

A NOTE ON THE FUSION OF TWO VERTICES IN A FUZZY GRAPH

By

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Abstract. The concept of fuzzy sets was introduced by L. A. Zadeh in 1965. The concept of fuzzy graph was introduced by A. Rosenfeld [5] in 1975. Many of the crisp graph concepts have been extended to fuzzy graph theory. Here we define the fusion of two vertices in a fuzzy graph and investigate some properties and also give fusion algorithm for effective connectedness and for adjacency matrix.

1 Preliminaries

A *fuzzy graph* $G = (\sigma, \mu)$ is a pair of functions $\sigma : S \rightarrow [0, 1]$ and $\mu : S \times S \rightarrow [0, 1]$ such that for all x, y in S we have $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ where S is the underlying (vertex) set. (In this paper we always consider S as a finite set.) For the fuzzy graph G , $G^* = (\sigma^*, \mu^*)$ is called the corresponding (crisp) *support graph* where $\sigma^* = \text{Support of } \sigma$ and $\mu^* = \text{Support of } \mu$. The fuzzy graph $H = (\tau, \nu)$ is called a (partial) *fuzzy subgraph* of G if $\tau(x) \leq \sigma(x)$ and $\nu(x, y) \leq \mu(x, y)$ for all x, y in S . The fuzzy graph $H = (\tau, \nu)$ is called a *spanning fuzzy subgraph* of G if $\tau(x) = \sigma(x)$ for all x in S . A fuzzy graph $G = (\sigma, \mu)$ is called a *complete fuzzy graph* if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all x and y ; it is called *quasi complete* if G^* is complete (clearly complete implies quasi complete). A complete fuzzy graph with n vertices is denoted by σ_n .

We shall call a fuzzy graph $G = (\sigma, \mu)$, *bipartite** if the graph $G^* = (\sigma^*, \mu^*)$ is bipartite where $\sigma^* = \text{Supp } \sigma$ and $\mu^* = \text{Supp } \mu$. We call a fuzzy graph $G = (\sigma, \mu)$, *fuzzy bipartite** if it has a spanning fuzzy subgraph $F = (\sigma, \nu)$ which is bipartite* and for all arcs (x, y) not in F (i.e. $\nu(x, y) = 0$), we have, $\mu(x, y) < \nu^\infty(x, y)$.

A fuzzy graph $G = (\sigma, \mu)$ is called *complete bipartite** if $G^* = (\sigma^*, \mu^*)$ is a complete bipartite graph. G is called *complete fuzzy bipartite** *fuzzy graph* if G

has a spanning fuzzy subgraph $F = (\sigma, \mu')$ which is complete bipartite* and for all arcs (x, y) not in F , $\mu(x, y) < \mu'^{\infty}(x, y)$.

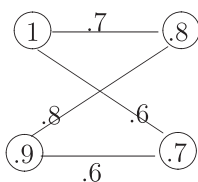
An edge (x, y) of a fuzzy graph $G = (\sigma, \mu)$ is called an *effective edge* if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$. A path in G is called an *effective path* if all the edges in that path are effective edges. A pair of vertices of G are called *effectively connected* if there is an effective path between them. A fuzzy graph is called *effectively connected* if every pair of vertices are effectively connected.

The terms undefined here may be seen in [1], [2] & [5].

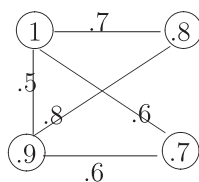
2 Fusion of Vertices in A Fuzzy Graph

DEFINITION 2.1. A fuzzy graph $G = (\sigma, \mu)$ is called *strong complete fuzzy bipartite* fuzzy graph* if G is complete fuzzy bipartite* and $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all (x, y) in F and it is denoted by $\sigma_{m,n}$.

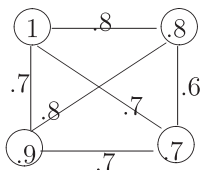
EXAMPLE 2.2.



complete bipartite*



complete fuzzy bipartite*



strong complete fuzzy bipartite*

DEFINITION 2.3. Let $G = (\sigma, \mu)$ be a fuzzy graph and let $u, v \in \sigma^*$. By the *fusion* of two vertices u and v we mean the following:

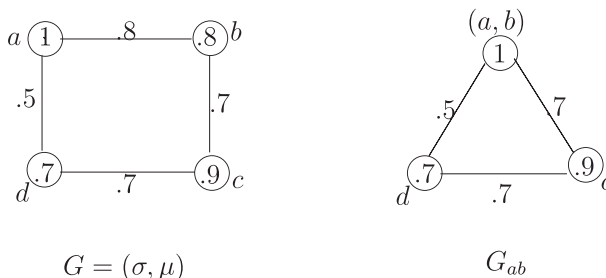
- (i) Fuse the vertices u and v as uv in the corresponding crisp graph $G^* = (\sigma^*, \mu^*)$ and then consider its underlying simple graph.
- (ii) The resulting fuzzy graph is $G_{uv} = (\sigma_{uv}, \mu_{uv})$ where

$$\sigma_{uv}(x) = \begin{cases} \max[\sigma(u), \sigma(v)], & \text{if } x = uv \\ \sigma(x), & \text{if } x \neq uv \end{cases}$$

and

$$\mu_{uv}(x, y) = \begin{cases} \max[\mu(x, u), \mu(x, v)], & \text{if } y = uv \\ \max[\mu(u, y), \mu(v, y)], & \text{if } x = uv \\ \mu(x, y), & \text{if } x \neq uv \text{ and } y \neq uv \end{cases}$$

EXAMPLE 2.4.



PROPOSITION 2.5. *The fusion of two vertices of K_n results in K_{n-1} .*

The analogous result in fuzzy graphs is as follows:

PROPOSITION 2.6. *The fusion of two vertices of a complete fuzzy graph on n vertices results in a complete fuzzy graph.*

(i.e.) *The fusion of two vertices of σ_n results in σ_{n-1} .*

PROOF. Let $G = (\sigma, \mu)$ be a complete fuzzy graph with n vertices. (i.e.) $G = \sigma_n$. Suppose x and y are fused together and get $G_{xy} = (\sigma_{xy}, \mu_{xy})$. By definition, $\sigma_{xy}(x, y) = \max[\sigma(x), \sigma(y)]$. Without loss of generality assume that $\sigma(x) < \sigma(y)$. Therefore, $\sigma_{xy}(xy) = \sigma(y)$. Suppose z is any vertex other than xy in G_{xy} . Hence $\sigma_{xy}(z) = \sigma(z)$. Then

$$\begin{aligned} \mu_{xy}(xy, z) &= \max[\mu(x, z), \mu(y, z)] \\ &= \max[\sigma(x) \wedge \sigma(z), \sigma(y) \wedge \sigma(z)] \\ &= (\sigma(x) \wedge \sigma(z)) \vee (\sigma(y) \wedge \sigma(z)) \\ &= (\sigma(x) \vee \sigma(y)) \wedge \sigma(z) \\ &= \sigma_{xy}(xy) \wedge \sigma_{xy}(z) \end{aligned}$$

By definition,

$$\begin{aligned} \mu_{xy}(u, v) &= \mu(u, v) \quad \text{for all } u, v \text{ where } u \neq (x, y) \text{ and } v \neq (x, y) \\ &= \sigma(u) \wedge \sigma(v) \\ &= \sigma_{xy}(u) \wedge \sigma_{xy}(v) \end{aligned}$$

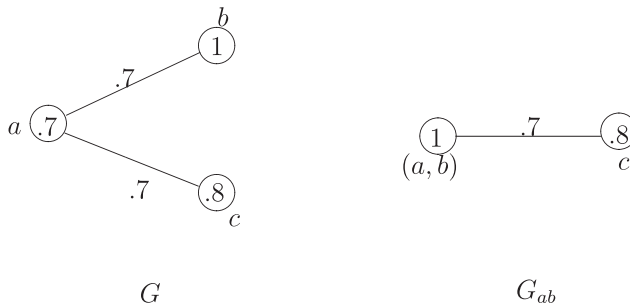
Hence $G_{xy} = \sigma_{n-1}$. \diamond

PROPOSITION 2.7. *The fusion of any two vertices of $K_{1,n}$ results in $K_{1,n-1}$.*

The above result need not be true in fuzzy graphs. The fusion of two vertices of $\sigma_{1,n}$ with both vertices in V_1 or in V_2 results in $\sigma_{1,n-1}$; fusion with one vertex in V_1 and another in V_2 need not result in $\sigma_{1,n-1}$. (i.e.)

PROPOSITION 2.8. *The fusion of two vertices of $\sigma_{1,n}$ with both the vertices in same set (V_1 or V_2) results in $\sigma_{1,n-1}$.*

EXAMPLE 2.9.

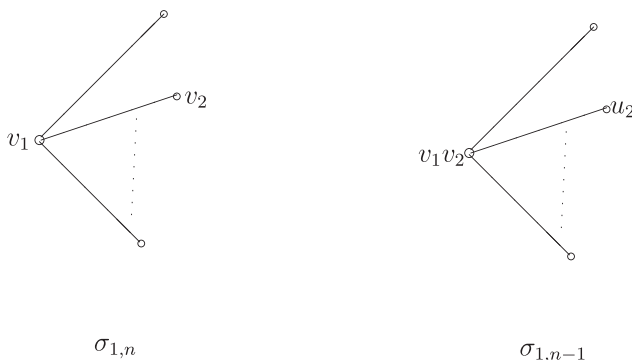


Clearly $G = \sigma_{1,2}$ but $G_{ab} \neq \sigma_{1,1}$.

PROPOSITION 2.10. *The fusion of two vertices of $\sigma_{1,n}$ with one vertex in V_1 and another in V_2 results in $\sigma_{1,n-1}$ iff for every pair of vertices u_2 and v_2 of V_2 ,*

$$\sigma(v_1) \geq \text{Min}[\sigma(u_2), \sigma(v_2)] \quad \text{where } V_1 = \{v_1\} \tag{1}$$

PROOF.



Suppose the inequality (1) is true. Let v_1 be fused with v_2 . Then $\sigma_{v_1v_2}(v_1v_2) = \text{Max}[\sigma(v_1), \sigma(v_2)]$. Let u_2 be any vertex other than v_2 in V_2 . Then

$$\begin{aligned} \mu_{v_1v_2}(v_1v_2, u_2) &= \mu(v_1, u_2) \\ &= \sigma(v_1) \wedge \sigma(u_2) \end{aligned} \tag{2}$$

Case-(i)

Suppose $\sigma(u_2) < \sigma(v_1) < \sigma(v_2)$. Then (2) implies

$$\begin{aligned} \mu_{v_1v_2}(v_1v_2, u_2) &= \sigma(u_2) \\ &= \sigma_{v_1v_2}(u_2) \\ &= \sigma_{v_1v_2}(v_1v_2) \wedge \sigma_{v_1v_2}(u_2) \end{aligned}$$

Case-(ii)

Suppose $\sigma(v_2) < \sigma(v_1) < \sigma(u_2)$. Then (2) implies

$$\begin{aligned} \mu_{v_1v_2}(v_1v_2, u_2) &= \sigma(v_1) \\ &= \sigma_{v_1v_2}(v_1v_2) \wedge \sigma_{v_1v_2}(u_2) \end{aligned}$$

Case-(iii)

Suppose $\sigma(u_2) < \sigma(v_2) < \sigma(v_1)$. Then (2) implies

$$\begin{aligned} \mu_{v_1v_2}(v_1v_2, u_2) &= \sigma(u_2) \\ &= \sigma_{v_1v_2}(u_2) \\ &= \sigma_{v_1v_2}(v_1v_2) \wedge \sigma_{v_1v_2}(u_2) \end{aligned}$$

Case-(iv)

Suppose $\sigma(v_2) < \sigma(u_2) < \sigma(v_1)$. Then (2) implies

$$\begin{aligned}\mu_{v_1v_2}(v_1v_2, u_2) &= \sigma(u_2) \\ &= \sigma_{v_1v_2}(u_2) \\ &= \sigma_{v_1v_2}(v_1v_2) \wedge \sigma_{v_1v_2}(u_2)\end{aligned}$$

Hence in all cases, we have $G_{v_1v_2}$ is strong complete fuzzy bipartite*. Therefore $G_{v_1v_2} = \sigma_{1,n-1}$.

Conversely, suppose the fusion of two vertices of $\sigma_{1,n}$ with one vertex in V_1 and another in V_2 results in $\sigma_{1,n-1}$.

To prove that (1) is true. Suppose not. Then,

$$\sigma(v_1) < \text{Min}[\sigma(u_2), \sigma(v_2)] \quad \text{for some pair of vertices } u_2, v_2 \text{ of } V_2. \quad (3)$$

Without loss of generality, assume that

$$\text{Min}[\sigma(u_2), \sigma(v_2)] = \sigma(u_2)$$

From (3), $\sigma(v_1) < \sigma(u_2) \leq \sigma(v_2)$. If we fuse u_2 with v_1 then

$$\begin{aligned}\sigma_{v_1u_2}(v_1u_2) &= \sigma(u_2) \\ \mu_{v_1u_2}(v_1u_2, v_2) &= \mu(v_1, v_2) \\ &= \sigma(v_1) \wedge \sigma(v_2) \\ &= \sigma(v_1) \\ &< \sigma(u_2) \\ &= \sigma_{v_1u_2}(v_1u_2) \wedge \sigma_{v_1u_2}(v_2)\end{aligned}$$

Which is a contradiction. Hence (1) is true. \diamond

PROPOSITION 2.11. *If $m \neq 1 \neq n$, then the fusion of two vertices of $K_{m,n}$ with both vertices in V_1 or in V_2 results in $K_{m-1,n}$ or $K_{m,n-1}$.*

The analogous result in fuzzy graphs is as follows:

PROPOSITION 2.12. *The fusion of two vertices of $\sigma_{m,n}$ ($m \neq 1 \neq n$) with both the vertices in V_1 or in V_2 results in $\sigma_{m-1,n}$ or $\sigma_{m,n-1}$. The proof is as similar as that of Proposition 2.6.*

In crisp graph theory, there is an algorithm to compute the adjacency matrix of the resulting graph after fusing two adjacent vertices. In a similar way we give algorithm to compute the adjacency matrix of a fuzzy graph after fusion of two adjacent vertices.

Algorithm to compute the adjacency matrix of a fuzzy graph after fusion of two adjacent vertices

Let $G = (\sigma, \mu)$ be a fuzzy graph whose adjacency matrix [1] is denoted by $A(G)$. Suppose u and v are adjacent. (i.e.) $\mu(u, v) > 0$. Suppose we fuse u and v . Then the adjacency matrix of G_{uv} is denoted by $A(G_{uv})$ and it can be obtained by the following steps:

Step-1

Change u 's row to the maximum of u 's row and v 's row and change u 's column to the maximum of u 's column and v 's column.

Step-2

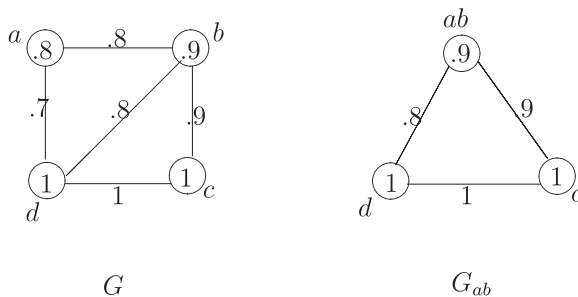
Delete the row and column corresponding to v .

Step-3

Name u 's row and u 's column as uv and put the entry as 0 corresponding to (uv, uv) .

Then the resulting matrix is the adjacency matrix of the new fuzzy graph G_{uv} .

Illustration



$$A(G) = \begin{matrix} & \begin{matrix} a & b & c & d & \sigma \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0 & .8 & 0 & .7 & .8 \\ .8 & 0 & .9 & .8 & .9 \\ 0 & .9 & 0 & 1 & 1 \\ .7 & .8 & 1 & 0 & 1 \end{pmatrix} \end{matrix}$$

Step-1

$$\begin{array}{c} a \quad b \quad c \quad d \quad \sigma \\ a \begin{pmatrix} .8 & .8 & .9 & .8 & .9 \\ .8 & 0 & .9 & .8 & .9 \\ .9 & .9 & 0 & 1 & 1 \\ .8 & .8 & 1 & 0 & 1 \end{pmatrix} \\ b \\ c \\ d \end{array}$$

Step-2

$$\begin{array}{c} a \quad c \quad d \quad \sigma \\ a \begin{pmatrix} .8 & .9 & .8 & .9 \\ .9 & 0 & 1 & 1 \\ .8 & 1 & 0 & 1 \end{pmatrix} \\ c \\ d \end{array}$$

Step-3

$$A(G_{ab}) = \begin{array}{c} ab \quad c \quad d \quad \sigma \\ ab \begin{pmatrix} 0 & .9 & .8 & .9 \\ .9 & 0 & 1 & 1 \\ .8 & 1 & 0 & 1 \end{pmatrix} \\ c \\ d \end{array}$$

The fusion algorithm for effective connectedness

Let $G = (\sigma, \mu)$ be a fuzzy graph whose adjacency matrix is denoted by $A(G)$.

Step-1

Fuse vertex v_1 to the first of the vertices v_2, v_3, \dots, v_n with which it is effectively adjacent. In which the new fuzzy graph is also denoted by G and the new vertex is also denoted by v_1 . (The above 3-step process gives the adjacency matrix $A(G)$).

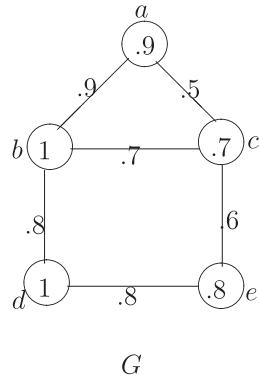
Step-2

Repeat step-1 until v_1 is not effectively adjacent to any of the other vertices.

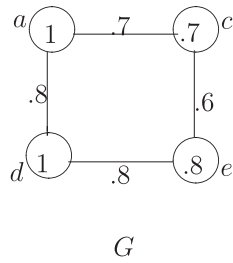
Step-3

Repeat steps-1&2 on all the remaining vertices v_i of the resulting fuzzy graphs. The final fuzzy graph is (effectively) empty and the number of its (effectively isolated) vertices is the number of effectively connected components of the given fuzzy graph G .

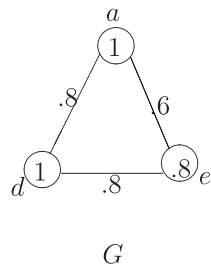
Illustration



Step-1



Step-2



Step-3



References

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