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HERMITIAN JACOBI FORMS OF INDEX ONE

By

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Introduction

The theory of Jacobi forms was established by Eichler and Zagier [2], and, using it, they clarified and extended Maass' results [7] on the "Saito-Kurokawa conjecture" and related works given by several mathematicians.

The most basic examples of Jacobi forms are Fourier-Jacobi coefficients of Siegel modular forms of degree 2. Replacing Siegel modular forms by hermitian modular forms associated with the ring of Gaussian integers, we get analogues to Jacobi forms, which we call hermitian Jacobi forms.

In the present paper, using wonderful methods developed by Eichler and Zagier, we shall discuss hermitian Jacobi forms. Though the Eisenstein Jacobi forms play the fundamental role in their work, theta functions take that role in this paper.

Our main thema is to invesigate the following three spaces:

1. $J_{k,1}$: the space of hermitian Jacobi forms of weight k and index 1,
2. $M_{k-1}^{+}(4)$: the space of modular forms, on the congruence subgroup $\Gamma_1(4)$, whose Fourier expansions $\sum_{n=0}^{\infty} a_n q^n$ have the property: $a(n) = 0$ for $n \equiv 1 \pmod 4$. This is an analogue to Kohnen's "+" space.
3. $\mathscr{M}_k$: the space of hermitian modular forms, of weight k and of degree 2, whose Fourier coefficients satisfy certain relations. This is an analogue to Maass' *Spezialchar*.

In the case of cusp forms, Kojima [6] proved that these three spaces are isomorphic. However the main part of his proof is too technical to work in the general modular forms.

With the aid of Eichler and Zagier's work and the theory of theta functions, we shall prove elementarily that these three spaces are isomorphic to each others in general. Moreover we shall determine the structure of $J_{k,1}$ with the first Fourier-Jacobi coefficients of hermitian modular forms of degree 2.

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Unfortunately, Hecke's theory for hermitian Jacobi forms is still untouched, however we hope that this paper will contribute to a future consideration of this theory.

§ 1. Basic Properties of Hermitian Jacobi Forms

We start with the definition of hermitian Jacobi forms. We denote by $\mathcal{O}$ the ring of Gaussian integers and by $\mathcal{O}^{\times}$ the group of units in $\mathcal{O}$. The hermitian modular group is denoted by $\Gamma_1(\mathcal{O})$, which is an extension of the modular group $\mathrm{SL}_2(\mathbf{Z})$ by the group $\mathcal{O}^{\times}$:

$$\Gamma_1(\mathcal{O}) = \{\varepsilon M \in \mathrm{Mat}_2(\mathcal{O}) \mid \varepsilon \in \mathcal{O}^{\times}, M \in \mathrm{SL}_2(\mathbf{Z})\}.$$

The group $\Gamma_1(\mathcal{O})$ and the additive group $\mathcal{O}^2$ act on the space of functions $\phi : \mathbf{H}_1 \times \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ in the following way. We fix integers k and m and define

$$(\phi|_{k,m}M)(\tau, z, z') = (c\tau + d)^{-k}\mathbf{e}\left(\frac{-mczz'}{c\tau + d}\right)\phi\left(M\langle\tau\rangle, \frac{\det(M)z}{c\tau + d}, \frac{z'}{c\tau + d}\right)$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(\mathcal{O})$ and

$$(\phi|_m(\lambda, \mu))(\tau, z, z') = \mathbf{e}(m(\tau\lambda\bar{\lambda} + z\bar{\lambda} + z'\lambda))\phi(\tau, z + \tau\lambda + \mu, z' + \tau\bar{\lambda} + \bar{\mu})$$

for $(\lambda, \mu) \in \mathcal{O}^2$. Here $\mathbf{e}(a) = \exp(2\pi i a)$.

A *hermitian Jacobi form* of weight k and index m $(k, m \in \mathbf{N})$ relative to $\Gamma_1(\mathcal{O})$ is a holomorphic function $\phi : \mathbf{H}_1 \times \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ satisfying

$$\phi|_{k,m}M = \phi, \quad \forall M \in \Gamma_1(\mathcal{O}), \tag{1}$$

$$\phi|_m(\lambda, \mu) = \phi, \quad \forall (\lambda, \mu) \in \mathcal{O}^2, \tag{2}$$

and ϕ has the Fourier expansion of the form

$$\sum_{n=0}^{\infty} \sum_{r \in \mathcal{O}} c(n, r) q^n \zeta^{\bar{r}} (\zeta')^r$$

with $c(n, r) = 0$ unless $4nm - |r|^2 \geq 0$. Here

$$q = \mathbf{e}(\tau), \quad \zeta = \mathbf{e}(z/2), \quad \zeta' = \mathbf{e}(z'/2).$$

If ϕ satisfies $c(n, r) = 0$ unless $4nm - |r|^2 > 0$, it is called a (hermitian Jacobi) cusp form. We denote by $J_{k,m}$ the vector space of hermitian Jacobi forms of weight k and index m.

The following is a hermitian version of Th. 2.2 in [2]:

Lemma 1. *Let ϕ be a hermitian Jacobi form of weight k and index m with the Fourier expansion $\sum c(n,r)q^n\zeta^{\bar{r}}(\zeta')^r$. Then we have the following*:

1. $c(n,r)$ *depends only on* $4nm-|r|^2$ *and* $r(\mathrm{mod}\ 2m)$.
2. $\varepsilon^k c(n,r) = c(n,\bar{\varepsilon}r)$, *for any unit* $\varepsilon \in \mathcal{O}^\times$.
3. *If* $m=1$ *and* $k \equiv 0 \pmod 4$, *then* $c(n,r)$ *depends only on* $4n-|r|^2$.
4. *If* $m=1$ *and* k *is odd, then* ϕ *is identically zero.*

Proof. Assume $r \equiv r' \pmod{2m}$, $4nm-|r|^2 = 4n'm-|r'|^2$. If $r' = r+2m\lambda$ ($\lambda \in \mathcal{O}$), then we have

$$4n'm-|r'|^2 = 4n'm-|r|^2-2m(\lambda\bar{r}+\bar{\lambda}r)-4m^2|\lambda|^2.$$

Therefore we get $n' = n+\lambda\bar{r}/2+\bar{\lambda}r/2$. By (2), we have

$$c(n,r) = c(n+\lambda\bar{r}/2+\bar{\lambda}r/2+m|\lambda|^2, r+2m\lambda) = c(n',r').$$

Thus we have the first assertion.

The second one comes from (1) for $M = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$.

If $m=1$ and $4n-|r|^2 = 4n'-|r'|^2$, then $r \equiv r'$ or $r \equiv ir' \pmod 2$. If the first case occurs, we have $c(n,r) = c(n',r')$ by 1. If the second case occurs, we have

$$c(n,r) = c(n',\sqrt{-1}r') = (-\sqrt{-1})^k c(n',r').$$

If $k \equiv 0 \pmod 4$, then $c(n,r) = c(n',r')$. If k is odd, then, by 1, 2, we have $-c(n,r) = c(n,-r) = c(n,r)$ and $c(n,r) = 0$. □

For a residue class $\mu \in \mathcal{O}/2m\mathcal{O}$ and a non-negative integer N, we define $c_\mu(N)$ by

$$c_\mu(N) = c\left(\frac{N+|r|^2}{4m}, r\right),$$

where r is any Gaussian integer satisfying $\mu \equiv r \pmod{2m}$. We extend the definition to all non-negative integers N by setting $c_\mu(N) = 0$ unless $4m$ divides $N+|r|^2$. We define a holomorphic function $h_\mu : \mathbf{H}_1 \to \mathbf{C}$ by

$$h_\mu(\tau) = \sum_{N=0}^{\infty} c_\mu(N)\mathbf{e}\left(\frac{N}{4m}\tau\right), \tag{3}$$

and a holomorphic function $\theta_{m,\mu} : \mathbf{H}_1 \times \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ by

$$\theta_{m,\mu}(\tau,z,z') = \sum_{r \in \mathcal{O},\, \mu = r \,(\mathrm{mod}\, 2m)} \mathbf{e}\left(\frac{|r|^2}{4m}\tau + \frac{\bar{r}}{2}z + \frac{r}{2}z'\right). \tag{4}$$

Then

$$\phi(\tau, z, z') = \sum_{\mu \in \mathcal{O}/2m\mathcal{O}} h_\mu(\tau)\theta_{m,\mu}(\tau, z, z'), \tag{5}$$

and the function $\theta_{m,\mu}$ is a product of usual theta functions:

$$\theta[a](\tau|z) = \sum_{p \in \mathbf{Z}} \mathbf{e}\left(\frac{1}{2}\tau(a'+p)^2 + (a'+p)(z+a'')\right),$$

where $a = {}^t(a', a'') \in \mathbf{R}^2$. In fact if $\mu = a_1 + ia_2 \pmod{2m}$ with $a', a'' \in \mathbf{Z}$, then

$$\theta_{m,\mu}(\tau, z, z') = \theta\begin{bmatrix} \frac{a_1}{2m} \\ 0 \end{bmatrix}(2m\tau \mid m(z'+z))\theta\begin{bmatrix} \frac{a_2}{2m} \\ 0 \end{bmatrix}(2m\tau \mid im(z'-z)). \tag{6}$$

For fundamental properties of theta functions, we refer to [4] and [8]. By (6) and the basic properties of theta functions, we get

$$\theta_{m,\mu}(\tau, iz, -iz') = \theta_{m,-i\mu}(\tau, z, z')$$

and

$$\theta_{m,\mu}(\tau+1, z, z') = \mathbf{e}\left(\frac{|\mu|^2}{4m}\right)\theta_{m,\mu}(\tau, z, z').$$

Therefore, by (1), (2) and (4), we get

$$i^k h_{-i\mu}(\tau) = h_\mu(\tau), \quad h_\mu(\tau+1) = \mathbf{e}\left(-\frac{|\mu|^2}{4m}\right)h_\mu(\tau). \tag{7}$$

Moreover, by the transformation formula of theta functions, we get

$$\theta_{m,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}, \frac{z'}{\tau}\right) = \frac{\tau}{2mi}\mathbf{e}\left(\frac{mzz'}{\tau}\right)\sum_{\nu \in \mathcal{O}/2m\mathcal{O}} \mathbf{e}\left(\frac{-1}{2m}\operatorname{Re}(\mu\bar{\nu})\right)\theta_{m,\nu}(\tau, z, z') \tag{8}$$

and

$$h_\mu(-\tau^{-1}) = \frac{i}{2m}\tau^{k-1}\sum_{\nu} \mathbf{e}\left(\frac{1}{2m}\operatorname{Re}(\mu\bar{\nu})\right)h_\nu(\tau). \tag{9}$$

In the sequel, we treat only hermitian Jacobi form of weight k and index 1, and take the set $\{0, 1, i, 1+i\}$ as a set of complete set of representatives for $\mathcal{O}/2\mathcal{O}$.

The following lemma is an immediate consequence of (7):

LEMMA 2. *If* $k \equiv 0 \pmod 4$, *then*

$$h_i(\tau) = h_1(\tau).$$

If $k \equiv 2 \pmod 4$, *then*

$$h_0(\tau) = h_{1+i}(\tau) = 0, \quad h_i(\tau) = -h_1(\tau).$$

§ 2. Hermitian Modular Forms and Their Fourier-Jacobi Coefficients

We begin with recalling the definition of hermitian modular forms. For details we refer to [1]. The hemitian half-space $\mathbf{H}_n$ of degree n is defined by

$$\mathbf{H}_n = \left\{ W \in \mathrm{Mat}_n(\mathbf{C}) \,\middle|\, \frac{W - {}^t\overline{W}}{2i} \right\}.$$

The hermitian modular group, of degree n,

$$\Gamma_n(\mathcal{O}) = \{ M \in \mathrm{M}_{2n}(\mathcal{O}) \mid {}^t\overline{M} J_{2n} M = J_{2n} \}, \quad J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

associated with the ring $\mathcal{O}$ of Gaussian integers, acts on $\mathbf{H}_n$ by

$$W \mapsto M \cdot W = (AW + B)(CW + D)^{-1}$$

for $W \in \mathbf{H}_n$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n(\mathcal{O})$.

A holomorphic function $F(W)$ on $\mathbf{H}_n$ is called a hermitian modular form of weight k if it satisfies the functional equations:

$$F(M \cdot W) = \det(CW + D)^k F(W) \tag{10}$$

for all $W \in \mathbf{H}_n$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n(\mathcal{O})$ and it has a Fourier expansion of the form

$$F(W) = \sum_S A(S)\mathbf{e}(\mathrm{tr}(SW))$$

where S runs over the set of positive semi-integral hermitian matrices. Here a hermitian matrix S is said to be semi-integral if the diagonal entries are rational integers and the off-diagonal entries are contained in $\frac{1}{2}\mathcal{O}$. We denote by $A_k(\Gamma_n(\mathcal{O}))$ the complex vector space of Hermitian modular forms of weight k.

From now on we assume $n = 2$, then we can write W as $\begin{pmatrix} \tau & z \\ z' & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathbf{H}_1$, $z, z' \in \mathbf{C}$, $\mathrm{Im}(\tau)\,\mathrm{Im}(\tau') + |z - \bar{z}'|^2 > 0$, and write S as $\begin{pmatrix} n & r/2 \\ \bar{r}/2 & m \end{pmatrix}$ with $n, m \in \mathbf{Z}$, $n, m \geq 0$, $r \in \mathcal{O}$, $4nm - |r|^2 \geq 0$.

We write $F(\tau, z, z', \tau')$ for a hermitian modular form $F(W)$ of weight k, and $A(n, r, m)$ for $A(S)$, so the Fourier expansion of $F(W)$ becomes

$$F(\tau, z, z', \tau') = \sum_{n, m \in \mathbf{Z}, r \in \mathcal{O}} A(n, r, m)\mathbf{e}\left(n\tau + \frac{1}{2}rz' + \frac{1}{2}\bar{r}z + m\tau'\right).$$

Moreover F has the Fourier-Jacobi expansion

$$F(\tau, z, z', \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z, z')\mathbf{e}(m\tau'),$$

where

$$\phi_m(\tau, z, z') = \sum_{4nm - |r|^2 \geq 0} A(n, r, m)\mathbf{e}\left(n\tau + \frac{1}{2}rz' + \frac{1}{2}\bar{r}z\right).$$

Applying (10), we see that $\phi_m(\tau, z, z')$ is a hermitian Jacobi form of weight k and index m.

In his paper [3], Freitag determines the structure of the ring of hermitian modular forms

$$A(\Gamma_2(\mathcal{O})) = \bigoplus\nolimits_{k=0}^{\infty} A_k(\Gamma_2(\mathcal{O})),$$

in which theta series play the central role. In this case we have not enough Eisenstein series. Now we recall the definition of the theta series and some typical hermitian modular forms.

For a vector $a = {}^t(a_1, a_2, a_3, a_4)$ with $a_i = 0$ or 1, we put $a' = {}^t(a_1, a_2)$, $a'' = {}^t(a_3, a_4)$. Such a vector a is said to be *even*, if ${}^t a' a'' \equiv 0 \pmod 2$. Let W be a point in $\mathbf{H}_2$, then the series

$$\Theta[a](W) = \sum_{n \in \mathcal{O}^2} \mathbf{e}\left(\frac{1}{2}\left(n + \frac{1+i}{2}a'\right)^* W\left(n + \frac{1+i}{2}a'\right) + \mathrm{Re}\left(\frac{1+i}{2}{}^t a'' n\right)\right),$$

where $\alpha^* = {}^t\bar{\alpha}$, represents a holomorphic function on $\mathbf{H}_2$, it is called the hermitian theta function of characteristic a. Following [3], we set

$$\varphi_{4k}(W) = \sum_{a:\,\mathrm{even}} \Theta[a](W)^{4k}, \quad k = 1, 2, 3, 4,$$

$$\chi_{10}(W) = \prod_{a:\,\mathrm{even}} \Theta[a](W).$$

Then these are hermitian modular forms of weight $4, 8, 12, 16$ and 10, respectively.

To write down the Fourier-Jacobi coefficients of these modular forms, we use the following notation:

$$x = \theta_{00}(\tau), \quad y = \theta_{01}(\tau), \quad z = \theta_{10}(\tau).$$

Here

$$\theta_{ij}(\tau) = \theta_{ij}(\tau|0) = \theta\begin{bmatrix} i/2 \\ j/2 \end{bmatrix}(\tau|0)$$

is the Jacobi's theta function. It is well known that these satisfy the equation

$$x^4 = y^4 + z^4, \tag{11}$$

and the Eisenstein series can be expressed in the following form

$$E_4 = \frac{1}{2}(x^8 + y^8 + z^8), \quad E_6 = \frac{1}{2}(x^4 + y^4)(x^4 + z^4)(y^4 - z^4). \tag{12}$$

Moreover we have

$$2^{-6}.3^{-3}(E_4^3 - E_6^2) = q\prod_{n=1}^{\infty}(1 - q^n)^{24} = 2^{-8}(xyz)^8. \tag{13}$$

We denote by $\phi_{4k,1}$ $(k = 1, 2, 3, 4)$ and $\chi_{10,1}$ the first Fourier-Jacobi coefficients of φ_{4k} $(k = 1, 2, 3, 4)$ and χ_{10}. Then by the properties of theta functions we see that these hermitian Jacobi forms can be written in the following form:

$$\phi_{4k,1}(\tau, z, z') = c_{4k} \sum_{a,b=0,1} h_{a+bi}^{(4k)}(\tau)\theta_{1,a+bi}(\tau, z, z'), \tag{14}$$

where

$$c_4 = 2^2.5.13, \quad c_8 = 2^6, \quad c_{12} = 2^5.3, \quad c_{16} = 2^7,$$

and

$$h_0^{(4)} = x^6 + y^6$$

$$h_1^{(4)} = z^6$$

$$h_{1+i}^{(4)} = x^6 - y^6,$$

$$h_0^{(8)} = x^{14} + y^{14} + 7((x^{12} + y^{12})(x^2 + y^2) + z^{12}(x^2 - y^2))$$

$$h_1^{(8)} = z^{14} + 7((x^{12} - y^{12})z^2 + z^{14})$$

$$h_{1+i}^{(8)} = x^{14} - y^{14} + 7((x^{12} + y^{12})(x^2 - y^2) + z^{12}(x^2 + y^2)),$$
$$h_0^{(12)} = x^{22} + y^{22} + 11((x^{20} + y^{20})(x^2 + y^2) + z^{20}(x^2 - y^2))$$
$$h_1^{(12)} = z^{22} + 11((x^{20} - y^{20})z^2 + z^{22})$$
$$h_{1+i}^{(12)} = x^{22} - y^{22} + 11((x^{20} - y^{20})(x^2 - y^2) + z^{20}(x^2 + y^2)), \tag{15}$$
$$h_0^{(16)} = x^{30} + y^{30} + 3.5((x^{28} + y^{28})(x^2 + y^2) + z^{28}(x^2 - y^2))$$
$$h_1^{(16)} = z^{30} + 3.5((x^{28} - y^{28})z^2 + z^{30})$$
$$h_{1+i}^{(16)} = x^{30} - y^{30} + 3.5((x^{28} - y^{28})(x^2 - y^2) + z^{28}(x^2 + y^2)).$$

$$\chi_{10,1}(\tau, z, z') = 3^4.2^{-2}x^6y^6z^6(\theta_{1,1}(\tau, z, z') - \theta_{1,i}(\tau, z, z')). \tag{16}$$

Among these hermitian Jacobi forms, $\chi_{10,1}$ is an only cusp form.

In order to determine the structure of the space of hermitian Jacobi forms of index 1, we define more simple forms by

$$\begin{aligned} \Phi_{4,1} &= (x^6 + y^6)\theta_0 + z^6(\theta_1 + \theta_i) + (x^6 - y^6)\theta_{1+i} \\ \Phi_{8,1} &= (x^{14} + y^{14})\theta_0 + z^{14}(\theta_1 + \theta_i) + (x^{14} - y^{14})\theta_{1+i} \\ \Phi_{12,1} &= (x^{22} + y^{22})\theta_0 + z^{22}(\theta_1 + \theta_i) + (x^{22} - y^{22})\theta_{1+i} \\ \Phi_{16,1} &= (x^{30} + y^{30})\theta_0 + z^{30}(\theta_1 + \theta_i) + (x^{30} - y^{30})\theta_{1+i}, \end{aligned} \tag{17}$$

$$\begin{aligned} \Psi_{8,1} &= -x^4y^4z^4(x^2 - y^2)\theta_0 + x^4y^4z^6(\theta_1 + \theta_i) - x^4y^4z^4(x^2 + y^2)\theta_{1+i} \\ \Psi_{12,1} &= -x^4y^4z^4(x^{10} - y^{10})\theta_0 + x^4y^4z^{14}(\theta_1 + \theta_i) \\ &\quad - x^4y^4z^4(x^{10} - y^{10})\theta_{1+i} \\ \Psi_{16,1} &= -x^4y^4z^4(x^{18} - y^{18})\theta_0 + x^4y^4z^{22}(\theta_1 + \theta_i) \\ &\quad - x^4y^4z^4(x^{18} - y^{18})\theta_{1+i} \quad (\theta_\mu = \theta_{1,\mu}(\tau, z, z')). \end{aligned} \tag{18}$$

These are in fact hermitian Jacobi forms, especially $\Psi_{k,1}$'s are cusp forms, because, using (11) and (12), we have the following relations:

$$\Phi_{4,1} = 2^{-2}.5^{-1}.13^{-1}\phi_{4,1}$$
$$\Phi_{8,1} = -2^{-3}.3^{-1}.5^{-1}.13^{-1}\phi_{8,1} + 7.2^{-3}.5^{-1}.13^{-1}E_4\phi_{4,1}$$

$$\Psi_{8,1} = E_4\Phi_{4,1} - \Phi_{8,1}$$

$$\Phi_{12,1} = 2^{-5}.3^{-1}.23^{-1}\phi_{12,1} - 5.11.23^{-1}E_4\Psi_{8,1}$$

$$\Psi_{12,1} = E_4\Phi_{8,1} - \Phi_{12,1}$$

$$\Phi_{16,1} = 2^{-7}.31^{-1}\phi_{16,1} + 3.5.7.31^{-1}E_4^2\Psi_{8,1},$$

$$\Psi_{16,1} = E_4\Phi_{12,1} - \Phi_{16,1}.$$

§ 3. The Space of Hermitian Jacobi Forms of Index One

In this section we shall determine the structure of the space of hermitian Jacobi forms of index 1. The idea is essentially due to [2], Ch. 1 § 3. However, not Eisenstein series but hermitian Jacobi forms given in the preceeding section play the central role.

We develop a hermitian Jacobi form $\phi(\tau, z, z')$ of weight k and index 1 in the Taylor expansion around $(z, z') = (0, 0)$:

$$\phi(\tau, z, z') = \sum_{\mu, \nu=0}^{\infty} \chi_{\mu,\nu}(\tau) z^{\mu}(z')^{\nu}. \tag{19}$$

Applying the transformation law (1)

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{\varepsilon z}{c\tau+d}, \frac{\varepsilon^{-1}z'}{c\tau+d}\right) = \varepsilon^k (c\tau+d)^k \mathbf{e}\left(\frac{czz'}{c\tau+d}\right)\phi(\tau, z, z'), \tag{20}$$

we get

$$\chi_{\mu,\nu}\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon^{k-\mu+\nu}(c\tau+d)^{k+\mu+\nu}\left(\chi_{\mu,\nu}(\tau) + \frac{2\pi ic}{c\tau+d}\chi_{\mu-1,\nu-1}(\tau)\right.$$
$$\left. + \frac{1}{2!}\left(\frac{2\pi ic}{c\tau+d}\right)^2 \chi_{\mu-2,\nu-2}(\tau) + \cdots\right).$$

By (19) and (20), we see $\chi_{\mu,\nu} = 0$ unless $\mu - \nu \equiv k \pmod 4$. The first several coefficients satisfy the following functional equations:

$$\chi_{0,0}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \chi_{0,0}(\tau),$$

$$\chi_{1,1}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2}\chi_{1,1}(\tau) + 2\pi ic(c\tau+d)^{k+1}\chi_{0,0}(\tau),$$

$$\chi_{\mu,\nu}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2}\chi_{\mu,\nu}(\tau) \quad (\mu,\nu) = (2,0), (0,2),$$

$$\chi_{\mu,\nu}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+4}\chi_{\mu,\nu}(\tau) \quad (\mu,\nu) = (4,0),(0,4),$$

$$\chi_{2,2}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+4}\chi_{2,2}(\tau) + 2\pi ic(c\tau+d)^{k+3}\chi_{1,1}$$
$$+ \frac{(2\pi ic)^2}{2!}(c\tau+d)^{k+2}\chi_{0,0}(\tau).$$

We denote by M_k (resp. S_k) the space of modular (resp. cusp) forms on the modular group $\mathrm{SL}_2(\mathbf{Z})$ of weight k.

By the above equations, we have $\chi_{0,0} \in M_k$. If the Fourier expansion of ϕ is

$$\phi(\tau,z,z') = \sum_{n,r} c(n,r)q^n\zeta^{\bar{r}}(\zeta')^r,$$

then we have

$$\chi_{\mu,\nu}(\tau) = \frac{(\pi i)^{\mu+\nu}}{\mu!\nu!}\sum_{n=0}^{\infty}\left(\sum_{r\in\mathcal{O},\,4n\geq|r|^2}\bar{r}^{\mu}r^{\nu}c(n,r)\right)q^n.$$

Therefore $\chi_{\mu,0},\chi_{0,\mu} \in S_{k+\mu}$ for $\mu = 2,4$.

Differentiating and arranging the above equations, we get cusp forms $\xi_{1,1}$ and $\xi_{2,2}$ of weight $k+2$ and $k+4$, respectively, which are defined by

$$\begin{aligned}\xi_{1,1}(\tau) &= \chi_{1,1}(\tau) - \frac{2\pi i}{k}\chi'_{0,0}(\tau),\\ \xi_{2,2}(\tau) &= \chi_{2,2}(\tau) - \frac{2\pi i}{k+2}\chi'_{1,1}(\tau) + \frac{(2\pi i)^2}{2(k+1)(k+2)}\chi''_{0,0}(\tau).\end{aligned} \tag{21}$$

By the heat equation

$$\frac{\partial^2}{\partial z^2}\theta_{ij}(\tau|z) = 4\pi\frac{\partial}{\partial\tau}\theta_{ij}(\tau|z), \tag{22}$$

the Taylor expansion of $\theta_{a,0}(2\tau\,|\,z+z')$ becomes

$$\theta_{a0}(2\tau) + \frac{2\pi i}{2!}\frac{d}{d\tau}\theta_{a0}(2\tau)(z+z')^2 + \frac{(2\pi i)^2}{4!}\frac{d^2}{d\tau^2}\theta_{a0}(2\tau)(z+z')^4 + \cdots.$$

Similarly we get the Taylor expansion of $\theta_{a0}(2\tau\,|\,i(z'-z))$. For simplicity, we put

$$\theta_{a0}(2\tau) = T_a, \quad 2\pi i\frac{d}{d\tau}\theta_{a0}(2\tau) = T'_a, \quad (2\pi i)^2\frac{d^2}{d\tau^2}\theta_{a0}(2\tau) = T''_a.$$

Hence we have

$$\begin{aligned}&\theta_{a0}(2\tau \mid z+z')\theta_{b0}(2\tau \mid i(z'-z))\\&\quad = T_aT_b + (T_aT_b' + T_a'T_b)zz' + \frac{1}{2}(T_a'T_b - T_aT_b')(z^2+(z')^2)\\&\quad + \frac{1}{4}(T_a''T_b + T_aT_b'' + 2T_a'T_b')z^2(z')^2\\&\quad + \frac{1}{4!}(T_a''T_b + T_aT_b'' - 6T_a'T_b')(z^4+(z')^4) + \cdots. \qquad (23)\end{aligned}$$

Assume $k \equiv 0 \pmod 4$. We write a hermitian Jacobi form ϕ as a linear combination of theta function:

$$\begin{aligned}\phi(\tau,z,z') &= \sum_{a,b=0,1} h_{a+bi}(\tau)\theta_{1,a+bc'}(\tau,z,z')\\&= \sum_{a,b=0,1} h_{a+bi}(\tau)\theta_{a0}(2\tau \mid z+z')\theta_{b0}(2\tau \mid i(z'-z)).\end{aligned}$$

Since $h_1 = h_i$, by the definition of $\chi_{\mu,\nu}$,

$$(\chi_{0,0},\chi_{1,1},\chi_{2,2},\chi_{4,0}+\chi_{0,4}) = (h_0,h_1,h_{1+i})$$

$$\times\begin{pmatrix} T_0^2 & 2T_0T_0' & \frac{1}{2}(T_0T_0''+(T_0')^2) & \frac{1}{4!}(2T_0T_0''-6(T_0')^2)\\ 2T_0T_1 & 2(T_0T_1'+T_0'T_1) & \frac{1}{2}(T_0T_1''+T_0''T_1+2T_0'T_1') & \frac{1}{4!}(T_0T_1''+T_0''T_1-6T_0'T_1')\\ T_1^2 & 2T_1T_1' & \frac{1}{2}(T_1T_1''+(T_1')^2) & \frac{1}{4!}(2T_1T_1''-6(T_1')^2)\end{pmatrix}.$$

Hence we have

$$\left(\chi_{0,0},\frac{1}{2}\chi_{1,1},\frac{1}{2}(\chi_{2,2}-6(\chi_{4,0}+\chi_{0,4})\right) = (h_0,h_1,h_{1+i})A, \qquad (24)$$

where

$$A = \begin{pmatrix} T_0^2 & T_0T_0' & (T_0')^2\\ 2T_0T_1 & T_0T_1'+T_0'T_1 & 2T_0'T_1'\\ T_1^2 & T_1T_1' & (T_1')^2\end{pmatrix}.$$

One can get the determinant $\det(A)$ by the addition formula of theta functions and the Jacobi's derivative formula:

$$\frac{\partial}{\partial z}\theta_{11}(\tau|z)|_{z=0} = -\pi\theta_{00}(\tau)\theta_{10}(\tau)\theta_{01}(\tau). \qquad (25)$$

In fact we have the following:

LEMMA 3. 1. $\det(A) = (T_0 T_1' - T_0' T_1)^3$.
2. $T_0 T_1' - T_0' T_1 = \frac{\pi^2}{2}(\theta_{00}(\tau)\theta_{10}(\tau)\theta_{01}(\tau))^2$.

PROOF. 1 is a simple caculation. Though we showed 2 in [9], §3, we give a proof of it for reader's covinient. By the addition formula, we have

$$\theta_{11}^2(\tau|z) = \theta_{10}(2\tau)\theta_{00}(2\tau\,|\,2z) - \theta_{00}(2\tau)\theta_{10}(2\tau\,|\,2z).$$

Differentiating with respect to z both sides twice and putting $z = 0$, we get

$$2\left(\frac{\partial}{\partial z}\theta_{11}(\tau|z)\right)^2\Big|_{z=0} = \theta_{10}(2\tau)\frac{\partial^2}{\partial z^2}\theta_{00}(2\tau\,|\,2z)|_{z=0} - \theta_{00}(2\tau)\frac{\partial^2}{\partial z^2}\theta_{10}(2\tau\,|\,2z)|_{z=0}.$$

By the heat equation (22) and the derivative formula (25), we get 2. □

For $l = 4, 8, 12$, let $\Phi_{l,1}$ be the hermitian Jacobi forms, of weight l and index 1, given in the preceeding section (17).

THEOREM 1. *Assume* $k \equiv 0 \pmod 4$. *Both of the linear maps*

$$\xi : J_{k,1} \to M_k \oplus S_{k+2} \oplus S_{k+4}, \quad \phi \mapsto (\chi_{0,0}, \xi_{1,1}, \xi_{2,2} - 6(\chi_{4,0} + \chi_{0,4}))$$

and

$$\eta : M_{k-4} \oplus M_{k-8} \oplus M_{k-12} \to J_{k,1}, \quad (f, g, h) \mapsto f\Phi_{4,1} + g\Phi_{8,1} + h\Phi_{12,1}$$

are isomorphisms. In particular, $\dim J_{k,1} = k/4$.

PROOF. First, we shall show the injectivity of ξ. If $\xi(\phi) = 0$, then $\chi_{0,0} = 0$, $\chi_{1,1} = 0$ and $\chi_{2,2} - 6(\chi_{4,0} + \chi_{0,4}) = 0$. By (24), we have

$$(h_0, h_1, h_{1+i})A = (0, 0, 0).$$

Since $\det(A) \neq 0$, we have $(h_0, h_1, h_{1+i}) = (0, 0, 0)$, hence $\phi = 0$.

Secondly we shall show the injectivity of η. Suppose $f\Phi_{4,1} + g\Phi_{8,1} + h\Phi_{12,1} = 0$. Then by (17) and the same argument as above we have

$$(f, g, h)H = (0, 0, 0),$$

where

$$H = \begin{pmatrix} x^6 + y^6 & z^6 & x^6 - y^6 \\ x^{14} + y^{14} & z^{14} & x^{14} - y^{14} \\ x^{22} + y^{22} & z^{22} & x^{22} - y^{22} \end{pmatrix}.$$

Since the determinant:

$$\det(H) = 2x^6y^6z^6(x^8 - y^8)(y^8 - z^8)(z^8 - x^8).$$

is not identically zero, hence $f = g = h = 0$.

Finally by the dimension formula we have

$$\dim M_{k-4} + \dim M_{k-8} + \dim M_{k-12} = \dim M_k + \dim S_{k+2} + \dim S_{k+4}.$$

Thus we see that both of ξ and η are isomorphisms. □

Let $J_{k,1}^{cusp}$ denote the space of cusp hermitian Jacobi forms of weight k and index 1. In this case we use cusp forms $\Psi_{k,1}$'s given in §2 (18).

Theorem 2. *Assume* $k \equiv 0 \pmod 4$. *Both of the linear maps*

$$\xi : J_{k,1}^{cusp} \to S_k \oplus S_{k+2} \oplus S_{k+4}, \quad \phi \mapsto (\chi_{0,0}, \xi_{1,1}, \xi_{2,2} - 6(\chi_{4,0} + \chi_{0,4}))$$

and

$$\eta^{cusp} : M_{k-8} \oplus M_{k-12} \oplus M_{k-16} \to J_{k,1}^{cusp}, \quad (f, g, h) \mapsto f\Psi_{8,1} + g\Psi_{12,1} + h\Psi_{16,1}$$

are isomorphisms. In particular, $\dim J_{k,1}^{cusp} = (k-4)/4$.

Proof. We shall show that η^{cusp} is injective. Assume

$$f\Psi_{8,1} + g\Psi_{12,1} + h\Psi_{16,1} = 0.$$

In this case we have, by (18),

$$(f, g, h)K = (0, 0, 0),$$

where

$$K = -x^4y^4z^4 \begin{pmatrix} x^2 - y^2 & -z^2 & x^2 + y^2 \\ x^{10} - y^{10} & -z^{10} & x^{10} + y^{10} \\ x^{18} - y^{18} & -z^{18} & x^{18} - y^{18} \end{pmatrix}.$$

The rest is the same as the proof of the previous theorem. □

Finally we discuss the case $k \equiv 2 \pmod 4$. In this case any hermitian Jacobi form can be written in the following form:

$$\phi(\tau, z, z') = h(\tau)(\theta_{1,1}(\tau, z, z') - \theta_{1,i}(\tau, z, z')).$$

Hence the coefficient of $z^2 + (z')^2$ in the Taylor expansion of ϕ becomes

$$h(\tau)(T_0T_1' - T_1T_0'),$$

which is a cusp form of weight $k+2$.

Theorem 3. *Assume* $k \equiv 2 \pmod 4$. *Then the linear map*

$$J_{k,1} \to S_{k+2}$$

defined by

$$\phi \mapsto h(\tau)(T_0T_1' - T_1T_0')$$

is an isomorphism. In particular $J_{2,1} = J_{6,1} = \{0\}$.

Proof. By Lemma 3, we see that this map is injective. Conversely if $k \geq 10$ any cusp form f of weight $k+2$ can be written as $f = g\Delta$ for some modular form g of weight $k-10$, where Δ is the cusp form of weight 12:

$$\Delta(\tau) = \left(\frac{1}{2}\theta_{00}(\tau)\theta_{10}(\tau)\theta_{01}(\tau)\right)^8.$$

Then the function

$$\frac{f}{T_0T_1' - T_1T_0'}(\theta_1(\tau,z,z') - \theta_i(\tau,z,z'))$$

is a hermitian Jacobi form and it goes to f by the map. □

§ 4. Hermitian Jacobi Forms and Modular Forms

In this section we shall discuss the relation between hermitian Jacobi forms of weight k and index 1 and modular forms of weight $k-1$, which is a hermitian version of a part of [2], II § 5.

The spaces of modular forms, of weight k, on congruence subgroups $\Gamma_0(4)$ and $\Gamma_1(4)$ are denoted by $M_k(\Gamma_0(4))$ and $M_k(\Gamma_0(4),\chi) = M_k(\Gamma_1(4))$, respectively, where χ is the non-trivial Diriclet character modulo 4.

Throughout this section we assume $k \equiv 0 \pmod 4$. Now we fix a hermitian Jacobi form

$$\phi(\tau,z,z') = \sum_{\mu\in\{0,1,i,1+i\}} h_\mu\theta_{1,\mu}(\tau,z,z')$$

of weight k and index 1. The following argument is essentially given by Kojima [6].

By the formulas (7) and (9), we have

$$(h_0, h_1, h_i, h_{1+i})|_{k-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (h_0, h_1, h_i, h_{1+i})S, \tag{26}$$

$$(h_0, h_1, h_i, h_{1+i})|_{k-1}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = (h_0, h_1, h_i, h_{1+i})S^3,$$

$$(h_0, h_1, h_i, h_{1+i})|_{k-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i(h_0, h_1, h_i, h_{1+i})T, \tag{27}$$

where

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^4\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{aligned}(h_0, h_1, h_i, h_{1+i})|_{k-1}\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} &= (h_0, h_1, h_i, h_{1+i})(iT)S^4(iT)(-1_4) \\ &= (h_0, h_1, h_i, h_{1+i}).\end{aligned} \tag{28}$$

On the other hand, since

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{aligned}(h_0, h_1, h_i, h_{1+i})|_{k-1}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} &= (h_0, h_1, h_i, h_{1+i})(iT)S^2(iT)(-1_4) \\ &= (h_0, h_1, h_i, h_{1+i})\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.\end{aligned} \tag{29}$$

LEMMA 4. *Assume* $k \equiv 0 \pmod 4$, *then* $h_0(\tau) \in M_{k-1}(\Gamma_0(4), \chi)$.

PROOF. The congruence subgroup $\Gamma_1(4)$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},$$

and $[\Gamma_0(4) : \Gamma_1(4)] = 2$. Therefore, for any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

$$h_0|_{k-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (-1)^{(d-1)/2} h_0 = \chi(d) h_0.$$

The set of cusps for $\Gamma_0(4)$ is $\{\infty, 1/2, 0\}$. The finiteness of h_0 at 0 and 1/2 comes from (27) and (29), respectively. Thus we see $h_0 \in M_{k-1}(\Gamma_0(4), \chi)$. □

Put

$$\omega_4 = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}.$$

Then

$$*|_{k-1}\omega_4 : M(\Gamma_0(4), \chi) \to M(\Gamma_0(4), \chi)$$

is an linear isomorphism and its square is -1.

By the preceeding lemma, $h_0 \in M_{k-1}(\Gamma_0(4), \chi)$; hence so is $h_0|\omega_4$.

$$\begin{aligned} h_0|_{k-1}\omega_4 &= h_0|_{k-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{i}{2^{k-2}}(h_0(4\tau) + h_1(4\tau) + h_i(4\tau) + h_{1+i}(4\tau)). \end{aligned}$$

Put $c(N) = c_0(N) + c_1(N) + c_i(N) + c_{1+i}(N)$, then we have

$$h_0|_{k-1}\omega_4 = \frac{i}{2^{k-2}}\sum_{N=0}^{\infty} c(N) q^N.$$

If $r \in \mathcal{O}$ satisfies $\mu \equiv r \pmod 2$, then

$$4n + 1 + |r|^2 \equiv 1 + |\mu|^2 \not\equiv 0 \pmod 4;$$

hence, by the definition of $c_\mu(N)$, $c_\mu(4n+1) = 0$.

The following lemma is a key of the next theorem. Its proof will be given in the following section 5.

LEMMA 5. *Let* $g(\tau) = \sum_{N \geq 0} c(N) q^N$ *be an element of* $M_{k-1}(\Gamma_0(4), \chi)$ *and*

$$g_j(4\tau) = \sum_{n=0}^{\infty} c(4n+j)q^{4n+j}, \quad j = 0, 1, 2, 3.$$

Then $c(N) = 0$ *for* $N \equiv 1 \pmod 4$ *if and only if*

$$g_0(-\tau^{-1}) + g_2(-\tau^{-1}) = i\tau^{k-1}(g_0(\tau) + g_2(\tau)).$$

We denote by $M_{k-1}^{+}(4, \chi)$ the subspace of $M_{k-1}(\Gamma_0(4), \chi)$ consisting of modular forms g whose Fourier expansions $g(\tau) = \sum_{N=0}^{\infty} c(N)q^N$ satisfy $c(N) = 0$ for $N \equiv 1 \pmod 4$.

Theorem 4.

$$\phi(\tau, z, z') = \sum_{\mu \in \mathcal{O}/2\mathcal{O}} h_\mu \theta_{1,\mu}(\tau, z, z')$$

$$\mapsto h_0|_{k-1}\omega_4 = \frac{i}{2^{k-2}}(h_0(4\tau) + h_1(4\tau) + h_i(4\tau) + h_{1+i}(4\tau))$$

gives an isomorphism between $J_{k,1}$ *and* $M_{k-1}^{+}(4, \chi)$.

Proof. In the above we showed $h_0|\omega_4 \in M_{k-1}^{+}(4, \chi)$. If $h_0|\omega_4 = 0$, $h_0 = -(h_0|\omega_4^2) = 0$ and $h_{1+i} = 0$ by (29); hence $h_1 + h_i = 2h_1 = 0$. Thus we get the injectivity.

Now we shall show the surjectivity. Let

$$g(\tau) = \sum_{N=0}^{\infty} c(N)q^N, \quad c(N) = 0 \text{ if } N \equiv 1 \pmod 4$$

be arbitrary element of $M_{k-1}^{+}(4, \chi)$. Set

$$h_0(\tau) = \sum_{n=0}^{\infty} c(4n)q^n$$

$$h_1(\tau) = h_i(\tau) = \sum_{n=0}^{\infty} c(4n+3)q^{(4n+3)/4}$$

$$h_{1+i}(\tau) = \sum_{n=0}^{\infty} c(4n+2)q^{(4n+2)/4},$$

and $h(\tau) = h_0(\tau) + h_1(\tau) + h_i(\tau) + h_{1+i}(\tau)$. Then they satisfy (26). To prove (27) is a corner stone of our proof.

Since $g(\tau) = h(4\tau) \in M_{k-1}(\Gamma_0(4), \chi)$, we have

$$g\left(\frac{\tau}{4\tau+1}\right) = (4\tau+1)^{k-1} g(\tau) = (4\tau+1)^{k-1} h(4\tau)$$
$$= h\left(\frac{4\tau}{4\tau+1}\right) = h\left(1 + \frac{-1}{4\tau+1}\right)$$
$$= h_0\left(\frac{-1}{4\tau+1}\right) - i\left(h_1\left(\frac{-1}{4\tau+1}\right) + h_i\left(\frac{-1}{4\tau+1}\right)\right) - h_{1+i}\left(\frac{-1}{4\tau+1}\right).$$

Replacing $4\tau+1$ by τ, we get

$$h(\tau-1) = \tau^{-(k-1)}\left(h_0\left(\frac{-1}{\tau}\right) - i\left(h_1\left(\frac{-1}{\tau}\right) + h_i\left(\frac{-1}{\tau}\right)\right) - h_{1+i}\left(\frac{-1}{\tau}\right)\right)$$
$$= h_0(\tau) + i(h_1(\tau) + h_i(\tau)) + h_{1+i}(\tau).$$

Replacing $-\tau^{-1}$ by τ in this formula, we get

$$h_0\left(\frac{-1}{\tau}\right) + i\left(h_1\left(\frac{-1}{\tau}\right) + h_i\left(\frac{-1}{\tau}\right)\right) - h_{1+i}\left(\frac{-1}{\tau}\right)$$
$$= \tau^{k-1}(-h_0(\tau) + i(h_1(\tau) + h_i(\tau)) + h_{1+i}(\tau)).$$

By these we have

$$h_0(-\tau^{-1}) - h_{1+i}(-\tau^{-1}) = 2i\tau^{k-1} h_1(\tau) = 2i\tau^{k-1} h_i(\tau), \tag{30}$$

$$h_1(-\tau^{-1}) = h_{1+i}(-\tau^{-1}) = \frac{i}{2}\tau^{k-1}(h_0(\tau) - h_{1+i}(\tau)). \tag{31}$$

Moreover, by the lemma above,

$$h_0(-1\tau^{-1}) + h_{1+i}(-\tau^{-1}) = i\tau^{k-1}\tau^{k-1}(h_0(\tau) + h_{1+i}(\tau)). \tag{32}$$

By (30), (31) and (32), we get (27).

Now we put

$$\phi(\tau, z, z') = \sum_{\mu} h_\mu(\tau)\theta_{1,\mu}(\tau, z, z').$$

By (26), (27) and the transformation formula of theta functions, we have

$$\phi|_{k,1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \phi, \quad \phi|_{k,1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \phi,$$

and $\phi_{k,1}(\lambda,\mu)=\phi$ for all (λ,μ); hence ϕ is a hermitian Jacobi form of weight k and index 1. Thus we completed the proof. □

§ 5. Proof

We start by recalling some classical results. The ring

$$A(\Gamma_0(4))=\sum_{k=0}^{\infty}M_{2k}(\Gamma_0(4))$$

of modular forms is generated by $\theta_{00}^4(2\tau)$ and $\theta_{10}^4(2\tau)$, which are algebraically independent. In particular the set

$$\{\theta_{00}(2\tau)^{4l}\theta_{10}(2\tau)^{4m}\mid l+m=k\}$$

forms a basis of the vector space $M_{2k}(\Gamma_0(4))$. Moreover the set

$$\{\theta_{00}(2\tau)^{4l+2}\theta_{10}(2\tau)^{4m}\mid l+m=k\}$$

forms a basis of

$$M_{2k+1}(\Gamma_1(4))=M_{2k+1}(\Gamma_0(4),\chi)=M_{2k+1}(4,\chi),$$

where χ is the Dirichlet character modulo 4.

For $t=1,2,4,8$, put $\theta_{00}^2(t\tau)=A_t$, $\theta_{10}^2(t\tau)=B_t$. Then by the addition formula of theta functions, we have

$$A_2=A_4+B_4,\quad B_2^2=4A_4B_4,\quad A_4=A_8+B_8,\quad B_4^2=4A_8B_8;$$

hence the set

$$\{(A_4+B_4)^{2l+1}(4A_4B_4)^m\mid l+m=2k+1\}$$

forms a basis of $M_{4k+3}(4,\chi)$.

Let

$$g(\tau)=\sum_{m=0}^{2k+1}c_m(A_4+B_4)^{4k-2m+3}(4A_4B_4)^m,\quad c_m\in\mathbf{C}$$

be an arbitrary element of $M_{4k+3}(4,\chi)$, and

$$g(\tau)=\sum_{n=0}^{\infty}a(n)q^n$$

its Fourier expansion. For $j = 0, 1, 2, 3$, put

$$g_j(4\tau) = \sum_{n=0}^{\infty} a(4n + j)q^{4n+j}.$$

Then $g(\tau)$ becomes

$$\sum_{\alpha=0}^{k} 4^{2\alpha} c_{2\alpha} \left(\sum_{s=0}^{2k-2\alpha+1} \binom{4k-4\alpha+3}{2s+1} A_4^{4k-2\alpha-2s+2} B_4^{2\alpha+2s+1} \right)$$
$$+ \sum_{\beta=0}^{k} 4^{2\beta+1} c_{2\beta+1} \left(\sum_{t=0}^{2k-2\beta} \binom{4k-4\beta+1}{2t} A_4^{4k-2\beta-2t+2} B_4^{2\beta+2t+1} \right)$$
$$+ \sum_{\alpha=0}^{k} 4^{2\alpha} c_{2\alpha} \left(\sum_{t=0}^{2k-2\alpha+1} \binom{4k-4\alpha+3}{2t} A_4^{4k-2\alpha-2t+3} B_4^{2\alpha+2t} \right)$$
$$+ \sum_{\beta=0}^{k} 4^{2\beta+1} c_{2\beta+1} \left(\sum_{s=0}^{2k-2\beta} \binom{4k-4\beta+1}{2s+1} A_4^{4k-2\beta-2s+1} B_4^{2\beta+2s+2} \right).$$

We define two polynomials $G(X, Y)$ and $H(X, Y)$ by

$$G(X, Y) = \sum_{\alpha=0}^{k} 4^{2\alpha} c_{2\alpha} \left(\sum_{s=0}^{2k-2\alpha+1} \binom{4k-4\alpha+3}{2s+1} X^{2k-\alpha-s+1} Y^{\alpha+s} \right)$$
$$+ \sum_{\beta=0}^{k} 4^{2\beta+1} c_{2\beta+1} \left(\sum_{t=0}^{2k-2\beta} \binom{4k-4\beta+1}{2t} X^{2k-\beta-t+1} Y^{\beta+t} \right),$$

and

$$H(X, Y) = \sum_{\alpha=0}^{k} 4^{2\alpha} c_{2\alpha} \left(\sum_{t=0}^{2k-2\alpha+1} \binom{4k-4\alpha+3}{2t} X^{2k-\alpha-t+1} Y^{\alpha+t} \right)$$
$$+ \sum_{\beta=0}^{k} 4^{2\beta+1} c_{2\beta+1} \left(\sum_{s=0}^{2k-2\beta} \binom{4k-4\beta+1}{2s+1} X^{2k-\beta-s} Y^{\beta+s+1} \right).$$

Then we have $g(\tau) = G(A_4^2, B_4^2)B_4 + A_4 H(A_4^2, B_4^2)$ obviously.

LEMMA 6.

$$H(Y, X) = G(X, Y), \quad G(Y - X, Y) = -G(X - Y, -Y).$$

Proof. By setting $4k-4\alpha-2t+2=2s$ and $4k-4\beta+1-(2s+1)=2t$ in the definition of $H(X,Y)$, we see $H(X,Y)=G(Y,X)$.

In the definition of $G(X,Y)$, we see that

$$(X-Y)^{2k-\alpha-s+1}(-Y)^{\alpha+s}=-(Y-X)^{2k-\alpha-s+1}Y^{\alpha+s}$$

and that

$$(X-Y)^{2k-\beta-t+1}(-Y)^{\beta+t}=-(Y-X)^{2k-\beta-t+1}Y^{\beta+t};$$

hence we get $G(Y-X,Y)=-G(X-Y,-Y)$. □

Now we shall recall Fourier expansion of theta functions.

$$A_4=\left(1+2\sum_{n=1}^{\infty}q^{2n^2}\right)^2,$$

$$B_4=\left(2\sum_{n=1}^{\infty}q^{2(n-1/2)^2}\right)^2,$$

$$A_8=\left(1+2\sum_{n=1}^{\infty}q^{4n^2}\right)^2=\sum_{n=0}^{\infty}b(n)q^n,$$

$$B_8=\left(2\sum_{n=1}^{\infty}q^{(2n-1)^2}\right)^2=\sum_{n=0}^{\infty}c(n)q^n.$$

Thus unless $n\equiv 0 \pmod 4$, $b(n)=0$ and unless $n\equiv 2 \pmod 4$, $c(n)=0$; hence the Fourier expansion of $B_4^2=4A_8B_8$ is the form

$$\sum_{n=0}^{\infty}d(2n)q^{2n},$$

so that the term in $g(\tau)$ of the form $A_4^rB_4^{2s}$ does not contribute with $g_1(4\tau)$ and $g_3(4\tau)$. The Fourier expansion of B_4 is the form

$$\sum_{n=0}^{\infty}e(4n+1)q^{4n+1}.$$

Thus we get the following:

Lemma 7.

$$g_0(4\tau)+g_2(4\tau)=A_4H(A_4^2,B_4^2),\quad g_1(4\tau)+g_3(4\tau)=B_4G(A_4^2,B_4^2).$$

THEOREM 5. *The following are equivalent*:

1. $g_1(4\tau) = 0$, *i.e.* $a(4n+1) = 0\ \forall n$.
2. *Any monomial of even degree with respect to* V *does not occur in the polynomial* $G((U+V)^2, 4UV)$ *of* U, V.
3. $G((U+V)^2, 4UV) = -G((U-V)^2, -4UV)$.
4. $G(X, Y) = -G(X-Y, -Y)$.
5. $G(X, Y) = G(Y-X, Y)$.
6. $H(X, Y) = H(X, X-Y)$.
7. $g_0(-\tau^{-1}) + g_2(-\tau^{-1}) = i\tau^{4k+3}(g_0(\tau) + g_2(\tau))$.

PROOF. By the argument before the Lemma and taking U, V as A_8, B_8, we get $1 \Leftrightarrow 2$. $2 \Leftrightarrow 3$ is obvious. Replacing $(U+V)^2$, $4UV$ by X, Y, we get $3 \Leftrightarrow 4$. By the Lemma, we have $4 \Leftrightarrow 5 \Leftrightarrow 6$.

Now we shall prove $6 \Leftrightarrow 7$. By the transformation of theta functions, we have

$$\theta_{00}(-\tau^{-1}) = -i\tau\theta_{00}(\tau)$$

$$\theta_{10}(-\tau^{-1}) = -i\tau\theta_{01}(\tau).$$

Since $g_0(\tau) + g_2(\tau) = A_1 H(A_1^2, B_1^2)$, it follows

$$g_0(-\tau^{-1}) + g_2(-\tau^{-1}) = i\tau^{4k+3} A_1 H(A_1^2, (B_1')^2),$$

where $B_1' = \theta_{01}^2(\tau)$. The theta relation $\theta_{00}^4(\tau) = \theta_{10}^4(\tau) + \theta_{01}^4(\tau)$ yields $(B_1')^2 = A_1^2 - B_1^2$. Therefore the last assertion is equivalent to $H(A_1^2, B_1^2) = H(A_1^2, A_1^2 - B_1^2)$, i.e. $H(X, Y) = H(X, X-Y)$. □

§ 6. Maass Subspace

In § 2, we recalled hemitian modular forms and noticed that the first Fourier-Jacobi coefficient of a hermitian modular form of degree 2 is a hermitian Jacobi form. Thus we get a linear map

$$\mathscr{F}_1 : A_k(\Gamma_2(\mathcal{O})) \to J_{k,1}, \quad F \mapsto \phi_1.$$

We define the Maass subspace $\mathscr{M}_k$ of $A_k(\Gamma_2(\mathcal{O}))$ by the set of modular forms $F(\tau, z, z', \tau)$ whose Fourier coefficients $A(n, r, m)$ are satisfying

$$A(n, r, m) = \sum_{a \mid (n, r, m)} a^{k-1} A\left(\frac{nm}{a^2}, \frac{r}{a}, 1\right), \quad \forall n, r, m. \tag{33}$$

In order to get an *inverse* of the map $\mathscr{F}_1 : \mathscr{M}_k \to J_{k,1}$, following [2] I.§4, we define the operator V_l $(l > 0)$ on functions $\phi : \mathbf{H}_1 \times \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ by

$$(\phi|_{k,m}V_l)(\tau, z, z') = l^{k-1}\sum_{M}(c\tau+d)^{-k}\mathbf{e}\left(\frac{-mlczz'}{c\tau+d}\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{lz}{c\tau+d}, \frac{lz'}{c\tau+d}\right), \tag{34}$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over a complete set of representatives for

$$\mathrm{SL}_2(\mathbf{Z})\backslash\{M \in \mathrm{Mat}_2(\mathbf{Z}) \mid \det(M) = l\}.$$

It is easily seen that the right hand side does not depend on the choice of a set of representatives; hence the operator V_l is well defined.

Moreover we define V_0 by

$$\phi|_{k,m}V_0 = c(0,0)\left(-\frac{2k}{B_{2k}} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right),$$

where B_{2k} is the $2k$-th Bernoulli number and

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Lemma 8. *If $\phi = \sum c(n,r)q^n\zeta^{\bar{r}}(\zeta')^r$ is an element of $J_{m,k}$, then $\phi|_{k,m}V_l \in J_{k,ml}$ and*

$$(\phi|V_l)(\tau, z, z') = \sum_{n,r}\sum_{a\,|\,(n,l,r)} a^{k-1}c\left(\frac{nl}{a^2}, r\right)\mathbf{e}\left(n\tau + \frac{\bar{r}z}{2} + \frac{rz'}{2}\right).$$

Proof. We take the standard set of representatives:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad a, d > 0, \quad b \pmod d, \quad ad = l.$$

Then

$$(\phi|V_l)(\tau, z, z') = l^{k-1}\sum_{ad=l,\, b \,(\mathrm{mod}\, d)} d^{-k}\phi\left(\frac{a\tau+b}{d}, \frac{lz}{d}, \frac{lz}{d}\right)$$

$$= l^{k-1}\sum_{ad=l} d^{-k}\sum_{b\,(\mathrm{mod}\, d)}\sum_{n,r} c(n,r)\mathbf{e}\left(\frac{na\tau}{d} + \frac{nb}{d} + \frac{\bar{r}az}{2} + \frac{raz'}{2}\right)$$

$$= l^{k-1} \sum_{ad=l} d^{1-k} \sum_{n, r, n \equiv 0 \, (\mathrm{mod}\, d)} c(n, r) \mathbf{e}\left(\frac{na\tau}{d} + \frac{\bar{r}az}{2} + \frac{raz'}{2}\right)$$

$$= \sum_{a|l} a^{k-1} \sum_{n, r} c\left(\frac{nl}{a}, r\right) \mathbf{e}\left(na\tau + \frac{\bar{r}az}{2} + \frac{raz'}{2}\right)$$

$$= \sum_{n, r} \sum_{a \,|\, (n, l, r)} a^{k-1} \sum_{n, r} c\left(\frac{nl}{a^2}, r\right) \mathbf{e}\left(n\tau + \frac{\bar{r}z}{2} + \frac{rz'}{2}\right). \qquad \square$$

The following is a hermitian version of [2] Th. 6.2.

Theorem 6. *Let ϕ be a hermitian Jacobi form of weight k and index 1. Then*

$$\mathscr{V}\phi(\tau, z, z', \tau') = \sum_{m=0}^{\infty} (\phi | V_m)(\tau, z, z') \mathbf{e}(m\tau')$$

is a hermitian modular form of weight k, and

$$\mathscr{F}_1 : M_k \to J_{k,1}, \quad \mathscr{V} : J_{k,1} \to M_k$$

are inverses to each other.

Proof. It is known that the modular group $\Gamma_2(\mathscr{O})$ is generated by the following matrices:

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \bar{\mu} & 0 \\ 0 & 0 & 1 & -\bar{\lambda} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(\mathscr{O})$ and $\lambda, \mu \in \mathscr{O}$ with $\lambda\mu = \bar{\lambda}\bar{\mu}$ and

$$M_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $\phi | V_m$ is a hermitian Jacobi form of weight k and index m for every m, $\mathscr{V}\phi$ satisfies the equation (10) for any M in the first and the second type of matrices. Put

$$\mathscr{V}\phi(\tau, z, z', \tau') = \sum_{n,m,r} A(n,r,m)\mathbf{e}(n\tau + \bar{r}z/2 + rz'/2 + m\tau').$$

Then

$$A(n,r,m) = \sum_{a\,|\,(n,r,m)} a^{k-1} c\left(\frac{nm}{a^2}, \frac{r}{a}\right)$$

$$= \sum_{a\,|\,(n,r,m)} a^{k-1} c\left(\frac{4nm - |r|^2}{a^2}\right).$$

Therefore $A(n,r,m)$'s satisfy the Maass' condition (33) and $A(n,r,m) = A(m,r,n)$; hence $\mathscr{V}\phi$ satisfies (10) for M_0. Thus $\mathscr{V}\phi$ is an element of the Maass subspace. Apparently both of $\mathscr{F}_1\mathscr{V}$ and $\mathscr{V}\mathscr{F}_1$ are the identities. $\square$

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