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A CONDITION FOR ALGEBRAS ASSOCIATED WITH A CYCLIC QUIVER TO BE SYMMETRIC

By

Takashi TESHIGAWARA

Abstract. Let K be a field, $f(x)$ a monic polynomial in $K[x]$ and $K\Gamma$ the path algebra of a cyclic quiver Γ with s vertices and s arrows. In this paper, we give a necessary and sufficient condition for the algebra $K\Gamma/(f(X))$ to be a symmetric algebra, where X is the sum of all arrows in $K\Gamma$.

1. Introduction

Let K be a field and Γ the cyclic quiver with $\{e_1, \dots, e_s\}$ as the set of vertices and $\{a_1, \dots, a_s\}$ as the set of arrows ($s \geq 2$) such that the start point and the end point of a_t are e_t and e_{t+1}, respectively. Let $K\Gamma$ be the path algebra of Γ. We denote the sum of all arrows by $X : X = a_1 + \cdots + a_s$. It is known by Erdmann and Holm [EH] that $K\Gamma/(X^p)$ is a symmetric algebra if and only if $p \equiv 1 \pmod{s}$. In this paper, we consider the K-algebra $A := K\Gamma/(f(X))$ where $f(x)$ is a monic polynomial over K. Our purpose is to give a necessary and sufficient condition for A to be a symmetric algebra.

We describe the brief way to get the main theorem. First we will show that the equation $(f(X)) = (X^c h(X))$ holds where $h(x)$ is a monic polynomial in $K[x^s]$ and c is an integer such that $0 \leq c \leq s-1$. Second we construct a left A-isomorphism $\mathrm{Hom}_K(A, K) \to A$ and also a right one (Propositions 2.3, 2.5). So we see that A is a Frobenius algebra. If $c = 0$ and the constant term of $h(x)$ is nonzero, then we have a certain left A-isomorphism $A \to A$ and also a right one (Lemma 3.3). By the above propositions and lemma, we have an isomorphism $\mathrm{Hom}_K(A, K) \to A$ of A-bimodules. Also if $c = 1$, then A is a symmetric algebra; if $2 \leq c \leq s-1$, then A is a nonsymmetric algebra (Proposition 3.5). Summarizing these statements we get the following main result; *A is a*

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symmetric algebra if and only if either $c = 0$ *and the constant term of* $h(x)$ *is nonzero or* $c = 1$ *holds* (Theorem 3.6). Moreover, by means of the decomposition of algebras, we can compute the Hochschild cohomology ring of A in principle (Remark 3.8).

2. A is a Frobenius Algebra

Let s be a positive integer ($s \geq 2$). By Γ we denote the cyclic quiver with $\{e_1, \dots, e_s\}$ as the set of vertices and $\{a_1, \dots, a_s\}$ as the set of arrows such that the start point and the end point of a_t are e_t and e_{t+1}, respectively. Let K be a field and $K\Gamma$ the path algebra of Γ. Here we regard the index t of e_t modulo s. Hence $a_t = e_{t+1}a_t e_t$ holds for $1 \leq t \leq s$ in $K\Gamma$. We denote the sum of all arrows by $X : X = a_1 + \cdots + a_s$. Then X^j is a sum of all paths of length j for $j \geq 0$.

Let $f(x)$ be a monic polynomial of degree m $(m \geq 1)$ over $K : f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{m-1}x^{m-1} + x^m$. We consider the K-algebra $A = K\Gamma/(f(X))$.

For each i $(0 \leq i \leq s-1)$, we set

$$f_i(x) = \alpha_i x^i + \alpha_{s+i}x^{s+i} + \alpha_{2s+i}x^{2s+i} + \cdots,$$

which is the sum of the all terms of $f(x)$ whose degree is congruent to i modulo s. Then we have $f(x) = f_0(x) + f_1(x) + \cdots + f_{s-1}(x)$. Let $g_i(x)$ be the polynomials whose constant term is nonzero such that $f_i(x) = x^{n_i s+i}g_i(x)$ $(n_i \geq 0)$ if $f_i(x) \neq 0$, and we set $g_i(x) = 0$ if $f_i(x) = 0$. Then $g(x) := \gcd(g_0(x), g_1(x), \dots, g_{s-1}(x))$ is in $K[x^s]$ since $g_i(x) \in K[x^s]$. If we set $d = \min\{n_i s + i \mid 0 \leq i \leq s-1, f_i(x) \neq 0\}$, then there exist an integer c $(0 \leq c \leq s-1)$ and a monic polynomial $h(x) \in K[x^s]$ such that $\gcd(f_0(x), f_1(x), \dots, f_{s-1}(x)) = x^d g(x) = x^c h(x)$. Note that c and $h(x)$ are uniquely determined by $f(x)$. Since $e_{t+i}f(X)e_t = e_{t+i}f_i(X)e_t$ $(1 \leq t \leq s, 0 \leq i \leq s-1)$, we have the following equation of ideals in $K\Gamma$

$$(f(X)) = (f_0(X)) + (f_1(X)) + \cdots + (f_{s-1}(X)) = (X^c h(X)).$$

Thus we have the following lemma.

LEMMA 2.1. *For the algebra* A, *there exist an integer* c $(0 \leq c \leq s-1)$ *and a monic polynomial* $h(x) \in K[x^s]$ *such that*

$$A = K\Gamma/(X^c h(X)).$$

EXAMPLE 2.2. Let K be the field of rationals $\mathbf{Q}$.

(i) Case $s = 2$. If $f(x) = x - 2x^2 + x^3$, then

$$f_0(x) = x^2 g_0(x) = x^2 \cdot (-2),$$

$$f_1(x) = x g_1(x) = x(1 + x^2).$$

Since $\gcd(f_0(x), f_1(x)) = x$, we have

$$\mathbf{Q}\Gamma/(f(X)) = \mathbf{Q}\Gamma/(X).$$

(ii) Case $s = 3$. If $f(x) = x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10}$, then

$$f_0(x) = x^3 g_0(x) = x^3(1 + x^3 + x^6),$$

$$f_1(x) = x^4 g_1(x) = x^4(1 + x^3 + x^6),$$

$$f_2(x) = x^2 g_2(x) = x^2(1 + x^3 + x^6).$$

Therefore $\gcd(f_0(x), f_1(x), f_2(x)) = x^2(1 + x^3 + x^6)$, so we have

$$\mathbf{Q}\Gamma/(f(X)) = \mathbf{Q}\Gamma/(X^2(1 + X^3 + X^6)).$$

(iii) Case $s = 4$. If $f(x) = x^5 - x^6 + x^7 + 2x^9 + 2x^{11} + 2x^{13} + x^{14} + 2x^{15} + x^{17} + 2x^{18} + 2x^{19} + x^{22} + x^{23} + x^{27}$, then we write $f(x)$ as follows:

$$f(x) = \underbrace{x^5 + 2x^9 + 2x^{13} + x^{17}}_{=f_1(x)} + \underbrace{(-x^6 + x^{14} + 2x^{18} + x^{22})}_{=f_2(x)}$$

$$+ \underbrace{x^7 + 2x^{11} + 2x^{15} + 2x^{19} + x^{23} + x^{27}}_{=f_3(x)}.$$

Each of the above polynomials $f_i(x)$ factors as follows:

$$f_1(x) = x^5 g_1(x) = x^5(1 + x^4)(1 + x^4 + x^8),$$

$$f_2(x) = x^6 g_2(x) = x^6(-1 + x^4 + x^8)(1 + x^4 + x^8),$$

$$f_3(x) = x^7 g_3(x) = x^7(1 + x^4 + x^{12})(1 + x^4 + x^8).$$

Therefore $\gcd(f_1(x), f_2(x), f_3(x)) = x^5(1 + x^4 + x^8)$, so we have

$$\mathbf{Q}\Gamma/(f(X)) = \mathbf{Q}\Gamma/(X^5(1 + X^4 + X^8))$$

$$= \mathbf{Q}\Gamma/(X(X^4 + X^8 + X^{12})).$$

Using the above notations, we set $h(x) = k_0 + k_1 x^s + \cdots + k_{n-1} x^{(n-1)s} + x^{ns} \in K[x^s]$. We will show that the K-algebra $A = K\Gamma/(X^c h(X))$ is a Frobenius algebra. In the rest of this paper, we use a representative elements instead of their residue classes. We take the set $\{X^j e_i \mid 1 \le i \le s, 0 \le j \le ns+c-1\}$ as a K-basis of A and also the dual basis $\{(X^j e_i)^* \in \mathrm{Hom}_K(A,K) \mid 1 \le i \le s, 0 \le j \le ns+c-1\}$ (cf. [FS]). Then we obtain the following proposition. On that occasion we set $k_n = 1$ in the following.

PROPOSITION 2.3. *We have a left A-isomorphism $\varphi : \mathrm{Hom}_K(A,K) \to A$ defined by*

$$\varphi((X^j e_i)^*) = \sum_{\ell=m+1}^{n} k_\ell e_i X^{\ell s+c-j-1} \quad \text{for } 1 \le i \le s,\ 0 \le j \le ns+c-1,$$

where m is the integer $(-1 \le m \le n-1)$ such that $j = ms+c+r$ $(0 \le r \le s-1)$. So A is a Frobenius algebra.

We prepare the following lemma for the proof of the proposition.

LEMMA 2.4. *Let i, j, t, t', u be integers with $1 \le i, u \le s$, $0 \le j \le ns+c-1$, $1 \le t \le n-1$ and $0 \le t' \le n-1$. Then for $(X^j e_i)^* \in \mathrm{Hom}_K(A,K)$, we have*

$$X(X^j e_i)^* = \begin{cases} \begin{cases} -k_0(X^{ns-1}e_{i+1})^* & \text{if } j=0,\ c=0, \\ 0 & \text{if } j=0,\ c\ne 0, \end{cases} \\ \begin{cases} (X^{ts-1}e_{i+1})^* - k_t(X^{ns-1}e_{i+1})^* & \text{if } j=ts,\ c=0, \\ (X^{t's+c-1}e_{i+1})^* - k_{t'}(X^{ns+c-1}e_{i+1})^* & \text{if } j=t's+c,\ c\ne 0, \end{cases} \\ (X^{j-1}e_{i+1})^* & \text{otherwise,} \end{cases}$$

$$e_u(X^j e_i)^* = \begin{cases} (X^j e_i)^* & \text{if } u=i, \\ 0 & \text{if } u \ne i. \end{cases}$$

PROOF. Case $c=0$; If $j=0$, then for $0 \le p \le ns-1$ and $1 \le q \le s$,

$$(\dagger) \qquad (X(e_i)^*)(X^p e_q) = (e_i)^*(X^p e_q X) = (e_i)^*(X^{p+1}e_{q-1}).$$

Here in case of $p+1 = ns$ and $q-1 \equiv i \pmod s$, since $X^{ns} = -k_0 - k_1 X^s - \cdots - k_{n-1}X^{(n-1)s}$ in A, we have $(e_i)^*(X^{ns}e_i) = (e_i)^*((-k_0 - \cdots - k_{n-1}X^{(n-1)s})e_i) = -k_0$. Therefore

$$(\text{equation } (\dagger)) = \begin{cases} -k_0 & \text{if } p+1 = ns \text{ and } q-1 \equiv i \pmod s, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand $-k_0(X^{ns-1}e_{i+1})^*(X^p e_q) = -k_0$ if $p = ns-1$ and $q \equiv i+1 \pmod s$, 0 otherwise. Thus we have $X(e_i)^* = -k_0(X^{ns-1}e_{i+1})^*$. If $j = ts$, then for

$0 \le p \le ns-1$ and $1 \le q \le s$, $(X(X^{ts}e_i)^*)(X^p e_q) = (X^{ts}e_i)^*(X^{p+1}e_{q-1}) = 1$ if $p+1 = ts$ and $q-1 \equiv i \pmod{s}$, $-k_t$ if $p+1 = ns$ and $q-1 \equiv i \pmod{s}$, 0 otherwise. Also $((X^{ts-1}e_{i+1})^* - k_t(X^{ns-1}e_{i+1})^*)(X^p e_q) = 1$ if $p = ts-1$ and $q \equiv i+1 \pmod{s}$, $-k_t$ if $p = ns-1$ and $q \equiv i+1 \pmod{s}$, 0 otherwise. Thus we have $X(X^{ts}e_i)^* = (X^{ts-1}e_{i+1})^* - k_t(X^{ns-1}e_{i+1})^*$.

Case $c \neq 0$; If $j = 0$, then for $0 \le p \le ns+c-1$ and $1 \le q \le s$, $(X(e_i)^*)(X^p e_q) = (e_i)^*(X^p e_q X) = (e_i)^*(X^{p+1}e_{q-1})$. Here in case of $p+1 = ns+c$ and $q-1 \equiv i \pmod{s}$, since $X^{ns+c} = -k_0X^c - k_1X^{s+c} - \cdots - k_{n-1}X^{(n-1)s+c}$ in A, we have $(e_i)^*(X^{ns+c}e_i) = (e_i)^*((-k_0X^c - k_1X^{s+c} - \cdots - k_{n-1}X^{(n-1)s+c})e_i) = 0$. Therefore $X(e_i)^* = 0$. The remaining cases are clear. Hence we have the equation for $X((X^je_i)^*)$.

Next we will show that the second equation of lemma holds. If $u = i$, then for $0 \le p \le ns+c-1$ and $1 \le q \le s$, $(e_i(X^je_i)^*)(X^pe_q) = (X^je_i)^*(X^pe_qe_i) = (X^je_i)^*(X^pe_i)$ if $q = i$, 0 if $q \neq i$. Also we have $(X^je_i)^*(X^pe_q) = (X^je_i)^*(X^pe_i)$ if $q = i$, 0 if $q \neq i$. Hence we have $e_u(X^je_i)^* = (X^je_i)^*$. If $u \neq i$, then for $0 \le p \le ns+c-1$ and $1 \le q \le s$, $(e_u(X^je_i)^*)(X^pe_q) = (X^je_i)^*(X^pe_q)$ if $q = u$, 0 if $q \neq u$. If $q = u$, then we have $q \neq i$ because $u \neq i$. Hence $(e_u(X^je_i)^*)(X^pe_q) = 0$ for $0 \le p \le ns+c-1$, $1 \le q \le s$. Therefore the proof of lemma is completed. □

By this lemma, we will prove the Proposition 2.3.

Proof of Proposition 2.3. It is clear to see that φ is an isomorphism of K-spaces. So it suffices to show that φ is a homomorphism of left A-modules. Hence we prove that

$$\varphi(X(X^je_i)^*) = X\varphi((X^je_i)^*), \quad \varphi(e_u(X^je_i)^*) = e_u\varphi((X^je_i)^*),$$

for $1 \le i, u \le s$ and $0 \le j \le ns+c-1$. First we will show that $\varphi(X(X^je_i)^*) = X\varphi((X^je_i)^*)$. We consider the case $c = 0$. If $j = 0$, then we have $X\varphi((e_i)^*) = X\sum_{\ell=1}^{n} k_\ell e_i X^{\ell s-1} = \sum_{\ell=1}^{n} k_\ell e_{i+1} X^{\ell s} = -e_{i+1}\sum_{\ell=0}^{n-1} k_\ell X^{\ell s} + \sum_{\ell=1}^{n-1} k_\ell e_{i+1} X^{\ell s} = -k_0e_{i+1}$, and $\varphi(X(e_i)^*) = \varphi(-k_0(X^{ns-1}e_{i+1})^*) = -k_0e_{i+1}$. If $j = ts$ $(1 \le t \le s-1)$, then we have $X\varphi((X^{ts}e_i)^*) = X\sum_{\ell=t+1}^{n} k_\ell e_i X^{(\ell-t)s-1} = \sum_{\ell=t+1}^{n} k_\ell e_{i+1} X^{(\ell-t)s}$, and $\varphi(X(X^{ts}e_i)^*) = \varphi((X^{ts-1}e_{i+1})^* - k_t(X^{ns-1}e_{i+1})^*) = \sum_{\ell=t}^{n} k_\ell e_{i+1} X^{(\ell-t)s} - k_te_{i+1} = \sum_{\ell=t+1}^{n} k_\ell e_{i+1} X^{(\ell-t)s}$. We consider the case $c \neq 0$. If $j = 0$, then we have $X\varphi((e_i)^*) = X\sum_{\ell=0}^{n} k_\ell e_i X^{\ell s+c-1} = \sum_{\ell=0}^{n} k_\ell e_{i+1} X^{\ell s+c} = 0$, and $\varphi(X(e_i)^*) = \varphi(0) = 0$. The remaining cases are clear. Therefore we have $\varphi(X(X^je_i)^*) = X\varphi((X^je_i)^*)$.

Second we will show that $\varphi(e_u(X^je_i)^*) = e_u\varphi((X^je_i)^*)$. If $u = i$, then we have $e_u\varphi((X^je_i)^*) = \sum_{\ell=m+1}^{n} k_\ell e_i X^{\ell s+c-j-1} = \varphi((X^je_i)^*) = \varphi(e_u(X^je_i)^*)$. If $u \neq i$, then we have $e_u\varphi((X^je_i)^*) = 0$ since $e_u \neq e_i$. Also $\varphi(e_u(X^je_i)^*) = \varphi(0) = 0$. Hence φ is an isomorphism of left A-modules. This completes the proof of the proposition. □

Similarly, considering the operation of A onto $\mathrm{Hom}_K(A,K)$ from the right, we get the following proposition.

Proposition 2.5. *We have a right A-isomorphism $\psi : \mathrm{Hom}_K(A,K) \to A$ defined by*

$$\psi((X^je_i)^*) = \sum_{\ell=m+1}^{n} k_\ell e_{i+c-1}X^{\ell s+c-j-1} \quad \text{for } 1 \le i \le s,\ 0 \le j \le ns+c-1,$$

where m is the integer $(-1 \le m \le n-1)$ such that $j = ms + c + r$ $(0 \le r \le s-1)$.

3. Main Theorem

In this section we give a necessary and sufficient condition for the algebra $A = K\Gamma/(X^c h(X))$ to be a symmetric algebra, where c is the integer such that $0 \le c \le s-1$ and $h(x) = k_0 + k_1x^s + \cdots + x^{ns}$. We prepare some lemmas for the proof of the main theorem.

The following fact is described in [EH].

Lemma 3.1. *$K\Gamma/(X^p)$ $(p \ge 1)$ is a symmetric algebra if and only if $p \equiv 1 \pmod s$.*

Proof. We denote $K\Gamma/(X^p)$ by B. We set $p = ns + c$ $(0 \le c \le s-1)$ and $h(x) = x^{ns}$. Then the above A coincides with B. If $p \equiv 1 \pmod s$, that is, $c = 1$, then φ of Proposition 2.3 coincides with ψ of Proposition 2.5. Hence B is a symmetric algebra. Conversely we assume that B is a symmetric algebra. We will use an indirect proof by assuming that $p \not\equiv 1 \pmod s$. Let ξ be an isomorphism of B-bimodules $\mathrm{Hom}_K(B,K) \to B$. Fix an i with $1 \le i \le s$. Let $\xi((e_i)^*) = \sum_{j=0}^{p-1}\sum_{\ell=1}^{s} k_{j,\ell}X^je_\ell$ for $k_{j,\ell} \in K$. Since ξ is an isomorphism of B-bimodules, the equation $\xi((e_i)^*)e_u = \xi((e_i)^*e_u)$ holds for any $1 \le u \le s$. The left hand side equals $\sum_{j=0}^{p-1}\sum_{\ell=1}^{s} k_{j,\ell}X^je_\ell e_u = \sum_{j=0}^{p-1} k_{j,u}X^je_u$ and the right hand side equals $\sum_{j=0}^{p-1}\sum_{\ell=1}^{s} k_{j,\ell}X^je_\ell$ if $i = u$, 0 if $i \neq u$. This implies that $k_{j,\ell} = 0$ for $1 \le \ell \le s$ such that $\ell \neq i$ and any $0 \le j \le p-1$. So we have $\xi((e_i)^*) = \sum_{j=0}^{p-1} k_{j,i}X^je_i$. Furthermore, the equation $e_u\xi((e_i)^*) = \xi(e_u(e_i)^*)$ holds for $1 \le u \le s$. The left hand side equals $\sum_{j=0}^{p-1} k_{j,i}X^je_{u-j}e_i = \sum_{j=0}^{p-1} k_{j,i}X^je_i$ if

$u \equiv i+j \pmod{s}$, 0 if $u \not\equiv i+j \pmod{s}$. The right hand side equals $\sum_{j=0}^{p-1} k_{j,i} X^j e_i$ if $i = u$, 0 if $i \neq u$. This implies that $k_{j,i} = 0$ for $0 \leq j \leq p-1$ such that $j \not\equiv 0 \pmod{s}$. So we have $\xi((e_i)^*) = \sum_{j=0}^{n} k_{js,i} X^{js} e_i$. Moreover, since $X\xi((e_i)^*) = \xi(X(e_i)^*)$ and $X(e_i)^* = 0$ (by Lemma 2.4), it follows that $\sum_{j=0}^{n} k_{js,i} X^{js+1} e_i = 0$ in B. Since the set $\{X^j e_i \mid 1 \leq i \leq s, 0 \leq j \leq p-1\}$ is a K-basis of B, we have $\xi((e_i)^*) = 0$ if $p \not\equiv 1 \pmod{s}$, a contradiction. Thus we have $p \equiv 1 \pmod{s}$ if B is symmetric. This completes the proof of the lemma. □

It is known by Furuya and Sanada [FS] that $Z(K\Gamma)$ equals to $K[X^s]$, where $Z(K\Gamma)$ is the center of $K\Gamma$. And an algebra isomorphism $K\Gamma/(p_1(X) \cdots p_m(X)) \simeq K\Gamma/(p_1(X)) \oplus \cdots \oplus K\Gamma/(p_m(X))$ where each $p_i(x) \in K[x^s]$ and $\gcd(p_i(x), p_j(x)) = 1$ for all $1 \leq i, j \leq m$ such that $i \neq j$ is given by [FS]. By the similar way, we have the following lemma.

Lemma 3.2. *If $p(x) \in K[x^s]$ and $p(x)$ is not divided by x, then we have the following decomposition of algebras for the algebra $K\Gamma/(X^r p(X))$ $(r \geq 1)$:*

$$K\Gamma/(X^r p(X)) \simeq K\Gamma/(X^r) \oplus K\Gamma/(p(X)).$$

Proof. Since x^r and $p(x)$ are relatively prime, we have $X^r u_1(X) + p(X)u_2(X) = 1$ in $K\Gamma$ for some $u_1(x), u_2(x) \in K[x]$. Let $z \in (X^r) \cap (p(X))$. If $p(X) \in K[X^s] = Z(K\Gamma)$, then there exist $v_1, v_2 \in K\Gamma$ such that $z = X^r v_1 = p(X)v_2$. So we have $z = z(X^r u_1(X) + p(X)u_2(X)) = v_2 X^r p(X) u_1(X) + X^r p(X) v_1 u_2(X) \in (X^r p(X))$. Thus we have $(X^r) \cap (p(X)) \subset (X^r p(X))$. The converse inclusion is clear. By Chinese remainder theorem, we have the decomposition of algebras

$$K\Gamma/(X^r p(X)) = K\Gamma/((X^r) \cap (p(X))) \simeq K\Gamma/(X^r) \oplus K\Gamma/(p(X)). \qquad \square$$

Lemma 3.3. *Let $c = 0$. If $k_0 \neq 0$, then we have a left A-isomorphism $\varphi' : A \to A$ defined by $\varphi'(e_i X^j) = e_i X^{j+1}$ and a right A-isomorphism $\psi' : A \to A$ defined by $\psi'(e_i X^j) = e_{i+1} X^{j+1}$ for $1 \leq i \leq s$, $0 \leq j \leq ns-1$.*

Proof. Since $k_0 \neq 0$, each K-linear maps is an isomorphism of K-spaces. It is easy to show that these maps are homomorphisms of A-modules. □

Proposition 3.4. *Let $c = 0$. Then A is a symmetric algebra if and only if $k_0 \neq 0$.*

PROOF. If $k_0 \neq 0$, then by Propositions 2.3, 2.5 and Lemma 3.3, we have the left A-isomorphism $\varphi' \circ \varphi : \mathrm{Hom}_K(A, K) \to A((X^j e_i)^* \mapsto \sum_{\ell=m+1}^{n} k_\ell e_i X^{\ell s-j})$ and also we have the right A-isomorphism $\psi' \circ \psi : \mathrm{Hom}_K(A, K) \to A((X^j e_i)^* \mapsto \sum_{\ell=m+1}^{n} k_\ell e_i X^{\ell s-j})$. Thus $\varphi' \circ \varphi$ coincides with $\psi' \circ \psi$, so this is the isomorphism of A-bimodules. This means that A is a symmetric algebra. Conversely we assume that $k_0 = 0$. Then there exists an integer t $(1 \leq t \leq n)$ such that $h(x) = x^{ts} h_0(x)$ where the constant term of $h_0(x)$ $(\in K[x^s])$ is nonzero. By Lemma 3.2, we have the following decomposition of A:

$$A \simeq K\Gamma/(X^{ts}) \oplus K\Gamma/(h_0(X)).$$

For the decomposition, $K\Gamma/(X^{ts})$ is a nonsymmetric algebra by Lemma 3.1. Hence A is a nonsymmetric algebra too ([EN, Proposition 1]). This completes the proof of the lemma. □

PROPOSITION 3.5. *If $c = 1$, then A is a symmetric algebra, and if $2 \leq c \leq s-1$, then A is a nonsymmetric algebra.*

PROOF. For the algebra A, there exists the integer t $(0 \leq t \leq n)$ such that $(X^c h(X)) = (X^{ts+c} h_0(X))$ where the constant term of $h_0(x) \in K[x^s]$ is nonzero. Then, by Lemma 3.2, we have the following decomposition:

$$A = K\Gamma/(X^{ts+c} h_0(X)) \simeq K\Gamma/(X^{ts+c}) \oplus K\Gamma/(h_0(X)).$$

By Proposition 3.4, $K\Gamma/(h_0(X))$ is a symmetric algebra. By Lemma 3.1, if $c = 1$, then $K\Gamma/(X^{ts+1})$ is a symmetric algebra, and if $2 \leq c \leq s-1$, then $K\Gamma/(X^{ts+c})$ is a nonsymmetric algebra. □

We summarize the above results as follows.

THEOREM 3.6. *A is a symmetric algebra if and only if either $c = 0$ and $k_0 \neq 0$ hold or $c = 1$ holds.*

EXAMPLE 3.7. In Example 2.2, the algebras of the cases (i), (iii) are symmetric algebras, but one of the case (ii) is a nonsymmetric algebra.

REMARK 3.8. We saw that there is a decomposition $A = K\Gamma/(X^{ts+c} h_0(X)) \simeq K\Gamma/(X^{ts+c}) \oplus K\Gamma/(h_0(X))$ where the constant term of $h_0(x) \in K[x^s]$ is nonzero and $0 \leq c \leq s-1$. For the decomposition of A, the Hochschild cohomology ring of the first term is given by [EH], and also one of the second term is given by [FS]. Therefore the Hochschild cohomology ring of A is obtained by these facts.

For example, we denote the $\mathbf{Q}\Gamma/(X^2(1+X^3+X^6))$ $(\simeq \mathbf{Q}\Gamma/(X^2) \oplus \mathbf{Q}\Gamma/(1+X^3+X^6))$ in Example 2.2 (ii) by C. We will compute the even Hochschild cohomology ring $\mathrm{HH}^{ev}(C) = \bigoplus_{i \geq 0} \mathrm{HH}^{2i}(C)$. By [EH, Section 4.8], the even Hochschild cohomology ring $\mathrm{HH}^{ev}(\mathbf{Q}\Gamma/(X^2))$ is isomorphic to $\mathbf{Q}[y_2, y_6]/(y_2^2, y_2y_6)$ where $\deg y_2 = 2$ and $\deg y_6 = 6$. Also, by [FS, Propositions 3.2, 3.7], the even Hochschild cohomology ring $\mathrm{HH}^{ev}(\mathbf{Q}\Gamma/(1+X^3+X^6))$ is isomorphic to $\mathbf{Q}[z_0]/(1+z_0+z_0^2)$ where $\deg z_0 = 0$. Thus we have

$$\mathrm{HH}^{ev}(C) \simeq \mathbf{Q}[y_2, y_6]/(y_2^2, y_2y_6) \oplus \mathbf{Q}[z_0]/(1+z_0+z_0^2).$$

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Department of Mathematics
Tokyo University of Science
Wakamiya-cho 26, Shinjuku-ku
Tokyo 162-0827, Japan
E-mail address: teshigawara@ma.kagu.tus.ac.jp