_{kj})_{1\leqslant k\leqslant m<j\leqslant m+n}$, $C = (x_{i\ell})_{1\leqslant i\leqslant m<\ell\leqslant m+n}$, $D = (x_{k\ell})_{m+1\leqslant k,\ell\leqslant m+n}$.

The even part of $\mathbf{GL}(m|n)$ is given as follows

$$\mathbf{GL}(m|n)_{\mathrm{ev}} \cong \mathrm{GL}_m \times \mathrm{GL}_n.$$

Example 3.1.5. Let $\mathbf{Q}(n)$ be the subsupergroup of $\mathbf{GL}(n|n)$ such that

$$\mathbf{Q}(n)(R) := \{\left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right) \in \mathbf{GL}(n|n)(R) \mid A \in \mathrm{Mat}_n(R_{\bar{0}}),\ B \in \mathrm{Mat}_n(R_{\bar{1}})\},$$

where R is a commutative superalgebra. Obviously, this $\mathbf{Q}(n)$ is a closed subsupergroup of $\mathbf{GL}(n|n)$. The even part is GL_n.

Example 3.1.6 (cf. [29])**.** For a fixed $m \geqslant 1$, we let $\mathbf{G}_{\mathrm{a}}^{-m}$ denote the algebraic supergroup such that $\mathbf{G}_{\mathrm{a}}^{-m}(R)$ is the additive group $R_{\bar{1}}^m$, where R is a commutative superalgebra. The representing object is the exterior algebra $\wedge(\xi_1, \dots, \xi_m)$ generated by non-zero odd primitives $\xi_1, \dots, \xi_m$.

3.2 Lie superalgebras of supergroups

Assume that $\Bbbk$ is 2-torsion free.

Let $\mathbf{G}$ be an affine supergroup. Set $A := \mathcal{O}(\mathbf{G})$. Then the following is easy to see.

Lemma 3.2.1. *For homogeneous elements $a, b \in A^+$, we have*

$$\Delta(ab) \equiv 1 \otimes ab + ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a$$

modulo $A^+ \otimes (A^+)^2 + (A^+)^2 \otimes A^+$.

Set $\mathfrak{d} := A^+/(A^+)^2$. This is a supermodule. The *Lie superalgebra*

$$\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$$

of $\mathbf{G}$ is the dual supermodule $\mathfrak{d}^*$ of $\mathfrak{d}$. Explicitly,

$$\mathrm{Lie}(\mathbf{G}) := (A^+/(A^+)^2)^*.$$

Note that, A^* is the dual superalgebra of the supercoalgebra A. Regard $\mathfrak{g}$ as a subsupermodule of A^* through the natural embedding $\mathfrak{g} \subset \Bbbk \oplus \mathfrak{d}^* = (A/(A^+)^2)^* \subset A^*$. By definition we have

$$\mathfrak{g}_{\bar{1}} = (W^A)^*.$$

Proposition 3.2.2. *The superlinear endomorphism* $\mathrm{id}_{A^*\otimes A^*} - c_{A^*,A^*}$ *on* $A^* \otimes A^*$*, composed with the product on* A^**, restricts to a map,* $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$*, with which* $\mathfrak{g}$ *is indeed a Lie superalgebra. This satisfies (A3) in Definition 2.5.2.*

Proof. By Lemma 3.2.1 it follows that $(\mathrm{id}_{A^*\otimes A^*} - c_{A,A}) \circ \Delta$ induces a super-linear map

$$\delta : \mathfrak{d} \longrightarrow \mathfrak{d} \otimes \mathfrak{d}, \qquad \text{(I.3.2.1)}$$

which is seen to satisfy

$$(\mathrm{id}_{\mathfrak{d}\otimes\mathfrak{d}} + c_{\mathfrak{d},\mathfrak{d}}) \circ \delta = 0, \quad (\mathrm{id}_{\mathfrak{d}\otimes\mathfrak{d}\otimes\mathfrak{d}} + c_{\mathfrak{d},\mathfrak{d}\otimes\mathfrak{d}} + c_{\mathfrak{d}\otimes\mathfrak{d},\mathfrak{d}}) \circ (\delta \otimes \mathrm{id}_{\mathfrak{d}}) \circ \delta = 0.$$

Therefore, δ is dualized to a map $[\ ,\ ]$ such as above, which satisfies (i), (iii) and (iv) required to super-brackets; see Section 2.5. Let $v \in \mathfrak{g}_{\bar{1}}$. Then it follows from Lemma 3.2.1 that given a, b as in the lemma, we have

$$v^2(ab) = v(a)v(b) + (-1)^{|a||b|}v(b)v(a) = 0,$$

since $v(a)v(b) = 0$ unless $|a| = |b| = 1$. Therefore, $v^2 \in \mathfrak{g}_{\bar{0}}$ and $[v, v] = 2v^2$. Thus (A3) in Definition 2.5.2 is satisfied. The remaining (ii) is satisfied, since $[[v, v], v] = 2[v^2, v] = 0$. ☐

Set $G := \mathbf{G}_{\mathrm{ev}}$. Then $\overline{A} = \mathcal{O}(G)$. We have the Lie algebra $\mathrm{Lie}(G) = (\overline{A}^+/(\overline{A}^+)^2)^*$ of G.

Lemma 3.2.3. *The natural embedding* $\overline{A}^* \subset A^*$ *induces an isomorphism* $\mathrm{Lie}(G) \simeq \mathfrak{g}_{\bar{0}}$ *of Lie algebras.*

Proof. One sees that this is the dual of the canonical isomorphism

$$A_{\bar{0}}^+/((A_{\bar{0}}^+)^2 + A_{\bar{1}}^2) \simeq (A_{\bar{0}}^+/A_{\bar{1}}^2)/\big(((A_{\bar{0}}^+)^2 + A_{\bar{1}}^2)/A_{\bar{1}}^2\big).$$

Thus we are done. ☐

Example 3.2.4. For algebraic supergroups $\mathbf{GL}(m|n)$ and $\mathbf{Q}(n)$, one can easily see that its Lie superalgebras are given by $\mathrm{Lie}(\mathbf{GL}(m|n)) = \mathfrak{gl}(m|n)$ and $\mathrm{Lie}(\mathbf{Q}(n)) = \mathfrak{q}(n)$.

Chapter 4

Harish-Chandra Pairs Constructions

In this chapter, we assume that $\Bbbk$ is 2-torsion free,

4.1 Universal enveloping superalgebras

Recall that given an algebra S, an *S-ring* [1, p.195] is an algebra given an algebra map from S.

Let $\mathfrak{g}$ be an admissible Lie superalgebra. The inclusion $\mathfrak{g}_{\bar{0}} \subset \mathfrak{g}$ induces a Hopf superalgebra map

$$\mathcal{U}(\mathfrak{g}_{\bar{0}}) \longrightarrow \mathcal{U}(\mathfrak{g}),$$

by which we will regard $\mathcal{U}(\mathfrak{g})$ as a $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-ring, and in particular as a left and right $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module.

Proposition 4.1.1. $\mathcal{U}(\mathfrak{g})$ *is free as a left as well as right* $\mathcal{U}(\mathfrak{g}_{\bar{0}})$*-module. In fact, if* $\mathfrak{X}$ *is an arbitrary* $\Bbbk$*-free basis of* $\mathfrak{g}_{\bar{1}}$ *given a total order* $\leqslant$*, then the products*

$$x_1 \cdots x_n, \qquad x_i \in \mathfrak{X}, \quad x_1 < \cdots < x_n, \quad n \geq 0$$

in $\mathcal{U}(\mathfrak{g})$ *form a* $\mathcal{U}(\mathfrak{g}_{\bar{0}})$*-free basis, where* x_i *in the product denotes the image of the element under the canonical map* $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$*.*

This is proved in [24, Lemma 11], in the generalized situation treating dual Harish-Chandra pairs, but over a field of characteristic $\neq 2$. Our proof of the proposition will confirm the proof of the cited lemma in our present situation. To use the same notation as in [24] we set

$$J := \mathcal{U}(\mathfrak{g}_{\bar{0}}), \quad V := \mathfrak{g}_{\bar{1}}.$$

Then the right adjoint action

$$\mathrm{ad}_{\mathrm{R}}(u)(v) = [v, u], \quad u \in \mathfrak{g}_{\bar{0}}, \ v \in V \tag{I.4.1.1}$$

by $\mathfrak{g}_{\bar{0}}$ on V gives rise to the right J-module structure on V, which we denote by $v \triangleleft a$, where $v \in V$, $a \in J$. If $i : V \to \mathcal{U}(\mathfrak{g})$ denotes the canonical map, we have

$$i(v \triangleleft a) = \mathcal{S}(a_1)\, i(v)\, a_2, \quad v \in V, \ a \in J \tag{I.4.1.2}$$

in $\mathcal{U}(\mathfrak{g})$. Indeed, this follows by induction on the largest length r, when we express a as a sum of elements $u_1 \cdots u_r$, where $u_i \in \mathfrak{g}_{\bar{0}}$.

Lemma 4.1.2. *The right J-module structure on V and the super-bracket $[\ ,\ ] : V \otimes V \to \mathfrak{g}_{\bar{0}} \subset J$ restricted to V make (J, V) into a dual Harish-Chandra pair [24, Definition 6], or explicitly we have*

(a) $[u \triangleleft a_1, v \triangleleft a_2] = \mathcal{S}(a_1)[u, v]a_2,$

(b) $[u, v] = [v, u]$ *and*

(c) $v \triangleleft [v, v] = 0$

for all $u, v \in V$, $a \in J$. Properties (b), (c) implies

(d) $u \triangleleft [v, w] + v \triangleleft [w, u] + w \triangleleft [u, v] = 0, \quad u, v, w \in V.$

We remark that (a) is an equation in $\mathfrak{g}_{\bar{0}}$, and the product of the right-hand side is computed in J, which is possible since $\mathfrak{g}_{\bar{0}} \subset J$.

Proof of Lemma 4.1.2. One verifies (a), just as proving (I.4.1.2). Properties (b), (c) are those of Lie superalgebras. One sees that (b), applied to $u + v + w$ and combined with (c), implies (d). □

Proof of Proposition 4.1.1. We will prove only the left J-freeness. The result with the antipode applied shows the right J-freeness.

Let $\mathfrak{X}$ be a totally ordered basis of V. We confirm the proof of [24, Lemma 11] as follows. First, we introduce the same order as in the proof into all words in the letters from $\mathfrak{X} \cup \{*\}$, where $*$ stands for any element of J. Second, we see by using (I.4.1.2) that the J-ring $\mathcal{U}(\mathfrak{g})$ is generated by $\mathfrak{X}$, and is defined by the reduction system consisting of

(i) $xa \to a_1(x \triangleleft a_2), \quad x \in \mathfrak{X},\ a \in J,$

(ii) $xy \to -yx + [x, y], \quad x, y \in \mathfrak{X},\ x > y,$

(iii) $x^2 \to \frac{1}{2}[x, x], \quad x \in \mathfrak{X},$

where we suppose that in (i), $x \triangleleft a_2$ is presented as a $\Bbbk$-linear combination of elements in $\mathfrak{X}$. Third, we see that the reduction system satisfies the assumptions required by Bergman's Diamond Lemma [1, Proposition 7.1], indeed its opposite-sided version.

To prove the desired result from the Diamond Lemma, it remains to verify the following by using the properties (a)–(d) in Lemma 4.1.2: the overlap ambiguities which may occur when we reduce the words

(iv) $xya, \quad x \geq y$ in $\mathfrak{X},\ a \in J,$

(v) $xyz, \quad x \geq y \geq z$ in $\mathfrak{X}$

are all resolvable. The proof of [24, Lemma 11] verifies the resolvability only when x, y and z are distinct, and the same proof works now as well.

As for the remaining cases (omitted in the cited proof), first let xya be a word from (iv) with $x = y$. This is reduced on the one hand as

$$xxa \to xa_1(x \triangleleft a_2) \to a_1(x \triangleleft a_2)(x \triangleleft a_3),$$

and on the other hand as

$$\begin{aligned} xxa \to \Big(\frac{1}{2}[x,\, x]\Big)a &= a_1\mathcal{S}(a_2)\Big(\frac{1}{2}[x,\, x]\Big)a_3 \\ &= a_1\Big(\frac{1}{2}[x \triangleleft a_2,\, x \triangleleft a_3]\Big). \end{aligned}$$

Let $b \in J$. The last equality holds since $\mathcal{S}(b_1)(\frac{1}{2}[x,\, x])b_2$ and $\frac{1}{2}[x \triangleleft b_1,\, x \triangleleft b_2]$ coincide since their doubles do by (a). For the desired resolvability it suffices to see that the two polynomials

$$(x \triangleleft b_1)(x \triangleleft b_2), \quad \frac{1}{2}[x \triangleleft b_1,\, x \triangleleft b_2] \tag{I.4.1.3}$$

are reduced to the same one. For this, suppose

$$(x \triangleleft b_1) \otimes (x \triangleleft b_2) = \sum_{i,j=1}^{n} t_{ij}\, x_i \otimes x_j \quad \text{in } V \otimes V,$$

where $t_{ij} \in \Bbbk$, and $x_1 < \cdots < x_n$ in $\mathfrak{X}$. Note that $t_{ij} = t_{ji}$ since J is cocommutative. Then the first polynomial in (I.4.1.3) is reduced as

$$\sum_{i<j} t_{ij}(x_i x_j + x_j x_i) + \sum_i t_{ii}\, x_i x_i \to \sum_{i<j} t_{ij}[x_i,\, x_j] + \sum_i t_{ii}\Big(\frac{1}{2}[x_i,\, x_i]\Big).$$

This last and the second polynomial in (I.4.1.3) coincide since by (b), their doubles do. This proves the desired result.

Next, let xyz be a word from (v), and suppose $x = y > z$. Note that if $(u, w) = ([x,\, z], x)$ or $(\frac{1}{2}[x,\, x], z)$, then u is primitive, and so we have the reduction $wu \to uw + w \triangleleft u$ given by (i). Then it follows that $xyz = xxz$ is reduced as

$$\begin{aligned} xxz &\to -xzx + x[x,\, z] \to zxx - [x,\, z]x + [x,\, z]x + x \triangleleft [x,\, z] \\ &\to z\Big(\frac{1}{2}[x,\, x]\Big) + x \triangleleft [x,\, z] \to \Big(\frac{1}{2}[x,\, x]\Big)z + z \triangleleft \Big(\frac{1}{2}[x,\, x]\Big) + x \triangleleft [x,\, z]. \end{aligned}$$

The word is alternatively reduced as

$$xxz \to \Big(\frac{1}{2}[x,\, x]\Big)z.$$

These two results coincide, since the element $z \triangleleft (\frac{1}{2}[x,\, x]) + x \triangleleft [x,\, z]$, whose double is zero by (d), is zero. The ambiguity for the word xyz is thus resolvable when $x = y > z$. One proves similarly the resolvability in the remaining cases, $x > y = z$ and $x = y = z$, using (d) and (c), respectively. $\square$

The proposition just proven shows the following.

Corollary 4.1.3. *If $\mathfrak{g}$ is an admissible Lie superalgebra, then there exists a unit-preserving, left $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module super-coalgebra isomorphism*

$$\mathcal{U}(\mathfrak{g}_{\bar{0}}) \otimes \wedge(\mathfrak{g}_{\bar{1}}) \xrightarrow{\simeq} \mathcal{U}(\mathfrak{g}).$$

Here, "unit-preserving" means that the isomorphism sends $1 \otimes 1$ to 1.

4.2 Harish-Chandra pairs

Let G be an algebraic group over $\Bbbk$. For a right $\mathcal{O}(G)$-comodule M, we write its structure map $M \to M \otimes \mathcal{O}(G)$; $m \mapsto m_0 \otimes m_1$. The corresponding left G-module structure is given by

$$G(S) \longrightarrow \mathrm{Aut}_S(M \otimes S); \quad \gamma \longmapsto (m \otimes 1_S \mapsto m_0 \otimes \gamma(m_1)),$$

where S is a commutative algebra over $\Bbbk$ (cf. Part II, Section 2.1). For simplicity, this left (resp. right) G-module structure is represented as

$$^{\gamma}m \qquad (\text{resp. } m^{\gamma}) \tag{I.4.2.1}$$

for $m \in M$ and $\gamma \in G$. Let W be a $\Bbbk$-finite projective module. A left G-module structure on W is *transposed* to W^* so that

$$\langle v^g, w \rangle := \langle v, {}^g w \rangle \tag{I.4.2.2}$$

for $v \in W^*$, $w \in W$, $g \in G(S)$, where S is a commutative algebra. Actually, this notational convention will be applied only when G is an affine (algebraic) group.

The Lie algebra $\mathrm{Lie}(G)$ of G is naturally embedded into $\mathcal{O}(G)^*$, and the embedding gives rise to an algebra map $\mathcal{U}(\mathrm{Lie}(G)) \to \mathcal{O}(G)^*$. The associated pairing

$$\langle\ ,\ \rangle : \mathcal{U}(\mathrm{Lie}(G)) \times \mathcal{O}(G) \longrightarrow \Bbbk \tag{I.4.2.3}$$

is a Hopf pairing. Therefore, given a left G-module structure (= right $\mathcal{O}(G)$-comodule structure) on a $\Bbbk$-module M, there is induced a left $\mathcal{U}(\mathrm{Lie}(G))$-module structure on M defined by

$$x \rightharpoonup m := m_0 \, \langle x, m_1 \rangle, \tag{I.4.2.4}$$

where $x \in \mathcal{U}(\mathrm{Lie}(G))$ and $m \in M$.

The right adjoint action by G on itself is dualized to the right co-adjoint coaction

$$\mathrm{coad}_{\mathrm{R}} : \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(G); \quad a \longmapsto a_2 \otimes \mathcal{S}(a_1)a_3. \tag{I.4.2.5}$$

This induces on $\mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2$ a right $\mathcal{O}(G)$-comodule (or left G-module) structure. We assume

(B1) $\mathcal{O}(G)/(\mathcal{O}(G)^+)^2$ is $\Bbbk$-finite projective.

This is necessarily satisfied if $\Bbbk$ is a field. Under the assumption, the left G-module structure on $\mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2$ just obtained is transposed to a right G-module structure on $\mathrm{Lie}(G)$. The induced right $\mathcal{U}(\mathrm{Lie}(G))$-module structure coincides with the right adjoint action $\mathrm{ad}_{\mathrm{R}}(u)(v) = [v,\, u]$, $u, v \in \mathrm{Lie}(G)$, as is seen by using the fact that the pairing above satisfies

$$\langle u,\, ab\rangle = \langle u,\, a\rangle\, \varepsilon(b) + \varepsilon(a)\, \langle u,\, b\rangle, \quad \langle u,\, \mathcal{S}(a)\rangle = -\langle u,\, a\rangle \tag{I.4.2.6}$$

for $u \in \mathrm{Lie}(G)$, $a, b \in \mathcal{O}(G)$.

Let G be an algebraic group which satisfies (B1), and let $\mathfrak{g}$ be a Lie superalgebra such that $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$. Note that $\mathfrak{g}_{\bar{0}}$ is $\Bbbk$-finite projective and so $\Bbbk$-flat; it is a right G-module, as was just seen. We assume in addition,

(B2) $\mathfrak{g}_{\bar{1}}$ is $\Bbbk$-finite free, and $\mathfrak{g}$ is admissible, and

(B3) G is flat, i.e., $\mathcal{O}(G)$ is $\Bbbk$-flat.

Assuming (B1) we see that (B2) is equivalent to that $\mathfrak{g}_{\bar{1}}$ is $\Bbbk$-finite free, and $\mathfrak{g}$ satisfies (A3).

Definition 4.2.1 (cf. [6, Definition 3.1], [27, Definition 4.4])**.** Let G be an algebraic group satisfying (B1) and (B3), and let $\mathfrak{g}$ be a Lie superalgebra satisfying (B2). The pair $(G, \mathfrak{g})$ is called a *Harish-Chandra pair* if it satisfies the following conditions

(i) The Lie algebra $\mathrm{Lie}(G)$ of G coincides with $\mathfrak{g}_{\bar{0}}$,

(ii) there is a right G-module structure on $\mathfrak{g}_{\bar{1}}$ such that the induced right $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module structure coincides with the right adjoint $\mathfrak{g}_{\bar{0}}$-action given by (I.4.1.1), and

(iii) the super-bracket $[\ ,\ ] : \mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ restricted to $\mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}$ is right G-equivariant.

A *morphism* $(G, \mathfrak{g}) \to (G', \mathfrak{g}')$ between Harish-Chandra pairs is a pair (α, β) of a morphism $\alpha : G \to G'$ of affine groups and a morphism $\beta = \beta_{\bar{0}} \oplus \beta_{\bar{1}} : \mathfrak{g} \to \mathfrak{g}'$ Lie superalgebras, such that

(iv) the Lie algebra map $\mathrm{Lie}(\alpha)$ induced from α coincides with $\beta_{\bar{0}}$, and

(v) $\beta_{\bar{1}}(v^{\gamma}) = \beta_{\bar{1}}(v)^{\alpha(\gamma)}$ for $\gamma \in G,\ v \in \mathfrak{g}_{\bar{1}}$.

The Harish-Chandra pairs and their morphisms form a category HCP.

Remark 4.2.2. By convention (see (I.4.2.1)) the equation (ii) above should read

$$(\beta_{\bar{1}} \otimes \mathrm{id}_S)((v \otimes 1)^{\gamma}) = ((\beta_{\bar{1}} \otimes \mathrm{id}_S)(v \otimes 1))^{\alpha_S(\gamma)},$$

where S is a commutative algebra, $\alpha_S : G(S) \to G'(S)$, $v \in \mathfrak{g}_{\bar{1}}$, and $\gamma \in G(S)$.

Remark 4.2.3. Suppose that $\Bbbk$ is a field of characteristic $\neq 2$. In this situation the notion of Harish-Chandra pairs was defined by [24, Definition 7] in purely Hopf algebraic terms. It is remarked by

[24, Remark 9 (2)] that if the characteristic char $\Bbbk$ of $\Bbbk$ is zero, there is a natural category anti-isomorphism between our HCP defined above and the category of the Harish-Chandra pairs as defined by [24, Definition 7]. But this is indeed the case without the restriction on char $\Bbbk$. A key fact is the following: once we are given an algebraic group G, a finite-dimensional right G-module V and a right G-equivariant linear map $[\ ,\ ] : V \otimes V \to \mathrm{Lie}(G)$, then the pair $(\mathcal{O}(G), V^*)$, accompanied with $[\ ,\ ]$, is a Harish-Chandra pair in the sense of [24], if and only if the direct sum $\mathfrak{g} := \mathrm{Lie}(G) \oplus V$ is a Lie superalgebra (in our sense), with respect to the grading $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$, $\mathfrak{g}_{\bar{1}} = V$, and with respect to the super-bracket which uniquely extends

(a) the bracket on $\mathrm{Lie}(G)$,

(b) the map $[\ ,\ ]$, and

(c) the right adjoint $\mathrm{Lie}(G)$-action on V which is induced from the right G-action on V.

See [24, Remark 2 (1)], but note that in [24], the notion of Lie superalgebras is used in a restrictive sense when char $\Bbbk = 3$; indeed, to define the notion, the article excludes Condition (ii) from our axioms given in the beginning of Section 2.5.

Remark 4.2.4. Our definition of Harish-Chandra pairs looks different from those definitions given in [5, Section 7.4] and [6, Section 3.1] which require that the whole super-bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is G-equivariant. But this follows from the weaker requirement of ours that the restricted super-bracket

$$[\ ,\ ]|_{\mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}} : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}}$$

is G-equivariant, since $[\ ,\ ]|_{\mathfrak{g}_{\bar{0}} \otimes \mathfrak{g}_{\bar{0}}}$ is obviously G-equivariant, and $[\ ,\ ]|_{\mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{0}}}$ is, too, as will be seen below. For $\gamma \in G$, $u \in \mathfrak{g}_{\bar{0}}$ and $v \in \mathfrak{g}_{\bar{1}}$, one sees that

$$\langle u, a_1 \rangle \, \gamma(a_2) = \gamma(a_1) \, \langle u^{\gamma}, a_2 \rangle, \quad a \in \mathcal{O}(G).$$

Then the common requirement for the induced $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module structure on $\mathfrak{g}_{\bar{1}}$ shows that $[v, u]^{\gamma} = [v^{\gamma}, u^{\gamma}]$.

4.3 From ASG to HCP

A Hopf superalgebra is said to be *affine* if it is commutative and finitely generated as a superalgebra.

Definition 4.3.1. We define AHSA to be the full subcategory of the category of affine Hopf superalgebras which consists of the affine Hopf superalgebras A such that

(C1) A is split (see Definition 3.1.1),

(C2) $\overline{A}$ is flat, and

(C3) $\overline{A}/(\overline{A}^{+})^2$ is finite projective.

Note that the affinity and (C1) imply that W^A is finite free.

Definition 4.3.2 ([27, p.13])**.** We define ASG to be the full subcategory of the category of algebraic supergroups which consists of the algebraic supergroups $\mathbf{G}$ such that

(D1) $\mathcal{O}(\mathbf{G})$ is split, and

(D2) $\mathbf{G}_{\mathrm{ev}}$ satisfies (B1), (B3) (see Section 4.2).

One sees that this category ASG is anti-isomorphic to the category AHSA.

Remark 4.3.3. If $\Bbbk$ is a field of characteristic $\neq 2$, then the category AHSA is precisely the category of all affine Hopf superalgebras and the category ASG is precisely the category of all algebraic supergroups.

Let $\mathbf{G} \in$ ASG. Set

$$A := \mathcal{O}(\mathbf{G}), \quad G := \mathbf{G}_{\mathrm{ev}}, \quad \mathfrak{g} := \mathrm{Lie}(\mathbf{G}).$$

Then $A \in$ AHSA, and $\mathcal{O}(G)\,(=\overline{A})$ satisfies (B1), (B3). By Proposition 3.2.2, $\mathfrak{g}$ satisfies (B2). By Lemma 3.2.3 we have a natural isomorphism $\mathrm{Lie}(G) \simeq \mathfrak{g}_{\bar{0}}$, through which we will identify the two, and suppose $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$. Just as was seen in (I.4.2.5), the right co-adjoint $\overline{A}$-coaction defined by

$$\mathrm{coad}_{\mathrm{R}} : A \longrightarrow A \otimes \overline{A}; \quad a \longmapsto a_2 \otimes \mathcal{S}(\overline{a}_1)\overline{a}_3, \tag{I.4.3.1}$$

induces on $A^+/(A^+)^2$ a right $\overline{A}$-supercomodule structure; by (C3), it is transposed to a left $\overline{A}$-supercomodule structure on $\mathfrak{g}$, which is restricted to $\mathfrak{g}_{\bar{1}}$.

Lemma 4.3.4. *Given the restricted right G-module structure on $\mathfrak{g}_{\bar{1}}$, the pair $(G, \mathfrak{g})$ forms a Harish-Chandra pair, and so $(G, \mathfrak{g}) \in$ HCP.*

Proof. The right G-module structure on $\mathfrak{g}_{\bar{1}}$ induces the right adjoint $\mathfrak{g}_{\bar{0}}$-action, as is seen by using (I.4.2.6). Since one sees that the map δ given in (I.3.2.1) is G-equivariant, so is its dual, [,]. □

We denote this object in HCP by

$$\mathbf{P}(\mathbf{G}) := (G, \mathfrak{g}).$$

Proposition 4.3.5. $\mathbf{G} \mapsto \mathbf{P}(\mathbf{G})$ *gives a functor* $\mathbf{P} :$ ASG $\to$ HCP*.*

Proof. Indeed, the constructions of G and of $\mathfrak{g}$ are functorial. □

4.4 From HCP to ASG

Let $(G, \mathfrak{g}) \in$ HCP. Modifying the construction of $A(C, W)$ given in [24], we construct an object $A(G, \mathfrak{g})$ in AHSA. To be close to [24] for notation we set

$$J := \mathcal{U}(\mathfrak{g}_{\bar{0}}), \quad C := \mathcal{O}(G), \quad W := \mathfrak{g}_{\bar{1}}^*.$$

Then W is finite free. It is a right C-comodule, or a left G-module, with the right G-module structure on $\mathfrak{g}_{\bar{1}}$ transposed to W.

Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the semigroup of non-negative integers. A supermodule is said to be $\mathbb{N}$-*graded*, if it is $\mathbb{N}$-graded as a $\Bbbk$-module and if the original $\mathbb{Z}_2$-grading equals the $\mathbb{N}$-grading modulo 2.

Definition 4.4.1 ([24, Definition 1])**.** A Hopf superalgebra is said to be $\mathbb{N}$-*graded*, if it is $\mathbb{N}$-graded as an algebra and coalgebra and if the original $\mathbb{Z}_2$-grading equals the $\mathbb{N}$-grading modulo 2.

Recall from Example 2.1.3 that the tensor algebra $T(\mathfrak{g}_{\bar{1}}) = \bigoplus_{n \geqslant 0} T^n(\mathfrak{g}_{\bar{1}})$ on $\mathfrak{g}_{\bar{1}}$ is a cocommutative Hopf superalgebra; this is $\mathbb{N}$-graded. Recall that $\mathfrak{g}_{\bar{0}}$ acts on $\mathfrak{g}_{\bar{1}}$ by the right adjoint; see (I.4.1.1). This uniquely extends to a right J-module-algebra structure on $T(\mathfrak{g}_{\bar{1}})$, with which is associated the smash-product algebra [35, p.155]

$$\mathcal{H} := J \ltimes T(\mathfrak{g}_{\bar{1}}).$$

Given the tensor-product coalgebra structure on $J \otimes T(\mathfrak{g}_{\bar{1}})$, this $\mathcal{H}$ is a cocommutative Hopf superalgebra, which is $\mathbb{N}$-graded so that $\mathcal{H}(n) = J \otimes T^n(\mathfrak{g}_{\bar{1}})$, $n \in \mathbb{N}$; see [24, Section 3.2]. Set

$$\mathcal{U} := \mathcal{U}(\mathfrak{g}).$$

Since we see that $\mathcal{H}$ is the quotient Hopf superalgebra of $T(\mathfrak{g})$ divided by the Hopf super-ideal generated by

$$zw - wz - [z,\, w], \quad z \in \mathfrak{g},\ w \in \mathfrak{g}_{\bar{0}},$$

it follows that $\mathcal{U} = \mathcal{H}/\mathcal{I}$, where $\mathcal{I}$ is the Hopf super-ideal of $\mathcal{H}$ generated by the even primitives

$$1 \otimes (uv + vu) - [u,\, v] \otimes 1, \quad 1 \otimes v^2 - \frac{1}{2}[v,\, v] \otimes 1, \tag{I.4.4.1}$$

where $u, v \in \mathfrak{g}_{\bar{1}}$.

Let $T_c(W)$ denote the *tensor coalgebra* on W, as given in [24, Section 4.1]; this is a commutative $\mathbb{N}$-graded Hopf superalgebra. In fact, this equals the tensor algebra $T(W) = \bigoplus_{n \geqslant 0} T^n(W)$ as an $\mathbb{N}$-graded module, and is the *graded dual* $\bigoplus_{n \geqslant 0} T^n(\mathfrak{g}_{\bar{1}})^*$ of $T(\mathfrak{g}_{\bar{1}})$ (see [35, p.231]) as an algebra and coalgebra. Suppose that $T^0(W) = \Bbbk$ is the trivial right C-comodule, and $T^n(W)$ for $n > 0$, is the n-fold tensor product of the right C-comodule W. Then $T_c(W)$ turns into a right C-comodule coalgebra. The associated smash coproduct

$$C \blacktriangleright\!\!< T_c(W),$$

given the tensor-product algebra structure on $C \otimes T_c(W)$, is a commutative $\mathbb{N}$-graded Hopf superalgebra. Explicitly, the coproduct and the counit is given by

$$\Delta(c \blacktriangleright\!\!< d) = \big(c_1 \blacktriangleright\!\!< (d_1)_0\big) \otimes \big((d_1)_1 c_2 \blacktriangleright\!\!< d_2\big), \quad \varepsilon(c \blacktriangleright\!\!< d) = \varepsilon(c)\,\varepsilon(d), \tag{I.4.4.2}$$

where $c \in C$, $d \in T_c(W)$, and $T_c(W) \to T_c(W) \otimes C$; $d \mapsto d_0 \otimes d_1$ denotes the right C-comodule structure on $T_c(W)$.

In general, given an $\mathbb{N}$-graded supermodule $\mathcal{A} = \bigoplus_{n \geqslant 0} \mathcal{A}(n)$, we suppose that it is given the linear topology defined by the the descending chains of super-ideals

$$\bigoplus_{i>n} \mathcal{A}(n), \qquad n = 0, 1, \ldots .$$

The completion $\widehat{\mathcal{A}}$ coincide with the direct product $\prod_{n=0}^{\infty} \mathcal{A}(n)$. This is not $\mathbb{N}$-graded any more, but is still a supermodule. Given another $\mathbb{N}$-graded supermodule $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is naturally an $\mathbb{N}$-graded supermodule. The complete tensor product $\widehat{\mathcal{A}} \widehat{\otimes} \widehat{\mathcal{B}}$ coincides with the completion of $\mathcal{A} \otimes \mathcal{B}$. We regard $\Bbbk$ as a trivially $\mathbb{N}$-graded supermodule, which is discrete. Suppose that $\mathcal{A}$ is an $\mathbb{N}$-graded Hopf superalgebra. The structure maps on $\mathcal{A}$, being $\mathbb{N}$-graded and hence continuous, are completed to

$$\widehat{\Delta} : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{A}} \widehat{\otimes} \widehat{\mathcal{A}}, \quad \widehat{\varepsilon} : \widehat{\mathcal{A}} \longrightarrow \Bbbk, \quad \widehat{\mathcal{S}} : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{A}}.$$

Satisfying the axiom of Hopf superalgebras with $\otimes$ replaced by $\widehat{\otimes}$, this $\widehat{\mathcal{A}}$ may be called a *complete topological Hopf superalgebra*. If $\mathcal{A}$ is commutative, then $\widehat{\mathcal{A}}$ is, too. See [24, Section 2.3].

Applying the construction above to $C \blacktriangleright\!\!< T_c(W)$, we suppose

$$\mathcal{A} = C \blacktriangleright\!\!< T_c(W), \qquad \widehat{\mathcal{A}} = \prod_{n=0}^{\infty} C \otimes T^n(W)$$

in what follows. We let

$$\pi : \widehat{\mathcal{A}} \longrightarrow C \otimes T^0(W) = C \tag{I.4.4.3}$$

denote the natural projection.

We regard C as a left J-module by

$$x \rightharpoonup c := c_1 \langle x, c_2 \rangle, \quad x \in J, \ c \in C,$$

where $\langle \ , \ \rangle : J \times C \to \Bbbk$ denotes the canonical Hopf pairing; see (I.4.2.3).

Let $\mathrm{Hom}_J(\mathcal{H}, C)$ denote the set of all left J-module maps from $\mathcal{H}$ to C. We regard $\mathrm{Hom}_J(\mathcal{H}, C)$ as the completion of the $\mathbb{N}$-graded supermodule $\bigoplus_{n \geqslant 0} \mathrm{Hom}_J(J \otimes T^n(\mathfrak{g}_{\bar{1}}), C)$. The canonical isomorphisms

$$C \otimes T^n(W) = \mathrm{Hom}(T^n(\mathfrak{g}_{\bar{1}}), C) \xrightarrow{\simeq} \mathrm{Hom}_J(J \otimes T^n(\mathfrak{g}_{\bar{1}}), C), \quad n \geqslant 0 \tag{I.4.4.4}$$

altogether amount to a superlinear homeomorphism

$$\xi : \widehat{\mathcal{A}} \xrightarrow{\simeq} \mathrm{Hom}_J(\mathcal{H}, C). \tag{I.4.4.5}$$

Tensoring the canonical pairings $J \times C \to \Bbbk$ and $T(\mathfrak{g}_{\bar{1}}) \times T_c(W) \to \Bbbk$, we define

$$\langle \ , \ \rangle : \mathcal{H} \times \mathcal{A} \longrightarrow \Bbbk, \quad \langle x \otimes y, c \otimes d \rangle := \langle x, c \rangle \langle y, d \rangle, \tag{I.4.4.6}$$

where $x \in J$, $y \in T(\mathfrak{g}_{\bar{1}})$, $c \in C$, $d \in T_c(W)$. This is a Hopf pairing, as was seen in [24, Proposition

17].

Lemma 4.4.2. *ξ is determined by*

$$\xi(a)(x) = \pi(a_1)\,\langle x, a_2\rangle \tag{I.4.4.7}$$

for $a \in \mathcal{A}, x \in \mathcal{H}$.

Proof. Note that if $a = c \otimes d$, where $c \in C$, $d \in T_c(W)$, then

$$\pi(a_1) \otimes a_2 = c_1 \otimes (c_2 \otimes d).$$

Then the lemma follows since ξ is the completion of the $\mathbb{N}$-graded linear map

$$\mathcal{A} = C \otimes (\bigoplus_{n \geqslant 0} T^n(W)) \longrightarrow \mathrm{Hom}_J(J \otimes (\bigoplus_{n \geqslant 0} T^n(\mathfrak{g}_{\bar{1}})), C)$$

given by $c \otimes d \mapsto \big(x \otimes y \mapsto xc\,\langle y, d\rangle\big)$, and this last element equals $c_1\,\langle x \otimes y, c_2 \otimes d\rangle$. □

Remark 4.4.3. Recall that $\langle \mathcal{H}(n), \mathcal{A}(m)\rangle = 0$ unless $n = m$. Therefore, the pairing (I.4.4.6) uniquely extends to

$$\langle\ ,\ \rangle : \mathcal{H} \times \widehat{\mathcal{A}} \to \Bbbk \tag{I.4.4.8}$$

so that for each $x \in \mathcal{H}$, $\langle x, -\rangle : \widehat{\mathcal{A}} \to \Bbbk$ is continuous. Using this pairing one sees that the value $\xi(a)$ at $a \in \widehat{\mathcal{A}}$ is given by the same formula as (I.4.4.7), with $\pi(a_1) \otimes a_2$ understood to be $(\pi \mathbin{\widehat{\otimes}} \mathrm{id}) \circ \widehat{\Delta}(a)$.

We aim to transfer the structures on $\widehat{\mathcal{A}}$ to $\mathrm{Hom}_J(\mathcal{H}, C)$ through ξ; see Proposition 4.4.6 below.

Definition 4.4.4. Let G be an affine group, in general. A $\Bbbk$-supermodule M is called a *left* (resp. *right*) *G-supermodule* if M is a left (resp. right) G-module such that each component M_ϵ, $\epsilon \in \mathbb{Z}_2$ is G-stable. Let ${}_G\mathsf{SMod}$ (resp. SMod_G) denote the category of left (resp. right) G-supermodules. This forms naturally a tensor category, and is symmetric with respect to the supersymmetry (see (I.2.1.1)).

Recall from Section 4.3 that $\mathfrak{g}_{\bar{0}}$ is a right G-module. Combined with the given right G-module structure on $\mathfrak{g}_{\bar{1}}$, it results that $\mathfrak{g} \in \mathsf{SMod}_G$. Moreover, $\mathfrak{g}$ is a Lie-algebra object in SMod_G, since the super-bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is G-equivariant, as was proved in Remark 4.2.4.

We regard $\mathcal{A}$ as a right C-supercomodule, or an object in ${}_G\mathsf{SMod}$, with respect to the right co-adjoint coaction

$$\mathcal{A} \longrightarrow \mathcal{A} \otimes C, \quad a \longmapsto a_2 \otimes \mathcal{S}(\pi(a_1))\,\pi(a_3). \tag{I.4.4.9}$$

Lemma 4.4.5. *We have the following.*

(1) The right G-supermodule structure on $\mathfrak{g}$ uniquely extends to that on $\mathcal{H}$ so that $\mathcal{H}$ turns into an algebra object in SMod_G. In fact, $\mathcal{H}$ turns into a Hopf-algebra object in SMod_G.

(2) With the structure above, $\mathcal{A}$ turns into a Hopf-algebra object in ${}_G\mathsf{SMod}$.

(3) The resulting structures are dual to each other in the sense that

$$\langle x^\gamma, a\rangle = \langle x, {}^\gamma a\rangle, \quad \gamma \in G,\ x \in \mathcal{H},\ a \in \mathcal{A}. \tag{I.4.4.10}$$

Proof. (1) The right G-supermodule structure on $\mathfrak{g}$ uniquely extends to that on $T(\mathfrak{g})$ so that $T(\mathfrak{g})$ turns into an algebra object in SMod_G. The extended structure factors to $\mathcal{H}$, since we have $[z, w]^\gamma = [z^\gamma, w^\gamma]$, where $\gamma \in G$, $z \in \mathfrak{g}$ and $w \in \mathfrak{g}_{\bar{0}}$. One sees easily that the resulting structure on $\mathcal{H}$ is such as mentioned above.

(2) This is easy to see.

(3) Let $a \in C$, and let $x = u_1 \cdots u_r$ be an element of J with $u_i \in \mathfrak{g}_{\bar{0}}$. One sees by induction on r that (I.4.4.10) holds for these x and a, using the fact that G-actions preserve the algebra structure on J and the coalgebra structure on C.

We see from (I.4.4.2) that the left G-module structure on $\mathcal{A}$, restricted to $T_c(W) = \Bbbk \otimes T_c(W)$, is precisely what corresponds to the original right C-comodule structure on $T_c(W)$. It follows that (I.4.4.10) holds for $x \in T(\mathfrak{g}_{\bar{1}})$, $a \in T_c(W)$.

The desired equality now follows from the definition (I.4.4.6) together with the fact that the G-actions preserve the products on $\mathcal{H}$ and on $\mathcal{A}$. □

For each $n \geq 0$ we have a natural linear isomorphism (see (I.4.4.4)) from

$$\bigoplus_{i+j=n} \mathrm{Hom}_J(J \otimes T^i(\mathfrak{g}_{\bar{1}}),\, C) \otimes \mathrm{Hom}_J(J \otimes T^j(\mathfrak{g}_{\bar{1}}),\, C)$$

onto the $\Bbbk$-module

$$\bigoplus_{i+j=n} \mathrm{Hom}_{J\otimes J}((J \otimes T^i(\mathfrak{g}_{\bar{1}})) \otimes (J \otimes T^j(\mathfrak{g}_{\bar{1}})),\, C \otimes C)$$

which consists of left $J \otimes J$-module maps. The direct product $\prod_{n=0}^{\infty}$ of the isomorphisms gives the super-linear homeomorphism

$$\mathrm{Hom}_J(\mathcal{H},\, C) \,\widehat{\otimes}\, \mathrm{Hom}_J(\mathcal{H},\, C) \xrightarrow{\simeq} \mathrm{Hom}_{J\otimes J}(\mathcal{H} \otimes \mathcal{H},\, C \otimes C),$$

which is indeed the completion of the continuous map

$$f \otimes g \longmapsto (x \otimes y \mapsto f(x) \otimes g(y)),$$

where $f, g \in \mathrm{Hom}_J(\mathcal{H}, C)$, $x, y \in \mathcal{H}$. This homeomorphism will be used later.

Proposition 4.4.6. *Suppose that $f, g \in \mathrm{Hom}_J(\mathcal{H},\, C)$, $x, y \in \mathcal{H}$ and $\gamma, \delta \in G(S)$, where S is an arbitrary commutative algebra.*

(1) The product, the identity, the counit $\widehat{\varepsilon}$ and the antipode $\widehat{\mathcal{S}}$ on $\widehat{\mathcal{A}}$ are transferred to $\mathrm{Hom}_J(\mathcal{H},\, C)$

through ξ so that

$$\begin{aligned}(fg)(x) &= f(x_1)g(x_2),\\ \xi(1)(x) &= \varepsilon(x)1,\\ \widehat{\varepsilon}(f) &= \varepsilon(f(1)),\\ \langle\gamma,\ \widehat{\mathcal{S}}(f)(x)\rangle &= \langle\gamma^{-1},\ f(\mathcal{S}(x)^{\gamma^{-1}})\rangle.\end{aligned}$$

(2) Through ξ and $\xi \mathbin{\widehat{\otimes}} \xi$, the coproduct on $\widehat{\mathcal{A}}$ is translated to

$$\begin{aligned}\widehat{\Delta} : \mathrm{Hom}_J(\mathcal{H}, C) \longrightarrow &\mathrm{Hom}_J(\mathcal{H}, C) \mathbin{\widehat{\otimes}} \mathrm{Hom}_J(\mathcal{H}, C)\\ &\approx \mathrm{Hom}_{J\otimes J}(\mathcal{H}\otimes\mathcal{H}, C\otimes C)\end{aligned}$$

so that

$$\langle(\gamma,\delta),\ \widehat{\Delta}(f)(x\otimes y)\rangle = \langle\gamma\delta,\ f(x^{\delta}\, y)\rangle.$$

Here, $\langle\gamma^{\pm1}, -\rangle$, $\langle\gamma\delta, -\rangle$ and $\langle(\gamma,\delta), -\rangle$ denote the functor points in $G(S)$ and in $(G\times G)(S)$, respectively.

The formulas are essentially the same as those given in [24, Proposition 18 (2), (3)]. One will see below that the proof here, using Lemma 4.4.2, is simpler.

Proof. (1) Let $a \in \mathcal{A}$, and write as $\pi(a) = \overline{a}$. Then one has

$$^{\gamma}a = \langle\gamma^{-1}, \overline{a}_1\rangle\, a_2\, \langle\gamma, \overline{a}_3\rangle, \quad \gamma\in G. \tag{I.4.4.11}$$

To prove the last formula we may suppose $f = \xi(a)$, since we evaluate f, $\widehat{\mathcal{S}}(f)$ on $\mathcal{H}$. By using Lemma 4.4.2 we see that

$$\begin{aligned}\mathrm{LHS} &= \langle x,\, \mathcal{S}(a_1)\rangle\, \langle\gamma,\, \mathcal{S}(\overline{a}_2)\rangle = \langle\mathcal{S}(x),\, a_1\rangle\, \langle\gamma^{-1},\, \overline{a}_2\rangle\\ &= \langle\gamma^{-1},\, \overline{a}_1\rangle\, \langle\gamma,\, \overline{a}_2\rangle\, \langle\mathcal{S}(x),\, a_3\rangle\, \langle\gamma^{-1},\, \overline{a}_4\rangle\\ &= \langle\gamma^{-1},\, \overline{a}_1\rangle\, \langle\mathcal{S}(x),\, {}^{\gamma^{-1}}a_2\rangle = \mathrm{RHS}.\end{aligned}$$

The rest is easy to see.

(2) As above we may suppose $f = \xi(a)$, $a\in\mathcal{A}$. Then

$$\begin{aligned}\mathrm{LHS} &= \langle\gamma,\, \overline{a}_1\rangle\, \langle x,\, a_2\rangle\, \langle\delta,\, \overline{a}_3\rangle\, \langle y,\, a_4\rangle\\ &= \langle\gamma,\, \overline{a}_1\rangle\, \langle\delta,\, \overline{a}_2\rangle\, \langle\delta,\, \mathcal{S}(\overline{a}_3)\rangle\, \langle x,\, a_4\rangle\, \langle\delta,\, \overline{a}_5\rangle\, \langle y,\, a_6\rangle\\ &= \langle\gamma,\, \overline{a}_1\rangle\, \langle\delta,\, \overline{a}_2\rangle\, \langle x,\, {}^{\delta}a_3\rangle\, \langle y,\, a_4\rangle = \mathrm{RHS}.\end{aligned}$$

□

Recall from (I.4.4.1) that $\mathcal{I}$ is the Hopf super-ideal of $\mathcal{H}$ such that $\mathcal{H}/\mathcal{I} = \mathcal{U}$. Note that by the $\Bbbk$-flatness assumption (B3), the following statement makes sense.

Lemma 4.4.7. *$\mathcal{I}$ is G-stable, or in other words, it is C-costable. Therefore, $\mathcal{U} \in \mathsf{SMod}_G$.*

Proof. Since $[\ ,\ ] : \mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ is G-equivariant, it follows that the elements $uv + vu - [u,\, v]$ from (I.4.4.1) generate in $\mathcal{H}$ a C-costable $\Bbbk$-submodule.

Let $\rho : \mathcal{H} \to C \otimes \mathcal{H}$ be the left C-comodule structure on $\mathcal{H}$. Let $v \in \mathfrak{g}_{\bar{1}}$, and suppose $\rho(v) = \sum_i c_i \otimes v_i$. By (B3), $C \otimes \mathfrak{g}_{\bar{0}}$ is 2-torsion free. Therefore, we can conclude that

$$\rho(\frac{1}{2}[v, v]) = \sum_i c_i^2 \otimes \frac{1}{2}[v_i, v_i] + \sum_{i<j} c_i c_j \otimes [v_i, v_j], \tag{I.4.4.12}$$

by seeing that the doubles of both sides coincide. It follows that

$$\rho(v^2 - \frac{1}{2}[v, v]) = \sum_i c_i^2 \otimes (v_i^2 - \frac{1}{2}[v_i, v_i]) + \sum_{i<j} c_i c_j \otimes (v_i v_j + v_j v_i - [v_i, v_j]).$$

Since this is contained in $C \otimes \mathcal{I}$, the lemma follows. □

Since $\mathfrak{g}$ is admissible, it follows by Corollary 4.1.3 that there is a unit-preserving left J-module super-coalgebra isomorphism

$$\phi : J \otimes \wedge(\mathfrak{g}_{\bar{1}}) \xrightarrow{\simeq} \mathcal{U}. \tag{I.4.4.13}$$

We fix this ϕ for use in what follows.

Corollary 4.4.8. $\mathrm{Hom}_J(\mathcal{U},\, C)$ *is a discrete super-subalgebra of* $\mathrm{Hom}_J(\mathcal{H},\, C)$*, and is stable under* $\widehat{\mathcal{S}}$*. Moreover, the map* $\widehat{\Delta}$ *given in Proposition 4.4.6 (2) sends* $\mathrm{Hom}_J(\mathcal{U},\, C)$ *into* $\mathrm{Hom}_{J\otimes J}(\mathcal{U} \otimes \mathcal{U},\, C \otimes C)$*.*

Proof. Since $\mathcal{U}$ is finitely generated as a left J-module by (I.5.3.7), we have

$$\mathrm{Hom}_J(\mathcal{U},\, C) \subseteq \mathrm{Hom}_J(J \otimes \big(\bigoplus_{i<n} T^i(\mathfrak{g}_{\bar{1}})\big),\, C)$$

for n large enough. This means that $\mathrm{Hom}_J(\mathcal{U},\, C)$ is discrete. The rest follows easily from Lemma 4.4.7. □

Given a Harish-Chandra pair $(G, \mathfrak{g})$ as above, we define

$$\mathbf{A}(G, \mathfrak{g})$$

to be the $\Bbbk$-submodule of $\widehat{\mathcal{A}}$ such that the homeomorphism ξ given in (I.4.4.5) restricts to a linear isomorphism

$$\eta : \mathbf{A}(G, \mathfrak{g}) \xrightarrow{\simeq} \mathrm{Hom}_J(\mathcal{U},\, C). \tag{I.4.4.14}$$

In what follows we set $A := \mathbf{A}(G, \mathfrak{g})$.

Lemma 4.4.9. *We have the following.*

(1) A is a discrete super-subalgebra of $\widehat{\mathcal{A}}$, which is stable under $\widehat{\mathcal{S}}$.

(2) The canonical map $A \otimes A \to \widehat{\mathcal{A}} \widehat{\otimes} \widehat{\mathcal{A}}$ is an injection. Regarding this injection as an inclusion, we have $\widehat{\Delta}(A) \subset A \otimes A$.

(3) $(A, \widehat{\Delta}|_A, \widehat{\varepsilon}|_A, \widehat{\mathcal{S}}|_A)$ is a commutative Hopf superalgebra.

Proof. (1) This follows from Corollary 4.4.8.

(2) By using η, the canonical map above is identified with the composite of the canonical map

$$\mathrm{Hom}_J(\mathcal{U}, C) \otimes \mathrm{Hom}_J(\mathcal{U}, C) \to \mathrm{Hom}_{J \otimes J}(\mathcal{U} \otimes \mathcal{U}, C \otimes C) \tag{I.4.4.15}$$

with the embedding $\mathrm{Hom}_{J\otimes J}(\mathcal{U} \otimes \mathcal{U}, C \otimes C) \subset \mathrm{Hom}_{J\otimes J}(\mathcal{H} \otimes \mathcal{H}, C \otimes C)$. By using ϕ, the map (I.4.4.15) is identified with the canonical map

$$\mathrm{Hom}(\wedge(\mathfrak{g}_{\bar{1}}), C) \otimes \mathrm{Hom}(\wedge(\mathfrak{g}_{\bar{1}}), C) \longrightarrow \mathrm{Hom}(\wedge(\mathfrak{g}_{\bar{1}}) \otimes \wedge(\mathfrak{g}_{\bar{1}}), C \otimes C),$$

which is an isomorphism since $\wedge(\mathfrak{g}_{\bar{1}})$ is $\Bbbk$-finite free. This proves the desired injectivity. The rest follows from Corollary 4.4.8.

(3) Just as above the canonical map $A \otimes A \otimes A \to \widehat{\mathcal{A}} \widehat{\otimes} \widehat{\mathcal{A}} \widehat{\otimes} \widehat{\mathcal{A}}$ is seen to be an injection. From this we see that $\widehat{\Delta}|_A$ is coassociative. The rest is easy to see. □

The restriction $\pi|_A$ of the projection (I.4.4.3) to A is a Hopf superalgebra map, which we denote by

$$A \longrightarrow C, \quad a \longmapsto \overline{a}. \tag{I.4.4.16}$$

This notation is consistent with (I.3.1.2), as will be seen from Theorem 4.4.11 (2). We see from Remark 4.4.3 that the pairing (I.4.4.8) induces

$$\langle\ ,\ \rangle : \mathcal{U} \times A \longrightarrow \Bbbk, \tag{I.4.4.17}$$

and the following lemma holds.

Lemma 4.4.10. *The map η is given by essentially the same formula as* (I.4.4.7) *so that*

$$\eta(a)(x) = \overline{a}_1 \langle x, a_2 \rangle$$

for $a \in A$, $x \in \mathcal{U}$.

Define a map ϱ to be the composite

$$\varrho : A \xrightarrow{\eta} \mathrm{Hom}_J(\mathcal{U}, C) \cong \mathrm{Hom}(\wedge(\mathfrak{g}_{\bar{1}}), C) \xrightarrow{\varepsilon_*} \wedge(\mathfrak{g}_{\bar{1}})^* = \wedge(W), \tag{I.4.4.18}$$

where the second isomorphism is the one induced from the fixed ϕ (see (I.5.3.7)), and the following ε_* denotes $\mathrm{Hom}(\wedge(\mathfrak{g}_{\bar{1}}), \varepsilon)$.

Theorem 4.4.11. *We have the following.*

(1) The map

$$\psi : A \longrightarrow C \otimes \wedge(W); \quad a \longmapsto \overline{a}_1 \otimes \varrho(a_2)$$

is a counit-preserving isomorphism of left C-comodule superalgebras.

(2) We have natural isomorphisms

$$\overline{A} \simeq C, \qquad W^A \simeq W = \mathfrak{g}_{\bar{1}}^* \tag{I.4.4.19}$$

of Hopf algebras and of $\Bbbk$-modules, respectively.

Proof. (1) Compose the isomorphism $\mathrm{Hom}_J(\mathcal{U}, C) \simeq \mathrm{Hom}(\wedge(\mathfrak{g}_{\bar{1}}), C)$ in (I.4.4.18) with the canonical one $\mathrm{Hom}(\wedge(\mathfrak{g}_{\bar{1}}), C) \simeq C \otimes \wedge(W)$. Through the composite we will identify as $\mathrm{Hom}_J(\mathcal{U}, C) = C \otimes \wedge(W)$. Since $\langle x, a\rangle = \varepsilon(\eta(a)(x))$, $a \in A$, $x \in \mathcal{U}$, one sees that ψ is identified with η, whence it is a bijection. The desired result follows since ϱ is a counit-preserving superalgebra map.

(2) We see from the isomorphism just obtained that the Hopf superalgebra map (I.4.4.16) induces $\overline{A} \simeq C$, and the pairing (I.4.4.17), restricted to $\mathfrak{g}_{\bar{1}} \times A$, induces $W^A \simeq \mathfrak{g}_{\bar{1}}^*$. □

The lemma shows the following.

Proposition 4.4.12. $\mathbf{A}(G, \mathfrak{g}) \in \mathsf{AHSA}$.

We let

$$\mathbf{G}(G, \mathfrak{g}) \tag{I.4.4.20}$$

denote the object in ASG which corresponds to $\mathbf{A}(G, \mathfrak{g})$. Namely, $\mathbf{G}(G, \mathfrak{g}) := \mathrm{SSp}\,(\mathbf{A}(G, \mathfrak{g}))$.

Proposition 4.4.13. $(G, \mathfrak{g}) \mapsto \mathbf{G}(G, \mathfrak{g})$ *gives a functor* $\mathbf{G} : \mathsf{HCP} \to \mathsf{ASG}$.

Proof. This follows since the constructions of $\widehat{\mathcal{A}}$, $\mathrm{Hom}_J(\mathcal{H}, C)$ and $\mathrm{Hom}_J(\mathcal{U}, C)$ are all functorial, and the homeomorphism ξ is natural. □

Proposition 4.4.14. *The Harish-Chandra pair* $\mathbf{P}(\mathbf{G}(G, \mathfrak{g}))$ *associated with* $\mathbf{G}(G, \mathfrak{g})$ *is naturally isomorphic to the original* $(G, \mathfrak{g})$.

To prove this we need a lemma. Set $A := \mathbf{A}(G, \mathfrak{g})$, again. Then A is an object (indeed, a Hopf-algebra object) in ${}_G\mathsf{SMod}$, being defined by the same formula as (I.4.4.11). Recall from Lemma 4.4.7 that $\mathcal{U} \in \mathsf{SMod}_G$.

Lemma 4.4.15. *The pairing* (I.4.4.17) *is a Hopf pairing such that*

$$\langle x^\gamma, a\rangle = \langle x, {}^\gamma a\rangle \tag{I.4.4.21}$$

for $x \in \mathcal{U}$, $a \in A$.

Proof. Note that the co-adjoint coaction $\mathcal{A} \to \mathcal{A} \otimes C$ given in (I.4.4.9) is completed to $\widehat{\mathcal{A}} \to \widehat{\mathcal{A}} \otimes C$, by which $\widehat{\mathcal{A}}$ is a left G-supermodule including A as a G-subsupermodule. One sees that the pairing (I.4.4.8) satisfies the same formula as (I.4.4.10) for $a \in \widehat{\mathcal{A}}$. The resulting formula shows (I.4.4.21).

The rest follows since the pairing (I.4.4.8) satisfies the formulas in Definition 2.3.2 required to Hopf pairings. Here we understand that for $x, y \in \mathcal{H}$ and $a \in \widehat{\mathcal{A}}$, $\langle x, a_1\rangle \langle y, a_2\rangle$ represents $\langle x \otimes y, \widehat{\Delta}(a)\rangle$; this last denotes the pairing on $(\mathcal{H} \otimes \mathcal{H}) \times (\widehat{\mathcal{A}} \widehat{\otimes} \widehat{\mathcal{A}})$ which is obtained naturally from the pairing on $(\mathcal{H} \otimes \mathcal{H}) \times (\mathcal{A} \otimes \mathcal{A})$, just as (I.4.4.8) is obtained from (I.4.4.6). □

Proof of Proposition 4.4.14. We see from the definition of ψ that the pairing $\langle\ ,\ \rangle : \mathcal{U} \times A \to \Bbbk$ given in (I.4.4.17) satisfies

$$\langle \phi(x \otimes y),\, a\rangle = \langle x,\, \overline{a}_1\rangle\, \langle y,\, \varrho(a_2)\rangle$$

for $x \in J$, $y \in \wedge(\mathfrak{g}_{\bar{1}})$, $a \in A$. What appear on the right-hand side are the canonical pairings on $J \times C$ and on $\wedge(\mathfrak{g}_{\bar{1}}) \times \wedge(W)$. It follows that the pairing induces a non-degenerate pairing $\mathfrak{g} \times A^+/(A^+)^2 \to \Bbbk$. Lemma 4.4.15 shows that the last pairing induces an isomorphism $\mathrm{Lie}(\mathbf{G}) \simeq \mathfrak{g}$ of Lie superalgebras, where $\mathbf{G} := \mathbf{G}(G, \mathfrak{g})$. In addition, the isomorphism $W^A \simeq \mathfrak{g}_{\bar{1}}^*$ obtained in (I.4.4.19) is indeed G-equivariant. It follows that the Lie superalgebra isomorphism together with $\overline{A} \simeq C$ give the desired isomorphism of Harish-Chandra pairs. It is natural since the construction of (I.4.4.17) is functorial. □

Remark 4.4.16. One sees that the construction above gives an affine (not necessarily algebraic) supergroup, more generally, starting with a pair $(G, \mathfrak{g})$ such that

(i) G is a flat affine group i.e., $\mathcal{O}(G)$ is $\Bbbk$-flat,

(ii) $\mathfrak{g}$ is an admissible Lie superalgebra with $\mathfrak{g}_{\bar{1}}$ $\Bbbk$-finite (free),

(iii) $\mathfrak{g}$ is given a right G-supermodule structure such that the super-bracket on $\mathfrak{g}$ is G-equivariant, and

(iv) there is given a bilinear map $\langle\ ,\ \rangle : \mathfrak{g}_{\bar{0}} \times \mathcal{O}(G) \to \Bbbk$ such that

  (iv-a) $\langle x, ab\rangle = \langle x, a\rangle\, \varepsilon(b) + \varepsilon(a)\langle x, b\rangle$,

  (iv-b) $\langle x^\gamma, a\rangle = \langle x, {}^\gamma a\rangle$,

  (iv-c) $[z, x] = \langle x, z_{-1}\rangle\, z_0$,

  where $x \in \mathfrak{g}_{\bar{0}}$, $a, b \in \mathcal{O}(G)$, $\gamma \in G$, $z \in \mathfrak{g}$, and $\mathfrak{g} \to \mathcal{O}(G) \otimes \mathfrak{g}$; $z \mapsto z_{-1} \otimes z_0$ denotes the left $\mathcal{O}(G)$-supercomodule structure on $\mathfrak{g}$ which corresponds to the given right G-supermodule structure.

Here we do not assume (B1) or G being algebraic.

Remark 4.4.17. Given a super Lie group, say $\mathfrak{G}$, we have in mind as G and $\mathfrak{g}$ above, the universal algebraic hull of the associated Lie group $\mathfrak{G}_{\mathrm{red}}$ and the Lie superalgebra $\mathrm{Lie}(\mathfrak{G})$ of $\mathfrak{G}$, respectively. See [24, Remark 11] for a similar construction in an alternative situation.

4.5 The category equivalence

The following is our main result.

Theorem 4.5.1. *We have a category equivalence* $\mathsf{ASG} \approx \mathsf{HCP}$*. In fact the functors* $\mathbf{P} : \mathsf{ASG} \to \mathsf{HCP}$ *and* $\mathbf{G} : \mathsf{HCP} \to \mathsf{ASG}$ *are quasi-inverse to each other.*

Since Proposition 4.4.14 shows that $\mathbf{P} \circ \mathbf{G}$ is naturally isomorphic to the identity functor id, it remains to prove $\mathbf{G} \circ \mathbf{P} \simeq \mathrm{id}$.

Let $\mathbf{G} \in \mathsf{ASG}$. Set

$$A := \mathcal{O}(\mathbf{G}), \quad \mathfrak{g} := \mathrm{Lie}(\mathbf{G}), \quad \mathcal{U} := \mathcal{U}(\mathfrak{g}), \quad G := \mathbf{G}_{\mathrm{ev}}.$$

Lemma 4.5.2. *The natural embedding $\mathfrak{g} \subset A^*$ uniquely extends to a superalgebra map $\mathcal{U} \to A^*$. The associated pairing $\langle\ ,\ \rangle : \mathcal{U} \times A \to \Bbbk$ is a Hopf pairing.*

Proof. The superalgebra map $T(\mathfrak{g}) \to A^*$ which extends $\mathfrak{g} \subset A^*$ kills the first elements in (I.2.5.1), by definition of the super-bracket. For $v \in \mathfrak{g}_{\bar{1}}$ it kills $2v^2 - [v,\, v]$, whence it does $v^2 - \frac{1}{2}[v,\, v]$ since A^* is 2-torsion free. This proves the first assertion.

As for the second it is easy to see $\langle x, 1\rangle = \varepsilon(x)$, $x \in \mathcal{U}$. It remains to prove

$$\langle x,\, ab\rangle = \langle x_1,\, a\rangle\, \langle x_2,\, b\rangle$$

for $x \in \mathcal{U}$, $a, b \in A$. We may suppose that x is of the form $x = u_1 \cdots u_r$, where u_i are homogeneous elements in $\mathfrak{g}$. Then the equation is proved by induction on the length r. □

Recall $A \in {}_G\mathsf{SMod}$, $\mathcal{U} \in \mathsf{SMod}_G$; see (I.4.3.1) or (I.4.4.11) as for A, and see Lemma 4.4.7 as for $\mathcal{U}$. Indeed, A and $\mathcal{U}$ are Hopf-algebra objects in the respective categories.

Lemma 4.5.3. *The Hopf pairing $\langle\ ,\ \rangle : \mathcal{U} \times A \to \Bbbk$ just obtained satisfies the same formula as* (I.4.4.21).

Proof. The G-module structure on $\mathfrak{g}$ is transposed from that on $A^+/(A^+)^2$. Therefore, the formula holds for every $x \in \mathfrak{g}$ and for any $a \in A$. The desired formula follows by induction, as in the last proof; see also the proof of Lemma 4.4.5 (3). □

Set

$$C := \mathcal{O}(G), \quad J := \mathcal{U}(\mathfrak{g}_{\bar{0}}).$$

Note $\mathbf{P}(\mathbf{G}) = (G, \mathfrak{g})$. We aim to show that the affine Hopf superalgebra $\mathbf{A}(G, \mathfrak{g})$, which is constructed from this last Harish-Chandra pair as in the previous subsection, is naturally isomorphic to the present A. By using the Hopf pairing above, we define

$$\eta' : A \to \mathrm{Hom}_J(\mathcal{U},\, C); \quad a \longmapsto (x \mapsto \overline{a}_1\, \langle x, a_2\rangle),$$

where $a \in A$, $x \in \mathcal{U}$. Note that $\mathrm{Hom}_J(\mathcal{U}, C)$ has the Hopf superalgebra structure which is transferred from $\mathbf{A}(G, \mathfrak{g})$ thorough η (see (I.4.4.14)), and which is presented by the formulas given in Proposition 4.4.6 with the obvious modification. We remark here that our η' above is essentially the same, up to sign, as the existing ones such as η^* in [5, p.133, lines 2–3]. See [24, Remark 1] for the difference of sign-rule.

Proposition 4.5.4. *η' is an isomorphism of Hopf superalgebras.*

Proof. Using Lemma 4.5.3 one computes in the same way as proving Proposition 4.4.6 (2) so that

$$\langle(\gamma,\delta),\ (\eta'(a_1)\otimes\eta'(a_2))(x\otimes y)\rangle = \langle\gamma\delta,\ \eta'(a)(x^{\delta}\,y)\rangle,$$

where $a\in A$, $\gamma,\delta\in G$, $x,y\in\mathcal{U}$. The right-hand side equals $\langle(\gamma,\delta),\ \Delta(\eta'(a))(x\otimes y)\rangle$, by the formula giving the coproduct on $\mathrm{Hom}_J(\mathcal{U},C)$. Therefore, η' preserves the coproduct. It is easy to see that η' preserves the remaining structure maps, and is hence a Hopf superalgebra map.

Set $W := W^A$. Choose ϕ such as in (I.5.3.7), and define $\varrho' : A\to\wedge(W)$ as ϱ in (I.4.4.18), with η replaced by η'. Then as was seen for η in the proof of Theorem 4.4.11 (1), η' is identified with

$$\psi' : A\to C\otimes\wedge(W);\quad a\longmapsto \overline{a}_1\otimes\varrho'(a_2). \tag{I.4.5.1}$$

Since one sees that this ψ' satisfies the assumption of Lemma 4.5.5 below, the lemma proves that ψ' and so η' are isomorphisms. □

Lemma 4.5.5. *In general, let A be a split affine Hopf superalgebra, and set $C := \overline{A}$, $W := W^A$. Let $\psi : A\to C\otimes\wedge(W)$ be a counit-preserving map of left C-comodule superalgebras. Assume that the composite $(\varepsilon\otimes\varpi)\circ\psi : A\to W$, where $\varpi:\wedge(W)\to W$ denotes the canonical projection, coincides with the canonical projection $A\to A_{\bar 1}/A_{\bar 0}^{+}A_{\bar 1} = W$. Then ψ is necessarily an isomorphism.*

Proof. Let $B := C\otimes\wedge(W)$. Set $\mathfrak{a} := (A_{\bar 1})$ and $\mathfrak{b} := (B_{\bar 1})$ $(= C\otimes\wedge(W)^{+})$ in A and in B, respectively. Since $\psi(\mathfrak{a}^n)\subset\mathfrak{b}^n$ for every $n\ge 0$, there is induced a counit-preserving, left C-comodule $\mathbb{N}$-graded algebra map

$$\mathrm{gr}\,\psi : \mathrm{gr}\,A = \bigoplus_{n=0}^{\infty}\mathfrak{a}^n/\mathfrak{a}^{n+1}\longrightarrow \mathrm{gr}\,B = \bigoplus_{n=0}^{\infty}\mathfrak{b}^n/\mathfrak{b}^{n+1}.$$

One sees that $\mathrm{gr}\,B = B = C\otimes\wedge(W)$. Since A is split, we have as in [22, Proposition 4.9 (2)], a canonical isomorphism $\mathrm{gr}\,A\simeq C\otimes\wedge(W)$, through which we will identify the two. Then $\mathrm{gr}\,\psi$ is a counit-preserving endomorphism of the left C-comodule $\mathbb{N}$-graded algebra $C\otimes\wedge(W)$. Being a counit-preserving endomorphism of the left C-comodule algebra C, $\mathrm{gr}\,\psi(0)$ is the identity on C. This together with the assumption above imply that $\mathrm{gr}\,\psi(1)$ is the identity on $C\otimes W$. It follows that $\mathrm{gr}\,\psi$ is an isomorphism. Since the affinity assumption implies $\mathrm{gr}\,A(n) = 0 = \mathrm{gr}\,B(n)$ for $n\gg 0$, one sees that ψ is an isomorphism. □

Proof of Theorem 4.5.1. Since we see that η and η' are both natural, it follows that $\mathbf{A}(G,\mathfrak{g})$ and A are naturally isomorphic. This proves $\mathbf{G}\circ\mathbf{P}\simeq\mathrm{id}$, as desired. □

Remark 4.5.6. Suppose that $\Bbbk$ is a field of characteristic $\neq 2$. Identify ASG with AHSA, through the obvious category anti-isomorphism. Identify our HCP defined by Definition 4.2.1 with that defined by [24, Definition 7], through the category anti-isomorphism given in Remark 4.2.3. Then the category equivalences **P** and **G** given by Theorem 4.5.1 are easily identified with those $A\mapsto(\overline{A},W^A)$ and $(C,W)\mapsto A(C,W)$ given by [24, Theorem 29].

Definition 4.5.7 ([27, Definition 6.3])**.** Let $(G,\mathfrak{g})$ be a Harish-Chandra pair over $\Bbbk$. Let K be a flat closed subgroup of G and $\mathfrak{k}$ be a free Lie subsuperalgebra of $\mathfrak{g}$ with $\mathrm{Lie}(K) = \mathfrak{k}_{\bar 0}$. The pair $(K,\mathfrak{k})$ is called a *sub-pair* of $(G,\mathfrak{g})$ if $\mathfrak{k}_{\bar 1}$ is K-stable in $\mathfrak{g}_{\bar 1}$.

In this case, $(K, \mathfrak{k})$ is a Harish-Chandra pair and the corresponding algebraic supergroup $\mathbf{G}(K, \mathfrak{k})$ is a closed subsupergroup of $\mathbf{G}(G, \mathfrak{g})$. Conversely, for a flat closed subsupergroup $\mathbf{K}$ of $\mathbf{G}$, the corresponding Harish-Chandra pair $(\mathbf{K}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{K}))$ is a sub-pair of $(\mathbf{G}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{G}))$. In this way, the map $\mathbf{K} \mapsto (\mathbf{K}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{K}))$ gives a bijection from the set of all flat closed supergroups of $\mathbf{G}$ to the set of all sub-pairs of $(\mathbf{G}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{G}))$.

4.6 Generalization using 2-operations

In this section, we work over an *arbitrary* non-zero commutative ring $\Bbbk$. We will refine Gavarini's category equivalence; see Theorem 4.6.9.

4.6.1 2-Operations and universal enveloping superalgebras

Let $\mathfrak{g}$ be a Lie superalgebra having a 2-operation $(-)^{\langle 2 \rangle}$, see Section 2.5.2. In this section, we let $\mathcal{U}(\mathfrak{g})$ denote the cocommutative Hopf superalgebra which is defined as in [12, Section 4.3.4]. This is the quotient Hopf superalgebra of the tensor superalgebra $T(\mathfrak{g})$ divided by the super-ideal generated by the homogeneous primitives

$$zw - (-1)^{|z||w|}wz - [z, w], \quad v^2 - v^{\langle 2 \rangle}, \tag{I.4.6.1}$$

where z and w are homogeneous elements in $\mathfrak{g}$, and $v \in \mathfrak{g}_{\bar{1}}$. The only difference from the definition given in Section 2.5.1 is that the second generators $v^2 - \frac{1}{2}[v, v]$ in (I.2.5.1) are here replaced (indeed, generalized) by $v^2 - v^{\langle 2 \rangle}$.

Lemma 4.6.1. *Suppose that the homogeneous components $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ are both $\Bbbk$-free, and choose their totally ordered bases $\mathfrak{X}_{\bar{0}}$ and $\mathfrak{X}_{\bar{1}}$. Then $\mathcal{U}(\mathfrak{g})$ has the following monomials as a $\Bbbk$-free basis,*

$$a_1^{r_1} \cdots a_m^{r_m} x_1 \cdots x_n,$$

where $a_1 < \cdots < a_m$ in $\mathfrak{X}_{\bar{0}}$, $r_i > 0$, $m \geq 0$, and $x_1 < \cdots < x_n$ in $\mathfrak{X}_{\bar{1}}$, $n \geq 0$.

Proof. To prove Proposition 4.1.1 we used the Diamond Lemma [1, Proposition 7.1] for R-rings. But we use here the Diamond Lemma [1, Theorem 1.2] for $\Bbbk$-algebras. We suppose that $\mathfrak{X}_{\bar{0}} \cup \mathfrak{X}_{\bar{1}}$ is the set of generators, and extend the total orders on $\mathfrak{X}_{\epsilon}$, $\epsilon = \bar{0}, \bar{1}$, to the set so that $a < x$ whenever $a \in \mathfrak{X}_{\bar{0}}$, $x \in \mathfrak{X}_{\bar{1}}$. The reduction system consists of the obvious reductions arising from the super-bracket, and

$$x^2 \to x^{\langle 2 \rangle}, \quad x \in \mathfrak{X}_{\bar{1}},$$

where the last $x^{\langle 2 \rangle}$ is supposed to be presented as a linear combination of elements in $\mathfrak{X}_{\bar{0}}$. It is essential to prove that the overlap ambiguities which may occur when we reduce the words

- $xxa, \quad x \in X_{\bar{1}}, a \in \mathfrak{X}_{\bar{0}}$,
- $xyz, \quad x = y \geq z$ or $x \geq y = z$ in $\mathfrak{X}_{\bar{1}}$

are resolvable. This is easily proved (indeed, more easily than was in the proof of Proposition 4.1.1), by using Condition (iii) in Definition 2.5.4. For example, the word xxa is reduced on the one hand as

$$xxa \to x[x,a] + xax \to x[x,a] + [x,a]x + ax^{\langle 2\rangle} \to [x,[x,a]] + ax^{\langle 2\rangle},$$

and on the other hand as

$$xxa \to x^{\langle 2\rangle}a.$$

The two results coincide by (iii). □

Remark 4.6.2. To use Condition (iii) as above, we cannot treat $\mathcal{U}(\mathfrak{g})$ as a $J = \mathcal{U}(\mathfrak{g}_{\bar{0}})$-ring as in the the proof of Proposition 4.1.1. Indeed, to reduce the word xxa with $a \in J$ in the proof, we are not allowed to present a as (a linear combination of) bc with $b \in \mathfrak{g}_{\bar{0}}$, $c \in J$, and to reduce as

$$xxa \to xxbc \to x[x,b]c + xbxc,$$

because by the first step, the lengths of words increase, $\mathrm{length}(xx*) < \mathrm{length}(xx**)$; see the proof of [24, Lemma 11].

Corollary 4.6.3 (cf. [12, (4.7)])**.** *If $\mathfrak{g}_{\bar{0}}$ is $\Bbbk$-finite projective and $\mathfrak{g}_{\bar{1}}$ is $\Bbbk$-free, then the same result as Corollary 4.1.3 holds, that is, there exists a unit-preserving, left $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module super-coalgebra isomorphism $\mathcal{U}(\mathfrak{g}_{\bar{0}}) \otimes \wedge(\mathfrak{g}_{\bar{1}}) \xrightarrow{\simeq} \mathcal{U}(\mathfrak{g})$.*

Proof. Choose a totally ordered basis $\mathfrak{X}$ of $\mathfrak{g}_{\bar{1}}$. Then the left $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module $\mathcal{U}(\mathfrak{g})$ is free with the free basis

$$x_1 x_2 \cdots x_n, \tag{I.4.6.2}$$

where $x_1 < x_2 < \cdots < x_n$ in $\mathfrak{X}$, $n \geqslant 0$. We define a left $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module (supercoalgebra) map $\phi : \mathcal{U}(\mathfrak{g}_{\bar{0}}) \otimes \wedge(\mathfrak{g}_{\bar{1}}) \to \mathcal{U}(\mathfrak{g})$ by

$$\phi(1 \otimes (x_1 \wedge \cdots \wedge x_n)) = x_1 \cdots x_n,$$

where $x_1 < \cdots < x_n$ in $\mathfrak{X}$, $n \geq 0$. To prove that this is bijective, it suffices to prove the localization $\phi_{\mathfrak{m}}$ at each maximal ideal $\mathfrak{m}$ of $\Bbbk$ is bijective. Note that $\mathfrak{g}_{\mathfrak{m}}$ is a $\Bbbk_{\mathfrak{m}}$-Lie superalgebra given a 2-operation by Proposition 2.5.6, and

$$\mathcal{U}(\mathfrak{g}_{\bar{0}})_{\mathfrak{m}} = \mathcal{U}((\mathfrak{g}_{\bar{0}})_{\mathfrak{m}}), \quad (\wedge(\mathfrak{g}_{\bar{1}}))_{\mathfrak{m}} = \wedge((\mathfrak{g}_{\bar{1}})_{\mathfrak{m}}), \quad \mathcal{U}(\mathfrak{g})_{\mathfrak{m}} = \mathcal{U}(\mathfrak{g}_{\mathfrak{m}}).$$

Since $(\mathfrak{g}_{\bar{0}})_{\mathfrak{m}}$ is $\Bbbk_{\mathfrak{m}}$-free under the assumption above, Lemma 4.6.1 shows that $\phi_{\mathfrak{m}}$ is bijective. □

Let $\mathbf{G}$ be an *affine supergroup*; see Section 3.1. Recall from Section 3.2

$$\mathrm{Lie}(\mathbf{G}) := (\mathcal{O}(\mathbf{G})^+/(\mathcal{O}(\mathbf{G})^+)^2)^*.$$

Note that the proof of Proposition 3.2.2 does not use the assumption that $\Bbbk$ is 2-torsion free. From the proposition and the proof one sees the following.

Proposition 4.6.4. *Let* $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$.

(1) $\mathfrak{g}$ *is naturally a Lie superalgebra.*

(2) Given $v \in \mathfrak{g}_{\bar{1}}$, *the square* v^2 *in* $\mathcal{O}(\mathbf{G})^*$ *is contained in* $\mathfrak{g}_{\bar{0}}$. *Moreover, the square map* $(-)^2 : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ *gives a* 2*-operation on* $\mathfrak{g}$.

We will suppose that $\mathrm{Lie}(\mathbf{G})$ is given this specific 2-operation. One sees that Lie gives a functor from the category of affine supergroups to the category of Lie superalgebras given 2-operations. A morphism of the latter category is a morphism in $\mathsf{SMod}_{\Bbbk}$ which preserves the super-bracket and the 2-operation.

Remark 4.6.5. Let $\mathfrak{g}$ be a Lie superalgebra. Note from Section 2.4 that the deformation ${}_{\sigma}\mathfrak{g}$ by σ is the object $\mathfrak{g}$ in $\mathsf{SMod}_{\Bbbk}$ given the super-bracket

$$_{\sigma}[z, w] := (-1)^{|z||w|}[z, w], \quad z, w \in \mathfrak{g}$$

deformed from the original super-bracket $[z, w]$. If $\mathfrak{g}$ is given a 2-operation, we suppose that ${}_{\sigma}\mathfrak{g}$ is given the deformed 2-operation

$$v^{\sigma\langle 2\rangle} := -v^{\langle 2\rangle}, \quad v \in \mathfrak{g}_{\bar{1}}.$$

This indeed defines a 2-operation on ${}_{\sigma}\mathfrak{g}$, as is easily seen.

Let $\mathbf{G}$ be an affine supergroup. As is seen from the last paragraph of Section 2.4, the Lie superalgebra $\mathrm{Lie}(\mathbf{G})$ given the 2-operation as defined above is different from that defined in [27, Appendix]. In fact, the two are the deformations of each other by σ.

4.6.2 Definitions of Gavarini's categories

Recall from [12, Definitions 3.2.6 and 4.1.2] the following definitions of two categories, $(\text{gss-fsgroups})_{\Bbbk}$, $(\text{sHCP})_{\Bbbk}$.

Let $(\text{gss-fsgroups})_{\Bbbk}$ denote the category of the affine supergroups $\mathbf{G}$ such that when we set $A := \mathcal{O}(\mathbf{G})$,

(E1) A is split (Definition 3.1.1),

(E2) $\overline{A}/(\overline{A}^+)^2$ is $\Bbbk$-finite projective, and

(E3) $W^A = A_{\bar{1}}/A_{\bar{0}}^+ A_{\bar{1}}$ is $\Bbbk$-finite (free).

The morphisms in $(\text{gss-fsgroups})_{\Bbbk}$ are the natural transformations of group-valued functors.

Let $(G, \mathfrak{g})$ be a pair of an affine group G and a Lie superalgebra $\mathfrak{g}$ given a 2-operation, such that $\mathfrak{g}_{\bar{1}}$ is $\Bbbk$-finite free and is given a right G-module structure. Suppose that this pair satisfies

(F1) $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$,

(F2) $\mathcal{O}(G)/(\mathcal{O}(G)^+)^2$ is $\Bbbk$-finite projective, so that $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$ is necessarily $\Bbbk$-finite projective, and it is naturally a right G-module (recall from Section 4.2 that the corresponding left $\mathcal{O}(G)$-comodule structure on $\mathrm{Lie}(G)$ is transposed from the right co-adjoint $\mathcal{O}(G)$-coaction on $\mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2$), (F3) the right $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module structure on $\mathfrak{g}_{\bar{1}}$ induced from the given right G-module structure coincides with the right adjoint $\mathfrak{g}_{\bar{0}}$-action on $\mathfrak{g}_{\bar{1}}$,

(F4) the restricted super-bracket $[\ ,\ ] : \mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ is G-equivariant, and

(F5) the diagram

$$\begin{array}{ccc} \mathfrak{g}_{\bar{1}} & \xrightarrow{(-)^{\langle 2 \rangle}} & \mathfrak{g}_{\bar{0}} \\ \downarrow & & \downarrow \\ \mathcal{O}(G) \otimes \mathfrak{g}_{\bar{1}} & \xrightarrow{(-)^{\langle 2 \rangle}_{\mathcal{O}(G)}} & \mathcal{O}(G) \otimes \mathfrak{g}_{\bar{0}} \end{array}$$

commutes, where the vertical arrows are the left $\mathcal{O}(G)$-comodule structures.

One sees that under (F4), Condition (F5) is equivalent to

$$(v_S^{\langle 2 \rangle})^{\gamma} = (v^{\gamma})_S^{\langle 2 \rangle}, \quad v \in \mathfrak{g}_{\bar{1}} \otimes S,\ \gamma \in G(S),$$

where S is an arbitrary commutative algebra.

Let $(\mathrm{sHCP})_{\Bbbk}$ denote the category of all those pairs $(G, \mathfrak{g})$ which satisfy Conditions (F1)–(F5) above. A morphism $(G, \mathfrak{g}) \to (G', \mathfrak{g}')$ in $(\mathrm{sHCP})_{\Bbbk}$ is a pair (α, β) of a morphism $\alpha : G \to G'$ of affine groups and a Lie superalgebra map $\beta = \beta_{\bar{0}} \oplus \beta_{\bar{1}} : \mathfrak{g} \to \mathfrak{g}'$, which satisfies Conditions (iv), (v) in Definition 4.2.1, and

(vi) $\beta_{\bar{0}}(v^{\langle 2 \rangle}) = \beta_{\bar{1}}(v)^{\langle 2 \rangle}, \quad v \in \mathfrak{g}_{\bar{1}}$.

Remark 4.6.6. One sees from Lemma 2.5.5 that if $\Bbbk$ is 2-torsion free, then our HCP and ASG (see Definition 4.2.1 and Section 4.3), roughly speaking, coincide with $(\mathrm{sHCP})_{\Bbbk}$ and $(\text{gss-fsgroups})_{\Bbbk}$, respectively. To be precise, ours are more restrictive in that for objects $(G, \mathfrak{g}) \in \mathsf{HCP}$, $\mathbf{G} \in \mathsf{ASG}$, the commutative Hopf algebras $\mathcal{O}(G)$ and $\mathcal{O}(\mathbf{G}_{\mathrm{ev}})$ are assumed to be affine and $\Bbbk$-flat.

We may remove the affinity assumption so long as (B1) and (C3) are assumed. But the assumption seems natural, since if $\Bbbk$ is a field of characteristic $\neq 2$, it ensures that (B1) and (C3) are satisfied, so that our Theorem 4.5.1 then coincides with the known category equivalence between all algebraic supergroups and the Harish-Chandra pairs; see Remark 4.5.6.

Note from (I.4.4.12) that under the $\Bbbk$-flatness assumption above, $\mathcal{O}(G) \otimes \mathfrak{g}_{\bar{1}}$ is 2-torsion free, and Condition (F5) for $v^{\langle 2 \rangle} = \frac{1}{2}[v, v]$ is necessarily satisfied. Recall that the condition is not contained in the axioms for objects in HCP.

4.6.3 A refinement of Gavarini's equivalence

Our category equivalences between $(\text{gss-fsgroups})_{\Bbbk}$ and $(\mathrm{sHCP})_{\Bbbk}$ will be presented differently from Gavarini's Φ_g, Ψ_g; see Remark 4.6.10. So, we will use different symbols, $\mathbf{P}'$, $\mathbf{G}'$, to denote them.

Let us construct a functor

$$\mathbf{P}' : (\text{gss-fsgroups})_{\Bbbk} \longrightarrow (\mathrm{sHCP})_{\Bbbk}.$$

Given $\mathbf{G} \in (\text{gss-fsgroups})_\Bbbk$, set $G := \mathbf{G}_{\text{ev}}$, $\mathfrak{g} := \text{Lie}(\mathbf{G})$. Recall from Proposition 4.6.4 and the following remark that $\mathfrak{g}$ is a Lie superalgebra given the square map as a 2-operation. As in Lemma 3.2.3 one has $\mathfrak{g}_{\bar{0}} \cong \text{Lie}(G)$, through which we will identify the two, and suppose $\mathfrak{g}_{\bar{0}} = \text{Lie}(G)$. Since $\mathfrak{g}$ is $\Bbbk$-finite projective by (E2), (E3), the co-adjoint $\mathcal{O}(G)$-coaction on $\mathcal{O}(\mathbf{G})^+/(\mathcal{O}(\mathbf{G})^+)^2$ (see (I.4.3.1)) is transposed to $\mathfrak{g}$, so that $\mathfrak{g}$ is a right G-supermodule. The restricted right G-module structure on $\mathfrak{g}_1$ satisfies (F3), (F4), as was seen in the proof of Lemma 4.3.4. To conclude $(G, \mathfrak{g}) \in (\text{sHCP})_\Bbbk$, it remains to prove the following.

Lemma 4.6.7. *The condition (F5) is satisfied.*

Proof. Let $v \mapsto \sum_i c_i \otimes v_i$ denote the left $\mathcal{O}(G)$-comodule structure $\mathfrak{g}_{\bar{1}} \to \mathcal{O}(G) \otimes \mathfrak{g}_{\bar{1}}$ on $\mathfrak{g}_{\bar{1}}$. Let $a \mapsto a_0 \otimes a_1$ denote the right co-adjoint $\mathcal{O}(G)$-coaction $\mathcal{O}(\mathbf{G}) \to \mathcal{O}(\mathbf{G}) \otimes \mathcal{O}(G)$ on $\mathcal{O}(\mathbf{G})$. Since $\mathfrak{g}$ is $\Bbbk$-finite projective, we have the canonical injection $\mathcal{O}(G) \otimes \mathfrak{g} = \text{Hom}_\Bbbk(\mathfrak{g}^*, \mathcal{O}(G)) \to \text{Hom}_\Bbbk(\mathcal{O}(\mathbf{G}), \mathcal{O}(G))$. Therefore, it suffices to prove

$$\langle v^2, a_0 \rangle\, a_1 = \sum_i c_i^2 \,\langle v_i^2, a \rangle + \sum_{i<j} c_i c_j \,\langle [v_i, v_j], a \rangle$$

for $v \in \mathfrak{g}_{\bar{1}}$, $a \in \mathcal{O}(\mathbf{G})$, where $\langle\ ,\ \rangle$ denotes the canonical pairing $\mathcal{O}(\mathbf{G})^* \times \mathcal{O}(\mathbf{G}) \to \Bbbk$. This is proved as follows.

$$\begin{aligned} \text{LHS} &= \langle v, (a_1)_0 \rangle \langle v, (a_2)_0 \rangle (a_1)_1 (a_2)_1 \\ &= \sum_{i,j} c_i c_j \langle v_i, a_1 \rangle \langle v_j, a_2 \rangle = \sum_{i,j} c_i c_j \langle v_i v_j, a \rangle = \text{RHS}. \end{aligned}$$

□

Let $\mathbf{P}'(\mathbf{G})$ denote the thus obtained object $(G, \mathfrak{g})$ in $(\text{sHCP})_\Bbbk$. As in Proposition 4.3.5, we see that $\mathbf{P}' : (\text{gss-fsgroups})_\Bbbk \to (\text{sHCP})_\Bbbk$ gives the desired functor, since the Lie superalgebra map induced from a morphism of affine supergroups obviously preserves the 2-operation.

Let us construct a functor

$$\mathbf{G}' : (\text{sHCP})_\Bbbk \longrightarrow (\text{gss-fsgroups})_\Bbbk.$$

Let $(G, \mathfrak{g}) \in (\text{sHCP})_\Bbbk$. Then the natural right G-module structure on $\mathfrak{g}_{\bar{0}} = \text{Lie}(G)$ and the given right G-module structure on $\mathfrak{g}_{\bar{1}}$ amount to a right G-supermodule structure on $\mathfrak{g}$, by which the super-bracket on $\mathfrak{g}$ is G-equivariant, as is seen as in Remark 4.2.4 by using (F3), (F4).

Remark 4.6.8. According to the original definition [12, Definition 4.1.2], the proved G-equivariance is assumed as an axiom for objects in $(\text{sHCP})_\Bbbk$. But it can be weakened to (F4), as was just seen.

Using (F5), one sees as in Lemma 4.4.7 (indeed, more easily) that the right G-supermodule structure on $\mathfrak{g}$ uniquely extends to $\mathcal{U}(\mathfrak{g})$, so that $\mathcal{U}(\mathfrak{g})$ turns into a Hopf-algebra object in SMod_G. By using an isomorphism $\mathcal{U}(\mathfrak{g}_{\bar{0}}) \otimes \wedge(\mathfrak{g}_{\bar{1}}) \simeq \mathcal{U}(\mathfrak{g})$ such as given by Corollary 4.6.3, we can trace the argument in Section 4.4, to construct a split commutative Hopf superalgebra, $A = \mathbf{A}(G, \mathfrak{g})$, such that

$$A \simeq \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}_{\bar{0}})}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G)), \qquad \overline{A} \simeq \mathcal{O}(G), \qquad W^A \simeq \mathfrak{g}_{\bar{1}}^*.$$

It follows that this A satisfies (E1)–(E3). We let $\mathbf{G}'(G, \mathfrak{g})$ denote the affine supergroup corresponding to A. Then one sees that $\mathbf{G}'(G, \mathfrak{g}) \in$ (gss-fsgroups)$_\Bbbk$, and

$$(G, \mathfrak{g}) \longmapsto \mathbf{G}'(G, \mathfrak{g})$$

gives the desired functor. As for the fuctoriality, note that Condition (iii) given just above Remark 4.6.6 is used to see that a morphism (α, β) in (sHCP)$_\Bbbk$ induces, in particular, a Hopf superalgebra map $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}')$; see the proof of Proposition 4.4.13.

Theorem 4.6.9 ([12, Theorem 4.3.14])**.** *We have a category equivalence*

$$\text{(gss-fsgroups)}_\Bbbk \approx \text{(sHCP)}_\Bbbk.$$

In fact the functors $\mathbf{P}'$ *and* $\mathbf{G}'$ *constructed above are quasi-inverse to each other.*

Proof. To prove $\mathbf{P}' \circ \mathbf{G}' \simeq \mathrm{id}$, $\mathbf{G}' \circ \mathbf{P}' \simeq \mathrm{id}$, we can trace the argument of Section 4.5 proving $\mathbf{P} \circ \mathbf{G} \simeq \mathrm{id}$, $\mathbf{G} \circ \mathbf{P} \simeq \mathrm{id}$, except in two points.

First, to prove $\mathbf{P}' \circ \mathbf{G}' \simeq \mathrm{id}$, we have to show that if $(G, \mathfrak{g}) \in$ (sHCP)$_\Bbbk$, and we set $\mathbf{G} := \mathbf{G}'(G, \mathfrak{g})$, then the natural Lie superalgebra isomorphism $\mathrm{Lie}(\mathbf{G}) \simeq \mathfrak{g}$ as given in the proof of Proposition 4.4.14 preserves the 2-operation. Note that we have a Hopf pairing $\mathcal{U}(\mathfrak{g}) \times \mathcal{O}(\mathbf{G}) \to \Bbbk$ as given in (I.4.4.17), and it restricts to a non-degenerate pairing $\mathfrak{g} \times \mathcal{O}(\mathbf{G})^+/(\mathcal{O}(\mathbf{G})^+)^2 \to \Bbbk$, which induces the isomorphism above. Therefore, we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\simeq} & \mathrm{Lie}(\mathbf{G}) \\ \downarrow & & \downarrow \\ \mathcal{U}(\mathfrak{g}) & \longrightarrow & \mathcal{O}(\mathbf{G})^*, \end{array}$$

where the arrow in the bottom is the map induced from the Hopf pairing above. Given $v \in \mathfrak{g}_1$, the composite $\mathfrak{g} \xrightarrow{\simeq} \mathrm{Lie}(\mathbf{G}) \hookrightarrow \mathcal{O}(\mathbf{G})^*$, which factors through $\mathcal{U}(\mathfrak{g})$ as above, sends $v^{\langle 2 \rangle}$ to v^2. This proves the desired result.

Second, to prove $\mathbf{G}' \circ \mathbf{P}' \simeq \mathrm{id}$, we should remark that Lemma 4.5.5 can apply, since the conclusion of the lemma holds so long as W^A is $\Bbbk$-finite, even if the split commutative Hopf superalgebra A is not finitely generated. □

Remark 4.6.10. In [12], details are not given for the following two.

(1) 2-*operations.* Condition (F5) is not explicitly given in [12]. The functor Φ_g : (gss-fsgroups)$_\Bbbk \to$ (sHCP)$_\Bbbk$ in [12] is almost the same as our $\mathbf{P}'$, but it does not specify the associated 2-operation; see [12, Proposition 4.1.3]. Accordingly, it is not proved that $\Phi_g(G_{\mathcal{P}}) \xrightarrow{\simeq} \mathcal{P}$ preserves the 2-operation on the associated Lie superalgebras; see the first paragraph of the proof of [12, Theorem 4.3.14].

(2) *Proof of* $\mathcal{U}(\mathfrak{g}_{\bar{0}})\otimes\wedge(\mathfrak{g}_{\bar{1}}) \simeq \mathcal{U}(\mathfrak{g})$. This isomorphism is what was proved by our Corollary 4.6.3. The proof of [12] given in the three lines above Eq. (4.7) is rather sketchy, and it might overlook the localization argument used in our proof. Note that, the argument uses Proposition 2.5.6; this last result or any equivalent one is not given in [12].

Chapter 5

Functor Points Constructions

In this chapter, we work over a non-zero commutative ring $\Bbbk$. Working over an arbitrary commutative ring, we should careful to define super-commutativity. Thus, in this chapter, we re-define the notion of commutative superalgebra. A superalgebra R is said to be *commutative* if $ab = (-1)^{|a||b|}ba$ for $a, b \in R$ and $a^2 = 0$ for $a \in R_{\bar{1}}$.

5.1 Base extension of abstract groups

Suppose that the quintuple

$$(\Sigma,\ F,\ G,\ i,\ \alpha)$$

consists of groups Σ, F and G, a group map $i : F \to G$, and anti-group map $\alpha : G \to \mathrm{Aut}(\Sigma)$ such that

(G1) F is a subgroup of Σ,

(G2) $\varphi^{i(f)} = f^{-1}\varphi f$ for all $f \in F$, $\varphi \in \Sigma$,

(G3) $f^g \in F$ and $i(f^g) = g^{-1}i(f)g$,

where $f \in F$, $g \in G$, $\varphi \in \Sigma$, and we let φ^g denote $\alpha(g)(\varphi)$. Suppose that F and G act on Σ and G, respectively, from the right by inner automorphisms. Then (G2) reads that i preserves the actions on Σ, while (G3) reads that F is G-stable, and i is G-equivariant.

Let $G \ltimes \Sigma$ be the semi-direct product given by α, and set

$$\Xi = \{\, (i(f), f^{-1}) \in G \ltimes \Sigma \mid f \in F \,\}.$$

Then one sees from (G2)–(G3) that Ξ is a normal subgroup of $G \ltimes \Sigma$. We let

$$\Gamma = \Gamma(\Sigma, F, G, i, \alpha)$$

denote the quotient group $G \ltimes \Sigma / \Xi$.

Lemma 5.1.1. *We have the following.*

(1) The composite $G \to G \ltimes \Sigma \to \Gamma$ of the inclusion with the quotient map is an injection, through which we will regard G as a subgroup of Γ.

(2) The composite $\Sigma \to G \ltimes \Sigma \to \Gamma$ of the inclusion with the quotient map induces a bijection $F\backslash\Sigma \to G\backslash\Gamma$ between the sets of right cosets.

Proof. Choose arbitrarily a set $X \subset F$ of representatives of $F\backslash\Sigma$. Then the product map $p : F \times X \to \Sigma$, $p(f,x) = fx$ is a bijection, through which we will identify Σ with $F \times X$. Then we have $G \ltimes \Sigma = (G \ltimes F) \times X$ as left $G \ltimes F$-sets. Note $\Xi \subset G \ltimes F$ and that the canonical map $G \to G \ltimes F/\Xi = \Xi\backslash G \ltimes F$ is an isomorphism. The direct product with id_X gives a left G-equivariant bijection $G \times X \to (\Xi\backslash G \ltimes F) \times X = \Gamma$. This implies the assertions. □

Taking into account the property shown in Part 2 above we say:

Definition 5.1.2. Γ is the *base extension* of Σ along $i : F \to (G, \alpha)$. Here one supposes i to be a morphism of groups acting on Σ, bearing in mind the action of F by inner automorphisms.

5.2 Construction of affine supergroups

5.2.1 The group $\Sigma(A)$

Let $\mathfrak{g}$ be a Lie superalgebra which satisfies the following conditions; see Corollary 4.6.3.

(i) $\mathfrak{g}_{\bar{0}}$ is $\Bbbk$-finite projective, and

(ii) $\mathfrak{g}_{\bar{1}}$ is $\Bbbk$-finite free.

Suppose that it is given a 2-operation $(-)^{\langle 2\rangle}$.

Let $A \in \mathsf{SAlg}_\Bbbk$ be a commutative superalgebra (over $\Bbbk$). We have the group $\mathsf{Gpl}(\mathcal{U}(\mathfrak{g})_A)$ of all even grouplikes in the Hopf superalgebra $\mathcal{U}(\mathfrak{g})_A = A \otimes \mathcal{U}(\mathfrak{g})$ over A. As is seen from the paragraph following Proposition 4.1.1, the canonical maps

$$\begin{aligned} A_{\bar{0}} \otimes \mathfrak{g}_{\bar{0}} \longrightarrow A \otimes \mathcal{U}(\mathfrak{g}_{\bar{0}}) \quad &\longrightarrow \quad A \otimes \mathcal{U}(\mathfrak{g}), \\ A \otimes \mathfrak{g}_{\bar{1}} \quad &\longrightarrow \quad A \otimes \mathcal{U}(\mathfrak{g}) \end{aligned}$$

are all injections, which we will regard as inclusions. We define even elements $e(a,v)$, $f(\epsilon,x)$ of $A \otimes \mathcal{U}(\mathfrak{g})$ by

$$e(a,v) = 1 \otimes 1 + a \otimes v, \qquad f(\epsilon,x) = 1 \otimes 1 + \epsilon \otimes x, \tag{I.5.2.1}$$

where $a \in A_{\bar{1}}$, $v \in \mathfrak{g}_{\bar{1}}$, $x \in \mathfrak{g}_{\bar{0}}$, and $\epsilon \in A_{\bar{0}}$ with $\epsilon^2 = 0$. Note that $e(\lambda a, v) = e(a, \lambda v)$, $f(\lambda\epsilon, x) = f(\epsilon, \lambda x)$ for $\lambda \in \Bbbk$.

Lemma 5.2.1. *The elements $e(a,v)$, $f(\epsilon,x)$ are contained in $\mathsf{Gpl}(\mathcal{U}(\mathfrak{g})_A)$, and we have*

$$e(a,v)^{-1} = e(-a,v), \quad f(\epsilon,x)^{-1} = f(-\epsilon,x), \quad e(0,v) = 1 = f(0,x).$$

Proof. This follows since $a \otimes v$ and $\epsilon \otimes x$ are even primitives z such that $z \otimes_A z = 0$. □

Lemma 5.2.2. *Let $a, b \in A_1$, $u, v \in \mathfrak{g}_{\bar{1}}$, $x, y \in \mathfrak{g}_{\bar{0}}$, and $\epsilon, \eta \in A_{\bar{0}}$ with $\epsilon^2 = \eta^2 = 0$. Then the following relations hold in $\mathsf{Gpl}(\mathcal{U}(\mathfrak{g})_A)$.*

(i) $e(a,u)\, e(b,v) = f(-ab, [u,v])\, e(b,v)\, e(a,u)$

(ii) $e(a,v)\, e(b,v) = f(-ab, v^{\langle 2 \rangle})\, e(a+b,v)$

(iii) $e(a,v)\, f(\epsilon,x) = f(\epsilon,x)\, e(a,v)\, e(\epsilon a, [v,x])$

(iv) $f(\epsilon,x)\, f(\eta,y) = f(\eta,y)\, f(\epsilon,x)\, f(\epsilon\eta, [x,y])$

Proof. These follow by direct computation. □

In particular, $e(a,u)$ and $e(b,v)$ (resp., $e(a,v)$ and $f(\epsilon,x)$; resp., $f(\epsilon,x)$ and $f(\eta,y)$) commute with each other if $ab = 0$ or $[u,v] = 0$ (resp., $\epsilon a = 0$ or $[v,x] = 0$; resp., $\epsilon\eta = 0$ or $[x,y] = 0$).

Let

$$\mathbf{\Sigma}(A)$$

denote the subgroups of $\mathsf{Gpl}(\mathcal{U}(\mathfrak{g})_A)$ generated by all the elements $e(a,v)$, $f(\epsilon,x)$ defined by (I.5.2.1). Let $F(A_{\bar{0}})$ denote the subgroup of $\mathbf{\Sigma}(A)$ generated by all $f(\epsilon,x)$.

Proposition 5.2.3. *We have the following.*

(1) $F(A_{\bar{0}}) = \mathbf{\Sigma}(A) \cap \mathcal{U}(\mathfrak{g}_{\bar{0}})_{A_{\bar{0}}}$.

(2) Choose arbitrarily a $\Bbbk$-free basis $v_1, \dots, v_n$ of $\mathfrak{g}_{\bar{1}}$. Then every element of $\mathbf{\Sigma}(A)$ is uniquely expressed of the form

$$f\, e(a_1,v_1)\, e(a_2,v_2) \cdots e(a_n,v_n), \tag{I.5.2.2}$$

where $f \in F(A_{\bar{0}})$, and $a_i \in A_{\bar{1}}$, $1 \le i \le n$.

Proof. If $v = \sum_{i=1}^n \lambda_i v_i$ with $\lambda_i \in \Bbbk$, then

$$e(a,v) = e(\lambda_1 a, v_1)\, e(\lambda_2 a, v_2) \cdots e(\lambda_n a, v_n).$$

Therefore, $\mathbf{\Sigma}(A)$ is generated by all $e(a, v_i)$, $1 \le i \le n$, and $f(\epsilon,x)$. To these generators associate the numbers i and 0, respectively. Given an element of $\mathbf{\Sigma}(A)$ expressed as a product of the last generators, associate naturally a word of the letters $0, 1, \dots, n$. For example, to $e(a,v_2)\, f(\epsilon,x)\, f(\eta,y)\, e(b,v_1)$, associated is the word 2001. Introduce the lexicographical well-order among the words. Suppose that we are given an element of $\mathbf{\Sigma}(A)$ expressed as above. Suppose that the associated word is not of the form

$$\underbrace{0 \dots 0}_{r}\, i_1 i_2 \dots i_s, \tag{I.5.2.3}$$

where $r \ge 0$ and $0 < i_1 < i_2 < \cdots < i_s \le n$, $s \ge 0$. This means that the expression includes

$$\text{(i) } e(a,v_j)\, e(b,v_i),\ i<j, \quad \text{(ii) } e(a,v_i)\, e(b,v_i) \quad \text{or} \quad \text{(iii) } e(a,v_i)\, f(\epsilon,x).$$

By using the relations (i)–(iii) in Lemma 5.2.2, the expression is reduced to another one with smaller associated word. Continuing such reductions it is reduced finally to an expression with associated word of the form (I.5.2.3). This proves that every element of $\mathbf{\Sigma}(A)$ is expressed of the form (I.5.2.2). Note that $F(A_{\bar{0}}) \subset A \otimes \mathcal{U}(\mathfrak{g}_{\bar{0}})$. Then the uniqueness of the expression and (1) follow, since we see from Proposition 4.1.1 that $\mathcal{U}(\mathfrak{g})_A$ has the elements given by (I.4.6.2) as left $A \otimes \mathcal{U}(\mathfrak{g}_{\bar{0}})$-free basis. □

5.2.2 The group $\mathbf{\Gamma}(A)$

Keep the situation as above.

Let G be an affine group. The right adjoint action $G \times G \to G$, $(h, g) \mapsto g^{-1}hg$ is dualized to the left G-module structure on $\mathcal{O}(G)$ defined by

$$ {}^{g}c := g^{-1}(c_1)\, c_2\, g(c_3), \qquad g \in G(S),\ c \in \mathcal{O}(G), \tag{I.5.2.4} $$

where $S \in \mathsf{Alg}_{\Bbbk}$ is a commutative algebra (over $\Bbbk$). This makes $\mathcal{O}(G)$ into a Hopf-algebra object in the symmetric tensor category ${}_G\mathsf{Mod}$ of left G-modules.

Recall that $\mathfrak{g}$ is a Lie superalgebra given a 2-operation, and it satisfies the assumptions stated in Proposition 4.1.1.

Definition 5.2.4. Let $\mathsf{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ denote the supergroup functor such that

$$ \mathsf{Aut}_{\mathrm{Lie}}(\mathfrak{g})(S) := \mathrm{Aut}_{S\text{-}\mathrm{Lie}}(\mathfrak{g}_S), $$

where $S \in \mathsf{Alg}_{\Bbbk}$. Here, $\mathrm{Aut}_{S\text{-}\mathrm{Lie}}(\mathfrak{g}_S)$ is the group of all S-Lie-superalgebra automorphisms preserving $(-)^{\langle 2 \rangle}_S$; see Proposition 2.5.6.

We are going to work in a more general situation than will be needed to discuss a category equivalence in the next subsection. For our motivation of this see Remark 5.2.14 below.

Suppose that we are given a pairing and an anti-morphism,

$$ \langle\ ,\ \rangle : \mathfrak{g}_{\bar{0}} \times \mathcal{O}(G) \longrightarrow \Bbbk, \qquad \alpha : G \longrightarrow \mathsf{Aut}_{\mathrm{Lie}}(\mathfrak{g}). \tag{I.5.2.5} $$

Let us write as $\rho_\alpha(z) = z_{-1} \otimes z_0$ for $z \in \mathfrak{g}$. We assume that

(H1) $[z, x] = \langle x, z_{-1} \rangle\, z_0$,

(H2) $\langle x, cd \rangle = \langle x, c \rangle\, \varepsilon(d) + \varepsilon(c)\, \langle x, d \rangle$, and

(H3) $\langle x^g, c \rangle_S = \langle x, {}^g c \rangle_S$,

where $x \in \mathfrak{g}_{\bar{0}}$, $z \in \mathfrak{g}$, $c, d \in \mathcal{O}(G)$ and $g \in G(S)$, $S \in \mathsf{Alg}_{\Bbbk}$.

By (H2) we have the map

$$ \mathfrak{g}_{\bar{0}} \longrightarrow \mathrm{Lie}(G)\ (\subseteq \mathcal{O}(G)^{\bar{*}}); \quad x \longmapsto \langle x, - \rangle. \tag{I.5.2.6} $$

This is a Lie algebra map, since we see from (H1) for even z and (H3) that

$$\begin{aligned}\langle [x,\, y],\; c\rangle &= \langle x,\; \mathcal{S}(c_1)c_3\rangle\,\langle y,\, c_2\rangle\\ &= \langle x\otimes y,\, \Delta(c)\rangle - \langle y\otimes x,\, \Delta(c)\rangle,\end{aligned}$$

where $x, y \in \mathfrak{g}_{\bar{0}}$, $c \in \mathcal{O}(G)$. Therefore, it uniquely extends to an algebra map $\mathcal{U}(\mathfrak{g}_{\bar{0}}) \to \mathcal{O}(G)^{\bar{*}}$, with which associated is the Hopf pairing

$$\langle\;,\;\rangle : \mathcal{U}(\mathfrak{g}_{\bar{0}}) \times \mathcal{O}(G) \longrightarrow \Bbbk \tag{I.5.2.7}$$

that uniquely extends the given pairing.

Recall $A \in \mathsf{SAlg}_{\Bbbk}$. By Lemma 2.3.3 the base extension to $A_{\bar{0}}$ of the last Hopf pairing gives rise to the group map

$$\mathsf{Gpl}(\mathcal{U}(\mathfrak{g}_{\bar{0}})_{A_{\bar{0}}}) \longrightarrow \mathsf{Alg}_{\Bbbk}(\mathcal{O}(G), A_{\bar{0}}) = G(A_{\bar{0}}), \quad g \longmapsto \langle g,\, -\rangle_{A_{\bar{0}}},$$

whose restriction to $F(A_{\bar{0}})$ we denote by

$$i_{A_{\bar{0}}} = i : F(A_{\bar{0}}) \to G(A_{\bar{0}}).$$

Lemma 5.2.5. *Let $S \in \mathsf{Alg}_{\Bbbk}$ and $g \in G(S)$. Then $\alpha_S(g) \in \mathrm{Aut}_{S\text{-Lie}}(\mathfrak{g}_S)$ uniquely extends to an automorphism of the Hopf superalgebra $\mathcal{U}(\mathfrak{g})_S$ over S.*

Proof. One sees that $\alpha_S(g)$ uniquely extends an automorphism of the S-Hopf superalgebra $T(\mathfrak{g})_S$. It is easy to see that the automorphism stabilizes the super-ideal of $T(\mathfrak{g})_S$ generated by the elements $zw - (-1)^{|z||w|}wz - [z,\, w]$ in (I.4.6.1). To see that it stabilizes the super-ideal generated by all elements in (I.4.6.1), let $v \in \mathfrak{g}_{\bar{1}}$, and suppose $v^g = \sum_i c_i \otimes v_i \in S \otimes \mathfrak{g}_{\bar{1}}$. Then the desired result will follow if one compares the following two.

$$\begin{aligned}(v^{\langle 2\rangle})^g &= (v^g)^{\langle 2\rangle}_S = \sum_i c_i^2 \otimes v_i^{\langle 2\rangle} + \sum_{i<j} c_ic_j \otimes [v_i, v_j],\\ (v^2)^g &= (v^g)^2 = \sum_i c_i^2 \otimes v_i^2 + \sum_{i<j} c_ic_j \otimes (v_iv_j + v_jv_i).\end{aligned}$$

□

The assignment of the above extended automorphism to $g \in G(R)$ gives rise to an anti-morphism, which we denote again by

$$\alpha : G \longrightarrow \mathsf{Aut}_{\mathrm{Hopf}}(\mathcal{U}(\mathfrak{g})),$$

from G to the automorphism group functor of $\mathcal{U}(\mathfrak{g})$. Given $g \in G(A_{\bar{0}})$, the base extension $(\alpha_{A_{\bar{0}}}(g))_A$ of $\alpha_{A_{\bar{0}}}(g) \in \mathrm{Aut}_{A_{\bar{0}}\text{-Hopf}}(\mathcal{U}(\mathfrak{g})_{A_{\bar{0}}})$ along $A_{\bar{0}} \to A$ is an automorphism of the Hopf superalgebra $\mathcal{U}(\mathfrak{g})_A$ over A. As before, we will write u^g for $(\alpha_{A_{\bar{0}}}(g))_A(u)$, where $u \in \mathcal{U}(\mathfrak{g})_A$, $g \in G(A_{\bar{0}})$. Since the action stabilizes $\mathbf{\Sigma}(A)$, as will be seen from the next lemma, it follows that there is induced a group map, which we denote by

$$\alpha_A : G(A_{\bar{0}}) \longrightarrow \mathrm{Aut}(\mathbf{\Sigma}(A)),$$

from $G(A_{\bar{0}})$ to the automorphism group of the group $\mathbf{\Sigma}(A)$.

Lemma 5.2.6. *Let $g \in G(A_{\bar{0}})$. Let $e(a,v)$ and $f(\epsilon,x)$ be as before. Suppose*

$$\rho_\alpha(v) = \sum_{i=1}^{n} c_i \otimes v_i \in \mathcal{O}(G) \otimes \mathfrak{g}_{\bar{1}}, \qquad \rho_\alpha(x) = \sum_{j=1}^{m} d_j \otimes x_j \in \mathcal{O}(G) \otimes \mathfrak{g}_{\bar{0}}.$$

Then we have

(1) $e(a,v)^g = e(ag(c_1),v_1)\, e(ag(c_2),v_2) \cdots e(ag(c_n),v_n)$,

(2) $f(\epsilon,x)^g = f(\epsilon g(d_1),x_1)\, f(\epsilon g(d_2),x_2) \cdots f(\epsilon g(d_m),x_m)$.

This is easy to see. We remark that $F(A_{\bar{0}})$ is $G(A_{\bar{0}})$-stable by Part 2.

Proposition 5.2.7. *The quintuple*

$$(\mathbf{\Sigma}(A),\ F(A_{\bar{0}}),\ G(A_{\bar{0}}),\ i_{A_{\bar{0}}},\ \alpha_A)$$

satisfies Conditions (G1)–(G3) given in Section 5.2.1.

Proof. Since the last remark shows that the first half of (G3) is satisfied, it remains to verify (G2) and the second half of (G3).

Choose $g \in G(A_{\bar{0}})$, and let $f = f(\epsilon,x)$. Note

$$i(f)(c) = \varepsilon(c)1 + \epsilon\langle x,\ c\rangle, \quad c \in \mathcal{O}(G).$$

Then by using (H3) we see

$$\begin{aligned} i(f^g)(c) &= \varepsilon(c)1 + \epsilon\,\langle x^g,\ c\rangle_{A_{\bar{0}}} = \varepsilon(c)1 + \epsilon\,\langle x,\ {}^g c\rangle_{A_{\bar{0}}} \\ &= \varepsilon(c)1 + \epsilon\, g^{-1}(c_1)\,\langle x,\ c_2\rangle\, g(c_3) \\ &= (g^{-1} i(f) g)(c), \end{aligned}$$

which verifies the second half of (G3). By using (H1) we see

$$\begin{aligned} e(a,v)^{i(f)} &= 1 \otimes 1 + a\, i(f)(v_{-1}) \otimes v_0 \\ &= 1 \otimes 1 + a \otimes v + \epsilon a \otimes \langle x,\ v_{-1}\rangle v_0 \\ &= 1 \otimes 1 + a \otimes v + \epsilon a \otimes [v,x] \\ &= e(a,v)\, e(\epsilon a, [v,x]), \end{aligned}$$

and similarly,

$$f(\eta,y)^{i(f)} = f(\eta,y)\, f(\epsilon\eta, [y,x]).$$

These, combined with (iii)–(iv) of Lemma 5.2.2, verify (G2). □

Definition 5.2.8. $\mathbf{\Gamma}(A)$ denotes the base extension of $\mathbf{\Sigma}(A)$ along $i_{A_{\bar{0}}} : F(A_{\bar{0}}) \to (G(A_{\bar{0}}), \alpha_A)$; see Definition 5.1.2.

In $\mathbf{\Gamma}(A)$, the natural images of $e(a, v)$ and of elements $g \in G(A_{\bar{0}})$ will be denoted by the same symbols.

Proposition 5.2.9. *Choose arbitrarily a* $\Bbbk$*-free basis* $v_1, \dots, v_n$ *of* $\mathfrak{g}_{\bar{1}}$*. Then every element of* $\mathbf{\Gamma}(A)$ *is uniquely expressed of the form*

$$g\, e(a_1, v_1)\, e(a_2, v_2) \cdots e(a_n, v_n),$$

where $g \in G(A_{\bar{0}})$*,* $a_i \in A_{\bar{1}}$*,* $1 \le i \le n$*.*

Proof. This follows from Proposition 5.2.3 (2) and the proof of Lemma 5.1.1 (2). □

Gavarini's original construction starts with constructing by generators and relation the group which shall be the functor points of the desired affine supergroup. Let us prove that the group, which is essentially the same as $\mathbf{\Gamma}'(A)$ below, is isomorphic to our $\mathbf{\Gamma}(A)$, though the result will not be used in the subsequent argument.

Lemma 5.2.10. *Choose arbitrarily a* $\Bbbk$*-free basis* $v_1, \dots, v_n$ *of* $\mathfrak{g}_{\bar{1}}$*. Let* $E(A_{\bar{1}})$ *denote the free group on the set of the symbols*

$$e_i(a), \quad 1 \le i \le n,\ a \in A_{\bar{1}},$$

and let $\mathbf{\Gamma}'(A)$ *denote the quotient group of the free product* $G(A_{\bar{0}}) * E(A_{\bar{1}})$ *divided by the relations*

(i) $e_j(a)\, e_i(b) = i(f(-ab, [v_i, v_j]))\, e_i(b)\, e_i(a)$*,* $i < j$*,*

(ii) $e_i(a)\, e_i(b) = i(f(-ab, v_i^{\langle 2 \rangle}))\, e_i(a + b)$*,*

(iii) $e_i(a)\, g = g\, e_1(ag(c_{i1})) \cdots e_n(ag(c_{in}))$*, where* $g \in G(A_{\bar{0}})$*, and we suppose* $\rho_\alpha(v_i) = \sum_{k=1}^{n} c_{ik} \otimes v_k$*.*

Then

$$e_i(a) \longmapsto e(a, v_i), \quad 1 \le i \le n,\ a \in A_{\bar{1}}$$

gives an isomorphism $\mathbf{\Gamma}'(A) \xrightarrow{\simeq} \mathbf{\Gamma}(A)$ *which is identical on* $G(A_{\bar{0}})$*.*

Proof. It is easy to see that the assignment gives an epimorphism. By Proposition 5.2.9,

$$g\, e(a_1, v_1) \cdots e(a_n, v_n) \longmapsto g\, e_1(a_1) \cdots e_n(a_n)$$

well defines a section. This section is surjective, which proves the lemma, since one sees just as proving Proposition 5.2.3 that every element of $\mathbf{\Gamma}'(A)$ is expressed of the form $g\, e_1(a_1) \cdots e_n(a_n)$, where $g \in G(A_{\bar{0}})$, $a_i \in A_{\bar{1}}$, $1 \le i \le n$. □

5.2.3 The affine supergroup $\mathbf{\Gamma}$

Keep the situation as above. One sees easily that

$$\mathsf{SAlg}_{\Bbbk} \longrightarrow \mathsf{Grp}; \quad A \longmapsto \mathbf{\Gamma}(A)$$

defines a group functor $\mathbf{\Gamma}$ defined on $\mathsf{SAlg}_{\Bbbk}$. Moreover, we see:

Proposition 5.2.11. *This $\mathbf{\Gamma}$ is an affine supergroup, represented by the commutative superalgebra*

$$\mathbf{O} := \mathcal{O}(G) \otimes \wedge(\mathfrak{g}_{\bar{1}}^*). \tag{I.5.2.8}$$

Proof. Choose a $\Bbbk$-free basis $v_1, \dots, v_n$ of $\mathfrak{g}_{\bar{1}}$, as above. Let $w_1, \dots, w_n$ denote the dual basis of $\mathfrak{g}_{\bar{1}}^*$. Proposition 5.2.9 gives the bijection

$$G(A) \times A_{\bar{1}}^n \xrightarrow{\simeq} \mathbf{\Gamma}(A); \quad (g, a_1, \dots, a_n) \longmapsto g\, e(a_1, v_1) \cdots e(a_n, v_n), \tag{I.5.2.9}$$

which is seen to be natural in A. To an element $(g, a_1, \dots, a_n) \in G(A) \times A_{\bar{1}}^n$, assign the superalgebra map $\phi : \mathbf{O} \to A$ determined by

$$\phi(c) = g(c),\ c \in \mathcal{O}(G), \quad \phi(w_i) = a_i,\ 1 \le i \le n.$$

This assignment is indeed a bijection

$$G(A) \times A_{\bar{1}}^n \xrightarrow{\simeq} \mathsf{SAlg}_{\Bbbk}(\mathbf{O}, A) \tag{I.5.2.10}$$

which is natural in A. This proves the proposition. □

Remark 5.2.12. Note that G, regarded as $A \mapsto G(A_{\bar{0}})$, is a subgroup functor of $\mathbf{\Gamma}$. Let $\mathbf{G}_{\mathrm{a}}^{-n}$ denote the functor which assigns to $A \in \mathsf{SAlg}_{\Bbbk}$ the additive group $A_{\bar{1}}^n$, which is indeed represented by $\wedge(\mathfrak{g}_{\bar{1}}^*)$; see Example 3.1.6. One sees that the bijection (I.5.2.9) gives rise to a left G-equivariant isomorphism

$$G \times \mathbf{G}_{\mathrm{a}}^{-n} \xrightarrow{\simeq} \mathbf{\Gamma}$$

of functors which preserves the identity element.

The superalgebra $\mathbf{O}$ has uniquely a Hopf superalgebra structure which makes the composite $\mathbf{\Gamma}(A) \xrightarrow{\simeq} \mathsf{SAlg}_{\Bbbk}(\mathbf{O}, A)$ of the bijections (I.5.2.9) and (I.5.2.10) into an isomorphism of group functors. In particular, the counit is the tensor product

$$\varepsilon \otimes \varepsilon : \mathcal{O}(G) \otimes \wedge(\mathfrak{g}_{\bar{1}}^*) \longrightarrow \Bbbk$$

of the counits of the Hopf superalgebras $\mathcal{O}(G)$ and $\wedge(\mathfrak{g}_{\bar{1}}^*)$, as is seen from Remark 5.2.12. It follows that

$$\mathbf{O}^+/(\mathbf{O}^+)^2 = \mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2 \oplus \mathfrak{g}_{\bar{1}}^*,$$

which is dualized to the identification

$$\mathrm{Lie}(\mathbf{\Gamma}) = \mathrm{Lie}(G) \oplus \mathfrak{g}_{\bar{1}}$$

of $\Bbbk$-supermodules.

Let $i' : \mathfrak{g}_{\bar{0}} \to \mathrm{Lie}(G)$ denote the Lie algebra map given by (I.5.2.6). Let $\mathrm{Der}(\mathfrak{g})$ denote the Lie algebra of $\Bbbk$-super-linear derivations on $\mathfrak{g}$. The morphism α given in (I.5.2.5) induces the anti-Lie algebra map

$$\alpha' : \mathrm{Lie}(G) \longrightarrow \mathrm{Der}(\mathfrak{g}); \quad x \longmapsto (z \mapsto x(z_{-1})\, z_0),$$

where $x \in \mathrm{Lie}(G)$, $z \in \mathfrak{g}$. We remark that by (H1), the composite $\alpha' \circ i' : \mathfrak{g}_{\bar{0}} \to \mathrm{Der}(\mathfrak{g})$ coincides with the right adjoint representation.

Proposition 5.2.13. *We have the following.*

(1) The super-bracket on $\mathrm{Lie}(\mathbf{\Gamma}) = \mathrm{Lie}(G) \oplus \mathfrak{g}_{\bar{1}}$ *is given by*

$$[(x, u), (y, v)] = \big([x, y] + i'([u, v]),\ \alpha'(y)(u) - \alpha'(x)(v)\big),$$

where $x, y \in \mathrm{Lie}(G)$, $u, v \in \mathfrak{g}_{\bar{1}}$.

(2) $i' \oplus \mathrm{id}_{\mathfrak{g}_{\bar{1}}} : \mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \to \mathrm{Lie}(G) \oplus \mathfrak{g}_{\bar{1}} = \mathrm{Lie}(\mathbf{\Gamma})$ *is a Lie superalgebra map which preserves the 2-operation.*

Proof. (1) We see from Remark 5.2.12 that $\mathcal{O}(G)$ is a quotient Hopf superalgebra of $\mathbf{O}$ through $\mathrm{id} \otimes \varepsilon : \mathbf{O} = \mathcal{O}(G) \otimes \wedge(\mathfrak{g}_{\bar{1}}^*) \to \mathcal{O}(G)$, and G is thus a closed super-subgroup of $\mathbf{\Gamma}$; see below the proof. It follows that $\mathrm{Lie}(G)$ is a Lie subsuperalgebra of $\mathrm{Lie}(\mathbf{\Gamma})$ through the inclusion $\mathrm{Lie}(G) \to \mathrm{Lie}(G) \oplus \mathfrak{g}_{\bar{1}}$.

It remains to compute $[v_1, v_2]$ in $\mathrm{Lie}(\mathbf{\Gamma})$, where $v_1, v_2 \in \mathfrak{g}_{\bar{1}}$, or $v_1 \in \mathfrak{g}_{\bar{1}}$, $v_2 \in \mathrm{Lie}(G)$. If elements $\tau \in A$ and $v \in \mathrm{Lie}(\mathbf{\Gamma})$ satisfy $\tau^2 = 0$ and $|\tau| = |v|$, then

$$g(\tau, v) : \mathbf{O} \longrightarrow A; \quad h \longmapsto \varepsilon(h)1 + \tau\, v(h)$$

is an element in $\mathbf{\Gamma}(A)$ with inverse $g(-\tau, v)$. This coincides with $e(\tau, v)$ if $|\tau| = |v| = 1$. Note that $g(\tau, v) = i(f(\tau, x))$, if $|\tau| = 0$ and $v = i'(x)$ with $x \in \mathfrak{g}_{\bar{0}}$. Given elements $g_1 = g(\tau_1, v_1)$, $g_2 = g(\tau_2, v_2)$ as above, then the commutator $(g_1, g_2) = g_1 g_2 g_1^{-1} g_2^{-1}$ coincides with

$$g((-1)^{|\tau_1||\tau_2|}\tau_1\tau_2,\ [v_1, v_2]),$$

from which we will see the desired vales of $[v_1, v_2]$.

First, suppose that $A = \wedge(\tau_1, \tau_2)$, where τ_i, $i = 1, 2$, are odd variables. Let $u, v \in \mathfrak{g}_{\bar{1}}$. Since we have $(e(\tau_1, u), e(\tau_2, v)) = g(-\tau_1\tau_2,\ i'([u, v]))$ by (i) of Lemma 5.2.2, it follows that

$$[(0, u), (0, v)] = (i'([u, v]), 0).$$

Next, suppose that $A = \Bbbk[\tau_1]/(\tau_1^2) \otimes \wedge(\tau_2)$, where τ_1 (resp., τ_2) is an even (resp., odd) variable. Let $y, v \in \mathrm{Lie}(\mathbf{\Gamma})$ with y even and v odd. Note $g(\pm\tau_1, y) \in G(A_{\bar{0}})$. Since we see from (1) of Lemma

5.2.6 that

$$(g(-\tau_1, y), e(\tau_2, v)) = e(\tau_2, v)^{g(\tau_1,y)}\, e(-\tau_2, v) = e(\tau_1\tau_2, \alpha'(y)(v)),$$

it follows that

$$[(0, v),\ (y, 0)] = (\alpha'(y)(v), 0).$$

(2) By Part 1 and the remark given above the proposition it remains to prove that the map preserves the 2-operation. Suppose again that $A = \wedge(\tau_1, \tau_2)$. Then we see from (ii) of Lemma 5.2.2 that

$$g(-\tau_1\tau_2,\ i'(v^{\langle 2\rangle})) = e(\tau_1, v)\, e(\tau_2, v)\, e(-(\tau_1 + \tau_2), v).$$

This last equals $g(-\tau_1\tau_2,\ v^2)$, which proves the desired result. □

Recall that a *closed subsupergroup* of an affine supergroup $\mathbf{G}$ is a subgroup functor of $\mathbf{G}$ which is represented by a quotient Hopf superalgebra of $\mathcal{O}(\mathbf{G})$.

Remark 5.2.14. To explain a motivation to have worked in a general situation as above, suppose that $\mathfrak{G}$ is a super Lie group over a complete valuation field of characteristic $\neq 2$. Let $\mathfrak{G}_{\mathrm{red}}$ be the associated Lie group. Let $\mathcal{R}(\mathfrak{G}_{\mathrm{red}})$ be the commutative Hopf algebra of all analytic representative functions on $\mathfrak{G}_{\mathrm{red}}$; this is not necessarily finitely generated. The corresponding affine group and the Lie superalgebra $\mathrm{Lie}(\mathfrak{G})$ of $\mathfrak{G}$ have a natural pairing and an anti-morphism as in (I.5.2.5), which satisfy (H1)–(H3). The resulting affine supergroup $\mathbf{\Gamma}$ may be called the *universal algebraic hull* of $\mathfrak{G}$ (see [15, p.1141]).

5.3 The category equivalence

5.3.1 Re-proving Gavarini's equivalence

Let $\mathbf{G}$ be an affine supergroup, and set $\mathbf{O} = \mathcal{O}(\mathbf{G})$. Recall from [27, Section 2.5], for example, that the *associated affine group* $\mathbf{G}_{\mathrm{ev}}$ is the restricted group functor $\mathbf{G}|_{\mathsf{Alg}_{\Bbbk}}$ defined on $\mathsf{Alg}_{\Bbbk}$, which is indeed represented by the largest purely even quotient Hopf superalgebra

$$\overline{\mathbf{O}} := \mathbf{O}/\mathbf{O}\mathbf{O}_{\bar{1}}\ (= \mathbf{O}_{\bar{0}}/\mathbf{O}_{\bar{1}}^2)$$

of $\mathbf{O}$. This $\mathbf{G}_{\mathrm{ev}}$ is also regarded as the closed super-subgroup of $\mathbf{G}$ which assigns to $A \in \mathsf{SAlg}_{\Bbbk}$ the group $\mathbf{G}(A_{\bar{0}})$. Let

$$W^{\mathbf{O}} := \mathbf{O}_{\bar{1}}/\mathbf{O}_{\bar{0}}^{+}\mathbf{O}_{\bar{1}},\ \text{where } \mathbf{O}_{\bar{0}}^{+} = \mathbf{O}_{\bar{0}} \cap \mathbf{O}^{+},$$

as in [22]. Since $\mathbf{O}_{\bar{0}}^{+}/((\mathbf{O}_{\bar{0}}^{+})^2 + \mathbf{O}_{\bar{1}}^2) \simeq \overline{\mathbf{O}}^{+}/(\overline{\mathbf{O}}^{+})^2$, we have

$$\mathbf{O}^{+}/(\mathbf{O}^{+})^2 \simeq \overline{\mathbf{O}}^{+}/(\overline{\mathbf{O}}^{+})^2 \oplus W^{\mathbf{O}},$$

which is dualized to

$$\mathrm{Lie}(\mathbf{G}) \simeq \mathrm{Lie}(\mathbf{G}_{\mathrm{ev}}) \oplus (W^{\mathbf{O}})^{*};$$

see [27, Lemma 4.3]. It follows that

$$\mathrm{Lie}(\mathbf{G})_{\bar{0}} \simeq \mathrm{Lie}(\mathbf{G}_{\mathrm{ev}}), \quad \mathrm{Lie}(\mathbf{G})_{\bar{1}} = (W^{\mathbf{O}})^*.$$

The former is the canonical Lie-algebra isomorphism induced from the embedding $\overline{\mathbf{O}}^* \subset \mathbf{O}^*$, through which we will identify as

$$\mathrm{Lie}(\mathbf{G})_{\bar{0}} = \mathrm{Lie}(\mathbf{G}_{\mathrm{ev}}).$$

Just as for (I.5.2.4), the right adjoint action $\mathbf{G} \times \mathbf{G}_{\mathrm{ev}} \to \mathbf{G}$, $(f, g) \mapsto g^{-1} f g$ is dualized to the left $\mathbf{G}_{\mathrm{ev}}$-supermodule structure on $\mathbf{O}$ defined by

$$^{g}h = g^{-1}(h_1)\, h_2\, g(h_3), \quad g \in \mathbf{G}_{\mathrm{ev}}(S),\ h \in \mathbf{O}, \tag{I.5.3.1}$$

where $S \in \mathsf{Alg}_{\Bbbk}$. This makes $\mathbf{O}$ into a Hopf-algebra object in the symmetric tensor category ${}_{\mathbf{G}_{\mathrm{ev}}}\mathsf{SMod}$ of left $\mathbf{G}_{\mathrm{ev}}$-supermodules.

Let us recall the definitions [12, Definitions 3.2.6, 4.1.2] of two categories, following mostly the formulation of [27, Appendix].

First, let $(\text{gss-fsgroups})_{\Bbbk}$ denote the category of the affine supergroups $\mathbf{G}$ such that when we set $\mathbf{O} = \mathcal{O}(\mathbf{G})$,

(E1) there exists a counit-preserving isomorphism $\mathbf{O} \simeq \overline{\mathbf{O}} \otimes \wedge(W^{\mathbf{O}})$ of left $\overline{\mathbf{O}}$-comodule superalgebras,

(E2) $\overline{\mathbf{O}}^+/(\overline{\mathbf{O}}^+)^2$ is $\Bbbk$-finite projective, and

(E3) $W^{\mathbf{O}}$ is $\Bbbk$-finite free.

Note that the conditions are the same as the conditions given in Section 4.6.2.

Remark 5.3.1. Let $\mathbf{G}$ be an affine supergroup with $\mathbf{O} = \mathcal{O}(\mathbf{G})$. Assume that $\overline{\mathbf{O}} = \mathcal{O}(\mathbf{G}_{\mathrm{ev}})$ is $\Bbbk$-flat. Then (E1) is necessarily satisfied, if (E2) and (E3) are satisfied.

A morphism in $(\text{gss-fsgroups})_{\Bbbk}$ is a natural transformation of group functors. The conditions above re-number those (E1)–(E3) given in [27, Appendix].

Next, to define the category $(\text{sHCP})_{\Bbbk}$, let $(G, \mathfrak{g})$ be a pair of an affine group G and a Lie superalgebra $\mathfrak{g}$ given a 2-operation. Suppose that $\mathfrak{g}_{\bar{1}}$ is $\Bbbk$-finite free, and is given a right G-module structure. Suppose in addition,

(F1) $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$,

(F2) $\mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2$ is $\Bbbk$-finite projective, so that $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$ is necessarily $\Bbbk$-finite projective, and has the right G-module structure (see (I.4.2.2), and (I.5.3.3) below) determined by

$$x^{g}(c) = x(^{g}c), \quad x \in \mathfrak{g}_{\bar{0}},\ c \in \mathcal{O}(G), \tag{I.5.3.2}$$

where $^{g}c = g^{-1}(c_1)\, c_2\, g(c_3)$, as in (I.5.2.4), (F3) the left $\mathcal{O}(G)$-comodule structure $\mathfrak{g}_{\bar{1}} \to \mathcal{O}(G) \otimes \mathfrak{g}_{\bar{1}}$, $v \mapsto v_{-1} \otimes v_0$ on $\mathfrak{g}_{\bar{1}}$ corresponding to the given right G-module structure satisfies

$$[v,\, x] = x(v_{-1})\, v_0, \quad v \in \mathfrak{g}_{\bar{1}},\ x \in \mathfrak{g}_{\bar{0}},$$

(F4) the restricted super-bracket $[\ ,\ ]|_{\mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}} : \mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ is G-equivariant, and

(F5) the right G-module structure preserves the 2-operation, or explicitly,

$$(v_S^{\langle 2\rangle})^g = (v^g)_S^{\langle 2\rangle}, \quad v \in (\mathfrak{g}_{\bar{1}})_S,\ g \in G(S),$$

where $S \in \mathsf{Alg}_{\Bbbk}$.

Note that the conditions are the same as the conditions given in Section 4.6.2.

Finally, let $(\mathrm{sHCP})_{\Bbbk}$ denote the category of all those pairs $(G, \mathfrak{g})$ which satisfy (F1)–(F5) above. A morphism $(G, \mathfrak{g}) \to (G', \mathfrak{g}')$ in $(\mathrm{sHCP})_{\Bbbk}$ is a pair (γ, δ) of a morphism $\gamma : G \to G'$ of affine groups and a Lie superalgebra map $\delta = \delta_{\bar{0}} \oplus \delta_{\bar{1}} : \mathfrak{g} \to \mathfrak{g}'$, such that

(F6) the Lie algebra map $\mathrm{Lie}(\gamma)$ induced from γ coincides with $\delta_{\bar{0}}$,

(F7) $(\delta_{\bar{1}})_S(v^g) = \delta_{\bar{1}}(v)^{\gamma_S(g)}$, $v \in \mathfrak{g}_{\bar{1}}$, $g \in G(S)$, where $S \in \mathsf{Alg}_{\Bbbk}$, and

(F8) $\delta_{\bar{0}}(v^{\langle 2\rangle}) = \delta_{\bar{1}}(v)^{\langle 2\rangle}$, $v \in \mathfrak{g}_{\bar{1}}$.

Let us reproduce from [12] functors between the two categories just defined,

$$\begin{aligned} \Phi &: (\mathrm{gss\text{-}fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}, \\ \Psi &: (\mathrm{sHCP})_{\Bbbk} \to (\mathrm{gss\text{-}fsgroups})_{\Bbbk}, \end{aligned}$$

which are denoted by Φ_g, Ψ_g in [12].

First, let $\mathbf{G}$ be an object in $(\mathrm{gss\text{-}fsgroups})_{\Bbbk}$. Set $\mathbf{O} = \mathcal{O}(\mathbf{G})$. Consider the pair

$$(G, \mathfrak{g}) := (\mathbf{G}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{G})),$$

giving to $\mathfrak{g}_{\bar{1}}$ the right G-module structure determined by

$$v^g(h) = v({}^g h), \quad v \in \mathfrak{g}_{\bar{1}},\ h \in \mathbf{O},\ g \in G(S), \tag{I.5.3.3}$$

where $S \in \mathsf{Alg}_{\Bbbk}$, and ${}^g h$ is as in (I.5.3.1). To see that this indeed defines a right G-module structure, note that the left G-module structure on $\mathbf{O}$ given by (I.5.3.1) induces such a structure on $\mathbf{O}^+/(\mathbf{O}^+)^2$, and in turn, it is transposed to $\mathfrak{g}$, since $\mathbf{O}^+/(\mathbf{O}^+)^2$ is $\Bbbk$-finite projective by (E2)–(E3); see (I.4.2.2). What is given by (I.5.3.3) is precisely the restriction to $\mathfrak{g}_{\bar{1}}$ of the transposed structure, while the restriction to $\mathfrak{g}_{\bar{0}}$ coincides with the one given by (I.5.3.2). It is now easy to see that the pair satisfies (F1)–(F4). Recall that $\mathfrak{g}$ is given the 2-operation which arises from the square map on $\mathbf{O}^{\bar{*}}$. Then one verifies (F5), using the fact that the G-module structure on $\mathbf{O}$ preserves the coproduct; cf. [27, Lemma A.9]. Therefore, $(G, \mathfrak{g}) \in (\mathrm{sHCP})_{\Bbbk}$. We let

$$\Phi(\mathbf{G}) = (\mathbf{G}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{G})).$$

One sees easily that this indeed defines a functor.

Remark 5.3.2. Following [12, Defintion 2.3.3], let $(\mathrm{fsgroups})_{\Bbbk}$ denote the category of those affine supergroup which satisfy (E2) and (E3). This includes $(\mathrm{gss\text{-}fsgroups})_{\Bbbk}$ as a full subcategory. Note that Condition (E1) was not used above, to define the functor Φ. In fact we have defined a functor $\Phi : (\mathrm{fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}$, as is formulated by [12, Proposition 4.1.3]. This last functor will be used to prove Theorem 5.3.7 in the next subsection.

Next, to construct Φ, we prove:

Lemma 5.3.3. *Let* $\mathbf{\Gamma}$ *be the affine supergroup constructed in Section 5.2, and set* $\mathbf{O} = \mathcal{O}(\mathbf{\Gamma})$*. Then we have*

$$\mathbf{\Gamma}_{\mathrm{ev}} = G, \quad W^{\mathbf{O}} = \mathfrak{g}_{\bar{1}}^{*},$$

where G *and* $\mathfrak{g}$ *are those given in Section 5.2 from which* $\mathbf{\Gamma}$ *is constructed. Moreover,* $\mathbf{\Gamma}$ *satisfies (E1) and (E3) above.*

Proof. From Remark 5.2.12 and the following argument we see that (I.5.2.8) gives an identification $\mathcal{O}(\mathbf{\Gamma}) = \mathcal{O}(G) \otimes \wedge(\mathfrak{g}_{\bar{1}}^{*})$ of left $\mathcal{O}(G)$-comodule superalgebras with counit. This implies the desired results. ☐

Finally, let $(G, \mathfrak{g}) \in (\mathrm{sHCP})_{\Bbbk}$. Choose these G and $\mathfrak{g}$ as those in Section 5.2. One sees by (F1)–(F2) that $\mathfrak{g}$ satisfies the assumption of Corollary 4.6.3. The given right G-module structure on $\mathfrak{g}_{\bar{1}}$, summed up with such a structure on $\mathfrak{g}_{\bar{0}}$ determined by (I.5.3.2), gives rise to an anti-morphism, say α, from G to $\mathsf{Aut}_{\mathrm{Lie}}(\mathfrak{g})$; see [27, Remark 4.5 (2)]. This α, together with the canonical pairing

$$\langle\ ,\ \rangle : \mathfrak{g}_{\bar{0}} \times \mathcal{O}(G) \longrightarrow \Bbbk, \quad \langle x, c\rangle = x(c),$$

satisfy (H1)–(H3), as is easily seen. We remark that Lie algebra map $i' : \mathfrak{g}_{\bar{0}} \to \mathrm{Lie}(G)$ given by (I.5.2.6) is now the identity. The construction of Section 5.2 gives an affine supergroup $\mathbf{\Gamma}$, which satisfies (E1)–(E3) by Lemma 5.3.3. Indeed, by (F2) it satisfies (E2) as well, since $\mathbf{\Gamma}_{\mathrm{ev}} = G$. Define $\Psi(G, \mathfrak{g})$ to be this $\mathbf{\Gamma}$ in $(\mathrm{gss\text{-}fsgroups})_{\Bbbk}$. As is easily seen, Ψ defines a functor.

Theorem 5.3.4 ([12, Theorem 4.3.14])**.** *We have a category equivalence*

$$(\mathrm{gss\text{-}fsgroups})_{\Bbbk} \approx (\mathrm{sHCP})_{\Bbbk}.$$

In fact the functors Φ *and* Ψ *constructed above are quasi-inverse to each other.*

Proof. Let $\mathbf{G} \in (\mathrm{gss\text{-}fsgroups})_{\Bbbk}$, and set

$$(G, \mathfrak{g}) = \Phi(\mathbf{G}), \quad \mathbf{\Gamma} = \Psi \circ \Phi(\mathbf{G}).$$

Just as for (I.5.2.7) we see that there uniquely exists a Hopf paring

$$\langle\ ,\ \rangle : \mathcal{U}(\mathfrak{g}) \times \mathcal{O}(\mathbf{G}) \to \Bbbk$$

such that $\langle z,\, h\rangle = z(h)$, $z \in \mathfrak{g}$, $h \in \mathcal{O}(\mathbf{G})$. Suppose $A \in \mathsf{SAlg}_\Bbbk$. Recall that $\mathbf{\Gamma}(A)$ is a quotient of the group $G(A_{\bar{0}}) \ltimes \mathbf{\Sigma}(A)$ of semi-direct product. Since $\mathbf{\Sigma}(A) \subset \mathsf{Gpl}(\mathcal{U}(\mathfrak{g})_A)$, the last pairing induces, after base extension to A, a group map

$$\mathbf{\Sigma}(A) \longrightarrow \mathsf{SAlg}_\Bbbk(\mathcal{O}(\mathbf{G}),\, A) = \mathbf{G}(A). \tag{I.5.3.4}$$

Lemma 5.2.6 gives the following equations in $\mathbf{\Sigma}(A)$:

$$e(a,v)^g = 1 \otimes 1 + a\, v^g, \quad f(\epsilon,x)^g = 1 \otimes 1 + \epsilon\, x^g, \quad g \in G(A_{\bar{0}}). \tag{I.5.3.5}$$

By definitions of Φ and Ψ, the G-actions on $\mathfrak{g}$ which appear on the right-hand sides are determined by

$$\langle z^g,\, h\rangle_{A_{\bar{0}}} = \langle z,\, {}^g h\rangle_{A_{\bar{0}}}, \quad z \in \mathfrak{g},\ h \in \mathcal{O}(\mathbf{G}),\ g \in G(A_{\bar{0}}),$$

where ${}^g h = g^{-1}(h_1)\, h_2\, g(h_3)$, as in (I.5.3.1). It follows that the group map (I.5.3.4) is right $G(A_{\bar{0}})$-equivariant, where we suppose that $G(A_{\bar{0}}) = \mathbf{G}(A_{\bar{0}})$ acts on $\mathbf{G}(A)$ by inner automorphisms. Therefore, the group map together with the embedding $G(A_{\bar{0}}) \to \mathbf{G}(A)$ uniquely extend to $G(A_{\bar{0}}) \ltimes \mathbf{\Sigma}(A) \to \mathbf{G}(A)$. It factors through $\mathbf{\Gamma}(A) \to \mathbf{G}(A)$, since $\mathbf{\Gamma}(A)$ is the quotient group of $G(A_{\bar{0}}) \ltimes \mathbf{\Sigma}(A)$ divided by the relations

$$(i(f(\epsilon,x)),1) = (1, f(\epsilon,x)), \quad x \in \mathfrak{g}_{\bar{0}},\ \epsilon \in A_{\bar{0}},\ \epsilon^2 = 0,$$

and $i : F(A_{\bar{0}}) \to G(A_{\bar{0}})$ is now the restriction of the canonical map $\mathsf{Gpl}(\mathcal{U}(\mathfrak{g}_{\bar{0}})_{A_{\bar{0}}}) \to G(A_{\bar{0}})$. The group map $\mathbf{\Gamma}(A) \to \mathbf{G}(A)$, being natural in A, gives rise to a morphism $\mathbf{\Gamma} \to \mathbf{G}$. This morphism is natural in $\mathbf{G}$, as is easily seen. In fact, it is a natural isomorphism by Lemma 4.5.5; see also Remark 5.3.12 below. Indeed, the assumptions required by the cited lemma are satisfied, since $\mathbf{\Gamma}$ and $\mathbf{G}$ satisfy (E1), the morphism $\mathbf{\Gamma} \to \mathbf{G}$ restricts to the identity $\mathbf{\Gamma}_{\mathrm{ev}} \to \mathbf{G}_{\mathrm{ev}}$, and the map $\mathfrak{g}_{\bar{1}} = \mathrm{Lie}(\mathbf{\Gamma})_{\bar{1}} \to \mathrm{Lie}(\mathbf{G})_{\bar{1}}$ induced from the pairing above is the identity. We conclude $\Psi \circ \Phi \simeq \mathrm{id}$.

Let $(G, \mathfrak{g}) \in (\mathrm{sHCP})_\Bbbk$, and set $\mathbf{\Gamma} = \Psi(G, \mathfrak{g})$. Recall that for this $\mathbf{\Gamma}$, the Lie algebra map $i' : \mathfrak{g}_{\bar{0}} \to \mathrm{Lie}(G)$ given by (I.5.2.6) is the identity. By Lemma 5.3.3 and Proposition 5.2.13 we have the natural identifications

$$G = \mathbf{\Gamma}_{\mathrm{ev}}, \quad \mathfrak{g} = \mathrm{Lie}(\mathbf{\Gamma})$$

of affine groups and of Lie superalgebras given 2-operations. Let $S \in \mathsf{Alg}_\Bbbk$. To conclude $\Phi \circ \Psi = \mathrm{id}$, we wish to prove that given $v \in \mathfrak{g}_{\bar{1}}$ and $g \in G(S)$, the result $v^g \in (\mathfrak{g}_{\bar{1}})_S$ by the G-action associated with the original $(G, \mathfrak{g})$ coincides the one given by (I.5.3.3) for $\mathbf{\Gamma}$. Suppose $A = S \otimes \wedge(\tau)$, where τ is an odd variable. Note $A_{\bar{0}} = S$. Just as in (I.5.3.5) we have $e(\tau, v)^g = 1 \otimes 1 + \tau\, v^g$ in $\mathbf{\Gamma}(A)$. This, evaluated at $h \in \mathcal{O}(\mathbf{\Gamma})$, gives $\tau\, v({}^g h) = \tau\, v^g(h)$, which shows the desired result. $\square$

Remark 5.3.5. Let $(G, \mathfrak{g}) \in (\mathrm{sHCP})_\Bbbk$, and recall that to this $\mathfrak{g}$ is given a 2-operation, say $(-)^{\langle 2\rangle}$.

Replace $(\mathfrak{g},\ (-)^{\langle 2\rangle})$ with the cocycle deformation $({}_\sigma\mathfrak{g},\ (-)^{\sigma\langle 2\rangle})$ by σ (see Remark 4.6.5), keeping the right G-module structure on the odd component unchanged. Then we see $(G,\ {}_\sigma\mathfrak{g}) \in (\mathrm{sHCP})_{\Bbbk}$, and that $(G,\mathfrak{g}) \mapsto (G,\ {}_\sigma\mathfrak{g})$ gives an involutory category isomorphism, which we denote by

$$(\mathrm{id},\ {}_\sigma(-)) : (\mathrm{sHCP})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}.$$

As was remarked in Introduction, Gavarini's category equivalence was re-proved in [27, Appendix], using an older construction of affine supergroups. Due to different choice of tensor products of pairings, the category equivalence $\mathbf{P}' : (\text{gss - fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}$ shown there is slightly different from the Φ above. In fact, we see

$$\mathbf{P}' = (\mathrm{id},\ {}_\sigma(-)) \circ \Phi. \tag{I.5.3.6}$$

Remark 5.3.6. The argument of Gavarini [12] seems incomplete at some points, as is pointed out below. See also Remark 4.6.10.

(1) To construct the functor $\Phi_g : (\text{gss - fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}$, and prove $\Phi_g \circ \Psi_g = \mathrm{id}$ in [12, Proposition 4.1.3, Theorem 4.3.14], the article takes no account of 2-operations or G-supermodule structures on Lie superalgebras.

(2) The functoriality of $\Psi_g : (\mathrm{sHCP})_{\Bbbk} \to (\text{gss - fsgroups})_{\Bbbk}$ (see [12, Proposition 4.3.9 (2)]) is proved, indeed, if one replaces the original definition of Ψ_g by the group $G_{\mathcal{P}}(A)\ (= \Psi_g(\mathcal{P}))$ given in [12, Definition 4.3.2] (and referred to before Lemma 5.2.10), with the definition by the alternative $G^{\bullet}_{\mathcal{P}}(A)$ given in [12, Remark 4.3.3 (c)]. Nevertheless, in view of the equations preceding our Lemma 5.2.1, the relation $(1 + (c\eta)Y) = (1 + \eta(cY))$, $c \in \Bbbk$, is missing to define the group $G^{\bullet}_{\mathcal{P}}(A)$ in the last cited remark.

5.3.2 Tensor product decomposition

Let $\mathbf{G}$ be an affine supergroup, and set $\mathbf{O} = \mathcal{O}(\mathbf{G})$. We prove the following theorem. Note that the conclusion is the same as (E1).

Theorem 5.3.7. *Assume that $\overline{\mathbf{O}}$ is $\Bbbk$-flat. There exists a counit-preserving isomorphism $\mathbf{O} \simeq \overline{\mathbf{O}} \otimes \wedge(W^{\mathbf{O}})$ of left $\overline{\mathbf{O}}$-comodule superalgebras, if*

(E2) $\overline{\mathbf{O}}^{+}/(\overline{\mathbf{O}}^{+})^2$ is $\Bbbk$-finite projective, and

(E3) $W^{\mathbf{O}}$ is $\Bbbk$-finite free.

Remark 5.3.8. (1) Let $(\text{gss - fsgroups})'_{\Bbbk}$ denote the category of the affine supergroups $\mathbf{G}$ which satisfy (E2), (E3) and

(E0) $\mathcal{O}(\mathbf{G}_{\mathrm{ev}})$ is $\Bbbk$-flat.

This category is a full subcategory of $(\text{gss - fsgroups})_{\Bbbk}$ by Theorem 5.3.7. Let $(\mathrm{sHCP})'_{\Bbbk}$ denote the full subcategory of $(\mathrm{sHCP})_{\Bbbk}$ which consists of the objects (G, V) such that

(F0) $\mathcal{O}(G)$ is $\Bbbk$-flat.

One sees that the category equivalence given by Theorem 5.3.4 restricts to

$$(\text{gss-fsgroups})'_{\Bbbk} \approx (\text{sHCP})'_{\Bbbk}.$$

(2) Suppose that $\Bbbk$ is 2-torsion free, or namely, $2 : \Bbbk \to \Bbbk$ is an injection. In this special situation, essentially the same category equivalence as given by Theorem 5.3.4 was proved by Theorem 4.5.1; one need not there refer to 2-operations. To be more precise, considered there is the category ASG of the algebraic supergroups $\mathbf{G}$ which satisfy (E0) as well as (E1)–(E3). However, (E1) can be removed from the last conditions, as is ensured by Theorem 5.3.7. To define ASG in Section 4.3, one can thus weaken the condition that $\mathbf{O} = \mathcal{O}(\mathbf{G})$ is *split* to the one that $W^{\mathbf{O}}$ is $\Bbbk$-free.

(3) Suppose that $\Bbbk$ is a field of characteristic $\neq 2$. Then the conclusion of Theorem 5.3.7 holds for any finitely generated super-commutative Hopf superalgebra $\mathbf{O}$, since the assumptions are then necessarily satisfied. The result was in fact proved by [22, Theorem 4.5] for any $\mathbf{O}$ that is not necessarily finitely generated. The proof uses Hopf crossed products, and is crucial when $\mathbf{O}$ is finitely generated. The proof below gives an alternative proof of the cited theorem in this crucial case.

The rest of this subsection is devoted to proving the theorem. The proof is divided into 3 steps.

Step 1

Recall from Remark 5.3.2 that the functor Φ is defined on the category $(\text{fsgroups})_{\Bbbk}$ including $(\text{gss-fsgroups})_{\Bbbk}$, which consists of the affine supergroup satisfying (E2) and (E3).

Let $\mathbf{G} \in (\text{fsgroups})_{\Bbbk}$, and set $\mathbf{\Gamma} = \Psi \circ \Phi(\mathbf{G})$, as in the proof of Theorem 5.3.4. The argument in the cited proof which shows that we have a natural morphism $\mathbf{\Gamma} \to \mathbf{G}$ of affine supergroups is valid. Let

$$\phi : \mathbf{\Gamma} \longrightarrow \mathbf{G} \tag{I.5.3.7}$$

denote the morphism. We will prove that this ϕ is an isomorphism, assuming that $\overline{\mathbf{O}}$ is $\Bbbk$-flat. This proves the theorem, since $\mathbf{\Gamma}$ satisfies (E1).

Step 2

We need some general Hopf-algebraic argument. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the semigroup of non-negative integers, as before. An $\mathbb{N}$-graded $\Bbbk$-module $V = \bigoplus_{n=0}^{\infty} V(n)$ is regarded as a $\Bbbk$-supermodule so that $V_{\bar{0}} = \bigoplus_{n:\,\text{even}} V(n)$, $V_{\bar{1}} = \bigoplus_{n:\,\text{odd}} V(n)$. The $\mathbb{N}$-graded $\Bbbk$-modules form a symmetric tensor category $\mathsf{GrMod}_{\Bbbk}$ with respect to the super-symmetry.

Let $\mathsf{ConnAlg}_{\Bbbk}$ denote the category of the commutative algebra objects $\mathbf{B}$ in $\mathsf{GrMod}_{\Bbbk}$ such that $\mathbf{B}(0) = \Bbbk$; the Conn expresses "connected", meaning $\mathbf{B}(0) = \Bbbk$.

Fix a commutative Hopf algebra O. Note that O is a commutative Hopf-algebra object in $\mathsf{GrMod}_{\Bbbk}$ which is trivially graded, $O(0) = O$. A *graded left O-comodule* is a left O-comodule object in $\mathsf{GrMod}_{\Bbbk}$. The graded left O-comodules form a symmetric tensor category O-$\mathsf{GrComod}$. Let O-$\mathsf{NGrComodAlg}$ denote the category of the commutative algebra objects $\mathbf{A}$ in O-$\mathsf{GrComod}$ such that $\mathbf{A}(0) = O$; the NGr expresses "neutrally graded", meaning $\mathbf{A}(0) = O$. Note that every such object is an (ordinary) left O-Hopf module [30, Page 15] with respect to the left multiplication by O.

Here, commutative algebra objects may not satisfy the condition that every odd elements should be square-zero.

Given $\mathbf{B} \in \mathsf{ConnAlg}_{\Bbbk}$, the tensor product

$$O \otimes \mathbf{B}$$

of graded algebras, given the left O-comodule structure $\Delta \otimes \mathrm{id}_{\mathbf{B}}$, is an object in O-$\mathsf{NGrComodAlg}$. Moreover, this constructs a functor

$$O \otimes - : \mathsf{ConnAlg}_{\Bbbk} \longrightarrow O\text{-}\mathsf{NGrComodAlg}.$$

Proposition 5.3.9. *This functor is a category equivalence.*

Proof. Given $\mathbf{A} \in O$-$\mathsf{NGrComodAlg}$,

$$\mathbf{A}/O^{+}\mathbf{A}$$

is naturally an object in $\mathsf{ConnAlg}_{\Bbbk}$. One sees that this constructs a functor. We wish to show that this is a quasi-inverse of the functor $O \otimes -$. We have to prove that the two composites of the functors are naturally isomorphic to the identity functors. For one composite this is easy. For the remaining, let $\mathbf{A} \in O$-$\mathsf{NGrComodAlg}$. Set $\mathbf{B} = \mathbf{A}/O^{+}\mathbf{A}$, and let $\pi : \mathbf{A} \to \mathbf{B}$ denote the natural projection. We see that the left O-comodule structure $\mathbf{A} \to O \otimes \mathbf{A}$, $a \mapsto a_{-1} \otimes a_0$ on $\mathbf{A}$ induces the morphism

$$\mathbf{A} \to O \otimes \mathbf{B}, \quad a \mapsto a_{-1} \otimes \pi(a_0)$$

in O-$\mathsf{NGrComodAlg}$ which is natural in $\mathbf{A}$. It remains to prove that this is an isomorphism. As was remarked before, $\mathbf{A}$ is a left O-Hopf module, and the morphism above is in fact a morphism of Hopf modules. The fundamental theorem for Hopf modules [30, 1.9.4, Page 15] holds over an arbitrary base ring $\Bbbk$, and can now apply to see that the morphism above is an isomorphism. [To be more precise, what we need here for later use is a variant of the isomorphism α given in the proof of the cited theorem, and is in fact the first adjunction given in the proof of [37, Theorem 1] (see Page 456, line −7) when the right coideal subalgebra B of [37] is the base field.] ☐

Let $\mathbf{O}$ be a super-commutative Hopf superalgebra. Set $O = \overline{\mathbf{O}}$, and assume that this O is $\Bbbk$-flat. Let O-SComod denote the symmetric tensor category of left O-super-comodules. The flatness assumption ensures that this category is abelian; see [16, Part I, 2.9]. Indeed, the $\Bbbk$-linear kernel Z of a morphism $V \to U$ turns to be a sub-object of V, since we have $O \otimes Z \subset O \otimes V$, and the composite $Z \hookrightarrow V \to O \otimes V$ of the inclusion with the structure on V factors through $O \otimes Z$.

Let $I = \mathbf{O}\mathbf{O}_{\bar{1}}$, so that we have $\mathbf{O}/I = O$. Note that $\mathbf{O}$ is naturally a commutative algebra object in O-SComod, and the super-ideals I^n, $n > 0$, are sub-objects of $\mathbf{O}$ in O-SComod. It follows that

$$\mathrm{gr}\,\mathbf{O} = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$$

is an object in O-$\mathsf{NGrComodAlg}$. To see this, note $\operatorname{gr}\mathbf{O}(0) = O$. Moreover, $I^n/I^{n+1} = \mathbf{O}_{\bar{1}}^n/\mathbf{O}_{\bar{1}}^{n+2}$, and so $\operatorname{gr}\mathbf{O}(n)$ is purely odd (resp., even) if n is odd (resp., even).

Let $\mathbf{B} = \operatorname{gr}\mathbf{O}/O^+(\operatorname{gr}\mathbf{O})$ denote the object in $\mathsf{ConnAlg}_{\Bbbk}$ which corresponds to $\operatorname{gr}\mathbf{O}$ through the category equivalence given in (the proof of) Proposition 5.3.9. It is easy to see the following (see [22, Proposition 4.3 (1)]):

Lemma 5.3.10. *The composite of natural maps*

$$W^{\mathbf{O}} = \mathbf{O}_{\bar{1}}/\mathbf{O}_{\bar{0}}^+\mathbf{O}_{\bar{1}} \longrightarrow \mathbf{O}_{\bar{1}}/\mathbf{O}_{\bar{1}}^3 = \operatorname{gr}\mathbf{O}(1) \longrightarrow \mathbf{B}(1)$$

is an isomorphism.

Step 3

Let $\mathbf{O}$ be a super-commutative Hopf superalgebra. Note that the constructions of the associated $\overline{\mathbf{O}}$ and $W^{\mathbf{O}}$ are functorial.

Assume that $\mathbf{O}$ satisfies (E1) and (E3). Assume that $\overline{\mathbf{O}}$ is $\Bbbk$-flat. Let $\mathbf{O}'$ be a super-commutative Hopf superalgebra, and let $\psi : \mathbf{O}' \to \mathbf{O}$ is a Hopf superalgebra map. It naturally induces

$$\overline{\psi} : \overline{\mathbf{O}'} \longrightarrow \overline{\mathbf{O}}, \quad W^{\psi} : W^{\mathbf{O}'} \longrightarrow W^{\mathbf{O}}.$$

Proposition 5.3.11. *If these two maps are bijections, then ψ is an isomorphism.*

Proof. We may suppose $\overline{\mathbf{O}'} = \overline{\mathbf{O}} = O$ and $\overline{\psi} = \mathrm{id}_O$, where O is a commutative $\Bbbk$-flat Hopf algebra. We see that ψ induces a morphism $\operatorname{gr}(\psi) : \operatorname{gr}\mathbf{O}' \to \operatorname{gr}\mathbf{O}$ in O-$\mathsf{NGrComodAlg}$. Let $\xi : \mathbf{B}' \to \mathbf{B}$ be the corresponding morphism between the corresponding objects in $\mathsf{ConnAlg}_{\Bbbk}$.

We wish to show that ξ is an isomorphism. By Lemma 5.3.10, $\xi(1) : \mathbf{B}'(1) \to \mathbf{B}(1)$ is identified with W^{ψ}. Since $\mathbf{O}$ satisfies (E1), we see that $\operatorname{gr}\mathbf{O} = O \otimes \wedge(W^{\mathbf{O}})$, and so $\mathbf{B} = \wedge(W^{\mathbf{O}})$. It follows that ξ has a unique section in $\mathsf{ConnAlg}_{\Bbbk}$, since $\xi(1)$ is an isomorphism, and $\mathbf{B}'$ is super-commutative, with the odd elements being square-zero. Note that $\mathbf{B}'$ is generated by $\mathbf{B}'(1)$, since $\operatorname{gr}\mathbf{O}'$ is generated by $O = \operatorname{gr}\mathbf{O}'(0)$ and $\operatorname{gr}\mathbf{O}'(1)$. This implies that the section is an isomorphism, proving the desired result.

It follows that $\operatorname{gr}(\psi)$ is an isomorphism, and $\operatorname{gr}\mathbf{O}'(n) = \operatorname{gr}\mathbf{O}(n) = 0$ for $n \gg 0$. Therefore, ψ is an isomorphism. □

Remark 5.3.12. In the situation of Proposition 5.3.11, suppose in addition that $\mathbf{O}'$ satisfies (E1), and remove the assumption that $\overline{\mathbf{O}}$ is $\Bbbk$-flat. Then the same result as the proposition follows easily from Lemma 4.5.5. The result was essentially used to prove Theorem 5.3.4 in the last paragraph of the proof.

Let us return to the natural morphism $\phi : \mathbf{\Gamma} \to \mathbf{G}$ in (I.5.3.7), assuming that $\overline{\mathcal{O}(\mathbf{G})}$ is $\Bbbk$-flat. Consider $\mathcal{O}(\phi) : \mathcal{O}(\mathbf{G}) \to \mathcal{O}(\mathbf{\Gamma})$. In view of the proof of Theorem 5.3.4 (see the last part of the first paragraph), the induced $\overline{O(\mathbf{G})} \to \overline{\mathcal{O}(\mathbf{\Gamma})}$ and $W^{\mathcal{O}(\mathbf{G})} \to W^{\mathcal{O}(\mathbf{\Gamma})}$ are both the identity maps. It follows that $\overline{\mathcal{O}(\mathbf{\Gamma})}$ is $\Bbbk$-flat. Since $\mathbf{\Gamma}$ satisfies (E1) and (E3), Proposition 5.3.11, applied to $\mathcal{O}(\phi)$, proves that ϕ is an isomorphism, as desired.

5.3.3 The category equivalence over a field

In what follows we suppose that $\Bbbk$ is a field of characteristic $\neq 2$.

We let $\mathsf{ASG}_\Bbbk$ denote the category of algebraic supergroups over $\Bbbk$. This coincides with the full subcategory of (gss - fsgroups)$_\Bbbk$ consisting of the objects which are algebraic supergroups. By [22, Theorem 4.5] every object in $\mathsf{ASG}_\Bbbk$ satisfies (E1), in particular.

Since every Lie superalgebra has the unique 2-operation defined by

$$v^{\langle 2\rangle} = \frac{1}{2}[v,\, v]$$

we may not refer to 2-operations on Lie superalgebras. The definition of (sHCP)$_\Bbbk$ then contains redundancy in (F1). In other words one can remove $\mathfrak{g}_{\bar{0}}$ from the definition since it is determined by G. We define Harish-Chandra pairs as follows, as in [25, 29]. In the next subsection one will see that our definition is suitable at least to describe sub-objects.

A *Harish-Chandra pair* is a pair (G, V) of an algebraic group G and a finite-dimensional right G-module V which is given a G-equivariant linear map $[\ ,\ ] : V \otimes V \to \mathrm{Lie}(G)$ such that

(i) $[v, v'] = [v', v], \quad v, v' \in V,$

(ii) $v \triangleleft [v, v] = 0, \quad v \in V.$

When we say that $[\ ,\ ]$ is G-equivariant, $\mathrm{Lie}(G)$ is regarded as a right G-module as was done in (I.5.3.2). In (ii), $\triangleleft$ represents the right $\mathrm{Lie}(G)$-Lie module structure on V defined by

$$v \triangleleft x = x(v_{-1})\, v_0, \quad v \in V,\ x \in \mathrm{Lie}(G), \tag{I.5.3.8}$$

where $V \to \mathcal{O}(G) \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$ denotes the left $\mathcal{O}(G)$-comodule structure corresponding to the right G-module structure on V. A *morphism* $(\phi, \psi) : (G_1, V_1) \to (G_2, V_2)$ of Harish-Chandra pairs consists of a morphism $\phi : G_1 \to G_2$ of algebraic groups and a linear map $\psi : V_1 \to V_2$ such that

(iii) ψ is G_1-equivariant, with V_2 regarded as a G_1-module through ϕ,

(iv) $[\psi(v), \psi(v')] = \mathrm{Lie}(\phi)([v, v']), \quad v, v' \in V.$

We let $\mathsf{HCP}_\Bbbk$ denote the category of Harish-Chandra pairs over $\Bbbk$.

This category $\mathsf{HCP}_\Bbbk$ is isomorphic to the full subcategory of (sHCP)$_\Bbbk$ consisting of the objects $(G, \mathfrak{g})$ in which G is an algebraic group. To describe an explicit category isomorphism, let $(G, V) \in \mathsf{HCP}_\Bbbk$. Define $\mathfrak{g} := \mathrm{Lie}(G) \oplus V$, and suppose $\mathfrak{g} \in \mathsf{SMod}_\Bbbk$ so that $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$, $\mathfrak{g}_{\bar{1}} = V$. Give to $\mathfrak{g}$ the bracket on $\mathrm{Lie}(G)$ and the structure $[\ ,\ ]$ of (G, V), and define $[v, x] := v \triangleleft x$ for v, x as in (I.5.3.8). Then $\mathfrak{g}$ turns into a Lie superalgebra. Keep $\mathfrak{g}_{\bar{1}} = V$ given the right G-module structure. One sees that $(G, V) \mapsto (G, \mathfrak{g})$ gives the desired category isomorphism. The inverse is given by $(G, \mathfrak{g}) \mapsto (G, \mathfrak{g}_{\bar{1}})$, where to $\mathfrak{g}_{\bar{1}}$ of the latter, the restricted super-bracket and the original G-module structure are given.

Now, let $\mathbf{G} \in \mathsf{ASG}_\Bbbk$. Then $\mathbf{G}_{\mathrm{ev}}$ is an algebraic group, and the Lie superalgebra $\mathrm{Lie}(\mathbf{G})$ is finite-dimensional. Regard the odd component $\mathrm{Lie}(\mathbf{G})_{\bar{1}}$ of the Lie superalgebra as the right $\mathbf{G}_{\mathrm{ev}}$-module defined by (I.5.3.3). Restrict the super-bracket on Lie($\mathbf{G}$) to the odd component, and give it to the pair $(\mathbf{G}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{G})_{\bar{1}})$. Then the pair turns into a Harish-Chandra pair, and it corresponds to $\Phi(\mathbf{G})$ in $(\mathrm{sHCP})_{\Bbbk}$. By Theorem 5.3.4 we have:

Theorem 5.3.13. $\mathbf{G} \mapsto (\mathbf{G}_{\mathrm{ev}}, \mathrm{Lie}(\mathbf{G})_{\bar{1}})$ *gives a category equivalence*

$$\mathsf{ASG}_{\Bbbk} \xrightarrow{\approx} \mathsf{HCP}_{\Bbbk}.$$

Essentially the same result was already given in [25, 29]; see Remark 5.3.14 below. As an advantage we have obtained an explicit quasi-inverse of the functor above, which is essentially the same as Ψ in Section 5.3.1. Therefore, every algebraic supergroup can be realized as $\mathbf{\Gamma}$ constructed in Section 5.2. This realization is useful when we discuss group-theoretical properties of algebraic supergroups, as will be shown in the next subsection.

Remark 5.3.14. A category equivalence between $\mathsf{ASG}_{\Bbbk}$ and $\mathsf{HCP}_{\Bbbk}$ is given by [25, Theorem 6.5] and [29, Theorem 3.2], which both reformulate the result [24, Theorem 29] formulated in purely Hopf-algebraic terms. Given $(G, V) \in \mathsf{HCP}_{\Bbbk}$, denote now it by $(G, V, [\ ,\ ])$, indicating the structure. Replacing $[\ ,\ ]$ with $-[\ ,\ ]$, we still have $(G, V, -[\ ,\ ]) \in \mathsf{HCP}_{\Bbbk}$. Moreover,

$$(G, V, [\ ,\ ]) \longmapsto (G, V, -[\ ,\ ])$$

gives an involutory category isomorphism $\mathsf{HCP}_{\Bbbk} \to \mathsf{HCP}_{\Bbbk}$. The category equivalence given by Theorem 5.3.13 coincides, up to the last category isomorphism, with the one cited above, just as was seen in (I.5.3.6).

5.4 Normalizers and centralizers

Let $\Bbbk$ be a field of characteristic $\neq 2$. Throughout in this subsection we let $\mathbf{G} \in \mathsf{ASG}_{\Bbbk}$, and let (G, V) be the associated Harish-Chandra pair. We suppose that $\mathbf{G}$ is realized as the $\mathbf{\Gamma}$ which is constructed as in Section 5.2 from G, $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$, the canonical pairing $\mathfrak{g}_{\bar{0}} \times \mathcal{O}(G) \to \Bbbk$ and the right G-supermodule structure on $\mathfrak{g}$ defined by (I.5.3.2) and (I.5.3.3).

Recall that a pair (K, W) of closed subgroup $K \subset G$ and a vector subspace $W \subset V$ is a *sub-pair* of the Harish-Chandra pair (G, V) (see Definition 4.5.7), if

(i) W is K-stable in V, and

(ii) $[W, W] \subset \mathrm{Lie}(K)$,

where $[\ ,\ ]$ is the structure of (G, V). If $\mathbf{K}$ is a closed subsupergroup of $\mathbf{G}$, then the associated Harish-Chandra pair (K, W), with the right K-module structure on W and the structure $[\ ,\ ]$ forgotten, is a sub-pair of (G, V). In this case we say that the sub-pair (K, W) *corresponds to* $\mathbf{K}$. The assignment $\mathbf{K} \mapsto (K, W)$ as above gives a bijection from the set of all closed subsupergroups of $\mathbf{G}$ to the set of all sub-pairs of (G, V).

Lemma 5.4.1. *Let (K, W) be the sub-pair of (G, V) corresponding to a closed subsupergroup $\mathbf{K} \subset \mathbf{G}$. Given $v \in V$, the following are equivalent:*

(a) $v \in W$;

(b) $e(a,v) \in \mathbf{K}(A)$ *for arbitrary* $A \in \mathsf{SAlg}_{\Bbbk}$ *and* $a \in A_{\bar{1}}$;

(c) $e(a,v) \in \mathbf{K}(A)$ *for some* $A \in \mathsf{SAlg}_{\Bbbk}$ *and* $a \in A_{\bar{1}}$ *with* $a \neq 0$.

Proof. We only prove (c) ⇒ (a), since the rest is obvious.

Suppose that $e(a,v) \in \mathbf{K}(A)$ with $a \in A_{\bar{1}}$, but $v \notin W$. Given an arbitrary basis $w_1, \dots, w_r$ of W, one can extend it, adding v and others, to a basis $w_1, \dots, w_r, v, \dots$ of V. By Proposition 5.2.9, $e(a,v)$, being an element in $\mathbf{K}(A)$, is expressed uniquely of the form

$$e(a,v) = h\, e(a_1, w_1) \cdots e(a_r, w_r), \tag{I.5.4.1}$$

where $h \in K(A_{\bar{0}})$ and $a_i \in A_{\bar{1}}$, $1 \leq i \leq r$. The cited proposition gives analogous expressions of elements of $\mathbf{G}(A)$ which use the extended basis. Regarding (I.5.4.1) as two such expressions of one element, we have $a = 0$. □

Just as in the non-super situation we define as follows, and obtain the next lemma; see [16, Part I, 2.6].

Let $\mathbf{K} \subset \mathbf{G}$ be a closed subsupergroup. The *normalizer* $\mathcal{N}_{\mathbf{G}}(\mathbf{K})$ (resp., the *centralizer* $\mathcal{Z}_{\mathbf{G}}(\mathbf{K})$) of $\mathbf{K}$ in $\mathbf{G}$ is the subgroup functor of $\mathbf{G}$ whose A-points consists of the elements $g \in \mathbf{G}(A)$ such that for every $A \to A'$ in $\mathsf{SAlg}_{\Bbbk}$, the natural image $g_{A'}$ of g in $\mathbf{G}(A')$ normalizes (resp., centralizes) $\mathbf{K}(A')$.

Lemma 5.4.2. *$\mathcal{N}_{\mathbf{G}}(\mathbf{K})$ and $\mathcal{Z}_{\mathbf{G}}(\mathbf{K})$ are closed subsupergroups of $\mathbf{G}$. Moreover, $\mathcal{N}_{\mathbf{G}}(\mathbf{K})$ (resp., $\mathcal{Z}_{\mathbf{G}}(\mathbf{K})$) is the largest closed subsupergroup of $\mathbf{G}$ whose A-points normalize (resp., centralize) $\mathbf{K}(A)$ for every $A \in \mathsf{SAlg}_{\Bbbk}$.*

Let $\mathbf{K} \subset \mathbf{G}$ be a closed subsupergroup, and let (K, W) be the corresponding sub-pair of (G, V).

Recall that the *stabilizer* $\mathrm{Stab}_G(W)$ (resp., the *centralizer* $\mathrm{Cent}_G(W)$) of W in G is the largest closed subgroup of G that makes W into a module (resp., a trivial module) over it.

Let $\rho_K : V \to \mathcal{O}(K) \otimes V$ denote the left $\mathcal{O}(K)$-comodule structure on V corresponding to the restricted right K-module structure on V. Define

$$\mathrm{Inv}_K(V/W) := \{ v \in V \mid \rho_K(v) - 1 \otimes v \in \mathcal{O}(K) \otimes W \}.$$

This is the largest K-submodule of V including W whose quotient K-module by W is trivial. The definition makes sense, replacing W with any K-submodule, say U, of V. We will use $\mathrm{Inv}_K(V) = \mathrm{Inv}_K(V/0)$ when $U = 0$.

When $L = \mathrm{Lie}(K)$ or 0, we define

$$(L : W) := \{ v \in V \mid [v, W] \subset L \},$$

where [,] is the structure of (G, V).

Theorem 5.4.3. *Let $\mathbf{K} \subset \mathbf{G}$ and $(K, W) \subset (G, V)$ be as above.*

(1) The sub-pair of (G, V) corresponding to $\mathcal{N}_{\mathbf{G}}(\mathbf{K})$ is

$$(\mathcal{N}_G(K) \cap \mathrm{Stab}_G(W),\ \mathrm{Inv}_K(V/W) \cap (\mathrm{Lie}(K) : W)).$$

(2) The sub-pair of (G, V) corresponding to $\mathcal{Z}_{\mathbf{G}}(\mathbf{K})$ is

$$(\mathcal{Z}_G(K) \cap \mathrm{Cent}_G(W),\ \mathrm{Inv}_K(V) \cap (0 : W)).$$

Proof. In each part let us denote by (F, Z) the desired sub-pair.

(1) First, we prove

$$F \subset \mathcal{N}_G(K) \cap \mathrm{Stab}_G(W), \quad Z \subset \mathrm{Inv}_K(V/W) \cap (\mathrm{Lie}(K) : W). \tag{I.5.4.2}$$

Note that F normalizes $\mathbf{K}$ in $\mathbf{G}$. Then it follows that F normalizes $K = \mathbf{K}_{\mathrm{ev}}$ in $G = \mathbf{G}_{\mathrm{ev}}$, whence $F \subset \mathcal{N}_G(K)$. It also follows that the right G-supermodule structure on $\mathrm{Lie}(\mathbf{G})$, restricted to a right F-supermodule structure, stabilizes $\mathrm{Lie}(\mathbf{K})$, whence $F \subset \mathrm{Stab}_G(W)$.

Since $[\mathrm{Lie}(\mathbf{K}), \mathrm{Lie}(\mathcal{N}_{\mathbf{G}}(\mathbf{K}))] \subset \mathrm{Lie}(\mathbf{K})$, we have $[W, Z] \subset \mathrm{Lie}(K)$, whence $Z \subset (\mathrm{Lie}(K) : W)$.

To prove $Z \subset \mathrm{Inv}_K(V/W)$, choose $z \in Z$. We may suppose $z \notin W$. Let $A = \mathcal{O}(K) \otimes \wedge(\tau)$ with τ an odd variable. We have an A-point $e(\tau, z)$ of $\mathcal{N}_{\mathbf{G}}(\mathbf{K})$ by Lemma 5.4.1. Given a basis $w_1, \ldots, w_r$ of W, we extend it, adding z and others, to a basis $w_1, \ldots, w_r$, z, $u_1, \ldots, u_s$ of V. Present $\rho_K(z)$ as

$$\rho_K(z) = \sum_{i=1}^{r} a_i \otimes w_i + b \otimes z + \sum_{i=1}^{s} c_i \otimes u_i \quad \in \mathcal{O}(K) \otimes V.$$

Then we have $b = 1$. Let $h \in K(A_{\bar{0}})$ be $\mathrm{id}_{\mathcal{O}(K)}$. Then

$$e(\tau, z)\, h\, e(\tau, z)^{-1} = h\, e(\tau a_1, w_1) \cdots e(\tau a_r, w_r)\, e(\tau c_1, u_1) \cdots e(\tau c_s, u_s). \tag{I.5.4.3}$$

Since this is contained in $\mathbf{K}(A)$, it follows by the same argument as proving Lemma 5.4.1 that $c_i = 0$, $1 \leq i \leq s$, whence $Z \subset \mathrm{Inv}_K(V/W)$. We have thus proved (I.5.4.2).

Next, to prove the converse inclusions, choose $\phi : A \to A'$ from $\mathsf{SAlg}_{\Bbbk}$.

Let g be an A-point of $\mathcal{N}_G(K) \cap \mathrm{Stab}_G(W)$. Then $g_{A'}$ normalizes $K(A')$. Given $a \in A'_{\bar{1}}$ and $w \in W$, we have

$$e(a, w)^{g_{A'}} = 1 \otimes 1 + a\, w^g \in \mathbf{K}(A'),$$

and the same result with g replaced by g^{-1} holds. This proves $g \in F(A)$.

Let $v \in \mathrm{Inv}_K(V/W) \cap (\mathrm{Lie}(K) : W)$ and $0 \neq a \in A_{\bar{1}}$. To see that $v \in Z$, we wish to prove, using Lemma 5.4.1, that $e(a, v)$ is an A-point of $\mathcal{N}_{\mathbf{G}}(\mathbf{K})$. Note that the A'-point $e(a, v)_{A'}$ of its image is $e(\phi(a), v)$. Given $h \in K(A')$, the same argument as proving (I.5.4.3) shows $e(a, v)_{A'}\, h\, e(a, v)_{A'}^{-1} \in \mathbf{K}(A')$, since $v^h - v \in W_{A'_{\bar{0}}}$. Given $w \in W$ and $b \in A'_{\bar{1}}$, we see by Lemma 5.2.2 (i) that

$$e(a, v)_{A'}\, e(b, w)\, e(a, v)_{A'}^{-1} = i(f(-\phi(a)b, [v, w]))\, e(b, w) \in \mathbf{K}(A'),$$

since $[v, w] \in \mathrm{Lie}(K)$. The last two conclusions prove the desired result.

(2) We only prove

$$K \subset \mathcal{Z}_G(K) \cap \mathrm{Cent}_G(W), \quad Z \subset \mathrm{Inv}_K(V) \cap (0 : W).$$

The converse inclusions follow by modifying slightly the second half of the proof of Part 1.

Since F centralizes $\mathbf{K}$ in $\mathbf{G}$. it follows that F centralizes K in G, whence $F \subset \mathcal{Z}_G(K)$. It also follows that the restricted right F-supermodule structure on $\mathrm{Lie}(\mathbf{G})$ centralizes $\mathrm{Lie}(\mathbf{K})$, whence $F \subset \mathrm{Cent}_G(W)$.

Since $[\mathrm{Lie}(\mathbf{K}), \mathrm{Lie}(\mathcal{Z}_{\mathbf{G}}(\mathbf{K}))] = 0$, we have $[W, Z] = 0$, whence $Z \subset (0 : W)$. The argument which proved $Z \subset \mathrm{Inv}_K(V/W)$ above, modified with W replaced by 0, shows $Z \subset \mathrm{Inv}_K(V)$. □

Suppose that $\mathbf{G} = \mathbf{K}$, and so $G = K$, $V = W$. Then Part 2 above reads:

Corollary 5.4.4. *Let* $\mathbf{G}$ *and* (G, V) *be as above. The sub-pair of* (G, V) *corresponding to the center* $\mathcal{Z}(\mathbf{G}) = \mathcal{Z}_{\mathbf{G}}(\mathbf{G})$ *of* $\mathbf{G}$ *is*

$$(\mathcal{Z}(G) \cap \mathrm{Cent}_G(V),\ \mathrm{Inv}_G(V) \cap (0 : V)).$$

The algebraic group component of this sub-pair was obtained by [29, Proposition 7.1], recently.

Part II

Representations of Quasireductive Supergroups

Chapter 1

Introduction

In this part, we study representations of quasireductive supergroups over a field $\Bbbk$ of characteristic not equal to 2.

Representations of (connected) algebraic supergroups $\mathbf{G}$ over $\mathbb{C}$ were fully studied. These representations are essentially the same as representations of their Lie superalgebra $\mathrm{Lie}(\mathbf{G})$. The classification of finite-dimensional simple Lie superalgebras over $\mathbb{C}$ was done by Kac [17] in 1977.

On the other hand, representations of algebraic supergroups over a field of positive characteristic has been less studied. Brundan and Kleshchev [3] studied representations of the algebraic supergroup $\mathbf{Q}(n)$ which have a close relationship to modular representations of spin symmetric groups. Moreover, the Mullineux conjecture, now the Mullineux theorem, was re-proven by Brundan and Kujawa [4] with their results on the general linear supergroup $\mathbf{GL}(m|n)$. Shu and Wang [34] classified irreducible representations of the ortho-symplectic supergroup $\mathbf{OSP}(m|n)$, described them in some combinatoric terms that are related to the Mullineux theorem. In positive characteristic, representation theory of algebraic supergroups can apply to representation theory of ordinary algebraic groups.

An algebraic (super)group over $\Bbbk$ is said to be linearly reductive if its representation category is semisimple. Linearly reductive groups are one of important classes of algebraic groups. On the other hand, linearly reductive supergroups $\mathbf{G}$ which are not ordinary algebraic groups are rather restricted. If $\Bbbk$ is an algebraically closed field of $\operatorname{char} \Bbbk = 0$, then Weissauer [41] showed that $\mathbf{G}$ is a semidirect product of a reductive group and a product $\prod_{r \geqslant 1} \mathbf{OSP}(1|2r)^{n_r}$, $n_r \geqslant 0$ of the ortho-symplectic supergroups. If $\Bbbk$ is a field of positive characteristic, then Masuoka [24] showed that all linearly reductive supergroups must be purely even, i.e., $\mathbf{G} = \mathbf{G}_{\mathrm{ev}}$.

Serganova [33] defined the notion of quasireductive supergroups over $\Bbbk$. A quasireductive supergroup $\mathbf{G}$ is an algebraic supergroup such that the associated ordinary group $\mathbf{G}_{\mathrm{ev}}$ is a reductive group which is split, i.e., has a maximal split torus. The supergroups $\mathbf{Q}(n)$, $\mathbf{GL}(m|n)$ and $\mathbf{OSP}(m|n)$ are quasireductive. Moreover, the Chevalley supergroups of classical type that Fioresi and Gavarini [9] constructed as a super-analogue of the Chevalley-Demazure groups are quasireductive. Therefore, quasireductive supergroups form a wide class of algebraic supergroups. Serganova studied structures and representations of quasireductive supergroups $\mathbf{G}$ over an algebraically closed field $\Bbbk$ of $\operatorname{char} \Bbbk = 0$ in terms of its Lie superalgebra $\mathrm{Lie}(\mathbf{G})$.

This part gives a characteristic-free study of those quasireductive supergroups $\mathbf{G}$ which are, roughly speaking, defined over $\mathbb{Z}$. We systematically construct their irreducible representations, extends Serganova's construction to arbitrary characteristic.

This part is organized as follows. In Chapter 2, first we give the definition of the hyper-superalgebra $\mathrm{hy}(\mathbf{G})$ of a given algebraic supergroup $\mathbf{G}$. Then we discuss supermodules over an algebraic supergroup $\mathbf{G}$, and those over the hyper-superalgebra $\mathrm{hy}(\mathbf{G})$, when $\mathbf{G}_{\mathrm{ev}}$ is a split reductive group. Let T be a split maximal torus of such $\mathbf{G}_{\mathrm{ev}}$. Theorem 2.4.8 shows, roughly speaking, an equivalence of $\mathbf{G}$-supermodules with $\mathrm{hy}(\mathbf{G})$-T-supermodules. When $\Bbbk$ is a field, the theorem gives Corollary 2.4.10 which generalizes the result which were proved in [3] for $\mathbf{Q}(n)$, [4] for $\mathbf{GL}(m|n)$ and [34] for $\mathbf{OSP}(m|n)$. In Chapter 3, we characterize the quasireductive supergroups over $\mathbb{Z}$ in terms of the correspondence Harish-Chandra pairs. As an application, we re-construct the Chevalley supergroups over $\mathbb{Z}$; see Section 3.3. One sees that Chevalley supergroups of classical type (for example $\mathbf{OSP}(m|n)$) as well as $\mathbf{GL}(m|n)$ and $\mathbf{Q}(n)$ are quasireductive supergroups over $\mathbb{Z}$. By the base extension to an arbitrary field $\Bbbk$, a quasireductive supergroup over $\mathbb{Z}$ terns to be a quasireductive supergroup over $\Bbbk$ as Serganova [33] defined. In Chapter 4, We show that if A is a Hopf superalgebra (not necessary commutative) over $\Bbbk$ having a "dence big cell" all the simple A-supercomodules are explicitly constructed; see Theorem 4.2.9. This is a super-analogue of a result of Parshall and Wang [32]. In Chapter 5, we construct all the irreducible supermodules of a given quasireductive supergroup $\mathbf{G}$ over $\Bbbk$. There is a special closed subsupergroup $\mathbf{T}$ of $\mathbf{G}$ such that $\mathbf{T}_{\mathrm{ev}}$ is a split maximal torus T of $\mathbf{G}_{\mathrm{ev}}$. Since $\mathbf{T}$ is non-abelian in general, irreducible representations of $\mathbf{T}$ are more complicated than irreducible representations of T. The construction of irreducible supermodules of $\mathbf{T}$ are done by using a general theory of Clifford algebras; see Theorem 5.1.5. Finally, by using the results in Chapter 4, we construct irreducible representations of $\mathbf{G}$. For a general linear supergroup $\mathbf{GL}(m|n)$, Zubkov [42] proves a super-analogue of the Kempf vanishing theorem. Essential in his proof is the existence of some special subsupergroup of $\mathbf{GL}(m|n)$. In Chapter 6, we abstract such a special subsupergroup as distinguished parabolic subsupergroups. We show that if quasireductive supergroup $\mathbf{G}$ has a distinguished parabolic subsupergroup, then the Kempf vanishing theorem holds for $\mathbf{G}$, generalizing Zubkov's result. In this case, we classify all the irreducible representations of $\mathbf{G}$.

Chapter 2

Representations of Algebraic Supergroups

Throughout in this section we suppose that $\Bbbk$ is an integral domain. Our assumption that $\Bbbk$ is 2-torsion free is equivalent to that $2 \neq 0$ in $\Bbbk$.

2.1 Representation of supergroups

Let $\mathbf{G}$ be an infinitesimally flat algebraic supergroup. A superspace V is said to be a *representation of* $\mathbf{G}$ (or $\mathbf{G}$-*supermodule*) if there is a natural transformation

$$\Phi : \mathbf{G} \longrightarrow \mathbf{GL}(V)$$

from $\mathbf{G}$ to $\mathbf{GL}(V)$. For a representation V of $\mathbf{G}$, we can define a right $\mathcal{O}(\mathbf{G})$-supercomodule structure on V such that

$$V \overset{\mathrm{id}_V \otimes 1_{\mathcal{O}(\mathbf{G})}}{\longrightarrow} V \otimes \mathcal{O}(\mathbf{G}) \overset{\Phi_{\mathcal{O}(\mathbf{G})}(\mathrm{id})}{\longrightarrow} V \otimes \mathcal{O}(\mathbf{G}),$$

where $1_{\mathcal{O}(\mathbf{G})}$ is the unit element of $\mathcal{O}(\mathbf{G})$ and

$$\Phi_{\mathcal{O}(\mathbf{G})} : \mathbf{G}(\mathcal{O}(\mathbf{G})) \longrightarrow \mathbf{GL}(V)(\mathcal{O}(\mathbf{G})).$$

Conversely, any right $\mathcal{O}(\mathbf{G})$-supercomodule can be regard as a representation of $\mathbf{G}$. In this way, we may identify $\mathsf{SRep}(\mathbf{G})$ and $\mathsf{SMod}^{\mathcal{O}(\mathbf{G})}$, where $\mathsf{SRep}(\mathbf{G})$ denotes the category of representations of $\mathbf{G}$.

Definition 2.1.1. An irreducible representation V of $\mathbf{G}$ is a simple $\mathcal{O}(\mathbf{G})$-supercomodule. Namely, V has no non-trivial $\mathcal{O}(\mathbf{G})$-supercomodule.

Let $\mathbf{K}$ be a closed subsupergroup of $\mathbf{G}$. For a representation V of $\mathbf{G}$, we let

$$V^{\mathbf{K}} := V^{\mathrm{co}\,\mathcal{O}(\mathbf{K})}. \tag{II.2.1.1}$$

Here, we regard V as a right $\mathcal{O}(\mathbf{K})$-supercomodule by the Hopf superalgebra quotient $\pi : \mathcal{O}(\mathbf{G}) \to \mathcal{O}(\mathbf{K})$. This $V^{\mathbf{K}}$ is called the $\mathbf{K}$-*fixed points* of V. Explicitly,

$$V^{\mathbf{K}} = \{v \in V \mid v_0 \otimes \pi(v_1) = v \otimes 1\},$$

where $V \to V \otimes \mathcal{O}(\mathbf{G})$; $v \mapsto v_0 \otimes v_1$ is the $\mathcal{O}(\mathbf{G})$-supercomodule structure of V.

2.2 Hyper-superalgebras

Let $\mathbf{G} \in \mathsf{ASG}$ (see Part I, Definition 4.3.2), and set $G := \mathbf{G}_{\mathrm{ev}}$. We let $A := \mathcal{O}(\mathbf{G})$, whence $\overline{A} = \mathcal{O}(G)$ (see (I.3.1.1)). We assume that G is *infinitesimally flat* [16, Part I, 7.4]. This means that

(I1) For every $n > 0$, $\overline{A}/(\overline{A}^{+})^n$ is $\Bbbk$-finite projective.

By the condition (C1) in Part I, it follows that for every $n > 0$, $A/(A^+)^n$ is $\Bbbk$-finite projective.

Recall that A^* is the dual superalgebra of the supercoalgebra A. We suppose $(A/(A^+)^n)^* \subset A^*$ through the natural embedding, and set

$$\mathrm{hy}(\mathbf{G}) := \bigcup_{n>0} (A/(A^+)^n)^*.$$

We call this the *hyper-superalgebra* of $\mathbf{G}$.

Remark 2.2.1. This is often denoted alternatively by $\mathrm{Dist}(\mathbf{G})$, called the *super-distribution algebra* of $\mathbf{G}$.

It is easy to see that $\mathrm{hy}(\mathbf{G})$ is a super-subalgebra of A^*. By (I1), each $(A/(A^+)^n)^*$ is the dual coalgebra of the algebra $A/(A^+)^n$. One sees that if $n < m$, then $(A/(A^+)^n)^* \subset (A/(A^+)^m)^*$ is a coalgebra embedding, so that all $(A/(A^+)^n)^*$, $n > 0$, form an inductive system of coalgebras.

Lemma 2.2.2. *Given the coalgebra structure of the inductive limit, the superalgebra* $\mathrm{hy}(\mathbf{G})$ *forms a cocommutative Hopf superalgebra such that the canonical pairing* $\mathcal{O}(\mathbf{G})^* \times \mathcal{O}(\mathbf{G}) \to \Bbbk$ *restricts to a Hopf pairing*

$$\langle\ ,\ \rangle : \mathrm{hy}(\mathbf{G}) \times \mathcal{O}(\mathbf{G}) \longrightarrow \Bbbk. \tag{II.2.2.1}$$

Proof. Let $H := \mathrm{hy}(\mathbf{G})$. Since each $(A/(A^+)^n)^*$ is cocommutative, so is H. The dual $\mathcal{S}^*$ of the antipode $\mathcal{S}$ of A stabilizes H. Denote $\mathcal{S}^*|_H$ by $\mathcal{S}$. Then we see that the restricted pairing satisfies (I.2.3.2), (I.2.3.3) and (I.2.3.4) for $R = \Bbbk$. It follows that H satisfies the compatibility required to super-bialgebras (see [24, Lemma 1]), and has $\mathcal{S} = \mathcal{S}^*|_H$ as an antipode. □

If $f : \mathbf{G} \to \mathbf{K}$ is a morphism from $\mathbf{G}$ to another infinitesimally flat algebraic supergroup $\mathbf{K}$, then it naturally induces a filtered supercoalgebra map $\mathrm{hy}(f) : \mathrm{hy}(\mathbf{G}) \to \mathrm{hy}(\mathbf{K})$. If $\mathbf{G}'$ is another infinitesimally flat algebraic supergroup, then the product $\mathbf{G} \times \mathbf{G}'$ is also infinitesimally flat. In this case, there is a natural isomorphism $\mathrm{hy}(\mathbf{G} \times \mathbf{G}') \cong \mathrm{hy}(\mathbf{G}) \otimes \mathrm{hy}(\mathbf{G}')$ of filtered supercoalgebras.

For a Hopf superalgebra H, we define $P(H)$ to be the set of all primitive elements of H, i.e.,

$$P(H) := \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\},$$

where Δ is the comultiplication of H. This $P(H)$ is a Lie superalgebra by the following super-bracket

$$[h,k] := hk - (-1)^{|h||k|}kh,$$

where h, k are homogeneous elements in $P(H)$. As in ordinary case, one can show that the following fact.

Proposition 2.2.3. $P(\mathrm{hy}(\mathbf{G})) \cong \mathrm{Lie}(\mathbf{G})$ *as Lie superalgebras.*

If $\Bbbk$ is a field of characteristic zero, then there is an isomorphism $\mathrm{hy}(\mathbf{G}) \cong \mathcal{U}(\mathrm{Lie}(\mathbf{G}))$ of Hopf superalgebras, where $\mathcal{U}(\mathrm{Lie}(\mathbf{G}))$ is the universal enveloping superalgebra of $\mathrm{Lie}(\mathbf{G})$.

The Hopf superalgebra quotient $\mathcal{O}(\mathbf{G}) \to \mathcal{O}(G)$ gives rise to a Hopf superalgebra embedding of the hyperalgebra $\mathrm{hy}(G)$ of G into $\mathrm{hy}(\mathbf{G})$. Let $W := W^A\,(= \mathfrak{g}_{\bar{1}}^*)$, and choose a counit-preserving isomorphism

$$\psi : \mathcal{O}(\mathbf{G}) \xrightarrow{\simeq} \mathcal{O}(G) \otimes \wedge(W)$$

of left $\mathcal{O}(G)$-comodule superalgebras.

Lemma 2.2.4. *There uniquely exists a unit-preserving isomorphism*

$$\phi : \mathrm{hy}(G) \otimes \wedge(\mathfrak{g}_{\bar{1}}) \xrightarrow{\simeq} \mathrm{hy}(\mathbf{G})$$

of left $\mathrm{hy}(G)$*-module supercoalgebras such that*

$$\langle \phi(z), a \rangle = \langle z, \psi(a) \rangle, \quad a \in \mathcal{O}(\mathbf{G}),\ z \in \mathrm{hy}(G) \otimes \wedge(\mathfrak{g}_{\bar{1}}),$$

where the right-hand side gives the tensor product of the canonical pairings

$$\mathrm{hy}(G) \times \mathcal{O}(G) \longrightarrow \Bbbk, \qquad \wedge(\mathfrak{g}_{\bar{1}}) \times \wedge(W) \longrightarrow \Bbbk. \tag{II.2.2.2}$$

Proof. We see that ψ^* restricts to $\mathrm{hy}(G) \otimes \wedge(\mathfrak{g}_{\bar{1}}) \xrightarrow{\simeq} \mathrm{hy}(\mathbf{G})$, and this isomorphism is such as mentioned above. □

2.3 Representations using hyper-superalgebras

We will identify as

$$\mathcal{O}(\mathbf{G}) = \mathcal{O}(G) \otimes \wedge(W), \qquad \mathrm{hy}(G) \otimes \wedge(\mathfrak{g}_{\bar{1}}) = \mathrm{hy}(\mathbf{G}) \tag{II.2.3.1}$$

through ψ, ϕ, respectively.

Let Q be the quotient field of $\Bbbk$, and let G_Q denote the base change of G to Q. In addition to (I1), we assume

(I2) G_Q is connected, or in other words, $\mathcal{O}(G_Q) = \mathcal{O}(G) \otimes Q$ contains no non-trivial idempotent.

This assumption ensures the following.

Lemma 2.3.1. *For every $r > 0$, the superalgebra map*

$$\mathcal{O}(\mathbf{G})^{\otimes r} \longrightarrow (\mathrm{hy}(\mathbf{G})^{\otimes r})^*$$

which is associated with the r-fold tensor product of the Hopf pairing (II.2.2.1) *is injective.*

Proof. By Lemma 2.2.4 it suffices to prove that the algebra map $\mathcal{O}(G)^{\otimes r} \to (\mathrm{hy}(G)^{\otimes r})^*$ similarly given is injective. By [36, Proposition 0.3.1(g)], (I2) ensures that the Q-algebra map $\mathcal{O}(G_Q)^{\otimes r} \to (\mathrm{hy}(G_Q)^{\otimes r})^*$ for G_Q is injective. Since $\mathrm{hy}(G_Q) = \mathrm{hy}(G) \otimes Q$, we have the canonical map

$$(\mathrm{hy}(G)^{\otimes r})^* \otimes Q \longrightarrow (\mathrm{hy}(G_Q)^{\otimes r})^*.$$

By (B3) we have $\mathcal{O}(G)^{\otimes r} \subset \mathcal{O}(G)^{\otimes r} \otimes Q$. The desired injectivity follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{O}(G)^{\otimes r} \otimes Q & \longrightarrow & (\mathrm{hy}(G)^{\otimes r})^* \otimes Q \\ \downarrow\simeq & & \downarrow \\ \mathcal{O}(G_Q)^{\otimes r} & \longrightarrow & (\mathrm{hy}(G_Q)^{\otimes r})^*. \end{array}$$

□

Let M be a supermodule. Given a left $\mathbf{G}$-supermodule (resp., G-module) structure on M, one defines by the formula

$$u \rightharpoonup m := m_0\langle u, m_1\rangle, \qquad u \in \mathrm{hy}(\mathbf{G}),\ m \in M, \tag{II.2.3.2}$$

using the Hopf pairing $\langle\ ,\ \rangle : \mathrm{hy}(\mathbf{G}) \times \mathcal{O}(\mathbf{G}) \to \Bbbk$ (II.2.2.1) (resp., the first one of (II.2.2.2)), a left $\mathrm{hy}(\mathbf{G})$-supermodule (resp., $\mathrm{hy}(G)$-module) structure on M. We see that in the super-situation, this indeed defines a map from

- the set of all left $\mathbf{G}$-supermodule structures on M

to

- the set of those locally finite, left $\mathrm{hy}(\mathbf{G})$-supermodule structures on M whose restricted (necessarily, locally finite) $\mathrm{hy}(G)$-module structures arise from left G-module structures.

Note that the left and the right $\mathbf{G}$-supermodule structures (resp., locally finite $\mathrm{hy}(\mathbf{G})$-supermodule structures with the property as above) on M are in one-to-one correspondence, since one can switch the sides through the inverse on $\mathbf{G}$ (resp., the antipode on $\mathrm{hy}(\mathbf{G})$). Therefore, we may replace "left" with "right" in the sets above, to prove the following proposition. Indeed, we do so, to make the argument fit in with our results so far obtained.

Proposition 2.3.2. *If M is $\Bbbk$-projective, the map above is a bijection.*

Proof. Since M is $\Bbbk$-projective, the injection given by Lemma 2.3.1, tensored with M, remains injective. In addition the canonical map $(\mathrm{hy}(\mathbf{G})^{\otimes r})^* \otimes M \to \mathrm{Hom}(\mathrm{hy}(\mathbf{G})^{\otimes r}, M)$ is injective. Let

$$\mu^{(r)} : \mathcal{O}(\mathbf{G})^{\otimes r} \otimes M \to \mathrm{Hom}(\mathrm{hy}(\mathbf{G})^{\otimes r}, M)$$

denote their composite, which is an injective super-linear map. We will use only $\mu^{(1)}$, $\mu^{(2)}$.

Suppose that we are given a structure from the second set; it is a *right* $\mathrm{hy}(\mathbf{G})$-supermodule structure, in particular. We claim that the super-linear map

$$\rho : M \longrightarrow \mathrm{Hom}_{\Bbbk}(\mathrm{hy}(\mathbf{G}), M), \quad \rho(m)(x) = mx$$

factorizes into $\mu^{(1)}$ and a uniquely determined map, $\rho' : M \to \mathcal{O}(\mathbf{G}) \otimes M$. To show this we use the identification (II.2.3.1). Then, ρ decomposes as

$$M \xrightarrow{\rho_1} \mathrm{Hom}_{\Bbbk}(\mathrm{hy}(G), M) \xrightarrow{(\rho_2)_*} \mathrm{Hom}_{\Bbbk}(\mathrm{hy}(G), \mathrm{Hom}_{\Bbbk}(\wedge(\mathfrak{g}_{\bar{1}}), M)),$$

where the first map is defined, just as ρ, by $\rho_1(m)(x) = mx$, and the second $(\rho_2)_*$ denotes $\mathrm{Hom}_{\Bbbk}(\mathrm{id}, \rho_2)$ induced by the map $\rho_2 : M \to \mathrm{Hom}_{\Bbbk}(\wedge(\mathfrak{g}_{\bar{1}}), M)$ similarly defined. We have the injections

$$\nu_1 : \mathcal{O}(G) \otimes M \to \mathrm{Hom}_{\Bbbk}(\mathrm{hy}(G), M),$$
$$\nu_2 : \mathcal{O}(G) \otimes \mathrm{Hom}_{\Bbbk}(\wedge(\mathfrak{g}_{\bar{1}}), M) \to \mathrm{Hom}_{\Bbbk}(\mathrm{hy}(G), \mathrm{Hom}_{\Bbbk}(\wedge(\mathfrak{g}_{\bar{1}}), M))$$

which are defined in the same way as $\mu^{(1)}$. Indeed, ν_2 is identified with $\mu^{(1)}$. The condition regarding the restricted $\mathrm{hy}(G)$-structures means that ρ_1 factorizes into ν_1 and a uniquely determined map, $\rho'' : M \to \mathcal{O}(G) \otimes M$. The composite $(\mathrm{id} \otimes \rho_2) \circ \rho''$ is identified with the desired map ρ', as is seen from the commutative diagram

$$\begin{array}{ccc} \mathcal{O}(G) \otimes M & \xrightarrow{\mathrm{id}\otimes\rho_2} & \mathcal{O}(G) \otimes \mathrm{Hom}_{\Bbbk}(\wedge(\mathfrak{g}_1), M) \\ \downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_2} \\ \mathrm{Hom}_{\Bbbk}(\mathrm{hy}(G), M) & \xrightarrow{(\rho_2)_*} & \mathrm{Hom}_{\Bbbk}(\mathrm{hy}(G), \mathrm{Hom}_{\Bbbk}(\wedge(\mathfrak{g}_1), M)). \end{array}$$

By using $\mu^{(2)}$, we see that the associativity of the $\mathrm{hy}(\mathbf{G})$-action on M implies that $\rho' : M \to \mathcal{O}(\mathbf{G}) \otimes M$ is coassociative. Similarly, the unitality of the action implies that ρ' is counital. Thus, ρ' is a left $\mathcal{O}(\mathbf{G})$-super-comodule structure on M. It is the unique such structure that gives rise to the originally given structure, as is easily seen. $\square$

2.4 Integral representations

Let $G_{\mathbb{Z}}$ be a split reductive algebraic group over $\mathbb{Z}$; see [16, p.153]. By saying a reductive algebraic group we assume that it is connected and smooth. Choose a split maximal torus $T_{\mathbb{Z}}$. The pair $(G_{\mathbb{Z}}, T_{\mathbb{Z}})$ naturally corresponds to a root datum

$$(\mathsf{X}, \mathsf{R}, \mathsf{X}^{\vee}, \mathsf{R}^{\vee}).$$

In particular, X equals the character group $\mathsf{X}(T_{\mathbb{Z}})$ of $T_{\mathbb{Z}}$. It is known that $\mathcal{O}(G_{\mathbb{Z}})$ is $\mathbb{Z}$-free, and $G_{\mathbb{Z}}$ is infinitesimally flat. Moreover, for any field K, the base change $(G_{\mathbb{Z}})_K$ is a split reductive (in particular, connected) algebraic group over K, and $(T_{\mathbb{Z}})_K$ is its split maximal torus. Conversely, every split reductive algebraic group over K and its split maximal torus are obtained uniquely (up to isomorphism) in this manner.

Recall that $\Bbbk$ is supposed to be an integral domain. Let

$$G = (G_{\mathbb{Z}})_{\Bbbk}, \qquad T = (T_{\mathbb{Z}})_{\Bbbk}$$

be the base changes to $\Bbbk$. Note that $\mathcal{O}(G)$ is $\Bbbk$-free. In addition, G satisfies (I1) (with $\overline{A}$ supposed to be $\mathcal{O}(G)$) and (I2).

We have the inclusion $\mathrm{hy}(G) \supset \mathrm{hy}(T)$ of hyperalgebras, which coincides with the base changes of the hyperalgebras $\mathrm{hy}(G_{\mathbb{Z}}) \supset \mathrm{hy}(T_{\mathbb{Z}})$ over $\mathbb{Z}$. Since $\Bbbk$ contains no non-trivial idempotent, the character group $\mathsf{X}(T)$ of T remains to be X.

Definition 2.4.1 ([16, p.171])**.** For a left (resp. right) $\mathrm{hy}(G)$-module M, we say that M is a *left (resp. right)* $\mathrm{hy}(G)$-T*-module*, if the restricted $\mathrm{hy}(T)$-module structure on M arises from some T-module structure on it.

This is equivalent to saying that M is a direct sum $M = \bigoplus_{\lambda \in \mathsf{X}} M_\lambda$ of $\Bbbk$-submodules M_λ, $\lambda \in \mathsf{X}$, so that

$$xm = \lambda(x)\, m, \quad x \in \mathrm{hy}(T),\ m \in M_\lambda,\ \lambda \in \mathsf{X},$$

where we have supposed that M is a *left* $\mathrm{hy}(T)$-module. One sees that the T-module structure above is uniquely determined if M is $\Bbbk$-torsion free. A $\mathrm{hy}(G)$-T-module is said to be *locally finite* if it is locally finite as a $\mathrm{hy}(G)$-module.

Let M be a $\Bbbk$-module. Given a left G-module structure on M, there arises, as before, a left $\mathrm{hy}(G)$-module structure on M; it is indeed a locally finite $\mathrm{hy}(G)$-T-module structure, as is easily seen. Thus we have a map from

- the set of all left G-module structures on M

to

- the set of all locally finite, left $\mathrm{hy}(G)$-T-module structure on M.

The structures in each set above are in one-to-one correspondence with the opposite-sided structures, as before. The following is known.

Theorem 2.4.2 ([16, Part II, 1.20, p.171])**.** *If M is $\Bbbk$-projective, the map above is a bijection.*

Remark 2.4.3. Let $\Bbbk = \mathbb{Z}$, and suppose that $G_{\mathbb{Z}}$ is semisimple, or equivalently $[\mathsf{X} : \mathbb{Z}\mathsf{R}] < \infty$; see [16, Part II, 1.6, p.158]. Then it is known (see [18, 38]) that

$$\mathcal{O}(G_{\mathbb{Z}}) = \mathrm{hy}(G_{\mathbb{Z}})^\circ. \tag{II.2.4.1}$$

It follows that every $\mathbb{Z}$-free, locally finite $\mathrm{hy}(G_{\mathbb{Z}})$-module is necessarily a $\mathrm{hy}(G_{\mathbb{Z}})$-$T_{\mathbb{Z}}$-module.

Given a Hopf algebra H over $\mathbb{Z}$, we let H° denote, just when working over a field (see [35, Section 6.0]), the union of the $\mathbb{Z}$-submodules $(H/I)^*$ in H^*, where I runs over the ideals of H such that H/I is $\mathbb{Z}$-finite. Since the canonical map $(H/I)^* \otimes (H/I)^* \to (H/I \otimes H/I)^*$ is an isomorphism, each $(H/I)^*$ is a ($\mathbb{Z}$-finite free) coalgebra, whence H° is a coalgebra, and is in fact a Hopf algebra.

Keep G, T as above. Let us consider objects $\mathbf{G} \in \mathsf{ASG}$ such that $\mathbf{G}_{\mathrm{ev}} = G$.

Remark 2.4.4. As will be seen Section 3.3.1, if $\Bbbk = \mathbb{Z}$, the *Chevalley $\mathbb{Z}$-supergroups of classical type* which were constructed by Fioresi and Gavarini [9] and by Gavarini [11] are examples of $\mathbf{G}$ as above. Therefore, their base changes are, as well.

Remark 2.4.5. Suppose that $\Bbbk$ is a field of characteristic $\neq 2$. Recall that every split reductive algebraic group is of the form G as above. Then it follows from Part I, Corollary 3.1.3 that the objects under consideration are precisely all algebraic supergroups $\mathbf{G}$ such that $\mathbf{G}_{\mathrm{ev}}$ is a split reductive algebraic group.

Let $\mathbf{G} \in \mathsf{ASG}$ such that $\mathbf{G}_{\mathrm{ev}} = G$.

Definition 2.4.6. For a left (resp. right) $\mathrm{hy}(\mathbf{G})$-supermodule M, we say that M is a *left (resp. right)* $\mathrm{hy}(\mathbf{G})$-T*-supermodule*, if the restricted $\mathrm{hy}(T)$-module structure on M arises from some T-module structure on it.

This is equivalent to saying that M is a $\mathrm{hy}(G)$-T-module, regarded as a $\mathrm{hy}(G)$-module by restriction. A $\mathrm{hy}(\mathbf{G})$-T*-supermodule* is said to be *locally finite* if it is so as a $\mathrm{hy}(\mathbf{G})$-supermodule, or equivalently, as a $\mathrm{hy}(G)$-module.

Remark 2.4.7. In [3, §5], a $\mathrm{hy}(\mathbf{G})$-T-supermodule is celled an *integrable* $\mathrm{hy}(\mathbf{G})$*-supermodule.*

Let M be a supermodule. Given a left $\mathbf{G}$-supermodule structure on M, there arises, as before, a left $\mathrm{hy}(\mathbf{G})$-supermodule structure on M; it is indeed a locally finite $\mathrm{hy}(\mathbf{G})$-T-supermodule structure, as is easily seen. Thus we have a map from

- the set of all left $\mathbf{G}$-supermodule structures on M

to

- the set of all locally finite, left $\mathrm{hy}(\mathbf{G})$-T-supermodule structures on M.

The structures in each set above are in one-to-one correspondence with the opposite-sided structures, as before. Proposition 2.3.2 and Theorem 2.4.2 prove the following.

Theorem 2.4.8. *If M is $\Bbbk$-projective, the map above is a bijection.*

Remark 2.4.9. Let $\Bbbk = \mathbb{Z}$, and suppose that $G_{\mathbb{Z}}$ is semisimple. Then by using the same argument as proving [24, Proposition 31], we see from (II.2.4.1) that $\mathcal{O}(\mathbf{G}) = \mathrm{hy}(\mathbf{G})^\circ$. It follows that every $\mathbb{Z}$-free, locally finite $\mathrm{hy}(\mathbf{G})$-supermodule is necessarily a $\mathrm{hy}(\mathbf{G})$-$T_{\mathbb{Z}}$-supermodule.

Theorem 2.4.8 can be reformulated as an isomorphism between the category of $\Bbbk$-projective, left $\mathbf{G}$-supermodules and the category of $\Bbbk$-projective, locally finite left $\mathrm{hy}(\mathbf{G})$-T-supermodules. When $\Bbbk$ is a field of characteristic $\neq 2$, the result is formulated as follows, in view of Remark 2.4.5.

Corollary 2.4.10. *Suppose that* $\Bbbk$ *is a field of characteristic* $\neq 2$*, and let* $\mathbf{G}$ *be an algebraic supergroup over* $\Bbbk$ *such that* $\mathbf{G}_{\mathrm{ev}}$ *is a split reductive algebraic group. Choose a split maximal torus* T *of* $\mathbf{G}_{ev}$*. Then there is a natural isomorphism between the category of left* $\mathbf{G}$*-supermodules and the category of locally finite, left* $\mathrm{hy}(\mathbf{G})$*-*T*-supermodules.*

This has been known only for some special algebraic supergroups with the property as above; see Brundan and Kleshchev [3, Corollary 5.7], Brundan and Kujawa [4, Corollary 3.5], and Shu and Wang [34, Theorem 2.8].

Chapter 3

Quasireductive Supergroups

3.1 Quasireductive Lie superalgebras

As in [33], we treat a special class of Lie superalgebras.

Definition 3.1.1. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ over $\mathbb{C}$ is said to be *quasireductive* if $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ decomposes as the direct sum of weight spaces for a fixed Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a quasireductive Lie superalgebra over $\mathbb{C}$. Let

$$\mathfrak{h} := \{X \in \mathfrak{g} \mid [\mathfrak{h}_{\bar{0}}, X] = 0\},$$

and let $\mathfrak{h}_{\bar{1}} := \mathfrak{h} \cap \mathfrak{g}_{\bar{1}}$. Then $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ is a Lie subsuperalgebra of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}_{\bar{0}}^*$, we define

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}_{\bar{0}}\}.$$

By definition, we have $\mathfrak{g}_{\alpha=0} = \mathfrak{h}$. As in ordinary case, set

$$\begin{aligned} \Delta_{\bar{0}} &:= \{\alpha \in \mathfrak{h}_{\bar{0}}^* \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0\} \setminus \{0\}, \\ \Delta_{\bar{1}} &:= \{\alpha \in \mathfrak{h}_{\bar{0}}^* \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0\}, \text{ and} \\ \Delta &:= \Delta_{\bar{0}} \cup \Delta_{\bar{1}}. \end{aligned}$$

In general, root spaces are *not* one-dimensional.

Example 3.1.2. Let $\mathfrak{g}$ be the simple Lie superalgebra of type $A(1,1)$, see Appendix A. Explicitly, $\mathfrak{g} = \mathfrak{sl}(2|2)/\Bbbk I_4$, where I_4 is the unit matrix of size 4. A Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ is We take the following two elements

$$X = \left(\begin{array}{cc|cc} 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{array}\right) \in \mathfrak{g}, \qquad H = \left(\begin{array}{cc|cc} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ \hline 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{array}\right) \in \mathfrak{h}_{\bar{0}}$$

For $i = 1, 2, 3, 4$, we let $\varepsilon_i \in \mathfrak{h}_{\bar{0}}^*$ such that $\varepsilon_i(H) := h_i$. Let $\alpha := \varepsilon_1 - \varepsilon_3$. Since $h_1 + h_2 = h_3 + h_4$, we have $[X, H] = \alpha(H)X$. Thus, we conclude that $\alpha \in \Delta_{\bar{1}}$ and $\dim \mathfrak{g}_\alpha = 2$.

We have a root space decomposition

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \bigoplus_{0 \neq \alpha \in \Delta} \mathfrak{g}_\alpha.$$

Lemma 3.1.3. *Suppose that there exists non-zero $K \in \mathfrak{h}_{\bar{1}}$. For $\alpha \in \Delta_{\bar{0}}$, if $[\mathfrak{g}_\alpha, K] \neq 0$, then $\alpha \in \Delta_{\bar{1}}$.*

Proof. We fix a element $X_\alpha \in \mathfrak{g}_\alpha$ such that $[X_\alpha, K] \neq 0$. By the Jacobi identity, we have

$$[H, [X_\alpha, K]] = \alpha(H)[X_\alpha, K]$$

for all $H \in \mathfrak{h}_{\bar{0}}$. Thus, we conclude that $[X_\alpha, K] \in \mathfrak{g}_{\bar{1}} \cap \mathfrak{g}_\alpha$. □

Definition 3.1.4. An element $H^{\mathrm{reg}} \in \mathfrak{h}_{\bar{0}}$ is said to be *regular* if the real part $\mathrm{Re}(\alpha(H^{\mathrm{reg}}))$ of $\alpha(H^{\mathrm{reg}})$ is non-zero for all $0 \neq \alpha \in \Delta$.

For a regular element $H^{\mathrm{reg}} \in \mathfrak{h}_{\bar{0}}$, we let

$$\Delta^{\pm} := \{\alpha \in \Delta \mid \ \pm\mathrm{Re}(\alpha(H^{\mathrm{reg}})) > 0\}, \tag{II.3.1.1}$$

and let $\Delta_\epsilon^{\pm} := \Delta^{\pm} \cap \Delta_\epsilon$, for $\epsilon = \bar{0}, \bar{1}$. We define some Lie subsuperalgebras of $\mathfrak{g}$ as follows.

$$\mathfrak{u}^{\pm} := \bigoplus_{\alpha \in \Delta^{\pm}} \mathfrak{g}_\alpha, \qquad \mathfrak{b}^{\pm} := \mathfrak{h} \oplus \mathfrak{u}^{\pm}. \tag{II.3.1.2}$$

As in [31, § 3.2], we call $\mathfrak{b}^+$ a *Borel-Penkov-Serganova subsuperalgebra* of $\mathfrak{g}$. In this case, $\mathfrak{g}$ admits a triangular decomposition

$$\mathfrak{g} = \mathfrak{u}^+ \oplus \mathfrak{h} \oplus \mathfrak{u}^- \tag{II.3.1.3}$$

depending on $H^{\mathrm{reg}} \in \mathfrak{h}_{\bar{0}}$.

Example 3.1.5. Let $\mathfrak{g} = \mathfrak{gl}(2|1)$. Since the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is $\mathfrak{gl}_2 \oplus \mathfrak{gl}_1$, one sees that $\mathfrak{g}$ is quasireductive. We take a Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ as follows

$$\mathfrak{h}_{\bar{0}} := \{H = \left(\begin{array}{cc|c} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ \hline 0 & 0 & h_3 \end{array}\right) \in \mathfrak{g} \mid h_1, h_2, h_3 \in \mathbb{C}\}.$$

Note that, $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$. For $i = 1, 2, 3$, we let $\varepsilon_i \in \mathfrak{h}_{\bar{0}}^*$ such that $\varepsilon_i(H) := h_i$. Then we have

$$\Delta = \{\underbrace{\varepsilon_1 - \varepsilon_2,\ -(\varepsilon_1 - \varepsilon_2)}_{\in \Delta_{\bar{0}}},\ \underbrace{\varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_3,\ -(\varepsilon_2 - \varepsilon_3),\ -(\varepsilon_1 - \varepsilon_3)}_{\in \Delta_{\bar{1}}}.\}$$

By definition, the set of all regular elements are given by

$$\{H^{\mathrm{reg}} = \left(\begin{array}{cc|c} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ \hline 0 & 0 & h_3 \end{array}\right) \in \mathfrak{h}_{\bar{0}} \mid h_1 \neq h_2,\ h_2 \neq h_3,\ h_1 \neq h_3\}.$$

Therefore, there are six variations of Δ^+.

(i) If $\mathrm{Re}(h_3) < \mathrm{Re}(h_2) < \mathrm{Re}(h_1)$, then $\Delta^+ = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_3\}$.

(ii) If $\mathrm{Re}(h_3) < \mathrm{Re}(h_1) < \mathrm{Re}(h_2)$, then $\Delta^+ = \{-(\varepsilon_1 - \varepsilon_2),\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_3\}$.

(iii) If $\mathrm{Re}(h_2) < \mathrm{Re}(h_3) < \mathrm{Re}(h_1)$, then $\Delta^+ = \{\varepsilon_1 - \varepsilon_2,\ -(\varepsilon_2 - \varepsilon_3),\ \varepsilon_1 - \varepsilon_3\}$.

(iv) If $\mathrm{Re}(h_2) < \mathrm{Re}(h_1) < \mathrm{Re}(h_3)$, then $\Delta^+ = \{\varepsilon_1 - \varepsilon_2,\ -(\varepsilon_2 - \varepsilon_3),\ -(\varepsilon_1 - \varepsilon_3)\}$.

(v) If $\mathrm{Re}(h_1) < \mathrm{Re}(h_3) < \mathrm{Re}(h_2)$, then $\Delta^+ = \{-(\varepsilon_1 - \varepsilon_2),\ \varepsilon_2 - \varepsilon_3,\ -(\varepsilon_1 - \varepsilon_3)\}$.

(vi) If $\mathrm{Re}(h_1) < \mathrm{Re}(h_2) < \mathrm{Re}(h_3)$, then $\Delta^+ = \{-(\varepsilon_1 - \varepsilon_2),\ -(\varepsilon_2 - \varepsilon_3),\ -(\varepsilon_1 - \varepsilon_3)\}$.

In the case of (i), we have

$$\mathfrak{u}^+ = \{\left(\begin{array}{cc|c} 0 & * & * \\ 0 & 0 & * \\ \hline 0 & 0 & 0 \end{array}\right)\},\quad \mathfrak{h} = \{\left(\begin{array}{cc|c} * & 0 & 0 \\ 0 & * & 0 \\ \hline 0 & 0 & * \end{array}\right)\},\quad \mathfrak{u}^- = \{\left(\begin{array}{cc|c} 0 & 0 & 0 \\ * & 0 & 0 \\ \hline * & * & 0 \end{array}\right)\}.$$

If $\mathfrak{g}$ has a $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$, then there exists $\mathbb{Z}$-Lie-subsuperalgebras $\mathfrak{h}_{\mathbb{Z}}$, $\mathfrak{u}^{\pm}_{\mathbb{Z}}$ and $\mathfrak{b}^{\pm}_{\mathbb{Z}}$ of $\mathfrak{g}_{\mathbb{Z}}$ such that their complexifications coincide with $\mathfrak{h}$, $\mathfrak{u}^{\pm}$ and $\mathfrak{b}^{\pm}$ respectively. Moreover, $\mathfrak{g}_{\mathbb{Z}}$ admits a triangular decomposition $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{u}^+_{\mathbb{Z}} \oplus \mathfrak{h}_{\mathbb{Z}} \oplus \mathfrak{u}^-_{\mathbb{Z}}$.

Example 3.1.6. The followings are quasireductive Lie superalgebras having a $\mathbb{Z}$-form.

(1) The simple Lie superalgebras over $\mathbb{C}$ of classical type, see Appendix A. An explicit $\mathbb{Z}$-basis was given by Fioresi and Gavarini [9, Theorem 3.7].

(2) The general linear Lie superalgebra $\mathfrak{gl}(m|n)$. In this case, we choose $\mathfrak{h}$ as the set of all diagonal matrices in $\mathrm{Mat}_{m+n,m+n}(\mathbb{C})$. In particular, $\mathfrak{h}_{\bar{1}} = 0$. Let $E_{i,j} \in \mathfrak{gl}(m|n)$ denote the elementary matrix with a 1 in position (i,j) and 0 elsewhere. A $\mathbb{Z}$-form of $\mathfrak{gl}(m|n)$ is spanned by $E_{i,j}$ for $1 \leqslant i,j \leqslant m+n$.

(3) The queer superalgebra $\mathfrak{q}(n)$. In this case, we choose $\mathfrak{h}$ as follows

$$\mathfrak{h} = \{\begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{q}(n) \mid A \text{ and } B \text{ are diagonal}\}.$$

Therefore, $\mathfrak{h} \neq \mathfrak{h}_{\bar{0}}$. A $\mathbb{Z}$-form of $\mathfrak{q}(n)$ is spanned by $E_{i,j} + E_{i+n,j+n}$ and $E_{i+n,j} + E_{i,j+n}$ for $1 \leqslant i,j \leqslant n$. One sees that $\Delta_{\bar{0}} \cup \{0\} = \Delta_{\bar{1}}$.

Remark 3.1.7. It is easy to see that all $\mathbb{Z}$-forms in the above examples are admissible, see Part I, Definition 2.5.2.

3.2 Quasireductive supergroups

In [33], Serganova introduced the notion of quasireductive supergroups over a field and studied its structures and representations in characteristic zero. An algebraic supergroup $\mathbf{G}$ is said to be *quasireductive* if the even part $\mathbf{G}_{\mathrm{ev}}$ is linearly reductive.

To study quasireductive supergroups over our $\Bbbk$, we first define a special class of supergroups over $\mathbb{Z}$ by using Harish-Chandra pairs.

Definition 3.2.1. Let $G_{\mathbb{Z}}$ be a split and connected reductive algebraic group over $\mathbb{Z}$ with a split maximal torus $T_{\mathbb{Z}}$. Let $\mathfrak{g}_{\mathbb{Z}}$ be an admissible Lie superalgebra over $\mathbb{Z}$. Suppose that $(G_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$ forms a Harish-Chandra pair. Note that, $\mathfrak{g}_{\mathbb{Z}}$ is necessarily $\mathbb{Z}$-finite and free. Then the pair is said to be *quasireductive* if it satisfies the following conditions.

(i) $\mathfrak{g} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ is quasireductive and a Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ coincides with $\mathrm{Lie}(T_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C}$, and

(ii) $\mathfrak{g}$ admits a triangular decomposition (II.3.1.3).

Let $(G_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$ be a quasireductive Harish-Chandra pair. Set $\mathfrak{h}_{\mathbb{Z}} := \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Z}}$. This is a Lie subsuperalgebra of $\mathfrak{g}_{\mathbb{Z}}$. Then one sees that $(\mathfrak{h}_{\mathbb{Z}})_{\bar{0}} = (\mathfrak{h}_{\bar{0}})_{\mathbb{Z}}$, where $(\mathfrak{h}_{\bar{0}})_{\mathbb{Z}} := \mathfrak{h}_{\bar{0}} \cap \mathfrak{g}_{\mathbb{Z}}$. Moreover, by definition, $(\mathfrak{h}_{\mathbb{Z}})_{\bar{0}} = \mathrm{Lie}(T_{\mathbb{Z}})$.

Remark 3.2.2. For a quasireductive Harish-Chandra pair $(G_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$, we let $(\mathsf{X}, \mathsf{R}, \mathsf{X}^{\vee}, \mathsf{R}^{\vee})$ denote the corresponding root datum of $T_{\mathbb{Z}} \subseteq G_{\mathbb{Z}}$. Then it is easy to see that

$$\mathsf{X} = X(T_{\mathbb{Z}}), \quad \mathsf{R} = \Delta_{\bar{0}}, \quad \mathsf{X}^{\vee} \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}_{\bar{0}}, \text{ and } \mathrm{hy}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{U}(\mathfrak{g}_{\bar{0}}),$$

where $X(T_{\mathbb{Z}})$ is the character group of $T_{\mathbb{Z}}$. Here, $\mathrm{hy}(G_{\mathbb{Z}})$ is called a *Kostant $\mathbb{Z}$-form* of $\mathcal{U}(\mathfrak{g}_{\bar{0}})$, see [18].

For a quasireductive Harish-Chandra pair $(G_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$, we let

$$\mathbf{G}_{\mathbb{Z}} := \mathbf{G}(G_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}),$$

where $\mathbf{G}$ is the functor defined in (I.4.4.20). This is a connected algebraic supergroup $\mathbf{G}_{\mathbb{Z}}$ over $\mathbb{Z}$ satisfying $\mathcal{O}(\mathbf{G}_{\mathbb{Z}}) \cong \mathcal{O}(G_{\mathbb{Z}}) \otimes \wedge(\mathfrak{g}_{\mathbb{Z}})^*$ as left $\mathcal{O}(G_{\mathbb{Z}})$-comodule superalgebras. Since $\mathcal{O}(G_{\mathbb{Z}})$ is $\mathbb{Z}$-free, so is $\mathcal{O}(\mathbf{G}_{\mathbb{Z}})$. Moreover, $\mathbf{G}_{\mathbb{Z}}$ is infinitesimally flat, since so is $G_{\mathbb{Z}}$, see [16, Part II, 1.12(1)]. The base change $\mathbf{G}_{\mathbb{Z}}$ to our ground field $\Bbbk$ is a quasireductive supergroup over $\Bbbk$ in the sense of Serganova [33].

Example 3.2.3. We consider some Lie superalgebras as in Example 3.1.6.

(1) Let $\mathfrak{g}$ be a simple Lie superalgebra of classical type with its $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$. By the Chevalley-Demazure construction, we get a split and connected reductive algebraic group $G_{\mathbb{Z}}$ such that $\mathrm{Lie}(G_{\mathbb{Z}}) = (\mathfrak{g}_{\mathbb{Z}})_{\bar{0}}$. Then the pair $(G_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$ is a quasireductive Harish-Chandra pair, see Section 3.3 below. The constructed supergroup $\mathbf{G}_{\mathbb{Z}}$ is a *Chevalley supergroup of classical type*, defined by Fioresi and Gavarini [9]. In particular, for $\mathfrak{g} = \mathfrak{osp}(\ell|2n)$ and $G_{\mathbb{Z}} = \mathrm{SO}_{\ell} \times \mathrm{Sp}_{2n}$, then $\mathbf{G}_{\mathbb{Z}}$ is the *ortho-symplectic supergroup* $\mathbf{OSP}(\ell|2n)$.

(2) For the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $G_{\mathbb{Z}} = \mathrm{GL}_m \times \mathrm{GL}_n$, the constructed supergroup $\mathbf{G}_{\mathbb{Z}}$ is $\mathbf{GL}(m|n)$ as in Part I, Example 3.1.4.

(3) For the queer superalgebra $\mathfrak{g} = \mathfrak{q}(n)$ and $G_{\mathbb{Z}} = \mathrm{GL}_n$, the constructed supergroup $\mathbf{G}_{\mathbb{Z}}$ is $\mathbf{Q}(n)$ as in Part I, Example 3.1.5.

By construction, we have $\mathrm{Lie}(\mathbf{G}_{\mathbb{Z}}) = \mathfrak{g}_{\mathbb{Z}}$ and $\mathrm{hy}(\mathbf{G}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{U}(\mathfrak{g})$. For $X \in \mathfrak{g}_{\bar{0}}$ and $H \in \mathfrak{h}_{\bar{0}}$, we define elements in the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\bar{0}})$ of $\mathfrak{g}_{\bar{0}}$ as follows

$$X^{(n)} := \frac{1}{n!}X^n, \quad \binom{H}{n} := \frac{H(H-1)\cdots(H-n+1)}{n!},$$

where $n \geqslant 0$. Set

$$\ell := \mathrm{rank}(\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}, \quad r := \mathrm{rank}(\mathfrak{h}_{\mathbb{Z}})_{\bar{1}}. \tag{II.3.2.1}$$

For $\gamma \in \Delta_{\bar{1}}$, we let $s(\gamma) := \mathrm{rank}(\mathfrak{g}_{\mathbb{Z}})_{\gamma}$. By definition, for $\alpha \in \Delta_{\bar{0}}$, $1 \leqslant i \leqslant \ell$, $\gamma \in \Delta_{\bar{1}}$ and $1 \leqslant t \leqslant r$, there are elements $X_\alpha \in (\mathfrak{g}_{\mathbb{Z}})_\alpha$, $H_i \in (\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}$, $X_{\gamma_p} \in (\mathfrak{g}_{\mathbb{Z}})_\gamma$ and $K_t \in (\mathfrak{h}_{\mathbb{Z}})_{\bar{1}}$ such that

$$\{X_\alpha\}_{\alpha\in\Delta_{\bar{0}}} \cup \{H_i\}_{i=1}^{\ell} \cup \{X_{\gamma_p} \mid 1 \leqslant p \leqslant s(\gamma)\}_{\gamma\in\Delta_{\bar{1}}} \cup \{K_t\}_{t=1}^{r}$$

forms a (homogeneous) $\mathbb{Z}$-basis of $\mathfrak{g}_{\mathbb{Z}}$. As in non-super situation, we can prove *PBW-like theorem* for $\mathrm{hy}(\mathbf{G}_{\mathbb{Z}})$.

Theorem 3.2.4. *Given any totally order $\preceq$ on $\Delta_{\bar{0}} \cup \{\gamma_p \mid 1 \leqslant p \leqslant s(\gamma)\}_{\gamma\in\Delta_{\bar{1}}} \cup \{i_1, \ldots, i_\ell\} \cup \{t_1, \ldots, t_r\}$, the set of all products of factors of type*

$$X_\alpha^{(n_\alpha)}, \ \binom{H_{i_j}}{n_{i_j}}, \ X_{\gamma_p} \text{ and } K_{t_k}$$

with $\alpha \in \Delta_{\bar{0}}$, $1 \leqslant j \leqslant \ell$, $\gamma \in \Delta_{\bar{1}}$, $1 \leqslant p \leqslant s(\gamma)$ and $1 \leqslant k \leqslant r$, taken in $\mathrm{hy}(\mathbf{G})$ *with respect to $\prec$ forms a $\mathbb{Z}$-basis of* $\mathrm{hy}(\mathbf{G}_{\mathbb{Z}})$.

Proof. Let $(\mathfrak{Y} \preceq)$ denote the given totally ordered set. The subset $\mathfrak{X} := \{\gamma_p \mid 1 \leqslant p \leqslant s(\gamma)\}_{\gamma\in\Delta_{\bar{1}}} \cup \{t_1, \ldots, t_r\}$ of $\mathfrak{Y}$ is also a totally ordered set with respect to $\preceq$. Then by Part I, Lemma 2.2.4, the set of all products of factors of type

$$X_{\gamma_p} \text{ and } K_{t_k} \quad \text{with } \gamma \in \Delta_{\bar{1}},\ 1 \leqslant p \leqslant s(\gamma) \text{ and } 1 \leqslant k \leqslant r,$$

with respect to $\prec$ forms a $\mathrm{hy}(G_{\mathbb{Z}})$-free basis of $\mathrm{hy}(\mathbf{G}_{\mathbb{Z}})$. On the other hand, it is known the set of all products of factors of type

$$X_\alpha^{(n_\alpha)} \text{ and } \binom{H_{i_j}}{n_{i_j}} \quad \text{with } \alpha \in \Delta_{\bar{0}} \text{ and } 1 \leqslant j \leqslant \ell,$$

with respect to $\prec$ forms a $\mathbb{Z}$-free basis of $\mathrm{hy}(G_{\mathbb{Z}})$, see [16, Part II, 1.12(4)]. Thus, we are done. □

Remark 3.2.5. Such a base (PBW base) was known; [9] for $\mathbf{G}_{\mathbb{Z}}$ = Chevalley supergroups, [3] $\mathbf{G}_{\mathbb{Z}} = \mathbf{Q}(n)$ etc.

3.3 Chevalley supergroups

Those finite-dimensional simple Lie superalgebras over the complex number field $\mathbb{C}$ which are not purely even were classified by Kac [17]. They are divided into classical type and Cartan type. A *Chevalley* $\mathbb{C}$*-supergroup of classical/Cartan type* is a connected algebraic supergroup $\mathbf{G}$ over $\mathbb{C}$ such that $\mathrm{Lie}(\mathbf{G})$ is a simple Lie superalgebra of classical/Cartan type. As was mentioned in Remark 2.4.4, Fioresi and Gavarini [9, 11] constructed natural $\mathbb{Z}$-forms of Chevalley $\mathbb{C}$-supergroups of classical type. Gavarini [10] accomplished the same construction for Cartan type. The resulting $\mathbb{Z}$-forms are called *Chevalley* $\mathbb{Z}$*-supergroups of classical/Cartan type*; they are indeed objects in our category ASG defined over $\mathbb{Z}$.

Based on our result Part I, Theorem 4.5.1, we will re-construct the Chevalley $\mathbb{Z}$-supergroups, by giving the corresponding Harish-Chandra pairs. Indeed, our construction depends on part of Fioresi and Gavarini's, but simplifies the rest; see Remarks 3.3.3 and 3.3.8.

3.3.1 Chevalley supergroups of classical type

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra over $\mathbb{C}$ which is of classical type. Then $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, and $\mathfrak{g}_{\bar{1}}$, with respect to the right adjoint $\mathfrak{g}_{\bar{0}}$-action, decomposes as the direct sum of weight spaces for a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$. Let $\Delta_{\bar{0}}$ (resp., $\Delta_{\bar{1}}$) denote the set of the even (resp., odd) roots, that is, the weights with respect to the adjoint $\mathfrak{h}$-action on $\mathfrak{g}_{\bar{0}}$ (resp., on $\mathfrak{g}_{\bar{1}}$).

Let

$$(\mathsf{X}, \mathsf{R}, \mathsf{X}^\vee, \mathsf{R}^\vee), \quad G_{\mathbb{Z}} \supset T_{\mathbb{Z}} \tag{II.3.3.1}$$

be a root datum and the corresponding split reductive algebraic $\mathbb{Z}$-group and split maximal torus. Suppose that $\mathfrak{g}_{\bar{0}} \supset \mathfrak{h}$ coincide with the complexifications of $\mathrm{Lie}(G_{\mathbb{Z}}) \supset \mathrm{Lie}(T_{\mathbb{Z}})$. Then one has

$$\mathsf{R} = \Delta_{\bar{0}}, \quad \mathsf{X}^\vee \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h} \quad \text{and} \quad \mathrm{hy}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{U}(\mathfrak{g}_{\bar{0}}).$$

Recall that $\mathrm{hy}(G_{\mathbb{Z}})$ is called a *Kostant form* of $\mathcal{U}(\mathfrak{g}_{\bar{0}})$. We assume

$$\Delta_{\bar{1}} \subset \mathsf{X}. \tag{II.3.3.2}$$

Theorem 3.3.1 (Fioresi, Gavarini)**.** *There exists a* $\mathbb{Z}$*-lattice* $V_{\mathbb{Z}}$ *of* $\mathfrak{g}_{\bar{1}}$ *such that*

(i) $\mathfrak{g}_{\mathbb{Z}} := \mathrm{Lie}(G_{\mathbb{Z}}) \oplus V_{\mathbb{Z}}$ *is a Lie-superalgebra* $\mathbb{Z}$*-form of* $\mathfrak{g}$*.*

(ii) This Lie superalgebra $\mathfrak{g}_{\mathbb{Z}}$ *over* $\mathbb{Z}$ *is admissible.*

(iii) $V_{\mathbb{Z}}$ *is* $\mathrm{hy}(G_{\mathbb{Z}})$*-stable in the right* $\mathcal{U}(\mathfrak{g}_{\bar{0}})$*-module* $\mathfrak{g}_{\bar{1}}$*.*

Fioresi and Gavarini [9] and Gavarini [11] introduced the notion of *Chevalley bases*, gave an explicit example of such a basis for each $\mathfrak{g}$, and constructed from the basis a natural Hopf-superalgebra $\mathbb{Z}$-form, called a *Kostant superalgebra*, of $\mathcal{U}(\mathfrak{g})$; the even basis elements coincide with the classical Chevalley basis for $\mathfrak{g}_{\bar{0}}$. They do not refer to root data. But, once an explicit Chevalley basis is given as in [9, 11], one can re-choose the basis so that it includes a $\mathbb{Z}$-free basis of $\mathsf{X}^\vee$, by replacing part of the original basis, $H_1, \dots, H_\ell$, with a desired $\mathbb{Z}$-free basis; this replacement is possible, since it effects only on the adjoint action on the basis elements X_α, and the new basis elements still act via the roots α. (The method of [9, Remark 3.8] attributed to the referee gives an alternative construction of the desired basis from the scratch.) One sees that the odd elements in the Chevalley basis generate the desired $\mathbb{Z}$-lattice $V_\mathbb{Z}$ as above; see [9, Sections 4.2, 6.1] and [11, Section 3.4], to verify Condition (ii), in particular.

Set $\mathfrak{g}_\mathbb{Z} := \mathrm{Lie}(G_\mathbb{Z}) \oplus V_\mathbb{Z}$ in $\mathfrak{g}$, as above. One sees from (iii) and (II.3.3.2) that $V_\mathbb{Z}$ is a right $\mathrm{hy}(G_\mathbb{Z})$-$T_\mathbb{Z}$-module, whence it is a right $G_\mathbb{Z}$-module by Theorem 2.4.2. The restricted super-bracket $[\ ,\ ] : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathrm{Lie}(G_\mathbb{Z})$, being $\mathrm{hy}(G_\mathbb{Z})$-linear, is $G_\mathbb{Z}$-equivariant. This proves the following.

Proposition 3.3.2. $(G_\mathbb{Z}, \mathfrak{g}_\mathbb{Z})$ *is a Harish-Chandra pair.*

We let

$$\mathbf{G}_\mathbb{Z} = \mathbf{G}(G_\mathbb{Z}, \mathfrak{g}_\mathbb{Z})$$

denote the algebraic $\mathbb{Z}$-supergroup in ASG which is associated with the Harish-Chandra pair just obtained. Since one sees that the category equivalences in Part I, Theorem 4.5.1 are compatible with base extensions, it follows that $\mathbf{G}_\mathbb{Z}$ is a $\mathbb{Z}$-form of the algebraic $\mathbb{C}$-supergroup associated with the Harish-Chandra pair $(G, \mathfrak{g})$, where G denotes the base change of $G_\mathbb{Z}$ to $\mathbb{C}$. Recall from Section 3.3 the definition of Chevalley $\mathbb{C}$-supergroups of classical type, and note that every such $\mathbb{C}$-supergroup is associated with some Harish-Chandra pair of the last form. We have thus constructed a natural $\mathbb{Z}$-form of every Chevalley $\mathbb{C}$-supergroups of classical type.

Remark 3.3.3. (1) After constructing Kostant superalgebras, Fioresi and Gavarini's construction, which is parallel to the classical construction of Chevalley $\mathbb{Z}$-groups, continues as follows;

(a) Choose a faithful rational representation $\mathfrak{g} \to \mathfrak{gl}_\mathbb{C}(M)$ on a finite-dimensional super-vector space M over $\mathbb{C}$,

(b) choose a $\mathbb{Z}$-lattice $M_\mathbb{Z}$ in M which is stable under the action of the Kostant superalgebra,

(c) construct a natural group-valued functor which is realized as subgroups of $\mathbf{GL}(M_\mathbb{Z})(R)$, where R runs over the commutative superalgebras over $\mathbb{Z}$, and

(d) prove that the sheafification, say $\mathbf{G}_\mathbb{Z}^{\mathrm{FG}}$, of the constructed group-valued functor is representable, and has desired properties, which include the property that $\mathcal{O}(\mathbf{G}_\mathbb{Z}^{\mathrm{FG}})$ is split; see [9, Corollary 5.20] and [11, Corollary 4.22] for the last property.

Our method of construction dispenses with these procedures.

(2) The algebraic group $(\mathbf{G}_\mathbb{Z}^{\mathrm{FG}})_{\mathrm{ev}}$ associated with Fioresi and Gavarini's $\mathbf{G}_\mathbb{Z}^{\mathrm{FG}}$ is a split reductive algebraic $\mathbb{Z}$-group. As was noted in an earlier version of the present paper, it was not clear for the authors whether the split reductive algebraic $\mathbb{Z}$-groups which correspond to all *possible* root data (namely, all relevant root data satisfying (II.3.3.2)) can be realized as $(\mathbf{G}_\mathbb{Z}^{\mathrm{FG}})_{\mathrm{ev}}$; note that by definition, those algebraic $\mathbb{Z}$-groups are realized as our $(\mathbf{G}_\mathbb{Z})_{\mathrm{ev}} = G_\mathbb{Z}$. See Erratum added to a new version of [10].

3.3.2 Chevalley supergroups of Cartan type

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra over $\mathbb{C}$ which is of Cartan type. Then $\mathfrak{g}_{\bar{0}}$ is a direct sum

$$\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{\bar{0}}^{\mathrm{r}} \ltimes \mathfrak{g}_{\bar{0}}^{\mathrm{n}}$$

of a reductive Lie algebra $\mathfrak{g}_{\bar{0}}^{\mathrm{r}}$ with a nilpotent Lie algebra $\mathfrak{g}_{\bar{0}}^{\mathrm{n}}$. With respect to the right adjoint $\mathfrak{g}_{\bar{0}}^{\mathrm{r}}$-action, $\mathfrak{g}_{\bar{0}}^{\mathrm{n}}$ and $\mathfrak{g}_{\bar{1}}$ decompose as direct sums of weight spaces for a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}^{\mathrm{r}}$; we let $\Delta_{\bar{0}}^{\mathrm{r}}$, $\Delta_{\bar{0}}^{\mathrm{n}}$ and $\Delta_{\bar{1}}$ denote the sets of the roots for $\mathfrak{g}_{\bar{0}}^{\mathrm{r}}$, $\mathfrak{g}_{\bar{0}}^{\mathrm{n}}$ and $\mathfrak{g}_{\bar{1}}$, respectively. The nilpotent Lie algebra $\mathfrak{g}_{\bar{0}}^{\mathrm{n}}$ acts on $\mathfrak{g}_{\bar{1}}$ nilpotently.

This time we assume that the root datum and the corresponding algebraic $\mathbb{Z}$-groups given in (II.3.3.1) are as follows: $\mathfrak{g}_{\bar{0}}^{\mathrm{r}} \supset \mathfrak{h}$ coincide with the complexifications of $\mathrm{Lie}(G_{\mathbb{Z}}) \supset \mathrm{Lie}(T_{\mathbb{Z}})$, and $\Delta_{\bar{0}}^{\mathrm{n}}$, $\Delta_{\bar{1}} \subset \mathsf{X}$.

Theorem 3.3.4 (Gavarini)**.** *There exist $\mathbb{Z}$-lattices $N_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ of $\mathfrak{g}_{\bar{0}}^{\mathrm{n}}$ and $\mathfrak{g}_{\bar{1}}$, respectively, such that*

(i) $\mathfrak{g}_{\mathbb{Z}} := \mathrm{Lie}(G_{\mathbb{Z}}) \oplus N_{\mathbb{Z}} \oplus V_{\mathbb{Z}}$ *is a Lie-superalgebra $\mathbb{Z}$-form of $\mathfrak{g}$.*

(ii) This Lie superalgebra $\mathfrak{g}_{\mathbb{Z}}$ over $\mathbb{Z}$ is admissible.

(iii) $V_{\mathbb{Z}}$ is $\mathrm{hy}(G_{\mathbb{Z}})$*-stable in the right $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{r}})$-module $\mathfrak{g}_{\bar{1}}$.*

(iv) $N_{\mathbb{Z}}$ contains a $\mathbb{Z}$-free basis $x_1, \dots, x_s$ such that

(iv-1) the $\mathbb{Z}$-submodule $H_{\mathbb{Z}}$ of $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})$ which is (freely) generated by

$$\frac{x_1^{n_1}}{n_1!} \cdots \frac{x_s^{n_s}}{n_s!}, \quad n_1 \geq 0, \dots, n_s \geq 0$$

is a $\mathbb{Z}$-subalgebra,

(iv-2) $V_{\mathbb{Z}}$ is $H_{\mathbb{Z}}$-stable in the right $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})$-module $\mathfrak{g}_{\bar{1}}$, and

(iv-3) $H_{\mathbb{Z}}$ is $\mathrm{hy}(G_{\mathbb{Z}})$*-stable in the right $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{r}})$-module $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})$.*

Gavarini's construction in [10] is parallel to those in [9, 11]. One sees that among Gavarini's Chevalley basis elements, the elements contained in $\mathfrak{g}_{\bar{0}}^{\mathrm{n}}$ and the odd elements generate the desired $\mathbb{Z}$-lattices $N_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$, respectively; the former are precisely the desired elements for (iv). See [10, Section 3.1] for (ii), and see [10, Section 3.3] for (iii), (iv). Note that the $\mathbb{Z}$-algebra $H_{\mathbb{Z}}$ given in (iv-1) is indeed a Hopf-algebra $\mathbb{Z}$-form of $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})$.

Recall from [8, IV, Sect. 2, 4.5] there uniquely exists a unipotent algebraic group F over $\mathbb{C}$ such that $\mathrm{Lie}(F) = \mathfrak{g}_{\bar{0}}^{\mathrm{n}}$. The corresponding Hopf algebra $\mathcal{O}(F)$ is the polynomial algebra $\mathbb{C}[t_1, \dots, t_s]$ such that

$$\langle\ ,\ \rangle : \mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}}) \times \mathcal{O}(F) \longrightarrow \mathbb{C}, \quad \langle \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_s^{n_s}}{n_s!},\ t_1^{m_1} \cdots t_s^{m_s} \rangle = \delta_{n_1,m_1} \cdots \delta_{n_s,m_s} \tag{II.3.3.3}$$

is a Hopf pairing. This induces a Hopf algebra isomorphism

$$\mathcal{O}(F) \xrightarrow{\simeq} \mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})'. \tag{II.3.3.4}$$

Here and in what follows, given a finitely generated Hopf algebra B over a field or $\mathbb{Z}$, we define

$$B' := \bigcup_{n>0} (B/(B^+)^n)^*,$$

as in [30, Section 9.2]. This is a Hopf subalgebra of B°. If B is the commutative Hopf algebra corresponding to an algebraic group, then B' is the hyperalgebra of the algebraic group.

Lemma 3.3.5. *$\mathbb{Z}[t_1, \ldots, t_s]$ is a Hopf-algebra $\mathbb{Z}$-form of $\mathcal{O}(F) = \mathbb{C}[t_1, \ldots, t_s]$. The Hopf pairing (II.3.3.3) over $\mathbb{C}$ restricts to a Hopf pairing $\langle\ ,\ \rangle : H_{\mathbb{Z}} \times \mathbb{Z}[t_1, \ldots, t_s] \to \mathbb{Z}$ over $\mathbb{Z}$, and it induces an isomorphism*

$$\mathbb{Z}[t_1, \ldots, t_s] \xrightarrow{\simeq} H'_{\mathbb{Z}}$$

of $\mathbb{Z}$-Hopf algebras.

Proof. It is easy to see that the Hopf algebra isomorphism (II.3.3.4) restricts to a $\mathbb{Z}$-algebra map $\mathbb{Z}[t_1, \ldots, t_n] \to H'_{\mathbb{Z}}$. We have the following commutative diagram which contains the isomorphism and the restricted algebra map.

$$\begin{array}{ccc}
\mathbb{Z}[t_1, \ldots, t_s] & \hookrightarrow & \mathcal{O}(F) = \mathbb{C}[t_1, \ldots, t_s] \\
\downarrow & & \downarrow \simeq \\
H'_{\mathbb{Z}} & \hookrightarrow & \mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})' \\
\downarrow & & \downarrow \\
H^*_{\mathbb{Z}} & \hookrightarrow & \mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})^*
\end{array}$$

Since $H^*_{\mathbb{Z}} \simeq \mathbb{Z}[\![t_1, \ldots, t_n]\!]$, $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})^* \simeq \mathbb{C}[\![t_1, \ldots, t_n]\!]$, we see that the outer big square is a pull-back. The lower square is a pull-back, too, as is easily seen. It follows that the upper square is a pull-back, whence $\mathbb{Z}[t_1, \ldots, t_n] \to H'_{\mathbb{Z}}$ is an isomorphism. This implies that $\mathbb{Z}[t_1, \ldots, t_n]$ is a Hopf-algebra $\mathbb{Z}$-form of $\mathcal{O}(F)$. The rest is now easy to see. ☐

Let $F_{\mathbb{Z}}$ denote the algebraic $\mathbb{Z}$-group corresponding to the $\mathbb{Z}$-Hopf algebra $\mathbb{Z}[t_1, \ldots, t_s]$. Then

$$\mathcal{O}(F_{\mathbb{Z}}) = \mathbb{Z}[t_1, \ldots, t_s], \quad \mathrm{hy}(F_{\mathbb{Z}}) = H_{\mathbb{Z}}, \quad \mathrm{Lie}(F_{\mathbb{Z}}) = N_{\mathbb{Z}}.$$

Note from (i) of Theorem 3.3.4 that $N_{\mathbb{Z}}$ is a Lie-algebra $\mathbb{Z}$-form of $\mathfrak{g}_{\bar{0}}^{n}$. From the first two equalities above or from Gavarini's original construction one sees that the construction of $H_{\mathbb{Z}}$ does not depend on the order of the basis elements.

Let $G \supset T$ denote the base changes of $G_{\mathbb{Z}} \supset T_{\mathbb{Z}}$ to $\mathbb{C}$. The right $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{r}})$-module structure on $\mathfrak{g}_{\bar{0}}^{\mathrm{n}}$, which arises from the right adjoint action, is indeed a $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{r}})$-$T$-module structure. Hence it gives rise to a right G-module structure, by which $\mathfrak{g}_{\bar{0}}^{\mathrm{n}}$ is a Lie-algebra object in the symmetric tensor category Mod_G of right G-modules. The structure uniquely extends to $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})$ so that $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})$ turns into a Hopf-algebra object in Mod_G. One sees that the structure just obtained is transposed through (II.3.3.3) to $\mathcal{O}(F)$, so that $\mathcal{O}(F)$ is a Hopf-algebra object in the symmetric category ${}_G\mathsf{Mod}$ of left G-modules. Thus, F turns into a right G-equivariant algebraic group. The associated semi-direct product $G \ltimes F$ of algebraic groups has $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{\bar{0}}^{\mathrm{r}} \ltimes \mathfrak{g}_{\bar{0}}^{\mathrm{n}}$ as its Lie algebra, as is easily seen. Note that $\mathfrak{g}_{\bar{1}}$ is a right $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{r}})$-$T$-module, and is such a right $\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})$-module that is annihilated by $(\mathcal{U}(\mathfrak{g}_{\bar{0}}^{\mathrm{n}})^+)^m$ for some m. Then it follows that $\mathfrak{g}_{\bar{1}}$ turns into a right G-module and F-module. Moreover, it is a right $G \ltimes F$-module, as is seen by using

(1) $\mathrm{Lie}(G \ltimes F) = \mathfrak{g}_{\bar{0}}^{\mathrm{r}} \ltimes \mathfrak{g}_{\bar{0}}^{\mathrm{n}}$,

(2) $G \ltimes F$ is connected, and

(3) $\mathfrak{g}_{\bar{1}}$ is a right $\mathcal{U}(\mathfrak{g}_{\bar{0}})$-module.

What were constructed in the last paragraph are all defined over $\mathbb{Z}$, as is seen from the following Lemma.

Lemma 3.3.6. *Keep the notation as above.*

(1) The right $\mathcal{O}(G)$-comodule structure $\mathcal{O}(F) \to \mathcal{O}(F) \otimes_{\mathbb{C}} \mathcal{O}(G)$ on $\mathcal{O}(F)$ restricts to $\mathcal{O}(F_{\mathbb{Z}}) \to \mathcal{O}(F_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{O}(G_{\mathbb{Z}})$, by which $F_{\mathbb{Z}}$ turns into a right $G_{\mathbb{Z}}$-equivariant algebraic group. Therefore, we have the associated semi-direct product $G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}$ of algebraic groups.

(2) $V_{\mathbb{Z}}$ is naturally a right $G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}$-module.

Proof. (1) One sees that the right $\mathrm{hy}(G_{\mathbb{Z}})$-module structure on $H_{\mathbb{Z}}$ which is given by (iv-3) of Theorem 3.3.4 is indeed a $\mathrm{hy}(G_{\mathbb{Z}})$-$T_{\mathbb{Z}}$-module structure. Hence it gives rise to a right $G_{\mathbb{Z}}$-module structure on $H_{\mathbb{Z}}$, by which $H_{\mathbb{Z}}$ turns into a Hopf-algebra object in $\mathsf{Mod}_{G_{\mathbb{Z}}}$. Since the isomorphism given in Lemma 3.3.5 is compatible with base extension, it follows that the last structure is transposed to a left $G_{\mathbb{Z}}$-module structure on $\mathcal{O}(F_{\mathbb{Z}})$, so that $\mathcal{O}(F_{\mathbb{Z}})$ is a Hopf-algebra object in ${}_{G_{\mathbb{Z}}}\mathsf{Mod}$. By construction the corresponding right $\mathcal{O}(G_{\mathbb{Z}})$-comodule structure on $\mathcal{O}(F_{\mathbb{Z}})$ is the restriction of the right $\mathcal{O}(G)$-comodule structure on $\mathcal{O}(F)$. This proves the first assertion. The rest is easy to see.

(2) Just as for $H_{\mathbb{Z}}$, we see from (iii) of the theorem that $V_{\mathbb{Z}}$ is a right $\mathrm{hy}(G_{\mathbb{Z}})$-$T_{\mathbb{Z}}$-module, whence it is a right $G_{\mathbb{Z}}$-module. We see from (iv-2) that $V_{\mathbb{Z}}$ is a right $H_{\mathbb{Z}}$-module, and it is indeed a right $H_{\mathbb{Z}}/(H_{\mathbb{Z}}^+)^m$-module for the same m as before. It follows by Lemma 3.3.5 that $V_{\mathbb{Z}}$ is a right $F_{\mathbb{Z}}$-module.

It remains to prove that

$$(vf)g = (vg)f^g, \quad v \in V_{\mathbb{Z}},\ f \in F_{\mathbb{Z}},\ g \in G_{\mathbb{Z}}.$$

Let S be a commutative ring. The equality in $S \otimes_{\mathbb{Z}} \mathbb{C}$-points follows from the analogous equality for $\mathfrak{g}_{\bar{1}}$, since $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} S \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{g}_{\bar{1}} \otimes_{\mathbb{C}} (S \otimes_{\mathbb{Z}} \mathbb{C})$. To prove the equality in S-points, we may suppose $S = \mathcal{O}(F_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{O}(G_{\mathbb{Z}})$, and so that S is $\mathbb{Z}$-flat. In this case the equality follows from the previous result since we then have $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} S \subset V_{\mathbb{Z}} \otimes_{\mathbb{Z}} S \otimes_{\mathbb{Z}} \mathbb{C}$. □

Recall that $\mathfrak{g}_{\mathbb{Z}}$ is a Lie-superalgebra $\mathbb{Z}$-form as given in (i) of Theorem 3.3.4. Its odd component $V_{\mathbb{Z}}$ is a right $G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}$-module by Lemma 3.3.6.

Proposition 3.3.7. *$(G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$ is a Harish-Chandra pair.*

Proof. As is easily seen, $\mathrm{Lie}(G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}})$ coincides with the even component $\mathrm{Lie}(G_{\mathbb{Z}}) \ltimes N_{\mathbb{Z}}$ of $\mathfrak{g}_{\mathbb{Z}}$. The restricted super-bracket $[\ ,\ ] : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathrm{Lie}(G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}})$, being $\mathrm{hy}(G_{\mathbb{Z}})$- and $H_{\mathbb{Z}}$-linear, is $G_{\mathbb{Z}}$- and $F_{\mathbb{Z}}$-equivariant. It is necessarily $G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}$-equivariant. □

We have thus the algebraic $\mathbb{Z}$-supergroup $\mathbf{G}(G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$ in ASG which is associated with the Harish-Chandra pair just obtained. It is a $\mathbb{Z}$-form of the algebraic $\mathbb{C}$-supergroup which is associated with the Harish-Chandra pair $(G \ltimes F, \mathfrak{g})$. Since every Chevalley $\mathbb{C}$-supergroup of Cartan type (see Section 3.3) is associated with some Harish-Chandra pair of the last form, we have constructed a natural $\mathbb{Z}$-form of every such $\mathbb{C}$-supergroup.

Remark 3.3.8. Just as in the classical-type case (see Remark 3.3.3 (1)), Gavarini's construction requires faithful representations of $\mathfrak{g}$, which, however, must satisfy more involved conditions as given in [10, Definition 3.14]; Proposition 3.16 of [10] proves that part of the conditions is satisfied if the representation is completely reducible. The required representations look thus rather restrictive. On the other hand, Theorem 4.42 of [10] implies that the required representations are many enough to ensure that our $\mathbb{Z}$-forms all are realized by Gavarini's construction. But the proof of the theorem is wrong. After the publication of [10], a corrected proof of the theorem, which uses the category equivalence [12, Theorem 4.3.14] (= Part I, Theorem 4.6.9), was given in Erratum added to a new version of [10]. As far as I understand, the proof is correct if the same argument as proving our Lemma 3.3.6 is added.

3.4 Torus, unipotent and Borel subsupergroups

Let $(G_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$ be a quasireductive Harish-Chandra pair. Recall that $\mathrm{Lie}(T_{\mathbb{Z}}) = (\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}$. Then obviously $(\mathfrak{h}_{\mathbb{Z}})_{\bar{1}}$ is $T_{\mathbb{Z}}$-stable. Thus, the pair $(T_{\mathbb{Z}}, \mathfrak{h}_{\mathbb{Z}})$ is a sub-pair of $(G_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$. We obtain a closed algebraic subsupergroup $\mathbf{T}_{\mathbb{Z}} := \mathbf{G}(T_{\mathbb{Z}}, \mathfrak{h}_{\mathbb{Z}})$ of $\mathbf{G}_{\mathbb{Z}}$.

Remark 3.4.1. The supergroup $\mathbf{T}_{\mathbb{Z}}$ is no longer abelian, in general. If $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$, then we have $\mathbf{T}_{\mathbb{Z}} = T_{\mathbb{Z}}$.

Example 3.4.2. For $\mathbf{G}_{\mathbb{Z}} = \mathbf{Q}(n)$, the supergroup $\mathbf{T}_{\mathbb{Z}}$ is given as follows

$$\mathbf{T}_{\mathbb{Z}}(R) := \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathbf{Q}(n)(R) \mid A \text{ and } B \text{ are diagonal}\},$$

where R is a commutative superalgebra. This is not abelian.

As in (II.3.1.2), we have two Lie subsuperalgebras $\mathfrak{u}^{\pm}_{\mathbb{Z}}$ of $\mathfrak{g}_{\mathbb{Z}}$ over $\mathbb{Z}$ whose even-part $(\mathfrak{u}^{\pm}_{\mathbb{Z}})_{\bar{0}}$ are nilpotent Lie subalgebras of $(\mathfrak{g}_{\mathbb{Z}})_{\bar{0}}$. Then one can construct two connected and unipotent subgroups $U^{\pm}_{\mathbb{Z}}$ of $G_{\mathbb{Z}}$ such that $\mathrm{Lie}(U^{\pm}_{\mathbb{Z}}) = (\mathfrak{u}^{\pm}_{\mathbb{Z}})_{\bar{0}}$, see [16, Part II, 1.7]. Let $\{X_\alpha \mid \alpha \in \Delta^+_{\bar{0}}\}$ denote a $\mathbb{Z}$-basis of $\mathrm{Lie}(U^{\pm}_{\mathbb{Z}})$. Given any order on $\Delta^+_{\bar{0}}$, then the monomials

$$\prod_{\alpha \in \Delta^+_{\bar{0}}} X_\alpha^{(n_\alpha)} \tag{II.3.4.1}$$

with $n_\alpha \geqslant 0$ forms a $\mathbb{Z}$-basis of $\mathrm{hy}(U_\mathbb{Z}^+)$. Similarly, $\prod_{\alpha \in \Delta_{\bar{0}}^-} X_\alpha^{(n_\alpha)}$ with $n_\alpha \geqslant 0$ forms a $\mathbb{Z}$-basis of $\mathrm{hy}(U_\mathbb{Z}^-)$.

Lemma 3.4.3. *The pairs $(U_\mathbb{Z}^\pm, \mathfrak{u}_\mathbb{Z}^\pm)$ are sub-pairs of $(G_\mathbb{Z}, \mathfrak{g}_\mathbb{Z})$.*

Proof. We concentrate on $(U_\mathbb{Z}^+, \mathfrak{u}_\mathbb{Z}^+)$. What we have to show is $(\mathfrak{u}_\mathbb{Z}^+)_{\bar{1}}$ is $U_\mathbb{Z}^+$-table in $(\mathfrak{g}_\mathbb{Z})_{\bar{1}}$.

First, we prepare general statement. Let F be a closed subgroup of some GL_m over $\mathbb{Z}$ such that $\mathcal{O}(F)$ is $\mathbb{Z}$-flat. Let B denote the hyperalgebra $\mathrm{hy}(F)$ of F. Then the canonical pairing $B \times \mathcal{O}(F) \to \mathbb{Z}$ induces a map $\mathcal{O}(F) \to B^*$ whose image is included in B°. Here,

$$B^\circ := \bigcup_I (B/I)^* \quad (\subseteq B^*)$$

is the dual Hopf algebra of B, where I runs through the ideals of B such that B/I is $\mathbb{Z}$-finite; see Remark 2.4.3. Now, suppose that F is infinitesimally flat and connected. Then by [36, Proposition 0.3.1(g)], the map $\mathcal{O}(F_\mathbb{Q}) \to (B \otimes_\mathbb{Z} \mathbb{Q})^*$ is injective, where $F_\mathbb{Q}$ is the base extension of F to $\mathbb{Q}$. Combine with the above result, we have an inclusion $\mathcal{O}(F) \hookrightarrow B^\circ$ of Hopf algebras.

In our case, $U_\mathbb{Z}^+$ satisfies the conditions stated above. Hence, we have an inclusion $\mathcal{O}(U_\mathbb{Z}^+) \hookrightarrow \mathrm{hy}(U_\mathbb{Z}^+)^\circ$ of Hopf algebras. Moreover, one sees that this is indeed an isomorphism, since $U_\mathbb{Z}^+$ is unipotent. Therefore, we conclude that for a $\mathbb{Z}$-module M, there is a one-to-one correspondence between the set of all $U_\mathbb{Z}^+$-module structures on M and the set of all locally finite $\mathrm{hy}(U_\mathbb{Z}^+)$-module structures on M. By (II.3.4.1), we see that $(\mathfrak{u}_\mathbb{Z})_{\bar{1}}$ is $\mathrm{hy}(U_\mathbb{Z}^+)$-stable. This completes the proof. □

By Lemma 3.4.3, we obtain two closed algebraic subsupergroups $\mathbf{U}_\mathbb{Z}^\pm := \mathbf{G}(U_\mathbb{Z}^\pm, \mathfrak{u}_\mathbb{Z}^\pm)$ of $\mathbf{G}_\mathbb{Z}$. We construct two supergroups

$$\mathbf{B}_\mathbb{Z}^\pm \ := \ \mathbf{T}_\mathbb{Z}\, \mathbf{U}_\mathbb{Z}^\pm,$$

where the product is taken in $\mathbf{G}_\mathbb{Z}$. These $\mathbf{B}_\mathbb{Z}^\pm$ are actually closed subsupergroups of $\mathbf{G}_\mathbb{Z}$. Since $\mathbf{T}_\mathbb{Z}$ is a closed subsupergroup of $\mathbf{B}_\mathbb{Z}^\pm$, we have surjections $\mathcal{O}(\mathbf{B}_\mathbb{Z}^\pm) \twoheadrightarrow \mathcal{O}(\mathbf{T}_\mathbb{Z})$ of Hopf superalgebras.

Proposition 3.4.4. *There are splittings $\mathcal{O}(\mathbf{T}_\mathbb{Z}) \hookrightarrow \mathcal{O}(\mathbf{B}_\mathbb{Z}^\pm)$ of Hopf superalgebras.*

Proof. We concentrate on showing that $\mathcal{O}(\mathbf{T}_\mathbb{Z}) \hookrightarrow \mathcal{O}(\mathbf{B}_\mathbb{Z}^+)$. We fix a commutative superalgebra R over $\mathbb{Z}$. By using the adjoint action

$$\mathbf{T}_\mathbb{Z}(R) \times \mathbf{U}_\mathbb{Z}^+(R) \longrightarrow \mathbf{U}_\mathbb{Z}^+(R); \quad (t,\, u) \longmapsto tut^{-1},$$

we see that $\mathbf{U}_\mathbb{Z}^+$ is a normal closed subsupergroup of $\mathbf{B}_\mathbb{Z}^+$. On the other hand, it is easy to see that $\mathbf{T}_\mathbb{Z}(R) \cap \mathbf{U}_\mathbb{Z}^+(R) = \{e\}$, where e is the unit element of $\mathbf{G}_\mathbb{Z}$. Hence, we have an isomorphism

$$m : \mathbf{T}_\mathbb{Z}(R) \ltimes \mathbf{U}_\mathbb{Z}^+(R) \xrightarrow{\cong} \mathbf{B}_\mathbb{Z}^+(R); \quad (t,\, u) \longmapsto tu \tag{II.3.4.2}$$

of groups. We let $\mathcal{O}(m)$ denote the corresponding Hopf superalgebra isomorphism. Since m is identical on $\mathbf{T}_\mathbb{Z}$, the map

$$\mathcal{O}(\mathbf{T}_\mathbb{Z}) \xrightarrow{\mathrm{id}\otimes 1} \mathcal{O}(\mathbf{T}_\mathbb{Z}) \otimes_\mathbb{Z} \mathcal{O}(\mathbf{U}_\mathbb{Z}^+) \xrightarrow{\mathcal{O}(m)^{-1}} \mathcal{O}(\mathbf{B}_\mathbb{Z}^+)$$

gives a desired Hopf superalgebra splitting. ☐

Example 3.4.5. Let $\mathbf{G} = \mathbf{GL}(2|1)$. According to (i)–(vi) in Example 3.1.5, we get six variations of $\mathbf{B}^+$ as follows.

$$\text{(i)}\quad \mathbf{B}^+ = \{\left(\begin{array}{cc|c} * & * & * \\ 0 & * & * \\ \hline 0 & 0 & * \end{array}\right)\},\qquad \text{(ii)}\quad \mathbf{B}^+ = \{\left(\begin{array}{cc|c} * & 0 & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right)\},\qquad \text{(iii)}\quad \mathbf{B}^+ = \{\left(\begin{array}{cc|c} * & * & * \\ 0 & * & 0 \\ \hline 0 & * & * \end{array}\right)\},$$

$$\text{(iv)}\quad \mathbf{B}^+ = \{\left(\begin{array}{cc|c} * & * & 0 \\ 0 & * & 0 \\ \hline * & * & * \end{array}\right)\},\qquad \text{(v)}\quad \mathbf{B}^+ = \{\left(\begin{array}{cc|c} * & 0 & 0 \\ * & * & 0 \\ \hline * & 0 & * \end{array}\right)\},\qquad \text{(vi)}\quad \mathbf{B}^+ = \{\left(\begin{array}{cc|c} * & * & 0 \\ 0 & * & 0 \\ \hline * & * & * \end{array}\right)\}.$$

Let $\pi : \mathcal{O}(\mathbf{G}_\mathbb{Z}) \twoheadrightarrow \mathcal{O}(\mathbf{B}^+_\mathbb{Z})$ and $\pi' : \mathcal{O}(\mathbf{G}_\mathbb{Z}) \twoheadrightarrow \mathcal{O}(\mathbf{B}^-_\mathbb{Z})$ denote the Hopf quotient maps.

Proposition 3.4.6. *The following superalgebra map is injective.*

$$(\pi \otimes \pi') \circ \Delta : \mathcal{O}(\mathbf{G}_\mathbb{Z}) \longrightarrow \mathcal{O}(\mathbf{B}^+_\mathbb{Z}) \otimes_\mathbb{Z} \mathcal{O}(\mathbf{B}^-_\mathbb{Z}),$$

where Δ is the comultiplication of $\mathcal{O}(\mathbf{G}_\mathbb{Z})$.

Proof. By the Hopf superalgebra isomorphism in (II.3.4.2), we have a inclusion $\mathcal{O}(\mathbf{U}^-_\mathbb{Z}) \hookrightarrow \mathcal{O}(\mathbf{B}^-_\mathbb{Z})$ of superalgebras. Therefore, to prove the claim, it is enough to see that $\mathcal{O}(\mathbf{G}_\mathbb{Z}) \to \mathcal{O}(\mathbf{B}^+_\mathbb{Z}) \otimes_\mathbb{Z} \mathcal{O}(\mathbf{U}^-_\mathbb{Z})$ is injective. The multiplication map $f : \mathbf{B}^+_\mathbb{Z} \times \mathbf{U}^-_\mathbb{Z} \to \mathbf{G}_\mathbb{Z}$ induces a supercoalgebra map $T(f) : \mathrm{hy}(\mathbf{B}^+_\mathbb{Z}) \otimes_\mathbb{Z} \mathrm{hy}(\mathbf{U}^-_\mathbb{Z}) \to \mathrm{hy}(\mathbf{G}_\mathbb{Z})$. By Theorem 3.2.4, we conclude that $T(f)$ is an isomorphism of supercoalgebras. Thus its $\mathbb{Z}$-linear dual $T(f)^*$ gives an isomorphism of superalgebras. Since $\mathrm{hy}(\mathbf{B}^+_\mathbb{Z})$ and $\mathrm{hy}(\mathbf{U}^-_\mathbb{Z})$ are both $\mathbb{Z}$-free, the canonical map $\mathrm{hy}(\mathbf{B}^+_\mathbb{Z})^* \otimes_\mathbb{Z} \mathrm{hy}(\mathbf{U}^-_\mathbb{Z})^* \to (\mathrm{hy}(\mathbf{B}^+_\mathbb{Z}) \otimes_\mathbb{Z} \mathrm{hy}(\mathbf{U}^-_\mathbb{Z}))^*$ is injective. Therefore, we have the following commutative diagram of superalgebras

$$\begin{array}{ccccc} \mathrm{hy}(\mathbf{G}_\mathbb{Z})^* & \xrightarrow[\cong]{T(f)^*} & (\mathrm{hy}(\mathbf{B}^+_\mathbb{Z}) \otimes_\mathbb{Z} \mathrm{hy}(\mathbf{U}^-_\mathbb{Z}))^* & \longleftarrow & \mathrm{hy}(\mathbf{B}^+_\mathbb{Z})^* \otimes_\mathbb{Z} \mathrm{hy}(\mathbf{U}^-_\mathbb{Z})^* \\ \uparrow & & \circlearrowleft & & \uparrow \\ \mathcal{O}(\mathbf{G}_\mathbb{Z}) & & \longrightarrow & & \mathcal{O}(\mathbf{B}^+_\mathbb{Z}) \otimes_\mathbb{Z} \mathcal{O}(\mathbf{U}^-_\mathbb{Z}). \end{array}$$

Since $\mathbf{G}_\mathbb{Z}$, $\mathbf{B}^+_\mathbb{Z}$ and $\mathbf{U}^-_\mathbb{Z}$ are connected, the vertical maps are injective. Hence, we conclude that $\mathcal{O}(\mathbf{G}_\mathbb{Z}) \to \mathcal{O}(\mathbf{B}^+_\mathbb{Z}) \otimes_\mathbb{Z} \mathcal{O}(\mathbf{U}^-_\mathbb{Z})$ is injective. ☐

We denote the base change of $\mathbf{G}_\mathbb{Z}, \mathbf{T}_\mathbb{Z}, \mathbf{U}^\pm_\mathbb{Z}, G_\mathbb{Z}, T_\mathbb{Z}, U^\pm_\mathbb{Z}, \ldots$ etc. to the ground field $\Bbbk$ by $\mathbf{G}, \mathbf{T}, \mathbf{U}^\pm, G, T, U^\pm, \ldots$ etc. and the base change of $\mathfrak{g}_\mathbb{Z}, \mathfrak{h}_\mathbb{Z}, \mathfrak{u}^\pm_\mathbb{Z}, \ldots$ etc. to $\Bbbk$ by $\mathfrak{g}, \mathfrak{h}, \mathfrak{u}^\pm, \ldots$ etc. Note that, $(\mathfrak{g}_\mathbb{Z})_\epsilon \otimes_\mathbb{Z} \Bbbk = (\mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} \Bbbk)_\epsilon$ for $\epsilon = \bar{0}, \bar{1}$. By definition, we have $\mathrm{Lie}(\mathbf{G}) = \mathfrak{g}, \mathrm{Lie}(\mathbf{T}) = \mathfrak{h}, \mathrm{Lie}(\mathbf{U}^\pm) = \mathfrak{u}^\pm, \ldots$ etc.

A supergroup $\mathbf{H}$ over $\Bbbk$ is said to be *unipotent* if the corresponding commutative Hopf superalgebra $\mathcal{O}(\mathbf{H})$ is irreducible. By [24, Theorem 41], it was shown that $\mathbf{H}$ is unipotent if and only if so is $\mathbf{H}_{\mathrm{ev}}$. Since $\mathbf{U}^\pm_{\mathrm{ev}} = U^\pm$ are unipotent algebraic groups, we have the following

Proposition 3.4.7. $\mathbf{U}^\pm$ *are unipotent.*

3.5 Representations of quasireductive supergroups

Recall that $T_{\mathbb{Z}}$ is a maximal split torus of $G_{\mathbb{Z}}$. The corresponding commutative Hopf algebra $\mathcal{O}(T_{\mathbb{Z}})$ is the Laurent polynomial ring $\mathbb{Z}[T_1^{\pm}, T_2^{\pm}, \ldots, T_\ell^{\pm}]$ whose Hopf algebra structure is given by

$$\Delta(T_i^{\pm}) = T_i^{\pm} \otimes_{\mathbb{Z}} T_i^{\pm}, \quad \varepsilon(T_i^{\pm}) = 1, \quad \mathcal{S}(T_i^{\pm}) = T_i^{\mp}$$

for $i = 1, \ldots, \ell$. Since T is the base extension of $T_{\mathbb{Z}}$ to $\Bbbk$, the corresponding commutative Hopf algebra is given by $\Bbbk[T_1^{\pm}, \ldots, T_\ell^{\pm}]$. We let Λ denote the character group $\mathsf{X} = \mathsf{X}(T_{\mathbb{Z}})$ of $T_{\mathbb{Z}}$, or equivalently, the set of all grouplike elements of $\mathcal{O}(T)$. Explicitly,

$$\Lambda = \left\{ T_1^{n_1} \cdots T_\ell^{n_\ell} \mid n_1, \ldots, n_\ell \in \mathbb{Z} \right\} \cong \mathbb{Z}^\ell.$$

It is easy to see that Λ coincides with the $\mathbb{Z}$-linear dual of $(\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}$.

Lemma 3.5.1. *The abelian group* Λ *coincides with* $\mathrm{Hom}_{\mathbb{Z}}((\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}, \mathbb{Z})$.

Proof. By the definition, we have

$$(\mathfrak{h}_{\mathbb{Z}})_{\bar{0}} \;=\; \mathrm{Lie}(T_{\mathbb{Z}}) \;=\; \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}(T_{\mathbb{Z}})^+/(\mathcal{O}(T_{\mathbb{Z}})^+)^2, \mathbb{Z}).$$

Therefore, we have an isomorphism $\mathrm{Hom}_{\mathbb{Z}}((\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}, \mathbb{Z}) \cong \mathcal{O}(T_{\mathbb{Z}})^+/(\mathcal{O}(T_{\mathbb{Z}})^+)^2$ as abelian groups. One sees that $T_1 - 1, \ldots, T_\ell - 1$ form a $\mathbb{Z}$-free basis of $\mathcal{O}(T_{\mathbb{Z}})^+/(\mathcal{O}(T_{\mathbb{Z}})^+)^2$. For $i = 1, \ldots, \ell$, we denote the partial derivation on $\mathcal{O}(T_{\mathbb{Z}}) = \mathbb{Z}[T_1^{\pm}, \ldots, T_\ell^{\pm}]$ by $\partial/\partial T_i$. Let

$$H_i := \varepsilon \circ \frac{\partial}{\partial T_i} : \mathcal{O}(T_{\mathbb{Z}}) \longrightarrow \mathbb{Z}, \tag{II.3.5.1}$$

for $i = 1, \ldots, \ell$. Then $H_1, \ldots, H_\ell$ form a $\mathbb{Z}$-free basis of $\mathrm{Lie}(T_{\mathbb{Z}})$. Since H_i is the dual basis of $T_i - 1$, the following is an isomorphism of abelian groups.

$$\Lambda \longrightarrow \mathcal{O}(T_{\mathbb{Z}})^+/(\mathcal{O}(T_{\mathbb{Z}})^+)^2; \quad T_1^{n_1} \cdots T_\ell^{n_\ell} \longmapsto \sum_{i=1}^{\ell} n_i (T_i - 1).$$

This proves the claim. □

Let V be a representation of $\mathbf{G}$. By Theorem 2.4.8, we regard V as a $\mathrm{hy}(\mathbf{G})$-T-supermodule. For $\lambda \in \Lambda$, its λ-weight superspace V_λ of V is given by

$$V_\lambda = \{ v \in V \mid \binom{H_i}{n} v = \binom{\langle \lambda, H_i \rangle}{n} v \text{ for all } 1 \leqslant i \leqslant \ell,\, n \geqslant 0 \},$$

where $\langle\, , \,\rangle : \Lambda \times (\mathfrak{h}_{\mathbb{Z}})_{\bar{0}} \to \mathbb{Z}$ is the canonical pairing.

Chapter 4

Constructions of Simple Supercomodules

Let $\Bbbk$ be a ground field of characteristic $\neq 2$.

In this chapter, we will get a super-analogue of the Parshall and Wang's result [32, Theorem 8.3.1] (see also Bichon and Riche [2, Theorem 2.6]).

4.1 Simple supercomodules

Let C be a supercoalgebra. A non-zero right C-supercomodule L is *simple* if L has no non-trivial C-subsupercomodule. If L is simple, then ΠL is also simple.

Definition 4.1.1. A right simple C-supercomodule L is of *type* $\mathtt{Q}$ if $L \cong \Pi L$ as right C-supercomodules and *type* $\mathtt{M}$ otherwise.

Let $\mathsf{Simp}(C)$ denote the set of isomorphism classes of simple right C-supercomodules. The functor Π naturally acts on $\mathsf{Simp}(C)$ as a permutation, say π, of order 2. Let $\mathsf{Simp}_{\Pi}(C)$ denote the set of $\langle\pi\rangle$-orbits in $\mathsf{Simp}(C)$. Therefore, two elements L, L' in $\mathsf{Simp}(C)$ coincides in $\mathsf{Simp}_{\Pi}(C)$ if and only if $L \cong L'$ or $L \cong \Pi L'$.

Proposition 4.1.2 (Schur's lemma)**.** *Suppose that $\Bbbk$ is an algebraically closed field. For a simple C-supercomodule L, we have an isomorphism of C-supercomodules*

$$\underline{\mathrm{End}}^C(L) \cong \begin{cases} \Bbbk, & \textit{if } L \textit{ is of type } \mathtt{M}, \\ \Bbbk\mathbb{Z}_2, & \textit{if } L \textit{ is of type } \mathtt{Q}. \end{cases} \tag{II.4.1.1}$$

In particular, for simple C-supercomodules L and L', L has the same type as L' if and only if $\underline{\mathrm{End}}^C(L) \cong \underline{\mathrm{End}}^C(L')$.

Proof. For $\varphi \in \underline{\mathrm{End}}^C(L)$, we write $\varphi = \varphi_{\bar{0}} + \varphi_{\bar{1}} \in \underline{\mathrm{End}}^C(L)_{\bar{0}} \oplus \underline{\mathrm{End}}^C(L)_{\bar{1}}$. Since $\varphi_{\bar{0}} : L \to L$ is a parity preserving map, $\varphi_{\bar{0}}$ is a morphism of SMod^C. By the simplicity of L and $\Bbbk = \overline{\Bbbk}$, $\varphi_{\bar{0}}$ coincides with $c\,\mathrm{id}_L$ for some $c \in \Bbbk$. On the other hand, if L is of type $\mathtt{Q}$, then there exists a C-supercomodule isomorphism $J : L \xrightarrow{\cong} \Pi L$. Since $J^{-1} \circ \varphi_{\bar{1}} : L \to L$ is preserve the parity, this is a morphism of

SMod^C. Therefore, we have $\varphi_{\bar{1}} = c'J$ for some $c' \in \Bbbk$. Moreover, J satisfies $J^2 = \mathrm{id}_L$. This completes the proof. □

We denote the *cosmash product* of $\Bbbk\mathbb{Z}_2$ and C by $\Bbbk\mathbb{Z}_2 \blacktriangleright\!\!< C$. This is a coalgebra whose underlying vector space is $\Bbbk\mathbb{Z}_2 \otimes C$ and the comodule structures are given by

$$\begin{aligned} \Delta(\epsilon \otimes c) &= (\epsilon \otimes c_1) \otimes (|c_1| + \epsilon) \otimes c_2, \\ \varepsilon(\epsilon \otimes c) &= \varepsilon(c), \end{aligned}$$

where $\epsilon \in \mathbb{Z}_2$ and $c \in C_{\bar{0}} \cup C_{\bar{1}}$. Then, there is a natural identification $\mathsf{SMod}^C \approx \mathsf{Mod}^{\Bbbk\mathbb{Z}_2 \blacktriangleright\!\!< C}$. Therefore, we get the following properties.

Lemma 4.1.3. *Any simple right C-supercomodule is finite dimensional.*

Lemma 4.1.4. *For a C-supercomodule L, L is simple if and only if the dual supercomodule L^* is simple.*

For a right C-supercomodule V, we let $\mathrm{soc}_C(V)$ denote the (direct) sum of all simple C-subsupercomodules of V. This $\mathrm{soc}_C(V)$ is called the *C-socle of V*. In particular, for $V = C$, $\mathrm{corad}(C) := \mathrm{soc}_C(C)$ is called the *coradical* of C.

Lemma 4.1.5. *For a right C-comodule V, $V \neq 0$ if and only if $\mathrm{soc}_C(V) \neq 0$.*

A Hopf superalgebra U is said to be *irreducible* if its coradical $\mathrm{corad}(U)$ is trivial or equivalently, $\mathsf{Simp}_\Pi(U) = \{\Bbbk\}$; see [24, Definition 2].

Lemma 4.1.6. *Let U be an irreducible Hopf superalgebra, and let V be a right U-supercomodule. Then $V \neq 0$ if and only if $V^{\mathrm{co}\,U} \neq 0$.*

Proof. Since U is irreducible, we have $V^{\mathrm{co}\,U} = \mathrm{soc}_U(V)$. The claim follows, by lemma 4.1.5. □

4.2 Constructions of simple supercomodules

Let A be Hopf superalgebra. Let B, B' be a quotient Hopf superalgebras of A. We denote the quotient maps by

$$\pi_B : A \twoheadrightarrow B, \quad \pi_{B'} : A \twoheadrightarrow B'.$$

Let H be a Hopf superalgebra with Hopf superalgebra surjections.

$$\varphi_H : B \twoheadrightarrow H, \quad \varphi'_H : B' \twoheadrightarrow H$$

such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\ \pi_{B'}\ } & B' \\ {\scriptstyle \pi_B}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle \varphi'_H} \\ B & \xrightarrow[\ \varphi_H\ ]{} & H. \end{array} \qquad \text{(II.4.2.1)}$$

In this subsection, we assume that the following conditions hold:

(1) There is a Hopf superalgebra splitting $i_H : H \hookrightarrow B$ and (resp. $i'_H : H \hookrightarrow B'$), i.e., $\varphi_H \circ i_H = \mathrm{id}_H$ (resp. $\varphi'_H \circ i'_H = \mathrm{id}_H$). We regard H as a B-supercomodule (resp. B'-supercomodule) through i_H (resp. i'_H).

(2) There is an irreducible Hopf superalgebra quotient U (resp. U') of B (resp. B') such that i_H (resp. i'_H) induces an isomorphism $H \cong B^{\mathrm{co}U}$ (resp. $H \cong B'^{\mathrm{co}U'}$) of superspaces.

(3) The map $(\pi_B \otimes \pi_{B'}) \circ \Delta : A \to A \otimes A \to B \otimes B'$ is injective.

Since $\Bbbk$ is purely even, the following map coincides with the comultiplication Δ_B of B

$$B \,\Box_U\, \Bbbk \overset{\Delta_B \otimes \mathrm{id}_\Bbbk}{\longrightarrow} (B \otimes B) \,\Box_U\, \Bbbk \overset{\mathrm{id}_B \otimes c_{B,\Bbbk}}{\longrightarrow} B \otimes \Bbbk \otimes B.$$

Hence, we can regard $B^{\mathrm{co}\,U}$ as a right B-supercomodule with respect to Δ_B. In this way, $B^{\mathrm{co}\,U}$ has a structure of right H-supercomodule. Since $\varphi_H \circ i_H = \mathrm{id}_H$, we have the following result.

Lemma 4.2.1. *The isomorphism $i_H : H \overset{\cong}{\to} B^{\mathrm{co}U}$ in the assumption (4.2) is left B- right H-colinear.*

By the assumption (4.2), there is a surjective Hopf superalgebra map $B \twoheadrightarrow U$. For a right B-supercomodule M, we have the restricted U-supercomodule $\mathrm{res}^B_U(M)$. Therefore, we can consider the U-coinvariants $M^{\mathrm{co}\,U}$ of M.

Lemma 4.2.2. *For a right B-supercomodule M, there is an isomorphism $M^{\mathrm{co}\,U} \cong M \,\Box_B\, H$ of superspaces.*

Proof. One can easily show that the isomorphism of superspaces

$$M \,\Box_B\, B \overset{\cong}{\longrightarrow} \mathrm{res}^B_U(M)$$

given in Lemma 2.6.2 (2.6.2) is indeed an isomorphism of left U-supercomodules. Taking $(-) \,\Box_U\, \Bbbk$ to both sides, we get an isomorphism of superspaces

$$(M \,\Box_B\, B) \,\Box_U\, \Bbbk \cong \mathrm{res}^B_U(M) \,\Box_U\, \Bbbk.$$

By definition, the right hand side is nothing but $M^{\mathrm{co}U}$. On the other hand, the left hand side is

$$(M \,\Box_B\, B) \,\Box_U\, \Bbbk = M \,\Box_B\, (B \,\Box_U\, \Bbbk) \cong M \,\Box_B\, H,$$

by the associativity Lemma 2.6.2 (2.6.2) and by Lemma 4.2.1. Thus, we are done. □

By Lemma 4.2.2, we can regard $M^{\mathrm{co}\,U}$ as right H-supercomodule. In this way, we get a functor $(-)^{\mathrm{co}\,U}$ from the category of right B-supercomodules SMod^B to the category of right H-supercomodules SMod^H.

Lemma 4.2.3. *The functor $(-)^{\mathrm{co}\,U}$ is right adjoint to the restriction functor $\mathrm{res}^H_B(-)$.*

Proof. By the proof of Lemma 4.2.2, one sees the functor $(-)^{\mathrm{co}\,U}$ and the functor $\mathrm{ind}_B^H(-)$ are naturally isomorphic. Therefore, by the Frobenius reciprocity Proposition 2.6.3, we are done. □

By the assumption (4.2), we can consider the restricted B-supercomodule $\mathrm{res}_B^H(N)$ for any H-supercomodule N. Since $\varphi_H \circ i_H = \mathrm{id}$, we have the following natural isomorphism of right H-supercomodules

$$\mathrm{res}_H^B \mathrm{res}_B^H(N) \cong N.$$

Suppose that $\mathsf{Simp}_\Pi(H)$ is parametrized by a set Λ. Let $\mathfrak{u}(\lambda)$ denote the simple H-supercomodule corresponding to $\lambda \in \Lambda$.

Lemma 4.2.4. *For $\lambda \in \Lambda$, the B-supercomodule $\mathrm{res}_B^H(\mathfrak{u}(\lambda))$ is simple. This gives a one-to-one corresponding between $\mathsf{Simp}_\Pi(B)$ and Λ.*

Proof. It is enough to show that for all simple B-supercomodule L, there exists $\lambda \in \Lambda$ such that $L = \mathrm{res}_B^H(\mathfrak{u}(\lambda))$ or $\Pi\,\mathrm{res}_B^H(\mathfrak{u}(\lambda))$. For simplicity, we concentrate the case $L = \mathrm{res}_B^H(\mathfrak{u}(\lambda))$. By Lemma 4.1.6, the U-coinvariants L^U of L is non-zero. Therefore, there exists $\lambda \in \Lambda$ such that $\mathrm{res}_H^B(L^U) \supseteq \mathfrak{u}(\lambda)$. By Lemma 4.2.3, we have

$$0 \neq \mathrm{Hom}^H\big(\mathfrak{u}(\lambda),\, L^U\big) \cong \mathrm{Hom}^B\big(\mathrm{res}_B^H(\mathfrak{u}(\lambda)),\, L\big).$$

Hence, we have a surjective B-supercomodule map $f : \mathrm{res}_B^H(\mathfrak{u}(\lambda)) \twoheadrightarrow L$. By applying the functor res_H^B to both sides, we get $\mathfrak{u}(\lambda) \twoheadrightarrow \mathrm{res}_H^B(L)$. Therefore, the B-supercomodule map f is indeed an isomorphism. □

Lemma 4.2.5. *For a right H-supercomodule N, the following map is an inclusion of B'-supercomodules*

$$\mathrm{ind}_B^A(\mathrm{res}_B^H(N)) \hookrightarrow \mathrm{ind}_H^{B'}(N). \qquad \text{(II.4.2.2)}$$

Proof. By taking $\mathrm{res}_B^H(N) \square_B (-)$ to $A \hookrightarrow B \otimes B'$ in the assumption (4.2), we get $\mathrm{res}_B^H(N) \square_B A \hookrightarrow \mathrm{res}_B^H(N) \square_B (B \otimes B')$. Since the right hand side is equal to $N \otimes B'$, the following is a left B'-colinear inclusion

$$\mathrm{id}_N \otimes \pi_{B'} : \mathrm{res}_B^H(N) \square_B A \hookrightarrow N \otimes B'.$$

By the assumption (4.2), the image of above map lies in $N \square_H B'$. This completes the proof. □

For $\lambda \in \Lambda$, we let

$$H^0(\lambda) := \mathrm{ind}_B^A\big(\mathrm{res}_B^H(\mathfrak{u}(\lambda))\big). \qquad \text{(II.4.2.3)}$$

An element of $H^0(\lambda)$ is of the form $\sum_j c^j \otimes a^j$ ($\in \mathfrak{u}(\lambda) \otimes A$) satisfying the following equation.

$$\sum_j c_0^j \otimes i_H(c_1^j) \otimes a^j = \sum_j c^j \otimes \pi_B(a_1^j) \otimes a_2^j = 0,$$

where $\mathfrak{u}(\lambda) \to \mathfrak{u}(\lambda) \otimes H;\ c^j \mapsto c_0^j \otimes c_1^j$ is the right H-supercomodule structure of $\mathfrak{u}(\lambda)$. Since functors ind and res preserve the parity, we have

$$\Pi H^0(\lambda) = \mathrm{ind}_B^A\big(\,\mathrm{res}_B^H(\Pi\mathfrak{u}(\lambda))\big). \tag{II.4.2.4}$$

Let

$$\Lambda^\dagger \;:=\; \left\{\lambda \in \Lambda \mid H^0(\lambda) \neq 0\right\}.$$

Lemma 4.2.6. *For $\lambda \in \Lambda^\dagger$, we have* $\mathrm{soc}_{B'}(H^0(\lambda)) = \mathrm{res}_{B'}^H(\mathfrak{u}(\lambda))$.

Proof. By Lemma 4.2.5, there is a B'-supercomodule inclusion

$$\mathrm{res}_{B'}^A(H^0(\lambda)) \;\hookrightarrow\; \mathrm{ind}_H^{B'}(\mathfrak{u}(\lambda)).$$

Thus, we have $\mathrm{soc}_{B'}(H^0(\lambda)) \subseteq \mathrm{soc}_{B'}\big(\mathrm{ind}_H^{B'}(\mathfrak{u}(\lambda))\big)$. To conclude the proof, it is enough to see that

$$\mathrm{soc}_{B'}\big(\mathrm{ind}_H^{B'}(\mathfrak{u}(\lambda))\big) = \mathrm{res}_{B'}^H(\mathfrak{u}(\lambda)). \tag{II.4.2.5}$$

By Lemma 4.2.4, any simple B'-subsupercomodule of $\mathrm{ind}_H^{B'}(\mathfrak{u}(\lambda))$ is ether (i) $\mathrm{res}_{B'}^H(\mathfrak{u}(\mu))$ or (ii) $\Pi\,\mathrm{res}_{B'}^H(\mathfrak{u}(\mu))$ for some $\mu \in \Lambda$. First, we consider the case (i). In this case, we have

$$\mathrm{Hom}^{B'}\big(\,\mathrm{res}_{B'}^H(\mathfrak{u}(\mu)),\, \mathrm{ind}_H^{B'}(\mathfrak{u}(\lambda))\big) \;\simeq\; \mathrm{Hom}^H\big(\mathfrak{u}(\mu),\, \mathfrak{u}(\lambda)\big),$$

by the Frobenius reciprocity. Since the left hand side is non-zero, there is a non-zero H-supercomodule map $\mathfrak{u}(\mu) \to \mathfrak{u}(\lambda)$. Therefore, μ must coincide with λ. Hence, the equation (II.4.2.5) holds. Next, we consider the case (ii). Similarly, by the Frobenius reciprocity, we have $\Pi\mathfrak{u}(\mu) \cong \mathfrak{u}(\lambda)$. Therefore, we conclude that $\mu = \lambda$ and $\mathfrak{u}(\lambda)$ is of type $\mathtt{Q}$. This proves the equation (II.4.2.5) holds. □

Proposition 4.2.7. *For $\lambda \in \Lambda^\dagger$, we have $H^0(\lambda)^{\mathrm{co}U'} = \mathfrak{u}(\lambda)$.*

Proof. We have an inclusion $H^0(\lambda) \hookrightarrow \mathfrak{u}(\lambda) \,\square_H\, B'$ of right B'-supercomodules, by Lemma 4.2.5. By taking the functor $- \,\square_{B'}\, H$ to both sides, we have

$$H^0(\lambda)^{\mathrm{co}U'} \hookrightarrow (\mathfrak{u}(\lambda) \,\square_H\, B') \,\square_{B'}\, H.$$

The right hand side coincides with $\mathfrak{u}(\lambda)$. Since $\mathfrak{u}(\lambda)$ is simple, this proves the claim. □

For $\lambda \in \Lambda^\dagger$, we let

$$L(\lambda) \;:=\; \mathrm{soc}_A(H^0(\lambda)). \tag{II.4.2.6}$$

Lemma 4.2.8. *For $\lambda \in \Lambda^\dagger$, $L(\lambda)$ is a unique simple A-subsupercomodule of $H^0(\lambda)$.*

Proof. If there are two simple A-subsupercomodules L and L' of $H^0(\lambda)$, then $\mathrm{soc}_{B'}(L), \mathrm{soc}_{B'}(L') = \mathrm{res}_{B'}^H(\mathfrak{u}(\lambda))$, by Lemma 4.2.6. Therefore, we get $\mathfrak{u}(\lambda) \subseteq L \cap L'$ $(\subseteq L,\, L')$. Since $\mathfrak{u}(\lambda) \neq 0$, we have $L = L'$. □

Theorem 4.2.9. *For any simple A-supercomodule L, there uniquely exists $\lambda \in \Lambda^\dagger$ such that L coincides with ether $L(\lambda)$ or $\Pi L(\lambda)$.*

Proof. For simplicity, assume that L is of type $\mathtt{Q}$, i.e., $L = \Pi L$. Since L is non-zero, there is a right B-supercomodule surjection $\mathrm{res}^A_B(L) \twoheadrightarrow \mathrm{res}^H_B(\mathfrak{u}(\lambda))$. By the Frobenius reciprocity, we have

$$0 \neq \mathrm{Hom}^B\big(\,\mathrm{res}^A_B(L), \mathrm{res}^H_B(\mathfrak{u}(\lambda))\big) \,\simeq\, \mathrm{Hom}^A\big(L,\, H^0(\lambda)\big). \tag{II.4.2.7}$$

Thus, there is an A-supercomodule inclusion $L \hookrightarrow H^0(\lambda)$. In particular, λ is indeed in $\Lambda^\dagger$. Therefore, we conclude that $L = L(\lambda)$, by Lemma 4.2.8. Similarly, we can show $L = \Pi L(\lambda)$ for the case (ii). ☐

By Theorem 4.2.9, we get a map

$$\Lambda^\dagger \longrightarrow \mathsf{Simp}_\Pi(A); \quad \lambda \longmapsto L(\lambda).$$

Corollary 4.2.10. *The above map gives a one-to-one correspondence between $\Lambda^\dagger$ and $\mathsf{Simp}_\Pi(A)$. Moreover, $L(\lambda)$ is of type* $\mathtt{M}$ *(resp.* $\mathtt{Q}$*) if and only if $\mathfrak{u}(\lambda)$ is of type* $\mathtt{M}$ *(resp.* $\mathtt{Q}$*).*

Proof. By (II.4.2.4) and $\Pi L(\lambda) = \mathrm{soc}_A(\Pi H^0(\lambda))$, we have $\Pi L(\lambda) \cong L(\lambda)$ if and only if $\mathfrak{u}(\lambda)$ is of $\mathtt{Q}$-type. Therefore, $L(\lambda)$ has the same type as $\mathfrak{u}(\lambda)$. ☐

Chapter 5

Irreducible Representations

Let $\Bbbk$ be a field of characteristic not equal to 2. We let $\mathbf{G}$, $\mathbf{T}$, $\mathbf{U}^{\pm}$ and $\mathbf{B}^{\pm}$ denote the algebraic supergroups over $\Bbbk$, defined in Section 3.2 and Section 3.4.

5.1 Irreducible representations of T

In this section, we construct irreducible representations of $\mathbf{T}$. For simplicity, we let $\mathsf{Irr}_{\Pi}(\mathbf{T}) := \mathsf{Simp}_{\Pi}(\mathcal{O}(\mathbf{T}))$, for the notation $\mathsf{Simp}_{\Pi}(-)$, see Section 4.1.

Recall that, $\mathrm{hy}(\mathbf{T})$ is the hyper-superalgebra of the supergroup $\mathbf{T}$. For a fixed $\lambda \in \Lambda$, let $\mathrm{hy}(\mathbf{T})_{\lambda}$ be the quotient superalgebra of $\mathrm{hy}(\mathbf{T})$ by the two-sided ideal generated by all

$$\binom{H}{m} - \binom{\langle \lambda, H\rangle}{m},$$

where $H \in (\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}$, $m \geqslant 0$. Recall that $r = \mathrm{rank}(\mathfrak{h}_{\mathbb{Z}})_{\bar{1}}$.

Lemma 5.1.1. $\mathrm{hy}(\mathbf{T})_{\lambda}$ *is a* 2^r*-dimensional space.*

Proof. Let $H_1, \ldots, H_{\ell}$ be a $\mathbb{Z}$-basis of $(\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}$ and let $K_1, \ldots, K_r$ be a $\mathbb{Z}$-basis of $(\mathfrak{h}_{\mathbb{Z}})_{\bar{1}}$. Then by Theorem 3.2.4, $\mathrm{hy}(\mathbf{T})$ has a $\Bbbk$-basis

$$\prod_{i=1}^{\ell} \binom{H_i}{m_i} \prod_{t=1}^{r} K_t^{\epsilon_t},$$

where $m_i \geqslant 0$, $\epsilon_t = 0$ or 1 for $i = 1, \ldots, \ell$, $t = 1, \ldots, r$. Therefore, $K_1^{\epsilon_1} \cdots K_r^{\epsilon_r}$ ($\epsilon_t = 0$ or 1 for $t = 1, \ldots, r$) form a $\Bbbk$-basis of $\mathrm{hy}(\mathbf{T})_{\lambda}$. □

We define a symmetric $\Bbbk$-bilinear form by

$$b_{\lambda} : \mathfrak{h}_{\bar{1}} \times \mathfrak{h}_{\bar{1}} \to \Bbbk\,; \quad b_{\lambda}(x, y) := \lambda([x, y]).$$

As in the previous section, we can construct the Clifford superalgebra $C(\mathfrak{h}_{\bar{1}}, b_{\lambda})$ over $\Bbbk$ for the quadratic space $(\mathfrak{h}_{\bar{1}}, b_{\lambda})$.

Note that $P(\mathrm{hy}(\mathbf{T})) = \mathfrak{h}$, by Proposition 2.2.3. We may regard $\mathfrak{h}_{\bar{1}}$ as a subspace of $\mathrm{hy}(\mathbf{T})$. By using this inclusion, we have the following algebra map.

$$f : T(\mathfrak{h}_{\bar{1}}) \longrightarrow \mathrm{hy}(\mathbf{T}) \twoheadrightarrow \mathrm{hy}(\mathbf{T})_\lambda.$$

This is indeed a surjective map.

Lemma 5.1.2. *The above map f induces a superalgebra isomorphism $C(\mathfrak{h}_{\bar{1}}, b_\lambda) \cong \mathrm{hy}(\mathbf{T})_\lambda$.*

Proof. By definition, we have $I(\mathfrak{h}_{\bar{1}}, b_\lambda) \subseteq \operatorname{Ker} f$. Hence f induces the following superalgebra surjection

$$C(\mathfrak{h}_{\bar{1}}, b_\lambda) \twoheadrightarrow \mathrm{hy}(\mathbf{T})_\lambda.$$

On the other hand, the dimension of $C(\mathfrak{h}_{\bar{1}}, b_\lambda)$ and $\mathrm{hy}(\mathbf{T})_\lambda$ are the same, by (III.2.0.2) and Lemma 5.1.1. Therefore, f is an isomorphism. ☐

The bilinear form b_λ on $\mathfrak{h}_{\bar{1}}$ can be extended to a bilinear form on the quotient space $\mathfrak{h}_{\bar{1}}/\mathrm{rad}(b_\lambda)$ which we denote the same symbol b_λ. Set

$$d_\lambda := \dim_{\Bbbk}(\mathfrak{h}_{\bar{1}}/\mathrm{rad}(b_\lambda)).$$

There exists a d_λ-dimensional subspace W of $\mathfrak{h}_{\bar{1}}$ such that $\mathfrak{h}_{\bar{1}} = \mathrm{rad}(b_\lambda) \perp W$. Let $x_1, x_2, \ldots, x_r$ be an orthogonal basis of $\mathfrak{h}_{\bar{1}}$ with respect to b_λ such that $x_1, x_2, \ldots, x_{d_\lambda}$ is a basis of W. Set

$$\delta_\lambda := (-1)^{d_\lambda(d_\lambda+1)/2}\, \lambda([x_1, x_1])\lambda([x_2, x_2]) \cdots \lambda([x_{d_\lambda}, x_{d_\lambda}]).$$

We let $\delta_\lambda := 0$, if $b_\lambda = 0$, this is equivalent to saying that $d_\lambda = 0$. For simplicity, we treat the integer 0 as an even number.

Proposition 5.1.3. *For each $\lambda \in \Lambda$, the superalgebra $\mathrm{hy}(\mathbf{T})_\lambda$ has a unique simple supermodule $\mathfrak{u}(\lambda)$ up to isomorphism and parity change. Moreover, $\mathfrak{u}(\lambda)$ is of type* `M` *if and only if d_λ is even and $\delta_\lambda \in \Bbbk^2$.*

Proof. First, suppose that $\delta_\lambda = 0$. By definition, $\mathrm{hy}(\mathbf{T})_\lambda = \wedge(r)$ as superalgebras. The Grassmann superalgebra $\wedge(r)$ has a unique one-dimensional purely even or odd supermodule. This is of type `M`.

Next, suppose that $\delta_\lambda \in \Bbbk^\times$. Let R_λ be the two-sided ideal of $\mathrm{hy}(\mathbf{T})_\lambda$ generated by $\mathrm{rad}(b_\lambda)$. By Lemma 5.1.2, we have

$$\mathrm{hy}(\mathbf{T})_\lambda/R_\lambda \cong C(\mathfrak{h}_{\bar{1}}/\mathrm{rad}(b_\lambda), b_\lambda). \qquad \text{(II.5.1.1)}$$

Therefore, $\mathrm{hy}(\mathbf{T})_\lambda/R_\lambda$ has a unique simple supermodule up to isomorphism and parity change, by Proposition B.0.4 Since the Jacobson radical of $\mathrm{hy}(\mathbf{T})_\lambda$ coincides with R_λ, we are done. ☐

Remark 5.1.4. If $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$, then $\mathrm{hy}(\mathbf{T})_\lambda = \Bbbk$. In this case, $\mathfrak{u}(\lambda) = \Bbbk_\lambda$ or $\mathfrak{u}(\lambda) = \Pi\Bbbk_\lambda$, where $\Bbbk_\lambda$ is the one-dimensional purely even left $\mathrm{hy}(T)$-supermodule of weight λ.

We regard the left $\mathrm{hy}(\mathbf{T})_\lambda$-supermodule $\mathfrak{u}(\lambda)$ as a left $\mathrm{hy}(\mathbf{T})$-supermodule via the quotient map $\mathrm{hy}(\mathbf{T}) \twoheadrightarrow \mathrm{hy}(\mathbf{T})_\lambda$. Actually, this is a locally finite left $\mathrm{hy}(\mathbf{T})$-T-supermodule.

Theorem 5.1.5. *For a simple locally finite left* $\mathrm{hy}(\mathbf{T})$*-*T*-supermodule* L*, there exists* $\lambda \in \Lambda$ *such that* $L \cong \mathfrak{u}(\lambda)$ *or* $\Pi\mathfrak{u}(\lambda)$*.*

Proof. Since L is a non-zero left $\mathrm{hy}(\mathbf{T})$-T-supermodule, there exists $\lambda \in \Lambda$ such that $L_\lambda \neq 0$. By the definition of $\mathfrak{h}$, we have

$$\binom{H}{n} K.v = K\binom{H}{n}.v = \binom{\langle \lambda, H\rangle}{n} K.v,$$

for $H \in (\mathfrak{h}_{\mathbb{Z}})_{\bar{0}}$, $n \geqslant 0$, $K \in \mathfrak{h}_{\bar{1}}$ and $v \in L_\lambda$. Therefore, L_λ is a left $\mathrm{hy}(\mathbf{T})_\lambda$-supermodule. Then by Lemma 5.1.3, L_λ contains $\mathfrak{u}(\lambda)$ (resp. $\Pi\mathfrak{u}(\lambda)$). We conclude that there is an inclusion $L \supseteq \mathfrak{u}(\lambda)$ (resp. $\Pi\mathfrak{u}(\lambda)$) of left $\mathrm{hy}(\mathbf{T})$-supermodules. Since L is simple, this completes the proof. □

There is a category equivalence between the category of locally finite left $\mathrm{hy}(\mathbf{T})$-T-supermodules and the category of $\mathbf{T}$-supermodules (i.e., right $\mathcal{O}(\mathbf{T})$-supercomodules), by Corollary 2.4.10. Therefore, we obtain the following well-defined map

$$\Lambda \longrightarrow \mathsf{Irr}_\Pi(\mathbf{T}); \quad \lambda \longmapsto \mathfrak{u}(\lambda).$$

Corollary 5.1.6. *The above map is bijective. Moreover,* $\mathfrak{u}(\lambda)$ *is of type* `M` *if and only if* d_λ *is even and* $\delta_\lambda \in \Bbbk^2$*.*

Remark 5.1.7. By using compositions of the canonical projections $\mathcal{O}(\mathbf{T}) \to \mathcal{O}(\mathbf{B}^+) \to \mathcal{O}(\mathbf{B}^+_{\mathrm{ev}})$, we may regard $\mathfrak{u}(\lambda)$ as an right $\mathcal{O}(\mathbf{B}^+_{\mathrm{ev}})$-comodule. Then $\mathfrak{u}(\lambda)$ coincides with the $\dim_\Bbbk \mathfrak{u}(\lambda)$ copies of $\Bbbk_\lambda$ as right $\mathcal{O}(\mathbf{B}^+_{\mathrm{ev}})$-comodules.

5.2 The case when the base field is algebraically closed

Suppose that $\Bbbk$ is an algebraically closed field. For a fixed $\lambda \in \Lambda$, we describe the simple supermodule $\mathfrak{u}(\lambda)$ more explicitly.

Let $\mathfrak{h}^\lambda_{\bar{1}}$ be a *maximal totally isotropic subspace* of $\mathfrak{h}_{\bar{1}}$ with respect to b_λ, i.e., a maximal subspace $\mathfrak{n}$ of $\mathfrak{h}_{\bar{1}}$ such that $b_\lambda(\mathfrak{n}, \mathfrak{n}) = 0$. Set

$$\mathfrak{h}^\lambda := \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}^\lambda_{\bar{1}}.$$

This is a Lie subsuperalgebra of $\mathfrak{h}$. It is obvious that the pair $(T, \mathfrak{h}^\lambda)$ is a sub-pair of $\mathbf{G}$. Let

$$\mathbf{T}^\lambda := \mathbf{G}(T, \mathfrak{h}^\lambda)$$

be the corresponding closed superalgebra of $\mathbf{G}$. This is indeed a closed subsupergroup of $\mathbf{T}$ and satisfies $\mathrm{Lie}(\mathbf{T}^\lambda) = \mathfrak{h}^\lambda$. By Part I, Lemma 2.2.4, we have

$$\mathrm{hy}(\mathbf{T}^\lambda) \cong \mathrm{hy}(T) \otimes \wedge(\mathfrak{h}^\lambda_{\bar{1}}).$$

We regard $\Bbbk_\lambda$ as a left $\mathrm{hy}(\mathbf{T}^\lambda)$-supermodule by letting $\mathfrak{h}^\lambda_{\bar{1}}\,\Bbbk_\lambda = 0$. As a subsuperspace of $\mathrm{hy}(\mathbf{T})$, we let

$$\mathrm{coind}^{\mathbf{T}}_{\mathbf{T}^\lambda}(\lambda) := \mathrm{hy}(\mathbf{T}) \otimes_{\mathrm{hy}(\mathbf{T}^\lambda)} \Bbbk_\lambda.$$

By definition, one sees that $\mathrm{coind}_{\mathbf{T}^\lambda}^{\mathbf{T}}(\lambda)$ is a left $\mathrm{hy}(\mathbf{T})_\lambda$-supermodule.

Note that, the dimension of the space $\mathrm{coind}_{\mathbf{T}^\lambda}^{\mathbf{T}}(\lambda)$ coincides with $2^{\dim_{\Bbbk}(\mathfrak{h}_{\bar{1}}/\mathfrak{h}_{\bar{1}}^{\lambda})}$. Since $\mathrm{rad}(b_\lambda) \subseteq \mathfrak{h}_{\bar{1}}^{\lambda}$, one sees that this is indeed a $\mathrm{hy}(\mathbf{T})$-T-supermodule. Therefore, we can regard $\mathrm{coind}_{\mathbf{T}^\lambda}^{\mathbf{T}}(\lambda)$ as a $\mathbf{T}$-supermodule.

Proposition 5.2.1. $\mathrm{coind}_{\mathbf{T}^\lambda}^{\mathbf{T}}(\lambda)$ *coincides with* $\mathfrak{u}(\lambda)$ *or* $\Pi\mathfrak{u}(\lambda)$*.*

Proof. Since $\mathrm{coind}_{\mathbf{T}^\lambda}^{\mathbf{T}}(\lambda) \neq 0$, there is an inclusion $\mathrm{coind}_{\mathbf{T}^\lambda}^{\mathbf{T}}(\lambda) \supseteq \mathfrak{u}(\lambda)$ or $\Pi\mathfrak{u}(\lambda)$ of $\mathbf{T}$-supermodules, by Corollary 5.1.6. We will show that the dimension of both sides are the same. Equivalently, we will show the following equation

$$\dim_{\Bbbk} \mathfrak{h}_{\bar{1}} - \dim_{\Bbbk} \mathfrak{h}_{\bar{1}}^{\lambda} \;=\; \lfloor (d_\lambda + 1)/2 \rfloor \,, \qquad \text{(II.5.2.1)}$$

see Remark B.0.5.

Let $m := \lfloor d_\lambda/2 \rfloor$ and let $V := \mathfrak{h}_{\bar{1}}/\mathrm{rad}(b_\lambda)$. Since $\Bbbk$ is an algebraically closed field and (V, b_λ) is a non-degenerate quadratic space, we can choose $2m$ vectors $x_1, \dots, x_m, y_1, \dots, y_m \in V$ so that

$$b_\lambda(x_i, x_j) = \delta_{i,j},\; b_\lambda(y_i, y_j) = -\delta_{i,j},\; b_\lambda(x_i, y_j) = 0, \quad 1 \leqslant i, j \leqslant m,$$

where $\delta_{i,j}$ is the Kronecker delta. For each i, we define the 2-dimensional subspace $\mathbb{H}_i$ of V spanned by x_i and y_i. The space $\mathbb{H}_i$ is called a *hyperbolic plane*. Then we have an orthogonal (Witt) decomposition $V = \mathbb{H}_1 \perp \cdots \perp \mathbb{H}_m \perp V_a$, where V_a satisfies $\dim_{\Bbbk} V_a = 0$ if d_λ is even, $\dim_{\Bbbk} V_a = 1$ otherwise. This m is called the *Witt index* of V. In general, it is known that the Witt index of V coincides with the dimension of maximal totally isotropic subspace of V, see [21, I Corollary 4.4], for example. Thus, we have the following equation

$$\dim_{\Bbbk} \mathfrak{h}_{\bar{1}}^{\lambda} - \dim_{\Bbbk} \mathrm{rad}(b_\lambda) \;=\; \lfloor d_\lambda/2 \rfloor \,.$$

One can easily see that this equation implies the equation (II.5.2.1). □

Remark 5.2.2. In addition, if $\Bbbk$ is characteristic zero, then our construction of simples of $\mathbf{T}$ is the same as Serganova's [33, §9].

5.3 Irreducible representations of G

For simplicity, we let $A := \mathcal{O}(\mathbf{G})$, $B := \mathcal{O}(\mathbf{B}^-)$, and $H := \mathcal{O}(\mathbf{T})$, as before.

Just as Chapter 4, we define

$$H^0(\lambda) \;:=\; \mathrm{ind}_B^A\big(\,\mathrm{res}_B^H(\mathfrak{u}(\lambda))\big),$$

and

$$L(\lambda) := \mathrm{soc}_A(H^0(\lambda)),$$

for each $\lambda \in \Lambda$. Let

$$\Lambda^\dagger := \{\lambda \in \Lambda \mid H^0(\lambda) \neq 0\}.$$

For simplicity, set $\mathsf{Irr}_\Pi(\mathbf{G}) := \mathsf{Simp}_\Pi(A)$.

Theorem 5.3.1. *The map*

$$\Lambda^\dagger \longrightarrow \mathsf{Irr}_\Pi(\mathbf{G}); \quad \lambda \longmapsto L(\lambda)$$

is well-defined and bijective. Moreover, $L(\lambda)$ is of type $\mathtt{M}$ *if and only if d_λ is even and $\delta_\lambda \in \Bbbk^2$.*

Proof. By Proposition 3.4.4, Proposition 3.4.6, Proposition 3.4.7 and Theorem 5.1.6, we can apply Corollary 4.2.10. □

Example 5.3.2. We determine the type of simple supermodules for some algebraic supergroups treated in Example 3.2.3.

(1) If $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$, then all irreducible $\mathbf{G}$ representations are of type $\mathtt{M}$. For example, the following algebraic supergroups satisfy this condition

  (a) the general linear supergroup $\mathbf{GL}(m|n)$, and

  (b) the Chevalley supergroups $\mathbf{G}$ of classical type such that its Lie superalgebra Lie($\mathbf{G}$) is different from the strange Lie superalgebra $Q(n)$ of type II, see Appendix A.

(2) The case of the algebraic supergroup $\mathbf{G} = \mathbf{Q}(n)$, see Part I, Example 3.1.5. Note that, $\Lambda \cong \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ as $\mathbb{Z}$-modules. Assume that $p :=$ char $\Bbbk > 2$. As in [3, p.13], for $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i \in \Lambda$, we define

$$h_{p'}(\lambda) := \#\{i \mid 1 \leqslant i \leqslant n,\ p \nmid \lambda_i\}.$$

Then one sees that $d_\lambda = h_{p'}(\lambda)$.

Recall that Δ^+ is the set of positive roots of $\mathfrak{g}$, see (II.3.1.1). For $\mu, \lambda \in \Lambda$, we define a partial order on Λ as follows

$$\mu \leqslant \lambda :\Longleftrightarrow \lambda - \mu \in \sum_{\alpha\in\Delta^+} \mathbb{N}\alpha,$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$.

Let W be the Weyl group of $\mathbf{G}_{\mathrm{ev}}$. As in the non-super case, for a weight supermodule $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, the Weyl group W acts on each weight superspace of M as follows

$$wM_\lambda = M_{w\lambda},$$

where $w \in W$ and $\lambda \in \Lambda$.

For $\lambda \in \Lambda^\dagger$, we have

$$H^0(\lambda)^{\mathbf{U}^+} \hookrightarrow H^0(\lambda)_\lambda, \tag{II.5.3.1}$$

where $H^0(\lambda)^{\mathbf{U}^+}$ is the $\mathbf{U}^+$-fixed points of $H^0(\lambda)$, see (II.2.1.1).

Proposition 5.3.3. *For $\lambda \in \Lambda^\dagger$, λ is a maximal weight of $H^0(\lambda)$ with respect to $\leqslant$ and $H^0(\lambda)_\lambda = \mathfrak{u}(\lambda)$.*

Proof. If μ is a maximal weight of $H^0(\lambda)$, then $H^0(\lambda)_\mu$ is included in the $\mathbf{U}^+$-invariant space $H^0(\lambda)^{\mathbf{U}^+}$ of $H^0(\lambda)$. By Proposition 4.2.7, we have $H^0(\lambda)_\mu \subseteq \mathfrak{u}(\lambda)$. By (II.5.3.1) and by considering weight space decomposition of $H^0(\lambda)$, we can conclude that $\mu = \lambda$. □

5.4 Induced representations

For simplicity, set $\bar{A} := \mathcal{O}(\mathbf{G}_{\mathrm{ev}})$, $\bar{B} := \mathcal{O}(\mathbf{B}^-_{\mathrm{ev}})$ and $\bar{H} := \mathcal{O}(\mathbf{T}_{\mathrm{ev}})$, as before. For $I_A := (A_{\bar{1}})$, the chain

$$A = I_A^0 \supseteq I_A^1 \supseteq I_A^2 \supseteq \cdots$$

of superideals defines the graded algebra $A^{\mathrm{gr}} := \bigoplus_{n \geqslant 0} I_A^n / I_A^{n+1}$. We regard A as a right $\bar{A}$-comodule, by the coadjoint action $\mathrm{coad}(a) := a_2 \otimes \mathcal{S}(\bar{a}_1)\bar{a}_3$ for $a \in A$, where $\bar{a}$ is the canonical image of $a \in A$. This induces a right $\bar{A}$-comodule structure on W^A. Therefore, we can construct the cosmash product

$$\bar{A} \blacktriangleright\!\!< \wedge(W^A)$$

of $\bar{A}$ and $\wedge(W^A)$. By [22, Prposition 4.9(2)], we have an isomorphism $A^{\mathrm{gr}} \xrightarrow{\cong} \bar{A} \blacktriangleright\!\!< \wedge(W^A)$ of graded Hopf algebras. Since $\bar{H}$ is cosemisimple, we see that A^{gr} coincides with A as right $\bar{H}$-comodules. Therefore, we have an isomorphism

$$A \xrightarrow{\cong} \bar{A} \blacktriangleright\!\!< \wedge(W^A) \tag{II.5.4.1}$$

of right $\bar{H}$-comodules.

Proposition 5.4.1. *For a right B-supercomodule V, there is an inclusion*

$$\mathrm{ind}_B^A(V) \hookrightarrow \mathrm{ind}_{\bar{B}}^{\bar{A}}(\mathrm{res}_{\bar{B}}^B(V)) \otimes \wedge(W^A)$$

of right $\bar{H}$-comodules.

Proof. Taking the functor $V \,\square_{\bar{B}}\, -$ to both sides in (II.5.4.1), we get

$$V \,\square_{\bar{B}}\, A \xrightarrow{\cong} (V \,\square_{\bar{B}}\, \bar{A}) \otimes \wedge(W^A).$$

On the other hand, we have $V \,\square_B\, A \hookrightarrow V \,\square_{\bar{B}}\, A$, by definition of cotensor. This completes the proof. □

Corollary 5.4.2. *For a finite dimensional right B-supercomodule V, $\mathrm{ind}_B^A(V)$ is finite. In particular, $H^0(\lambda)$ is finite.*

Proof. Since A is finitely generated, W^A is finite dimensional, see [22, Proposition 4.4]. On the other hand, it is known that $\mathrm{ind}_{\bar{B}}^{\bar{A}}(V)$ is finite, see [16, Part I, 5.12(c)]. The claim follows immediately from the above Proposition 5.4.1. □

For $\lambda \in \Lambda$, we regard $\Bbbk_\lambda$ as the trivial one-dimensional right $\bar{B}$-comodule through λ. We define $H^0_{\mathrm{ev}}(\lambda) := \mathrm{ind}_{\bar{B}}^{\bar{A}}(\Bbbk_\lambda)$. Let

$$\Lambda^+ := \left\{ \lambda \in \Lambda \mid H^0_{\mathrm{ev}}(\lambda) \neq 0 \right\}.$$

By Remark 5.1.7, we have $\mathrm{res}_{\bar{B}}^B(\mathfrak{u}(\lambda))$ coincides with the $n_\lambda := \dim_\Bbbk \mathfrak{u}(\lambda)$ copies of $\Bbbk_\lambda$. Therefore, we have the following result.

Corollary 5.4.3. *There is a right $\bar{H}$-colinear inclusion*

$$H^0(\lambda) \hookrightarrow H^0_{\mathrm{ev}}(\lambda)^{\oplus n_\lambda} \otimes \wedge(W^A).$$

In particular, we have an inclusion $\Lambda^\dagger \subseteq \Lambda^+$.

Remark 5.4.4. By the well-known fact, the elements in Λ^+ can be write down in terms of the root data of $\mathfrak{g}_{\bar{0}}$, i.e.,

$$\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geqslant 0 \text{ for all } \alpha \in \Delta^+_{\bar{0}}\},$$

see [16, p.178], for example. An element of Λ^+ is so called a *dominant weight*.

Example 5.4.5. Let $\mathfrak{g}$ be a classical simple Lie superalgebra of type B, C or D, see Appendix A. Suppose that $G_{\mathbb{Z}}$ is a Chevalley-Demazure group of *universal type* associated to $\mathfrak{g}_{\bar{0}}$. Then we can construct a Chevalley supergroup $\mathbf{G}$ over $\Bbbk$ such as in Example 3.2.3 (1). It is known that the elements in $\Lambda^\dagger$ for $\mathbf{G}$ are described in terms of combinatorial language. If char $\Bbbk > 2$, then Shu and Wang determined the explicit form of $\Lambda^\dagger$, see [34, Proposition 5.1 and Theorem 5.3].

Example 5.4.6. Let $\mathfrak{g} = \mathfrak{q}(n)$ be the queer superalgebra. As in Example 3.2.3 (3), we can construct an algebraic supergroup $\mathbf{G} = \mathbf{Q}(n)$ over $\Bbbk$. Brundan and Kleshchev determined $\Lambda^\dagger$ for $\mathbf{Q}(n)$ in positive characteristic, see [3, Theorem 6.11]. Combined with the known result in char $\Bbbk = 0$, we have

$$\Lambda^\dagger = \{\lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \in \Lambda^+ \mid \lambda_i = \lambda_{i+1} \Longrightarrow p' \mid \lambda_i\},$$

where $p' :=$ char $\Bbbk$. If $p' = 0$, then $p' \mid \lambda_i$ means that $\lambda_i = 0$.

Chapter 6

Quasireductive Supergroups having a Distinguished Parabolic Subsupergroup

Let $\Bbbk$ be a field of characteristic not equal to 2.

For a general linear supergroup $\mathbf{GL}(m|n)$, Zubkov [42] showed a super-analogue of the Kempf vanishing theorem. It was essential that the existence of a distinguished parabolic subsupergroup of $\mathbf{GL}(m|n)$ to his proof. In this chapter, we generalize his result and show a super-analogue of the Kempf vanishing theorem, and classify the irreducible representations of a quasireductive supergroup having a distinguished parabolic subsupergroup. As an application, we calculate the character of $H^0(\lambda)$ for $\lambda \in \Lambda^+$.

6.1 A version of splitting property

Before we start our main discussion, let us show the motivation with an example, see [42, Proposition 5.1]

Example 6.1.1. Set $\mathbf{G} := \mathbf{GL}(m|n)$. We define a closed subsupergroup $\mathbf{P}^-$ of $\mathbf{G}$ as follows

$$\mathbf{P}^-(R) := \{ \left(\begin{array}{c|c} A & 0 \\ \hline C & D \end{array} \right) \in \mathbf{G}(R) \}$$

for a commutative superalgebra R. Note that, $\mathbf{G}_{\text{ev}} = \mathbf{P}^-_{\text{ev}}$. Recall that the corresponding Hopf superalgebra $\mathcal{O}(\mathbf{G})$ is given by $A(m|n)_d$, where $A(m|n)$ is generated by the elements $\{x_{ij}\}_{1 \leqslant i,j \leqslant m+n}$. For the notation, see Part I, Example 3.1.4. Set $W^+ := \bigoplus_{1 \leqslant i \leqslant m < \ell} \Bbbk x_{i\ell}$. We may regard $\mathrm{SSp}\,(\wedge(W^+))$ as a subsupergroup of $\mathbf{G}$. In [42, Proposition 5.1, Remark 5.1], Zubkov showed that the multiplication map

$$\mathbf{P}^-(R) \times \mathrm{SSp}\,(\wedge(W^+))(R) \longrightarrow \mathbf{G}(R); \quad (\left(\begin{array}{c|c} A & 0 \\ \hline C & D \end{array} \right), \left(\begin{array}{c|c} 1 & B \\ \hline 0 & 1 \end{array} \right)) \longmapsto \left(\begin{array}{c|c} A & AB \\ \hline C & CB + D \end{array} \right) \tag{II.6.1.1}$$

is an isomorphism. Indeed, the inverse is given by

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \longmapsto (\left(\begin{array}{c|c} A & 0 \\ \hline C & D - CA^{-1}B \end{array}\right), \left(\begin{array}{c|c} 1 & A^{-1}B \\ \hline 0 & 1 \end{array}\right)).$$

We regard $\mathbf{P}^- \times \mathrm{SSp}\,(\wedge(W^+))$ as a right $\mathbf{P}^-$-supermodule by

$$m_{\mathbf{P}^-} \times \mathrm{id} : \mathbf{P}^- \times \mathbf{P}^- \times \mathrm{SSp}\,(\wedge(W^+)) \longrightarrow \mathbf{P}^- \times \mathrm{SSp}\,(\wedge(W^+)),$$

where $m_{\mathbf{P}^-}$ is the multiplication of $\mathbf{P}^-$. Then the isomorphism (II.6.1.1) is a left $\mathbf{P}^-$-supermodule map. This means that the corresponding superalgebra isomorphism

$$\mathcal{O}(\mathbf{G}) \xrightarrow{\cong} \mathcal{O}(\mathbf{P}^-) \otimes \wedge(W^+).$$

is left $\mathcal{O}(\mathbf{P}^-)$-colinear.

We want to generalize the above example.

In general, let A be a finitely generated commutative Hopf superalgebra over $\Bbbk$, and let P be a quotient Hopf superalgebra of A with the canonical projection $\pi : A \twoheadrightarrow \mathcal{P}$. We regard A as a left P-supercomodule by using π, as in (I.2.6.1). We define ${}^{\mathrm{co}\,P}A := \Bbbk \,\square_P\, A$ a left version of (II.2.1.1). This is a coideal subsuperalgebra of A.

Theorem 6.1.2. *Keep the notation as above. If $\bar{A} = \bar{P}$, then there is a left P-supercomodule algebra isomorphism $A \xrightarrow{\cong} P \otimes {}^{\mathrm{co}\,P}A$.*

Proof. Let $B := {}^{\mathrm{co}\,P}A$. This B is a Hopf subsuperalgebra of A and A is faithfully flat B-supermodule, see [22, Corollary 5.5]. Equivalently, any non-zero left A- right B- Hopf supermodule is a projective generator of the category of left B-modules. In this case, P coincides with $A/\!/B := A/AB^+$, where $B^+ := \mathrm{Ker}(\varepsilon : B \to \Bbbk)$. In particular A is a projective B-module. Hence, there exists B-supermodule map $\phi : A \to B$ such that the following diagram commutes.

$$\begin{array}{ccccc} P & \xrightarrow[\cong]{\mathrm{id}\otimes 1} & P \otimes B/B^+ & \xrightarrow{\varepsilon_P \otimes \mathrm{id}} & B/B^+ \\ {\scriptstyle\pi}\uparrow & & \circlearrowleft & & \uparrow \\ A & & \overset{}{\underset{\phi}{- - - - - - - - \dashrightarrow}} & & B, \end{array}$$

where the right vertical arrow is the canonical projection. Let

$$\psi : A \longrightarrow P \otimes B\,; \quad a \longmapsto \pi(a_1) \otimes \phi(a_2).$$

This ψ is left P-colinear and right B-linear.

Now we consider the quotient map

$$\overline{\psi} : A/\!/B \longrightarrow (P \otimes B)/\!/B.$$

Since $(P \otimes B)//B = (P \otimes B) \otimes_B B/B^+ = P \otimes B/B^+$, $\overline{\psi}$ is an isomorphism, by the definition of ϕ. By the assumption, we have ${}^{\mathrm{co}\,\bar{P}}A = {}^{\mathrm{co}\,\bar{A}}A$. Note that, ${}^{\mathrm{co}\,\bar{A}}A = \wedge(W^A)$. Hence, B^+ is a nilpotent ideal of B. Since the right B-module $P \otimes B$ is projective, this is indeed a flat B-module. Therefore, we can conclude that ψ is an isomorphism. $\square$

6.2 The Kempf vanishing theorem

Let $\mathfrak{g}$, $\mathfrak{b}^{\pm}$, $\mathbf{G}$, $\mathbf{B}^{\pm}$, ... as in Chapter 5.

Let $\mathfrak{p}^{\pm}$ be the proper subsuperspaces of $\mathfrak{g}$ such that

$$\mathfrak{p}^{\pm}_{\bar{0}} = \mathfrak{g}_{\bar{0}}, \qquad \mathfrak{p}^{\pm}_{\bar{1}} = \mathfrak{b}^{\pm}_{\bar{1}}.$$

In this section, we assume that $\mathfrak{p}^{\pm}$ form Lie subsuperalgebras of $\mathfrak{g}$. In other words, we assume that $\mathfrak{b}^{\pm}_{\bar{1}}$ is $\mathfrak{g}_{\bar{0}}$-stable under the adjoint action.

One sees that the pair $(\mathbf{G}_{\mathrm{ev}}, \mathfrak{p}^{\pm})$ are sub-pairs of $(\mathbf{G}_{\mathrm{ev}}, \mathfrak{g})$. Thus we can define two closed subsupergroups $\mathbf{P}^{\pm} := \mathbf{G}(G, \mathfrak{p}^{\pm})$ of $\mathbf{G}$. For the notation, see (I.4.4.20). Note that, $\mathrm{Lie}(\mathbf{P}^{\pm}) = \mathfrak{p}^{\pm}$.

Example 6.2.1. We show some examples.

(1) If $\mathfrak{g}$ is a classical simple Lie superalgebra of *type I* (see Appendix A), then there is a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$ satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \ (i, j = -1, 0, 1), \quad \mathfrak{g}_{\bar{0}} = \mathfrak{g}_0, \text{ and } \quad \mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1.$$

Note that, $\mathfrak{g}_{-2}, \mathfrak{g}_2 := 0$. Then $\mathfrak{p}^+ := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie subsuperalgebra of $\mathfrak{g}$ and has a $\mathbb{Z}$-form $\mathfrak{p}^+_{\mathbb{Z}}$.

(2) For the case of $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathbf{G} = \mathbf{GL}(m|n)$, such $\mathfrak{p}^{\pm}$ do exist. If we take $\mathfrak{b}^-$ as the set of all lower triangle matrices, then $\mathbf{P}^-$ is nothing but the supergroup considered in Example 6.1.1.

(3) For $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{q}(n)$, $\mathfrak{b}^{\pm}_{\bar{1}}$ is not $\mathfrak{g}_{\bar{0}}$-stable. Thus, such $\mathbf{P}^{\pm}$ does *not* exist.

For simplicity, we let $A := \mathcal{O}(\mathbf{G})$, $B := \mathcal{O}(\mathbf{B}^-)$ and $P := \mathcal{O}(\mathbf{P}^-)$. Set $\bar{A} := \mathcal{O}(\mathbf{G}_{\mathrm{ev}})$ and $\bar{B} := \mathcal{O}(\mathbf{B}^-_{\mathrm{ev}})$, as before.

Let V be a right B-supercomodule. Note that, the map $\mathrm{id}_V \otimes \varepsilon : V \,\square_B\, P \to V$ is right $\bar{B}$-colinear where $\varepsilon : P \to \Bbbk$ is the counit of P. Then by Frobenius reciprocity, we have the following well-defined right $\bar{A}$-comodule map

$$\mathcal{N}_V : V \,\square_B\, P \longrightarrow V \,\square_{\bar{B}}\, \bar{A}; \quad v \otimes p \longmapsto v \otimes \bar{p}, \tag{II.6.2.1}$$

where $\bar{p}$ is the canonical image of $p \in P$.

The following is a generalization of Zubkov's result [42, Proposition 5.2].

Proposition 6.2.2. *The above* $\mathcal{N} : \mathrm{res}^P_{\bar{A}} \mathrm{ind}^P_B(-) \to \mathrm{ind}^{\bar{A}}_{\bar{B}} \mathrm{res}^B_{\bar{B}}(-)$ *is a natural equivalence.*

Proof. First, we prove that $\mathcal{N}_V$ is an isomorphism for $V = B$. As a right $\bar{B}$-supercomodule, B is isomorphic to $\wedge(\mathfrak{b}_{\bar{1}}^-)^* \otimes \bar{B}$, by the right version of Part I, Definition 3.1.1. Therefore, we have

$$\mathcal{N}_B : B \,\square_B\, P \longrightarrow B \,\square_{\bar{B}}\, \bar{A} \cong \wedge(\mathfrak{b}_{\bar{1}}^-)^* \otimes \bar{A}.$$

One sees that this map coincides with the tensor decomposition $P \cong \wedge(\mathfrak{b}_{\bar{1}}^+)^* \otimes \bar{A}$ which coincides the right version of Part I, Definition 3.1.1. Hence, $\mathcal{N}_B$ is an isomorphism.

Since (II.6.2.1) does not depend on the supersymmetry, we see that $\mathcal{N}_V$ is also an isomorphism for $V = \Pi B$. Finally, we show that $\mathcal{N}_V$ is an isomorphism for all B-supercomodule V. It is known that a B-supercomodule is injective if and only if it is a direct summand of a direct sum of some copies of B and ΠB, see [42, Proposition 3.1]. Thus $\mathcal{N}_V$ is an isomorphism for any injective B-supercomodule V. By [32, Lemma 8.4.5], this shows that $\mathcal{N}$ is natural equivalence. ☐

Since the functor $\mathrm{ind}_B^A(-) : \mathsf{SMod}^B \to \mathsf{SMod}^A$ is left exact, we have its right derived functor $R^n\mathrm{ind}_B^A(-)$ for $n = 0, 1, 2, \dots$.

Theorem 6.2.3. *For a right B-supercomodule V, there is an isomorphism of superspaces*

$$R^n\mathrm{ind}_B^A(V) \cong R^n\mathrm{ind}_{\bar{B}}^{\bar{A}}(V) \otimes \wedge(\mathfrak{b}_{\bar{1}}^+)^*.$$

Proof. By [16, Part I, 4.1(2)] and Proposition 6.2.2, we have an isomorphism $R^n\mathrm{ind}_B^P(V) \cong R^n\mathrm{ind}_{\bar{B}}^{\bar{A}}(V)$ of right $\bar{A}$-comodules. Since $A\,(\cong P \otimes \wedge(\mathfrak{b}_{\bar{1}}^+)^*)$ is an injective object in the category of left P-supercomodules, the functor $\mathrm{ind}_P^A(-)$ is exact. Again by [16, Part I, 4.1(2)], we have $R^n\mathrm{ind}_B^A(V) \cong \mathrm{ind}_P^A R^n\mathrm{ind}_B^P(V)$ as right A-supercomodules. By Theorem 6.1.2, the right hand side coincides with $R^n\mathrm{ind}_B^P(V) \otimes \wedge(\mathfrak{b}_{\bar{1}}^+)^*$ as superspaces. Combine with the above result, we are done. ☐

In this case, we can classify the irreducible representations of $\mathbf{G}$.

Corollary 6.2.4. *We have $\Lambda^\dagger = \Lambda^+$.*

Example 6.2.5. For $\mathbf{G} = \mathbf{GL}(m|n)$, $\Lambda^\dagger = \Lambda^+$ is the set of all dominant weights of $\mathrm{GL}_m \times \mathrm{GL}_n$, i.e., $\Lambda \cong \bigoplus_{i=1}^{m+n} \mathbb{Z}\epsilon_i$ as a $\mathbb{Z}$-module and

$$\Lambda^\dagger = \{\sum_{i=1}^{m+n} \lambda_i\epsilon_i \in \Lambda \mid \lambda_1 \leqslant \cdots \leqslant \lambda_m,\ \lambda_{m+1} \leqslant \cdots \leqslant \lambda_{m+n}\}.$$

This fact was well-known.

As in [16, Part II 2.1], for $\lambda \in \Lambda$, we shall write

$$H^n(\lambda) := R^n\mathrm{ind}_B^A(\mathfrak{u}(\lambda)), \quad H^n_{\mathrm{ev}}(\lambda) := R^n\mathrm{ind}_{\bar{B}}^{\bar{A}}(\Bbbk_\lambda).$$

By the well-known Kempf's vanishing theorem, we have $H^n_{\mathrm{ev}}(\lambda) = 0$ for $\lambda \in \Lambda^+$ and $n > 0$. Hence, we have the following a super-analogue of Kempf's Vanishing Theorem.

Corollary 6.2.6. *For $\lambda \in \Lambda^+$, we have $H^n(\lambda) = 0$ for all $n > 0$.*

6.3 Character formulas

Let $\mathbb{Z}\Lambda$ be the group algebra of Λ over $\mathbb{Z}$, and let $\{e^\lambda \mid \lambda \in \Lambda\}$ be the standard basis of $\mathbb{Z}\Lambda$. Note that $e^\lambda e^\mu = e^{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$. For a finite T-supermodule M, we let $\mathrm{ch}(M)$ denote the *formal character* of M. Explicitly,

$$\mathrm{ch}(M) := \sum_{\lambda \in \Lambda} \dim(M_\lambda)\, e^\lambda \quad \in \mathbb{Z}\Lambda.$$

Recall that W is the Weyl group of $\mathbf{G}_{\mathrm{ev}}$. Set

$$\rho_{\bar{0}} := \frac{1}{2} \sum_{\alpha \in \Delta_{\bar{0}}^+} \alpha, \quad \rho_{\bar{1}} := \frac{1}{2} \sum_{\gamma \in \Delta_{\bar{1}}^+} \gamma \quad \in \Lambda_{\mathbb{Q}},$$

where $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. For $\mu \in \Lambda_{\mathbb{Q}}$, we let

$$A(\mu) := \sum_{w \in W} \det(w) e^{w\lambda}.$$

By *Weyl's character formula* [16, Part II, 5.10], we have

$$\mathrm{ch}(H^0_{\mathrm{ev}}(\lambda)) = \frac{A(\lambda + \rho_{\bar{0}})}{A(\rho_{\bar{0}})} \tag{II.6.3.1}$$

for each $\lambda \in \Lambda^+$.

Proposition 6.3.1. *The formal character of $H^0(\lambda)$ for $\lambda \in \Lambda^+$ is given as follows.*

$$\mathrm{ch}(H^0(\lambda)) = \frac{A(\lambda + \rho_{\bar{0}})}{A(\rho_{\bar{0}})}\, e^{-\rho_{\bar{1}}} \prod_{\gamma \in \Delta_{\bar{1}}^+} (e^{\gamma/2} - e^{-\gamma/2}).$$

Proof. Since there is an isomorphism $H^0(\lambda) \cong H^0_{\mathrm{ev}}(\lambda) \otimes \wedge(\mathfrak{b}_{\bar{1}}^+)^*$ of T-modules, what we have to know is the formal character of $\wedge(\mathfrak{b}_{\bar{1}}^+)^*$. In general, $\mathrm{ch}(M^*) = \sum_\mu \dim(M_\mu)\, e^{-\mu}$. Thus we have $\mathrm{ch}(\wedge(\mathfrak{b}_{\bar{1}}^+)^*) = \prod_{\gamma \in \Delta_{\bar{1}}^+} (1 - e^{-\gamma})$. □

Appendix A

Simple Lie Superalgebras

In this appendix, we work over $\mathbb{C}$.

A.1 Definitions

A Lie superalgebra $\mathfrak{g}$ is said to be *simple* if $\mathfrak{g}$ has no non-trivial (homogeneous) ideal.

Definition A.1.1. Let $\mathfrak{g}$ be a simple Lie superalgebra. Then $\mathfrak{g}$ is said to be *classical* if the representation

$$\mathfrak{g}_{\bar{0}} \longrightarrow \mathrm{End}(\mathfrak{g}_{\bar{1}}); \quad x \longmapsto (y \mapsto [x, y]) \tag{III.1.1.1}$$

of the even part $\mathfrak{g}_{\bar{0}}$ on the odd part $\mathfrak{g}_{\bar{1}}$ is completely reducible, where $[-, -]$ is the super-bracket of $\mathfrak{g}$.

It is known that a simple Lie superalgebra $\mathfrak{g}$ is classical if and only if $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. We define a *type* of classical Lie superalgebra.

Definition A.1.2. A classical Lie superalgebra $\mathfrak{g}$ is of *type II* if $\mathfrak{g}_{\bar{1}}$ is irreducible $\mathfrak{g}_{\bar{0}}$-module with respect to the representation (III.1.1.1). If $\mathfrak{g}_{\bar{1}}$ is the direct sum of two irreducible $\mathfrak{g}_{\bar{0}}$-modules, then $\mathfrak{g}$ is said to be of *type I*.

Let $\mathfrak{g}$ be a Lie superalgebra. A bilinear form $b(-, -)$ on $\mathfrak{g}$ is said to be *invariant* if it satisfies $b([x, y], z) = b(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$.

Definition A.1.3. A classical Lie superalgebra $\mathfrak{g}$ is said to be *basic* if $\mathfrak{g}$ has a non-degenerate invariant bilinear form; otherwise, $\mathfrak{g}$ is said to be *strange*.

If $\mathfrak{g}$ is a basic Lie superalgebra of type I, then $\mathfrak{g}$ admits a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ satisfying

$$\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0, \quad \mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1, \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j} \ (i, j = -1, 0, 1).$$

Here, we set $\mathfrak{g}_{-2} := \mathfrak{g}_2 := 0$.

A.2 Classification of classical simple Lie superalgebras

We define some Lie subsuperalgebras of $\mathfrak{gl}(m|n)$ as follows.

$$\mathfrak{sl}(m|n) := \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(m|n) \;\middle|\; \mathrm{str}(X) = 0 \right\},$$

where $\mathrm{str}(X) = \mathrm{tr}(A) - \mathrm{tr}(D)$ is the *supertrace* of X.

For $p = 2m+1$ and $q = 2n$, we let

$$\mathfrak{osp}(p|q) := \left\{ \left(\begin{array}{ccc|cc} A & B & u & X & X_1 \\ C & -{}^tA & v & Y & Y_1 \\ -{}^tv & -{}^tu & 0 & {}^tz & {}^tz_1 \\ \hline {}^tY_1 & {}^tX_1 & z_1 & D & E \\ -{}^tY & -{}^tX & -z & F & -{}^tD \end{array} \right) \in \mathfrak{gl}(p|q) \;\middle|\; \begin{array}{l} {}^tB = -B,\ {}^tC = -C, \\ {}^tE = E,\ {}^tF = F \end{array} \right\}.$$

Here, ${}^tA$ denote the transposed matrix of A. For $p = 2m$ and $q = 2m$, we let

$$\mathfrak{osp}(p|q) := \left\{ \left(\begin{array}{cc|cc} A & B & X & X_1 \\ C & -{}^tA & Y & Y_1 \\ \hline {}^tY_1 & {}^tX_1 & D & E \\ -{}^tY & -{}^tX & F & -{}^tD \end{array} \right) \in \mathfrak{gl}(p|q) \;\middle|\; \begin{array}{l} {}^tB = -B,\ {}^tC = -C, \\ {}^tE = E,\ {}^tF = F \end{array} \right\}.$$

We define the following Lie superalgebras:

(1) $A(m, n) := \mathfrak{sl}(m+1|n+1)$, for $m \neq n \geqslant 0$;

(2) $A(n, n) := \mathfrak{sl}(n+1|n+1)/\mathbb{C}I_{2(n+1)}$ for $n \geqslant 1$;

(3) $B(m, n) := \mathfrak{osp}(2m+1|2n)$, for $m \geqslant 0$, $n \geqslant 1$;

(4) $C(n) := \mathfrak{osp}(2|2n-2)$, for $n \geqslant 2$;

(5) $D(m, n) := \mathfrak{osp}(2m|2n)$, for $m \geqslant 2$, $n \geqslant 1$;

(6) $P(n) := \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in \mathfrak{gl}(n+1|n+1) \;\middle|\; \begin{array}{c} A \in \mathfrak{sl}(n+1), \\ {}^tB = B,\ {}^tC = -C \end{array} \right\}$, for $n \geqslant 2$;

(7) $Q(n) := \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{gl}(n+1|n+1) \;\middle|\; B \in \mathfrak{sl}(n+1) \right\} \Big/ \mathbb{C}I_{2(n+1)}$, for $n \geqslant 2$,

where I_N denote the identity matrix of size N.

The even part of the Lie superalgebras (1)–(6) are given as follows:

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$
$A(m,\, n)$	$\mathfrak{gl}_1 \oplus A_m \oplus A_n$
$A(n,\, n)$	$A_m \oplus A_n$
$B(m,\, n)$	$B_m \oplus C_n$
$C(n)$	$\mathfrak{gl}_1 \oplus C_n$
$D(m,\, n)$	$D_m \oplus C_n$
$P(n)$	A_n
$Q(n)$	A_n

In 1977, Kac classified the finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero. In particular, he showed the following result.

Theorem A.2.1 ([17])**.** *Let $\mathfrak{g}$ be a finite-dimensional classical simple Lie superalgebra over $\mathbb{C}$ such that $\mathfrak{g}_{\bar{1}} \neq 0$. Then $\mathfrak{g}$ is isomorphic to one of the following Lie superalgebras:*

	parameter	type
$A(m,\, n)$	$m \geqslant n \geqslant 0,\, m+n \neq 0$	basic type I
$B(m,\, n)$	$m \geqslant 0,\, n \geqslant 1$	basic type II
$C(n)$	$n \geqslant 3$	basic type I
$D(m,\, n)$	$m \geqslant 2,\, n \geqslant 1$	basic type II
$P(n)$	$n \geqslant 2$	strange type I
$Q(n)$	$n \geqslant 2$	strange type II
$F(4)$		basic type II
$G(3)$		basic type II
$D(2,\, 1;\, a)$	$-1, 0 \neq a \in \mathbb{C}$	basic type II

For the definition of the simple Lie superalgebras $F(4)$, $G(3)$ and $D(2,\, 1;\, a)$, see [17].

Appendix B

Clifford Algebras

Let V be a finite-dimensional vector space over $\Bbbk$ and let $b : V \times V \to \Bbbk$ a symmetric $\Bbbk$-bilinear form on V. The pair (V, b) is called a *quadratic space*. Let $I(V, b)$ be the two-sided ideal of the tensor algebra $T(V)$ generated by all $xy + yx - b(x, y)$, where $x, y \in V$. Set

$$C(V, b) := T(V)/I(V, b).$$

This is a superalgebra over $\Bbbk$ with each $v(\in V)$ odd, and is called the *Clifford superalgebra for* (V, b). Let $\delta_i := b(x_i, x_i)$ for $i = 1, \ldots, r$. Then $C(V, b)$ is generated by the odd elements $x_1, \ldots, x_r$, and is defined by the relations

$$x_i^2 - \delta_i,\ 1 \leqslant i \leqslant r; \quad x_i x_j + x_j x_i,\ 1 \leqslant i < j \leqslant r.$$

If $b = 0$, then $C(V, b)$ coincides with $\wedge(V)$, the exterior superalgebra on V.

Let

$$\mathrm{rad}(b) := \{v \in V \mid b(v, w) = 0 \text{ for all } w \in V\}$$

be the radical of b. We say that a quadratic space (V, b) is *non-degenerate* if $\mathrm{rad}(b) = 0$.

Given two quadratic spaces (V_1, b_1), (V_2, b_2), the orthogonal sum $(V_1, b_1) \perp (V_2, b_2)$ is a direct sum $(V_1 \oplus V_2, b)$ given the bilinear form

$$b((v_1, v_2), (v_1', v_2')) := b_1(v_1, v_1') + b_2(v_2, v_2'),$$

where $v_1, v_1' \in V_1$ and $v_2, v_2' \in V_2$. There is an isomorphism of superalgebras

$$C((V_1, b_1) \perp (V_2, b_2)) \xrightarrow{\cong} C(V_1, b_1) \otimes C(V_2, b_2) \qquad \text{(III.2.0.1)}$$

given by $(v_1, v_2) \mapsto v_1 \otimes 1 + 1 \otimes v_2$, where $v_1 \in V_1$, $v_2 \in V_2$.

Since the characteristic of $\Bbbk$ is not 2, We have an orthogonal basis $x_1, \ldots, x_r$ of V with respect to b, that is, a basis such that $b(x_i, x_j) = 0$ if $i \neq j$.

Example B.0.2. Suppose that (V, b) is non-degenerate.

1. If $r = 1$, then $C(V, b)$ is the 2-dimensional superalgebra generated by one odd indeterminate

x such that $x^2 = \delta_1$. This is obviously central simple superalgebra over $\Bbbk$. Let $C(\delta_1)$ denote the superalgebra $C(V, b)$, for simplicity. Note that $C(\delta_1)$ is *not* simple as an algebra.

2. If $r = 2$, then it is clear that the super center of $C(V, b)$ is $\Bbbk$. There is a superalgebra isomorphism
$$C(V, b) \otimes \bar{\Bbbk} \xrightarrow{\cong} \mathrm{Mat}_{1|1}(\bar{\Bbbk}).$$
Thus, $C(V, b)$ is a central simple superalgebra over $\Bbbk$. In particular, it is central simple over $\Bbbk$ as an ordinary algebra.

For a quadratic space (V, b), we have

$$\dim_{\Bbbk} C(V, b) = 2^r, \tag{III.2.0.2}$$

where $r := \dim_{\Bbbk} V$. Moreover, the dimension of $C(V, b)_\epsilon$ for $\epsilon = \bar{0}, \bar{1}$ is 2^{r-1}.

By [40, Theorem 2], central simple superalgebras over $\Bbbk$ are closed under the ($\mathbb{Z}_2$-graded) tensor product. Thus, for a non-degenerate (V, b), $C(V, b)$ is a central simple superalgebra over $\Bbbk$, by (III.2.0.1) and Example B.0.2. Moreover, if r is even, then $C(V, b)$ is simple as an algebra, by [40, Lemma 3].

Lemma B.0.3. *For an even-dimensional non-degenerate quadratic space (V, b), $C(V, b)$ is a central simple algebra over $\Bbbk$.*

For $z_0 := x_1 \cdots x_r$, the usual center of $C(V, b)_{\bar{0}}$ is $\Bbbk \oplus \Bbbk z_0$. Let

$$\delta := z_0^2 = (-1)^{r(r-1)/2} \delta_1 \cdots \delta_r. \tag{III.2.0.3}$$

Set $(\Bbbk^\times)^2 := \{x \in \Bbbk \mid a^2 = x \text{ for some } a \in \Bbbk^\times\}$. If $\delta \in (\Bbbk^\times)^2$, then the center is isomorphic to $\Bbbk \times \Bbbk$. If $\delta \notin (\Bbbk^\times)^2$, then the center is isomorphic to the quadratic field $\Bbbk(\sqrt{\delta})$.

Proposition B.0.4. *Let (V, b) be a non-degenerate quadratic space. There is a unique simple left $C(V, b)$-supermodule $\mathfrak{u}$ up to isomorphism and parity change such that*

$$\begin{cases} \mathfrak{u} \neq \Pi\mathfrak{u}, & \text{if } \dim_{\Bbbk} V \text{ is even and } \delta \in (\Bbbk^\times)^2, \\ \mathfrak{u} = \Pi\mathfrak{u}, & \text{otherwise.} \end{cases}$$

Proof. Let $A := C(V, b)$. Recall that, the category of A-supermodules is identified with the category of $A \rtimes \Bbbk\mathbb{Z}_2$-modules, by (I.2.2.2). It is easy to see that $A \rtimes \Bbbk\mathbb{Z}_2 \cong A \otimes C(1)$ as superalgebras. Thus, if $r = \dim_{\Bbbk} V$ is is odd, then $A \rtimes \Bbbk\mathbb{Z}_2$ is a central simple algebra over $\Bbbk$ by Lemma B.0.3.

Next, we assume that r is even. Then by Lemma B.0.3, A is isomorphic to $\mathrm{Mat}_n(D)$ as an algebra, where n is a positive integer and D is a central division algebra over $\Bbbk$. By Proposition 2.2.1, $A \rtimes \Bbbk\mathbb{Z}_2$ is Morita equivalent to $A_{\bar{0}}$. If $\delta \in (\Bbbk^\times)^2$, then by the structure theorem of central simple superalgebras [40, Theorem 1], we have

$$A_{\bar{0}} \cong \{\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix} \mid X \in \mathrm{Mat}_s(D),\ W \in \mathrm{Mat}_t(D)\},$$

where $s, t > 0$ and $s + t = n$. In this case, one sees that n must be even and $s = t = n/2$, since $\dim_{\Bbbk} A_{\bar{0}} = 2^{r-1}$. Therefore, A is isomorphic to $\mathrm{Mat}_{s|s}(D)$ as a superalgebra. Let

$$e_1 := \begin{pmatrix} 1_s \\ 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 \\ 1_s \end{pmatrix},$$

where 1_s is the column vector of size s consisting of 1's. Then $V_i = A_{\bar{0}}\, e_i$, $i = 1, 2$ are the distinct simple left $A_{\bar{0}}$-modules. The corresponding simple left A-supermodules are $M_i := A \otimes_{A_{\bar{0}}} V_i$ for $i = 1, 2$ with the $\mathbb{Z}_2$-grading $(M_i)_\epsilon = A_\epsilon\, e_i$ for $\epsilon = \bar{0}, \bar{1}$. It is easy to see that $\Pi M_1 = M_2$. Thus, we conclude that $M_1 \neq \Pi M_1$.

Finally, we consider the case if $\delta \notin (\Bbbk^\times)^2$. Then $A_{\bar{0}}$ is central simple algebra over $\Bbbk(\sqrt{\delta})$. Since $A_{\bar{0}} \otimes \bar{\Bbbk} \cong \mathrm{Mat}_s(\bar{\Bbbk}) \times \mathrm{Mat}_s(\bar{\Bbbk})$ is semisimple, we conclude that $A_{\bar{0}}$ is central simple algebra over $\Bbbk(\sqrt{\delta})$. Therefore, we are done. □

Remark B.0.5. If $\Bbbk$ is an algebraically closed, we have

$$\dim_{\Bbbk} \mathfrak{u} = 2^{\lfloor (r+1)/2 \rfloor}, \tag{III.2.0.4}$$

where $r = \dim_{\Bbbk} V$ and $\lfloor x \rfloor$ is the largest integer greater than x.

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