Some topics in Model theory: model companion, recursive saturation and Schröder-Bernstein property.

Yu-ichi Tanaka
Doctoral Program in Mathematics

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Abstract

In this dissertation, the author investigates three topics in model theory, namely model companions, recursive saturation and the Schröder-Bernstein property. In the finite topic, the author considers the problem of finding sufficient conditions for a theory to have a model companion. In the second, the author proposes a new construction of real closed fields using the concept of ultrapower and provides necessary and sufficient condition for a model of o-minimal theory to have the property of recursive saturation. Lastly, the author studies the Schröder-Bernstein property in the context of countable models.

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Chapter 1

Introduction

1.1 Introduction and Main theorems

Model theory is a branch of mathematical logic that studies mathematical structures, such as graphs, groups, fields, etc., from the point of view of logic. In this dissertation, the author investigates three different areas of model theory: (a) classical model theory, (b) computational model theory, and (c) stability theory. Concerning (a), the notion of model companion will be studied. It is one of the most important notions in classical model theory. If a theory T has a model companion, T is very close to be a model complete theory, and a model complete theory admits a weak version of quantifier elimination. The author gives some conditions for a theory of graphs deciding whether it is model companionable or not. Meanwhile, in computational model theory, one theme of research is on a constructive version of logical concepts. This direction is relevant to decidability and definability. In this regard, the author gives a result similar to Ressayer's theorem in computational model theory. On the other hand, stability theory tackles generalization of notions of linear independence and algebraic independence for structures. The author applies stability frameworks to come up with a model theoretic analogue of Schröder-Bernstein theorem.

In the following, we give some details on the results presented in the dissertation.

(a) In the first part, we study a problem on the existence of model companions of theories extending the graph axioms. We prove general results concerning the existence of model companion. Then, by applying these results to the case of graphs, we give several examples of theories with a model companion and theories without a model companion. The following is the main theorem of this part:

Theorem A. *Suppose that T admits a definable extension and that it is finitely axiomatized. Then T has a model companion.*

(b) The second part introduces a novel construction of real closed fields via an elementary extension of an ordered field with an integer part satisfying the Peano Arithmetic (PA). This method can be extended to a finite extension of an ordered field with an integer part satisfying PA. In general, a field obtained from our construction is either real closed or algebraically closed, so an analogy of Ostrowski's dichotomy holds. We also study recursive saturation of an o-minimal extension of a real closed field by finitely many function symbols. Here are the main results:

Theorem B. *Let K be a finite algebraic extension of Q and let K^* be a proper elementary extension of K. If F is the finite part of K^* and I is the infinitesimal part of K^*, then for $R = F/I$, the following dichotomy property holds:*

1. *If K is a real field, then R is a real closed ordered field.*
2. *Otherwise, R is an algebraically closed field.*

Theorem C. *Let T be an effectively model complete o-minimal L-theory extending RCOF by function symbols. Let R be a model of T. Suppose that R has an integer part $Z \models PA$, $Z \neq \mathbb{Z}$, for which each adding function is Q-definably approximated. Then R is recursively saturated.*

(c) In the last part, we study the Schröder-Bernstein property of countable models. We say a first-order theory has the Schröder-Bernstein (SB) property if whenever two models are elementarily bi-embeddable then they are isomorphic. We investigate how to construct non-isomorphic bi-embeddable countable models of a superstable multidimensional theory. Among others, we give a sufficient condition for small theory to have countably many such models. The following statements are the main theorems of this part:

Theorem D. *Let T be a stable complete first-order theory. Suppose that T has a multidimensional stationary regular type with finite dimension in some model of T and moreover, every formula can be extended to an isolated type. Then T does not have the SB property for countable models.*

Theorem E. *Let T be a small superstable theory. Suppose that T has a multidimensional non-atomic strongly regular type. Then there are infinitely many non-isomorphic countable universal models.*

This dissertation reflects the scholarly work of the author which are also published as journal articles. These published papers are the following [24], [25] and [26].

1.2 Preliminaries

In this article, we use usual notation in Model theory. We see fundamental concepts and results related to this article here.

Definition 1.2.1. *We give the following definitions.*

1. *We say that a set* L *is a language if it is a set consisting of constant symbols* $\{c_i\}_i$, *relational symbols* $\{R_j\}_j$ *and function symbols* $\{f_k\}_k$.

2. *We say that a tuple* $M = (M; c_i^M, R_j^M, f_k^M)$ *is a* L*-structure if the followings holds:*

    - *Let* L *be the language* $\{c_i, R_j, f_k : i, j, k\}$,
    - M *is a non-empty set,*
    - c_i^M *is a element of* M,
    - R_j^M *is a subset of* M^n *for some* n *depending on* R_j, *and*
    - f_k^M *is an* n*-array function over* M *for some* n *depended* f_k.

3. *We say that a sequence of symbol* $t(\bar{x})$ *is an* L*-term if one of followings holds:*

    - $t(\bar{x})$ *is a singleton* c_i *for some* i,
    - $t(\bar{x})$ *is a singleton* x *for some variable symbol* x, *and*
    - *There are* L*-term* $t_1(\bar{x}), \dots, t_n(\bar{x})$ *and an* n*-array function symbol* f_j *such that* $t(\bar{x})$ *equals a the sequence* $f_j(t_1(\bar{x}), \dots, t_n(\bar{x}))$.

    Where by the recursive definition with respect to a length of a sequence, terms are well defined.

4. *We say that a sequence* φ *of* L*-symbols is an* L*-formula if one of followings holds:*

    - *There are two terms* $t_1(\bar{x}), t_2(\bar{x})$ *such that* φ *is equal to* $t_1(\bar{x}) = t_2(\bar{x})$,
    - *There are terms* $t_1(\bar{x}), \dots, t_n(\bar{x})$ *and* n*-array relational symbol* R_j *such that* φ *is equal to* $R_j(t_1(\bar{x}), \dots, t_n(\bar{x}))$,
    - *There is a formula* ψ *such that* φ *is equal to* $\neg\phi$,
    - *There are two formula* ψ_1 *and* ψ_2 *such that* φ *is equal to* $\psi_1 \vee \psi_2$, $\psi_1 \wedge \psi_2$ *or* $\psi_1 \to \psi_2$,
    - *There are a variable symbol* y *is not contained in* $\bar{x}$ *and an* L*-formula* $\psi(y, \bar{x})$ *such that* φ *is equal to* $\exists y.\psi(y, \bar{x})$ *or* $\forall y.\psi(y, \bar{x})$.

5. *Let M be an L-structure and $\bar{a}$ a tuple of M. We will define an interpretation $t^M(\bar{a}) \in M$ w.r.t. a term $t(\bar{x})$ as follows:*

   - *if $t(\bar{x})$ is a singleton c_i for some i, then t^M is equal to c_i^M,*
   - *if $t(\bar{x})$ is a singleton x for some variable symbol x, then $t^M(\bar{a})$ is equal to $a \in \bar{a}$ where a is coressponding to a variable x of $\bar{x}$.*
   - *if $t(\bar{x})$ is $f_j(t_1(\bar{x}), \dots, t_n(\bar{x}))$, where $t_1(\bar{x}), \dots, t_n(\bar{x})$ are L-terms and f_j is an n-array function symbol, then $t^M(\bar{a})$ is equal to $f_j^M(t_1^M(\bar{a}), \dots, t_n^M(\bar{a}))$.*

6. *We say that an L-formula φ is a quantifier free formula if φ does not contain symbols $\exists$ and $\forall$. We say that an L-formula φ is a universal formula if there are variable symbols $\bar{y}$ and L-quantifier free formula $\psi(\bar{y}, \bar{x})$ such that φ is equal to $\forall \bar{y}.\psi(\bar{y}, \bar{x})$, where $\bar{y} = y_1 y_2 \cdots y_n$, and $\forall \bar{y}$ is an abbreviation of $\forall y_1 \forall y_2 \cdots \forall y_n$. We say that an L-formula φ is an existential formula if there are variable symbols $\bar{y}$ and L-quantifier free formula $\psi(\bar{y}, \bar{x})$ such that φ is equal to $\exists \bar{y}.\psi(\bar{y}, \bar{x})$.*

7. *Let L be a language, M is a L-structure, $\varphi(\bar{x})$ is an L-formula and $\bar{a}$ is a tuple from M with same length w.r.t. $\bar{x}$. We say that M satisfies $\varphi(\bar{a})$ if one of followings holds:*

   - *$\varphi(\bar{x})$ is equal to $t_1(\bar{x}) = t_2(\bar{x})$ and $t_1^M(\bar{a}) = t_2^M(\bar{a})$ holds,*
   - *$\varphi(\bar{x})$ is equal to $R_j(t_1(\bar{x}), \dots, t_n(\bar{x}))$ and $(t_1^M(\bar{a}), \dots, t_n^M(\bar{a})) \in R_j^M$ holds,*
   - *$\varphi(\bar{x})$ is equal to $\neg\psi(\bar{x})$ and M does not satisfy $\psi(\bar{a})$,*
   - *$\varphi(\bar{x})$ is equal to $\psi_1(\bar{x}) \vee \psi_2(\bar{x})$ and M satisfies either $\psi_1(\bar{a})$ or $\psi_2(\bar{a})$,*
   - *$\varphi(\bar{x})$ is equal to $\psi_1(\bar{x}) \wedge \psi_2(\bar{x})$ and M satisfies both $\psi_1(\bar{a})$ and $\psi_2(\bar{a})$,*
   - *$\varphi(\bar{x})$ is equal to $\psi_1 \to \psi_2$ and if M satisfies $\psi_1(\bar{a})$ then M satisfies $\psi_2(\bar{a})$,*
   - *$\varphi(\bar{x})$ is equal to $\exists y.\psi(y, \bar{x})$ and there is an element $b \in M$ such that M satisfies $\psi(b, \bar{a})$,*
   - *$\varphi(\bar{x})$ is equal to $\forall y.\psi(y, \bar{x})$ and for every element $b \in M$, M satisfies $\psi(b, \bar{a})$.*

   This is often denoted by $M \models \varphi(\bar{a})$

Definition 1.2.2. *We give the following definitions.*

1. *We say that an L-fomula φ is an L-sentence if every variables x in φ is bounded by quantifier $\forall x$ or $\exists x$.*

2. *Let Φ be a set of L-sentences. We say that Φ is consistent if there exists a model M such that M satisfies φ for all $\varphi \in \Phi$. Also a consistent set of L-sentences is called an L-theory.*

3. *Let T be an L-theory. We say that a L-structure M is a model of T if M satisfies all sentences of T.*

4. *We say that an L-theory T is complete if it does not have proper consistent extension of L-sentences.*

5. *Let A be a set of elements or variables. We use $L(A)$ for the language L augmented by the constants for elements in A.*

6. *Let $\Phi(\bar{x})$ be a set of L-formula with free variables from $\bar{x}$. We say that $\Phi(\bar{x})$ is a type if $\Phi(\bar{x})$ is consistent as a set of $L(\bar{x})$-sentences. For an L-theory T, we say that $\Phi(\bar{x})$ is a type of T if $\Phi(\bar{x}) \cup T$ is consistent.*

Definition 1.2.3. *We give the following definitions.*

1. *For structures $B, C \supset A$, we write $B \cong_A C$, if there is an isomorphism $\sigma : B \to C$ with $\sigma | A = \mathrm{id}_A$.*

2. *The diagram $\mathrm{Diag}(A)$ of A is the set of all quantifier-free $L(A)$-sentences that hold in A. If we want to clarify that A is a substructure of M, we write $\mathrm{Diag}_M(A)$ to denote the diagram of A.*

3. *For a finite $A = \{a_1, \ldots, a_n\}$, we sometimes say that $\varphi(x_1, \ldots, x_n)$ is the diagram of A, if it is the conjunction of all formulas $\psi(x_1, \ldots, x_n)$ with $\psi(a_1, \ldots, a_n) \in \mathrm{Diag}(A)$. The elementary diagram $\mathrm{Diag}_{\mathrm{el}}(A)$ of A is the set of all $L(A)$-sentences that hold in A.*

4. *When we fix a complete theory T, the type $\mathrm{tp}(a)$ is the set of $L(x)$-formulas $\{\varphi(x) : \varphi(a) \in \mathrm{Diag}_{el}(a)\}$ and we say $a \equiv b$ if $\mathrm{tp}(a) = \mathrm{tp}(b)$ holds. We use similar manner for sets.*

5. *We say $N \succ M$ if N is an elementary extension, i.e. for any L-formula φ and any elements $\bar{m} \in M$,*
$$M \vDash \varphi(\bar{m}) \longleftrightarrow N \vDash \varphi(\bar{m})$$

The following fact is fundamental. It is used in chapter 4.

Fact 1.2.4 (Tarski-Vaught Test)**.** *$M \succ N$ if and only if $M \subset N$ and the following holds: Whenever $\bar{a} \in M$, $b \in N$, $\varphi(x, \bar{y})$ is an L-formula and $N \vDash \varphi(b, \bar{a})$ holds, then there is $b' \in M$ such that $M \vDash \varphi(b', \bar{a})$.*

Proof. The only if part is clearly shown. The condition implies that for every L-formula φ and a tuple $\bar{m} \in M$, $N \vDash \varphi(\bar{m})$ implies $M \vDash \varphi(\bar{m})$. So it is sufficient to show that

$M \vDash \varphi(\bar{m})$ implies $N \vDash \varphi(\bar{m})$ for every L-formula φ and a tuple $\bar{m} \in M$. By using Skolem standard form, we can assume φ is a form of $Q_1 x_1, Q_2 x_2, \ldots Q_n x_n, \psi$ where each Q_i is a quantifier and ψ is a quantifier free formula. We will show it by the induction with respect to the number of quantifiers of φ. If φ is a quantifier free (without quantifier) formula, it is clear that $M \vDash \varphi(\bar{m})$ implies $N \vDash \varphi(\bar{m})$ since M is a substructure of N. Suppose that for every φ has at most n quantifiers, $M \vDash \varphi(\bar{m})$ implies $N \vDash \varphi(\bar{m})$. Let $\varphi(x, \bar{y})$ be a form $Qz.\psi(z, x, \bar{y})$ where Q is a quantifier and ψ has at most n quantifiers. Let $\bar{a}$ be a tuple in M and b is an element of N. Suppose that N satisfies $\varphi(\bar{m}) = Qz.\psi(z, \bar{m})$. If Q is equal to $\exists$, we can easily conclude that $M \vDash \varphi(\bar{m})$ because of the condition. Assume Q is equal to $\forall$. We will show that $M \vDash \forall z.\psi(z, \bar{m})$. Fix $m \in M$. Since $M \subset N$, N satisfies $\psi(m, \bar{m})$. By the induction hypothesis, M also satisfies $\psi(m, \bar{m})$. Since m is an arbitrary element of M, M satisfies $\forall z.\psi(z, \bar{m})$. □

Corollary 1.2.1 (Downword Löwenheim)**.** *Let λ be the cardinality of the set of all L-formulas where are variables from $\{y\} \cup \{x_i : i \in \omega\}$. Let N be an L-structure with power greater than λ. Then there exists an elementary submodel M of N with power at most λ.*

Proof. Let Φ be the set of all L-formulas where are variables from $\{y\} \cup \{x_i : i \in \omega\}$. Let A_0 be the empty set $\varnothing$. We will define a sequence $(A_i)_{i \in \lambda}$ of subsets of N in the following manner:

1. for successor ordinal $i = j + 1$, A_i is a union of A_j and all realizations of $\phi(y, \bar{m})$ in N for each formula $\varphi(y, \bar{x}) \in \Phi$ and tuples $\bar{m} \in A_j$ such that $N \vDash \exists y.\varphi(y, \bar{m})$.

2. for limit ordinal i, A_i is equal to $\bigcup_{j<i} A_j$.

Let M be the union of $(A_i)_i$. Then M satisfies the condition of Tarski-Vaught Test for N and the cardinality of M is at most λ. □

In chapter 3, we use the concept of ultrapower. It is defined as follows.

Definition 1.2.5. *Let L be the language $\{c_i, R_j, f_k\}$, I a set and for each $l \in I$, $M_l = (M_l; c_i^{M_l}, R_j^{M_l}, f_k^{M_l})$ an L-structure. We denote the family of L-structures $(M_l)_l$ by $\mathcal{M}$.*

1. *For a set A, A^I is a set of all functions from I to A. If $S = (S_l)_l$ is a I-indexed family of set, S^I means a set of all function from I to $\bigcup S$, where $\bigcup S$ means a union of all element of S, such that for each $l \in I$, the image of l is in S_l.*

2. *Let F be a subset of power set 2^I. F is a (non principal) filter of I if the followings holds: (1) $I \in F$ and $\{a\} \notin F (\forall a \in I)$ (2)if $A \in F$ and $B \in F$ then $A \cap B \in F$ (3) $S \subset F$ then $\bigcup S \in F$.*

3. *Let F be a filter. We say that F is a ultrafilter if F is a maximal filter, i.e. F has no proper extension as a filter. This maximality equivalents to the condition: F contains one of A and $I \setminus A$ and F does not contain the other.*

4. *Let F be a ultrafilter over I. The notation $\mathcal{M}^F$ means a structure defined by the following universe and interpretation:*

   - *Let $f, g \in \mathcal{M}^I$. We denote $f \approx_F g$ if the set $\{l \in I : f(l) = g(l)\}$ is in F. We use $[f]$ for the $\approx_F$-equivalent class of f.*
   - *The universe of $\mathcal{M}^F$ is a quotient set $\mathcal{M}^I / \approx_F$.*
   - *Let C_i be the constant function such that $C_i(l) = c_i^{M_l} (\forall l \in I)$. Define an interpretation of constant c_i of $\mathcal{M}^F$ by $[C_i]$.*
   - *We define an interpretation of n-array relation R_j of $\mathcal{M}^F$ by the following set:*

   $$\left\{([f_1], \dots, [f_n]) \in \mathcal{M}^F : \{l \in I : M_l \models R_j(f_1(l), \dots, f_n(l))\} \in F\right\}.$$

   - *We define an interpretation of n-array function f_k of $\mathcal{M}^F$ by the following map:*

   $$([g_1], \dots, [g_n]) \in (\mathcal{M}^F)^n \mapsto [f_k(g_1, \dots, g_n)] \in \mathcal{M}^F.$$

   $\mathcal{M}^F$ is called the ultraproduct of $\mathcal{M}$ by F.

5. *Let M be a structure and suppose that all structures $M_l (l \in I)$ are isomorphic to M. M^F denotes the ultraproduct $\mathcal{M}^F$, and it is called the ultrapower of the M.*

Fact 1.2.6. *For every infinite set I, there is an ultrafilter over I.*

Proof. Just apply the Zorn's lemma (derived Axiom of Choice) for the set of all filters of I. □

Fact 1.2.7 (Łos's theorem)**.** *Let M be a I-indexes set of L-structures and F a ultrafilter of an infinite set I. For every L-formula $\varphi(x_1, \dots, x_n)$ and elements $[f_1], \dots, [f_n] \in M^F$, the followings are equivalent:*

- $M^F \models \varphi([f_1] \dots, [f_n])$
- $\{l \in I : M_l \models \varphi(f_1(l), \dots, f_n(l))\} \in F$

Proof. By the induction with respect to the complexity of formulas. □

By using ultraproduct, we can deduce the compactness theorem.

Fact 1.2.8 (Compactness)**.** *Let $\Phi = \{\varphi_j : j \in J\}$ be a finitely consistent set of L-sentences, i.e. every finite subset of Φ is consistent. Then Φ is consistent.*

Proof. Let I be the set of all finite subset of J. By assumption, for each $i \in I$, there exists a model M_i satisfying $\wedge_{j \in i} \varphi_j$. Let F be an ultrafilter of I and $\mathcal{M}$ be the I-indexes set $(M_i)_{i \in I}$ of L-structures. Then the ultraproduct $\mathcal{M}^F$ satisfies all sentence in Φ because of Łos's theorem. □

Corollary 1.2.2 (Upperword Löwenheim)**.** *Let M be an L-structure and λ a cardinal greater than the cardinality of M. Then there exists an L-structure N of power λ such that $N \succ M$.*

Proof. Let $(c_i)_{i<\lambda}$ be a sequence of new constant symbols with length λ. Let Φ be the set $\mathrm{Diag}_{\mathrm{el}}(M) \cup \{c_i \neq c_j : i < j < \lambda\}$. By compactness, there is a model N of Φ. Then $N \succ M$ because N satisfies $\mathrm{Diag}_{\mathrm{el}}(M)$. Since elements $c_j^N (j < \lambda)$ are distinct each other, N has at least cardinality λ. By downword Löwenheim, we may assume N has the cardinality λ. □

By using compactness, for each L-structure M and a type in M, we always obtain a realization of the type, in some elementary extension of N. Furthermore by taking realizations of types iteratively, we obtain a monster model, an elementary extension of a given L-structure, in which types with parameters from a small subset are always realized in the model.

We use T to denote a first-order complete theory; L for the language of T. We work in the monster model of T. We use $M, N, \ldots$ for small elementary submodels of the monster; $a, b, \ldots$ for elements or finite tuples of elements; $A, B, \ldots$ for subsets of the monster; $p, q, \ldots$ for types. For a tuple $a = a_1, \ldots, a_n$, the notation $a \in A$ indicates that each a_i belongs to A. Finite tuples of variables are denoted by x, y, ... If x is a single variable, we write so explicitly. The union $A \cup B$ will often be written as AB, if there is no confusion.

Definition 1.2.9. *We give the following definitions.*

1. *We denote by* $\mathrm{tp}(a/A)$ *the complete type of a tuple a over a set A of T.*

$$\mathrm{tp}(a/A) = \{\varphi(x) : \varphi(a) \text{ is satisfied in the monster of } T\}$$

2. *We denote by $S^n(A)$ the set of all n-complete types over a set A of T.*

3. *We write $S^n(T)$ to mean $S^n(\varnothing)$, where $\varnothing$ is the empty set.*

Chapter 2

Model companions

2.1 Introduction

Let L be a language and T a consistent set of universal-existential sentences formulated in L. Consider a model M of T. If whenever a quantifier-free formula in $L(M)$ has a solution in an extension $N \vDash T$, it has a solution in M, we say that M is existentially closed. Any model can be extended to an existentially closed model using a chain of extensions. There are plenty examples of existentially closed models in different fields of mathematics. To name a few, an algebraically closed field, a dense linear order $(A, <)$ without minimum or maximum elements and a random graph are all existentially closed models of field axioms, linear order axioms and graph axioms, respectively. It is usual for existentially closed models, as in the examples mentioned, to be characterized by a first-order theory. This leads to the notion of model companions. A model companion S of T is an $L-$theory such that a model M satisfies $S \iff M$ is an existentially closed model of T. There are theories with model companions and there are those with none. For instance, the following theories have been proved to have model companions: differential fields with an automorphism and $\mathcal{D}$-fields which contain difference fields and differential fields (see [15], [11] and [16]). Meanwhile the following do not have model companions: any unstable NIP theory T plus an automorphism, the theory of pseudo-finite fields with an automorphism, and the theory of random graphs with an automorphism. Thus, the natural (yet seemingly difficult) problem of determining the existence of a model companion for a given theory arises. In this part of the dissertation, we focus on theories extending graph axioms. Note that the determination of the existence of model companions for theories of graphs can be deemed as challenging as for any other theory as every structure having a finite language can be bi-interpreted in a graph.

In the following, we sum up the contents of each section in this chapter. We remark that the sections, at the start, introduce results in the general setting which are then applied in the particular case of theories extending graph axioms. The first section is devoted in providing both necessary and sufficient conditions for a model companion to exist. Results specific to theories that extend graph axioms are discussed. Meanwhile, the second section is an exposition on some elementary results in model theory. Definitions pertinent to graph theory are given. In the third section, we study what we call an obstacle. An obstacle to an existential formula $\delta'(x)$ is an existential formula $\delta(x)$ when the set $T \cup \{\delta(x)\} \cup \{\delta'(x)\}$ is inconsistent. This definition is obviously symmetric. In Section 3, the following results are proved.

A. T has a model companion if and only if for any existential formula $\delta(x)$, there are essentially a finite number of obstacles to $\delta(x)$.

B. T has no model companion if and only if there is a consistent existential formula $\delta(x)$ and a set $\Gamma(x)$ such that, (a) in existentially closed models, every element realizing $\Gamma(x)$ satisfies $\delta(x)$, but (b) every finite $\Gamma_0 \subset \Gamma$ admits the omission of δ (i.e., for some δ-obstacle δ', $\Gamma_0(x) \wedge \delta'(x)$ is satisfied in an existentially closed model).

The former is a characterization result proved using similar techniques employed in [12, Section 8.3.]. The latter result, on the other hand, is used to exhibit non-existence of model companions for many theories. Specifically, we deduce that many theories that extend graph axioms have no model companion, among them the planar graphs. The notion of essential finiteness, appearing in the above result, is defined in Section 3. In Section 4, we give this important result.

C. If T admits a definable extension, then T has a model companion. (See Theorem A for a more precise expression.)

We say that $N = M \cup \{a\}$ is a definable extension of M if the set of all nodes in M adjacent to a is a definable subset of M. Moreover, if every extension of T can be "approximated" by a definable extension, we say T admits a definable extension.

We write T_K to denote the axioms of K-free graphs for a class K of finite graphs. By K-free graphs, we mean graphs whose induced subgraphs are not isomorphic to a member of K. We show that many theories having the form T_K have corresponding companion models using the theorem below. In addition, as a consequence to the theorem, we show that for a finite set K closed under adding edges, there exists a model companion for T_K. This means that if K is a finite class of finite graphs and T is the theory of graphs without subgraphs appearing in K, then there exists a model companion for T. To illustrate, T_K has a model companion when $K = \{K_4, K_4^-\}$ with K_4^- denoting the graph K_4 reduced by one edge.

In the last section, we provide theories T of random graphs such that T_σ have non-existent model companion. Here, T_σ denotes the axioms of graphs with an automorphism.

In [13], it is shown model companions do not exist for T_σ.

2.2 Preliminaries

From here onwards, a theory is always a consistent set of universal-existential sentences. Let T be a theory and M be a model of T. Given M, there is an extension to an existentially closed model N. This means that for any quantifier-free formula $\theta(x)$ with parameters from N, if N has an extension N' that satisfies T in which θ has a solution, then θ has a solution in N. We now give the definition of a model companion to a theory T. We say a theory S, formulated in the same language as T, is a model companion of T, provided the bi-condition is satisfied:

$$M \models S \iff M \text{ is an existentially closed model of } T.$$

From that fact that a model companion S is model complete, it follows that every formula is equivalent to an existential formula under S. Let $T_\forall$ be the universal consequences of T. We have the following.

1. A is a model of $T_\forall$ if and only if it is a substructure of some $M \models T$.

2. T has a model companion S if and only if $T_\forall$ has a model companion S.

Now, a graph G is taken to be an R-structure, where R^G is a binary relational being irreflexive and symmetric. The nodes, or vertices, of the graph are the elements of G and the edges are $\{a,b\}$, provided $(a,b) \in R^G$. In this framework, an induced subgraph of G corresponds to a model theoretic substructure of G. From hereon, we always take a subgraph as an induced one, unless specifically indicated on the contrary.

Definition 2.2.1. *1. Let $n \in \omega$. An n-path is any graph isomorphic to P_n, which is the graph whose vertex set is $\{0,\dots,n-1\}$ and whose edge set is . The length of P_n is length $n-1$*

2. For vertices a and b in G, $d_G(a,b)$ gives the minimum length of paths connecting the two nodes.

3. A graph $G = \{a_0,\dots,a_{n-1}\}$ with distinct a_is is called a cycle, if $R^G \supset \{\{a_i,a_{i+1}\} : i = 0,\dots,n-2\} \cup \{\{a_{n-1},a_0\}\}$. $C_n = \{0,\dots,n-1\}$ is a special kind of cycle with the edges $\{\{i,i+1\} : i = 0,\dots,n-2\} \cup \{\{n-1,0\}\}$. K_n is the complete graph with n nodes.

In what follows, we take K as a class of (isomorphism types) of finite graphs. A graph G is K-free if no member of K is a subgraph of G. Suppose $K = \{C_n : n \geq 3\}$. Then the K-free graphs coincide with the cycle-free graphs. It is evident that the class of K-free graphs is an elementary class axiomatized by universal sentences. Denote by T_K the theory of K-free graphs. We see that every universal theory (in the language of graphs) has the form T_K for some K. We can extend a T_K-model to an existentially closed T_K-model. Lastly, if K is finite, then so is T_K.

2.3 Characterization of theories having a model companion

Let T be a universal-existential theory. We assume this from here on unless stated otherwise.

Definition 2.3.1. *Let $\delta(x) = \exists y \delta_0(x,y)$ and $\delta'(x) = \exists z \delta_0'(x,z)$ be existential formulas where δ_0 and δ_0' are quantifier free formulas. An existential formula $\delta'(x)$ is called a δ-obstacle if $T \cup \{\delta'(x)\} \cup \{\delta(x)\}$ is inconsistent. In the above situation, if b and c are witnesses of $\models \delta(a)$ and $\models \delta'(a)$ respectively, then we say that ab is an obstacle to ac over a.*

Definition 2.3.2. *Let $\delta(x)$ be an existential formula and $\mathcal{O}$ is a set of all δ-obstacles. We say that a subset $\mathcal{O}_0 \subset \mathcal{O}$ is a basis of $\mathcal{O}$ if the following condition holds:*

- *(*) For any existentially closed model $M \models T$ and any $a \in M$, if $M \models \{\neg\delta'(a) : \delta' \in \mathcal{O}_0\}$ then $M \models \{\neg\delta'(a) : \delta' \in \mathcal{O}\}$.*

Remark 2.3.1. *We will denote the set of all δ-obstacles by $\mathcal{O}_\delta$. Let $\mathcal{O}_0 \subset \mathcal{O}_\delta$ be a basis and let M be an existentially closed model. Then the following holds: $M \ni a$ has an extension $N \supset M$ with $N \models T \cup \delta(a)$ if and only if $M \models \{\neg\delta'(a) : \delta' \in \mathcal{O}_0\}$. To see the if part, notice that $T \cup Diag(M) \cup \{\delta(a)\}$ is consistent, since otherwise there is a δ-obstacle $\delta'(a) \in Diag(M)$.*

Proposition 2.3.1. *For an existential formula $\delta(x)$, let $\mathcal{O}_\delta$ be the set of all δ-obstacles. The following are equivalent:*

1. *T has a model companion S.*
2. *For any existential formula $\delta(x)$ there is a finite basis of $\mathcal{O}_\delta$.*
3. *For any existential formula $\delta(x)$ and for any basis $\mathcal{O}_0$ of $\mathcal{O}_\delta$, there is a finite basis of $\mathcal{O}_0$.*

Proof. 1 ⇒ 2: Let $\delta(x)$ be an existential formula. Since S is model complete, there is an existential formula $\psi(x)$ that is equivalent to $\neg\delta(x)$ (under S). Then we have $S \cup \{\neg\psi(x)\} \vDash \{\neg\delta'(x) : \delta' \in \mathcal{O}_\delta\}$, because each $\delta'(x)$ is an δ-obstacle. Hence $\{\psi(x)\}$ is a basis of $\mathcal{O}_\delta$ since every existentially closed model of T satisfies S.

1 ⇒ 3: Since 1 implies 2, we may assume $\{\psi(x)\}$ is a finite basis of $\mathcal{O}_\delta$. Let $\mathcal{O}_0$ be an infinite basis of $\mathcal{O}_\delta$. Then

$$S \cup \{\neg\psi'(x) : \psi' \in \mathcal{O}_0\} \vDash \neg\psi(x).$$

By using compactness, we have a finite basis of $\mathcal{O}_0$.

3 ⇒ 2: This is clear, because $\mathcal{O}_\delta$ is a basis of itself.

2 ⇒ 1: Let $\delta_i(x)$ $(i \in \alpha)$ be an enumeration of all existential formulas. By our assumption, for each i, we can choose a basis $\{\psi_i(x)\}$ of $\mathcal{O}_{\delta_i}$. Now let S be the extension of T augmented by the stenteces χ_i:

$$\forall x(\neg\psi_i(x) \to \delta_i(x)).$$

Then S is a model companion of T: Let M be an existentially closed model of T and suppose $M \vDash \neg\psi_i(a)$. Since $\{\psi_i(x)\}$ is a basis of $\mathcal{O}_{\delta_i}$, $M \vDash \{\neg\delta'(a) : \delta' \in \mathcal{O}_{\delta_i}\}$ must hold. So there is an extension $N \vDash T$ of M with $N \vDash \delta_i(a)$ (See Remark **??**). Hence, by the existential closedness of M, we have $M \vDash \delta_i(a)$. This shows that χ_i holds in M. So M is a model of S.

Then suppose that M is a model of S. Let $N \supset M$ be a model of T. Suppose $N \vDash \delta_i(a)$ with $a \in M$ and an existential formula $\delta_i(x)$. Since $N \vDash \delta_i(a)$, we have $N \vDash \neg\psi_i(a)$. Hence $M \vDash \neg\psi_i(a)$ because $\neg\psi_i$ is an universal formula. So, by the definition of S, $M \vDash \delta_i(a)$. This shows that M is an existentially closed model of T. □

Proposition 2.3.2. *The following are equivalent:*

1. *There is a consistent existential formula $\delta(x)$ and a set $\Gamma(x)$ of formulas such that*
   - *(a) $M \vDash \Gamma(a) \Rightarrow M \vDash \delta(a)$, for all existentially closed model $M \vDash T$ and $a \in M$ (in this case we say that Γ forces δ in M);*
   - *(b) For every finite $\Gamma_0 \subset \Gamma(x)$, there is an existentially closed model M and a δ-obstacle $\delta'(x)$ such that $M \vDash \exists x(\Gamma_0(x) \wedge \delta'(x))$ (in this case we say that Γ_0 admits omitting δ).*
2. *T has no model companion.*

Proof. 1 ⇒ 2: By way of a contradiction, we assume that S is a model companion of T.

By condition (1a), $\Gamma(x) \cup S$ proves $\delta(x)$. So, there is a finite subset Γ_0 of Γ such that

$$\Gamma_0(x) \cup S \vDash \delta(x).$$

For this Γ_0, by condition (1b), choose $M \vDash S$, a δ-obstacle δ' and $a \in M$ such that $M \vDash \Gamma_0(a) \wedge \delta'(a)$. Then $M \vDash \delta(a) \wedge \delta'(a)$. This contradicts the fact that δ' is a δ-obstacle.

$2 \Rightarrow 1$: By Proposition 2.3.1, we can find an existential formula $\delta(x)$ such that there is no finite basis of $\mathcal{O}_\delta$. Let $\Gamma(x) = \{\neg\delta'(x) : \delta' \in \mathcal{O}_\delta\}$. Since δ' is a δ-obstacle, the condition (1a) clearly holds (see Remark 2.3.1). To confirm the condition (1b), suppose that (1b) is not the case for a finite $\Gamma_0 \subset \Gamma$. However, it means that $\mathcal{O}_0 = \{\delta'(x) : \neg\delta' \in \Gamma_0\}$ is a finite basis of $\mathcal{O}_\delta$, a contradiction. □

In specific situations with a theory of graphs, we obtain a useful corollary of the above result, which works for finding theories with no model companion.

Definition 2.3.3. *We say that a graph G of size ≥ 2 is 2-edge connected if G remains connected even if one edge is removed.*

Corollary 2.3.1. *Let K be a set of finite 2-edge connected graphs. Suppose that for any $n \in \omega$ there is a K-free graph G_n having a_n, b_n such that $n \leq d_M(a_n, b_n)$ holds for any K-free graph $M \supset G_n$. Then T_K has no model companion.*

Proof. For each $n \geq 3$, let $\delta_n(x, y)$ be a formula expressing that there is an n-path with the endpoints x and y. Let $\Gamma(x, y) = \{\neg\delta_n(x, y) : n > 3\}$. We will show that $\delta = \delta_3$ and Γ satisfy the conditions (1a) and (1b) of Proposition 2.3.2.

(1a): Suppose that $\Gamma(a, b)$ holds in an existentially closed model M containing a, b. Then a and b must be connected by a 3-path in M. Suppose not. Then a and b are not connected by a path. Let $N = Mc$, where c is a single element such that the edges of N are those in M plus the two edges ac and cb. Clearly N is K-free, since no new 2-edge connected subgraphs appear in N. Since M is existentially closed, there must be $c' \in M$ such that $ac'b$ forms a 3-path, a contradiction.

(1b): Let Γ_0 be a finite subset of Γ. We may assume $\Gamma_0 = \{\neg\delta_i : 3 < i < n\}$. Let M be an existentially closed model of T_K containing G_n, a_n, b_n. By $n \leq d_M(a_n, b_n)$, we have $M \vDash \Gamma_0(a_n, b_n)$. Moreover, $a_n b_n$ satisfies a δ_3-obstacle, since otherwise we would have $M \vDash \delta_3(a_n, b_n)$ by the existential closedness of M. □

This corollary gives us many examples having no model companion.

Example 2.3.1. *1. (Cycle free graphs) Let $K = \{C_n : n \geq 3\}$, where C_n is the n-cycle. Then T_K has no model companion: Notice that this K consists only of 2-edge* *connected graphs and that K-free graphs $G_n := P_n$ satisfy the required condition in Corollary 2.3.1.*

2. *(Planar graphs) Let K be the set of all 2-edge connected finite graphs having K_5 or $K_{3,3}$ (complete bipartite graph on $3+3$ vertices) as a minor. T_K is a first order axiomatization of the planer graphs by the well known Kuratowski-Wagner theorem. We will show that T_K has no model companion by using Corollary 2.3.1. For this, it is enough to show that, for each $n \in \omega$, there is a graph $G \models T_K$ and $a, b \in G$ such that $d_M(a,b) \geq n$ for all T_K-graphs $M \supset G$. (See Claim A below.)*

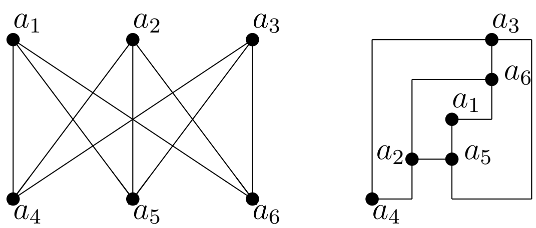


Figure 2.1: $K_{3,3}$ and $K_{3,3}^-$ embedded in G_2

Let $K_{3,3}^-$ be the graph obtained by deleting an edge from $K_{3,3}$ (see Figure 2.1, where the edge a_1a_4 is deleted), and let $G_n = \{b_{i,j} : -n \leq i,j \leq n\}$ be the square grid graph of size $(2n+1)^2$ with the edges $b_{i,j}b_{k,l}$ $(|i-k|+|j-l|=1)$. Notice that, for $b, b' \in G_n$, if $d_{G_n}(b,b')$ is large enough, then there is a (graph theoretic) subgraph $A \ni b, b'$ of G_{n+2} such that A is homeomorphic to $K_{3,3}^-$ and such that bb' corresponds the deleted edge.

Claim 2.3.1. *For every n there is a number k such that $d_M(b_{0,0}, b_{k,k}) \geq n$ holds for all T_K-graphs $M \supset G_{k+2}$.*

Take $k \gg n$ large enough and let $a = b_{0,0}$ and $b = b_{k,k}$. Towards a contradiction, suppose there is a path of length $< n$ connecting a and b in some extension $M \supset G_{k+2}$. Choose such a path

$$a = v_1, w_{1,1}, \dots, w_{1,n_1},\ v_2,\ \dots\ , w_{l-2,n_{l-2}},\ v_{l-1}, w_{l-1,1}, \dots, w_{l-1,n_{l-1}},\ v_l = b,$$

where v_i's are in G_{k+2} and $w_{i,j}$'s are not. Choose a minimum s such that $v_s \in G_{k+2} \setminus G_k$. Then choose $s_1 < s_2 \leq s$ such that $d_{G_{k+2}}(v_{s_1}, v_{s_2}) \geq k/s$, where k/s is a large number by $k \gg n$. G_{k+2} contains a subgraph $A \ni v_{s_1}, v_{s_2}$ that is homeomorphic to $K_{3,3}^-$, where $v_{s_1}v_{s_2}$ corresponds to the missing edge. This means M contains a subgraph (in graph theoretic meaning) that is homeomorphic to $K_{3,3}$, contradicting the assumption that M is a planer graph.

2.4 Theories admitting a definable extension

In this section, let T be a universal theory in a finite relational language. We are mainly interested in the case where T is finitely axiomatized.

Definition 2.4.1. *Let M be an L-structure and let $N = Mb$ be an extension of M as an L-structure.*

1. *We say that N is a $\varphi(x,a)$-extension of $M \ni a$ if $\mathrm{Diag}_N(ab) = \varphi(x,y)$.*
2. *We say that a $\varphi(x,a)$-extension N is a definable extension if the atomic type $\mathrm{atp}(b/M)$ is definable over a, i.e., for each atomic formula $P(x,y)$, there is a formula $dP(y,a)$ such that*

$$N \models P(b,c) \iff M \models dP(c,a), \text{ for all } c \in M.$$

We call the map $d : P(x,y) \mapsto dP(y,z)$ a defining schema of the extension and the extended structure N with d will be referred to by $(Mb)^{d,a}$.

Definition 2.4.2. *Let $\varphi(x,y)$ be a (complete) diagram of a finite T-structure.*

1. *We say that T admits a definable $\varphi(x,y)$-extension if there is a defining schema $d : P(x,y) \mapsto dP(y,z)$ such that for every existentially closed model M of T and $\varphi(x,a)$-extension $Mb \models T$ with $a \in M$, $(Mb)^{d,a} \models T \cup \{\varphi(b,a)\}$.*
2. *We say that T admits a definable extension if T admits a definable $\varphi(x,a)$-extension for all (complete) diagrams $\varphi(x,y)$.*

Remark 2.4.1. First notice that there are only finitely many atomic formulas (modulo the choice of variables), since L is finite and relational. Suppose that T admits a definable extension by a defining schema d. Then the L-structure $(Mb)^{d,a}$ is (uniformly) interpretable in $M \ni a$, i.e. for each L-formula $\psi(x,y)$, there is an L-formula $\theta_{\psi,d}(y,z)$ such that for all L-structures $M \ni a$,

$$(Mb)^{d,a} \models \psi(b,c) \iff M \models \theta_{\psi,d}(c,a), \text{ for all } c \in M.$$

This is clear, since $(Mb)^{d,a}$ is M^{eq}-definable, and since an eq-structure adds no new definable sets to the original sort. (See [17] or [10].)

Theorem A. *Suppose that T admits a definable extension and that it is finitely axiomatized. Then T has a model companion.*

Proof. For each (complete) diagram $\varphi(x,y)$, let d_φ be a defining scheme witnessing the definable φ-extendability of T. Let S be the theory

$$T \cup \left\{ \forall z \left[\theta_{\psi,d_\varphi}(z) \wedge \theta_{\varphi,d_\varphi}(z,z) \ \rightarrow \ \exists x \varphi(x,y) \right] \right\}_\varphi,$$

where $\theta_{\psi,d}$ is as in Remark 2.4.1 and $\psi = \bigwedge T$. Then S intuitively says that if there is a definable extension $(Mb)^{d,a}$ of M having a solution b to $\varphi(x,a)$ then there is a solution in M. So, it is clear that every existentially closed model of T is a model of S.

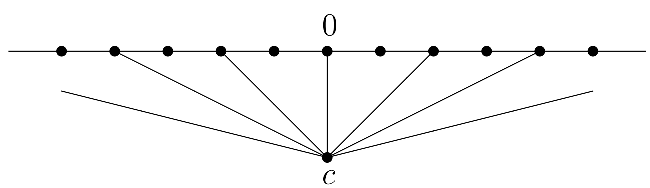
By our supposition that T admits a definable extension, if a formula $\varphi(x,a)$ has a solution in some extension of M, then there is a definable extension having a solution of φ. So every model of S is existentially closed. □

Corollary 2.4.1. Let K be a finite set of finite graphs that is closed under adding edges. (If B has the same vertexes as $A \in K$ and $R^B \supset R^A$, then $B \in K$.) Let T_K be the theory of K-free graphs. Then T_K has a model companion.

Proof. Let $M \ni a$ be an existentially closed model of T and Mb a T-extension of M. Since K is closed under adding edges, the models of T is closed under deleting edges. Therefore the free amalgam $M \oplus_a b$ is a model of T, this means T admits a definable extension. □

Remark 2.4.2. *In Corollary 2.4.1, we assumed that K is closed under adding edges. Example 2.4.1 below shows that this assumption is necessary. Example* **??** *gives an example which admits a definable extension but does not satisfy the assumption of Corollary 2.4.1.*

Example 2.4.1. *Let $M = \mathbb{Z} \cup \{c\}$ be the graph with the edges $R^M = \{\{n, n+1\} : n \in \mathbb{Z}\} \cup \{\{2n, c\} : n \in \mathbb{Z}\}$.*



Let K be the set of (isomorphism types of) all graphs X with $|X| \leq 6$ and such that X is not a substructure of M. We show that T_K has no model companion. By way of a contradiction, we assume that T_K has a model companion S. First notice that every cycle C_n with $n \geq 7$ is a T_K-graph. Also notice that neither C_3 nor C_5 is a T_K-graph.

Claim 2.4.1. *For every odd $n \geq 7$, C_n has no proper connected T_K-extension.*

Suppose otherwise and choose c' such that c' is connected to some element in C_n by an edge. By symmetry, we may assume $R(c', 0)$. Let us consider the six vertex *graph* $\{c', n-1\} \cup \{0, \dots, 3\}$. *Since the graph* $Y = \{a, b\} \cup \{d_1, \dots, d_4\}$ *having the edges* $ad_1, bd_1, d_1d_2, d_2d_3, d_3d_4$ *is a forbidden graph in* K, c' *must be connected to one of the vertexes* $1, 2, 3$ *by an edge. If* $c'1$ *is an edge, then the three element set* $\{c', 0, 1\}$ *forms a triangle (a contradiction). If* $c'3$ *is an edge, then the five element set* $\{c', 0, 1, 2, 3\}$ *forms* C_5 *(a contradiction). Therefore,* c' *must be connected to* 2 *by an edge. For a similar reason, we see that* c' *is also connected to* $2, 4, \dots, n-1$. *But, then* $\{c', n-1, 0\}$ *forms a triangle. This is a contradiction, and we conclude that* C_n *has no proper connected* T_K*-extension. (End of Proof of Claim A)*

Let N *be a countably saturated model of* S. *Since an arbitrarily large odd cycle* C_n *is a* T_K*-graph, by claim A,* N *contains an infinite line* Z, *whose vertices always have degree* 2 *in* N. *We identify* Z *with* $\mathbb{Z} \subset M$. M *can be embedded in* N *(over* $\mathbb{Z}$*), because* $M \cup N = N \cup \{c\}$ *is a* T_K*-graph and* N *is an existentially closed model of* T_K. *Then the degree of* $0 \in \mathbb{Z}$ *in* N *is at least three. This is a contradiction.*

2.5 Model companion of graphs having an automorphism

Let T be a universal-existential theory. We denote by T the model companion, if it exists, of T^*. Define $T_\sigma = T \cup \{$"σ is an automorphism"$\}$, where σ is a new unary function symbol. Observe that T_σ is also a universal-existential theory. Our main objective is to determine the existence of model companion of T_σ.

The following proposition is a consequence of Proposition 2.3.2.

Proposition 2.5.1. *Let* K *be a finite set of finite graphs such that if* $F \in K$, *then there are less than* $|F| - 2$ *non-adjacent pairs of vertices in* F, *that is, there are more than* $\binom{|F|}{2} - (|F| - 2)$ *edges in* F. *Then* $(T_K)_\sigma$ *does not have a model companion.*

In the next lines, we present a theory T having a model companion such that T_σ does not.

Example 2.5.1. *Let* K *be a finite set of graphs that is invariant under adding edges. Then* T_K *has a model companion. If additionally, we assume that* K *meets the hypothesis of Proposition 2.5.1. As such,* $(T_K)_\sigma$ *has no model companion.*

In [13], Kikyo proved that if a theory of random graphs has an automorphism, then it cannot have a model companion. In general,

Proposition 2.5.2 (Kikyo, Corollary 3.6 and Example 3.7 in)**.** *If* $T = T^{eq}$, *admits quantifier elimination, has the* $PAPA$ *and the fcp, then* T_σ *has no model companion.*

Note that the above proposition implies Example 2.5.1 (see Remark 2.5.2 and proceeding lines). Thus, we see some connections between the model companions of T_σ and $(T^*)_\sigma$. Here, the language of T may be a non-graph language.

Lemma 2.5.1. *Let T admit amalgamation over models. Let N be a model of T and let $M, M' \subset N$ be two submodels. Suppose that the mapping $\tau : M \to M'$ is an isomorphism. Then there is a model $(N^*, \sigma^{N^*}) \models T_\sigma$ such that (i) $N^* \supset N$ and (ii) $\sigma^{N^*} \supset \tau$.*

Proof. Let $M_0 = M$, $M'_0 = M'$, $N_0 = N$ and $\tau_0 = \tau$. We will inductively construct T-models M_i, M'_i, N_i and isomorphisms $\tau_i : M_i \to M'_i$ $(i \in \omega)$ such that for $i \in \omega$

1. $M_i M'_i \subset N_i,\ \ N_{i-1} \subset M_i \cap M'_i$;
2. $\tau_0 \subset \ldots \tau_{i-1} \subset \tau_i$.

Suppose we have defined M_i, M'_i, N_i and τ_i for all $i \leq n$. First choose a T-model N'_n and an isomorphism τ'_n extending τ_n such that

$$\tau'_n : N_n M_n \cong N'_n M'_n.$$

Since T admits amalgamation over models, there is an amalgam $M'_{n+1} \models T$ of N_n and N'_n over M'_i. For simplicity, we can assume $M'_{n+1} \supset N_n N'_n$. (The embedding of N_n into M'_{n+1} is an identity.) Then we choose M_{n+1} and an isomorphism $\tau_{n+1} \supset \tau'_n$ such that

$$\tau_{n+1} : M_{n+1} N_n \cong M'_{n+1} N'_n.$$

Let N_{n+1} be an amalgam of M_{n+1} and M'_{n+1} over N_n. Clearly M_{n+1}, M'_{n+1}, N_{n+1} and τ_{n+1} satisfy the required conditions 1 and 2. Thus we have defined M_i, M'_i, N_i and τ_i for all $i \in \omega$. Finally let $N^* = \bigcup_{i\in\omega} M_i = \bigcup_{i\in\omega} M'_i$ and $\sigma^{N^*} = \bigcup_{i\in\omega} \tau_i$. It is clear that $\sigma^{N^*} \supset \tau$ gives an automorphism of N^*. Since T is a universal-existential theory, N^* is a model of T.

□

Proposition 2.5.3. *Suppose that T_σ has a model companion and that T admits amalgamation over models. Then T has a model companion.*

Proof. Let $(T_\sigma)^*$ be a model companion of T_σ, and let S be the universal-existential L-consequences of $(T_\sigma)^*$. Clearly $S \models T$. We show that S is a model companion of T. For this, it is sufficient to show the following claims A and B.

Claim 2.5.1. *Let M be an existentially closed model of T. Then M is a model of S.*

Let $(N, \sigma^N) \supset (M, id_M)$ be an existentially closed model of T_σ. Then (N, σ^N) is a model of $(T_\sigma)^*$, and hence N is a model of S. Let $\forall x \exists y \varphi(x, y)$ be an arbitrary member of S, where φ is quantifier free. Then, for any $a \in M$, we have $N \models \exists y \varphi(a, y)$. Since M is an existentially closed model of T, and since $S \models T$, we have $M \models \exists y \varphi(a, y)$. Since $a \in M$ was arbitrary, we have $M \models \forall x \exists \varphi(x, y)$. Therefore $M \models S$.

Claim 2.5.2. *Let M be a model of S. Then M is an existentially closed model of T.*

Since $S \models T$, M is a model of T. For showing that it is existentially closed, let $N_0 \models T$ be an extension of M such that $N_0 \models \exists x \varphi(x)$, where φ is a quantifier free $L(M)$-formula. Recall that S is the set of universal-existential consequences of $(T_\sigma)^*$. Since M is a model of S, we can find a model $(N_1, \sigma^{N_1}) \models (T_\sigma)^*$ such that $M \prec_\exists N_1$ (i.e. $M \models \psi \iff N_1 \models \psi$, for all existential $L(M)$-sentences). Since T admits amalgamation over models, we can find an amalgam N of N_0 and N_1 over $M \models T$. By letting $M_0 = N_1$, $M_0' = \sigma^{N_1}(N_1)$, and $\tau_0 = \sigma^{N_1}$, we have the situation in Lemma 2.5.1. Thus there is $(N^*, \sigma^*) \supset (N_1, \sigma^{N_1})$ such that $N^* \models \exists x \varphi(x)$. Since (N_1, σ^{N_1}) is an existentially closed model of T_σ, we have $N_1 \models \exists x \varphi(x)$. Since $M \prec_\exists N_1$, we have $M \models \exists x \varphi(x)$. Therefore M is an existentially closed model of T. □

Remark 2.5.1. *Let T be a universal-existential theory having a model companion T^*. Suppose that T^* admits elimination of quantifiers. Then T admits amalgamation: Let AB_1 and AB_2 be two models of T, where A is the common substructure. We have T^*-models $N_i \supset AB_i$ ($i = 1, 2$). Since T^* admits quantifier elimination, it is substructure complete, so $N_1 \equiv N_2$ in the language $L(A)$. So there is a T^*-model $N^* \supset A$ having an isomorphic copy (over A) of N_1 and that of N_2.*

Proposition 2.5.4. *Let T be a universal existential theory with a model companion T^*. Suppose that T admits amalgamation over models. Then the following are equivalent:*

1. *(M, σ^M) is an existentially closed model of T_σ;*
2. *(M, σ^M) is an existentially closed model of $T_\sigma^* = (T^*)_\sigma$.*

Proof. $1 \Rightarrow 2$: Suppose 1. First we show that M is a model of T_σ^*. For this, it is sufficient to show that M is an existentially closed model of T. Let $N \supset M$ be a model of T and suppose that a quantifier free $L(M)$-formula $\varphi(x)$ has a solution in N. By Lemma 2.5.1, we can find $(N^*, \sigma^{N^*}) \models T_\sigma$ with $N \subset N^*$ and $\sigma^M \subset \sigma^{N^*}$. Clearly N^* has a solution of $\varphi(x)$. Since (M, σ^M) is an existentially closed model of T_σ, M has a solution of $\varphi(x)$.

Now it is clear that (M, σ) is an existentially closed model of T_σ^*, since a T_σ^*-extension of (M, σ) is a model of T_σ.

$2 \Rightarrow 1$: Suppose 2. Obviously, (M, σ^M) is a model of T_σ. Let $(M', \sigma^{M'}) \supset (M, \sigma^M)$ be a T_σ-extension and suppose that a quantifier free $L_\sigma(M)$-formula $\varphi(x)$ has a solution in

M'. By an iterative use of Lemma 2.5.1, we can define extension sequences $(N_i, \sigma^{N_i}) \models T_\sigma$ $(i \in \omega)$ and $M_i^* \models T^*$ $(i \in \omega)$ such that

$$M' \subset M_0^* \subset N_0 \subset \cdots \subset M_i^* \subset N_i \subset \cdots.$$

Then, by letting $N = \bigcup_{i\in\omega} N_i = \bigcup_{i\in\omega} M_i$ and $\sigma^N = \bigcup_{i\in\omega} \sigma^{N_i}$, (N, σ^N) is a model of T_σ^* extending (M, σ^M). So, by 2, there is a solution of $\varphi(x)$ in M. □

Remark 2.5.2. *From the above proposition, we know that if T admits amalgamation over models, then $(T_\sigma)_\forall = (T_\sigma^*)_\forall$: Let (M_0, σ^{M_0}) be a model of $(T_\sigma)_\forall$. Then there is an extension $(M, \sigma^M) \models T_\sigma$ of (M_0, σ^{M_0}). By extending M if necessary, we can assume (M, σ^M) is existentially closed. By Proposition 2.5.4, (M, σ^M) is an existentially closed model of T_σ^*. Therefore, $(T_\sigma)_\forall \subset (T_\sigma^*)_\forall$. The inclusion in the other direction can be proven similarly.*

As we will explicitly state in Question, we don't know if it is possible to erase the condition of amalgamation over models from Proposition 2.5.4.

Let T be the theory of graphs. It is well-known that T has a model companion T^*, the theory of random graphs. Moreover, T^* admits elimination of quantifiers and has the amalgamation property. So, by [13], T_σ^* does not have a model companion. By the remark above, we know also that T_σ does not have a model companion, though we've shown that T_σ does not have a model companion by a direct argument.

Question 2.5.1.

1. *Is there any theory T such that T does not have a model companion but T_σ has a model companion?*

2. *Is it always the case that a model companion of $(T^*)_\sigma$ is a companion of T_σ?*

Chapter 3

Construction of real closed fields

3.1 Introduction

Dedekind cuts and Cauchy sequences are well-known constructings of the ordered field $\mathbb{R}$ of all real numbers. A somewhat less well-known alternative construction using a nonprincipal ultrafilter is as follows: If we set I to be the set of all infinitesimals in Q^* and F to be the set of all finite elements, then the quotient field F/I is a complete ordered field. Moreover, F/I is isomorphic to $\mathbb{R}$.(See [1] for more details.)

We are interested in the quotient field F/I for the general case when Q^* is not necessarily an ultrapower of $\mathbb{Q}$. It is not easy to show that the quotient ordered field F/I satisfies $RCOF$, which is the theory of real closed ordered fields, when Q^* is a proper elementary extension of $\mathbb{Q}$. In this chapter, we investigate the properties of the quotient field F/I in a more general setting. For example, we consider the case where the base field Q is an ordered field containing an integer part Z whose non negative part satisfies PA (the theory of first-order Peano arithmetic) and $Q^* \succ Q$ where $\mathbb{Q}$ is bounded in Q^*. Even in such a case, one can prove that F/I satisfying $RCOF$ (Proposition 3.3.1). The key observation is that we are able to express a recursive procedure in a first-order way by using an integer part Z.

In Section 3, the base field K is a finite algebraic extension of Q as described above. We show roughly the following:

Theorem A. *Let K be a finite algebraic extension of Q and let K^* be a proper elementary extension of K. If F is the finite part of K^* and I is the infinitesimal part of K^*, then for $R = F/I$, the following dichotomy property holds:*

1. *If K is a real field, then R is a real closed ordered field.*

2. *Otherwise, R is an algebraically closed field.*

This is a first order analogue of Ostrowski's result. Where Ostrowski's result states that if a field K is complete with respect to an archimedean absolute value, then K is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

In Section 4, we investigate an expansion of real closed fields by adding new functions. In [2], they investigated $RCOF$ and found the sufficient and necessary condition of recursively saturated model of $RCOF$. Here, we define well-approximated function over R and prove the following.

Theorem B. *Let T be an effectively model complete o-minimal L-theory extending $RCOF$ by function symbols. Let R be a model of T. Suppose that R has a non-archimedean integer part $Z \vDash PA$ for which new functions are well-approximated. Then R is recursively saturated.*

By using F and I, we obtain a model R with properties in above theorem.

In Section 2, we review some of the facts necessary for our results.

3.2 Preliminaries

PA is the theory of first-order Peano arithmetic formulated in the language $\{0, 1, +, \cdot\}$.

Definition 3.2.1. *The first-order theory PA is consisted form following axioms:*

1. $\forall a, a + 0 = a$
2. $\forall a, \forall b, a + b = b + a$
3. $\forall a, \forall b, \forall c, a + (b + c) = (a + b) + c$
4. $\forall a, a \cdot 1 = a$
5. $\forall a, \forall b, a \cdot b = b \cdot a$
6. $\forall a, \forall b, \forall c, a \cdot (b \cdot c) = (a \cdot b) \cdot c$
7. $\forall a, \forall b, \forall c, a \cdot (b + c) = a \cdot b + a \cdot c$
8. $\nexists\, a, 1 + a = 0$
9. $\forall a, 1 + a = 1 \rightarrow a = 0$
10. $\forall \bar{y}, \{[\varphi(0, \bar{y})) \wedge (\forall a, \varphi(a, \bar{y}) \rightarrow \varphi(a + 1, \bar{y}))] \rightarrow \forall a, \varphi(a, \bar{y})\}$ *, for every formula* $\varphi(x, \bar{y})$.

The set of natural numbers is denoted by $\mathbb{N}$ or ω. A model M of PA which is not isomorphic to $\mathbb{N}$ is called a nonstandard model of PA. The standard part of $M \models PA$ is $\mathbb{N}$. We may regard M as an integral domain by adding imaginary elements to M We sometimes say that an ordered integral domain is a model of PA if the set of all nonnegative elements is a model of PA. In a model M of PA, we are able to code finite sequences of numbers in M. In this way, "recursive functions" is definable functions in M. More precisely, for a given recursive function f, there exists a definable function g on M such that the restriction $g|\mathbb{N}$ is f.

Let Q be an ordered field. A discretely ordered subring Z of Q is called an integer part of Q if every $a \in Q$ belongs to an interval $[n, n+1)$ for some $n \in Z$ and $1 \in Z$ is the minimum positive element in Z. For an archimedean Q, its integer part and its rational part are unique. For a non-archimedean Q, its integer part is not uniquely determined.

$RCOF$ is the set of axioms for real closed ordered fields. There are a number of equivalent conditions for an ordered field to be a real closed ordered field. We adopt the following definition:

$$K \text{ is a real closed ordered field} \iff K \text{ is an ordered field on which the intermediate value theorem holds for all polynomials over } K.$$

An o-minimal theory is a complete theory having $<$ in the language such that the universe is linearly ordered by $<$ and every definable subset of the universe is a finite union of intervals (and finite points). By the Tarski–Seidenberg Theorem ([8]), $RCOF$ is known to have elimination of quantifiers. From this, we know that $RCOF$ is an o-minimal theory.

Definition 3.2.1. *Let M be an L-structure, where L is a recursive set. A set $\Sigma(x, a)$ of formulas in $L(a)$, $a \in M$, is called a recursive type in M if $\Sigma(x, a)$ is finitely satisfiable in M and $\Sigma(x, y)$ is a recursive set.*

In $M \models PA$, for each $n \in \omega$, there exists a partial truth definition $Tr_n(x)$ such that $M \models \forall x(\varphi(x) \leftrightarrow Tr_n(\ulcorner \varphi(\bar{x}) \urcorner)$ holds for all Σ_n-formulas φ. From this we know that any nonstandard $M \models PA$ has (restricted) recursive saturation for recursive types consisting only of Σ_n-formulas.

3.3 Nonstandard models and field extensions

Let $L_{ring} = \{0, 1, +, -, \cdot\}$ and $L_{or} = L_{ring} \cup \{<\}$ denote the languages for rings and for ordered rings, respectively. We can formulate the axiom of ordered fields using L_{or}. Let Q be an ordered field and Z its integer part satisfying PA. We denote the quotient field of Z as $QF(Z)$. Let Q^* be a proper elementary extension of Q where Q is bounded in Q^*. From Q^*, we introduce two subsets which corresponds to the finite part and infinitesimal part of Q^*. Define $F = F_{Q^*} = \{a \in Q^* : \exists q \in Q^+(|a| < q)\}$ and $I = I_{Q^*} = \{a \in Q^* : \forall q \in Q^+(|a| < q)\}$. Notice that F and I are not definable. We see that $R = F/I$ is an ordered field. Indeed, F is a subring of Q^* while I is a convex maximal ideal of F. Let $a \in F$. Define $[a] = \{b \in Q^* : a - b \in I\}$. We see that $[a]$ is an equivalence class under $\sim$, where we put $a \sim b$ whenever $a - b \in I$ for $b \in F$. We now introduce a notation for comparing two equivalence classes $[a]$ and $[b]$. We write $[a] < [b]$ to mean $\iff$ $a \lnsim b$, i.e., $a < b$ and $[a] \neq [b]$. With these, we see that $R = \{[a] : a \in F\}$.

3.3.1 Real closed fields constructed from nonstandard models of Q

We begin with a simple proposition.

Proposition 3.3.1. *Let Q be an ordered field with an integer part Z which satisfies PA. Let $Q^* \succ Q$ be a proper elementary extension of Q such that Q is bounded in Q^*. Let F be a finite part of Q^* and I be an infinitesimal part of I. Then the quotient field $R = F/I$ is a real closed ordered field.*

Proof. Let (Q^{**}, Z^{**}) be a sufficiently saturated elementary extension of (Q, Z), where Z is considered as an additional unary predicate. Then Z^{**} is an integer part of Q^{**} satisfying PA. We can assume that Q^* is an L_{or}-elementary restriction of Q^{**}, because Q^{**} is an elementary extension of Q in the sense of L_{or} In this proof, we will work in Q^{**}.

Let $\bar{f}(X) = \sum_{m=0}^{n} [a_m] X^m$ be a polynomial in $R[X]$. Let $[b]$ and $[c]$ is an element of R such that $\bar{f}(X)$ takes a positive value at $[b] \in R$ and a negative value at $[c] \in R$. We will prove that there is a zero of $\bar{f}(X)$ in R. We assume that b is less than c. Let $f(x)$ be the polynomial $\sum_{m=0}^{n} a_m X^m$. We can take an element q of Q^+ such that $f(b) > q$ and $f(c) < -q$. Take a positive element ε of I. First notice that we can briefly find uniformly an approximation of f in $QF(Z^{**})[X]$ in the following sense:

Claim 3.3.1. *There is an n-degree polynomial $g(X) \in QF(Z^{**})[X]$ and $b', c' \in QF(Z^{**})$ with $b < b' < c' < c$ such that*

1. *$|f(x) - g(x)| < \varepsilon$ holds for all $x \in [b, c]$;*
2. *$g(b') > q$ and $g(c') < -q$ hold.*

Then we will use the binary search method for finding a zero of g between b' and c'.

Claim 3.3.2. *There exists $d \in Q^{**}$ such that $|f(d)| < 2\varepsilon$.*

Let b', c', and g be as chosen in Claim 3.3.1. Let h_0 and h_1 be recursive functions from $Z^{**}_{\geq 0}$ to $QF(Z^{**})$ such that $h_0(0) = c'$, $h_1(0) = b'$ and the followings hold:

1. h_0 is increasing and $g(h_0(k)) \leq 0$ holds for all k;

2. h_1 is decreasing and $g(h_1(k)) \geq 0$ holds for all k;

3. $|h_0(k) - h_1(k)| = 2^{-k} \cdot |c' - b'|$ holds for all k.

There exists a positive element δ of $QF(Z^{**})$ such that if $|x-y| < \delta$ then $|g(x) - g(y)| < \varepsilon$ for all $x, y \in [b', c']$, because g is a uniformly continuous function on $[b', c']$. Let n^* be an element of Z^{**} such that $|h_0(n^*) - h_1(n^*)|$ is less than δ and let d equal $h_0(n^*)$. Since $g(h_0(n^*)) \leq 0 \leq g(h_1(n^*))$, $|g(d)| = |g(h_0(n^*))|$ is less than ε. By triangular inequality and the choice of g, b' and c', the inequality $|f(d)| \leq |f(d) - g(d)| + |g(d)| \leq 2\varepsilon$ holds. (End of Proof of Claim 3.3.2)

Finally, we will find a zero of $\bar{f}$ in R. By Claim 3.3.2, the following statement holds in Q^{**}:

$$\left|\sum_{m=0}^{n} a_m d^m\right| < 2\varepsilon \wedge b < d < c.$$

Since $Q^{**} \succ Q^*$ and $a_0, \ldots a_n, b, c, 2\varepsilon \in Q^*$, we can find an element d^* of Q^* such that $|\sum_{m=0}^{n} a_m {d^*}^m| < 2\varepsilon \wedge b < d^* < c$. The inequality $b < d^* < c$ implies that d^* is contained in F_{Q^*}. Since $|f(d^*)| < 2\varepsilon \in I$ and I is convex, the equality $\bar{f}([d^*]) = [f(d^*)] = [0]$ holds.

Thus, we have shown that R satisfies $RCOF$. □

3.3.2 Nonstandard models of $Q(\alpha)$

Let $(K, <)$ be an ordered field. Let v_K be the standard absolute value of K which is a function from $K^\times$ to K^+ defined by a formula $v_K(a) := |a|$. In this subsection, we say that (K, v) is a valued field if v is a function from $K^\times$ to $\{0\} \cup v(K)$, where $v(K)$ is a multiplicative ordered abelian group and 0 is an imaginary minimum element of $v(K)$, such that for all $x, y \in K$ the followings hold,

- $v(x) = 0 \iff x = 0$,
- $v(x) = v(-x)$,
- $v(x \cdot y) = v(x) \cdot v(y)$,
- $v(x + y) \leq \max(v(2x), v(2y))$.

The standard absolute value v_K satisfies the conditions above. Let (K, v) be a valued field. We say that v induces an order on K if there exists an order $<$ on K such that v is equivalent to its standard absolute value, i.e., $|x| < |y| \iff v(x) <_{v(K)} v(y)$ holds for all nonzero x, y.

Suppose $K = Q(\alpha)$ for some ordered field Q having an integer part satisfying PA and algebraic element α. With the Q-linearly independent elements $1, \alpha, \ldots, \alpha^n$, we express K as

$$K = Q \oplus \alpha Q \oplus \cdots \oplus \alpha^n Q.$$

We now consider $rcl(Q)(\sqrt{-1})$ where $rcl(Q)$ is a real closure of Q. This is algebraically closed. An absolute value in $rcl(Q)(\sqrt{-1})$ is given by $|x + y\sqrt{-1}| := \sqrt{x^2 + y^2}$. As there is an embedding from K to $rcl(Q)(\sqrt{-1})$, we can view the restriction of the above absolute value as an absolute value in K. Let us now fix an absolute value $v : K \to v(K)$ which extends the standard absolute value $v_Q : x \mapsto |x|$. We now look at the two-sorted structure $(K, v) := (K \sqcup v(K), v)$. We see that either

- (Orderable Case) v induces an order on K; or
- (Non-orderable Case) v does not induce an order on K.

In either possibility, we define a quotient field R in the following manner. Let K^* be a proper elementary extension of (K, v). For simplicity, we also refer as v the extension of v in K^*. As before, we define two subsets of K^*:

- (Finite Part) $F = \{a \in K^* : \exists q \in K^\times(v(a) < v(q))\}$;
- (Infinitesimal Part) $I = \{a \in K^* : \forall q \in K^\times(v(a) < v(q))\}$.

We have that $R = F/I$ is a field because I is a maximal ideal of F. The result below states that R is either real or algebraically closed.

Theorem B. *Let Q be an ordered field with an integer part Z which satisfies PA. Let K be a finite algebraic extension $Q(\alpha)$ of Q with an absolute value $v \supset v_Q$. Let (K^*, v) be a proper elementary extension of (K, v) such that the set $v(K)$ is a bounded set of $v(K^*)$. Let F and I be defined as above. Then the following dichotomy holds for $R = F/I$:*

- *(Orderable Case) R is a real closed field.*
- *(Non-orderable Case) R is an algebraically closed field.*

Proof. Let $(K_1; Q_1, v, Z_1)$ be an elementary extension of $(K; Q, v, Z)$, where the language is expanded. We further assume that it is sufficiently saturated. From this assumption, we can see that K^* is an elementary restriction of (K_1, v) (in the original language). Additionally let $(K_2; Q_2, v, Z_2)$ be a proper elementary extension of $(K_1; Q_1, v, Z_1)$. Define:

- (Finite Part) $F_{K_2,K_1} = \{a \in K_2 : \exists q \in K_1^\times(v(a) < v(q))\}$;
- (Infinitesimal Part) $I_{K_2,K_1} = \{a \in K_2 : \forall q \in K_1^\times(v(a) < v(q)\}$.

Let C be the quotient field $F_{K_2,K_1}/I_{K_2,K_1}$. We prove the theorem on C formulated as follows.

Claim 3.3.3. *C is algebraically closed or real closed.*

Let F_{Q_2,Q_1} and I_{Q_2,Q_1} be the finite and infinitesimal parts, respectively. That is, F_{Q_2,Q_1} is the set of Q_1-finite elements in Q_2. From Proposition 3.3.1, the quotient field $R^* = F_{Q_2,Q_1}/I_{Q_2,Q_1}$ is a real closed field. We can prove that F_{Q_2,Q_1} equals $F_{K_2,K_1} \cap Q_2$ and I_{Q_2,Q_1} equals $I_{K_2,K_1} \cap Q_2$ because $v(Q_1)$ is cofinal in $v(K_1)$. Then C has R^* as a subfield. If C equals R^*, C is a real closed field. We can assume that $C \neq R^*$. In this case, $C \subset R^*([\alpha])$ because F_{K_2,K_1} is a finite extension $\bigoplus_{i=0}^{n} \alpha^i F_{Q_2,Q_1}$ of F_{Q_2,Q_1} (as rings) . Thus C is an algebraic extension over R^*. So $C$is an algebraically closed field. (End of Proof of Claim 3.3.3)

Both real closed fields and algebraically closed fields are axiomatized by formulas which state the existence of zeros of polynomials satisfying certain conditions. We find zeros of polynomial of R in R. Let $P(X)$ be a monic polynomial with coefficients $[a_0], \dots, [a_m]$ from R, where $a_0, \dots, a_m \in F$ and $a_m = 1$. Suppose that C is algebraically closed because the other case can be proved similarly. We will show the existence of a zero of P. (In case C is a real closed field, we additionally assume that there exist $[a] < [b] \in R$ such that $P([a]) < 0 < P([b])$ and find a zero between $[a]$ and $[b]$.) Let $p(X)$ be the polynomial $\sum_{i=0}^{m} a_i X^i$ and $\bar{p}(X)$ be the polynomial $\sum_{i=0}^{m} \bar{a}_i X^i$, where the notation $\bar{x}$ is equal to an element $x + I_{K_2,K_1}$ of C. Then $\bar{p}(X)$ is a polynomial with coefficients from C because F is a subset of F_{K_2,K_1}. We obtain a root $\bar{\beta}$ of the polynomial $\bar{p}(X)$ in C with $\beta \in F_{K_2,K_1}$. We will show that there exists an element d of K^* such that $v(\beta) < v(d)$ to see that the polynomial $P(X)$ has a zero in R. From the definition of absolute value, we can prove the following:

Claim 3.3.4. *In an absolute valued field (K, v), the v-value of a root of a polynomial $\sum_{i=0}^{m} a_i X^i$ is bounded by:*

$$\max\left\{v\left(2^m \cdot \frac{a_0}{a_m}\right), \dots, v\left(2^m \cdot \frac{a_{m-1}}{a_m}\right), v(1)\right\}.$$

From the claim above, we obtain that $v(\beta)$ is less than or is equal to the maximum value among $v\left(2^m \cdot (a_0 - p(\beta))\right)$,$v\left(2^m \cdot a_1\right)$,…,$v\left(2^m \cdot a_{m-1}\right), v(1)$. Then $v(a_0 - p(\beta))$ is less than $\max(v(2a_0), v(1))$ because $p(\beta)$ is an element of I_{K_2,K_1}. We can take $d \in K^*$ such that $v(\beta)$ is less than $v(d)$. Fix an element $\varepsilon \in I_{K^*} \setminus \{0\} \subset K_1$. Then $v(p(\beta))$ is less than $v(\varepsilon)$ because K^* is a bounded set of K_2. So $K_2 \models v(p(\beta)) < v(\varepsilon) \wedge v(\beta) < v(d)$. Since (K^*, v) is an elementary restriction of (K_2, v), the statement $K^* \models v(p(\beta')) < v(\varepsilon) \wedge v(\beta') < v(d)$ is holds for some $\beta' \in K^*$. So $[\beta']$ is a root of $P(X)$ in R. Thus, we have proved that R is algebraically closed or real closed.

If R is a real closed then field clearly (K, v) is orderable. To complete the proof, it is sufficient to prove that (K, v) is not orderable when R is an algebraically closed field. If R is an algebraically closed field, then there exists an element $[\gamma] \in R$ such that $[\gamma]^2 + 1$ is equal to 0 where $\gamma \in K^*$. By contradiction, suppose that (K, v) is orderable. It is easy to prove that an order $<_K$ of (K, v) induced by v is first-order definable in (K, v) in the following manner: $x <_K y \iff v(y - x) < v(1 + y - x) \wedge v(1) < v(1 + y - x)$. Then (K^*, v) is orderable. Since $\gamma^2 >_{K^*} 0$, $\gamma^2 + 1 >_{K^*} 1$ holds. This is a contradiction. Then (K, v) is not orderable. □

We will consider the case $Q = \mathbb{Q}$. It is known that the set of all integers $\mathbb{Z}$ is definable in $K = \mathbb{Q}(\alpha) \subset \mathbb{C}$ (see [7]). The real part and the imaginary parts of α are both algebraic numbers, and so there exist definable functions $g, h : \mathbb{N} \to \mathbb{Q}$ such that g converges to $Re(\alpha)$ and h converges to $Im(\alpha)$. Write $(g(x) + h(x)\sqrt{-1})^i$ as $g_i(x) + h_i(x)\sqrt{-1}$. For example, $h_2(x) = 2g(x)h(x)$. Then g_i converges to $Re(\alpha^i)$ and h_i converges to $Im(\alpha^i)$. Now let $|x|$ be the standard absolute value of x (in $\mathbb{C}$). Then for $x, y \in K$, $|x| < |y|$ can be expressed by the following statement: There exist $\varepsilon \in \mathbb{Q}^+$, $q_0, \dots, q_n \in \mathbb{Q}$, $r_0, \dots, r_n \in \mathbb{Q}$, and $m \in \mathbb{N}$ such that, for any $k > m$,

$$x = \sum_{i \le n} q_i \alpha^i \ \wedge \ y = \sum_{i \le n} r_i \alpha^i$$
$$\wedge \ \Big(\sum_{i \le n} q_i g_i(k)\Big)^2 + \Big(\sum_{i \le n} q_i h_i(k)\Big)^2 + \varepsilon < \Big(\sum_{i \le n} r_i g_i(k)\Big)^2 + \Big(\sum_{i \le n} r_i h_i(k)\Big)^2.$$

This shows that $|x| < |y|$ is a definable relation in K, and so the absolute value $v(x) = |x|$ on K is also definable in the sense of K^{eq}.

By applying Theorem B to the case of $Q = \mathbb{Q}$, where v is definable, we obtain the following:

Corollary 3.3.1. *Let $K = \mathbb{Q}(\alpha)$ be a finite extension of $\mathbb{Q}$ and $v : K^\times \to v(K)$ be an absolute value that is definable with parameters from $\{\alpha\}$ in the sense of K^{eq}. Let $K^* \succ K$ be a proper L_{or}-elementary extension of K. K^* has an absolute value extending v. The finite part F and the infinitesimal part I are defined as above. If we let $R = F/I$, then R is either real closed or algebraically closed.*

3.4 Recursive saturation and o-minimality

Let R be a real closed ordered field having a non-standard model of PA as its positive part. Then R is recursively saturated (see [2]). Applying this fact to $R = F/I$ of Proposition 3.3.1, we see that if there is a subset Z^* in Q^* having the property that $(Q^*, Z^*) \succ (Q, Z)$, then R must be recursively saturated. We consider an analog of this result in our setting.

First we give some definitions.

Definition 3.4.1. *Let T is an L-theory where L contains an order relation $<$. We say that T is an o-minimal theory if the following holds:*

- *The order relation $<$ is a totally linear order under T;*
- *For every L-formula $\phi(x, \bar{y})$, every model M of T and every elements $\bar{b} \in M$, there is a finite tuple $(c_i, d_i)_i$ of $M \cup \{-\infty, \infty\}$ such that $M \models \forall x, (\phi(x) \leftrightarrow \bigvee_i x \in (c_i, d_i))$, where (a, b) is an interval in M and (a, a) means a singleton $\{a\}$.*

Definition 3.4.2. *Let R be a real closed ordered field and denote its integer part as Z. Set the quotient field of Z to be $Q \subset R$. Suppose that PA holds for the nonnegative part N of Z. Consider a continuous function $E : R^n \to R$. Then E is Q-definably approximated if we can define a Q-definable continuous function F from $N \times Q^n$ to Q such that on any closed bounded box $B \subset Q^n$, $\{F(m, \bar{x}) : m \in N\}$ converges uniformly to $E(\bar{x})$.*

Definition 3.4.3. *We say that a theory T is effectively model complete if we have a recursive procedure for finding an existential formula that is logically T-equivalent to a given formula.*

Remark 3.4.1. *1. The (uniform) convergence is understood in the sense of N (not $\mathbb{N}$).*

2. All decidable model complete theory are effectively model complete.

Theorem C. *Let $h_0, h_1, \ldots, h_k$ be function symbols. Let $L = L_{or} \cup \{h_0, \ldots, h_k\}$. Let T be an L-theory extending $RCOF$ which is effectively model complete o-minimal. Let R be a model of T. Suppose that R has an integer part $Z \models PA$, $Z \neq \mathbb{Z}$, for which each h_i^R is Q-definably approximated. Then R is recursively saturated.*

Before we prove this theorem, we introduce lemmas where we keep the meaning of L, T, R, Q, and N as defined previously. Also, we take k_0 such that for $i \leq k$, $h_i(\bar{x})$ is Q-definably approximated by a Σ_{k_0}-formula.

Lemma 3.4.1. *Every L-term is Q-definably approximated by Σ_{k_0}-formulas.*

Proof. For $i \le k$, let F_i be Σ_{k_0}-functions where $F_i(m, \bar{x}) = y$ Q-definably approximates $h_i(\bar{x})$. Using induction, we show the general case. Suppose $len(\bar{x}) = 1$. We prove that $h_0(h_1(x))$ is Q-definably approximated by $F_0(n, F_1(n, x))$. Choose an arbitrary closed bounded subset I of Q. Let $\varepsilon \in Q^+$. As h_1 is continuous and o-minimal, $h_1(I)$ is closed and bounded. Taking a subsequence of the approximation, we can assume that there is a closed and bounded set J containing $\bigcup_{n \in N} F_1(n, I)$. The existence of $n_0 \in N$ satisfying the inequality below follows from the uniform convergence of $F_1(n, q) \to h_1(q)$ on I and the uniform continuity of h_0 on J:

$$|h_0(h_1(q)) - h_0(F_1(n, q))| < \varepsilon \quad (q \in I,\ n \ge n_0).$$

Since $F_0(n, r) \to h_0(r)$ in J, we find $n_1 \in N$ with $n_1 \ge n_0$ such that

$$|h_0(F_1(n, q)) - F_0(n, F_1(n, q))| < \varepsilon \quad (q \in I,\ n \ge n_1).$$

Therefore, for all $(q \in I)$, $|h_0(h_1(q)) - F_0(n, F_1(n, q))| < 2\varepsilon$ where $n \ge n_1$. It follows that $h_0(h_1(x))$ is Q-definably approximated by the Σ_{k_0}-definable function $F_0(n, F_1(n, x))$. □

Lemma 3.4.2. *Given an existential L-formula $\varphi(\bar{x})$, we can find effectively an L-formula $\varphi_0(\bar{x})$ and an L_{or}-formula $\varphi'(\bar{x})$ satisfying $R \vDash \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \varphi_0(\bar{x}))$ and $R \vDash \varphi_0(\bar{b}) \iff Q \vDash \varphi'(\bar{b})$, for all $\bar{b} \in Q$.*

Proof. The existential formula $\varphi(\bar{x})$ has the form

$$\exists y_1, \dots, y_n \big[f(\bar{y}, \bar{x}) = 0 \wedge \bigwedge_{i \le k} g_i(\bar{y}, \bar{x}) > 0 \big],$$

where f and g_i $(i \le k)$ are L-terms. Consequently, φ shares this form. To simplify the discussion, we suppose that the form of φ is

$$\exists y_1, \dots, y_n \big[f(\bar{y}, \bar{x}) = 0 \wedge g(\bar{y}, \bar{x}) > 0 \big].$$

We now let $\varphi_0(\bar{x})$ be the L-formula expressing that there exist $u_0, v_0, \dots, u_{n-1}, v_{n-1}$ with $u_i < v_i$ $(i < n)$ such that

1. $\forall \varepsilon > 0[\exists \bar{y}(\bigwedge_{i<n} u_i \le y_i \le v_i \wedge |f(\bar{y}, \bar{x})| < \varepsilon)]$,
2. $\exists \varepsilon > 0[\forall \bar{y}(\bigwedge_{i<n} u_i \le y_i \le v_i \to g(\bar{y}, \bar{x}) > \varepsilon]$.

The formulas φ and φ_0 must be equivalent in R since f and g are continuous. Take F an approximation of f, likewise G for g. Let $\varphi'(\bar{x})$ be the L_{or}-formula expressing (in Q) that there exist $u_0, v_0, \dots, u_{n-1}, v_{n-1}$ with $u_i < v_i$ $(i < n)$ such that

3. $\forall k \exists \bar{r}\big[\bigwedge_{i<n} u_i \le r_i \le v_i \wedge \exists l \in N \forall m \in N\big(m \ge l \to |F(m;\bar{r},\bar{x})| < 1/k\big)\big]$,

4. $\exists k \forall \bar{r}\big[\bigwedge_{i<n} u_i \le r_i \le v_i \to \exists l \in N \forall m \in N\big(m \ge l \to G(m;\bar{r},\bar{x}) > 1/k\big)\big]$.

We have $R \models \varphi_0(\bar{q})$ is equivalent to $Q \models \varphi'(\bar{q})$, for $\bar{q} \in Q$ as implied by the uniform convergence of the approximations F and G on closed boxes. □

Thus, the φ' in Lemma 3.4.2 is a Σ_{k_0+4}-formula.

Lemma 3.4.3. *Let existential formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ be such that $R \models \forall \bar{x}(\varphi \to \psi)$. From Lemma 3.4.2, we get formulas φ' and ψ'. Then $Q \models \forall x(\varphi' \to \psi')$.*

Proof. We want to show $Q \models \forall x(\varphi' \to \psi')$. Suppose otherwise. That is, assume $R \models \forall \bar{x}(\varphi \to \psi)$ and $Q \models \exists x(\varphi' \wedge \neg\psi')$. Then we can find $\bar{q}$ in Q with $Q \models \varphi'(\bar{q}) \wedge \neg\psi'(\bar{q})$. As a consequence, $R \models \varphi_0(\bar{q}) \wedge \neg\psi_0(\bar{q})$ and $R \models \varphi(\bar{q}) \wedge \neg\psi(\bar{q})$. Contradiction. □

Lemma 3.4.4. *Let $\bar{a} \in R$. The subset* $\mathrm{dcl}(\bar{a})$ *of R is bounded.*

Proof. Let x and y be free variables. Consider a recursive enumeration $\{\varphi_i(x,y) : i \in \omega\}$ of formulas in x and y. Designate the formula

$$(\exists! y \varphi_i(x,y) \to \varphi_i(x,y)) \wedge (\neg \exists! y \varphi_i(x,y) \to y = 1)$$

by $\psi_i(x,y)$. The $\psi_i(x,y)$ $(i \in \omega)$ gives all $\varnothing$-definable functions. We write $f_i(x) = y$ for $\psi_i(x,y)$.

Claim 3.4.1. *For some $m^* \in N$,* $\mathrm{dcl}(\varnothing) < m^*$.

Let the elements of $\{\theta_i\}$ be given by$\theta_i(z) \coloneqq f_i(0) < z$ $(i \in \omega)$. This set is taken to be a recursive set of existential formulas by invoking the effective model completeness assumption. To each θ_i, we set θ'_i the formula we get in Lemma 3.4.2 and collect these formulas into

$$\Gamma(z) \coloneqq \{\theta'_i(z) : i \in \omega\} \cup \{z \in N\}.$$

We see that $\Gamma(z)$ is a recursive type in Q and the $\theta'_i(z)$'s are Σ_{k_0+4}-formulas. So, there exists an $m^* \in N$ that realizes Γ in Q. This means that for $i \in \omega$, $f_i(0) < m^*$ in R.

Claim 3.4.2. *Every $\varnothing$-definable function is continuous on the interval $[p,q]$, for some $p,q \in Q$ with $p < a < q$.*

For each function f_i, we collect all its finitely many discontinuous points into the subset D_i of R, $i \in \omega$. We now define g_i as the minimum of the set $\{|x-y| : x \neq y \in D_i\}$ where we set $g_i = \infty$ whenever $|D_i| < 2$ so that $g_i \neq 0$. Then $1/g_i$ is a $\varnothing$-definable element of R. Claim 3.4.1 implies that $1/g_i < m^*$ for some element m^* of N. Clearly, $0 < 1/m^* < g_i$

for all $i \in \omega$. Let us take initially $p_0, q_0 \in Q$ such that $p_0 < a < q_0$ and $|q_0 - p_0|$ is no more than $1/m^*$. Let $D := (\bigcup_{i\in\omega} D_i) \cap [p_0, q_0]$. Suppose there are two points $r < s$ in D. Take functions f_i and f_j which are discontinuous at r and s, respectively. Then the definable function $f_k = f_i \cdot f_j$ is discontinuous at r and s, implying that $g_k \leq |r - s| \leq |q_0 - p_0| < 1/m^*$. Contradiction. Hence, $|D| < 2$. We now choose $p_0 < p < q < q_0$ such that $[p, q]$ does not intersect D. The claim now follows.

Claim 3.4.3. *For some $n^* \in N$,* $\mathrm{dcl}(a) < n^*$.

Let $p, q \in Q$ be the elements we get in the previous claim. Recall that every f_i is continuous on $[p, q]$. So, $\forall x(p < x < q \to f_i(x) < y)$is consistent and corresponds to an end segment of R. As in Claim 3.4.1 , the type $\{\forall x(p < x < q \to f_i(x) < y) : i \in \omega\}$ is realized by some $n^* \in N$. For all $i \in \omega$, we then have $f_i(a) < n^*$.This proves the claim.

The lemma holds even when a is a finite tuple. We can show this by induction on the length of a.

□

We are now in the position to prove Theorem C using the above lemmas.

Proof of Theorem C. Fix $\bar{a} \in R$ and let k denote its length. Define the recursive type $\Sigma(x, \bar{a}) = \{\varphi_i(x, \bar{a}) : i \in \omega\}$. Suppose $R \vDash \varphi_{i+1}(x, \bar{a}) \to \varphi_i(x, \bar{a})$. Without loss of generality, we suppose $\bar{a} \in R \setminus \mathrm{dcl}(\varnothing)$ and $\bar{a}$ is mutually non-algebraic, as the arguments for the other cases follows similarly. We associate to each $\varphi_i(x, \bar{a}) \in \Sigma$ the formula $\theta_i(u_0, u_1, \bar{v}_0, \bar{v}_1) = \theta_i(u_0, u_1, v_{00}, \ldots, v_{0,k-1}, v_{1k}, \ldots, v_{1,k-1})$ given by

$$\forall x\bar{y}\left(u_0 < x < u_1 \land \bigwedge_{j<k} v_{0j} < y_j < v_{1j} \ \to\ \varphi_i(x, \bar{y})\right).$$

From the cell decomposition theorem, we have that $\exists u_0 u_1(u_0 < u_1 \land \bigwedge_{j<k} v_{0j} < a_j < v_{1j} \land \theta_i(u_0, u_1, \bar{v}_0, \bar{v}_1))$ is satisfied in R. Moreover, using the assumption on model completeness, we may suppose θ_i is an existential formula. Since the theory is o-minimal, there are lexicographically minimal and maximal values which we denote as $\bar{b}_0$ and $\bar{b}_1$, respectively, such that $\exists u_0 u_1(u_0 < u_1 \land \bigwedge_{j<k} b_{0j} < a_j < b_{1j} \land \theta_i(u_0, u_1, \bar{b}_0, \bar{b}_1))$ has a realization in R. It follows that $\bar{b}_0, \bar{b}_1 \in \mathrm{dcl}(\bar{a}) \cup \{\pm\infty\}$. We now take n^* with $\mathrm{dcl}(\bar{a}) < n^*$ as in Lemma 3.4.4. We select $\bar{c}_0, \bar{c}_1 \in Q$ satisfying $\sum_{j<k} |c_{0j} - c_{1j}| < 1/n^*$ where $b_{0j} < c_{0j} < a_j < c_{1j} < b_{1j}$ $(j < k)$. We see that $\exists u_0 u_1(u_0 < u_1 \land \theta_i(u_0, u_1, \bar{c}_0, \bar{c}_1))$is satisfied in R independent of the chosen $i \in \omega$.

Using Lemma 3.4.2, recall that there corresponds to each θ_i a θ'_i with R satisfies $\forall u_0 u_1 \bar{v}(\theta_i \leftrightarrow \theta_{i,0})$ and also R satisfies $\theta_{i,0}(q_0, q_1, \bar{r}, \bar{s})$ if and only if Q satisfies $\theta'_i(q_0, q_1, \bar{r}, \bar{s})$ for all $q_0, q_1, \bar{r}, \bar{s} \in Q$

Note that $\exists u_0 u_1(u_0 < u_1 \wedge \theta_{i,0}(u_0, u_1, \bar{c}_0, \bar{c}_1))$ is satisfied in R. Likewise, $\exists u_0 u_1(u_0 < u_1 \wedge \theta'_i(u_0, u_1, \bar{c}_0, \bar{c}_1))$ is satisfied in Q because the elements u_0, u_1 can be taken from Q. We know from Lemma 3.4.3 that the recursive type $\{\theta'_i(u_0, u_1, \bar{c}_0, \bar{c}_1) : i \in \omega\}$ is Σ_{k_0+4}-type in Q Now the existence of $(d_1, d_2) \in Q^2$ where $Q \vDash \bigwedge_{i \in \omega} \theta'_i(d_0, d_1, \bar{c}_0, \bar{c}_1)$ follows from the Σ_{k_0+4}-recursive saturation of Q. Finally, all elements between d_0 and d_1 satisfy the type $\Sigma(x, \bar{a})$ in R. □

Corollary 3.4.1. *Let $Q \equiv \mathbb{Q}$ be a non-archimedean field and let Q^* be a proper elementary extension of Q such that Q is a bounded subset of Q^*. Define F, I, and the quotient $R = F/I$ as before. Let $m \mapsto q_m \in \mathbb{Q}$ be a recursive function. Define $h_0(x) = \sum_{m \in \mathbb{N}} q_m x^m$ be a function on $\mathbb{R}$. Assume that h_0 be an analytic function on $\mathbb{R}$. Let $n^* \in N^* \setminus F$ be a nonstandard element and $H_0(x)$ be the (definable) function on Q^* defined by*

$$H_0(x) = \sum_{m \le n^*} q_m x^m.$$

Define $h : R \to R$ by $h([x]) = [H_0(x)]$. Assume that the theory of (R, h) is effectively model complete and o-minimal. Then (R, h) is recursively saturated.

Proof. For $n \in N$, let $F(n, x)$ be $\sum_{m \le n} q_m x^m$. Then F is a definable on Q^*. For all $p, q \in \mathbb{Q}$ and $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$\mathbb{Q} \vDash \forall n, n' \in \mathbb{N}\, \forall x \in [p, q]\, (n, n' \ge n_0 \to |F(n, x) - F(n', x)| < \varepsilon).$$

Replacing $\mathbb{Q}$ by Q or Q^*, we get that for all $p, q \in Q$ and $\varepsilon \in Q^+$, there exists $n_0 \in N$ such that

$$Q^* \vDash \forall n \in N^*\, \forall x \in [p, q]\, (n \ge n_0 \to |F(n, x) - F(n^*, x)| < \varepsilon).$$

Assume $[x] = [y]$ $(x, y \in Q^*)$ is inside in $[p, q]$. Then $|F(n^*, x) - F(n^*, y)| < |F(n_0, x) - F(n_0, y)| + 2\varepsilon$. Because $|F(n_0, x) - F(n_0, y)| \in I_{Q^*}$ and letting ε tend to 0, we see $H_0(x) = F(n^*, x) \sim F(n^*, y) = H_0(y)$. Then we have well-definedness of h. Likewise, h is Q-definably approximated by $F(n, x)$. The result now follows from Theorem C. □

Definition 3.4.4. *Let $\mathbb{Q}^*$ be a proper elementary extension of the ordered ring $\mathbb{Q}$. We say that $\mathbb{Q}^*$ is a full extension of $\mathbb{Q}$ if every cut of $\mathbb{Q}$ is realized in $\mathbb{Q}^*$. For pairs $Q^* \succ Q$ of nonstandard models, we say that Q^* is a full extension of Q if $(Q^*, Q) \equiv (\mathbb{Q}^*, \mathbb{Q})$ for some full extension $\mathbb{Q}^*$ of $\mathbb{Q}$, where $(\mathbb{Q}^*, \mathbb{Q})$ is the ordered ring $\mathbb{Q}^*$ with the unary predicate designating $\mathbb{Q}$.*

Assume $\mathbb{Q}^*$ is full extension of $\mathbb{Q}$. Then $\mathbb{R}$ and the quotient field $R = F/I$ are isomorphic. The structure $(\mathbb{Q}^*, \mathbb{Q})$ contains F and I as definable subsets. Consequently, R is also definable. This means that all functions which are $\mathbb{Q}$-definably approximated are definable in $(\mathbb{Q}^*, \mathbb{Q})$. Let $n^* \in \mathbb{N}^* \setminus F$ be nonstandard. Define the definable function on $\mathbb{Q}^*$ by

$$E_0(x) := \sum_{n \le n^*} \frac{1}{n!} x^n.$$

Define on R the function $E([x]) := [E_0(x)]$. Then $E([x])$ is $\mathbb{Q}$-definably approximated. Moreover, (R, E) is isomorphic to $(\mathbb{R}, exp)$. For the general situation, we can look at a nonstandard $Q \succ \mathbb{Q}$. We can define E_0 and E in a similar way. In this situation, (R, E) satisfies $Th(\mathbb{R}, exp)$. Wilkie ([9]) showed that $Th(\mathbb{R}, exp)$ is model complete and o-minimal. Moreover, Macintyre and Wilkie ([5]) showed that if Schanuel's conjecture is true, then $Th(\mathbb{R}, exp)$ is decidable and so effectively model complete. Hence, if Schanuel's conjecture is true, then (R, E) is recursively saturated.

Chapter 4

Schröder-Bernstein property for countable models

4.1 Introduction

Schröder-Bernstein's theorem is an elementary result of set theory which states that every bi-embeddable pair of sets has a bijective map from one to the other. We can consider a model theoretic analogue of Schröder-Bernstein's theorem: when is a bi-elementary embeddable pair of models isomorphic. Given a class $\mathcal{K}$ of models of a theory, we say that $\mathcal{K}$ has the Schröder-Bernstein (SB) property if whenever two models in $\mathcal{K}$ are elementarily bi-embeddable, then they are isomorphic. If $\mathcal{K}$ is the class $Mod(T)$ of all models of T, we simply say that T has the SB property.

It is easy to find a complete theory without the SB property. For example, the theory of equivalence relation of infinitely many infinite classes does not have SB property. In 1988, Nurmagambetov showed that a countable ω-stable theory T has the SB property if and only if T does not have a multi-dimensional type ([23]). Goodrick and Laskowski investigated equivalence conditions of the SB property for weakly minimal cases ([21]).

Goodrick ([20]) introduced the notion of a nomadic type (a variation of a multidimensional type). Goodrick and Laskowski ([22]) showed that a necessary and sufficient condition for the class of all a-models of a superstable theory to have the SB property is the omission of a nomadic type.

Since an a-model has a certain kind of saturation, Goodrick's construction ([22]) of counter examples have large cardinality. In this article, we are interested in a category of small (countable) models. For the statement of the main theorems, we define:

Definition 4.1.1. Let T be a first-order theory. We say that T has the Schröder-Bersnstein (SB) property for countable models if whenever M and N are bi-embeddable countable models of T then $M \cong N$.

We show the following two theorems in this article:

Theorem D. *Let T be a stable complete first-order theory. Suppose that T has a multi dimensional stationary regular type with finite dimension in some model of T and moreover, every formula can be extended to an isolated type. Then T does not have the SB property for countable models.*

Theorem E. *Let T be a small superstable theory. Suppose that T has a multidimensional non-atomic strongly regular type. Then there are infinitely many non-isomorphic countable universal models.*

In section 2, we present notation and basic concepts in stability theory. In section 3, we prove the first theorem. In section 4, we give a simple proof of first theorem under strict assumptions and exhibit the second theorem.

4.2 Preliminary

We assume the reader knows some basic facts on stability theory (see [19]). In particular, the reader is assumed to know the notion of forking.

Definition 4.2.1. *We give the following definitions.*

1. *We say that T is λ-stable if $|S_A(T)| = \lambda$ for any set A of power λ.*
2. *T is stable if there is an infinite cardinal λ such that T is λ-stable.*
3. *T is superstable if T is λ-stable for all sufficiently large cardinals λ.*
4. *T is ω-stable if T is λ-stable for every cardinal λ.*

Definition 4.2.2. *We give the following definitions.*

1. *A set of formulas $\{\varphi_i(x) : i \in \mathbb{N}\}$ is n-inconsistent if for any n-choices $i(0), i(1), \dots, i(n-1)$ of indexes, the followings holds:*

$$\vDash \neg \exists x. \bigwedge_{j<n} \varphi_{i(j)}(x).$$

2. *We say that an L- formula $\varphi(x; a)$ is divide over A if there is an $n \in \mathbb{N}$ and seaquence $a_i(i \in \mathbb{N})$, such that (1) $a \equiv_A a_i(\forall i \in \mathbb{N})$ (2) $\{\varphi(x; a_i) : i \in \mathbb{N}\}$ is n-inconsistent.*

3. *We say that an L- formula $\varphi(x;a)$ forks over A if there are dividing formulas $\psi_0(x,b_0)$, ..., $\psi_{n-1}(x,b_{n-1})$ over A such that the followings holds:*

$$\vDash \bigvee_{i<n} \psi_i(x,b_i) \to \varphi(x,a)$$

A type p forks over A if every formula contained in p is forks over A.

4. *Let p be a complete type and q be a extension of p. Then q is a nonforking extension if q does not fork over parameters of p.*

5. *We will write $a \downarrow_B c$ if $\operatorname{tp}(a/B \cup \{c\})$ does not fork over B.*

Fact 4.2.3. *Fix T to be a stable theory and work on T here. Let a and b be tuples, A, B and C be sets.*

1. *(Transitivity) If $A \subset B \subset C$, $a \downarrow_A B$ and $a \downarrow_B C$ then $a \downarrow_A C$ holds.*

2. *(Monotonicity) If $a \downarrow_A C$ and $B \subset C$ then $a \downarrow_A B$ holds.*

3. *(Symmetry) $a \downarrow_A b$ implies $b \downarrow_A a$*

4. *(Local character) if $a \not\downarrow_A B$ then there is a finite tuple $b \in B$ such that $a \not\downarrow_A b$.*

5. *(Triviality) $a \downarrow_A a$ holds.*

6. *(Existence) For each a, A and B, there is a b such that $a \equiv_A b$ and $b \downarrow_A B$.*

7. *(Conjugacy) Let $p \in S(A)$ and $A \subset B$. Then p has at most $2^{|T|}$ nonforking extensions in $S(B)$.*

Definition 4.2.4. *We give the following definitions.*

1. *We say T is a small theory if for each n, $|S^n(T)| \leq |T|$.*

2. *(a) Let $p \in S(A)$ and $q \in S(B)$. We say p is orthogonal to q and write $p \perp q$ if the following holds. For every C containing $A \cup B$ and every a realizing p, b realizing q, if $a \downarrow_A C$ and $b \downarrow_B C$ then $a \downarrow_C b$. (b) Let $p \in S(A)$ and B a set. Then p is a orthogonal to B (written $p \perp B$) iff $p \perp q$ for each $q \in S(B)$.*

3. *We say $p \in S(A)$ is stationary if there is a unique nonforking extension of p over B for each set B containing A.*

4. *A non-algebraic type $p \in S(A)$ is regular if and only if every extension of p does not fork over A or is orthogonal to p.*

5. *We say* $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$ *if* $\models E(a,b)$ *holds for ever* $L(A)$*-definable finite equivalence relation* $E(x,y)$ *under* T*.*

6. *A type* $p \in S(a)$ *is multidimensional if and only if there exists a tuple* b *such that* $\mathrm{stp}(a) = \mathrm{stp}(b)$*,* $a \downarrow b$ *and* $p \perp p_b$ *where* $p_b \in S(b)$ *is a copy of type* p*.*

Remark 4.2.5. *1. Let* T *be a complete countable small theory. Then we have the following easy results. (a)* T *has a universal model. (b) Every formula can be extended to an isolated type.*

2. (The Open Mapping Theorem; see [19]) Let $A \subset B$ *and* $\phi(x,b) \in L(B)(b \in B)$*. Then there is a formula* $\psi(y) \in L(A)$ *such that for all* $p \in S(A)$*,* $\psi(y) \in p$ *if and only if there is a* $q \in S(B)$ *such that* $\phi(x,b) \in q$ *and* q *is a nonforking extension of* p*.*

We use the following machinery for analyzing small theory.

Definition 4.2.6. *1. A finite tuple* a *is said to be almost atomic over* A *if for each finite set* $A_0 \subset A$*, there is a finite* $A_1 \subset A$ *with* $A_0 \subset A_1$ *such that* $\mathrm{tp}(a/A_1)$ *is an isolated type.*

2. A set B *is said to be almost atomic over* A *if every finite tuple* $a \in B$ *is almost atomic over* A*.*

Lemma 4.2.7. *Let* $B \supset A$ *be a countable set which is almost atomic over* A *and* T *be a countable small theory. Let* $\varphi(x)$ *be a consistent* $L(B)$*-formula. Then there is an element* b *satisfying* $\varphi(x)$ *such that* Bb *is almost atomic over* A*.*

Proof. First choose an increasing sequence $\{B_i : i \in \omega\}$ of finite subsets of B such that $B = \bigcup_{i\in\omega} B_i$. We assume B_0 contains the parameters of φ. Then choose an increasing sequence $\{A_i : i \in \omega\}$ of finite subsets of A such that $\mathrm{tp}(B_i/A_i)$ is atomic. This is possible by the assumption that B is almost atomic over A. Next we find atomic types $p_i \in S(A_iB_i)$ such that (i) $\varphi(x) \in p_0$ and (ii) $p_i \subset p_j$ if $i < j$. Let $b \models \bigcup_{i\in\omega} p_i$.

Claim 4.2.8. *Bb is almost atomic over A.*

Let $X \subset B$ be a finite set. Choose $i \in \omega$ such that $Xb \subset A_iB_ib$. Then, by transitivity, $\mathrm{tp}(A_iB_ib/A_i)$ is atomic. So $\mathrm{tp}(X/A_i)$ is atomic. □

From Lemma 4.2.7, we deduce

Proposition. 4.2.9. *Let* A *be a countable set and* T *a countable small theory. Then there is a countable model* $M \supset A$ *that is almost atomic over* A*.*

Proof. Let $\{\varphi_i : i \in \omega\}$ be an enumeration of L-formulas with free variables from $\{x, y_0, y_1, \ldots.\}$. We assume that the free variables in φ_i are contained in $\{x, y_0, \ldots, y_{i-1}\}$ and that every formula appears infinitely many times in the enumeration. Starting from $A_0 = A$, we can inductively define A_i's such that

1. $A_i = A_{i-1} \cup \{a_{i-1}\}$;

2. $\varphi_i(a_i, A_i)$ if $\varphi_i(x, A_i)$ is consistent;

3. A_i is almost atomic over A.

Then $M = A \cup \{a_i : i \in \omega\}$ is a model satisfying the required condition. □

4.3 Proof of Theorem D

In this section, we prove Theorem D by constructing two countable models which are bi-elementary embeddable and non-isomorphic. Let p_a be a multidimensional stationary regular type in $S(a)$.Let a_0, a_1 be a Morley sequence of $\mathrm{tp}(a)$.

Define tuples $< M_l, N_l, f_l >_{l\in\omega}$ which consist of finite fragments of models and elementary maps $f_l : N_{l-1} \to M_l$. We construct the tuples such that $N = \bigcup_l N_l$ and $M = \bigcup_l M_l$ are models of T, and $f = \bigcup_l f_l$ is an elementary embedding. Moreover, we want these tuples to satisfy the following properties:

- $a_0 \downarrow M_l$,
- $N_l = a_0 M_l \bar{c}$ where $\mathrm{tp}(\bar{c}/a_0 M_l)$ is isolated,
- there exists a realization of $p_b|M_l$ in M_{l+1} for each $b \in M_l$ with $\mathrm{tp}(b) = \mathrm{tp}(a)$.

Let $\{\phi_l(z, X)\}_l$ be an enumeration of all formulas with $|X| < l$. M_0 equals $\{a_1\}$ and N_0 equals $\{a_0, a_1\}$. Let f_0 be the empty function. Suppose that a tuple $< N_l, M_l, f_l >$ is already defined. Then N_l equals $a_0 M_l \bar{c}$ where $\mathrm{tp}(\bar{c}/a_0 M_l)$ is isolated.

Next we define M_{l+1} and f_{l+1}. Let $\bar{x}$ be a tuple consisting of realizations of types in $\{p_b|M_l : b \in M_l, \mathrm{tp}(a) = \mathrm{tp}(b)\}$ and satisfying $\bar{x} \downarrow_{M_l} a_0$. Let $\bar{y}$ be a tuple satisfying $\mathrm{tp}(\bar{y}, f_l(N_{l-1})) = \mathrm{tp}(N_l - N_{l-1}, N_{l-1})$ and $\bar{y} \downarrow_{M_l \bar{x}} a_0$. Let d be a realization of the formula $\phi_l(z, M_l)$ which satisfies $d \downarrow_{M_l \bar{x}\bar{y}} a_0$. Since $\mathrm{tp}(\bar{c}/a_0 M_l)$ is isolated and every formula can be extended to an isolated type, there is a formula ψ with parameters from $a_0 M_l \bar{x}\bar{y}d$ which isolates a type over $a_0 M_l \bar{x}\bar{y}d$ containing $\mathrm{tp}(\bar{c}/a_0 M_l)$. Fix a realization $\bar{e}$ of ψ. By using an automorphism fixing $a_0 M_l$, there are tuples $\bar{x'}\bar{y'}d'$ such that $\mathrm{tp}(\bar{x'}\bar{y'}d'\bar{c}/a_0 M_l) = \mathrm{tp}(\bar{x}\bar{y}d\bar{e}/a_0 M_l)$. Then we define M_{l+1} by $M_l \bar{x'}\bar{y'}d'$ and f_{l+1} by an extension of f_l with the correspondence between $N_l - N_{l-1}$ and $\bar{y'}$.

Next we define N_{l+1}. By definition, $a_0 \downarrow M_{l+1}$ holds and $\text{tp}(\bar{c}/a_0 M_{l+1})$ is isolated. Since every formula can be extended to an isolated type, we can take a realization c' of isolated type over $a_0 M_{l+1}\bar{c}$ containing $\phi_l(z, N_l)$. Then we define N_{l+1} by $a_0 M_{l+1}\bar{c}c'$.

Clearly $< N_{l+1}, M_{l+1}, f_{l+1} >$ satisfies the desired properties. This completes our construction. We see that N is a model of T; M is an elementary submodel of N; $f : N \to M$ is an elementary embedding; and for each $b \in M$, $\text{tp}(b) = \text{tp}(a)$ implies $\dim(p_b, M) = \omega$. Without loss of generality, we may assume $\dim(p_{a_0}, M[a_0]) = 0$ where $M[a_0]$ is a prime model over $\{a_0\}$. Finally we prove the next claim.

Claim 4.3.1. $\dim(p_{a_0}, N) = 0$.

Proof. Suppose not. There is a realization $i \in N$ of p_{a_0}. So there is an $l \in \omega$ such that $i \in N_l$ i.e. $\text{tp}(i/a_0 M_l)$ is isolated. Since $\dim(p_{a_0}, M[a_0]) = 0$ and using the open mapping theorem, $i \not\downarrow_{a_0} M_l$ holds. On the other hand, $i \downarrow_{a_0} M_l$ follows from multidimensionality of p_{a_0} and $a_0 \downarrow M_l$. This is a contradiction. □

We have shown that M and N are not isomorphic.

4.4 Proof of Theorem E

We first give a result on a small theory.

Proposition. 4.4.1. *Let T be a small theory with a non-atomic multidimensional type. Then T does not have the SB property for countable models.*

Proof. Let M be a countable saturated model. Let $p_a(x)$ be a non-atomic type witnessing the multidimensionality. We can assume that a and M are independent. Let N be a countable almost atomic model over Ma. Since M is saturated, M and N are bi-embeddable.

Claim 4.4.2. *$p_a(x)$ is not realized in N.*

By way of contradiction, suppose that $d \in N$ realizes p_a. Since N is almost atomic over Ma, there is a finite set $M_0 \subset M$ such that $\text{tp}(d/M_0 a)$ is an atomic type. By the fact that M_0 and a are independent, using $p_a \perp \varnothing$, we deduce that $\text{tp}(d/M_0 a)$ does not fork over a. By the open mapping theorem, we have that $p_a = \text{tp}(d/a)$ is an atomic type. This is a contradiction.(End of Claim 4.4.2) □

Definition 4.4.3. Let $\varphi(x)$ be a formula (with parameters) and let $p(x) \in S(A)$ be a type containing $\varphi(x)$. We say that (p, φ) is a strongly regular type if whenever $A \subset B$, $p^*(x) \in S(B)$ and $\varphi(x) \in p^*(x)$ then either (i) $p^*(x)$ is a nonforking extension of $p(x)$ or (ii) $p^*(x) \perp p(x)$.

We now begin proving Theorem 3. Let M be a countable saturated model. Notice that any countable model (elementarily) extending M is a universal model. So we want to construct infinitely many countable models extending M. The following claim is shown by exactly the same argument as before.

Claim 4.4.4. *Let $p_a(x)$ be a non-atomic type such that $p_a \perp \varnothing$. Let $a_0, \dots, a_{n-1}$ be elements such that $\mathrm{tp}(a_i) = \mathrm{tp}(a)$ $(i < n)$ and that the set $\{M, a_0, \dots, a_{n-1}\}$ is independent. Let N be an almost atomic model over $Ma_0 \dots a_{n-1}$. Then $p_{a_i}(x)$ is not realized in N.*

From now on, we assume (p_a, φ) is a strongly regular type with $p_a \perp \varnothing$.

Claim 4.4.5. *Let N be a model. Let $a, b \in N$ be elements with $\mathrm{tp}(b) = \mathrm{tp}(a)$ and $a \downarrow b$. Let d be any element realizing $p_b|N$. Let N' be an almost atomic model over Nd. Then $\dim(p_a, N) = \dim(p_a, N')$.*

Let $I \subset N$ be an independent set witnessing the dimension $\dim(p_a, N)$. By way of contradiction, we assume $\dim(p_a, N) < \dim(p_a, N')$. Then there is $e \in N'$ realizing $p_a|(aI)$. Clearly $\mathrm{tp}(e/N)$ does not fork over $aI \cup \varphi^N$. Notice that any element in φ^N does not realize $p_a|(aI)$. So, by the strong regularity, we can inductively show that e and $\{b_i\}_{i<n}$ are independent over aI, where $\{b_i\}_{i\in\omega}$ is an enumeration of the set φ^N. So $\mathrm{tp}(e/aI \cup \varphi^N)$ does not fork over aI. Now by the transitivity of nonforking, e realizes $p_a|N$. By the multidimensionality, $p_a|N \perp p_b|N$, so d and e are independent over N. In other words, $\mathrm{tp}(e/Nd)$ is a nonforking extension of p_a. Since $\mathrm{tp}(e/Nd)$ is almost atomic, for sufficiently large $N_0 \subset N$, $\mathrm{tp}(e/N_0 d)$ is an atomic type. Thus, by the open mapping theorem, we conclude that p_a is an atomic type. This is a contradiction. (End of Claim 4.4.5)

Let $\{a_i : i \in \omega\}$ be a generically M-independent set with $\mathrm{tp}(a_i) = \mathrm{tp}(a)$ $(i \in \omega)$. For each $n \in \omega$, let N_n be a countable almost atomic model over $M \cup \bigcup_{i<n} a_i$. Then $\dim(p_{a_i}, N_n) = 0$ $(i < n)$. Now we apply Claim 4.4.5 countably many times and we get a countable model $N_n^* \supset N_n$ with the following properties:

1. $\dim(p_{a_i}, N_n^*) = 0$ $(i < n)$;
2. $\dim(p_{a'}, N_n^*) = \omega$, if $a' \in N_n^*$, $a' \equiv a$ and $a' \downarrow \{a_0, \dots, a_{n-1}\}$.

Claim 4.4.6. *Among the N_n^* $(n \in \omega)$, there are infinitely many non-isomorphic models.*

Let $m = \mathrm{w}t(a)$, where $\mathrm{w}t$ is the abbreviation of weight. By superstability, m is finite. We show that N_n^* and N_k^* are not isomorphic if $k \geq mn + 1$. Suppose otherwise and let $\sigma : N_n^* \to N_{mn+1}^*$ be isomorphism between the two models. Since $\dim(p_{a_i}, N_{mn+1}^*) = 0$ $(i = 0, \dots, mn)$, by the property 2 above, each a_i is dependent with $b = \sigma(a_0), \dots, \sigma(a_{n-1})$.

This means that the weight of b is $\geq mn + 1$. On the other hand, by a weight calculation, we have $\mathrm{wt}(b) = \mathrm{wt}(a_0, \ldots, a_{n-1}) = n \cdot \mathrm{wt}(a) = mn$. This is a contradiction. (End of Claim 4.4.6)

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