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ON GRAPHS LIKE HYPERCUBES

By

Juhani Nieminen, Matti Peltola and Pasi Ruotsalainen

Abstract. By defining new concepts like pseudocomplements in graphs a new class of graphs is obtained. They have very many properties in common with hypercubes and therefore they are called pseudocubes. Pseudocubes are Hasse diagram graphs (covering garphs) of finite lattices, where pseucomplements constitute a sublattice. As an application, the routing and fault tolerance properties of certain pseudocubes are determined.

1. Introduction

Hypercubes constitute a very remarkable class of graphs especially for transmitting communication and therefore the graphs having many properties in common with hypercubes are an alternative for hypercubes. When looking after properties of hypercubes one has to find a model the properties of which one can translate into graphs. The model of this paper is the class of finite Boolean lattices, the properties of which we are monitoring and translating into graphs. As well known, the Hasse diagram graph (the covering graph) of a finite Boolean lattice is a hypercube, and conversely each finite hypercube can be identified as a finite Boolean lattice by using e.g. the well known labeling of vertices by 0,1-strings. Each Boolean lattice is distributive and complemented, and by weakening these properties we try to find alternatives for hypercubes. Each vertex of a finite hypercube has the same degree and thus a good generalization must have this property. An arbitrary vertex of finite hypercube can be identified as a least element of a finite Boolean lattice, and thus this kind of symmetry must exist in a good generalization. Moreover, in a finite Boolean lattice each vertex is on a shortest path between the least and greatest element, and also this property must

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exist in a good generalization. The complementarity of Boolean lattices seem to be a too weak property to affect to all vertices of a graph but its weakened form called pseudocomplementarity will order the vertices of a graph rather throughout, and thus a good generalization must be pseudocomplemented. After this list we are able to define pseudocubes, the graphs like hypercubes. Before the definition, we need some notations.

2. Weak Pseudocubes

The graphs $G = (V, E)$ of this paper are finite, connected and undirected with vertex set V and edge set E. A vertex set $A \subset V$ is a convex, if $x, y \in A$ and z on a shortest $x - y$ path (on an $x - y$ geodesic) imply that $z \in A$. Clearly a non-empty intersection $A \cap B$ of two convexes is a convex, too. By $\langle D \rangle$ we denote the least convex containing the vertex set D: $\langle D \rangle = \bigcap\{C \mid C$ is a convex and $D \subset C\}$. The least convex containing the vertices x and y is briefly denoted by $\langle x, y \rangle$, and by $[x, y]$ we denote the set of all vertices locating on an $x - y$ geodesic. Clearly the convex $\langle x, y \rangle$ also contains vertices on an $x - y$ geodesic and hence $[x, y] \subset \langle x, y \rangle$. In the covering graph of a finite distributive lattice, $[x, y] = \langle x, y \rangle$ for each two vertices x and y, and thus in each (finite) hypercube $\langle x, y \rangle \subset [x, y]$ for all $x, y \in V$. In the covering graph of a finite Boolean lattice, if x' is the complement of x, then $[x, x'] = V$. In a graph G, a vertex x has a complement y, if $[x, y] = V$. In a finite lattice L with a least element 0 an element a has a pseudocomplement a^*, if $a \wedge a^* = 0$ and $a \wedge x = 0$ imply $x \leq a^*$. In a graph G a vertex x has a pseudocomplement x_z^* relative to a vertex z if and only if $\langle x_z^*, z \rangle$ is the greatest convex such that $\langle x_z^*, z \rangle \cap \langle z, x \rangle = \{z\}$. This means that if $\langle b, z \rangle \cap \langle z, x \rangle = \{z\}$ then $\langle z, b \rangle \subset \langle x_z^*, z \rangle$.

DEFINITION 1. *A finite graph $G = (V, E)$ is called a weak pseudocube, if*

(i) *for each convex A of G there are vertices $a, b \in A$ such that $[a, b] = A$, especially we have $\langle a, b \rangle = [a, b]$ for each pair a, b of vertices in G.*

(ii) *for each vertex $a \in V$ there is a unique vertex a' such that $V = [a, a']$;*

(iii) *for each triple a, b, c of vertices in G there is a unique vertex a_b^* such that $[a_b^*, b] \cap [a, b] = \{b\}$ and the relation $[a, b] \cap [b, c] = \{b\}$ implies the relation $[b, c] \subset [b, a_b^*]$.*

As one can easily check, each cycle C_{2n} with even number of vertices is a weak pseudocube but an odd cycle is not: if a is a vertex of C_{2n-1}, there is no vertex a' such that $[a, a'] = V$.

We prove some results of weak pseudocubes.

LEMMA 1. *In a weak pseudocube* $WP = (V, E)$, *if vertices* s *and* z *are adjacent*, *then*
(i) $d(a,z) = d(a,s) + 1$ *for all* $a \in [s, z']$;
(ii) $[s, z'] \cap [s', z] = \emptyset$.

PROOF. (i) Clearly $|d(a,s) - d(a,z)| \le 1$ for all $a \in V$. If $a \in [s, z'] = [z', s]$, then $[z', a] \subseteq [z', s] \subseteq V = [z', z]$. Thus there exists a shortest $z' - z$ path which goes through z', a, s, z in this order and the last edge of the path is sz.

(ii) Assume that $a \in [s, z'] \cap [s', z]$. Since $a \in [s', z]$, we have $[a, z] \subseteq [s', z]$, thus $s \in [s', z]$, which is absurd.

LEMMA 2. *In a weak pseudocube* $WP = (V, E)$ *diameter of* WP, $diam(WP) = d(a, a')$ *for all* $a \in V$.

PROOF. Assume that s and z are adjacent vertices of V. Since $z' \in [s, z']$ and $s \in [z', s]$ the previous lemma implies that $d(z', z) = d(z', s) + 1$ and $d(s, s') = d(s, z') + 1$. Thus we have $d(s, s') = d(z, z')$ and since weak pseudocubes are clearly connected, this implies that $d(a, a') = d(b, b')$ for each pair of vertices a and b. Since $V = [a, a'] = [b, b']$, the assertion follows.

LEMMA 3. *In a weak pseudocube* $WP = (V, E)$, *if* s *and* z *are adjacent*, *then* s' *is adjacent to* z', $s' = s_z^*$, *and if* $y (\neq s)$ *is adjacent to* z, *then* $y \in [z, s']$. *Moreover* $[z, s'] \cup [z', s] = V$.

PROOF. Since $[s, z'] \subseteq [s, s']$, there exists a shortest $s - s'$ path which goes through z', thus $d(s, s') = d(s, z') + d(z', s')$. By the proof of the previous lemma we have $d(s, z') = d(s, s') - 1$, which implies that s' and z' are adjacent.

Since $z \notin [s, z']$, we have $[s, z'] \cap [s, z] = [s, z'] \cap \{s, z\} = \{s\}$. By the definition of the weak pseudocube, we have $[s, z'] \subseteq [s, z_s^*]$, thus there exists a shortest $s - z_s^*$ path through z'. Since $d(s, z') = d(s, s') - 1$, then $z_s^* \in \{z', s'\}$. If $z_s^* = s'$, then we have $\{s, z\} = [z, s] = [s, s'] \cap [z, s] = [s, z_s^*] \cap [z, s] = \{s\}$, which is absurd. Thus $z_s^* = z'$.

Assume now that y is adjacent to z, $y \neq s$ and $y \notin [s', z]$. This implies that $[s', z] \cap [y, z] = \{z\}$, whence $y_z^* = s'$. On the other hand, $s' = s_z^*$, and analogously, $y' = y_z^*$. This means that the vertex $y' = s'$ has two complements y and s, which is absurd. Hence, if y is adjacent to z and $y \neq s$, then $y \in [z, s']$.

Assume now that $y \notin [z, s']$. Then $z \notin [s, y]$, since otherwise there exists a shortest $s - s'$ path which goes through z and y on this order. Thus $[s, y] \cap [s, z] = \{s\}$. Then $[s, y] \subseteq [s, z_s^*] = [s, z']$, so $y \in [z', s]$, which implies that $[z, s'] \cup [z', s] = V$.

A convex K is a prime convex, if also the set $V \backslash K$ is a convex in G. A graph G is a prime convex intersection graph if each convex D of G is the intersection of prime convexes containing D. As known, each (finite) hypercube is a prime convex intersection graph.

By the previous lemmas we have proved the following result.

THEOREM 1. *In a weak pseudocube, if vertices z and s are adjacent, then $[z, s']$ and $[s, z']$ are prime convexes.*

THEOREM 2. *Weak pseudocubes are bipartite, i.e. each of a cycle of weak pseudocube has an even length.*

PROOF. Assume that a weak pseudocube $WP = (V, E)$ contains an odd cycle P. Then there exist three vertices s, z, a of P such that s and z are adjacent and $d(s, a) = d(z, a)$. Since by the previous lemmas we have $V = [s, z'] \cup [s', z]$ and $[s, z'] \cap [s', z] = \emptyset$, we may assume that $a \in [s, z']$. Since $V = [z, z']$, there exists a shortest $z - z'$ path which goes through a. Then $d(z, z') = d(z, a) + d(a, z') = d(s, a) + d(a, z') = d(s, z') < d(s, s')$, which is absurd by Lemma 2.

THEOREM 3. *In a weak pseudocube $WP = (V, E)$, $deg(z) = deg(z')$ for each vertex z.*

PROOF. Let y and s be adjacent to z and $y \neq s$. By Lemma 3, $y \in [z, s']$. By (iii) of Definition 1, each vertex y has a unique complement y', and because $[y, y'] = V \neq [z, s']$, y' cannot belong to the convex $[z, s']$. Because $[z, s'] \cup [s, z'] = V$, the vertex y' belongs to the convex $[s, z']$, and by Lemma 3, y' is adjacent to z'. Thus for each vertex x adjacent to z there is a unique vertex x' adjacent to z', which shows that $deg(z) = deg(z')$.

THEOREM 4. *Weak pseudocubes are prime convex intersection graphs.*

PROOF. Assume that A is a convex of a weak pseudocube $WP = (V, E)$ and let $B = \bigcap\{PC \mid PC$ is a prime convex, A a convex and $A \subset PC\}$. If B contains a vertex z such that $z \notin A$, we can choose z so that z is adjacent to a vertex $s \in A$. If there is another vertex $t \in A$ which also is adjacent to z, then t and s are adjacent, because otherwise z is on a shortest $t-s$ path and thus belongs to convex A. But since weak pseudocube is bipartite, this is absurd. Thus z is adjacent to a single vertex (here s) of A.

By (i) in Definition 1, $A=[u,w]$, and thus s is on a shortest $u-w$ path. If $u,w \in [s,z']$, then $A=[u,w] \subset [s,z']$, where $[s,z']$ is a prime convex containing A and not z. Thus $z \notin B$, and hence we must assume that the vertices u and w cannot simultaneously belong to one of the prime convexes $[s,z']$ and $[z,s']$: in the following we assume that $u \in [z,s']$ and $w \in [s,z']$. Let t be the vertex of A nearest to s on a shortest $u-z$ path; in an extreme case $t=u$. If $d(s,t)<d(z,t)$, then $d(z,t) \geq d(z,s)+d(s,t)=1+d(s,t)$, and thus $s \in [z,t] \subseteq [z,s']$, which is absurd. If $d(s,t)>d(z,t)$, we similarly see that $z \in [s,z']$, which is absurd, too. Hence $d(z,t)=d(s,t)=a$. Because t is on a shortest $u-z$ path, it also is on a shortest $z-s'$ path, and we denote $d(t,s')=h$. On the other hand, t also is on a shortest $z-z'$ path and we denote $d(t,z')=k$. By using these notations, we have $d(z,z')=d(z,t)+d(t,z')=a+k$ and $d(s,s')=d(s,t)+d(t,s')=a+h$. By Lemma 2, $d(s,s')=d(z,z')$, and thus $h=k$. But then $d(z,s')=d(z,t)+d(t,s')=a+h=d(z,z')$, which is absurd. This implies that u and w must simultaneously belong to one of the prime convexes $[z,s']$ and $[s,z']$, whence $\bigcap\{PC \mid PC$ is a prime convex and $A \subset PC\}=A$ and each weak pseudocube is a prime convex intersection graph.

3. Pseudocubes

In Figure 1a there is a weak pseudocube which cannot be mapped onto a hypercube with respect to vertex b. Thus some more properties are needed.

Definition 2. *A weak pseudocube is called a pseudocube P, if for each three vertices a, b, c of P there is a vertex d such that $[a,c] \cap [b,c]=[d,c]$.*

Consider the weak pseudocube of Figure 1a, where $[a,b] \cap [c,b]=[d,e]$, which shows that the graph of the figure is not a pseudocube.

Because for each vertex z of a pseudocube $P=(V,E)$ there is a unique vertex z' such that $[z,z']=V$, we can order the vertices of P such that z is the least and z' the greatest element of P. Since $[a,z] \cap [b,z]=[d,z]$, we see that for any two elements/vertices a and b of P there is a greatest lower bound d, and thus the vertices of P constitute a meet-semilattice with z as least element. Because P is finite and there is a greatest element z', the vertices of P constitute a lattice with z' as a greatest element, $a \wedge b = d$ if and only if $[a,z] \cap [b,z] = [d,z]$, and $a \vee b = e$ if and only if $[e,z] = \bigcap\{[x,z] \mid [a,z],[b,z] \subset [x,z]\}$. Thus $[a,z] \cup [b,z] \subset [a \vee b, z]$. Denote by $L(z,z',P)$ the lattice generated by P with z as the least element and z' as the greatest element. One can now see that a pseudocomplement a_z^* of a vertex a in pseudocube P is the same element/vertex as the pseudocomplement a^* of the element a in the lattice $L(z,z',P)$, since a^* is the greatest element of $L(z,z',P)$ having the property $a \wedge a^*$ is the least element of the lattice.

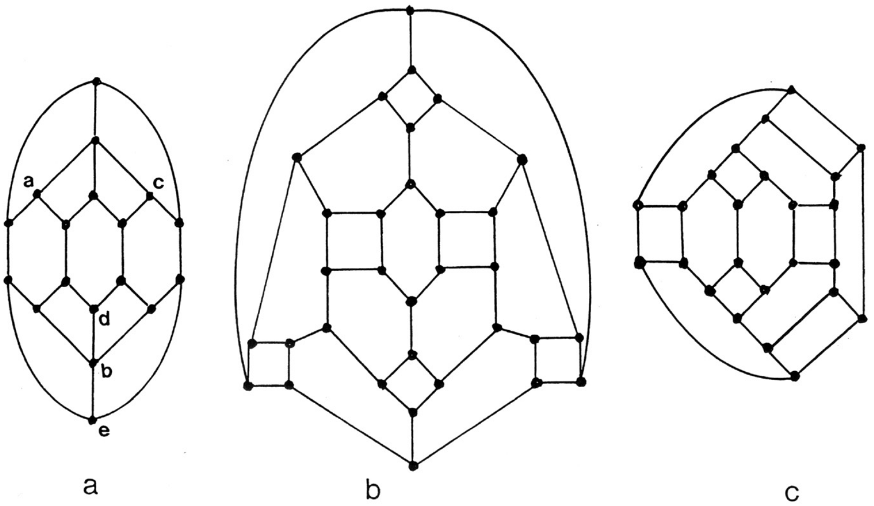

Figure 1: a) a weak pseudocube, b) a pseudocube c) a strong pseudocube.

THEOREM 5. *A vertex b of a pseudocube P is a pseudocomplement p_z^* with respect to a vertex z if and only if the convex $[b,z]$ has an expression as an intersection of the prime convexes $[s',z]$, where s is adjacent to z, i.e. $[p_z^*,z] = [b,z] = \bigcap\{[s',z] \mid s$ is adjacent to z and $b \in [s',z]\}$.*

PROOF. Let $b = p_z^*$, $S = \{s_1,\dots,s_n\}$ the set of all vertices adjacent to z, and let $\{s_{1p},\dots,s_{up}\} = [p,z] \cap S$. Because $s_{ip} \in [p,z]$ and $[p,z] \cap [p_z^*,z] = \{z\}$, we have $[s_{ip},z] \cap [p_z^*,z] = \{z\}$. As stated in Theorem 1, the convex $[s'_{ip},z]$ is a prime convex not containing the vertex s_{ip}, and thus $[s_{ip},z] \cap [s'_{ip},z] = \{z\}$ as well as $p_z^* \in [s'_{ip},z]$ for each $s_{ip} \in [p,z]$. This implies that $[p_z^*,z] \subset \bigcap\{[s'_{ip},z] \mid s_{ip} \in [p,z]\}$. If $[p,z] \cap (\bigcap\{[s'_{ip},z] \mid s_{ip} \in [p,z]\}) \neq \{z\}$, this intersection must contain also a vertex adjacent to z, which is absurd, and hence $[p,z] \cap (\bigcap\{[s'_{ip},z] \mid s_{ip} \in [p,z]\}) = \{z\}$.

By the definition, $[p_z^*, z]$ is the greatest convex containg each convex $[x, z]$ with the property $[p, z] \cap [x, z] = \{z\}$, and thus $[b, z] = [p_z^*, z] = \bigcap\{[s'_{ip}, z] \mid s_{ip} \in [p, z]$ and s_{ip} is adjacent to $z\}$.

Conversely, let $[b, z] = \bigcap\{[s'_{jb}, z] \mid s_{jb} \in S_b$ and S_b is a subset of $S\}$. Denote $S_{cb} = S \backslash S_b$. Because $[s, z']$, $[s', z]$ is a pair of prime convexes for each vertex $s \in S$, we have $s \notin [s', z]$ and $S\backslash\{s\} \subset [s', z]$. Thus $S_{cb} \subset \bigcap\{[s'_{jb}, z] \mid s_{jb} \in S_b\}$ and $S_b \subset \bigcap\{[s'_{kcb}, z] \mid s_{kcb} \in S_{cb}\}$, and this implies, because $S_b \cap S_{cb} = \emptyset$, that $(\bigcap\{[s'_{jb}, z] \mid s_{jb} \in S_b\}) \cap (\bigcap\{[s'_{kcb}, z] \mid s_{kcb} \in S_{cb}\}) = \{z\}$. By the proof of the first part of this theorem, $[b_z^*, z] = \bigcap\{[s', z] \mid s \in S\backslash[b, z] = S_{cb}\}$. By the same reason $[(b_z^*)_z^*, z] = \bigcap\{[s', z] \mid s \in S\backslash[b_z^*, z] = S_b\}$, and the theorem follows.

Note that not each convex of a pseudocube P is the intersection of the prime convexes $[s', z]$ with s as a vertex adjacent to z. Thus the lattice $L(z, z', P)$ need not be distributive and also not Boolean. Although each convex $[x, z]$ of P is the intersection of prime convexes of P, these prime convexes need not simultaneously be prime ideals of the lattice $L(z, z', P)$ (see the pseudocube of Figure 1b). This is the reason for the essential difference between the graph P and the lattice $L(z, z', P)$.

Theorem 6. *Let P be a pseudocube. The following equation holds in the lattice $L(z, z', P) : (a \vee b)^* = a_z^* \wedge b_z^*$.*

Proof. Let S be the set of all vertices adjacent to z, $S \cap [a, z] = S_a$ and $S \cap [b, z] = S_b$. By Theorem 5, $[a_z^*, z] = \bigcap\{[s', z] \mid s \in S_a\}$ and $[b_z^*, z] = \bigcap\{[s', z] \mid s \in S_b\}$. By the definition of the meet in the lattice $L(z, z', P)$, $[a_z^* \wedge b_z^*, z] = \bigcap\{[s', z] \mid s \in S_a \cup S_b\}$. This implies by Theorem 5 that $a_z^* \wedge b_z^*$ is a pseudocomplement with respect to z in P. Because $[a_z^*, z] \cap [a, z] = \{z\}$ and $[b_z^*, z] \cap [b, z] = \{z\}$ we also have $[a_z^* \wedge b_z^*, z] \cap [a, z] = \{z\} = [a_z^* \wedge b_z^*, z] \cap [b, z] = \{z\}$. By the definition of a pseudocomplement, $[p_z^*, z]$ is the greatest convex containing all convexes $[x, z]$ with the property $[p, z] \cap [x, z] = \{z\}$, and thus the equation $[a_z^* \wedge b_z^*, z] \cap [a, z] = \{z\} = [a_z^* \wedge b_z^*, z] \cap [b, z] = \{z\}$ implies $[a \vee b, z] \subset [(a_z^* \wedge b_z^*)_z^*, z]$, whence $[a \vee b, z] \cap [a_z^* \wedge b_z^*, z] = \{z\}$ and $[a_z^* \wedge b_z^*, z] \subset [(a \vee b)_z^*, z]$. By Theorem 5, $[(a \vee b)_z^*, z] = \bigcap\{[s', z] \mid s \in [a \vee b, z] \cap S\}$, and because $[a \vee b, z] \cap S \supset S_a \cup S_b$, we have $[(a \vee b)_z^*, z] \subset \bigcap\{[s', z] \mid s \in S_a \cup S_b\} = [a_z^* \wedge b_z^*, z]$, and thus $(a \vee b)_z^* = a_z^* \wedge b_z^*$ as asserted.

Theorem 7. *Let P be a pseudocube. The relation θ, where $(a, b) \in \theta \Leftrightarrow a_z^* = b_z^*$ is a congruence relation in the lattice $L(z, z', P)$.*

Proof. Clearly the relation θ is reflexive and symmetric, and thus it remains to show that it is compatible with respect to the lattice operations $\wedge$ and $\vee$. Let $(a,b),(c,d) \in \theta$ and thus $a_z^* = b_z^*$ and $c_z^* = d_z^*$. By Theorem 6, $(a \vee c)_z^* = a_z^* \wedge c_z^* = b_z^* \wedge d_z^* = (b \vee d)_z^*$, whence $(a \vee c, b \vee d) \in \theta$. We use the brief expression p_z^{**} instead of the long one $(p_z^*)_z^*$. By the definition of the pseudocomplement in P, we have $p_z^{***} = p_z^*$ in P as well as in $L(z,z',P)$. Now the equation of Theorem 6 implies $(p \wedge q)_z^* = (p \wedge q)_z^{***} = ((p \wedge q)_z^*)_z^{**} = (p_z^* \vee q_z^*)_z^{**}$. Thus the relation $(a,b),(c,d) \in \theta$ implies $(a \wedge c)_z^* = (a \wedge c)_z^{***} = ((a \wedge c)_z^*)_z^{**} = (a_z^* \vee c_z^*)_z^{**} = (b_z^* \vee d_z^*)_z^{**} = ((b \wedge d)_z^*)_z^{**} = (b \wedge d)_z^{***} = (b \wedge d)_z^*$, whence also $(a \wedge c, b \wedge d) \in \theta$ as asserted.

The congruence θ of Theorem 7 shows that each lattice $L(z,z',P)$ is homomorphic to the lattice of pseudocomplements of $L(z,z',P)$, and as known (see for example Theorem I.4.6 in Grätzer's book [2]), the lattice of pseudocomplements is a Boolean lattice (a hypercube). This is the reason for calling the graphs P of this paper pseudocubes. By Lemma 3, each vertex s' adjacent to the vertex z' is the pseudocomplement of a vertex s adjacent to the vertex z. Thus if $deg(z) = n$, then the lattice homorphism φ generated by the congruence relation θ on the lattice $L(z,z',P)$ maps $L(z,z',P)$ onto a Boolean lattice the covering graph of which is the n-dimensional hypercube Q_n.

4. Strong Pseudocubes

Definition 3. *A pseudocube P is a strong pseudocube SP if for any pair of prime convexes $[s',z]$, $[s,z']$, where s and z are adjacent vertices of P, there is a (graph) isomorphism f from the graph induced by $[s',z]$ onto the graph induced by $[s,z']$ such that if y and x are two adjacent vertices belonging to disjoint prime convexes (e.g. $x \in [s',z]$ and $y \in [s,z']$) then f maps x onto y and vice versa.*

As well known, each hypercube Q_n satisfies the isomorphism condition of Definition 3 but it does not hold for all pseudocubes. The pseudocube of Figure 1c is a strong one but the pseudocube of Figure 1b is not. In the case of strong pseudocubes we can prove

Theorem 8. *Strong pseudocubes are regular.*

Proof. Let SP be a strong pseudocube. Let s and z be adjacent vertices. Because $f(z) = s$ and $f(s') = z'$ in the (graph) isomorphism between the subgraphs induced by the convexes $[z, s']$ and $[s, z']$, we have $deg(z) = deg(s)$ and $deg(s') = deg(z')$. On the other hand, by Theorem 3 $deg(z) = deg(z')$, and thus $deg(z) = deg(s) = deg(z') = deg(s')$ for each two adjacent vertices z and s. But this implies that $deg(u) = deg(s) = deg(z)$ for each vertex u adjacent to s. Because SP is connected we see by induction that each vertex of SP has the same degree.

As the pseudocube P of Figure 1b) shows, each vertex of a not strong pseudocube may have the same degree. We have not succeeded to find a pseudocube having at least two vertices x and y with $deg(x) \neq deg(y)$ nor to prove that each vertex of a pseudocube has the same degree.

The cartesian product $G_1 \times G_2 = (V_{G_1 \times G_2}, E_{G_1 \times G_2})$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph, where $V_{G_1 \times G_2}$ is the cartesian product $V_1 \times V_2$ of the sets V_1 and V_2 and where $((a_1, a_2), (b_1, b_2)) \in E_{G_1 \times G_2}$ if and only if (a_1, a_2) and (b_1, b_2) are vertices in $G_1 \times G_2$ and either $a_1 = b_1$ and $(a_2, b_2) \in E_2$ or $(a_1, b_1) \in E_1$ and $a_2 = b_2$. Consider a path $(c_{10}, c_{20}), (c_{11}, c_{21}), (c_{12}, c_{22}), (c_{13}, c_{23}), \ldots, (c_{1n}, c_{2n})$ of the cartesian product $G_1 \times G_2$. The path has two projections: one on the graph G_1 and another on the graph G_2. The projection on G_1 is $c_{10}, c_{11}, c_{12}, c_{13}, \ldots, c_{1n}$ and the projection on G_2 is $c_{20}, c_{21}, c_{22}, c_{23}, \ldots, c_{2n}$. By removing the multiple vertices from the projections we obtain two paths and by the definition of $G_1 \times G_2$ we see that the length n of the path $(c_{10}, c_{20}), (c_{11}, c_{21}), (c_{12}, c_{22}), (c_{13}, c_{23}), \ldots, (c_{1n}, c_{2n})$ is the sum of the lengths of the projections in G_1 and G_2, respectively. Thus a path $(c_{10}, c_{20}), (c_{11}, c_{21}), (c_{12}, c_{22}), (c_{13}, c_{23}), \ldots, (c_{1n}, c_{2n})$ in $G_1 \times G_2$ is a shortest path between (c_{10}, c_{20}) and (c_{1n}, c_{2n}) if and only if the corresponding projections are shortest paths in G_1 and G_2, respectively. This implies that if A is a convex in $G_1 \times G_2$, then its respective projections in A_1 and A_2 in G_1 and G_2 are convexes too. Now, if the graphs G_1 and G_2 are weak pseudocubes, then $A_1 = [a_1, b_1]$ and $A_2 = [a_2, b_2]$, which imply that $A = [(a_1, a_2), (b_1, b_2)]$. By the definition one can also see that the intersection of vertex sets in $G_1 \times G_2$ reduces to the intersection of projected vertex sets in G_1 and G_2, respectively. This implies the validity of the following theorem

Theorem 9. *If the graphs G_1 and G_2 are*

(i) *weak pseudocubes then the cartesian product $G_1 \times G_2$ is a weak pseudocube*;

(ii) *pseudocubes then the cartesian product $G_1 \times G_2$ is a pseudocube*;

(iii) *strong pseudocubes then the cartesian product $G_1 \times G_2$ is a strong pseudocube.*

The isomorphism f of Definition 3 implies an automorphism of SP. Because an arbitrary vertex of SP can be chosen as the vertex z and an arbitrary vertex adjacent to z as the vertex s, there is an automorphism of SP mapping any two adjacent vertices onto each other. If $(a_1, a_2), (a_2, a_3) \in E_{SP}$, then the automorphisms of SP from a_1 to a_2 and from a_2 to a_3 imply an automorphism from a_1 to a_3. Because SP is connected, we see by induction that there is an automorphism of SP mapping an arbitary vertex a to another arbitrary vertex b (a property which holds in every hypercube). Thus we have

THEOREM 10. *Strong pseudocubes are vertex transitive, i.e. for any two vertices a and b of a strong pseudocube SP there exists an automorphism of SP mapping a to b.*

5. An Application

Finally we consider shortly fault tolerance in strong pseucocubes.

Interconnection networks are usually modeled by graphs in which the vertices represent processors and the edges communication links. The message delivery system must find a route along with which to send each message to its destination, where a route is a path from one vertex to another. The problem is greatly simplified if one chooses a route in advance for each source/destination pair and uses that route for all messages. Such choice of routes is called routing, and if the routing is computed only once for a given graph, considerable effort can be put into its computation.

For a graph $G = (V, E)$ a routing ρ is a function which assigns to each pair $x, y \in V$, $x \neq y$, a fixed $x - y$ path. The routing ρ is called a shortest path routing/geodetic routing, if its each path/route is a shortest path/geodesic in G. A fault in G is either a vertex or an edge in G. Let F be a set of faults in G. An $x - y$ route $\rho(x, y)$ is said to avoid F if no fault is contained in it. Given a set F of faults in G, the fault free routing ρ/F is defined to be a reduction of ρ to fault free routes.

The efficiency of fault tolerance of a fixed routing ρ in a graph G is measured by counting how many fault free shortest paths of routing ρ one must use in order to transmit a message from a vertex v to another vertex y. Dolev et al. [1] have proved in Theorem 1 that in a hypercube Q_n one must use at most three fault free paths of an arbitrary shortest path routing ρ to transmit a message from an arbiray vertex v to another arbitrary vertex y if the hypercube contains at most $n - 1$ faults; see also the paper [3] by Opatrny, Srinivasan and Alagar.

Denote by SP_n a strong pseudocube where $deg(a) = n$ for each vertex a. Let Q_n be the Boolean lattice/hypercube image of the lattice $L(z, z', SP_n)$ under the homorphism φ determined by the congruence relation θ $((a, b) \in \theta \Leftrightarrow a_z^* = b_z^*)$ on $L(z, z', SP_n)$. We say that a vertex v of Q_n is fault free if and only if the congruence class C_v in $L(z, z', SP_n)$ corresponding to v is fault free, and an edge (v, u) in Q_n is fault free if and only if all edges joining the vertices of congruence classes C_v and C_u in $L(z, z', SP_n)$ corresponding to v and u are fault free.

Theorem 11. *Let SP_n be a strong pseudocube where $deg(a) = n$ for each vertex a. If SP_n contains at most $n - 1$ faults then for any two vertices a and b from any two fault free congruence classes of the congruence θ $((e, c) \in \theta \Leftrightarrow e_z^* = c_z^*$ in the lattice $L(z, z', SP_n)$ with freely chosen vertex $z)$ can be joined by using at most three fault free shortest paths of an arbitrary shortest path routing ρ on SP_n.*

Proof. Let Q_n be the Boolean lattice/hypercube image of the lattice $L(z, z', SP_n)$ of the strong pseudocube SP_n under the homorphism φ determined by the congruence relation θ $((a, b) \in \theta \Leftrightarrow a_z^* = b_z^*)$ on $L(z, z', SP_n)$. Consider a convex (note that each convex of Q_n determines a convex sublattice and vice versa) of Q_n. In a finite lattice Q_n each convex can be expressed as a set $[a_{Ql}, b_{Qg}]$, where $a_{Ql} \le b_{Qg}$ and where a_{Ql} is the least and b_{Qg} the greatest element of the convex $[a_{Ql}, b_{Qg}]$. Let $[a_{1Ll}, b_{1Lg}]$ and $[a_{2Ll}, b_{2Lg}]$ be the convex sublattices of $L(z, z', SP_n)$ for which $\varphi(c) = a_{Ql}$ for each $c \in [a_{1Ll}, b_{1Lg}]$ and $\varphi(c) = b_{Ql}$ for each $c \in [a_{2Ll}, b_{2Lg}]$. Because of the properties $\varphi(c \vee e) = \varphi(c) \vee \varphi(e)$, $\varphi(c \wedge e) = \varphi(c) \wedge \varphi(e)$ for all $c, e \in L(z, z', SP_n)$ and $\varphi(c) \le \varphi(e)$ for $c \le e$ in $L(z, z', SP_n)$ of the homorphism φ, the convex $[a_{Ql}, b_{Qg}]$ induces the convex $[a_{1Ll}, b_{2Lg}]$ in $L(z, z', SP_n)$. Moreover, for any vertex $c \in [a_{1Ll}, b_{2Lg}]$ we have $\varphi(c) \in [a_{Ql}, b_{Qg}]$, and because $\varphi(c) \in [a_{Ql}, b_{Qg}]$, c belongs to one of the congruence classes the images of which are the vertices of the convex $[a_{Ql}, b_{Qg}]$. This means that a convex in Q_n corresponds to a convex of congruence classes in $L(z, z', SP_n)$.

By the definition of fault free elements in Q_n, a fault free convex $[a_{Ql}, b_{Qg}]$ in Q_n implies a fault free convex $[a_{1Ll}, b_{2Lg}]$ in $L(z, z', SP_n)$. The proof of Theorem 1 in [Dolev et al] constructs for each two vertices v, y of Q_n with at most $n - 1$ faults three fault free convexes $[v, u]$, $[u, w]$ and $[w, y]$ in Q_n so that some of the vertices or even all three of the vertices u, w and y may coincide. The shortest $x - s$ path of any shortest path routing belongs to the convex $[x, s]$. Thus the construction implies that that any two vertices x, y of Q_n can be joined by using at most three fault free shortest paths of any arbitrary shortest path routing on

Q_n. If SP_n contains at most $n-1$ faults, then the homorphic image Q_n of $L(z, z', SP_n)$ also contains at most $n-1$ faults. Thus if the vertices a and b of SP_n belong to fault free congruence classes, there are in Q_n vertices $\varphi(a)$, u, w and $\varphi(b)$ such that the convexes $[\varphi(a), u]$, $[u, w]$, $[w, \varphi(b)]$ are fault free and imply fault free convexes in SP_n, whence the vertices a and b can be joined in SP_n by using at most three fault free paths of any shortest path routing ρ on SP_n.

6. Open Problems

Is there any other way than the cartesian product of graphs to create large pseudocubes?

Improve Theorem 11 for any two vertices of a strong pseudocube.

Is there any special shortest path routing so that one can join vertices by using at most two fault free shortest paths of the special shortest path routing (see Theorem 2 in [1]).

Construct a pseudocube with at least two vertices having unequal degrees.

References

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Division of Mathematics
Faculty of Technology
Department of Electrical and Information Engineering
University of Oulu
PL 4500 FIN-90014 OULU
Finland
E-mail: matti.peltola@ee.oulu.fi