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CHARACTERIZATIONS AND PROPERTIES OF STRATIFIABLE SPACES

By

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Abstract. In this paper, we prove some properties and characterizations of stratifiable spaces and the following theorem:

Theorem. The following are equivalent:

1 (Y, τ) is a stratifiable space.

2 There is a zero-dimension submetric stratifiable space (X, μ) with M_3-structures and an irreducible perfect map $f : (X, \mu) \to (Y, \tau)$.

A stratifiable space (X, μ) is said to have *an M_3-structure* if (X, μ) satisfies the following conditions A and B:

A. There is a countable collection $\mathscr{H} = \bigcup_n \mathscr{H}_n$ of ϱ-closed sets such that:

1 $H(n', i') \subset H(n, i)$ or $\varrho(H(n, i), H(n', i')) = r > 0$ if $H(n, i)$, $H(n', i') \in \mathscr{H}$ with $n' > n$.

2 $\mathscr{H}_n$ is a partition of X for each $n \in N$.

B. There is a g-function $\mathscr{W}$ such that:

1 $\bigcap_n W(n, x) = \{x\}$.

2 $x \in W(n, x_n)$, then $\{x_n : n \in N\}$ converges to x.

3 If H is closed and $x \notin H$, $x \notin Cl_\mu(\bigcup\{W(n, x') : x' \in H\})$ for some n.

4 $x' \in W(n, x)$ implies $W(n, x') \subset W(n, x)$.

5 $H(n, i) \cap (\bigcup \mathscr{W}_{nj}) = \varnothing$ if $j > i$.

6 $W(n, x) \subset W(n-1, x)$.

7 Each $\mathscr{W}_{nm}$ is a ϱ-discrete ϱ-clopen collection.

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8 $W(n,x) \subset c(n,x) \in \mathscr{C}$ for each $x \in X$.
Here $\mathscr{C}$ is a g-function of the stratifiable space (X,μ).

1. Introduction

Ceder [3] defined M_i-spaces, $i = 1,2,3$ and proved $M_1 \Rightarrow M_2 \Rightarrow M_3$. It is an interesting problem that whether or not these implications can be reversed. Recall that a space X is an M_1-*space* if X has a σ-closure preserving base $\mathscr{B}$. Recall that a collection $\mathscr{B}$ is a *quasi-base* for X if for each open set U of X and a point $x \in U$, there is $B \in \mathscr{B}$ such that $x \in Int\, B \subset B \subset U$. A space X is an M_2-*space* if X has a σ-closure preserving *quasi-base* and an M_3-*space* if X has a σ-cushioned pair-base.

Borges [1] gave some important results on M_3-spaces and renamed M_3-spaces as stratifiable spaces. Gruenhage [4] and Junnila [8] independently proved that stratifiable spaces are M_2-spaces. This is an important progress to the problem since stratifiable spaces have been shown to have many useful properties and are preserved by countable products, closed images, arbitrary subspaces; M_1-spaces have a simple and natural definition. Itō and Tamano [7] using closed mappings got interesting results. T. Mizokami got some important progresses on the problem in [10], [11] and [12]. Also there are many important results about stratifiable spaces commended by surveys of Tamano [15], Gruenhage [5] and [6], Burke and Luter [2].

We are going to show characterizations of stratifiable space (Y,τ). To do it we prove some properties of stratifiable spaces (Y,τ) in section 2. In section 3, we construct a stratifiable space (X,μ) which has a 0-dimensional submetric and an irreducible perfect map f from (X,μ) to (Y,τ). Section 4 contains two g-functions of stratifiable space (X,μ) and countably many partitions $\bigcup_n \mathscr{H}_n$ of X. A g-function $\mathscr{C}$ is used for relating (X,μ) and (Y,τ) and the another g-function $\mathscr{W}$ has closed and open images. We show characterizations and raise a problem in section 5.

In this paper, the letter N denotes the set of positive integers and ω denotes the first infinite ordinal. i, j, k, l, m and n are used to denote members in ω and N. If there are signs and definitions which have not been defined in this paper, we can see it in [5] or [15] in topology and in [9] in set theory.

2. Properties of (Y,τ)

An useful characterization of stratifiable spaces was given by Theorem 5.25, Lemma 5.26 and Theorem 5.27 in [5] as the following.

THEOREM 2.1. *A space Y is a stratifiable space if and only if there is a g-function $g : \omega \times Y \to \tau$ such that*

i $\{y\} = \bigcap_n g(n, y)$;
ii $y \in g(n, y_n) \Rightarrow y_n \to y$;
iii $y \notin Cl_\tau \cup \{g(n, y) : y \in H\}$ *for some* $n \in \omega$ *if* H *is closed and* $y \notin H$.
iv $y \in g(n, x)$ *then* $g(n, y) \subset g(n, x)$.
v $g(n+1, y) \subset g(n, y)$ *for each* y.

Let $\mathscr{G}_n = \{g(n, y) : y \in Y\}$ and $\mathscr{G} = \bigcup_n \mathscr{G}_n$.

DEFINITION 2.2. *A locally finite collection $\mathscr{B}$ of open sets of (Y, τ) is called a tangent cover of Y if $\mathscr{B}$ is pairwise disjoint with $\bigcup\{Cl\ B : B \in \mathscr{B}\} = Y$.*

PROPOSITION 2.3. *Let (Y, τ) be a stratifiable space. Then there is a submetric $\rho \subset \tau$ and countably many tangent cover $\mathscr{B} = \bigcup_n \mathscr{B}_n$ such that*:

1 $B_{n\alpha}$ *is a ρ-open set and* $\rho(B_{n\alpha}) < 1/n$ *for each* $n \in N$ *and each* $B_{n\alpha} \in \mathscr{B}_n$.
2 $Cl_\rho(\bigcup \mathscr{B}') = Cl_\tau(\bigcup \mathscr{B}')$ *for each* n *and each* $\mathscr{B}' \subset \mathscr{B}_n$.
3 $B_{n\alpha} \subset B_{m\beta}$ *or* $B_{n\alpha} \cap B_{m\beta} = \emptyset$ *if* $n > m$ *for* $B_{n\alpha} \in \mathscr{B}_n$ *and* $B_{m\beta} \in \mathscr{B}_m$.

PROOF. Let $\mathscr{B}' = \bigcup_n \mathscr{B}'_n$ be a σ-discrete base of submetric ρ with $\rho(B_{n\alpha}) < 1/n$ for each $n \in N$ and each $B_{n\alpha} \in \mathscr{B}_n$. Let $\mathscr{O} = \bigcup_n \mathscr{O}_n$ be a σ-locally finite cover of Y with $\rho(O) < 1/n$ for each $n \in N$ and each $O \in \mathscr{O}_n$. Let $|\mathscr{O}_n| = \aleph(n)$.

Pick an $n \in N$. Let $B_0 = \bigcup \mathscr{B}'_n$. If $Cl_\tau\ B_0 = Y$, let $\mathscr{B}_n = \mathscr{B}'_n$. If $Y - Cl_\tau\ B_0 \neq \emptyset$, then there is an $O \in \mathscr{O}_n$ with $O - Cl_\tau\ B_0 \neq \emptyset$. Let $B_1 = O - Cl_\tau\ B_0$.

Assume that, for $\alpha < \aleph(n)$, we have had B_β for each $\beta < \alpha$. If $Y = \bigcup_{\beta<\alpha} Cl_\tau\ B_\beta$, we take $\mathscr{B}_n = \mathscr{B}'_n \cup \{B_\beta : 0 < \beta < \alpha\}$. Otherwise $Y - \bigcup_{\beta<\alpha} Cl_\tau\ B_\beta \neq \emptyset$. Then there is an $O \in \mathscr{O}_n$ such that $O - \bigcup_{\beta<\alpha} Cl_\tau\ B_\beta \neq \emptyset$. Let $B_\alpha = O - \bigcup_{\beta<\alpha} Cl_\tau\ B_\beta$. Then B_α is closed since $\mathscr{O}_n$ is a locally finite cover of Y.

Then, by induction, there is a $\delta \leq \aleph(n)$ such that $\mathscr{B}_n = \mathscr{B}'_n \cup \{B_\alpha : 0 < \alpha < \delta\}$ and $Y = Cl_\tau(\bigcup_n \mathscr{B}_n)$. So we may assume $\mathscr{B}_n = \{B_\alpha : \alpha < \aleph(n)\}$ is a tangent cover of Y.

Let $\mathscr{B} = \bigcup_n \mathscr{B}_n$. Pick an n from N. Let

$$\mathscr{B}''_n = \{\textstyle\bigcap_{i\leq n} B_{i\alpha} : B_{i\alpha} \in \mathscr{B}_i \text{ for } i \leq n \text{ if } \bigcap_{i\leq n} B_{i\alpha} \neq \emptyset\}.$$

Then $\bigcup_n \mathscr{B}''_n$ satisfies 1 and 3 since each $\mathscr{B}_n$ is a tangent cover of Y.

Notice each $\mathscr{B}''_n$ is locally finite. Then $\mathscr{B}''_n$ is closure preserving. So, by Lemma 2.21 of Tamano [15], there is a submetric ρ_1 such that $\rho \subset \rho_1 \subset \tau$ and $\mathscr{B}''_n$ is also a closure preserving closed collection of (Y, ρ_1). Then ρ_1 and $\mathscr{B}'' = \bigcup_n \mathscr{B}''_n$ is desired. We denote ρ_1 and $\mathscr{B}'' = \bigcup_n \mathscr{B}''_n$ by ρ and $\mathscr{B} = \bigcup_n \mathscr{B}_n$ still. □

We call $\mathscr{B} = \bigcup_n \mathscr{B}_n$ a *decreasing σ-tangent cover* and use the decreasing σ-tangent cover $\mathscr{B}$ to construct a perfect pre-image of (Y, τ) and an irreducible perfect map in the following section.

For the above g-function in Theorem 2.1, we have $\mathscr{G}_n = \{g(n, y) : y \in Y\}$ for each n. Let $B_{n\alpha} = Y - \bigcup \mathscr{G}_\alpha$ for each $\mathscr{G}_\alpha \subset \mathscr{G}_n$. Let $\mathscr{B}'_n = \{B_{n\alpha} : \mathscr{G}_\alpha \subset \mathscr{G}_n\}$. Then, by Theorem 5.25 in [5], $\mathscr{B}' = \bigcup_n \mathscr{B}'_n$ is a τ σ-closure preserving τ-closed quasi-base of (Y, τ).

Then, by Lemma 2.21 of Tamano [15], there is a metrizable topology $\rho \subset \tau$ on Y such that each $\mathscr{B}'_n$ is a collection ρ-closure preserving ρ-closed sets. Denote the submetric topology by (Y, ρ). So, by Theorem 5.25 in [5], we may assume that $g(n, y)$ is ρ-open set for each $n \in N$ and each $y \in Y$.

Let $K'_{ni} = \{x \in Y : \bigcup\{g(n+i, y) \in \mathscr{G}_{n+i} : x \in g(n+i, y)\} \subset g(n, x)\}$.

Let $K_{ni} = \{x \in Y : x \in g(n+i, y) \Rightarrow y \in g(n, x)\}$. We have the following proposition.

PROPOSITION 2.4. *$K_{ni} = K'_{ni}$ and K_{ni} is ρ-closed.*

PROOF. Pick an $x \in K'_{ni}$. Then $\bigcup\{g(n+i, y) : x \in g(n+i, y)\} \subset g(n, x)$. So $x \in g(n+i, y)$ implies $g(n+i, y) \subset g(n, x)$. Then $y \in g(n, x)$ implies $x \in K_{ni}$.

Let $x \in K_{ni}$. Pick a $g(n+i, y) \in \mathscr{G}$ with $x \in g(n+i, y)$. Then $y \in g(n, x)$. Then $g(n+i, y) \subset g(n, y) \subset g(n, x)$ by iv of Theorem 2.1. So $x \in K'_{ni}$.

Pick an $x \in Cl_\rho(K_{ni})$. Let $g(n+i, y) \in \mathscr{G}$ with $x \in g(n+i, y)$. Then $x \in g(n+i, y) \cap g(n+i, x)$. Then $g(n+i, y) \cap g(n+i, x)$ is a ρ-open neighborhood of x since both $g(n+i, y)$ and $g(n+i, x)$ are ρ-open. So there is a $z \in (g(n+i, y) \cap g(n+i, x)) \cap K_{ni}$. Then $g(n, y) \subset g(n, z)$ since $z \in g(n+i, y)$ and $z \in K_{ni}$. $z \in g(n+i, x) \subset g(n, x)$ implies $g(n, z) \subset g(n, x)$. Then we have $g(n+i, y) \subset g(n, y) \subset g(n, z) \subset g(n, x)$. So $y \in g(n, x)$. This implies $x \in K_{ni}$. So K_{ni} is ρ-closed. □

PROPOSITION 2.5. *$K_{ni} \subset K_{ni+1}$ for each i, n and $\bigcup_{i \in \omega} K_{ni} = Y$.*

PROOF. Pick an $x \in K_{ni}$. Then $x \in g(n+i, y)$ implies $y \in g(n, x)$. So $g(n, y) \subset g(n, x)$. Let $x \in g(n+i+1, y)$. We have $g(n+i+1, y) \subset g(n, y)$ by (iv) in Theorem 2.1. So $g(n+i+1, y) \subset g(n, x)$ and $x \in K_{ni+1}$.

Suppose that there is an $x \in Y - \bigcup_{i \in \omega} K_{ni}$. Then x is not in K_{ni} for each $i \in \omega$. So there is a $g(n+i, y_i)$ such that $x \in g(n+i, y_i)$ and y_i is not in $g(n, x)$. Notice $\{y_i : i \in \omega\}$ τ-converges to x and $x \in g(n, x)$. Then $\{y_i : i \in \omega\}$ is eventually in $g(n, x)$, a contradiction. □

PROPOSITION 2.6. *$K_{n+1i} \subset K_{ni}$ for each i, n.*

PROOF. Pick an $x \in K_{n+1i}$. Then $\bigcup\{g((n+1)+i, y) \in \mathscr{G}_{n+1} : x \in g((n+1)+i, y)\} \subset g(n+1, x) \subset g(n, x)$. So $x \in K_{ni}$. □

Now we construct another g-function by induction. It is similar to Lemma 5.26 in [5] by using the above g-function $\mathscr{G}$ and $\mathscr{K} = \{K_{ni} : n \in N \text{ and } i \in \omega\}$.

Construction 2.

A. For $k = 1$, we take $\mathscr{K}_1 = \{K_{1i} : i \in \omega\} \subset \mathscr{K}$.

A.1. We take K_{10} from $\mathscr{K}_1$ for $i = 0$. Then $\mathscr{G}_1$ is an ρ-open cover of ρ-closed set K_{10}. So there is a ρ-locally finite ρ-open refinement $\mathscr{Q}_{10}$ of $\mathscr{G}_1$. Pick an $x \in K_{10}$. Let $\mathscr{Q}_{10}(x) = \{Q \in \mathscr{Q}_{10} : x \in Q\}$ and $g(1, 0, x) = \bigcap \mathscr{Q}_{10}(x)$. Let $\mathscr{G}_{10} = \{g(1, 0, x) : x \in K_{10}\}$.

A.2. Assume that we have had $\mathscr{G}_{1i}$ for $i < m$. Take K_{1m} from $\mathscr{K}_1$. Then $\mathscr{G}_{1+m} = \{g(1+m, x) : x \in K_{1m}\}$ is a ρ-open cover of ρ-closed set K_{1m}. So there is a ρ-locally finite ρ-open refinement $\mathscr{Q}_{1m}$. Let $\mathscr{Q}'_{1m} = \bigcup_{i<m} \mathscr{G}_{1i}$. Let $\mathscr{Q}^*_{1m} = \mathscr{Q}_{1m} \cup \mathscr{Q}'_{1m}$.

Pick an $x \in K_{1m} - K_{1m-1}$. Let $\mathscr{Q}^*_{1m}(x) = \{Q \in \mathscr{Q}^*_{1m} : x \in Q\}$ and $g(1, m, x) = (\bigcap \mathscr{Q}^*_{1m}(x)) - K_{1m-1}$. Then $g(1, m, x)$ is ρ-open since $\mathscr{Q}^*_{1m}$ is a ρ-locally finite collection of ρ-open sets and K_{1m-1} is ρ-closed set. Let $\mathscr{G}_{1m} = \{g(1, m, x) : x \in K_{1m} - K_{1m-1}\}$. Then $\mathscr{G}_{1m}$ is a ρ-locally finite collection of ρ-open sets since $\mathscr{Q}^*_{1m}$ is a ρ-locally finite collection of ρ-open sets. Then, by induction, we have $\mathscr{G}_{1m}$ for each m. Let $\mathscr{G}'_1 = \bigcup_m \mathscr{G}_{1m}$.

B. Assume that we have had $\mathscr{G}'_k$ for $k < n$ such that:

a. $(\bigcup \mathscr{G}_{ki}) \cap K_{ni-1} = \varnothing$ for each $i \in N$.

b. $\mathscr{G}_{ki}$ is a ρ-locally finite ρ-open collection for each $i \in N$.

For $k = n$, we take $\mathscr{K}_n = \{K_{ni} : i \in \omega\}$.

B.1. Take $K_{n0} \in \mathscr{K}_n$. Then $\mathscr{G}_n = \{g(n, x) : x \in Y\}$ is a ρ-open cover of ρ-closed set K_{n0}. So there is a ρ-locally finite ρ-open refinement $\mathscr{Q}_{n0}$ of $\mathscr{G}_n$.

Let $\mathscr{Q}'_{n0} = \bigcup_{j<n} \mathscr{G}_{j0}$. Let $\mathscr{Q}^*_{n0} = \mathscr{Q}_{n0} \cup \mathscr{Q}'_{n0}$. Pick an $x \in K_{n0}$. Let $\mathscr{Q}^*_{n0}(x) = \{Q \in \mathscr{Q}^*_{n0} : x \in Q\}$ and $g(n, 0, x) = \bigcap \mathscr{Q}^*_{n0}(x)$. Let $\mathscr{G}_{n0} = \{g(n, 0, x) : x \in K_{n0}\}$.

B.2. Assume that we have had $\mathscr{G}_{ni}$ for $i < m$. Take K_{nm} from $\mathscr{K}$. Then $\mathscr{G}_{n+m} = \{g(n+m, x) : x \in K_{nm}\}$ is a ρ-open cover of ρ-closed set K_{nm}. So there is a ρ-locally finite ρ-open refinement $\mathscr{Q}_{nm}$ of $\mathscr{G}_{n+m}$. Let $\mathscr{Q}'_{nm} = \bigcup_{i \le m} \bigcup_{k<n} \mathscr{G}_{ki}$. Let $\mathscr{Q}''_{nm} = \bigcup_{i<m} \mathscr{G}_{ni}$. Let $\mathscr{Q}^*_{nm} = \mathscr{Q}_{nm} \cup \mathscr{Q}'_{nm} \cup \mathscr{Q}''_{nm}$. Pick an $x \in K_{nm} - K_{nm-1}$. Let $\mathscr{Q}^*_{nm}(x) = \{Q \in \mathscr{Q}^*_{nm} : x \in Q\}$ and $g(n, m, x) = (\bigcap \mathscr{Q}^*_{nm}(x)) - K_{nm-1}$. Then $g(n, m, x)$ is ρ-open since $\mathscr{Q}^*_{nm}$ is a collection of ρ-locally finite ρ-open sets and K_{nm-1} is

ρ-closed set. Let $\mathscr{G}_{nm} = \{g(n,m,x) : x \in K_{nm} - K_{nm-1}\}$. Then $\mathscr{G}_{nm}$ is a collection of ρ-locally finite ρ-open sets.

Then, by induction, we have $\mathscr{G}_{nm}$ for $m \in \omega$. Let $\mathscr{G}'_n = \bigcup_{m\in\omega} \mathscr{G}_{nm}$.

Then, by induction, we have $\mathscr{G}'_n$ for each $n \in N$. Let $\mathscr{G}' = \bigcup_{n\in\omega} \mathscr{G}'_n$.

We have the following theorem by the above Construction 2.

THEOREM 2.7. *(Y,τ) is a stratifiable space if and only if there is a g-function $\mathscr{G}$ of (Y,τ) such that:*

1 $\bigcap_n g(n,i,y) = \{y\}$.

2 *If* $x \in g(n,i,y_n)$, $\{y_n : n \in N\}$ *τ-converges to* y.

3 *If H is τ-closed and $y \notin H$, $y \notin Cl_\tau(\bigcup\{g(n,i,y) : y \in H\})$ for some n.*

4 $y \in g(n,i,x)$ implies $g(n,i',y) \subset g(n,i,x)$ for some i'.

5 $g(n+1,i',x) \subset g(n,i,x)$.

6 $K_{ni-1} \cap (\bigcup \mathscr{G}'_{ni}) = \varnothing$.

7 *Each $\mathscr{G}'_{ni}$ is a ρ-locally finite ρ-open collection.*

8 $g(n,i,y) \subset g(n,y)$ *for* $y \in Y$.

PROOF. We prove 8 at first. To do it giving an n, pick a $y \in Y = \bigcup_i K_{ni}$. Then there is an i with $y \in K_{ni}$. We have $g(n,i,y) \subset g(n+i,z)$ for some z since $\mathscr{Q}_{ni}$ is a refinement of $\mathscr{G}_{n+i}$. Then $z \in g(n,y)$ since $y \in g(n+i,z)$ and $y \in K_{ni}$. So $g(n,z) \subset g(n,y)$. We have $g(n,i,y) \subset g(n+i,z) \subset g(n,z) \subset g(n,y)$.

PROOF OF 1. Notice $y \in g(n,i_n,y)$ and $g(n,i_n,y) \subset g(n,y)$ for $y \in Y$ and $n \in N$. Then $\bigcap_n g(n,i_n,y) \subset \bigcap_n g(n,y) = \{y\}$.

PROOF OF 2. In fact, $y \in g(n,i_n,y_n) \subset g(n,y_n)$ implies that $\{y_n : n \in N\}$ τ-converges to y.

PROOF OF 3. In fact, $g(n,i,y) \subset g(n,y)$ implies $\bigcup\{g(n,i,y) : y \in H\} \subset \bigcup\{g(n,y) : y \in H\}$.

PROOF OF 6. Notice that $g(n,m,x) = (\bigcap \mathscr{Q}^*_{nm}(x)) - K_{nm-1}$ for each $g(n,m,x) \in \mathscr{G}'_{nm}$ by B.2) in Construction 2.

PROOF OF 5. In fact, take $g(n+1,i,x)$ from $\mathscr{G}'_{n+1i}$. This implies $x \in K_{n+1i} \subset K_{ni}$ by Proposition 2.7. Then $g(n,i,x) \in \mathscr{Q}^*_{n+1i}$ by B.2) in Construction 2. So $g(n+1,i,x) = (\bigcap \mathscr{Q}^*_{n+1i}(x)) - K_{n+1i-1} \subset g(n,i,x)$.

Proof of 4. In fact, pick $x \in g(n,i,y) = (\bigcap \mathcal{Q}^*_{ni}(y)) - K_{ni-1} \subset \bigcap \mathcal{Q}^*_{ni}(y)$. Let $x \in K_{ni'} - K_{ni'-1}$. Then $i' \geq i$ by the above 6. Then $\mathcal{Q}^*_{ni}(y) \subset \mathcal{Q}^*_{ni'}(x)$ by B.2) in Construction 2. So $\bigcap \mathcal{Q}^*_{ni'}(x) \subset \bigcap \mathcal{Q}^*_{ni}(y)$. Then $(\bigcap \mathcal{Q}^*_{ni'}(x)) - K_{ni'-1} \subset (\bigcap \mathcal{Q}^*_{ni}(y)) - K_{ni-1}$ since $i \leq i'$ implies $K_{ni-1} \subset K_{ni'-1}$.

Proof of 7. Notice that $\mathcal{Q}^*_{ni}$ is a ρ-locally finite collection of ρ-open sets. Then $\mathcal{G}_{ni}$ is a ρ-locally finite collection of ρ-open sets by the definition of $g(n,i,x)$ in B.2) in Construction 2. □

Notice that, for each $n \in N$ and $y \in Y$, there is unique $g(n,m,y) \in \mathcal{G}'_n$ for some $m \in \omega$. So denote $g(n,y)$ by $g(n,m,y)$ sometimes. We still use $\mathcal{G}$ and $\mathcal{G}_n$ $(n \in N)$ to express the constructed collection of g-function in Construction 2. In the following sections, we'll use the g-function $\mathcal{G}$ and $\mathcal{K} = \{K_{ni} : n \in N$ and $i \in \omega\}$.

Corollary 2.8. *$\mathcal{G}$ is a σ-locally finite base of (Y,ρ).*

Give an n. Let $B'_{n\alpha} = Y - \bigcup \mathcal{G}_\alpha$ for each $\mathcal{G}_\alpha \subset \mathcal{G}_n$. Let $\mathcal{B}'_n = \{B'_{n\alpha} : \mathcal{G}_\alpha \subset \mathcal{G}_n\}$. Then, by Theorem 5.25 in [5], $\mathcal{B}' = \bigcup_n \mathcal{B}'_n$ is a ρ σ-closure preserving ρ-closed quasi-base of (Y,τ).

Let $\mathcal{B}$ be the decreasing σ-tangent cover in Proposition 2.3 with $\mathcal{B} = \bigcup_{n \in N} \mathcal{B}_n$ and $\mathcal{B}_n = \{B_\alpha : \alpha \in \aleph(n)\}$. Let

$$\bar{\mathcal{B}}_n = \{\bar{B}_\alpha : \alpha \in \aleph(n)\}.$$

Here $\bar{B}_\alpha = Cl_\rho(\bigcup \mathcal{B}') = Cl_\tau(\bigcup \mathcal{B}')$ by 2 of Proposition 2.3. Let

$$\bar{\mathcal{B}} = \bigcup_{n \in N} \bar{\mathcal{B}}_n.$$

In the following section we'll use the collections $\bar{\mathcal{B}}$ and $\mathcal{B}'$.

3. To Construct X and f

At first we construct a metric space (X,ϱ) and a perfect map $f : (X,\varrho) \to (Y,\rho)$. This method belongs to Michael [13]. To do it take $\bar{\mathcal{B}} = \bigcup_n \bar{\mathcal{B}}_n$ and give $\aleph(n)$ a discrete topology. Then countable product $\Pi_n \aleph(n)$ is a metric space. Let $M = \Pi_n \aleph(n)$. Pick $x' = (\alpha'_n)$ and $x'' = (\alpha''_n)$ from M. Let $\varrho(x',x'') = 1/n$ if n is the first index with $\alpha'_n \neq \alpha''_n$. Then ϱ is a metric of M. Let $X \subset M$ be all (α_n)'s with $\bigcap_n \bar{B}_{\alpha_n} \neq \emptyset$ and $\bigcap_{i \leq n} B_{\alpha_i} \neq \emptyset$ for each $n \in N$. Then (X,ϱ) is a metric space.

Giving an $n \in N$, let $V(\alpha_1, \dots, \alpha_n) = \{x \in X : P_i(x) = \alpha_i \text{ for } i \le n\}$. Then

$$V(\alpha_1, \dots, \alpha_n) = (\{\alpha_1\} \times \cdots \times \{\alpha_n\} \times \Pi_{i>n}\aleph(i)) \cap X.$$

Let $\mathscr{V}_n = \{V(\alpha_1, \dots, \alpha_n) : \alpha_i \in \aleph(i) \text{ for } i \le n\}$. Here $P_n : \Pi_{n>0}\aleph(n) \to \aleph(n)$ is a projection with product topology. Then $\mathscr{V} = \bigcup_{n \in N} \mathscr{V}_n$ is a σ-discrete base of (X, ϱ). We call σ-discrete base $\mathscr{V} = \bigcup_{n \in N} \mathscr{V}_n$ of (X, ϱ) *standard base.*

Let $H \subset X$ be a ϱ-closed set. Let $B(H, 1/n) = \bigcup \{V(\alpha_1, \dots, \alpha_n) \in \mathscr{V}_n : x = (\alpha_i) \in H\}$. Then $B(H, 1/n)$ is a ϱ-clopen ball with $\bigcap_n B(H, 1/n) = H$. Let $R(H, 1/n) = B(H, 1/n) - B(H, 1/(n+1))$.

Proposition 3.1. *(X, ϱ) is a closed 0-dimensional subspace of (M, ϱ) such that*:

1. *$V \cap V' \ne \emptyset$ implies $V \subset V'$ or $V' \subset V$ for $V, V' \in \mathscr{V}$.*
2. *$R(H, 1/n) = \{x : \varrho(x, H) = 1/(n+1)\}$.*
3. *$\varrho(R(H, 1/n), R(H, 1/(n+1))) = r > 0$*

Proof. We prove that (X, ϱ) is a closed subspace of (M, ϱ) at first.

To see it take a sequence $S = \{x_n = (\alpha(n1), \alpha(n2), \dots) : n \in N\}$ from (X, ϱ). Assume S converges to $x = (\alpha(n))$ with $\varrho(x_n, x) = 1/(n+1)$. Let $f(x_n) = y_n$ and $f(x) = y$. Giving an $i \ge 1$, we have $\alpha(ni) = \alpha(i)$ for each $n \ge i$ by definition of ϱ. So $\{y_n : n \ge i\} \subset \bar{B}_{\alpha(ni)} = \bar{B}_{\alpha(i)}$. $\rho(\bar{B}_{\alpha(n)}) \le 1/n$ implies $\{y_n : n \in N\}$ converging to $y \in \bar{B}_{\alpha(i)}$. Notice that $\mathscr{B}_n$ is a tangent cover. Then $B_\alpha \cap B_\beta = \emptyset$ if $\alpha \ne \beta$ for B_α and B_β in $\mathscr{B}_n$. This implies $\bar{B}_\alpha \ne \bar{B}_\beta$. So $\alpha(ni) = \alpha(i)$ implies $\bar{B}_{\alpha(ni)} = \bar{B}_{\alpha(i)}$ and $B_{\alpha(ni)} = B_{\alpha(i)}$. Then $\bigcap_{i \le n} B_{\alpha(i)} \ne \emptyset$ since $\bigcap_{i \le n} B_{\alpha(ni)} \ne \emptyset$. Then $x = (\alpha(i)) \in X$. So (X, ϱ) is a closed subspace of (M, ϱ).

And then we prove 2 only. To see it pick a $t = (\alpha_i) \in R(H, 1/n)$. Then $t \in B(H, 1/n)$. So there is an $x' = (\alpha'_i) \in H$ with $t = (\alpha_i) \in V(\alpha'_1 \cdots \alpha'_n)$. Then $\alpha'_i = \alpha_i$ for $i \le n$. So, for each $x'' = (\alpha''_i) \in H \cap V(\alpha'_1 \cdots \alpha'_n)$, we have $\alpha''_i = \alpha'_i = \alpha_i$ for $i \le n$.

On the another hand, $t \in R(H, 1/n)$ implies $t \notin B(H, 1/(n+1))$. Then $t \notin \bigcup \{V(\alpha''_1 \cdots \alpha''_{n+1}) \in \mathscr{V}_{n+1} : x'' = (\alpha''_i) \in H \cap V(\alpha'_1 \cdots \alpha'_n)\}$. Then $t \notin V(\alpha''_1 \cdots \alpha''_{n+1})$ for each $x'' = (\alpha''_i) \in H \cap V(\alpha'_1 \cdots \alpha'_n)$. Then $\alpha''_{n+1} \ne \alpha_{n+1}$. So $\min\{i : \alpha''_i \ne \alpha_i\} = n+1$ for $t = (\alpha_i)$ and $x'' = (\alpha''_i)$. Then $\varrho(t, x'') = 1/(n+1)$ for each $x'' \in H \cap V(\alpha'_1 \cdots \alpha'_n)$. So $\varrho(t, H) \le 1/(n+1)$.

Pick an $x'' \in H - (H \cap V(\alpha'_1 \cdots \alpha'_n))$. Then $x'' \notin V(\alpha'_1 \cdots \alpha'_n)$. Then there is an $i \le n$ with $\alpha''_i \ne \alpha'_i = \alpha_i$. So $j = \min\{l : \alpha''_l \ne \alpha_l\} \le i \le n$. Then $\varrho(t, x'') = 1/j \ge 1/n > 1/(n+1)$. Then $\varrho(t, H) \ge 1/(n+1)$.

This implies $\varrho(t, H) = \inf\{\varrho(t, x'') : x'' \in H\} = 1/(1+n)$. □

We'll use the standard base $\mathscr{V} = \bigcup_{n\in N} \mathscr{V}_n$ and denote subcollection of the standard base by $\mathscr{V}_\star$ or $\mathscr{V}(\star)$ and set belonging to the standard base by $V(\star)$ always.

Pick a $(\alpha_n) \in X$. Then $\bigcap_{n>0} \bar{B}_{n\alpha_n}$ is a single point set to say $\{y\}$ since $\rho(\bar{B}_{n\alpha_n}) \le 1/n$. So we may define $f : X \to Y$ with $f((\alpha_n)) = y$ if $\bigcap_{n>0} \bar{B}_{n\alpha_n} = \{y\}$.

Proposition 3.2. *$f : (X, \varrho) \to (Y, \rho)$ is an irreducible perfect map.*

Proof. 1 $f : (X, \varrho) \to (Y, \rho)$ is a continuous onto map.

It is easy to prove $f : (X, \varrho) \to (Y, \rho)$ is an onto map since $\bigcup_n \mathscr{B}_n$ is a decreasing tangent cover.

Let $S = \{x_n = (\alpha(n1), \alpha(n2), \ldots) : n \in N\}$ ϱ-converge to $x = (\alpha(n))$ with $\varrho(x_n, x) = 1/n$. Let $f(x_n) = y_n$ and $f(x) = y$. Giving an $i \ge 1$, we have $\alpha(ni) = \alpha(i)$ for each $n \ge i$ by definition of ϱ. So $\{y_n : n \ge i\} \subset \bar{B}_{\alpha(ni)} = \bar{B}_{\alpha(i)}$. $\rho(\bar{B}_{\alpha(n)}) \le 1/n$ implies $\{y_n : n \in N\}$ ρ-converging to y.

2 $f^{-1}(y)$ is a ϱ-compact.

In fact, let $\Lambda(n, y) = \{\alpha \in \Lambda(n) : y \in \bar{B}_\alpha \in \bar{\mathscr{B}}_n\}$. Then $\Lambda(n, y)$ is finite since $\bar{\mathscr{B}}_n$ is locally finite. Then $\Pi_{n>0}\Lambda(n, y)$ is ρ-compact. Notice $f^{-1}(y) = X \cap \Pi_{n>0}\Lambda(n, y)$. Then $f^{-1}(y)$ is ρ-compact since X is ρ-closed by Proposition 3.1.

3 $f : (X, \varrho) \to (Y, \rho)$ is a closed map.

Proof. Let $H \subset X$ be a ϱ-closed set. Let $y_n \in f(H)$ converge to y. Let $\bar{\mathscr{B}}_n(y) = \{\bar{B}_{n\alpha(i)} \in \bar{\mathscr{B}}_n : y \in \bar{B}_{n\alpha(i)}\} = \{\bar{B}_{n\alpha(i)} \in \bar{\mathscr{B}}_n : i \le i(n)\}$ for each $n \in N$. Then $\bigcup \bar{\mathscr{B}}_n(y)$ is a neighborhood by definition of tangent cover. Then we may assume $S_n = \{y_i : i \ge n\} \subset Int_\rho \bigcup \bar{\mathscr{B}}_n(y) \subset \bigcup \bar{\mathscr{B}}_n(y)$. This implies $\bar{\mathscr{B}}_n(y_i) \subset \bar{\mathscr{B}}_n(y)$ if $i \ge n$ for $n \in N$ since $\mathscr{B}_n$ is a tangent cover.

A. $S_1 = \{y_i : i \ge 1\} \subset Int_\rho \bigcup \bar{\mathscr{B}}_1(y) \subset \bigcup \bar{\mathscr{B}}_1(y)$ implies $\Lambda(1, y_i) \subset \Lambda(1, y)$ for $i \ge 1$.

Notice that $\Lambda(1, y)$ is finite. Then there is an $N(1)' \subset N$ and a $\Lambda(1, y)' \subset \Lambda(1, y)$ such that, for each $i \in N(1)'$, we have $P_1(f^{-1}(y_i)) = \Lambda(1, y_i) = \Lambda(1, y)'$. So, for each $i \in N(1)'$, $P_1(f^{-1}(y_i) \cap H) \subset \Lambda(1, y_i) = \Lambda(1, y)'$ and $P_1(f^{-1}(y_i) \cap H) \ne \emptyset$. So there is an $\alpha(1) \in \Lambda(1, y)'$ and an infinite subset $N(1) \subset N(1)'$ such that, for each $i \in N(1)$, there is an $x_i \in f^{-1}(y_i) \cap H$ with $P_1(x_i) = \alpha(1)$. Let $S_1' = \{x_i : i \in N(1)\}$.

B. Assume we have had an $\alpha(k-1) \in \Lambda(k-1, y)'$ and an infinite subset $N(k-1) \subset N(k-1)'$ such that, for each $i \in N(k-1)$, $x_i \in S_{k-1}' = \{x_i : i \in N(k-1)\}$ with $P_{k-1}(x_i) = \alpha(k-1)$. Since $\mathscr{B}$ is a decreasing tangent cover, for each $i \in N(k-1)$, we have $\Lambda(k, y_i)' \subset \Lambda(k, y)$. Then there is an infinite subset $N(k)' \subset N(k-1)$ and a finite subset $\Lambda(k,y)' \subset \Lambda(k,y)$ such that, for each $i \in N(k)'$, $\Lambda(k,y_i) = \Lambda(k,y)' \subset \Lambda(k,y)$ and $P_k(f^{-1}(y_i)) \in \Lambda(k,y_i) = \Lambda(k,y)'$. Notice $\Lambda(k,y)'$ is finite set. Then there is an infinite subset $N(k) \subset N(k)'$ and an $\alpha(k) \in \Lambda(k,y)'$ such that $P_k(x_i) = \alpha(k)$ for each $i \in N(k)$. Let $S_k' = \{x_i \in S_{k-1}' : i \in N(k)\}$. Then $S_k' \subset S_{k-1}'$. Then, by induction, there are S_k' $(k \in N)$ such that $S_1' \supset S_2' \supset \cdots \supset S_k' \supset \cdots$.

Take an $x_{i(n)} \in S_n'$ for each $n \in N$. Then $P_n(x_{i(n)}) = \alpha(n) \in \Lambda(n,y)'$. If $k < n$, $x_{i(n)} \in S_n \subset S_k$ implies $P_k(x_{i(n)}) = \alpha(k) \in \Lambda(k,y)'$. Let $x = (\alpha(n))$. Then $x \in X$ and $f(x) = y$. So $S'' = \{x_{i(n)} : n \in N\}$ ϱ-converges to x. Then $x \in H$ since H is ϱ-closed and $x_{i(n)} \in X$ for each n. This implies $y \in f(H)$ and $f(H)$ being closed.

4 $f : (X,\varrho) \to (Y,\rho)$ is an irreducible map.

To see it take an open set $O \subset X$. Then there is a $V(\alpha_1,\dots,\alpha_n) \subset O$. Then $B_{\alpha 1} \supset B_{\alpha 2} \supset \cdots \supset B_{\alpha n}$. Pick a $y \in B_{\alpha n}$. Then, for each $\alpha' \in \Lambda(i)$, $\alpha' \neq \alpha_i$ implies y is not in $\bar{B}_{\alpha'}$ for each $i \le n$. Then $\Lambda(i,y) = \{\alpha_i\}$. So $f^{-1}(y) \subset V(\alpha_1,\dots,\alpha_n) \subset O$. This implies f is irreducible. □

In the following, we construct a stratifiable space (X,μ) with $\mu \supset \rho$ and a perfect map $f : (X,\mu) \to (Y,\tau)$. Here (Y,τ) is the stratifiable space in section 2 with g-function $\mathscr{G}$, quasi-base $\mathscr{B}'$ and collection $\bar{\mathscr{B}}$. Notice that

$$f^{-1}(\mathscr{G}_{ni}) = \{f^{-1}(g(n,i,y)) : g(n,i,y) \in \mathscr{G}_{ni}\}$$

is a ϱ-locally finite ϱ-open sets collection since $f : (X,\varrho) \to (Y,\rho)$ is a perfect map and $\mathscr{G}_{ni}$ is a ϱ-locally finite ϱ-open sets collection. $\mathscr{V} = \bigcup_{n \in N} \mathscr{V}_n$ is standard base of (X,ϱ). Here $\mathscr{V}_n = \{V(\alpha_1 \cdots \alpha_n) : \alpha_i \in \Lambda(i) \text{ for } i \le n\}$ and $V(\alpha_1 \cdots \alpha_n) = \{x \in X : P_i(x) = \alpha_i \text{ for } i \le n\}$.

Take an $f^{-1}(g(n,i,y')) \in f^{-1}(\mathscr{G}_{ni})$ and an $x' = (\alpha_i) \in f^{-1}(y') \subset f^{-1}(g(n,i,y'))$. Let

$$c(n,i,x') = f^{-1}(g(n,i,y')) \cap V(\alpha_1 \cdots \alpha_n).$$

Then $c(n,i,x')$ is a ϱ-open set. Let

$$\mathscr{C}_{ni} = \{c(n,i,x') : x' = (\alpha_i) \in f^{-1}(y') \subset f^{-1}(g(n,i,y')) \text{ and } V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}_n\}.$$

Then $\mathscr{C}_{ni}$ is a collection of ϱ-locally finite ϱ-open sets. Let $\mathscr{C}_n = \bigcup_i \mathscr{C}_{ni}$. Then $\mathscr{C}_n$ is point finite since $\mathscr{G}_n$ is point finite in Y. We may assume $\mathscr{V}_n \subset \mathscr{C}_n$. Let

$$\mathscr{C} = \bigcup_n \mathscr{C}_n.$$

We'll prove $\mathscr{C}$ is a g-function of some stratifiable space in the next section.

Let

$$P_{n\alpha} = X - (\bigcup \mathscr{C}_\alpha)$$

for each $\mathscr{C}_\alpha \subset \mathscr{C}_n$. Let

$$\mathscr{P}_n = \{P_{n\alpha} : \mathscr{C}_\alpha \subset \mathscr{C}_n\}.$$

Then $\mathscr{P}_n$ is a collection of ϱ-closure preserving ϱ-closed sets. Let

$$\mathscr{P} = \bigcup_{n \in N} \mathscr{P}_n.$$

We'll prove $\mathscr{P}$ is a σ-closure preseving quasi-base of some stratifiable space in Claim 3.7.

Take quasi-base $\mathscr{B}'$ of (Y, τ). Pick an $n \in N$. Let $O_{n\alpha} = Int_\tau(B_{n\alpha})$ for each $B_{n\alpha} \in \mathscr{B}'_n$. Let $\mathscr{O}_n = \{O_{n\alpha} : B_{n\alpha} \in \mathscr{B}'_n\}$. Let

$$U(n, \alpha, x) = f^{-1}(O_{n\alpha}) \cap V(\alpha_1 \cdots \alpha_n)$$

for each $x = (\alpha_i) \in f^{-1}(O_{n\alpha})$ and each $O_{n\alpha} \in \mathscr{O}_n$. Let

$$\mathscr{U}_n = \{U(n, \alpha, x) : V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}_n \text{ and } O_{n\alpha} \in \mathscr{O}_n\}.$$

Let $\mathscr{U} = \bigcup_n \mathscr{U}_n$.

Proposition 3.3. *$\mathscr{U}$ is a base of some topology (X, μ).*

Proof. Take an $U(n, \alpha, x') \in \mathscr{U}_n$ and an $U(m, \beta, x'') \in \mathscr{U}_m$. Let $m \geq n$. Pick an $x = (\alpha_i) \in U(n, \alpha, x') \cap U(m, \beta, x'')$. Then $f(x) \in O_{n\alpha} \cap O_{m\beta}$. So there is an $l > m + n$ and a $B_{l\delta} \in \mathscr{B}'_l$ with $f(x) \in O_{l\delta} = Int_\tau\, B_{l\delta} \subset B_{l\delta} \subset O_{n\alpha} \cap O_{m\beta}$ by definition of quasi-base $\mathscr{B}'$ of (Y, τ). So $x \in f^{-1}(f(x)) \subset f^{-1}(O_{l\delta})$. Take $U(l, \delta, x) = f^{-1}(O_{l\delta}) \cap V(\alpha_1 \cdots \alpha_l)$ from $\mathscr{U}_l$. Then $x \in U(l, \delta, x) \subset U(n, \alpha, x') \cap U(m, \beta, x'')$ since $l > m + n$ implies $x = (\alpha_i) \in V(\alpha_1 \cdots \alpha_l) \subset V(\alpha_1 \cdots \alpha_n) \cap V(\alpha_1 \cdots \alpha_m)$. □

Theorem 3.4. *(X, μ) is a stratifiable space with $\mu \supset \rho$ and $f : (X, \mu) \to (Y, \tau)$ is an irreducible perfect map.*

Proof. We prove it by the following claims. □

Claim 3.5. *$f : (X, \mu) \to (Y, \tau)$ is a continuous map.*

Proof. Take an $O_{n\alpha} \in \mathscr{O}_n$. Notice $X = \bigcup \mathscr{V}_n$. Then

$$f^{-1}(O_{n\alpha}) = \bigcup \{U(n, \alpha, x) = f^{-1}(O_{n\alpha}) \cap V(\alpha_1 \cdots \alpha_n) : x = (\alpha_i) \in f^{-1}(O_{n\alpha})\}.$$

So $f^{-1}(O_{n\alpha})$ is μ-open. □

CLAIM 3.6. *$f^{-1}(y)$ is μ-compact for each $y \in Y$.*

PROOF. Let $\mathscr{U}' \subset \mathscr{U}$ with $f^{-1}(y) \subset \bigcup \mathscr{U}'$ and $U(n,\alpha,x) \cap f^{-1}(y) \neq \emptyset$ for each $U(n,\alpha,x) \in \mathscr{U}'$. $U(n,\alpha,x) \cap f^{-1}(y) \neq \emptyset$ implies $f^{-1}(O_{n\alpha}) \cap f^{-1}(y) \neq \emptyset$. So $f^{-1}(y) \subset f^{-1}(O_{n\alpha})$ for each $U(n,\alpha,x) \in \mathscr{U}'$.

Notice that $f^{-1}(y) = X \cap \Pi_{n>0}\Lambda(n,y)$ is ϱ-compact and $f^{-1}(y) \subset \bigcup \mathscr{U}' \subset \bigcup\{V(\alpha_1 \cdots \alpha_n) : U(n,\alpha,x) \in \mathscr{U}'\}$. Then there is a finite subcollection $\mathscr{V}(y) = \{V_{\alpha_i} : i \leq n\}$ of $\{V_\alpha : U(n,\alpha,x) \in \mathscr{U}'\}$ with $f^{-1}(y) \subset \bigcup \mathscr{V}(y)$. We take $f^{-1}(O_{k_i\alpha_i})$ with relation to V_{α_i} for $i \leq n$. Then

$$f^{-1}(y) \subset U = \bigcap_{i \leq n} f^{-1}(O_{k_i\alpha_i})$$

since $f^{-1}(y) \subset f^{-1}(O_{k_i\alpha_i})$ for each $i \leq n$. So

$$f^{-1}(y) \subset U \cap (\bigcup \mathscr{V}(y)) = \bigcup\{U \cap V_{\alpha_i} : i \leq n\} \subset \bigcup\{f^{-1}(O_{k_i\alpha_i}) \cap V_{\alpha_i} : i \leq n\}.$$

So $f^{-1}(y)$ is μ-compact. □

CLAIM 3.7. *(X,μ) is a stratifiable space with a submetric $\varrho \subset \mu$ and a σ closure preserving quasi-base $\mathscr{P} = \bigcup_m \mathscr{P}_m$.*

PROOF. Notice $V(\alpha_1 \cdots \alpha_n) = \bigcup\{f^{-1}(O_{n\alpha}) \cap V(\alpha_1 \cdots \alpha_n) : O_{n\alpha} \in \mathscr{O}_n\}$. Then $V(\alpha_1 \cdots \alpha_n)$ is μ-open. So $\varrho \subset \mu$.

Now we prove that (X,μ) is a stratifiable space. To see it pick an $U(n,\alpha,x) \in \mathscr{U}$ and an $x = (\alpha_i) \in U(n,\alpha,x) = f^{-1}(O_{n\alpha}) \cap V(\alpha_1 \cdots \alpha_n)$. Then $U(n,\alpha,x) \subset f^{-1}(O_{n\alpha})$ implies $f(x) \in O_{n\alpha} \subset Y$. Notice that $\mathscr{B}' = \bigcup_n \mathscr{B}'_n$ is a quasi-base of (Y,τ). Then there is an $m > n$ and a $B_{m\beta} = Y - \bigcup \mathscr{G}_{m\beta} \in \mathscr{B}'_m$ with $f(x) \in Int_\tau\, B_{m\beta} = O_{m\beta} \subset B_{m\beta} \subset O_{n\alpha}$. Here $\mathscr{G}_{m\beta} = \{g(m,y) : y \in H\}$ for some τ-closed set H. Then $x \in f^{-1}(f(x)) \subset f^{-1}(O_{m\beta}) \subset f^{-1}(Y - \bigcup \mathscr{G}_{m\beta}) = X - f^{-1}(\bigcup \mathscr{G}_{m\beta}) \subset f^{-1}(O_{n\alpha})$. Take $V(\alpha_1 \cdots \alpha_m) = V'$ from $\mathscr{V}_m$ since $x = (\alpha_i)$. Then $x \in f^{-1}(O_{m\beta}) \cap V' \subset T^* \subset f^{-1}(O_{n\alpha}) \cap V' \subset f^{-1}(O_{n\alpha}) \cap V(\alpha_1 \cdots \alpha_n)$ since $m > n$.

Here $T^* = f^{-1}(Y - \bigcup \mathscr{G}_{m\beta}) \cap V' = (X - \bigcup\{f^{-1}(g(m,y)) : y \in H)\}) \cap V'$. Let $\mathscr{V}' = \{V \in \mathscr{V}_m : V \neq V'\}$,

$$\mathscr{C}'_{m\beta} = \{f^{-1}(g(m,y)) \cap V' : y \in H \text{ with } f^{-1}(g(m,y)) \cap V' \neq \emptyset\} \quad \text{and}$$

$$\mathscr{C}_{m\beta} = \mathscr{C}'_{m\beta} \cup \mathscr{V}'.$$

Then $\mathscr{C}_{m\beta} \subset \mathscr{C}_m$ and $T^* = V' - \bigcup \mathscr{C}'_{m\beta} = X - (\bigcup \mathscr{V}') \cup (\bigcup \mathscr{C}'_{m\beta}) = X - (\bigcup \mathscr{C}_{m\beta})$.

Notice that $f^{-1}(O_{m\beta}) \cap V'$ is μ-open. So

$$x \in f^{-1}(O_{m\beta}) \cap V' \subset Int_\mu(X - (\bigcup \mathscr{C}_{m\beta})) \subset X - (\bigcup \mathscr{C}_{m\beta}) = T^* \subset U(n,\alpha,x).$$

On the other hand, $X - (\bigcup \mathscr{C}_{m\beta}) = P_{m\beta} \in \mathscr{P}_m$ and $\mathscr{P}_m$ is ϱ-closure preserving ϱ-closed. Then $\mathscr{P}$ is a ϱ σ-closure preserving ϱ-closed quasi-base of (X, μ). So (X, μ) is a stratifiable space. □

Claim 3.8. *$f : (X, \mu) \to (Y, \tau)$ is a closed map.*

Proof. Let $H \subset X$ be a μ-closed set. Pick a $y \in Cl_\tau(f(H))$. Take collection $\bar{\mathscr{B}}$ in section 2. Giving an $n \in N$, let

$$\bar{\mathscr{B}}_{n(y)} = \{\bar{B}_{n\alpha} \in \bar{\mathscr{B}}_n : y \in \bar{B}_{n\alpha}\} = \{\bar{B}_{n\alpha} : \alpha \in \Lambda(n, y)\}.$$

Here $\Lambda(n, y)$ is a finite set since $\bar{\mathscr{B}}_n$ is a ρ-locally finite collection. Let $\mathscr{N}_y = \{U_n : n \in N\}$ be a collection of ρ-open neighborhood of the point y such that $U_n \supset Cl_\rho(U_{n+1})$, $Int_\rho(\bigcup \bar{\mathscr{B}}_n(y)) \supset U_n$ and $\rho(Cl_\rho(U_n)) \le 1/n$ for each $n \in N$. We call it a decreasing ρ-open neighborhood base of the point y.

Subclaim 3.9. *Let $\mathscr{N}_y = \{U_n : n \in N\}$ is a decrease ρ-open neighborhood base of point y, $\{f^{-1}(U_n) : n \in N\}$ is a ϱ-open ϱ-neighborhood base of $f^{-1}(y)$.*

Proof. $f : (X, \varrho) \to (Y, \rho)$ is a perfect map. □

We construct a ω-tree to prove $f : (X, \mu) \to (Y, \tau)$ is a closed map by using the collections $\bar{\mathscr{B}}$ and $\mathscr{B}'$ in section 2. Let

$$\begin{aligned}\mathscr{V}_n(y) &= \{V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}_n : y \in f(V(\alpha_1 \cdots \alpha_n))\}\\ &= \{V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}_n : \alpha_i \in \Lambda(i, y) \text{ for } i \le n\}\\ &= \{V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}_n : (\alpha_1 \cdots \alpha_n) \in \Lambda(1, y) \times \cdots \times \Lambda(n, y)\}\end{aligned}$$

for each $n \in N$. Take ρ-open neighborhood U_n in Subclaim 3.9.

Subclaim 3.10. *$f^{-1}(Cl_\rho(U_n)) \subset \bigcup \mathscr{V}_n(y)$.*

Proof. Suppose there is a $V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}_n$, an $i \le n$ and an $\alpha_i \notin \Lambda(i, y)$ such that $V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(Cl_\rho(U_n)) \ne \varnothing$. Then $f(V(\alpha_1 \cdots \alpha_n)) \cap Cl_\rho(U_n) \ne \varnothing$. Then $(\bigcap_{i \le n} \bar{B}_{\alpha_i}) \cap Cl_\rho(U_n) \ne \varnothing$. Then $\bar{B}_{\alpha_i} \cap Cl_\rho(U_n) \ne \varnothing$ for each $i \le n$ since $\bigcap_{i \le n} \bar{B}_{\alpha_i} \subset \bar{B}_{\alpha_i}$. So $\bar{B}_{\alpha_i} \cap Cl_\rho(U_i) \ne \varnothing$ for each $i \le n$ since $Cl_\rho(U_n) \subset Cl_\rho(U_i)$. Notice $Int_\rho(\bigcup \bar{\mathscr{B}}_i(y)) \supset Cl_\rho(U_i)$. Then

$$\bar{B}_{\alpha_i} \cap Cl_\rho(U_i) \ne \varnothing \quad \text{implies} \quad \bar{B}_{\alpha_i} \cap Int_\rho(\bigcup \bar{\mathscr{B}}_i(y)) \ne \varnothing.$$

Then $\bar{B}_{\alpha_i} \in \bar{\mathscr{B}}_i(y)$ since $\mathscr{B}$ is a decreasing tangent cover of (Y, ρ). This implies $\alpha_i \in \Lambda(i, y)$ for each $i \le n$, a contradiction to $\alpha_i \notin \Lambda(i, y)$ for some $i \le n$. So, for each $V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}_n$, we have $V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(Cl_\rho(U_n)) = \varnothing$ if $\alpha_i \notin \Lambda(i, y)$ for some $i \le n$. This implies $f^{-1}(Cl_\rho(U_n)) \subset \bigcup \mathscr{V}_n(y)$. □

Recall that a *tree* T in K. Kunen [9] is a partial order such that for each $x \in T$, $\{y \in T : y < x\}$ is a well-ordered by $<$.

Let T be a tree.

a If $x \in T$, the *height* of x in T, or $ht(x, T)$, is type $(\{y \in T : y < x\})$.

b For each ordinal α, the α-th *level* of T, or $Lev_\alpha(T)$, is $\{x \in T : ht(x, T) = \alpha\}$.

c The *height* of T, or $ht(T)$, is the least α such that $Lev_\alpha(T) = 0$.

d A *chain* in T is a set $C \subset T$ which is totally ordered by $<$.

e A *ω-tree* is a tree T of height ω such that $|Lev_n(T)| < \omega$ for each $n < \omega$.

PROOF OF CLAIM 3.8 (continued). Take quasi-base $\mathscr{B}'$ in section 2. Giving an n, let $\mathscr{B}'_n(y) = \{B : B \in \bigcup_{i \le n} \mathscr{B}'_i$ and $y \in Int_\tau(B)\}$. We construct a ω-tree $(\mathscr{V}(y), \supset)$ by induction.

Let $\mathscr{V}(n, y) = \{V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}_n : (\alpha_1 \cdots \alpha_n) \in \Lambda(1, y) \times \cdots \times \Lambda(n, y)$ with $V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(B) \cap H \neq \varnothing$ for each $B \in \mathscr{B}'_n(y)\}$.

We may prove $\mathscr{V}(n, y) \neq \varnothing$ for each n.

Suppose $\mathscr{V}(n, y) = \varnothing$ for some n. Then, for each $(\alpha_1 \cdots \alpha_n) \in \Lambda(1, y) \times \cdots \times \Lambda(n, y)$, there is a $B(\beta_1 \cdots \beta_n) \in \mathscr{B}'_n(y)$ with $V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(B(\beta_1 \cdots \beta_n)) \cap H = \varnothing$. Let

$$B_n = \bigcap \{B(\beta_1 \cdots \beta_n) : (\alpha_1 \cdots \alpha_n) \in \Lambda(1, y) \times \cdots \times \Lambda(n, y)\}.$$

Then B_n is a τ-neighborhood of y in Y since $\Lambda(1, y) \times \cdots \times \Lambda(n, y)$ finite implies related collection $\{B(\beta_1 \cdots \beta_n) : (\alpha_1 \cdots \alpha_n) \in \Lambda(1, y) \times \cdots \times \Lambda(n, y)\}$ finite. So $U_n \cap B_n$ is a τ-neighborhood of y in Y. Then $(U_n \cap B_n) \cap f(H) \neq \varnothing$ since $y \in Cl_\tau f(H)$.

Then, for each $(\alpha_1 \cdots \alpha_n) \in \Lambda(1, y) \times \cdots \times \Lambda(n, y)$, we have

$$V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(B_n) \cap H \subset V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(B(\beta_1 \cdots \beta_n)) \cap H = \varnothing.$$

So

$$\begin{aligned}
\varnothing &= \bigcup \{V(\alpha_1 \cdots \alpha_n) \cap (f^{-1}(B_n) \cap H) : (\alpha_1 \cdots \alpha_n) \in \Lambda(1, y) \times \cdots \times \Lambda(n, y)\} \\
&= \left(\bigcup \{V(\alpha_1 \cdots \alpha_n) : (\alpha_1 \cdots \alpha_n) \in \Lambda(1, y) \times \cdots \times \Lambda(n, y)\}\right) \cap (f^{-1}(B_n) \cap H) \\
&= \left(\bigcup \mathscr{V}_n(y)\right) \cap (f^{-1}(B_n) \cap H) \supset f^{-1}(U_n) \cap (f^{-1}(B_n) \cap H) \\
&\neq \varnothing,
\end{aligned}$$

a contradiction. Let $\mathscr{V}(y) = \bigcup_{n \in N} \mathscr{V}(n, y)$.

Take a $V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}(y)$. Then $V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(B) \cap H \neq \varnothing$ for each $B \in \mathscr{B}'_n(y)$. Then $V(\alpha_1 \cdots \alpha_k) \cap f^{-1}(B) \cap H \neq \varnothing$ for each $k \leq n$ and each $B \in \mathscr{B}'_k(y)$ since $k \leq n$ implies $V(\alpha_1 \cdots \alpha_n) \subset V(\alpha_1 \cdots \alpha_k)$ and $\mathscr{B}'_k(y) \subset \mathscr{B}'_n(y)$. So $V(\alpha_1 \cdots \alpha_k) \in \mathscr{V}(y)$. This implies $(\mathscr{V}(y), \supset)$ is a tree.

Giving an $n \in N$, the *n-th level* of $\mathscr{V}(y)$ is finite since $Lev_n(\mathscr{V}(y)) = \mathscr{V}(n, y)$. The *height* of $\mathscr{V}(y)$ is ω. So, by the Köning Lemma (to see page 69 in [9]), $(\mathscr{V}(y), \supset)$ has an infinite *chain* $\mathscr{C}$. Let the chain be $V(\alpha_1) \supset V(\alpha_1\alpha_2) \supset \cdots \supset V(\alpha_1 \cdots \alpha_n) \supset \cdots$. Then $V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}(n, y)$. Then $V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(B) \cap H \neq \varnothing$ for each $B \in \mathscr{B}'_n(y)$. Let $x = (\alpha_i)$. Then $x \in X$ and $f(x) = y$ since X is ϱ-closed. Let $\mathscr{V}(x) = \{V(\alpha_1 \cdots \alpha_n) : n \in \omega\}$.

Now we prove $x \in H$. Notice $\mathscr{U} = \bigcup_n \mathscr{U}_n$ in Proposition 3.3 is a base of (X, μ). Take an $U(n, \alpha, x) = f^{-1}(O_{n\alpha}) \cap V(\alpha_1 \cdots \alpha_n)$ from $\mathscr{U}$ with $x \in U(n, \alpha, x)$. Then $V(\alpha_1 \cdots \alpha_n) \in \mathscr{V}(x)$ and $y \in O_{n\alpha} \in \mathscr{O}_n$. So there is an $m > n$ and a $B \in \mathscr{B}'_m(y)$ with $y \in Int_\tau(B) \subset B \subset O_{n\alpha}$. Then $V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(B) \cap H \neq \varnothing$. This implies $V(\alpha_1 \cdots \alpha_n) \cap f^{-1}(O_{n\alpha}) \cap H \neq \varnothing$. So $U(n, \alpha, x) \cap H \neq \varnothing$. Then $x \in H$ since H is μ-closed and $\mathscr{U}$ is a base of (X, μ).

So $y = f(x) \in f(H)$. Then $f(H)$ is τ-closed. Then $f : (X, \mu) \to (Y, \tau)$ is a closed map.

Claim 3.11. *$f : (X, \mu) \to (Y, \tau)$ is an irreducible map.*

Proof. Take an open set $U \subset X$. Then there is an $U(n, \alpha, x) = f^{-1}(O_{n\alpha}) \cap V(\alpha_1 \cdots \alpha_n) \subset U$. Then $f(U(n, \alpha, x)) = O_{n\alpha} \cap \bar{B}_{\alpha_n} \neq \varnothing$. Then $O_{n\alpha} \cap B_{\alpha_n} \neq \varnothing$ by 2 of Proposition 2.3. Pick a $y \in O_{n\alpha} \cap B_{\alpha_n}$. Then $f^{-1}(y) \subset V(\alpha_1 \cdots \alpha_n)$ by 4 of Proposition 3.2. Then $f^{-1}(y) \subset f^{-1}(O_{n\alpha}) \cap V(\alpha_1 \cdots \alpha_n)$. This implies $f : (X, \mu) \to (Y, \tau)$ is irreducible. □

This completes our proof of Theorem 3.4. □

4. g-functions of (X, μ)

We prove that collection $\mathscr{C} = \bigcup_n \mathscr{C}_n$ in section 3 is a g-function of (X, μ).

Notice $c(n, i, x) = f^{-1}(g(n, i, y)) \cap V(\alpha_1 \cdots \alpha_n)$ for $x = (\alpha_i) \in f^{-1}(y) \subset X$ and $c(n, i, x) \in \mathscr{C}_{ni} \subset \mathscr{C}$.

We have the following proposition.

Proposition 4.1. 1 *$x' \in c(n, i, x)$ implies $c(n, i', x') \subset c(n, i, x)$.*
2 $c(n+1, i', x) \subset c(n, i, x)$.

Proof. 1 $x' \in c(n,i,x) = f^{-1}(g(n,i,y)) \cap V(\alpha_1 \cdots \alpha_n)$ implies

$$f(x') \in g(n,i,y) \quad \text{and} \quad f(x) = y.$$

So $g(n,i',f(x')) \subset g(n,i,y)$. Then

$$f^{-1}(g(n,i',f(x'))) \cap V(\alpha_1 \cdots \alpha_n) \subset f^{-1}(g(n,i,y)) \cap V(\alpha_1 \cdots \alpha_n) = c(n,i,x).$$

Then $f^{-1}(g(n,i',f(x'))) \cap V(\alpha_1 \cdots \alpha_n) = c(n,i',x')$ since $x' \in f^{-1}(f(x'))$ and $P_i(x) = P_i(x') = \alpha_i$ for $i \le n$ by $x' \in V(\alpha_1 \cdots \alpha_n)$.

2 $c(n+1,i',x) = f^{-1}(g(n+1,i',f(x))) \cap V(\alpha_1 \cdots \alpha_{n+1}) \subset f^{-1}(g(n,i,f(x))) \cap V(\alpha_1 \cdots \alpha_n) = c(n,i,x)$. □

Let $H'_{ni} = \{x \in X : \bigcup\{c(n+i,i',x') \in \mathscr{C}_{n+i} : x \in c(n+i,i',x')\} \subset c(n,i,x)\}$ and $H_{ni} = \{x \in X : x \in c(n+i,i',x') \Rightarrow x' \in c(n,i,x)\}$.

Proposition 4.2. *$f^{-1}(K_{ni}) = H_{ni} = H'_{ni}$ for each n, i.*

Proof. Proving $H_{ni} = H'_{ni}$ is similar to proving $K_{ni} = K'_{ni}$ in Proposition 2.4 by Proposition 4.1.

Giving an n, we prove $f^{-1}(K_{ni}) = H_{ni}$ by induction.

A. $f^{-1}(K_{n0}) = H_{n0}$ for $i = 0$.

To see it pick an $x = (\alpha_i) \in f^{-1}(K_{n0})$. Then $y = f(x) \in K_{n0}$ implies $x \in f^{-1}(y) \subset f^{-1}(g(n,0,y))$. So $c(n,0,x) = f^{-1}(g(n,0,y)) \cap V(\alpha_1 \cdots \alpha_n)$ since $x \in V(\alpha_1 \cdots \alpha_n)$. Let $x \in c(n+0,0,x') = f^{-1}(g(n+0,0,y')) \cap V(\alpha'_1 \cdots \alpha'_n)$. Here $x' = (\alpha'_i) \in f^{-1}(y')$. $f(x) = y \in g(n+0,0,y')$ implies $g(n+0,0,y) \subset g(n+0,0,y')$ by 5 of Theorem 2.7. Then $y \in g(n+0,0,y') \subset g(n,y')$. So $g(n,y) \subset g(n,y')$. Notice $P_i(x') = P_i(x) = \alpha'_i = \alpha_i$ for $i \le n$ since $x = (\alpha_i) \in V(\alpha'_1 \cdots \alpha'_n)$. So $y' \in g(n,0,y)$ since $f(x) = y \in g(n+0,0,y')$ and $y \in K_{n0}$ by definition K_{n0}. Then $g(n,y') \subset g(n,y)$. This implies $g(n,0,y) = g(n,y')$. Then $x' \in f^{-1}(y') \subset f^{-1}(g(n,0,y')) = f^{-1}(g(n,0,y))$ and $x' \in V(\alpha_1 \cdots \alpha_n)$. Then $x' \in f^{-1}(g(n,0,y)) \cap V(\alpha_1 \cdots \alpha_n) = c(n,0,x)$. This implies $x \in H_{n0}$. So $f^{-1}(K_{n0}) \subset H_{n0}$.

To see $H_{n0} \subset f^{-1}(K_{n0})$ pick an $x = (\alpha_i) \in H_{n0}$. Then $x \in c(n+0,0,x')$ implies $x' = (\alpha'_i) \in c(n,0,x)$. Then $x \in f^{-1}(g(n+0,0,y')) \cap V(\alpha'_1 \cdots \alpha'_n)$ implies $x' \in f^{-1}(g(n,0,y)) \cap V(\alpha_1 \cdots \alpha_n)$. Then $f(x) = y \in g(n+0,0,y')$ implies $f(x') = y' \in g(n,0,y)$. So $y \in K_{n0}$. Then $x \in f^{-1}(y) \subset f^{-1}(K_{n0})$.

B. Assume that we have had $f^{-1}(K_{ni}) = H_{ni}$ for each $i < m$. Then

$$\begin{aligned} f^{-1}(K_{nm}) &= f^{-1}((K_{nm} - K_{nm-1}) \cup K_{nm-1}) \\ &= f^{-1}(K_{nm-1}) \cup f^{-1}((K_{nm} - K_{nm-1}) = H_{nm-1} \cup f^{-1}(K_{nm} - K_{nm-1}) \end{aligned}$$

by induction assumption. We can prove $f^{-1}(K_{nm} - K_{nm-1}) = H_{nm} - H_{nm-1}$ just as the same proof of the above A. So we have $f^{-1}(K_{nm}) = H_{nm}$. □

Corollary 4.3. 1 $H_{ni} \subset H_{ni+1}$ *for each* n, i.
2 $H_{n+1i} \subset H_{ni}$ *for each* n, i.
3 H_{ni} *is* ϱ*-closed for each* n, i.
4 $\bigcup_i H_{ni} = X$.

Proof. $f : (X, \varrho) \to (Y, \rho)$ is a perfect map. □

Proposition 4.4. $\mathscr{C}$ *is a g-function of stratifiable space* (X, μ) *such that*:
1 $\bigcap_n c(n, i(x), x) = \{x\}$.
2 $x \in c(n, i_n, x_n)$, *then* $\{x_n : n \in N\}$ μ*-converges to* x.
3 *If* H *is* μ*-closed and* $x \notin H$, *then* $x \notin Cl_\mu(\bigcup\{c(n, i, x') : x' \in H\})$.
4 $x' \in c(n, i, x)$ *implies* $c(n, i', x') \subset c(n, i, x)$.
5 $H_{ni-1} \cap (\bigcup \mathscr{C}_{ni}) = \varnothing$.
6 $c(n+1, i', x) \subset c(n, i, x)$.
7 *Each* $\mathscr{C}_{ni}$ *is a* ϱ*-locally finite collection of* ϱ*-open sets.*

Proof. It is easy to prove 1 since $x = (\alpha_i) \in c(n, i, x) \subset V(\alpha_1 \cdots \alpha_n)$ implies $x \in \bigcap_n V(\alpha_1 \cdots \alpha_n) = \{x\}$.

Proof of 2. $x = (\alpha_i) \in c(n, i_n, x_n) = f^{-1}(g(n, i_n, y_n)) \cap V(\alpha_1 \cdots \alpha_n)$ implies $f(x) \in g(n, i_n, y_n)$. So $S = \{y_n : n \in N\}$ τ-converges to $f(x)$. Then $f^{-1}(S \cup f(x))$ is μ-compact since $f : (X, \mu) \to (Y, \tau)$ is a perfect map. So $S' = \{x_n : n \in N\}$ μ-converges to x.

Proof of 4 and 6. It is Proposition 4.1.

Proof of 5. 5 implies from $g(n, i, f(x)) \cap K_{ni-1} = \varnothing$ and $f^{-1}(K_{ni-1}) = H_{ni-1}$.

Proof of 7. Notice that $f^{-1}(\mathscr{G}_{ni})$ is ϱ-locally finite and $\mathscr{V}_n$ is ϱ-discrete.

Proof of 3. Let $B \subset X$ be μ-closed with $x = (\alpha_i) \notin B$. Then $x \in O = X - B$. So there is a μ-open set $U(l, \delta, x) = f^{-1}(O_{l\delta}) \cap V(\alpha_1 \cdots \alpha_l) \in \mathscr{U}_l$ such that $x \in U(l, \delta, x) \subset O = X - B$. Then there is an $n > l$ and $P_{n\alpha} \in \mathscr{P}_n$ with $x \in Int_\mu(P_{n\alpha}) \subset P_{n\alpha} \subset U(l, \delta, x) \subset O = X - B$ by Claim 3.7. Notice $P_{n\alpha} = X - \bigcup \mathscr{C}_{n\alpha}$ for some $\mathscr{C}_{n\alpha} \subset \mathscr{C}_n$. Then $x \notin X - Int_\mu(P_{n\alpha}) \supset \bigcup \mathscr{C}_{n\alpha} \supset B$. Notice that $B \subset \bigcup \mathscr{C}_{n\alpha}$ implies $\bigcup\{c(n, i, x') \in \mathscr{C}_n : x' \in B\} \subset \bigcup \mathscr{C}_{n\alpha}$ by 4 and 6 in Proposition 4.4. So $x \notin X - Int_\mu(P_{n\alpha}) \supset Cl_\mu(\bigcup \mathscr{C}_{n\alpha}) \supset Cl_\mu(\bigcup\{c(n, i, x') \in \mathscr{C}_n : x' \in B\})$. □

Notice that, for each $n \in N$ and $x \in X$, there is unique $c(n,i,x) \in \mathscr{C}$ for some $i \in \omega$. So denote $c(n,x)$ by $c(n,i,x)$ sometimes. We still use $\mathscr{C}$ and $\mathscr{C}_n$ $(n \in N)$ to express the collection of g-function in Proposition 4.4.

Proposition 4.5. *Let H be a ϱ-closed set and $\mathcal{O}$ be a point finite ϱ-open cover. Then there is a ϱ-discrete ϱ-clopen refinement $\mathscr{V}'$ of $\mathcal{O}$ with $H \subset \bigcup \mathscr{V}'$ and $\mathscr{V}' \subset \bigcup_{j>n} \mathscr{V}_j$ for a given n.*

Proof. Take a ϱ-closed set $H \subset X$. Let $\mathcal{O}$ be a collection of point finite ϱ-open sets. Pick an $x \in H$. Let $\mathcal{O}(x) = \{O \in \mathcal{O} : x \in O\}$. Then $\bigcap \mathcal{O}(x) = O(x)$ is ϱ-open. Let n be the first number such that there is an $x = (\alpha_i) \in H$ and an $O(x)$ with $x \in V(\alpha_1 \cdots \alpha_n) \subset O(x)$. Let $\mathscr{V}_n' = \{V(\alpha_1 \ldots \alpha_n) \in \mathscr{V}_n : x = (\alpha_i) \in H$ with $x \in V(\alpha_1 \cdots \alpha_n) \subset O(x)\}$. Then $\mathscr{V}_n'$ is a ϱ-discrete ϱ-clopen collection.

Assume we have had $\mathscr{V}_j'$ for $n \le j < m$ such that $(\bigcup \mathscr{V}_j') \cap (\bigcup \mathscr{V}_i') = \varnothing$ if $n \le j < i < m$. Let $\mathscr{V}_m' = \{V(\alpha_1 \cdots \alpha_m) \in \mathscr{V}_m : x = (\alpha_i) \in H - (\bigcup \bigcup_{n \le j < m} \mathscr{V}_j')$ with $V(\alpha_1 \cdots \alpha_m) \subset (\bigcap \mathcal{O}(x)) - (\bigcup \bigcup_{j<m} \mathscr{V}_j)\}$. Then $\mathscr{V}_m'$ is a ϱ-discrete ϱ-clopen collection.

Then, by induction, we have $\mathscr{V}_m'$ for each $m \ge n$. Let $\mathscr{V}' = \bigcup_m \mathscr{V}_m'$.

1 $H \subset \bigcup \mathscr{V}'$.

Proof. Pick an $x = (\alpha_i) \in H \subset \bigcup \mathcal{O}$. Then $\bigcap \mathcal{O}(x)$ is ϱ-open since $\mathcal{O}$ is point finite. Let m be the least index such that $x \in V(\alpha_1 \cdots \alpha_m) \subset \bigcap \mathcal{O}(x)$. If $V(\alpha_1 \cdots \alpha_m) \cap (\bigcup \bigcup_{n \le j < m} \mathscr{V}_j') = \varnothing$, we have $V(\alpha_1 \cdots \alpha_m) \in \mathscr{V}_m'$ by definition of $\mathscr{V}_m'$. If $V(\alpha_1 \cdots \alpha_m) \cap (\bigcup \bigcup_{n \le j < m} \mathscr{V}_j') \neq \varnothing$, there is a $V \in \bigcup_{n \le j < m} \mathscr{V}_j$ with $V \cap V(\alpha_1 \cdots \alpha_m) \neq \varnothing$. So $V(\alpha_1 \cdots \alpha_m) \subset V$ by $j < m$ and property 1 in Proposition 3.3. Then $x \in V \subset \bigcup \mathscr{V}'$.

2 $\mathscr{V}'$ is ϱ-discrete.

Proof. Pick an $x = (\alpha_i) \in Cl_\varrho(\bigcup \mathscr{V}')$. Let $\mathscr{V}_x = \{V(\alpha_1 \cdots \alpha_m) : m \in N\}$. Take a $V(\alpha_1 \cdots \alpha_m) \in \mathscr{V}_x$. Then $V(\alpha_1 \cdots \alpha_m) \cap (\bigcup \mathscr{V}_j') \neq \varnothing$ for infinitely many $\mathscr{V}_j'$'s. So there is a $j > m$ and a $V \in \mathscr{V}_j'$ with $V \cap V(\alpha_1 \cdots \alpha_m) \neq \varnothing$. Then $V \subset V(\alpha_1 \cdots \alpha_m)$ by $j > m$. Let $V \in \mathscr{V}_j'$. Then $V = V(\alpha_1 \cdots \alpha_m \beta_{m+1} \cdots \beta_j)$ since $V \subset V(\alpha_1 \cdots \alpha_m)$. Then $V \cap H \neq \varnothing$ since $V \in \mathscr{V}_j'$. Pick an $x_m = (\alpha_1, \ldots, \alpha_m, \beta, \ldots) \in V \cap H$ since $V \subset V(\alpha_1 \cdots \alpha_m)$. Then $x \in H$ since H is ϱ-closed and $S = \{x_m : m \in N\}$ ϱ-converges to x. So $\mathscr{V}'$ is ϱ-discrete since $H \subset \bigcup \mathscr{V}'$ by the above 1. □

We give X a partition $\mathscr{H}_n'$ by $X = \bigcup_l H_{nl}$ for each n.

Take open ball $B(n,m,1/l) = \bigcup\{V(\alpha_1 \cdots \alpha_l) \in \mathscr{V}_l : x = (\alpha_i) \in H_{nm-1}\}$. Then $\bigcap_l B(n,m,1/l) = H_{nm-1}$. Let $R(n,m,1/l) = B(n,m,1/l) - B(n,m,1/(l+1))$. Then $R(n,m,1/l) = \{x \in X : \varrho(x, H_{nm-1}) = 1/(l+1)\}$ by property 2 in Proposition 3.1. Let $H(n,m,l) = R(n,m,1/l) \cap H_{nm}$. Then each $H(n,m,l)$ is ϱ-closed and $\varrho(H(n,m,l), H(n,m',l')) = r > 0$ if $H(n,m',l') \neq H(n,m,l)$ by property 3 in Proposition 3.1. Then $H_{nm} - H_{nm-1} = \bigcup_{l \in N} H(n,m,l)$. Let

$$\mathscr{H}_n' = \{H(n,m,l) : m, l \in N\} = \{H'(n,i) : i \in N\}.$$

Then $\mathscr{H}_n'$ is a partition of X. Let $\mathscr{H}' = \bigcup_n \mathscr{H}_n'$.

Pick an $n \in N$. Let

$$\mathscr{H}_n = \{\textstyle\bigcap_{i \leq n} H'(i, j(i)) : H'(i, j(i)) \in \mathscr{H}_i' \text{ for } i \leq n \text{ if } \bigcap_{i \leq n} H'(i, j(i)) \neq \varnothing\}.$$

Then $\mathscr{H}_n$ is countable. Let

$$\mathscr{H}_n = \{H(n,i) : i \in N\} \quad \text{and} \quad \mathscr{H} = \textstyle\bigcup_n \mathscr{H}_n.$$

Then $\mathscr{H}_n$ is a partition of X for each n.

Proposition 4.6. *There is a countable collection $\mathscr{H} = \bigcup_n \mathscr{H}_n$ of ϱ-closed sets such that*:

1 $H(n,i) \subset H(n',i')$, $H(n',i') \subset H(n,i)$ *or* $\varrho(H(n,i), H(n',i')) = r > 0$ *if* $H(n,i), H(n',i') \in \mathscr{H}$.

2 $\mathscr{H}_n$ *is a partition for each* $n \in N$.

Proof. Let $H(n,m,l)$ and $H(n,k,l')$ in $\mathscr{H}_n'$ with $H(n,m,l) \neq H(n,k,l')$.

Case 1, $m = k$. Then $l \neq l'$. Then there is an H_{nm} with $H(n,m,l) \cup H(n,k,l') \subset H_{nm}$. Then $\varrho(H(n,m,l), H(n,k,l')) = r > 0$ by property 3 in Proposition 3.1.

Case 2, $m \neq k$. Then we may assume $m > k$. Let $H(n,m,l) \subset H_{nm} - H_{nm-1}$ and $H(n,k,l) \subset H_{nk} - H_{nk-1}$. Then $H(n,k,l) \subset H_{nk} \subset H_{nm-1}$. Then $\varrho(H(n,m,l), H(n,k,l')) \geq \varrho(H(n,m,l), H_{nm-1}) = 1/(l+1) = r > 0$.

So $\varrho(H(n,i), H(n,j)) \geq \varrho(H(n,m,l), H(n,k,l')) = r > 0$ if $i \neq j$. When $n \neq m$, we assume $m > n$. Then $H(m,i') = \bigcap_{i \leq m} H'(i, j(i)) \subset H(n,i)$. Then $\varrho(H(m,i'), H(n,j)) \geq \varrho(H(n,i), H(n,j)) \geq \varrho(H(n,m,l), H(n,k,l')) = r > 0$. □

Construction 4.

We use partitions $\mathscr{H}$ to construct a g-function $\mathscr{W}$ of (X, μ) by induction.

A. At first we take partition $\mathscr{H}_1$ to construct $\mathscr{W}_1$ for $k = 1$.

A.a. We take ϱ-closed set $H(1,0)$ from $\mathscr{H}_1$. Then $H(1,0) = H_{10}$ and $\mathscr{C}_{1+0}$ is point finite ϱ-open cover of $H(1,0)$. Then there is a ϱ-discrete ϱ-clopen refinement

$\mathscr{W}_{10} \subset \bigcup_{i>2(1+1)} \mathscr{V}_i$ of $\mathscr{C}_{1+0}$ with $H(1,0) \subset \bigcup \mathscr{W}_{10}$ by Proposition 4.5. Let $W(1,0,x) = W$ if $x \in W \cap H(1,0)$ for each $W \in \mathscr{W}_{10}$. Then $\mathscr{W}_{10} = \{W(1,0,x) : x \in H(1,0)\}$.

A.b. Assume, for each $i < m$, we have had $\mathscr{W}_{1i}$ such that:

1 $(\bigcup \mathscr{W}_{1i}) \cap H(1, i-1) = \varnothing$ for each $i < m$.

2 $\mathscr{W}_{1i} \subset \bigcup_{j>i^*} \mathscr{V}_j$ is ϱ-discrete ϱ-clopen collection.

3 $x \in W(1,i',x')$ implies $W(1,i,x) \subset W(1,i',x')$.

4 $W(1,i,x) \subset c(1,i',x)$ for each $x \in H(1, m-1)$.

Take ϱ-closed set $H(1,m)$ from $\mathscr{H}_1$. Let $H(1,m)' = \bigcup_{i<m} H(1,i)$. Then we have $\varrho(H(1,m)', H(1,m)) = r > 0$ by Proposition 4.6. $H(1,m) \in \mathscr{H}_1$ implies that there is an $H_{1l} \in \mathscr{H}_1'$ with $H(1,m) \subset H_{1l}$. Let $m^* = \max\{2(1+m+l), 2/r\}$. Notice that $\mathscr{C}_{1+l}$ is a point finite ϱ-open cover of $H(1,m)$. Then, by Proposition 4.5, there is a ϱ-discrete ϱ-clopen refinement $\mathscr{W}_{1m}'$ of $\mathscr{C}_{1+l}$ with $H(1,m) \subset \bigcup \mathscr{W}_{1m}'$ and $\mathscr{W}_{1m}' \subset \bigcup_{j>m^*} \mathscr{V}_j$. Let $\mathscr{W}_{1m}'' = \bigcup_{i<m} \mathscr{W}_{1i}$ and $\mathscr{W}_{1m}^* = \mathscr{W}_{1m}' \cup \mathscr{W}_{1m}''$. Let $\mathscr{W}_{1m}^*(x) = \{W \in \mathscr{W}_{1m}^* : x \in W\}$ and $W(1,m,x) = \bigcap \mathscr{W}_{1m}^*(x)$ for each $x \in H(1,m)$. Let $\mathscr{W}_{1m} = \{W(1,m,x) : x \in H(1,m)\}$.

Then, by induction, we have $\mathscr{W}_{1m}$ for each $m \in \omega$. Let $\mathscr{W}_1 = \bigcup_m \mathscr{W}_{1m}$.

B. Assume we have had $\mathscr{W}_k$ for each $k < n$ such that:

1 $(\bigcup \mathscr{W}_{ki}) \cap H(k, i-1) = \varnothing$ for each $k < n$ and each $i \in \omega$.

2 $\mathscr{W}_{ki} \subset \bigcup_{j>i^*} \mathscr{V}_j$ is ϱ-discrete ϱ-clopen collection for each $k < n$ and each $i \in \omega$. Here $\bigcup_j \mathscr{V}_j$ is standard base of (X, ϱ).

3 $x \in W(k,i',x')$ implies $W(k,i,x) \subset W(k,i',x')$ for each $k < n$.

4 $W(k,i,x) \subset c(k,i',x)$ for each $k < n$ and each $x \in X$.

Now we take partition $\mathscr{H}_n$ to construct $\mathscr{W}_n$ for $k = n$.

B.a. Pick $H(n,0) \in \mathscr{H}_n$. Then there is an H_{nl} and $H(n-1,j) \in \mathscr{H}_{n-1}$ with $H(n,0) \subset H_{nl} \cap H(n-1,j)$ by definition of $\mathscr{H}_n$. Then $\mathscr{C}_{n+l}$ is a point finite ϱ-open collection, and $\mathscr{W}_{n-1j}$ is ϱ-discrete ϱ-clopen collection by induction assumption. Let $\mathscr{W}_{n0}'$ be a ϱ-discrete ϱ-clopen refinement of $\mathscr{C}_{n+l}$ with $H(n,0) \subset \bigcup \mathscr{W}_{n0}'$ and $\mathscr{W}_{n0}' \subset \bigcup_{j>2(n+l)} \mathscr{V}_j$ by Proposition 4.5. Let $\mathscr{W}_{n0}^* = \mathscr{W}_{n0}' \cup \mathscr{W}_{n-1j}$ and $\mathscr{W}_{n0}^*(x) = \{W \in \mathscr{W}_{n0}^* : x \in W\}$ for each $x \in H_{n0}$. Let $W(n,0,x) = \bigcap \mathscr{W}_{n0}^*(x)$ and $\mathscr{W}_{n0} = \{W(n,0,x) : x \in H(n,0)\}$. Then $\mathscr{W}_{n0}$ is a ϱ-discrete ϱ-clopen collection since $\mathscr{W}_{n0}^*$ is an union of finitely many ϱ-discrete ϱ-clopen collections. Then $\mathscr{W}_{n0} \subset \bigcup_{j>2(n+l)} \mathscr{V}_j$ since both $\mathscr{W}_{n0}'$ and $\mathscr{W}_{n-1j}$ are subsets of standard base $\mathscr{V}$.

B.b. Assume, for each $i < m$, we have had $\mathscr{W}_{ni}$ such that:

1 $(\bigcup \mathscr{W}_{ni}) \cap H(n, i-1) = \varnothing$ for each $i < m$.

2 $\mathscr{W}_{ni} \subset \bigcup_{j>i^*} \mathscr{V}_j$ is ϱ-discrete ϱ-clopen collection.

3 $x \in W(n,i',x')$ implies $W(n,i,x) \subset W(n,i',x')$.

4 $W(n,i,x) \subset c(n,i',x)$ for each $x \in X$.

We construct $\mathscr{W}_{nm}$.

Take $H(n,m)$ from $\mathscr{H}_n$. Let $H(n,m)' = \bigcup_{i<m} H(n,i)$. Then, by Proposition 4.6, we have $\varrho(H(n,m)', H(n,m)) = r > 0$. $H(n,m) \in \mathscr{H}_n$ implies that there is an $H_{nl} \in \mathscr{H}_n'$ with $H(n,m) \subset H_{nl}$. Let

$$m^* = \max\{2(n+m+l), 2/r\}.$$

Notice that $\mathscr{C}_{n+l}$ is a point finite ϱ-open cover of $H(n,m)$. Then, by Proposition 4.5, there is a ϱ-discrete ϱ-clopen refinement $\mathscr{W}_{nm}'$ of $\mathscr{C}_{n+l}$ with $H(n,m) \subset \bigcup \mathscr{W}_{nm}'$ and

$$\mathscr{W}_{nm}' \subset \bigcup_{j>m^*} \mathscr{V}_j.$$

We take $\mathscr{W}_{n-1i'}$ since $H(n,m) = H_{nl} \cap H(n-1,i') \subset H(n-1,i')$. Let $\mathscr{W}_{nm}'' = \bigcup_{i<m} \mathscr{W}_{ni}$. Let $\mathscr{W}_{nm}^* = \mathscr{W}_{nm}'' \cup \mathscr{W}_{nm}' \cup \mathscr{W}_{n-1i'}$ and $\mathscr{W}_{nm}^*(x) = \{W \in \mathscr{W}_{nm}^* : x \in W\}$ for each $x \in H(n,m)$. Let $W(n,m,x) = \bigcap \mathscr{W}_{nm}^*(x)$ and $\mathscr{W}_{nm} = \{W(n,m,x) : x \in H(n,m)\}$. Then $\mathscr{W}_{nm}$ is a ϱ-discrete ϱ-clopen collection since $\mathscr{W}_{nm}^*$ is an union of finitely many ϱ-discrete ϱ-clopen collections. Then $\mathscr{W}_{nm} \subset \bigcup_{j>m^*} \mathscr{V}_j$ since $\mathscr{W}_{nm}^*$ is a subset of standard base $\mathscr{V}$.

Then, by induction, we have $\mathscr{W}_{nm}$ for each $m \in \omega$. Let $\mathscr{W}_n = \bigcup_m \mathscr{W}_{nm}$. Then we have $\mathscr{W}_n$ for $k = n$. Then, by induction, we have $\mathscr{W}_n$ for each $n \in N$. Notice that, for each $x \in X$ and $n \in N$, $W(n,i,x) \in \mathscr{W}_n$ is unique. So we denote $W(n,x)$ by $W(n,i,x)$ sometimes. Let $\mathscr{W} = \bigcup_n \mathscr{W}_n$.

Proposition 4.7. *Stratifiable space (X,μ) satisfies the following conditions* A *and* B:

A. *There is a countable collection $\mathscr{H} = \bigcup_n \mathscr{H}_n$ of ϱ-closed sets such that*:

1 $H(n',i') \subset H(n,i)$ *or* $\varrho(H(n,i), H(n',i')) = r > 0$ *if* $H(n,i), H(n',i') \in \mathscr{H}$ *with* $n' > n$.

2 $\mathscr{H}_n$ *is a partition of* X *for each* $n \in N$.

B. *There is a g-function* $\mathscr{W}$ *such that*:

1 $\bigcap_n W(n,x) = \{x\}$.

2 $x \in W(n,x_n)$, *then* $\{x_n : n \in N\}$ *μ-converges to* x.

3 *If* H *is μ-closed and* $x \notin H$, $x \notin Cl_\mu(\bigcup\{W(n,x') : x' \in H\})$ *for some* n.

4 $x' \in W(n,x)$ *implies* $W(n,x') \subset W(n,x)$.

5 $H(n,i) \cap (\bigcup \mathscr{W}_{nj}) = \varnothing$ *if* $j > i$.

6 $W(n,x) \subset W(n-1,x)$.

7 *Each* $\mathscr{W}_{nm}$ *is a ϱ-discrete ϱ-clopen collection.*

8 $W(n,x) \subset c(n,x) \in \mathscr{C}$ *for each* $x \in X$.

Here $\mathscr{C}$ *is a g-function of stratifiable space* (X,μ) *satisfying Proposition* 4.4.

Proof. Condition A follows from Proposition 4.6. In the following we prove Condition B by Construction 4.

Proof of 8. Pick an $x \in H(n,m)$. Then there is an $H_{nl} \in \mathscr{H}_n'$ with $x \in H(n,m) \subset H_{nl}$. Then $x \in c(n,l,x)$. Notice that $\mathscr{W}_{nm}'$ is a refinement of $\mathscr{C}_{n+l}$ with $H(n,m) \subset \bigcup \mathscr{W}_{nm}'$. Then there is a $W' \in \mathscr{W}_{nm}'$ and a $c(n+l,l',x') \in \mathscr{C}_{n+l}$ with $x \in W' \subset c(n+l,l',x')$. This implies $x \in c(n+l,l',x') \cap H_{nl}$ by $H(n,m) \subset H_{nl}$. So $x \in c(n+l,l',x') \subset c(n,l,x)$ by definition of H_{nl}. Notice that $x \in W' \in \mathscr{W}_{nm}'$ implies $W' \in \mathscr{W}_{nm}^*(x)$. So $W(n,m,x) = \bigcap \mathscr{W}_{nm}^*(x) \subset W' \subset c(n+l,l',x') \subset c(n,l,x)$.

Proof of 1. Pick an $x \in X$. We have $W(n,x) \subset c(n,x)$ for each $n \in \omega$ by the above 8. So $\bigcap_n W(n,x) \subset \bigcap_n c(n,x) = \{x\}$ by 1 in Proposition 4.4.

Proof of 2. Pick an $x \in X$. Let $x \in W(n,x_n)$. We have $W(n,x_n) \subset c(n,x_n)$ for each $n \in \omega$ by the above 8. Then $\{x_n : n \in N\}$ μ-converges to x by 2 in Proposition 4.4.

Proof of 3. Let H be a μ-closed set with $x \notin H$. Then $x \notin Cl_\mu(\bigcup\{c(n,x') : x' \in H\})$ for some n by 3 in Proposition 4.4. Then $x \notin Cl_\mu(\bigcup\{W(n,x') : x' \in H\})$ by the above 8.

Then $\mathscr{W}$ is a g-function of (X,μ).

Proof of 5. Take $\mathscr{W}_{nm}$ in B.b in Construction 4. Then $\varrho(H(n,m)', H(n,m)) = r > 2/m^* > 0$. Here $H(n,m)' = \bigcup_{i<m} H(n,i)$. Notice $m^* = \max\{2(n+m+l), 2/r\}$ and $\mathscr{W}_{nm}' \subset \bigcup_{j>m^*} \mathscr{V}_j$. Then, for each $x \in \bigcup \mathscr{W}_{nm}$, we have $\varrho(x, H(n,m)) < 1/m^* \le r/2$. Then $(\bigcup \mathscr{W}_{nm}) \cap H(n,m)' = \varnothing$. Then $H(n,m-1) \subset H(n,m)'$ implies $H(n,m-1) \cap (\bigcup \mathscr{W}_{nm}) = \varnothing$.

Proof of 7. Notice that $\mathscr{W}_{nm}^*$ is an union of finitely many ϱ-discrete ϱ-clopen collections such that each collection of finitely many ϱ-discrete ϱ-clopen collections is a subset of the standard base by B.b of Construction 4. Pick an $x = (\alpha_i) \in H(n,m)$. Then, for each collection, there is a $V(\alpha_1 \cdots \alpha_{n(i)})$ in the collection with $x \in V(\alpha_1 \cdots \alpha_{n(i)})$. So $\mathscr{W}_{nm}^*(x) = \{V(\alpha_1 \cdots \alpha_{n(i)}) : i \le l(m)\}$. Then $\bigcap \mathscr{W}_{nm}^*(x) \in \mathscr{V}$ is ϱ-clopen. So $\mathscr{W}_{nm}$ a ϱ-discrete ϱ-clopen collection.

Proof of 4. Pick an $x' \in W(n,m,x)$. Let $x' \in H(n,m')$. Then $m' \ge m$ by the above proof of 5. If $m' = m$, we have $W(n,m',x') = W(n,m,x)$ since $\mathscr{W}_{nm}$ a ϱ-

discrete ϱ-clopen collection by the above 7. If $m' > m$, $W(n,m,x) \in \mathscr{W}^*_{nm}(x')$ since $x' \in W(n,m,x) \in \mathscr{W}_{nm}$. So $W(n,m',x') = \bigcap \mathscr{W}^*_{nm}(x') \subset W(n,m,x)$.

Proof of 6. Notice $H(n,m) = H_{nl} \cap H(n-1,i') \subset H(n-1,i')$. Then $\mathscr{W}_{n-1i'} \subset \mathscr{W}^*_{nm}$ by definition of $\mathscr{W}^*_{nm}$ in B.b in Construction 4. Then $W(n-1,i',x) \in \mathscr{W}^*_{nm}(x)$. So $W(n,m,x) = \bigcap \mathscr{W}^*_{nm}(x) \subset W(n-1,i',x)$. □

A stratifiable space (X,μ) is said to have *an M_3-structure* if (X,μ) satisfies conditions A and B in Proposition 4.7.

5. Results and Problems

Theorem 5.1. *The following are equivalent*:

1 *(Y,τ) is a stratifiable space.*

2 *There is a zero-dimension submetric stratifiable space (X,μ) with M_3-structures and an irreducible perfect map $f:(X,\mu) \to (Y,\tau)$.*

Proof. It easy to prove $2 \Rightarrow 1$. We prove $1 \Rightarrow 2$. By Theorem 3.4, there is a stratifiable space (X,μ) and an irreducible perfect map $f:(X,\mu) \to (Y,\tau)$. Then (X,μ) has a zero-dimension submetric and an M_3-structure by Proposition 4.6 and 4.7. □

Theorem 5.1 gives a part answer to a problem in Tamano [15] page 407 and Nagami [14] also.

Corollary 5.2. *The following are equivalent*:

1 *Each stratifiable space (Y,τ) is a M_1-space.*

2 *Each zero-dimension submetric stratifiable space (X,μ) with M_3-structures is M_1-space.*

Proof. $f:(X,\mu) \to (Y,\tau)$ is an irreducible perfect map. □

Theorem 5.1 and Corollary 5.2 raise the following Problem 1 which is equivalent to $M_3 \Rightarrow M_1$.

Problem 1. Is each zero-dimension submetric stratifiable space (X,μ) with M_3-structures an M_1-space?

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