Inradius collapsed manifolds

Zhilang Zhang

February 2016

Inradius collapsed manifolds

Zhilang Zhang
Doctoral Program in Mathematics

Submitted to the Graduate School of
Pure and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Science

at the

University of Tsukuba

Acknowledgement

I would like to express my sincere gratitude to Prof. Takao Yamaguchi, my supervisor. I would like to thank him for proposing this interesting and challenging problem of collapsing manifolds with boundary to me, and giving many helpful and enlightening advices to me on how to proceed with my work. Other than math, his encouragement and valuable advices on doing research in geometry also benefited me a lot. Thank you so much!

I would like to express my gratitude to Prof. Hiroyuki Tasaki, my supervisor, for his constant support on my doing research and he comments on our work.

I would like to express my gratitude to Prof. Koichi Nagano for the inspiring conversations on math.

I would like to thank my friends Daruhan Wu, Hanbiao Yang and Dongxiao Yu for their helps on my adapting to the life and study in University of Tsukuba.

Abstract

In this dissertation, we study collapsed manifolds with boundary, where we assume a lower sectional curvature bound, a two-sided bound on second fundamental forms of boundaries and an upper diameter bound. Our main concern is the case when inradii of manifolds converge to zero. This is a typical case of collapsing manifolds with boundary. Actually we show that the inradius collapse occurs when the limit space is a topological closed manifold, for instance. In the general case, we determine the limit spaces of inradius collapsed manifolds as Alexandrov spaces with curvature uniformly bounded below. When the limit space has co-dimension one, we completely determined the topology of inradius collapsed manifold in terms of singular I-bundles. General inradius collapse to almost regular spaces are also characterized. In the case of unbounded diameters, we prove that the number of boundary components of inradius collapsed manifolds is at most two. The main results in this dissertation are due to a joint work with Professor Takao Yamaguchi [YZ15].

Contents

Chapter 0

Introduction

In this chapter, we introduce the background of our work. It is well known that the understanding of relationship between geometry and topology in Riemannian manifolds is one of the central themes in geometry. Ever since the concept of Gromov-Hausdorff distance was introduced, there has been increasing interest in studying the relationship between geometry and topology in Gromov-Hausdorff convergence. If a precompact subset $\mathcal{M}$ of all compact length spaces with respect to Gromov-Hausdorff distance is given, it is natural to consider the following problem.

Problem 0.0.1. Let X_i be a sequence of elements in $\mathcal{M}$ converging to a compact length space X with respect to the Gromov-Hausdorff distance.

(1) Characterize the structure of X;

(2) Find geometric and topological relations between X_i and X for large enough i.

Given a Riemannian manifold M with boundary, the *inradius* of M is defined as

$$\mathrm{inrad}(M) := \sup_{x \in M} d(x, \partial M).$$

Inradius collapsed manifolds M_i mean a sequence of Riemannian manifolds with boundary satisfying $\mathrm{inrad}(M_i) \to 0$. In this dissertation, we mainly study Problem 0.0.1 for the inradius collapsed manifolds where we assume a lower sectional curvature bound, a two-sided bound on second fundamental forms of boundaries and an upper diameter bound. The main results in this dissertation are due to a joint work with Prof. Takao Yamaguchi [YZ15].

0.1 Background

Before we introduce our work, we review some results of Riemannian manifolds with or without boundary related to Problem 0.0.1. Since our work relates to sectional curvature bounded below, the introduction of background emphasizes this condition.

0.1.1 Manifolds without boundary

In this subsection, we review results on Problem 0.0.1 for manifolds without boundary.

Let $\mathcal{M}'(n,\kappa,d)$ (resp. $\mathcal{M}'(n,r^-,d)$) denote the collection of isometry classes of compact n-dimensional Riemannian manifolds without boundary, with a two-sided bound κ (resp. r^-) on sectional (resp. Ricci) curvatures and an upper diameter bound d. Let $\mathcal{M}(n,\kappa,d)$ (resp. $\mathcal{M}(n,r^-,d)$) denote the collection of isometry classes of compact n-dimensional Riemannian manifolds without boundary, with a lower sectional (resp. Ricci) curvature bound κ (resp. r^-) and an upper diameter bound d.

By Gromov's precompact theorem, the collection $\mathcal{M}(n,r^-,d)$ is precompact with respect to Gromov-Hausdorff distance. It is natural to consider Problem 0.0.1 in the cases of $\mathcal{M}=\mathcal{M}'(n,\kappa,d)$, $\mathcal{M}'(n,r^-,d)$, $\mathcal{M}(n,\kappa,d)$, and $\mathcal{M}(n,r^-,d)$ respectively.

1. The case of bounded sectional (Ricci) curvature

Now we review the studies of Problem 0.0.1 for $\mathcal{M}=\mathcal{M}'(n,\kappa,d)$ and $\mathcal{M}'(n,r^-,d)$.

For the case of **non-collapse**, by [G'bk, Kat85], if M_i in $\mathcal{M}'(n,\kappa,d)$ Gromov-Hausdorff converges to a space M with the same dimension, then M is a differentiable manifold and M_i is diffeomorphic to M for large enough i. This result implies the finiteness of the diffeomorphism classes in $\mathcal{M}'(n,\kappa,d)$ with a lower positive volume bound. Moreover, by [G'bk, GW88, Peters87], if $M_i\in\mathcal{M}'(n,\kappa,d)$ Gromov-Hausdorff converges to an n-dimensional manifold, then there exists a subsequence converges to a Riemannian manifold of $C^{1,\alpha}$ class. These results give answers to Problem 0.0.1 (1) and (2).

For the case of **collapse** in Problem 0.0.1 (2), let us first recall the Almost flat manifolds Theorem. A closed manifold M is called an *almost flat manifold* [G:AFM] if for each positive ε, there exists a metric g_ε on M such that

$$|sec_{(M,g_\varepsilon)}|\le 1 \text{ and } \operatorname{diam}(M,g_\varepsilon)<\varepsilon,$$

where $|sec_{(M,g_\varepsilon)}|$ is a positive two-sided bound on the sectional curvature. This theorem is a starting point of many studies of collapse phenomenon.

Theorem 0.1.1 (Almost flat manifolds Theorem [G:AFM],[Ruh82])**.** *Let M be a closed Riemannian manifold,*

(1) M is an infranil manifold if and only if it is an almost flat manifold,

(2) If M is an almost flat manifold, there exists a nilpotent subgroup of $\pi_1(M)$ with finite index.

Note that Margulis lemma played an essential role in the proof.

There are **two approaches** in extending Gromov's idea in this theorem. One came from Gromov-Cheeger's studies [CG86, CG90], which introduced the F**-structure** on a manifold and studied its relationship with the existence of metrics on this manifold such that the injectivity radii with respect to them converge to zero while the sectional curvatures are uniformly bounded. Another came from Fukaya's paper [Fu86], which established a **fibration theorem** that if M_i in $\mathcal{M}(n,\kappa,d)$ Gromov-Hausdorff converges to a manifold N with $|sec_N| \leq 1$, then M_i is a fiber bundle over N with an infranil fiber. Later, the approach of collapsing theory in [CFG92] combined and generalized Cheeger-Gromov's and Fukaya's ones.

2. The case of a lower bound on sectional and (respect. Ricci) curvature

Now we review the studies of Problem 0.0.1 for $\mathcal{M} = \mathcal{M}(n,\kappa,d)$ and $\mathcal{M}(n,r^-,d)$.

For the case of **non-collapse** in Problem 0.0.1 (2), Perelman's stability theorem [Pr94], cf. [Ka07], implies a homeomorphic version of Gromov's corresponding result above. Clearly, it implies the topological finiteness of Riemannian manifolds in $\mathcal{M}(n,\kappa,d,v)$, the set of all elements $M \in \mathcal{M}(n,\kappa,d)$ having volume $\mathrm{vol}(M) \geq v > 0$. Before Perelman's result was proved, it was known that $\mathcal{M}(n,\kappa,d,v)$ contains finite many homotopy types by Grove-Peterson in 1988 [GP88] and it contains at most finitely many homeomorphism types when $n \neq 3$, and only finitely many diffeomorphism types if in addition $n \neq 4$ by Grove-Petersen-Wu [GPW]. For the case of Ricci curvature, Cheeger-Colding [CC97] showed that given a closed smooth n-manifold M^n, there exists an $\varepsilon = \varepsilon(M) > 0$ such that if N^n is an n-manifold with $\mathrm{Ricci}_N > -(n-1)$ and $d_{GH}(M,N) < \varepsilon$, then N is diffeomorphic to M.

It is well known that the Gromov-Hausdorff limit X of a sequence of manifolds in $\mathcal{M}(n,\kappa,d)$ is an Alexandrov space [GP91]. Let S denote the set of singularities in X. By Otsu-Shioya's work [OS94], S has hausdorff dimension at most $n-1$ and there exists a natural $C^{1/2}$-Riemannian structure on a full-measure subset of $X \setminus S$, where the metric induced by this Riemannian structure coincides with the original metric in X. This fact gives an answer to Problem 0.0.1 (1).

For the case of **collapse** in Problem 0.0.1 (2), let us first review some results about manifolds collapsing to a point. The works in [FY92, Ya91] are their starting points.

In [FY92], Fukaya-Yamaguchi studied an important type of manifolds named *almost non-negatively curved manifolds*. A closed Riemannian manifold M^n is called an *almost non-negatively curved manifold* if and only if it possesses a metric g_ε for each given positive ε such that

$$K_{M_{g_\varepsilon}} \geq -1 \text{ and } \mathrm{diam}\,(M,g_\varepsilon) < \varepsilon,$$

where $K_{M_{g_\varepsilon}}$ denotes the lower sectional curvature bound of (M,g_ε). Obviously, the concept of almost non-negatively curved manifold is a generalization of almost flat manifold.

Fukaya-Yamaguchi proved that there exists a nilpotent subgroup in $\pi_1(M)$ with finite index C by a version of Margulis lemma with sectional curvature bounded below. Thus their theorem can be considered as a version of Gromov's almost flat manifolds theorem. In the year 2010, Petrunin-Kapovich-Tuschmann[KPT10] proved that the index C in Fukaya-Yamaguchi's result above is a constant depends only on $\dim M$. The gradient flow on Alexandrov spaces plays a key role in the proofs of [KPT10]. For the case of Ricci curvature, Cheeger-Colding [CC96] and Kapovich-Wilking [KW11] proved that Petrunin-Kapovich-Tuschmann's result is true even if a lower sectional curvature bound is replaced by a lower Ricci curvature bound by applying their generalized Margulis lemma with Ricci curvature bounded below. Their proofs are based on the structure results of Cheeger and Colding for limit spaces of manifolds with lower Ricci curvature bounds.

Next we review the results about manifolds collapsing to a manifold. In [Ya91] Yamaguchi established a general **fibration theorem** for collapsing manifolds with a lower sectional curvature bound, which generalizes Fukaya's fibration theorem [Fu86] and relates the topological rigitity to the structure of the first betti number of the fiber. This theorem implies that the almost non-negatively curved manifold M^n with first betti number $b_1(M) = n$ is diffeomorphic to a torus $\mathbb{T}^n$. Yamaguchi's work indicates that the study of almost nonnegatively curved manifolds is very important since they possess similar topological properties as the fibers of collapsing manifolds with a lower sectional curvature bound over their limits space. Another importance of this theorem is it opened a door to a series of studies on the structure of collapsing Alexandrov space (resp. Riemannian manifolds) with a lower curvature (resp. sectional curvature) bound. In 1996, a fibration theorem for Alexandrov spaces was established [Ya96] where the limit space and converging spaces are assumed to be almost regular. Later, this result was greatly generalized to a Equivariant Fibration-Capping Theorem 4.2.2 by improving the previous fibration theorems and applying Perelman's fibration theorem and Siebenmann's theory [Pr94, Sie72] and the theory of group action. This theorem was applied to study the classifications of three and four dimensional collapsing manifolds [SY00, Ya02]. Note that for the study of the structure of three dimensional volume collapsed manifolds, cf. [SY00, SY05], played an important role in Perelman's resolution of Thurston's Geometrization Conjecture [Pe02, Pe03]. The generalized fibration theorem also plays an important role in classification of collapsing three dimensional closed Alexandrov spaces [MY15].

For the case of convergence of manifolds with a lower or two-sided Ricci curvature bound, it is very different from $\mathcal{M}(n, \kappa, d)$ and $\mathcal{M}'(n, \kappa, d)$ in the sense that there is no fibration theorem for two close manifolds in $\mathcal{M}(n, r^-, d)$ in Gromov-Hausdorff distance.

0.1.2 Manifolds with boundary

In order to study Problem 0.0.1 for manifolds with boundary, we need a collection $\mathcal{M}$ of compact manifolds with boundary which is precompact with respect to Gromov-Hausdorff distance. Thus, in this subsection, we review various precompactness theorems with assumptions on sectional curvature or Ricci curvature. We also review the studies related to Problem 0.0.1 (1) and (2) for manifolds with boundary.

Various precompactness theorems

1. The case of sectional curvature

Kodani was the first to study the precompactness theorem for Riemannian manifolds with boundary with respect to Gromov-Hausdorff distance. Precisely, he derived a precompactness theorem for n-dimensional Riemannian manifolds with boundary with a two-sided bound on the sectional curvatures, a non-negative two-sided bound on the second fundamental forms of the boundaries, an upper bound on the diameters and a positive lower bound on the volumes [Kod90].

Next, we review Wong's precompactness theorem. Let $\mathcal{M}(n,\kappa,\lambda,d)$ denote the isometry classes of compact n-dimensional Riemannian manifolds with smooth boundary with a lower sectional curvature bound κ, a two-sided bound λ on the second fundamental forms of the boundaries and an upper diameter bound d. Applying Kosovskii's gluing theorem [Kos02], Wong was able to prove the precompactness of $\mathcal{M}(n,\kappa,\lambda,d)$ [Wo08], which is a generalization of Kodani's result. This result is the basic point of our work.

Remark 0.1.2. We don't know whether $\mathcal{M}(n,\kappa,\lambda,d)$ is still precompact if the assumption of the upper bound of the second fundamental forms of the boundaries is removed.

2. The case of Ricci curvature

In [Wo08], Wong also obtained a precompactness theorem for Riemannian manifolds with boundary with a lower Ricci curvature bound, which implies that, the collection of the isometry classes of compact n-dimensional Riemannian manifolds with smooth boundary with a lower bound on Ricci curvature, a two-sided bound on the second fundamental forms of the boundaries and an upper diameter bound, is precompact with respect to Gromov-Hausdorff distance. In [Pera], Perales proved some precompactness theorems for non-collapsing compact Riemannian manifolds with boundary with non-negative lower Ricci curvature bound and other assumptions related to the boundary and volume.

Structure theorems with a lower sectional curvature bound

To the author's knowledge, there exists no structure result for Gromov-Hausdorff convergent sequence of Riemannian manifolds with boundary with a lower or two-sided Ricci curvature bound. For Riemannian manifolds with boundary with a two-sided bound or lower bound on sectional curvature, there exist but not many results. We mainly introduce some results which relate to our main results.

The following consequence is an answer to Problem 0.0.1 (1) .

Theorem 0.1.3 ([Wo10]). *Let M_i be a sequence of Riemannian manifolds with boundary. Suppose M_i Gromov-Hausdorff converges to some metric space X. If $K_{M_i} \geq K^-$, $\tau(1/i) \leq II_{\partial M_i} \leq \lambda^+$, then X is an Alexandrov space with a lower curvature bound K^-.*

Obviously, the boundary of M_i increasingly becomes locally convex as i increases. It means that M_i increasingly looks like an Alexandrov space. It is natural to ask what the structure of the limit space of M_i is, when the lower bound of the second fundamental form of the boundary is replaced by a negative number. We will study this problem in the dissertation.

The following result comes from Wong [Wo10] and Yamaguchi [Ya96], which is an answer to Problem 0.0.1 (2) for collapsing manifolds with boundary which is also an Alexandrov space.

Theorem 0.1.4 ([Wo10], Theorem 2(i)). *Let M_i be a sequence of Riemannian manifolds with boundary and $M_i \xrightarrow{GH} N$, where N is a closed C^1 manifold. If $K_{M_i} \geq K^-$, $II_{\partial M_i} \geq 0$, then there exists a local trivial fiber bundle*

$$F_i \to M_i \to N,$$

where F_i is a manifold with boundary, almost non-negatively curved in the generalized sense [Ya91].

Remark 0.1.5. Notice that M_i is an Alexandrov space since $II_{\partial M_i} \geq 0$. Theorem 0.1.4 is a direct consequence of Yamaguchi's fibration theorem for Alexandrov space [Ya96].

It is natural to ask whether M_i fibers over its limit space N if M_i is not an Alexandrov space. We will study this problem in the dissertation.

The following theorem is a structure theorem for inradius collapsed manifolds.

Theorem 0.1.6 ([AB98], Theorem 1.1). *There exists a dimension-independent constant c such that if a complete connected Riemannian manifold M satisfies*

$$\operatorname{inrad}(M)^2 \sup\{|K_M|, |II_{\partial M}|^2\} < c,$$

then either M is diffeomorphic to the product of a manifold without boundary and an interval or M can be doubly covered by such a product.

Remark 0.1.7. The proof in [AB98] depends on the discussion of the cut locus and the integral curves of gradient fields generated by distance functions to the boundaries, where cut locus is the collection of all points p in M such that there is a geodesic segment γ realizing $d(p, \partial M)$, but no extension of γ realizes distance to ∂M.

For the structure results for non-collapsing manifolds in Problem 0.0.1, [Wo08] implies that if $M_i \in \mathcal{M}(n,\kappa,\lambda,d)$ Gromov-Hausdorff converges to an n-dimensional metric space N, then M_i is homeomorphic to M_j for i,j large enough. However, it is different from Perelman's stability theorem that M_i is not necessarily homeomorphic to the limit space N. Consider a an annulus with outer boundary fixed and inner boundary continuously moves to the outer boundary. The limit space is even not a manifold with boundary!

0.2 Main results

In this section we introduce our main results. All of them are due to a joint work with Prof. Takao Yamaguchi [YZ15]. The first one is about the structure of limit spaces, which give an answer to Problem 0.0.1 (1).

Recall that $\mathcal{M}(n,\kappa,\lambda,d)$ is the collection of all isometry classes of compact n-dimensional Riemannian manifolds with smooth boundary and with a lower sectional curvature bound κ, a two-sided bound λ on the second fundamental forms of the boundaries and an upper diameter bound d.

For a sequence of Riemannian manifolds M_i with boundary, we say M_i *inradius collapses* to a compact metric space N if and only if $M_i \xrightarrow{GH} N$ and $\operatorname{inrad}(M_i) \to 0$.

Theorem 0.2.1 ([YZ15])**.** *Let $M_i \in \mathcal{M}(n,\kappa,\lambda,d)$ inradius collapse to a length space N with respect to the Gromov-Hausdorff distance. Then N is an Alexandrov space with curvature $\geq c(\kappa,\lambda)$, where $c(\kappa,\lambda)$ is a constant depending only on κ and λ.*

In fact, N is isometric to the Gromov-Hausdorff limit of ∂M_i modulo an isometric $\mathbb{Z}_2$-action, therefore $c(\kappa,\lambda)$ is the same uniform lower curvature bound as $K_{\partial M_i}$. It should be noted that M_i may not be Alexandrov spaces unless $\Pi_{\partial M_i} \geq 0$, and that the constant $c(\kappa,\lambda)$ really depends on κ and λ (see Example 2.3.1 and 2.3.2).

Let $\mathcal{M}(n,\kappa,\lambda)$ denote the set of all isometry classes of n-dimensional complete Riemannian manifolds M with smooth boundary satisfying

$$K_M \geq \kappa, \ |\Pi_{\partial M}| \leq \lambda.$$

This family is also precompact with respect to the pointed Gromov-Hausdorff convergence [Wo08]. Theorem 0.2.1 actually holds true for the limit of manifolds in $\mathcal{M}(n,\kappa,\lambda)$ with respect to the pointed Gromov-Hausdorff convergence (see Theorem 5.0.7).

Next we discuss the topological structure of inradius collapsed manifolds. First we consider the case of inradius collapse of codimension one. where we define two types of models of the singularities around boundary component of the limit space, *the product or the twisted singular I-fiber bundle* (see Definition 4.1.1). We can give a complete characterization of codimension one inradius collapsed manifolds as follows, which is an answer to Problem 0.0.1 (2).

Theorem 0.2.2 ([YZ15]). *Let $M_i \in \mathcal{M}(n,\kappa,\lambda,d)$ inradius collapse to an $(n-1)$-dimensional Alexandrov space N. Then there is a singular I-fiber bundle:*

$$I \to M_i \xrightarrow{\pi} N.$$

More precisely,

(1) If N has no boundary, then M_i is homeomorphic to a product $N \times I$ or a twisted product $N \tilde{\times} I$;

(2) If N has non-empty boundary, each component $\partial_\alpha N$ of ∂N has a neighborhood V such that $\pi^{-1}(V)$ is homeomorphic to either the product or the twisted singular I-fiber bundle around $\partial_\alpha N$;

(3) If $\pi^{-1}(V)$ is homeomorphic to the product singular I-fiber bundle for some component $\partial_\alpha N$, then M_i is homeomorphic to

$$D(N) \times [-1,1]/(x,t) \sim (r(x),-t),$$

where r is the canonical reflection of the double $D(N)$.

Next, we consider the case of inradius collapse to an almost regular space, which is an answer to Problem 0.0.1 (2). We say that an Alexandrov space N is *almost regular* if any point of N has the space of directions whose volume is close to $\operatorname{vol}\mathbb{S}^{\dim N-1}$.

Theorem 0.2.3 ([YZ15]). *Let M_i in $\mathcal{M}(n,\kappa,\lambda,d)$ inradius collapse to an almost regular Alexandrov space N, then the topology of M_i can be classified into two types*

(a) There exists a local trivial fiber bundle

$$F_i \times I \to M_i \to N,$$

where F_i is a closed almost non-negatively curved manifold in a generalized sense as in [Ya91].

(b) There exists a local trivial fiber bundle

$$\mathrm{Cap}_i \to M_i \to N,$$

where Cap_i (resp. $\partial\,\mathrm{Cap}_i$) is a closed almost non-negatively curved manifold in a generalized sense as in [Ya91] with boundary (resp. without boundary).

Combined with Proposition 6.0.19, Theorem 0.2.3 yields the following.

Corollary 0.2.4 ([YZ15]). *Let a sequence M_i in $\mathcal{M}(n,\kappa,\lambda,d)$ Gromov-Hausdorff converge to a closed Riemannian manifold N of class C^1, then the topology of M_i can be classified into two types*

(a) There exists a local trivial fiber bundle

$$F_i \times I \to M_i \to N,$$

where F_i is a closed almost non-negatively curved manifold in a generalized sense as in [Ya91].

(b) There exists a local trivial fiber bundle

$$\mathrm{Cap}_i \to M_i \to N,$$

where Cap_i (resp. $\partial\,\mathrm{Cap}_i$) is a closed almost non-negatively curved manifold in a generalized sense as in [Ya91] with boundary (resp. without boundary).

Corollary 0.2.4 provides an extension of Theorem 0.1.4, and Theorem 0.2.3 solves a problem raised in [Wo10], p.297, without assuming the upper sectional curvature bound.

Next we discuss the number of boundary components of inradius collapsed manifolds, where we do not assume the diameter bound.

Theorem 0.2.5 ([YZ15]). *There exists a positive number $\epsilon=\epsilon_n(\kappa,\lambda)$ such that if M in $\mathcal{M}(n,\kappa,\lambda)$ satisfies $\mathrm{inrad}(M)<\epsilon$, then*

(1) the number k of connected components of ∂M is at most two;

(2) if $k=2$, then M is diffeomorphic to $W\times[0,1]$, where W is a component of ∂M.

Theorem 0.2.5 (1) was stated in [Wo10], Theorem 5. The following argument employed in the proof there is unclear to the authors: if $k\geq 3$ and if $p\in M$ is the furthest point from ∂M, then $B(p,\mathrm{inrad}(M))$, touches ∂M at least three points.

Theorem 0.2.5 might be considered as a generalization of the main theorem in [AB98], where an I-bundle structure was found for an inradius collapsed manifold under bounded sectional curvature. According to Wong's remark of Theorem 8 in [Wo10], all the conditions in this theorem are sharp.

0.3 The ideas of fibration theorems

Since the ideas of the fibration theorems 0.2.2 and 0.2.3 are very important in the dissertation, we shall introduce them in this section.

Let $M_i \in \mathcal{M}(n,\kappa,\lambda,d)$, N a compact metric space and $M_i \xrightarrow{GH} N$. Naturally, one expects to find a fiber bundle structure on M_i over N as the case of collapsing Riemannian manifolds without boundary, cf. [Ya91]. Unfortunately, since there exists not a fibration theorem for collapsing Riemannian manifolds with boundary, we have no tools to use directly. However, we can overcome this difficulty by extending M_i in a nice way.

Note that by gluing some special cylinders to the boundaries of all M_i by applying Wong's method, cf. section 1.3, we can extend all M_i to Alexandrov spaces $\tilde{M}_i$ with totally geodesic boundary and a uniform lower curvature bound, which are homeomorphic to M_i respectively. Therefore it is natural to study the convergence of $\tilde{M}_i$ instead of M_i. By Gromov's compactness theorem for Alexandrov spaces, there exists an Alexandrov space Y such that passing to a subsequence, we have

$$M_i \approx \tilde{M}_i \xrightarrow{GH} Y.$$

In order to understand the relation between M_i and N, we have to investigate the relation between Y and N which is the core of Theorem 0.2.2 and 0.2.3. We can see that Y is also a gluing space based on N. Notice that if M_i inradius collapses to N,

$$\dim N + 1 = \dim Y.$$

This fact motivates us to ask whether Y is an I-bundle over N. After a detailed study of the tangent cones at gluing points and their relation via the differential of gluing map, multiplicity of the gluing maps and singularities of N, we can see directly that Y is in general a singular I-bundle over N, cf. Theorem 3.3.4 (2), without applying other deep fibration theories. Here, the argument on the local structure of Y depends on Yamaguchi's result about collar neighborhood 1.2.7. Then we have the following relations

$$M_i \approx \tilde{M}_i \xrightarrow{GH} Y \xrightarrow{\pi} N,$$

where π is a singular fibration.

Notice that in the case of co-dimension one collapse, $\dim M_i = \dim Y$. The singular I-fiber bundle structure in Theorem 0.2.2 follows from Perelman's stability theorem.

In the case of general inradius collapse. Since N is almost regular, so is Y with almost regular boundary. Thus Y is an I-bundle over N. Since the multiplicity of the gluing map is at most two, there exist two possible I-bundle structures on Y over N. Note that the double $D(\tilde{M}_i)$ of $\tilde{M}_i$ is an Alexandrov space without boundary with the same lower curvature bound as $\tilde{M}_i$ by [Pr94] for each i. We can consider the Gromov-Hausdorff convergence of $D(\tilde{M}_i)$. Since $\partial D(\tilde{M}_i) = \emptyset$, $D(\tilde{M}_i)$ as well as its limit space possess symmetric structures and Y possesses two possible I-bundle structures, we can apply Yamaguchi's Equivariant Fibration-Capping Theorem 4.2.2 to derived two possible fiber bundle structures on M_i over its limit space N, cf. Theorem 0.2.3. The ideas of doubling $\tilde{M}_i$ and applying 4.2.2 came from [MY:pre].

0.4 Organization of the dissertaion

The organization and the outline of the proofs are as follows.

In Chapter 1, we first recall basic notions and facts on the Gromov-Hausdorff convergence and Alexandrov spaces with curvature bounded below. Then we focus on Wong's extension procedure of a Riemannian manifold with boundary by gluing a warped cylinder along their boundaries. By [Kos02], the result of the gluing is a $C^{1,\alpha}$-manifold with C^0 Riemannian metric, and becomes an Alexandrov spaces with curvature uniformly bounded below. This construction is quite effective and used in an essential way in our dissertation.

In Chapter 2, we describe the limit spaces of glued Riemannian manifolds with boundary in several aspects. The limit spaces also have gluing structure. In this section we focus on the topological structure of gluing, estimates of multiplicities of gluing, and intrinsic metric structure of the limit space.

In Chapter 3, we determine the metric structure of limit spaces. First we study the spaces of directions of the limit space at gluing points, and prove that the gluing map preserves the length of curves. This implies that the gluing in the limit space is done metrically in a natural manner, and yields significant structure results (see Theorem 3.3.4) on the limits including Theorem 0.2.1.

Those structure results are applied in Chapter 4 to obtain the fiber structures of inradius collapsed manifolds. Theorems 0.2.2 and 0.2.3 are proved there. To prove Theorem 0.2.2, we need to analyze the singularities of the singular I-fiber bundle in details. To prove Theorem 0.2.3, we apply the Equivariant Fibration-Capping Theorem [Ya02].

To prove Theorem 0.2.5, we consider the case of unbounded diameters in Chapter 5. Applying the results in Section 3, we obtain basically three types on local connectedness of the boundary of an inradius collapsed complete manifold, accdording to the types of the local limit spaces. After such local obsrvation, Theorem 0.2.5 follows from a monodromy argument.

In Chapter 6, we consider the convergence where the inradii have a positive lower bound. Then we prove a proposition which indicates that inradius collapse is typical.

Chapter 1

Preliminaries

In order to make the presented paper more accessible, we fix some basic definition, notations and conventions.

- $\tau(\delta)$ is a function depends on δ such that $\lim_{\delta\to 0}\tau(\delta)=0$.
- For topological spaces X and Y, $X\approx Y$ means X is homeomorphic to Y.
- The distance between two points x,y in a metric space is denoted by $d(x,y)$, $|x,y|$ or $|xy|$.
- For a point x and a subset A of a metric space X, $B(x,r)=B^X(x,r)$ and $B(A,r)=B^X(A,r)$ denote open r-balls in X around x and A respectively.
- For a metric space (X,d), and $r>0$, the rescaling metric space (X,rd) is denoted by rX.
- The Euclidean cone $K(\Sigma)$ over a metric space (Σ,ρ) is $\Sigma\times[0,\infty)$ equipped with the metric d defined as

  $$d((x_1,t_1),(x_2,t_2))=(t_1^2,t_2^2-2t_1t_2\cos(\min\{\rho(x_1,x_2),\pi\}))^{1/2},$$

  for any two points $(x_1,t_1),(x_2,t_2)\in\Sigma\times[0,\infty)$.
- For a subspace M of a metric space $(\tilde{M},d_{\tilde{M}})$, M^{ext} denotes $(M,d_{\tilde{M}})$, which is called the exterior metric of M.
- The metric d of a connected metric space (X,d) induces a length metric d_{int} of X defined as the infimum of the length of all curves joining two given points. We denote by X^{int} the new metric space (X,d_{int}).
- The length of a curve γ is denoted by $L(\gamma)$.

1.1 The Gromov-Hausdroff convergence

A (not necessarily continuous) map $f : X \to Y$ between two metric spaces X and Y is called an *ε-approximation* if it satisfies

1. $|d(x,y) - d(f(x), f(y))| < \varepsilon$, for all $x, y \in Y$,
2. $f(X)$ is ε-dense in Y, i.e., $B(f(X), \varepsilon) = Y$.

The Gromov-Hausdorff distance $d_{GH}(X,Y)$ is defined as the infimum of those ε such that there are ε-approximations $f : X \to Y$ and $g : Y \to X$.

A map $f : (X, x) \to (Y, y)$ between two pointed metric spaces is called a *pointed ε-approximation* if it satisfies

1. $f(x) = y$,
2. $|d(x,y) - d(f(x), f(y))| < \varepsilon$, for all $x, y \in B^X(x, 1/\varepsilon)$,
3. $f(B^X(x, 1/\varepsilon))$ is ε-dense in $B^Y(y, 1/\varepsilon)$.

The pointed Gromov-Hausdorff distance $d_{pGH}((X,x),(Y,y))$ is defined as the infimum of those ε such that there are pointed ε-approximations $f : (X,x) \to (Y,y)$ and $g : (Y,y) \to (X,x)$.

Consider a pair (X, Λ) of a metric space X and a group Λ of isometries of X. For such pairs (X,Λ), (Y,Γ), a triple (f, φ, ψ) of maps $f : X \to Y$, $\varphi : \Lambda \to \Gamma$ and $\psi : \Gamma \to \Lambda$ is called an *equivariant ε-approximation* from (X,Λ) to (Y,Γ) if the following holds

1. f is an ε-approximation,
2. if $\lambda \in \Lambda$, $x \in X$, then $d(f(\lambda x), (\varphi\lambda)(fx)) < \varepsilon$,
3. if $\gamma \in \Gamma$, $y \in Y$, then $d(f(\psi(\gamma)x), \gamma(fx)) < \varepsilon$.

The *equivariant Gromov-Hausdorff distance* $d_{eGH}((X,\Lambda),(Y,\Gamma))$ is defined as the infimum of those ε such that there are ε-approximations from (X,Λ) to (Y,Γ) and from (Y,Γ) to (X,Λ).

The following proposion is a locally compactness version of ([Wo06], Proposition B.2.1). Its proof is similar to Wong's. We just outline it here.

Proposition 1.1.1. *Let $(X_i, x_i), (Y_i, y_i), (X, x), (Y, y)$ be pointed metric spaces, where bounded subsets are precompact. $i \in \mathbb{N}$. Suppose $f_i : (X, x) \to (X_i, x_i)$ and $g_i : (Y_i, y_i) \to (Y, y)$ be pointed ε_i-approximations for all $i \in \mathbb{N}$, and there exists L-Lipschitz maps $\varphi_i : X_i \to Y_i$, where L is a fixed number. Then there exists an L-Lipschitz map $\varphi : X \to Y$. If in addition all φ_i are surjective, then φ is surjective.*

Proof. Since any bounded subset in X is precompact, there exists a countable dense subset $\mathfrak{D} \subset X$. Let

$$h_i := g_i \circ \varphi_i \circ f_i : X \to Y.$$

By the definition of pointed ε_i-approximations, for any given $x_\alpha \in \mathfrak{D}$,

$$\lim_{k\to\infty} \sup_{i\geq k} |h_i(x_\alpha)h_i(x)| \leq L|x_\alpha x|.$$

Since any bounded subset in X is precompact, by the diagonal argument, there exists a $y_\alpha \in Y$ such that

$$\lim_{i\to\infty} h_i(x_\alpha) = y_\alpha,$$

for all $\alpha \in \mathbb{N}$ without loss of generality. Then we define a map $\tilde{\varphi} : \mathfrak{D} \to Y$ by $\tilde{\varphi}(x_\alpha) = y_\alpha$. Extending $\tilde{\varphi}$ to X, we get $\varphi : X \to Y$.

□

Remark 1.1.2. Under the notations of this proposition, if X_i is L-bi-Lipschitz homeomorphic to Y_i, then X is L-bi-Lipschitz homeomorphic to Y.

1.2 Alexandrov spaces

In this section, we review some basic notions and results in Alexandrov geometry. We refer to [BBI], [BGP], [Pr94] and [Pr94'] for further details.

Let X be a geodesic metric space, where any two points can be joined by a shortest geodesic. For a fixed real number κ and a geodesic triangle Δpqr in X with vertices p, q and r, denote by $\tilde{\Delta} pqr$ a *comparison triangle* in the simply connected model surface M_κ^2 with constant curvature κ. This means that $\tilde{\Delta} pqr$ has the same side lengths as the corresponding ones in Δpqr. Here we suppose that the perimeter of Δpqr is less than $2\pi/\sqrt{\kappa}$ if $\kappa > 0$. The metric space X is called an *Alexandrov space with curvature* $\geq \kappa$, sometimes Alexandrov space for short if we do not emphasize the lower curvature bound, if each point of X has a neighborhood U satisfying the following: For any geodesic triangle in U with vertices p, q and r and for any point x on the segment qr, we have $|px| \geq |\tilde{p}\tilde{x}|$, where $\tilde{x}$ is the point on $\tilde{q}\tilde{r}$ corresponding to x. From now on we assume that an Alexandrov space is always finite dimensional.

For an Alexandrov space X with curvature bounded below by κ, let $\alpha : [0, s_0] \to X$ and $\beta : [0, t_0] \to X$ be two geodesics starting from a point x. The *angle* between α and β is defined by $\angle(\alpha, \beta) = \lim_{s,t\to 0} \tilde{\angle}\alpha(s)x\beta(t)$, where $\tilde{\angle}\alpha(s)x\beta(t)$ denotes the angle of a comparison triangle $\tilde{\Delta}\alpha(s)x\beta(t)$ at the point $\tilde{x}$. Two geodesics α, β from $x \in X$ is called *equivalent* if $\angle(\alpha, \beta) = 0$. We denoted by $\Sigma'_x(X)$ the set of equivalent classes of geodesics emanating from x. The *space of directions* at x, denoted by $\Sigma_x = \Sigma_x(X)$, is the completion of $\Sigma'_x(X)$ with the angle metric. Let X be n-dimensional. Then Σ_x is an $(n-1)$-dimensional compact Alexandrov space with curvature ≥ 1.

A point $x \in X$ is called *regular* if Σ_x is isometric to $\mathbb{S}^{n-1}$. Otherwise we call x a singular point. We denote by $X^{\rm reg}$ (resp. $X^{\rm sing}$) the set of all regular points (resp. singular points) of X.

The *tangent cone* at $x \in X$, denoted by $T_x(X)$, is the Euclidean cone $K(\Sigma_x)$ over Σ_x. It is known that $T_x(M) = \lim_{r\to 0}\left(\frac{1}{r}M, x\right)$.

For a closed subset A of X and $p \in A$, the space of directions $\Sigma_p(A)$ of A at p is defined as the set of all $\xi \in \Sigma_x(X)$ which can be written as the limit of directions from p to points p_i in A with $|p, p_i| \to 0$. For $x, y \in X \setminus A$, consider a comparison triangle on M^2_κ having the side-lengths $(|A, x|, |x, y|, |y, A|)$ whenever they exist. Then $\tilde\angle Axy$ denotes the angle of this comparison triangle at the vertex corresponding to x.

For $x, y, z \in X$, we denote by $\angle xyz$ (resp. $\tilde\angle xyz$) the angle between the geodesics yx and yz at x (resp. the geodesics $\tilde y\tilde x$ and $\tilde y\tilde z$ at $\tilde x$).

Let X be an n-dimensional Alexandrov space with curvature bounded below by κ. For $\delta > 0$, a system of n pairs of points, $\{a_i, b_i\}_{i=1}^n$ is called an (n,δ)-*strainer* at $x \in X$ if it satisfies

$$\tilde\angle a_i x b_i > \pi - \delta, \quad \tilde\angle a_i x a_j > \pi/2 - \delta,$$
$$\tilde\angle b_i x b_j > \pi/2 - \delta, \quad \tilde\angle a_i x b_j > \pi/2 - \delta,$$

for every $1 \leq i \neq j \leq n$. If $x \in X$ has a (n,δ)-strainer, then we say x is (n,δ)-strained. In this case, we call x δ-*regular*. We call X *almost regular* if every point of X is δ_n-regular for some $\delta_n \leq 1/100n$. It is known that a small neighborhood of any almost regular point is almost isometric to an open subset in $\mathbb{R}^n$.

Inductively on the dimension, the boundary ∂X is defined as the set of points $x \in X$ such that Σ_x has non-empty boundary $\partial\Sigma_x$. We denote by $D(X)$ the double of X, which is also an Alexandrov space with curvature $\geq \kappa$ (see [Pr94]). By definition, $D(X) = X \amalg_{\partial X} X$, where two copies of X are glued along their boundaries.

A boundary point $x \in \partial X$ is called δ-*regular* if x is δ-regular in $D(X)$. We say that X is *almost regular with almost regular boundary* if every point of X is δ-regular for $\delta < 1/100n$.

In Section 4.1, we need the following result on the dimension of the interior singular point sets.

Theorem 1.2.1 ([BGP], cf. [OS94])**.**

$$\dim_H(X^{\rm sing} \cap {\rm int}X) \leq n-2, \ \dim_H(\partial X)^{\rm sing} \leq n-2.$$

Theorem 1.2.2 ([Pr94], cf.[Ka07])**.** *If a sequence X_i of n-dimensional compact Alexandrov spaces with curvature $\geq \kappa$ Gromov-Hausdorff converges to an n-dimensional compact Alexandrov space X, then X_i is homeomorphic to X for large enough i.*

Definition 1.2.3. A n-dimensional metric space X is said to have a *topological stratification* if there exists subsets $X^{(n)}, X^{(n-1)}, X^{(n-2)}, \cdots, X^{(0)}, X^{-1}$ of X such that

(1) $X = X^{(n)} \supset X^{(n-1)} \supset \ldots X^{(0)} \supset X^{(-1)} = \emptyset$

(2) $X^{(k)} \setminus X^{(k-1)}$ is a topological k-manifold without boundary for all $k = 0, 1, 2, \cdots, n$.

A subset E of an Alexandrov space X is called *extremal* ([PP93]) if every distance function $f = \mathrm{dist}_q$, $q \in M \setminus E$ has the property that if $f|_E$ has a local minimum at $p \in E$, then $df_p(\xi) \leq 0$ for every $\xi \in \Sigma_p(E)$. Extremal subsets possess quite important properties.

Theorem 1.2.4 ([PP93])**.** *Let E be an extremal subset of X.*

1. *For every $p \in E$, $\Sigma_p(E)$ is an extremal subset of $\Sigma_p(X)$;*
2. *E is totally quasigeodesic in the sense that any nearby two points of E can be joined by a quasigeodesic (see [PP:pre]).*
3. *E has a topological stratification.*

Theorem 1.2.4(1), (2) implies the following

Corollary 1.2.5. *For an extremal subset E of X and $p \in E$,* $\dim \Sigma_p(E) \leq \dim E - 1$.

Suppose that a compact group G acts on X as isometries. Then the quotient space X/G is an Alexandrov space ([BGP]). Let F denote the set of G-fixed points.

Proposition 1.2.6 ([PP93])**.** *$\pi(F)$ is an extremal subset of X/G, where $\pi : X \to X/G$ is the projection.*

Boundaries of Alexandrov spaces are typical examples of extremal subsets.

Proposition 1.2.7 ([Ya02] Prop 5.10)**.** *The boundary ∂X of any finite dimensional Alexandrov space X has a collar neighborhood.*

Definition 1.2.8 ([Pr94])**.** Let $f = (f_1, ..., f_m) : U \to \mathbb{R}^m$ be a map on an open subset U of a finite dimensional Alexandrov space X defined by $f_i = d(A_i, \cdot)$ for compact subsets $A_i \subset X$. f is said to be (c, ε)-*regular* at $p \in U$ if there is a point $w \in X$ such that:

(1) $\angle((A_i)'_p, (A_j)'_p) > \pi/2 - \varepsilon$.

(2) $\angle(w'_p, (A_i)'_p) > \pi/2 + c$.

Theorem 1.2.9 (Perelman's fibration theorem)**.** *Let X be a finite dimensional Alexandrov space, $U \subset X$ an open subset, and let f be a map as above. Suppose f is proper and is (c, ε)-regular at each point of U. Then f is a local trivial fiber bundle if $\varepsilon \ll c$.*

An n-dimensional Alexandrov space is *smoothable* if and only if it is a Gromov-Hausdorff limit of n-dimensional closed Riemannian manifolds with a lower sectional curvature bound.

Theorem 1.2.10 ([Ka02]). *Let X be a smoothable Alexandrov space. Then for any $p \in X$, every iterated space of directions*

$$\Sigma_{\xi_k}(\Sigma_{k-1}(\cdots(\Sigma_{\xi_1}(\Sigma_p(X))\cdots)),$$

is homeomorphic to a sphere, where

$$\xi_1 \in \Sigma_p(X),\ \xi_2 \in \Sigma_{\xi_1}(X), \dots,\ \xi_k \in \Sigma_{\xi_{k-1}}(\cdots(\Sigma_{\xi_1}(\Sigma_p(X))\cdots)).$$

1.3 Manifolds with boundary and gluing

In this section, we consider a Riemannian manifold M with boundary in $\mathcal{M}(n,\kappa,\lambda,d)$. First, we recall some fundamental properties of ∂M, which were derived by Wong[Wo08]. We also recall Wong's cylindrical extension procedure based on Kosovskii's Gluing theorem [Kos02].

Let M be a Riemannian manifold with boundary, and ∂M^α denote a boundary component of ∂M. $(\partial M^\alpha)^{\text{int}}$ means ∂M^α with intrinsic length metric.

The following is a immediate consequence of the Gauss equation.

Proposition 1.3.1. *For every $M \in \mathcal{M}(n,\kappa,\lambda)$, ∂M has a uniform lower sectional curvature bound: $K_{\partial M} \geq K$, where $K = K(k,\lambda)$.*

Proof. It is sufficient to proof that there exists a constant $K = K(\kappa,\lambda)$ such that, for any point $p \in \partial M$ and any two orthonormal tangent vectors X_p, $Y_p \in T_p\partial M$, we have

$$K_{\partial M}(X_p, Y_p) \geq K.$$

Given arbitrary point $p \in \partial M$, consider the shape operater $S_\nu : T_p\partial M \to T_p\partial M$ defined by $S(x) = -(\nabla_x \nu)^T$, where $\nu \in (T_p\partial M)^\perp$ is inward normal. Suppose $\dim M = n$, $\{e_1, \dots, e_{n-1}\}$ is an orthonormal basis of $T_p\partial M$ such that $S(e_i) = \lambda_i e_i$, $i = 1, \dots, n-1$, where $\lambda_1, \dots, \lambda_{n-1}$ are the eigenvalues of S. From now on, we use Einstein sum convention. Let $X = x^i e_i$, $Y = y^j e_j$ be two orthonormal tangent vectors in $T_p\partial M$. Clearly, $\Sigma_{i=1}^{n-1}|x_i|^2 = 1$ and $\Sigma_{i=1}^{n-1}|y_j| = 1$. Then, by Gauss's Formula, we have

$$\begin{aligned} K_{\partial M}(X,Y) &= K_M(X,Y) + \langle S(X), X\rangle\langle S(Y), Y\rangle - (\langle S(X), Y\rangle)^2 \\ &= K_M(X,Y) + (x^i y^j)^2 \lambda_i \lambda_j - (x^i y^i \lambda_i)^2 \\ &\geq k - (x^i y^j)^2 \lambda^2 - (x^i y^j)^2 \lambda^2 \\ &= k - 2\lambda^2 \end{aligned}$$

□

Proposition 1.3.2 ([Wo08]). *Let $M \in \mathcal{M}(n, \kappa, \lambda, d)$.*

1. *There exists a constant $D = D(n, \kappa, \lambda, d)$ such that any boundary component ∂M^α has intrinsic diameter bound*

$$\operatorname{diam}\left((\partial M^\alpha)^{\text{int}}\right) \leq D;$$

2. *∂M has at most J components, where $J = J(n, \kappa, \lambda, d)$;*

It follows from Propositions 1.3.1, 1.3.2 that every boundary component of $M \in \mathcal{M}(n, \kappa, \lambda, d)$ is an Alexandrov space with curvature $\geq c$ and diameter $\leq D$, where $c = c(\kappa, \lambda)$, $D = D(n, \kappa, \lambda, d)$

In general, a Riemannian manifold with boundary is not necessarily an Alexandrov space. Wong ([Wo08]) carried out a gluing of warped cylinders and M along their boundaries in such a way that the resulting manifold becomes an Alexandrov space having totally geodesic boundary. Wong's method is based on Kosovskii's gluing theorem:

Theorem 1.3.3 ([Kos02]). *Let M_0 and M_1 be Riemannian manifolds with boundaries Γ_0 and Γ_1 respectively with sectional curvature $K_{M_i} \geq \kappa$ for $i = 0, 1$. Assume that there exists an isometry $\phi : \Gamma_0 \to \Gamma_1$, and let M denote the space with length metric obtained by gluing M_0 and M_1 along their boundaries via ϕ. Let L_i, $i = 0, 1$, be the second fundamental form of $\Gamma := \Gamma_0 \cong_\phi \Gamma_1 \subset M$ with respect to the normal inward to M_i. Then M is an Alexandrov space with curvature $\geq \kappa$ if and only if the sum $L := L_1 + L_2$ is positive semidefinite.*

Idea of the proof Roughly speaking, Kosovskii's idea of the proof is as follow, for every $\delta > 0$, construct a Riemmanian metric $\langle \cdot . \cdot \rangle_\delta$ on M, the resulting space is denoted by M_δ, such that the curvature of M_δ is bounded below by $\kappa(\delta)$ and Gromov-Hausdorff converge to M, where $\kappa(\delta) \to \kappa$ as $\delta \to 0$.

In order to introduce Kosovskii's idea of gluing two Riemannian manifolds with boundary in detail, we present some preliminaries here. For further details, see [Kos02] .

Recall that L is the sum of the the second fundamental form on the hypersurface $\Gamma \subset M$ relative to the inward normals corresponding to M_0 and M_1, respectively. $L \geq 0$ by assumption. Let $\mathbf{L}$ denote the selfadjoint operator corresponding to the quadratic form L. Using the parallel translation, one can show that in a small neighborhood of F, the selfadoint operator $\mathbf{L}$ extends to TM_0 so that $\mathbf{L}N = 0$ and $\nabla_N \mathbf{L} = 0$, where N is a local vector field Γ_0 in M_0 orthogonal to the hypersurface equidistant to Γ and ∇ is the Levi-Civita connection on M_0 corresponding to the metric $\langle \cdot, \cdot \rangle_0$. Let

$$\mathbf{I}^{n-1} : TM|_{\Gamma(d)} \to T\Gamma(d)$$

be the projection operator onto the tangent space of the hypersurface $\Gamma(d)$, where $\Gamma(d)$ denotes the hypersurface in M_0 which has a small distance d away from Γ.

Other than these notations, we need some auxiliary functions to define a new metric on M_0. First, we define a C^∞ function $f_\delta : [0,\infty) \to \mathbb{R}$ such that

$$\begin{aligned} f_\delta(x) = 1 - \frac{x}{\delta^4} & \quad \text{if } x \in [0, \delta^4], \\ -\delta \le f_\delta(x) \le 0 \text{ and } f'_\delta(x) \le \delta & \quad \text{if } x \in [\delta^4, \delta], \\ f_\delta(x) = 0 & \quad \text{if } x \in [\delta, +\infty], \\ \int_0^\delta f_\delta(x)dx = 0. \end{aligned}$$

Clearly, such a function f_δ exists for all small $\delta > 0$. Next we define

$$F_\delta(x) = \int_0^x f_\delta(t)dt, \mathcal{F}_\delta(x) = \int_0^x F_\delta(t)dt.$$

Then we can define a selfadjoint operator which plays a crucial role in the new metric on M_0

$$\mathbf{G}_\delta : \mathbf{I}_p + 2F_\delta(|p\Gamma|)\mathbf{L}_p - 2C\mathcal{F}_\delta(|p\Gamma|)\mathbf{I}_p^{n-1}$$

where $\mathbf{I}$ is the identity map and C is a large constant depend on the manifold M. Then we define a new metric on M_0 by

$$\langle \cdot, \cdot \rangle_\delta := \langle \cdot, \mathbf{G}_\delta \cdot \rangle_0.$$

Notice that $g_\delta \to g_0$ as $\delta \to 0$. Then we can prove that there exists a Riemannian metric $g_{(\delta)}$ with $W^{2,\infty}_{\text{loc}}$ coefficients on M, which coinsides with g_1 on M_1 and with g_δ on M_0. Since the coefficients of $g_{(\delta)}$ is not necessarily C^2, we can not talk about its sectional curvature. However, we can approximate the coefficients of $g_{(\delta)}$ by the Sobolev averging $g^h_{(\delta)}$ defined as follow

Definition 1.3.4. Sobolev averging of the Riemmannian metric $g_{ij(\delta)}$ on domain Ω is defined by

$$g^h_{ij(\delta)}(x) = \frac{1}{h}\int_\Omega g_{ij(\delta)}(u)\omega\Big(\frac{x-u}{h}\Big)du,$$

where $\omega : \mathbb{R}^n \to \mathbb{R}$ is the averaging kernel, which is a function satisfies the following properties,

- the support of ω is contained in the unit ball $B_0(1)$,
- the function ω is C^∞,
- $\int_{\mathbb{R}}^n \omega(x)dx = 1$.

Clearly, $g^h_{(\delta)} \to g_{(\delta)}$ as $h \to 0$. Suppose $(M, g^h_{(\delta)})$ has a lower sectional curvature bound $\kappa^h_{(\delta)}$. By some complicate computations, we can show that $\lim_{\delta\to 0, h\to 0} \kappa^h_{(\delta)} \geq \kappa$ and $\lim_{h\to 0} \kappa^h_{(\delta)} \to \kappa_{(\delta)}$. Clearly, $(M, g^h_{(\delta)}) \xrightarrow{GH} (M, g_{(\delta)})$ as $h \to 0$. Thus all $(M, g_{(\delta)})$ are Alexandrov space with uniform lower curvature bound. Clearly, $(M, g_{(\delta)}) \xrightarrow{GH} M$. Therefore M with length metric is an Alexandrov space with a lower curvature bound κ.

Next we recall the extension construction in [Wo08].

Suppose M is an n-dimensional compact Riemannian manifold with

$$K_M \geq \kappa,\ \lambda^- \leq II_{\partial M} \leq \lambda^+.$$

Let $\overline{\lambda} := \min\{0, \lambda^-\}$. Then for arbitrarily $t_0 > 0$ and $0 < \varepsilon_0 < 1$ there exists a monotone non-increasing function $\phi : [0, t_0] \to \mathbb{R}^+$ satisfying

$$\begin{gathered}\phi''(t) + K\phi(t) \leq 0,\ \phi(0) = 1,\ \phi(t_0) = \varepsilon_0, \\ -\infty < \phi'(0) \leq \overline{\lambda},\ \phi'(t_0) = 0,\end{gathered}$$

for some constant $K = K(\lambda, \varepsilon_0, t_0)$. Now consider the warped product metric on $\partial M \times [0, t_0]$ defined by

$$g(x, t) = dt^2 + \phi^2(t) g_{\partial M}(x)$$

where $g_{\partial M}$ is the Riemannian metric of ∂M induced from that of M. We denote by $\partial M \times_\phi [0, t_0]$ the warped product. It follows from the construction that

- $II_{\partial M\times\{0\}} \geq |\min\{0, \lambda^-\}|$;
- $II_{\partial M\times\{t_0\}} \equiv 0$;
- the sectional curvature of $\partial M \times_\phi [0, t_0]$ is greater than a constant $c(\kappa, \lambda^\pm, \varepsilon_0, t_0)$.

Clearly, $\partial M \times \{0\}$ in $\partial M \times_\phi [0, t_0]$ is canonically isometric to ∂M. Thus we can glue M and $\partial M \times_\phi [0, t_0]$ together along ∂M and $\partial M \times \{0\}$. The resulting space

$$\tilde{M} := M \amalg_{\partial M} (\partial M \times_\phi [0, t_0])$$

carries the structure of differentiable manifold of class $C^{1,\alpha}$ with C^0-Riemannian metric ([Kos02]). Obviously M is diffeomorphic to $\tilde{M}$.

Proposition 1.3.5 ([Wo08])**.** *For $M \in \mathcal{M}(n, \kappa, \lambda, d)$, we have*

1. *$\tilde{M}$ is an Alexandrov space with curvature $\geq \tilde{\kappa}$ and with diameter $\leq \tilde{d}$, where $\tilde{\kappa} = \tilde{\kappa}(\kappa, \lambda)$ and $\tilde{d} = d + 2t_0$;*
2. *the exterior metric M^{ext} is L-bi-Lipschitz homeomorphic to M for the uniform constant $L = 1/\varepsilon_0$.*

The notion of warped product also works for metric spaces.

Let X and Y be metric spaces, and $\phi : Y \to \mathbb{R}_+$ a positive continuous function. Then the warped product $X \times_\phi Y$ is defined as follows (see [Wo06]). For a curve $\gamma = (\sigma, \nu) : [a, b] \to X \times Y$, the length of γ is defined as

$$L_\phi(\gamma) = \sup_{|\Delta|\to 0} \sum_{i=1}^{k} \sqrt{\phi^2(\nu(s_i))|\sigma(t_{i-1}), \sigma(t_i)|^2 + |\nu(t_{i-1}), \nu(t_i)|^2},$$

where $\Delta : a = t_0 < t_1 < \cdots < t_k = b$ is a division of $[a, b]$ and s_i is any element of $[t_{i-1}, t_i]$. The warped product $X \times_\phi Y$ is defined as the topological space $X \times Y$ equipped with the length metric induced from L_ϕ.

Proposition 1.3.6 ([Wo06], Proposition B.2.6)**.** *Let X_i be a convergent sequence of length spaces. If Y is a compact length space, we have*

$$\lim_{GH}(X_i \times_\phi Y) = (\lim_{GH} X_i) \times_\phi Y.$$

whenever the limits exist.

Chapter 2

Descriptions of limit spaces

Under the notations in section 1.3, throughout this chapter, unless otherwise stated, we assume $M_i \in \mathcal{M}(n, k, \lambda, d)$ inradius collapses to a compact length space N. Let $\tilde{M}_i$ converge to a compact Alexandrov space Y, and M_i^{ext} converge to a compact subset X of Y under the convergence $\tilde{M}_i \to Y$.

In this chapter, we first study the topological structure of Y and show that Y possesses a singular I-bundle structure C/η_0 over N. (Proposition 2.1.3). We then discuss the intrinsic structure of X and prove that X^{int} is isometric to N (Proposition 2.2.2). All results in this chapter are due to a joint work with Prof. Takao Yamaguchi [YZ15].

Here we fix some notations used later on.

- C_i denotes $\partial M_i \times_\phi [0, t_0]$;
- C_{it} denotes the subspace $\partial M_i \times_\phi \{t\}$ in C_i;
- For $C_i \subset \tilde{M}_i$, C_i^{ext} denotes $(C_i, d_{\tilde{M}_i})$.

2.1 Topological structure of Y

In this section, we will prove that Y is homeomorphic to a singular-I-bundle (Proposition 2.1.3).

Under the notation presented in the begining of this section, in view of Proposition 1.3.2, passing to a subsequence, we may assume that C_i converges to some compact Alexandrov space C with cuvrvature $\geq K = K(\kappa, \lambda)$. Here C_i is not necessarily connected, and therefore the convergence $C_i \to C$ should be understood componentwisely. Note that

$$C = C_0 \times_\phi [0, t_0], \ C_0 = \lim_{i\to\infty} (\partial M_i)^{\mathrm{int}},$$

where $(\partial M_i)^{\rm int}$ denotes ∂M_i endowed with length metric induced by its original metric. Usually for simplicity we denote

$$C_0 := C_0 \times \{0\},\ C_t := C_0 \times \{t\}.$$

Since the identity map $\iota_i : C_i \to C_i^{\rm ext}$ is 1-Lipschitz, we can define a 1-Lipschitz map $\eta : C \to Y$ in the limits. More precisely, define $\eta : C \to Y$ by

$$\eta = \lim_{i\to\infty} g_i \circ \iota_i \circ f_i,$$

where $f_i : C \to C_i$, $g_i : \tilde{M}_i \to Y$ are component-wise ε_i-approximations with $\lim_{i\to\infty} \varepsilon_i = 0$. We will prove that η is a quotient map.

From now on, $\eta|_{C_0\times\{0\}}$ is denoted by η_0.

The following two lemmas are obvious.

Lemma 2.1.1. *The map $\eta : C \setminus C_0 \to Y \setminus X$ is a componentwise isometry.*

Lemma 2.1.2. *For $(p,t) \in C \setminus C_0$, we have $|\eta(p,t), X| = t$.*

Let C/η_0 denote the quotient space $C/p \sim q$, where $p \sim q$ if and only if $\eta_0(p) = \eta_0(q)$ for $p, q \in C_0$. By the above results, we obtain a singular-I-bundle structure on Y as follows.

Proposition 2.1.3. *Y and X are homeomorphic to the quotient spaces C/η_0 and C_0/η_0 respectively.*

Proof. Since C and C_0 are compact, it suffices to prove the following:

(a) $\eta : C \to Y$ is surjective,

(b) $\eta(C_0) = X$,

(c) $\eta : C \setminus C_0 \to Y \setminus X$ is injective.

Since $C_i^{\rm ext}$ is ϵ_i-dense in $\tilde{M}_i$ with $\lim_{i\to\infty} \varepsilon_i \to 0$, surjective 1-Lipschitz map $\iota_i : C_i \to C_i^{\rm ext}$ converges to the surjective 1-Lipschitz map $\eta : C \to Y$. This shows (a).

Since $(\partial M_i)^{\rm ext}$ is ϵ_i-dense in M_i with $\lim_{i\to\infty} \varepsilon_i \to 0$, surjective 1-Lipschitz map $(\partial M_i)^{\rm int} \to (\partial M_i)^{\rm ext}$ converges to the surjective 1-Lipschitz map $\eta_0 : C_0 \to X$. This shows (b).

Note that C is simply covered by the minimal geodesics from the points of C_{t_0} to C_0. Since η is injective on C_{t_0} and since η carries those minimal geodesics to minimal geodesics from $\eta(C_{t_0})$ to X, non-branching properties of geodesics in Y implies the injectivity of $\eta : C \setminus C_0 \to Y \setminus X$. This shows (c). $\square$

Corollary 2.1.4. *If M_i inradius collpases to a compact length space N, then it actually collapses to N. Namely we have* $\dim M_i > \dim N$.

Proof. From Propositions 1.3.5, 2.1.3 and Lemma 2.2.1, we have

$$\begin{aligned}\dim M_i = \dim \tilde{M}_i &\geq \dim Y \\ &\geq \dim X + 1 = \dim N + 1.\end{aligned}$$

□

Remark 2.1.5. Wong proved $\dim M_i > \dim N$ in ([Wo10], Lemma 1) under the condition that N is an absolute Poincaré duality space. By Proposition 6.0.19, we shall show that if N is a closed topological manifold or a closed Alexandrov space, then M_i inradius collapses. Hence Corollary 2.1.4 give another version of Wong's result.

Recall that Y is homeomorphic to the identification space C/η_0. We now study the multiplicities of the gluing map η_0.

Lemma 2.1.6. $\#\eta_0^{-1}(x) \leq 2$ *for every* $x \in X$.

Proof. Suppose that $\#\eta_0^{-1}(x) \geq 2$. Take $p_1, p_2 \in \eta_0^{-1}(x)$, and let $y_i := \eta(p_i, t)$, $i = 1, 2$, for some $t > 0$. We show that $|y_i, y_2| = 2t$ or equivalently, $\tilde{\angle} y_1 p y_2 = \pi$ for $t < \phi(t_0)|p_1 p_2|_{C_0}/2$.

Let γ be a minimal geodesic in Y joining y_1 and y_2 . If p_1 and p_2 are contained in distinct connected components of C_0, γ must meet X, and therefore $|y_i, y_2| = 2t$. Suppose that p_1 and p_2 are contained in the same connected component of C_0. Let $\tilde{\gamma} = (\sigma, \lambda) : [0, 1] \to C$ be a shortest path in C such that $\tilde{\gamma}(0) = (p_1, t)$ and $\tilde{\gamma}(1) = (p_2, t)$. Then we have

$$\begin{aligned}L(\gamma) = L(\tilde{\gamma}) &= \int_0^1 \sqrt{\phi^2(\nu(t))|\dot{\sigma}(t)|^2 + |\dot{\nu}(t)|^2}\, dt. \\ &= \int_0^1 \phi(t_0)|\dot{\sigma}(t)|\, dt \geq \phi(t_0)|p_1, p_2|_{C_0}.\end{aligned}$$

Thus we have

$$|(p_1, t), (p_2, t)|_C \geq \phi(t_0)|p_1 p_2|_{C_0}.$$

If γ does not meet X, the geodesic $\tilde{\gamma} = \eta^{-1} \circ \gamma$ joining (p_1, t) and (p_2, t) in C has the length $L(\gamma) \leq 2t < \phi(t_0)|p_1 p_2|_{C_0}$. This is a contradiction, and therefore γ meets X and $|y_i, y_2| = 2t$ □

Definition 2.1.7. In view of Lemma 2.1.6 and Proposition 2.2.2, we make an identification $N = X^{\text{int}}$ and set for $i = 1, 2$,

$$\begin{aligned}N_i = X_i &:= \{x \in X | \#\eta_0^{-1}(x) = i\}, \\ C_0^i &:= \{p \in C_0 \,|\, \eta_0(p) \in X_i\}.\end{aligned}$$

Next we construct a good approximation map $\tilde{M}_i \to Y$, which helps us to grasp a whole picture on the several convergences.

Let $\psi_i : \partial M_i = C_{i0} \to C_0$ be an ϵ_i-approximation with $\lim_{i\to\infty} \epsilon_i = 0$.

Lemma 2.1.8 ([Wo06])**.** *The map $\Psi_i : C_i \to C$ defined by*

$$\Psi_i(p,t) = (\psi_i(p), t)$$

is an ϵ'_i-approximation with $\lim_{i\to\infty} \epsilon'_i = 0$. Actually, for any approximation map $\Psi'_i : C_i \to C$ there is a $\psi_i : \partial M_i = C_{i0} \to C_0$ such that $|\Psi_i(p,t), \Psi'_i(p,t)| < \epsilon'_i$ for $\Psi_i = (\psi_i, \mathrm{id})$.

Proof. This follows from Proposition 1.3.6. □

Recall that $\eta : C \setminus C_0 \to Y \setminus X$ is a componentwise isometric bijection. In particular for every $y = (p, t_0) \in C_{t_0} \subset Y$, there is a unique minimal geodesic $\gamma_y : [0, t_0] \to Y$ between X and y such that $\gamma_y(0) \in X$, $\gamma(t_0) = y$. Actually γ_y is defined as $\gamma_y(t) = \eta(p,t)$. Define $g_i^* : C_i^{\mathrm{ext}} \to Y$ by

$$g_i^*(p,t) = \eta \circ \Psi_i \circ \iota_i^{-1}(p,t) = \eta(\psi(p), t). \tag{2.1}$$

Proposition 2.1.9. *The map $g_i^* : C_i^{\mathrm{ext}} \to Y$ defined above provides an ϵ'_i-approximation.*

Let $g_i : C_i^{\mathrm{ext}} \to Y$ be any ϵ_i-approximation such that $g_i = g_i^*$ on C_{it_0}, namely $g_i(p,t_0) = g_i^*(p,t_0)$.

For the proof of Proposition 2.1.9, it suffices to show the following.

Lemma 2.1.10. $|g_i(p,t), g_i^*(p,t)| < \epsilon'_i$ *for all* $(p,t) \in C_i^{\mathrm{ext}}$.

Proof. We have to show that

$$\lim_{i\to\infty} \sup_{(p,t)\in C_i} |g_i(p,t), g_i^*(p,t)| = 0.$$

Suppose the contrary. Then there are subsequence $\{j\} \subset \{i\}$ and $(p_j, t_j) \in C_j$ such that

$$|g_j(p_j,t_j), g_j^*(p_j,t_j)| \geq c > 0, \tag{2.2}$$

for some constant c independent of j. Passing to a subsequence, we may assume that $(\psi_j(p_j), t_j)$ converges to $(p_\infty, t_\infty) \in C$. Let $\gamma_j(t) = (p_j, t)$, $0 \leq t \leq t_0$, which is a minimal geodesic in C_j^{ext} between ∂M_j and C_{jt_0}. Now $g_j^* \circ \gamma_j(t) = \eta(\psi_j(p_j), t)$ converges to a minimal geodesic $\gamma_\infty(t) = \eta(p_\infty, t)$ realizing the distance between X and $(p_\infty, t_0) \in C_{t_0} \subset Y$.

Since g_j is ϵ_j-approximation, any limit of $g_j \circ \gamma_j$, say $\hat{\gamma}$, must also be a minimal geodesic between X and (p_∞, t_0). From the uniqueness of such geodesic, we have $\gamma_\infty(t) = \hat{\gamma}_\infty(t)$, which contradicts (2.2). □

2.2 Intrinsic structure of X

In this section, we determine the intrinsic structure of X, and prove Proposition 2.2.2 below, which will be crucial in classification of Y in terms of N.

Let $X \subset Y$ be the limit of a subsequence of M_i^{ext} under the convergence $\tilde{M}_i \to Y$. Since $d_{M_i} \geq d_{M_i^{\text{ext}}}$ and M_i is L-bi-Lipschitz homeomorphic to M_i^{ext} for a uniform constant L, we have

The identity $\iota_i : M_i \to M_i^{\text{ext}}$ is a L-bi-Lipschitz homeomorphism. Therefore, by Proposition 1.1.1, passing to a subsequnce, we have that

Lemma 2.2.1. $\iota_i : M_i^{\text{ext}} \to M_i$ *converges to an L-bi-Lipschitz homeomorphism $h : X \to N$.*

Let $N \cup_{h\circ\eta_0} C_0 \times_\phi [0, t_0]$ denote the length space obtained by gluing two length spaces N and $C_0 \times_\phi [0, t_0]$ by the map $h \circ \eta_0 : C_0 \times 0 \to N$. Then we have

Proposition 2.2.2. *Y is isometric to the length space*

$$N \cup_{h\circ\eta_0} C_0 \times_\phi [0, t_0].$$

In particular, X^{int} is isometric to N.

Proof. Let $Z = N \cup_{h\circ\eta_0} C_0 \times_\phi [0, t_0]$ be a length space. Clearly, $(Z \setminus N)^{\text{int}}$ is isometric to $C_0 \times_\phi [0, t_0] \setminus C_0$. Define a map $\Phi : Y \to Z$ by $\Phi(y) = y$ if $y \in Z \setminus N$ and $\Phi(y) = h(y)$ if $y \in X$. Then Φ is a homeomorphism. It suffice to prove that both Φ and Φ^{-1} are 1-Lipschitz maps.

Claim (1). Φ is a 1-Lipschitz map.

Given two points y_1, $y_2 \in Y$, we shall prove that $|\Phi(y_1)\Phi(y_2)|_Z \leq |y_1y_2|_Y$. Let γ be a geodesic in Y joining y_1 and y_2. Clearly, γ consists of subgeodesics intersect X only on their two end points and ones completely contained in X. Since $Y \setminus X$ is componentwise isometric to $C \setminus C_0$, if a segment σ' in Y intersects X only on its two end point, then $|\sigma'|_Y = |\Phi(\sigma')|_Z$. Suppose a segment $\sigma : [0, 1] \to Y$ of γ is completely contained in X. Let $\sigma_i : [0, 1] \to \tilde{M}_i$ be a shortest path in $\tilde{M}_i$ such that $\sigma_i \xrightarrow{GH} \sigma$ under $\tilde{M}_i \xrightarrow{GH} Y$ and $\sigma_i(0)$ and $\sigma_i(1)$ are contained in ∂M_i. Clearly, $\sigma_i = (\bigcup_{j=1}^{n_i} \sigma_{ij}) \cup (\bigcup_{j'=1}^{m_i} \alpha_{ij'})$, where $\sigma_{ij} \subset M_i \subset \tilde{M}_i$ and $\alpha_{ij'} \subset C_i \subset \tilde{M}_i$ are subsegments of σ_i for all $i, j, j' \in \mathbb{N}$ and n_i and m_i are fixed numbers (maybe infinity) depending on i. Clearly $|\sigma_{ij}|_{\tilde{M}_i} = |\sigma_{ij}|_{M_i}$ and $|\alpha_{ij'}|_{\tilde{M}_i} = |\alpha_{ij'}|_{C_i}$. By the definition of warped product,

$$|\alpha_{ij'}|_{\tilde{M}_i} = |\alpha_{ij'}|_{C_i} \geq \frac{\phi(t_i)}{\phi(0)} |P_1^i \circ \alpha_{ij'}|_{\partial M_i} \geq \frac{\phi(t_i)}{\phi(0)} |P_1^i \circ \alpha_{ij'}(0) P_1^i \circ \alpha_{ij'}(1)|_{\partial M_i},$$

where $P_1^i : C_i^{\text{ext}} \to \partial M_i$ is the projection of the points in C_i^{ext} to the first component, and ϕ is a strictly decreasing function in $M_i \times_\phi [0, t_0]$ satisfies $\phi(0) = 1$, and

$$t_i = \max\{|\alpha_{ij'}(t)\partial M_i|_{\tilde{M}_i} | t \in \alpha_{ij'}, j' \leq m_i\}.$$

$t_i \to 0$ since both $\sigma_i \xrightarrow{GH} \sigma$ and $\partial M_i^{\rm ext} \xrightarrow{GH} X$ under $\tilde{M}_i \xrightarrow{GH} Y$. Hence

$$\begin{aligned}
|\sigma(0)\sigma(1)|_Y &= |\sigma(0)\sigma(1)|_X = L(\sigma) \\
&= \lim_{i\to\infty} \Sigma_{j=1}^{n_i} |\sigma_{ij}|_{\tilde{M}_i} + \lim_{i\to\infty} \Sigma_{j'=1}^{m_i} |\alpha_{ij'}|_{\tilde{M}_i} \\
&= \lim_{i\to\infty} \Sigma_{j=1}^{n_i} |\sigma_{ij}|_{\tilde{M}_i} + \lim_{i\to\infty} \Sigma_{j'=1}^{m_i} |\alpha_{ij'}|_{C_i} \\
&\ge \lim_{i\to\infty} \Sigma_j |\sigma_{ij}|_{M_i} + \lim_{i\to\infty} \Sigma_{j'} \frac{\phi(t_i)}{\phi(0)} |P_1^i \circ \alpha_{ij'}|_{\partial M_i} \\
&\ge \lim_{i\to\infty} \frac{\phi(t_i)}{\phi(0)} |\sigma_i(0)\sigma_i(1)|_{M_i} \\
&= |h(\sigma(0))h(\sigma(1))|_N \ge |\Phi(\sigma(0))\Phi(\sigma(1))|_Z,
\end{aligned}$$

Therefore, $|\Phi(y_1)\Phi(y_2)|_Z \le |y_1 y_2|_Y$.

Claim (2). $\Phi^{-1} : Z \to Y$ is also 1-Lipschitz.

For any two points z_1, $z_2 \in Z$, there exists a shortest path γ joining z_1 and z_2. γ consists of the subsegments with interior contain in $Y \setminus X \subset Z$ and the subsegments completely contained in $N \subset Z$. Let γ_1 be the shortest path in Z intersects N at most at its two end points. Since $Z \setminus N$ is locally isometric to $Y \setminus X$, γ_1 in $Y \setminus X$ has the same length as in $Z \setminus N$. Moreover, for the shortest path γ_2 completely contained in N, $|\gamma_2|_N \ge |h^{-1} \circ \gamma_2|_X$ since $d_{M_i} \ge d_{M_i^{\rm ext}}$. Hence

$$|\Phi^{-1}(\gamma_2(0))\Phi^{-1}(\gamma_2(1))|_Y = |h^{-1} \circ \gamma_2|_X \le |\gamma_2(0)\gamma_2(1)|_N = |\gamma_2(0)\gamma_2(1)|_Z.$$

Therefore, $|\Phi^{-1}(z_1)\Phi^{-1}(z_2)|_Y \le |z_1 z_2|_Z$.

In the course of the proof above, it is easy to see that $X^{\rm int}$ is isometric to N. □

2.3 Examples

We exhibit some examples of collapse of manifolds with boundary. All the examples except Example 2.3.6 and 2.3.7 are inradius collapses.

Example 2.3.1. Let

$$\mathbb{S}^{n-1}(r) := \{\, x \in \mathbb{R}^n \mid \sum_{i=1}^{n} (x_i)^2 = r^2 \,\}.$$

For $\epsilon > 0$, define M_ϵ as the closed domain in $\mathbb{R}^n$ bounded by $\mathbb{S}^{n-1}(r+\epsilon)$ and $\mathbb{S}^{n-1}(r)$. Then $K_{M_\epsilon} \equiv 0$ and $|\Pi_{\partial M_\epsilon}| \le 1/r$, and M_ϵ inradius collapses to $N := \mathbb{S}^{n-1}(r)$, where the limit space is an Alexandrov space with curvature $\ge r^{-2}$. Note that $N_2 = N$, and that the limit Y of $\tilde{M}_\epsilon$ is isometric to the form

$$Y = (\mathbb{S}^{n-1}(r) \amalg \mathbb{S}^{n-1}(r)) \times_\phi [0, t_0]/(f(x), 0) \sim (x, 0),$$

where $f : \mathbb{S}^{n-1}(r) \amalg \mathbb{S}^{n-1}(r) \to \mathbb{S}^{n-1}(r) \amalg \mathbb{S}^{n-1}(r)$ is the canonical involution. Equivalently Y is isometric to the warped product

$$\mathbb{S}^{n-1}(r) \times_{\tilde{\phi}} [-t_0, t_0],$$

where $\tilde{\phi}(t) = \phi(|t|)$.

This example shows that the lower Alexandrov curvature bound of the limit in Theorem 0.2.1 really depends on the bound $\lambda \geq |\Pi_{\partial M}|$.

Example 2.3.2 ([Wo10])**.** Let $N \subset \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$ be a non-convex domain with smooth boundary, and let M'_ϵ denote the closure of ϵ-neighborhood of N in $\mathbb{R}^3$. After a slight smoothing of M'_ϵ, we obtain a flat Riemannian manifold M_ϵ with boundary such that $\Pi_{\partial M_\epsilon} \geq -\lambda$ for some $\lambda > 0$ independent of ϵ. Note that M_ϵ inradius collapses to N, where N has no lower Alexandrov curvature bound.

This example shows that Theorem 0.2.1 does not hold if one drops the upper bound $\lambda \geq \Pi_{\partial M}$.

Example 2.3.3. Let $\pi : P \to N$ be a Riemannian double covering between closed Riemannian manifolds with the deck transformation $\varphi : P \to P$. Define

$$\Phi : P \times [-\epsilon, \epsilon] \to P \times [-\epsilon, \epsilon]$$

by

$$\Phi(x, t) = (\varphi(x), -t),$$

and consider $M_\epsilon := P \times [-\epsilon, \epsilon]/\Phi$, which is a twisted I-bundle over N. Note that $M_\epsilon \in \mathcal{M}(n, \kappa, 0, d)$ for some κ and d, and that M_ϵ inradius collapses to N as $\epsilon \to 0$. In this case, we have $N_2 = N$. Note that the limit Y of $\tilde{M}_\epsilon$ is isometric to the form

$$Y = P \times_\phi [0, t_0]/(\varphi(x), 0) \sim (x, 0),$$

or equivalently Y is doubly covered by the warped product

$$P \times_{\tilde{\phi}} [-t_0, t_0].$$

A typical example of this example is the shrinking of a cylinder and a Mobius strip.

Example 2.3.4. Let N be a convex domain in $\mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^{n+1}$ with smooth boundary. Let M'_ϵ denote the intersection of the boundary of ϵ-neighborhood of N in $\mathbb{R}^{n+1}$ with the upper half space $H_+ = \{(x_1, \ldots, x_{n+1}) \,|\, x_{n+1} \geq 0\}$. After a slight smoothing of M'_ϵ, we obtain a nonnegatively curved Riemannian manifold M_ϵ with totally geodesic boundary. Note that M_ϵ inradius collapses to N as $\epsilon \to 0$. Note also that $(\partial M_\epsilon)^{\rm int}$, a smooth approximation of the boundary of ϵ-neighborhood of N in $\mathbb{R}^n$, converges to the double

$D(N)$ of N. It follows that $N_1 = \partial N$ and $N_2 = N \setminus \partial N$, and that the limit Y of $\tilde{M}_\epsilon$ is isometric to the form

$$Y = D(N) \times_\phi [0, t_0]/(r(x), 0) \sim (x, 0),$$

where $r : D(N) \to D(N)$ denotes the canonical reflection of $D(N)$.

This example implies that the case and both N_1 and N_2 are not empty may happen.

Next let us consider more general examples. The following ones come from Example 1.2 in [Ya91], where general examples of collapse of closed manifolds were given.

Example 2.3.5. Let $\hat{\pi} : M \to N$ be a fiber bundle over a closed manifold N with fiber F having non-empty boundary and with the structure group G such that

1. G is a compact Lie group;

2. F has a G-invariant metric g_F of nonnegative curvature which smoothly extends to the double $D(F)$;

Fix a bi-invariant metric b on G and a metric h on N. Let $\pi : P \to N$ be the principal G-bundle associated with $\hat{\pi} : M \to N$. Define G-invariant metric g_ϵ on P by

$$g_\epsilon(u, v) = h(d\pi(u), d\pi(v)) + \epsilon^2 b(\omega(u), \omega(v)),$$

where ω is a G-connection on P. Define a metric $\tilde{g}_\epsilon$ on $P \times D(F)$ as

$$\tilde{g}_\epsilon = g_\epsilon + \epsilon^2 g_F.$$

For the G-action on $P \times D(F)$ defined by

$$(p, f) \cdot g = (pg, g^{-1} f),$$

$\tilde{g}_\epsilon$ is G-invariant and invariant under the action of reflection of $D(F)$. Therefore it induces a metric $g_{D(M),\epsilon}$ on $D(M) = P \times D(F)/G$. Since $g_{D(M),\epsilon}$ is invariant under the action of reflection of $D(M)$, it induces a metric $g_{M,\epsilon}$ on M with totally geodesic boundary such that $(M, g_{M,\epsilon})$ inradius collapses to (N, h) under a lower sectional curvature bound.

Example 2.3.6. Let M be a compact manifold with boundary, and suppose that a compact Lie group of positive dimension effectively act on M which extends to the action on $D(M)$. Suppose that $D(M)$ has G-invariant and reflection-invariant smooth metric g. As in Example 1.2 of [Ya91], one can construct a metric $g_{D(M),\epsilon}$ on $(D(M)$ which collapses to $(D(M), g_{D(M),\epsilon})/G$ under a lower curvature bound. It follows that M with metric induced by $g_{D(M),\epsilon}$ also collapses to $(M, g_{M,\epsilon})/G$ under a lower curvature bound. Note that $(M, g_{M,\epsilon})$ has totally geodesic boundary.

A typical example of this example is as follows.

Example 2.3.7 (Manifolds collapse but their *inradii* do not converge to zero)**.** Let

$$I' = [-1, 1] \times \{0\} \times \{0\}, I = [0, 1] \times \{0\} \times \{0\}$$

in $\mathbb{R}^3$, $\partial B(\varepsilon_i)$ is the boundary of the ε_i-neighborhood of I' in $\mathbb{R}^3$. Let

$$\partial^+ B(\varepsilon_i) = \partial B(\varepsilon_i) \cap ([0, +\infty) \times \mathbb{R}^2).$$

Clearly, $\partial^+ B(\varepsilon_i)$ collapses to I and $\partial^+ B(\varepsilon_i) \in \mathcal{M}(2, 0, \lambda, 2)$, where $\lambda = 0$. Note that $\mathrm{inrad}(M_i) = 1 + \pi\varepsilon_i/2 \to 1$.

Chapter 3

Metric structure of limit spaces

Under the notations in last chapter, the main purpose of this chapter is to show that Y and N are actually isometric to C/η_0 and C_0/η_0 respectively. To study how this gluing is made, we first analyze the tangent cones of C, C_0, Y and X at gluing points, and their relations via the the differential $d\eta_0$ of the gluing map η_0. It turns out that the identification map η_0 preserves length of curves. Finally, we see that N is isometric to a quotient of $C_0^{\rm int}$ by an isometric $\mathbb{Z}_2$ action. (see Proposition 3.3.3), which implies Theorems 0.2.1 and 3.3.4. All results in this chapter are due to a joint work with Prof. Takao Yamaguchi [YZ15].

3.1 Spaces of directions and differentials

In this subsection, we study the the spaces of directions of C, C_0, Y and X at the points where the gluing is done, and the relation between them. We also study the differential of the gluing map η_0 at those points.

Let $\tilde{\pi} : C \to C_0$ and $\pi : Y \to X$ be the projections, which are surjective Lipschitz maps. To be precise,

$$\pi(y) := \eta_0 \circ \tilde{\pi}(\eta^{-1}(y)).$$

For simplicity, we use the same notation

$$C_t := \{q \in C \,|\, d(C_0, x) = t\},\ C_t := \{y \in Y \,|\, d(X, y) = t\}$$

for every $t \in (0, t_0]$. We also denote by

$$\tilde{\pi}_t : C \to C_t,\ \pi_t : Y \setminus X \to C_t$$

the canonical projections. Recall that

$$X_1 = \{x \in X | \#\eta_0^{-1}(x) = 1\},\ X_2 = \{x \in X | \#\eta_0^{-1}(x) = 2\},$$
$$C_0^i = \{p \in C_0 \,|\, \eta_0(p) \in X_i\},\ i = 1, 2.$$

Lemma 3.1.1. *For every $p \in C_0$, let $\tilde{\xi}_+ \in \Sigma_p(C)$ be the direction of the mimimal geodesic $\tilde{\gamma}_+$ from p to C_{t_0}. Then $\Sigma_p(C)$ is isometric to the half-spherical suspension $\{\tilde{\xi}_+\} * \Sigma_p(C_0)$.*

Proof. Since $C = C_0 \times_\phi [0, t_0]$, obviously we have $T_p(C) = T_p(C_0) \times [0, \infty)$, which implies the conclusion. □

Lemma 3.1.2. *For $x \in X_1$, let γ_+ be the shortest geodesic from x to C_{t_0}, and let $\xi_+ \in \Sigma_x(Y)$ be the direction of γ_+. Then*

1. *for every $\xi \in \Sigma_x(Y)$, there is a unique $v \in \Sigma_x(X)$ such that*
$$\angle(\xi_+, \xi) + \angle(\xi, v) = \angle(\xi_+, v) = \pi/2; \tag{3.1}$$
2. *conversely, every $v \in \Sigma_x(X)$ satisfies $\angle(\xi_+, v) = \pi/2$;*
3. *there exists a unique limit*
$$\lim_{\delta \to 0} (\frac{1}{\delta} X, x) = (T_x(X), o_x) = (K(\Sigma_x(X)), o_x).$$
under the convergence $\lim_{\delta \to 0}(\frac{1}{\delta}Y, x) = (T_x(Y), o_x)$

Proof. (1) Let γ_+ be the shortest geodesic from x to C_{t_0}, and let $\xi_+ \in \Sigma_x(Y)$ be the direction of γ_+ respectively. First consider an arbitrary minimal geodesic $\gamma : [0, \ell] \to Y$ starting from x with $\angle(\gamma, \Sigma_x(X)) > 0$. Set $\sigma(t) := \pi(\gamma(t))$, and let $\gamma_t : [0, t_0] \to Y$ be the minimal geodesic from $\sigma(t)$ to C_{t_0} through $\gamma(t)$. The limit γ_0 of γ_t as $t \to 0$ coincides with γ_+. Let $v \in \Sigma_x(X)$ be a direction defined by the curve σ. Let $\tilde{\gamma}$, $\tilde{\gamma}_+$ (resp. $\tilde{\sigma}$) be geodesics (resp. a curve) in C such that $\eta(\tilde{\gamma}) = \gamma$, $\eta(\tilde{\gamma}_+) = \gamma_+$ (resp. $\eta_0(\tilde{\sigma}) = \sigma$). Since γ is minimal, so is $\tilde{\gamma}$. Note that $\tilde{\sigma}(t) = \tilde{\pi}(\tilde{\gamma}(t))$. Put $p := \tilde{\gamma}(0)$. Let $\tilde{\xi}$ and $\tilde{\xi}_+$ be the directions at p defined by $\tilde{\gamma}$ and $\tilde{\gamma}_+$ respectively. Let $\tilde{v}$ be the directions at p defined by $\tilde{\sigma}$. Note that $\tilde{v}$ is uniquely determined since $\tilde{\sigma}$ is a shortest curve. From Lemma 3.1.1, we have
$$\angle(\tilde{\xi}_+, \tilde{\xi}) + \angle(\tilde{\xi}, \tilde{v}) = \angle(\tilde{\xi}_+, \tilde{v}) = \pi/2. \tag{3.2}$$
By the first variation formula, we have $\angle(\xi_+, v) \geq \pi/2$. Now we show (3.1). Consider the rescaling limits,
$$T_x(Y) = \lim_{\delta \to 0} \left(\frac{1}{\delta} Y, x \right), \ T_p(C) = \lim_{\delta \to 0} \left(\frac{1}{\delta} C, p \right).$$
Let σ_∞ and $\tilde{\sigma}_\infty$ be the limits of the Lipschitz curves σ and $\tilde{\sigma}$ under these convergence. It should also be noted that the geodesic γ_δ (resp. $\tilde{\gamma}_\delta$) converges to a geodesic ray $\gamma_{\infty 1}$ (resp. $\tilde{\gamma}_{\infty 1}$) starting from $\sigma_\infty(1)$ (resp. $\tilde{\sigma}_\infty(1)$) and perpendicular to $T_x(X) := \lim_{\delta \to 0} \left(\frac{1}{\delta} X, x \right) \subset T_x(Y)$ (resp. $T_p(C)$) under those convergences. We set
$$\rho(t) = |C_0, \tilde{\gamma}(t)| = |X, \gamma(t)|.$$

Notice that $\gamma_{\infty 1}$ meet ξ at distance $\rho'(0)$ from $\sigma_\infty(1)$. Thus we have

$$\angle(\xi, v) = \operatorname{Arc}\sin \rho'(0) = \angle(\tilde{\xi}, \tilde{v}). \tag{3.3}$$

Since

$$\begin{aligned} |\gamma(\delta), \gamma_+(\rho(\delta))|_{(C_{\rho(\delta)})^{int}}/\delta &\le |\tilde{\gamma}(\delta), \tilde{\gamma}_+(\rho(\delta))|_{(C_{\rho(\delta)})^{int}}/\delta \\ &= \phi(\rho(\delta))|p, \tilde{\sigma}(\delta)|_{C_0^{int}}/\delta, \end{aligned}$$

as $\delta \to 0$, we obtain $\sin\angle(\xi_+, \xi) \le \sin\angle(\tilde{\xi}_+, \tilde{\xi})$, and hence $\angle(\xi_+, \xi) \le \angle(\tilde{\xi}_+, \tilde{\xi})$. From

$$\begin{aligned} \pi/2 \le \angle(\xi_+, v) &\le \angle(\xi_+, \xi) + \angle(\xi, v) \\ &\le \angle(\tilde{\xi}_+, \tilde{\xi}) + \angle(\tilde{\xi}, \tilde{v}) = \pi/2. \end{aligned}$$

we conclude that (3.1) holds for $\xi \in \Sigma_x(Y) \setminus \Sigma_x(X)$.

Note that (3.1) shows that v is uniquely determined.

(2) For every $v \in \Sigma_x(X)$ take a sequence $x_i \in X$ with $x_i \to x$, and let $\mu_i : [0, t_i] \to Y$ denote a minimal geodesic from x to x_i with $v_i := \dot{\mu}_i(0) \to v$. Let $\lambda_i : [0, t_0] \to Y$ be a minimal geodesic from x_i to C_{t_0}. We may assume that $\lambda_i(t_0) \to \gamma_+(t_0)$. Take a point $y_i \in \lambda_i$ such that $|\angle(\xi_+, \xi_i) - \pi/4| < \epsilon_i$ with $\lim \epsilon_i = 0$, where $\xi_i := \uparrow_x^{y_i}$. Let $\gamma_i : [0, s_i] \to Y$ be a minimal geodesic from x to y_i, and set

$$\sigma_i(t) := \pi(\gamma_i(t)), \ \tilde{\gamma}_i = \eta^{-1}(\gamma_i), \ \tilde{\sigma}_i = \tilde{\pi}(\tilde{\gamma}_i).$$

From (1), σ_i defines a direction $\hat{v}_i \in \Sigma_x(X)$ such that

$$\angle(\xi_+, \xi_i) + \angle(\xi_i, \hat{v}_i) = \angle(\xi_+, \hat{v}_i) = \pi/2.$$

Note that $x_i = \sigma_i(s_i)$. Consider the convergence

$$\left(\frac{1}{t_i} Y, x\right) \to (T_x(Y), o_x), \ \left(\frac{1}{t_i} C, p\right) \to (T_p(C), o_p), \ t_i = |x, x_i|.$$

Then x_i converges to $v \in \Sigma_x(X) \subset T_x(Y)$ under the above convergence. We may assume that ξ_i converges to some $\xi \in \Sigma_x(Y) \subset T_x(Y)$.

Passing to a subsequence, we may assume that

(a) s_i/t_i converges to $s_\infty > 0$;

(b) $\gamma_i(t_i s)$ and $\sigma_i(t_i s)$ converge to geodesic $\gamma_\infty(s)$ and a Lipschitz curve $\sigma_\infty(s)$ respectively;

(c) $\tilde{\gamma}_i(t_i s)$ and $\tilde{\sigma}_i(t_i s)$ converge to geodesics $\tilde{\gamma}_\infty(s)$ and $\tilde{\sigma}_\infty(s)$.

We show that σ_∞ is a minimal geodesic. We may also assume that

$$\eta_i = \eta : \left(\frac{1}{t_i}C, p\right) \to \left(\frac{1}{t_i}Y, x\right)$$

converges to a 1-Lipschitz map

$$\eta_\infty : (T_p(C), o_p) \to (T_x Y, o_x),$$

with $\eta_\infty(\tilde\sigma_\infty(s)) = \sigma_\infty(s)$. Consider the geodesic triangles

$$\Delta_{o_x} := \Delta o_x \gamma_\infty(s_\infty)\sigma_\infty(s_\infty) \subset T_x(Y),$$
$$\Delta_{o_p} := \Delta o_p \tilde\gamma_\infty(s_\infty)\tilde\sigma_\infty(s_\infty) \subset T_p(C).$$

Obviously, we obtain

$$|o_x, \gamma_\infty(s_\infty)| = |o_p, \tilde\gamma_\infty(s_\infty)|,$$
$$|\gamma_\infty(s_\infty), \sigma_\infty(s_\infty)| = |\tilde\gamma_\infty(s_\infty), \tilde\sigma_\infty(s_\infty)|$$

Note that $\Sigma_{\gamma_\infty(s_\infty)}(T_x(Y)$ and $\Sigma_{\tilde\gamma_\infty(s_\infty)}(T_p(C)$ have the suspension structures and that from construction

$$|\gamma_\infty(s), \sigma_\infty(s)| = |\tilde\gamma_\infty(s), \tilde\sigma_\infty(s)|.$$

Together with argument in (1), this implies that

$$\angle o_x \gamma_\infty(s_\infty)\sigma_\infty(s_\infty) = \angle o_p \tilde\gamma_\infty(s_\infty)\tilde\sigma_\infty(s_\infty). \tag{3.4}$$

By the Euclidean cone structure, Δ_{o_x} and Δ_{o_p} span flat triangles isometric to ones in $\mathbb{R}^2$. From the above equalities, we conclude that $|o_x, \sigma_\infty(s_\infty)| = |o_p, \tilde\sigma_\infty(s_\infty)|$. Since $L(\sigma_\infty) \le L(\tilde\sigma_\infty)$, this implies that σ_∞ is a minimal geodesic, and $\angle(\xi_+, v) = \pi/2$ as required. This also show that (1) holds true for every $\xi \in \Sigma_x(Y)$.

(3) As observed above, for every $v \in \Sigma_x(X)$ and for every $\epsilon > 0$, one can find a Lipschitz curve σ in X starting from x such that σ determines a well-defined direction $\dot\sigma(0) \in \Sigma_x(X)$ satisfying $\angle(v, \dot\sigma(0)) < \epsilon$. Now (c) follows from a standard argument. □

Lemma 3.1.3. *For $x \in X_2$ let $\gamma_\pm$ be the two shortest geodesics from x to C_{t_0}, and let $\xi_\pm \in \Sigma_x(Y)$ be the directions of $\gamma_\pm$) respectively. Then we have*

(1) for every $\xi \in \Sigma_x(Y)$, there is a unique $v \in \Sigma_x(X)$ such that

$$\angle(\{\xi_\pm\}, \xi) + \angle(\xi, v) = \angle(\{\xi_\pm\}, v) = \pi/2; \tag{3.5}$$

(2) $\Sigma_x(Y)$ is isometric to the spherical suspension $\{\xi_\pm\} * \Sigma_x(X)$;

(3) *there exists a unique limit*

$$\lim_{\delta\to 0}(\frac{1}{\delta}X, x) = (T_x(X), o_x) = (K(\Sigma_x(X)), o_x).$$

under the convergence $\lim_{\delta\to 0}(\frac{1}{\delta}Y, x) = (T_x(Y), o_x)$.

Remark 3.1.4. The suspension structure in Lemma 3.1.3 (2) also follows from the proof of Lemma 2.1.6

Proof. (1), 2) In a way similar to (3.1), we have (3.5). Let $\Sigma_\pm$ denote the union of all minimal geodesics joining $\xi_\pm$ to the elements of $\Sigma_x(X)$. We see that $\Sigma_x(Y)$ is the union of Σ_+ and Σ_- glued along $\Sigma_x(X)$. Therefore $\angle(\xi_+, \xi_-) = \pi$ and $\Sigma_x(Y)$ is isometric to the spherical suspension $\{\xi_\pm\} * \Sigma_x(X)$.

(3) also follows in a way similar to the proof of Lemma 3.1.2. □

For any $p \in C_0$, since η_0 is 1-Lipschitz, $\eta_0 : (\frac{1}{r}C_0, p) \to (\frac{1}{r}X, x)$ subconverges to a 1-Lipschitz map $(d\eta_0)_p : T_p(C_0) \to T_x(X)$, which is called a *differential* of η_0 at p.

Proposition 3.1.5. *For every $p \in C_0$, any differential $d\eta_0 : T_p(C_0) \to T_x(X)$ satisfies*

$$|d\eta_0(\tilde{v})| = |\tilde{v}|.$$

for every $\tilde{v} \in T_p(C_0)$. In particular, $\eta_0 : C_0 \to X$ preserves the length of Lipschitz curves in C_0.

Proof. For every $\tilde{v} \in \Sigma_p(C_0)$, let $\tilde{\xi}$ be the midpoint of a minimal geodesic in $\Sigma_p(C_0)$ joining $\tilde{\xi}_+$ and $\tilde{v}$. Let $\tilde{\gamma}(t)$ be the geodesic starting from p in the direction $\tilde{\xi}$. Put $\tilde{\sigma}(t) = \tilde{\pi}(\tilde{\gamma}(t))$, $\sigma(t) = \eta_0(\tilde{\sigma}(t))$. Then from (3.3) in the proof of Lemma 3.1.3, we obtain

$$\tilde{\sigma}'(0) = \frac{\sqrt{2}}{2}\tilde{v}, \;\; \sigma'(0) = \frac{\sqrt{2}}{2}v,$$

which implies that $|d\eta_0(\tilde{v})| = |\tilde{v}|$. □

By Proposition 3.1.5, $d\eta_0$ provides a surjective 1-Lipschitz map $d\eta_0 : \Sigma_p(C_0) \to \Sigma_x(X)$.

Remark 3.1.6. By Lemma 3.1.3, $x \in X_2$ is a regular point of Y if and only if the tangent cone $T_x(X)$ is isometric to $\mathbb{R}^{m-1}$, where $m = \dim Y$. From this reason, in that case we call x a *regular point* of X, and set $X^{reg} := X \cap Y^{reg}$. Later we show that every $x \in X_1$ is a *singular point* of X unless $X = X_1$.

Proposition 3.1.7. *For every $p \in C_0^2$, we have*

(1) *any differential $d\eta_p$ provides an isometry $d\eta_p : T_p(C) \to T_x^+(Y)$ which preserves the half suspension structures of both $\Sigma_p(C) = \{\xi_+\} * \Sigma_p(C_0)$ and $\Sigma_x^+(Y) := \{\xi_+\} * \Sigma_x(X)$, where $T_x^+(Y) = T_x(X) \times \mathbb{R}_+$;*

(2) $p \in C_0^{reg}$ if and only if $x \in X^{reg}$. In this case, $(d\eta_0)_p : T_p(C_0) \to T_x(X)$ is a linear isometry.

Proof. (1) We show that $d\eta_0 : T_p(C_0) \to T_x(X)$ preserves norm. We use the notation in the proof of Lemma 3.1.3. Recall that $\eta_0(\tilde\sigma(t)) = \sigma(t)$, and $\tilde v \in \Sigma_p(C_0)$, $v \in \Sigma_x(X)$ denote the directions determined by $\tilde\sigma$, σ respectively. From (3.3), we have

$$|\tilde\sigma'(0)| = \cos\angle(\tilde\xi, \tilde v) = \cos\angle(\xi, v) = |\sigma'(0)|,$$

which implies that $|d\eta_0(\tilde v)| = |\tilde v|$.

For every $\tilde v_1, \tilde v_2 \in \Sigma_p(C_0)$, put $v_i := d\eta_0(\tilde v_i)$. We show that $\angle(\tilde v_1, \tilde v_2) = \angle(v_1, v_2)$. Let $\tilde\xi_i$ (resp. ξ_i) be the midpoint of the geodesic joining $\tilde\xi_+$ to $\tilde v_i$ (resp. ξ_+ to v_i). Note that $d\eta(\tilde\xi_i) = \xi_i$. We may assume that there are geodesics $\tilde\gamma_i(t)$ with $\tilde\gamma_i'(0) = \tilde\xi_i$, and set $\gamma_i(t) := \eta(\tilde\gamma_i(t))$. Since $T_x(Y) = T_x(X) \times \mathbb{R}$, any minimal geodesic joining $\tilde\gamma_1(t)$ and $\tilde\gamma_2(t)$ does not meet X for any small $t > 0$. It follows from the fact that $\eta : C \setminus C_0 \to Y \setminus X$ is a componentwise isometry that

$$|\tilde\gamma_1(t), \tilde\gamma_2(t)| = |\gamma_1(t), \gamma_2(t)|,$$

which implies that $\angle(\tilde\xi_1, \tilde\xi_2) = \angle(\xi_1, \xi_2)$. From the suspension structures, we conclude that $\angle(\tilde v_1, \tilde v_2) = \angle(v_1, v_2)$.

(2) is an immediate consequence of (1). □

3.2 Gluing maps

Using the results of the last subsection, we study the metric properties of the gluing map.

From Lemma 2.1.6, we can define a map $f : C_0 \to C_0$ as follows: For an arbitrary point $p \in C_0$, let $f(p) := q$ if $\{p, q\} = \eta_0^{-1}(\eta_0(p))$, where q may be equal to p if $\eta_0(p) \in X_1$. Recall that $C_0^i := \{p \in C_0 \,|\, \eta_0(p) \in X_i\}$, $i = 1, 2$. Note that f is an involutive map, i.e., $f^2 = id$. Moreover

Lemma 3.2.1. *$f : C_0 \to C_0$ is a homeomorphism.*

Proof. Since f is involutive, it suffices to prove that f is continuous. Let a sequence p_i converges to a point p in C_0. Passing to a subsequence, we assume that $q_i := f(p_i)$ converges to a point q in C_0. We have to prove that $f(p) = q$. We observe that

$$\eta_0(p) = \eta_0(q). \tag{3.6}$$

First consider the case $p \in C_0^1$, or equivalently $f(p) = q$. From (3.6), we certainly obtain $p = q = f(p)$.

Next consider the case $p \in C_0^2$. Suppose $f(p) \neq q$. Since $f(p) \neq p$, we then have $p = q$. Let $\tilde{\gamma}_i$, $\tilde{\gamma}_+$, $\tilde{\gamma}_-$ and $\tilde{\gamma}_i'$ be mimimal geodesics to C_{t_0} starting from p_i, p, $f(p)$ and q_i respectively. Put $\gamma_i := \eta(\tilde{\gamma}_i)$, $\gamma_+ := \eta(\tilde{\gamma}_+)$, $\gamma_- := \eta(\tilde{\gamma}_-)$ and $\gamma_i' := \eta(\tilde{\gamma}_i')$. Set $x_i := \eta_0(p_i) = \eta_0(q_i)$, $x := \eta_0(p) = \eta_0(q)$. By Lemma 3.1.3, we have

$$\tilde{\angle}\gamma_+(s_0)x\gamma_-(s_0) > \pi - \tau(s_0) \tag{3.7}$$

By Lemma 5.6 in [BGP],

$$|\angle xx_i\gamma_\pm(s_0) - \tilde{\angle} xx_i\gamma_\pm(s_0)| < \tau(s_0) + o_i \tag{3.8}$$

$$|\angle xx_i\gamma_\pm(s_0) - \pi/2| < \tau(s_0) + o_i, \tag{3.9}$$

where $\lim_{i\to\infty} o_i = 0$. Let $\sigma_i^\pm : [0, \ell_i^\pm] \to Y$ be minimal geodesic joining x_i to $\gamma_\pm(s_0)$.

Suppose that $p_i \in C_0^2$. Since $\Sigma_{x_i}(Y)$ is isometric to the spherical suspension

$$\{\dot{\gamma}_i(0), \dot{\gamma}_i'(0)\} * \Sigma_x(X),$$

in view of (3.9), we may assume that

$$\angle(\dot{\gamma}_i'(0), \dot{\sigma}_i^-(0)) < \tau(s_0) + o_i.$$

This implies that $|\gamma_i'(s_0), \gamma_-(s_0)| < (\tau(s_0) + o_i)s_0$. Since $|\gamma_i'(s_0), \gamma_+(s_0)| < o_i$, this yields a contradiction to (3.7).

Finally suppose that $p_i \in C_0^1$. Let $\gamma_t^\pm$ be the minimal geodesic from $\pi(\sigma_i^\pm(t))$ to C_{t_0}. As $t \to 0$, $\gamma_t^\pm$ converges to minimal geodesics $\gamma_0^\pm$ from x_i to C_{t_0}. Since $|\gamma_0^\pm(s_0), \gamma_\pm(s_0)| < o_i$, it follows that $|\gamma_0^+(s_0), \gamma_0^-(s_0)| > 2s_0 - o_i$. In particular we have $\gamma_0^+ \neq \gamma_0^-$, which contradicts to $p_i \in C_0^1$. □

Corollary 3.2.2. *$\eta_0|_{C_0^2} : C_0^2 \to X_2$ is a double covering space and X_2 is open in X.*

Proof. For $x \in X_2$ set $\eta_0^{-1}(x) = \{p_1, p_2\}$, and take a neighborhhd D_1 of p_1 in C_0 such that $D_1 \cap f(D_1)$ is empty. We set $D_2 = f(D_1)$. We show that $E := \eta_0(D_i)$ is open in X. Suppose that E is not open, and take $y \in E$ for which there are $y_i \in X \setminus E$ converging to y. Choose any $q_i \in \eta_0^{-1}(y_i)$. Since X is compact, passing to a subsequence, we may assume that q_i converges to a point q. It turns out that $\eta_0^{-1}(y)$ contains at least three points q_1, q_2 and q, where $q_i \in D_i$, $q \notin D_1 \cup D_2$. Since this is a contradiction, E is open. Now it is immediate that each restriction $\eta_0|_{D_i} : D_i \to E$ is a homeomorphism. □

Corollary 3.2.3. *If the inradius of $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ converges to zero, then the number of components of ∂M_i is at most two for large enough i.*

Proof. Suppose C_0 has more than two components. Let C_0^α be one component of C_0. Then $f(C_0^\alpha)$ is one component of C_0, since $f : C_0 \to C_0$ is an involutive homeomorphism.

Since C_0 has more than two components, we can pick a component C_0^β of C_0 such that neither C_0^α nor $f(C_0^\alpha)$ is equal to C_0^β. Thus C_0^α/f is a proper, open and closed subset of C_0/f. It means that C_0/f is not connected. On the other hand, since X is connected and homeomorphic to C_0/f by Proposition 2.1.3, C_0/f is connected which is a contradiction. Since the number of the boundary components of M_i is equal to the number of components of C eventually, the number of components of ∂M_i is at most two for i sufficiently large. This completes the proof. □

Remark 3.2.4. In Theorem 0.2.5, we remove the diameter bound to get the diameter free result.

Lemma 3.2.5. $\eta_0|_{C_0^2} : (C_0^2)^{\rm int} \to X_2^{\rm int}$ *is a local isometry.*

Proof. Since $\eta_0|_{C_0^2} : C_0^2 \to X_2$ is a covering by Corollary 3.2.2, we can find relatively compact open subsets D and E of C_0^2 and X^2 respectively such that $\eta_0 : D \to E$ is a homeomorphism. We must show that $\eta_0 : D \to E$ is an isometry with respect to the interior distances of C_0 and X respectively. Since η_0 is 1-Lipschitz, it suffices to show that $g := \eta_0^{-1} : E \to D$ is 1-Lipschitz. We may assume that D is small enough so as to satisfy that for every $x, y \in E$, there is a minimal geodesic $\gamma : [0,1] \to X_2$ joining x to y. We do not know if $g \circ \gamma$ is a Lipschitz curve yet. However by Proposition 3.1.5, $g \circ \gamma$ has the speed $v_{g\circ\gamma}(t)$ (see [BBI])

$$v_{g\circ\gamma}(t) = \lim_{\epsilon\to 0} \frac{|g\circ\gamma(t), g\circ\gamma(t+\epsilon)|}{|\epsilon|},$$

which is equal to the speed $v_\gamma(t)$ of γ, and therefore

$$|x,y| = L(\gamma) = \int_0^1 v_{g\circ\gamma}(t)dt = L(g\circ\gamma) \geq |g(x), g(y)|.$$

This completes the proof. □

Lemma 3.2.6. *If X_1 has non-empty interior in X, then $X = X_1$ and*

$$\eta_0 : (C_0)^{\rm int} \to X^{\rm int}$$

is an isometry.

Proof. If the interior U of X_1 is non-empty, then $V := \eta_0^{-1}(U) \subset C_0^1$ is open in C_0. From the non-branching property of geodesics in Alexandrov spaces, we have $V = C_0$ and $X = X_1$. An argument similar to the proof of Lemma 3.2.5 shows that $\eta_0 : (C_0)^{\rm int} \to X^{\rm int}$ is an isometry. □

Proposition 3.2.7. $f : (C_0)^{\rm int} \to (C_0)^{\rm int}$ *is an isometry.*

Proof. For $x \in X_2$ with $\eta_0^{-1}(x) = \{p_1, p_2\}$,by lemma 3.2.5, we can take disjoint open sets $p_i \in D_i$, $i = 1, 2$, and E such that $\eta_0^i = \eta_0|_{D_i} : D_i \to E$ are isometry. Thus $f|_{D_1} = (\eta_0^2)^{-1} \circ \eta_0^1 : D_1 \to D_2$ is an isometry with respect to the interior distances. Note that f is identity on C_0^1, and by Lemma 3.2.5, $f : (C_0^2)^{\mathrm{int}} \to (C_0^2)^{\mathrm{int}}$ is a locally isometry. For every $p_1, p_2 \in C_0$ we show that $|f(p_1), f(p_2)| = |p_1, p_2|$. This is obvious if $p_1, p_2 \in C_0^1$. Let $\gamma : [0, 1] \to C_0$ be a minimal geodesic joining p_1 to p_2. If $p_1, p_2 \in C_0^2$, applying Lemma 3.2.5, we may assume that γ meets C_0^1. Let $t_0 \in (0, 1)$ be the smallest parameter with $\gamma(t_0) \in C_0^1$. By Lemma 3.2.5, we have $|f(p_1), f(\gamma(t_0))| = |p_1, \gamma(t_0)|$. Therefore the non-branching property of geodesics in Alexandrov space implies that $\gamma \cap C_0^1$ consists of only the single point $\gamma(t_0)$, and therefore we also have $|f(p_2), f(\gamma(t_0))| = |p_2, \gamma(t_0)|$. It follows that

$$\begin{aligned}|f(p_1), f(p_2)| &\le |f(p_1), f(\gamma(t_0))| + |f(\gamma(t_0)).f(p_2)| \\ &\le |p_1, \gamma(t_0)| + |\gamma(t_0), p_2| = |p_1, p_2|.\end{aligned}$$

Repeating this, we also have $|p_1, p_2| \le |f(p_1), f(p_2)|$, and $|f(p_1), f(p_2)| = |p_1, p_2|$. The case of $p_1 \in C_0^1$ and $p_2 \in C_0^2$ is similar, and hence is omitted. This completes the proof. □

3.3 Structure theorems

In this subsection, making use of the results on gluing maps in the last subsection, we obtain structure results for limit spaces.

We begin with

Lemma 3.3.1. *X_2 is convex in X.*

Proof. Suppose this is not the case. Then we have a minimal geodesic $\gamma : [0, 1] \to X$ joining points $x, y \in X_2$ such that γ is not entirely contained in X_2. Let t_1 be the first parameter with $\gamma(t_1) \in X_1$. Set $z := \gamma(t_1)$. By Lemma 3.2.5, for any $p \in \eta_0^{-1}(x)$, there exists a unique geodesic $\tilde\gamma : [0, t_1] \to C_0$ such that $\tilde\gamma(0) = p$ and $\eta_0 \circ \tilde\gamma(t) = \gamma(t)$, for every $t \in [0, t_1]$. Put $\tilde z := \tilde\gamma(t_1) \in C_0^1$, and take $\tilde v \in \Sigma_{\tilde z}(C_0)$ such that

$$(d\eta_0)_{\tilde z}(\tilde v) = \frac{d}{dt}\gamma(t_0) \in \Sigma_z(X).$$

Let

$$\tilde\gamma_1 : [0, t_1] \to C_0 \text{ and } \gamma_1 : [0, t_1] \to X$$

be the reversed geodesic to $\tilde\gamma$ and $\gamma_{[0,t_1]}$: $\tilde\gamma_1(t) = \tilde\gamma(t_0 - t)$, $\gamma_1(t) = \gamma(t_1 - t)$, and set $\tilde\gamma_2(t) := f(\tilde\gamma_1(t))$. Since $(d\eta_0)_{\tilde z}$ preserves norm and is 1-Lipschitz, we have

$$\angle(\tilde v, \tilde\gamma_i'(0)) \ge \angle\left(\frac{d}{dt}\gamma(t_1), \frac{d}{dt}\gamma_1(0)\right) = \pi,$$

for $i = 1, 2$. Since $\tilde\gamma_1'(0) \ne \tilde\gamma_2'(0)$, this is impossible in the Alexandrov space C_0. □

Lemma 3.3.2. *For every $x, y \in X$, let $\gamma : [0,1] \to X$ be a minimal geodesic joining x to y, and let $p \in C_0$ be such that $\eta_0(p) = x$. Then there exists a unique minimal geodesic $\tilde{\gamma} : [0,1] \to C_0$ starting from p such that $\eta_0 \circ \tilde{\gamma} = \gamma$.*

In particular, if X_1 is not empty, then C_0 is connected.

Proof. From Lemmas 3.3.1 and the discussion there using non-branching property of geodesics in Alexandrov spaces, we have only the following possibilities:

1. γ is included in X_1 or X_2;

2. only one end point of γ is contained in X_1 and the other part of γ is included in X_2.

The conclusion follows immediately from Lemmas 3.2.5 and 3.2.6. □

Proposition 3.3.3. *$X^{\rm int}$ is isometric to $C_0^{\rm int}/f$.*

Proof. In the case of $X = X_1$ or $X = X_2$, the conclusion follows from Lemma 3.2.6 or Proposition 3.2.5 respectively. Next assume that both X_1 and X_2 are non-empty. We set $Z := C_0^{\rm int}/f$, which is an Alexandrov space, and decompose Z as

$$Z = Z_1 \cup Z_2, \;\; Z_i := C_0^i/f, \;\; i = 1, 2.$$

For every $[p] \in Z_1$, $\Sigma_{[p]}(Z)$ is isometric to $\Sigma_p(C_0)/f_*$, where $f_* : \Sigma_p(C_0) \to \Sigma_p(C_0)$ is an isometry induced by f. Since X_1 is a proper subset of X, f_* defines a non-trivial isometric $\mathbb{Z}_2$-action on $\Sigma_p(C_0)$. Thus $[p]$ is a single point of Z: $[p] \in Z^{sing}$, and therefore $Z_1 \subset Z^{sing}$. Thus $Z^{\rm reg} \subset Z_2$. Now by Proposition 3.1.7, there exists an isometry $F_0 : Z_2 \to X_2^{\rm int}$. Since $Z^{\rm reg}$ is convex in Z (see [Pet98]), F_0 defines a 1-Lipschitz map $F_1 : (Z^{\rm reg})^{\rm ext} \to X$ which extends to a 1-Lipschitz map $F : Z \to X$, where $(Z^{\rm reg})^{\rm ext}$ denotes the exterior metric of $Z^{\rm reg}$.

Conversely since X_2 is convex in X by Lemma 3.3.1, F_0^{-1} defines a 1-Lipschitz map $G_1 : (X_2)^{\rm ext} \to Z_2$ which extends to a 1-Lipschitz map $G : X \to Z$ satisfying $G \circ F = 1_Z$. Therefore X must be isometric to Z. □

Proof of Theorem 0.2.1. By Proposition 3.2.7, $f : C_0^{\rm int} \to C_0^{\rm int}$ is an involutive isometry. By Propositions 2.2.2 and 3.3.3, N is isometric to $C_0^{\rm int}/f$. Since $C_0^{\rm int}$ is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$, so is N. □

Theorem 3.3.4. *Let $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to a compact length space N. Let $\tilde{M}_i$ Gromov-Hausdroff converge to Y, and $M_i^{\rm ext}$ converge to $X \subset Y$ under the convergence $\tilde{M}_i \to Y$. Then*

(1) $X^{\rm int}$ is isometric to N;

(2) *Y is isometric to $C_0^{\rm int} \times_\phi [0, t_0]/(f(x), 0) \sim (x, 0)$, or equivalently, isometric to the following quotient by an isometric involution $\tilde f = (f, -{\rm id})$.*

$$C_0^{\rm int} \times_{\tilde\phi} [-t_0, t_0]/\tilde f,$$

where $\tilde\phi(t) = \phi(|t|)$.

In particular, Y is a singular I-bundle over N, where singular fibers occur exactly on X_1 unless $X = X_1$.

Compare Examples 2.3.1, 2.3.3 and 2.3.4.

Proof of Theorem 3.3.4. (1) is just Proposition 2.2.2. (2) follows immediately from Propositions 2.2.2 and 3.3.3. □

Proposition 3.3.5. *If $x \in X_1$, then $\Sigma_x(X)$ is isometric to the quotient space $\Sigma_p(C_0)/f_*$, and $\Sigma_x(Y)$ is isometric to the quotient space $\Sigma_p(C)/f_*$, where $f_* : \Sigma_p(C_0) \to \Sigma_p(C_0)$ is an isometry induced by f.*

Proof. Take an f-invariant neighborhood U_p of p in C_0, where $\eta_0(p) = x$. It is easy to check that $V_x := \eta_0(U_p)$ is a neighborhood of x isometric to U_p/f. The conclusion of (2) follows immediately. □

Corollary 3.3.6. *Let $\dim N = m$. Suppose that both X_1 and X_2 is non-empty. Then every element $x \in X_1$ satisfies that*

$$\mathrm{vol}\,\Sigma_x(X) \le \frac{1}{2}\mathrm{vol}\,\mathbb{S}^{m-1}.$$

Proof. For $x \in X_1$, take $p \in C_0$ with $\eta_0(p) = x$. Note that C_0 is connected by Lemma 3.3.2. If $f_* : \Sigma_p(C_0) \to \Sigma_p(C_0)$ is the identity, then the non-branching property of geodesics in Alexandrov spaces implies that f is the identity on C_0. Therefore f_* must be non-trivial on $\Sigma_p(C_0)$. The conclusion follows since

$$\mathrm{vol}\,\Sigma_x(X) = (1/2)\mathrm{vol}\,\Sigma_p(C_0) \le (1/2)\mathrm{vol}\mathbb{S}^{m-1}.$$

□

By Corollary 3.3.6, if every $x \in X$ satisfies that

$$\mathrm{vol}\Sigma_x(X) > (1/2)\mathrm{vol}\mathbb{S}^{m-1},$$

then $X = X_1$ or $X = X_2$.

Next let us consider such a case. If $X = X_1$, then by Lemma 3.2.6, η_0 is an isometry. If $X = X_2$, then by Lemma 3.2.5, η_0 is a locally isometric double covering. Therefore it is straightforward to see the following.

Corollary 3.3.7. *If $X = X_1$ or X_2, then Y can be classified by N as follows.*

(1) if $X = X_1$, then Y is isometric to $N \times_\phi [0, t_0]$.

(2) if $X = X_2$, then either Y is isometric to the gluing

$$N \times_{\tilde{\phi}} [-t_0, t_0],$$

with length metric, or else, Y is a nontrivial I-bundle over N, and is doubly covered by

$$C_0^{\mathrm{int}} \times_{\tilde{\phi}} [-t_0, t_0],$$

where $\tilde{\phi}(t) = \phi(|t|)$.

Compare Examples 2.3.1 and 2.3.3.

From now, we write for simplicity as $C_0 := C_0^{\mathrm{int}}$.

Chapter 4

Inradius collapsed manifolds

In this chapter, we investigate the structure of inradius collapsed manifolds M_i by applying the structure results for limit spaces in Chapter 3. First we study the case of inradius collapses of codimension one to determine the manifold structure. To carry out this, some additional considerations on the limit spaces are needed to determine the singularities of singular I-fibered spaces. In the second part of this chapter, we study inradius collapses to almost regular spaces. All results in this chapter are due to a joint work with Prof. Takao Yamaguchi [YZ15].

4.1 Inradius collapses of codimension one

We consider $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapses to an $(n-1)$-dimensional Alexandrov space N. Then by Theorem 1.2.2, M_i is homeomorphic to Y for large enough i, and by Theorem 3.3.4, we have

$$Y = C_0 \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f}, \; N = C_0/f,$$

where

$$\tilde{f} = (f, -\mathrm{id})$$

is an isometric involution. and the singular locus of the singular I-bundle structure on Y defined by the above form coincides with C_0^1 unless $X \neq X_1$. Later in Lemma 4.1.5, we show that $\eta_0(C_0^1) = \partial N$.

Assuming that N has non-empty boundary, we begin with construction of singularity models of singular I-fibered spaces around each boundary component of the limit space N.

By Proposition 1.2.7, each component $\partial_\alpha N$ of ∂N has a collar neighborhood V_α. Let

$$\varphi : V_\alpha \to \partial_\alpha N \times [0, 1)$$

be a homeomorphism. Let $\pi : Y \to N$ be the projection. Then I-fiber structure on $\pi^{-1}\varphi^{-1}(\{p\} \times [0,1)$ is isomorphic to the form

$$R_{t_0} := [0,1) \times [-t_0, t_0]/(0,y) \sim (0,-y),$$

with the projection $\pi : R_{t_0} \to [0,1)$ indecued by $(x,y) \to x$. Therefore $\pi^{-1}(V_\alpha)$ is an R_{t_0}-bundle over $\partial_\alpha N$.

Now we define two singularity models for the singular I-bundle $\pi^{-1}(V_\alpha)$: one is the case when $\pi^{-1}(V_\alpha)$ is a trivial R_{t_0}-bundle over $\partial_\alpha N$, and the other one is the case of non-trivial R_{t_0}-bundle.

Definition 4.1.1. (1). First, set

$$\mathcal{U}_1(\partial_\alpha N) := \partial_\alpha N \times R_{t_0},$$

and define $\pi : \mathcal{U}_1(\partial_\alpha N) \to \partial_\alpha N \times [0,1)$ by $\pi(p,x,y) = (p,x)$ for $(p,x,y) \in \partial_\alpha N \times R_{t_0}$. This gives $\mathcal{U}_1(\partial_\alpha N)$ the structure of a singular I-bundle over $\partial_\alpha N \times [0,1)$ whose singular locus is $\partial_\alpha N \times 0$. We call this *the product singular I-bundle model* around $\partial_\alpha N$.

(2). For the second model, suppose that $\partial_\alpha N$ admits a double covering

$$\rho : P_\alpha \to \partial_\alpha N$$

with the deck transformation φ. Let

$$\mathcal{U}_2(\partial_\alpha N) := (P_\alpha \times R_{t_0})/\Phi,$$

where Φ is the isometric involution on $P_\alpha \times R_{t_0}$ defined by $\Phi = (\varphi, r)$, where $r : R_{t_0} \to R_{t_0}$ is defined as $(x,y) \to (x,-y)$. Define

$$\pi : \mathcal{U}_2(\partial_\alpha N) \to \partial_\alpha N \times [0,1)$$

by

$$\pi([p,x,y)]) = (\rho(p), x)$$

for $(p,x,y) \in P_\alpha \times R_\epsilon$. This gives $\mathcal{U}_2(\partial_\alpha N)$ the structure of a singular I-bundle over $\partial_\alpha N \times [0,1)$ whose singular locus is $\partial_\alpha N \times 0$. The second model is a twisted one, and is doubly covered by the first model $\mathcal{U}_1(P_\alpha) = P_\alpha \times R_\epsilon$. We call this the *twisted singular I-bundle model* around $\partial_\alpha N$.

Example 4.1.2. Let us consider the codimension one inradius collapses in Example 2.3.4. Recall that the limit space Y of $\tilde{M}_\epsilon$ is isometric to the form

$$Y = D(E) \times_{\tilde{\phi}} [-t_0, t_0]/(x,t) \sim (r(x), -t),$$

where $r : D(E) \to D(E)$ denotes the canonical reflection of $D(E)$. If $\pi : Y \to E$ denotes the projection, then $\pi^{-1}(V)$ is isomorphic to the product singular I-bundle model around ∂E, where V is any collar neighborhood of ∂E.

Example 4.1.3. Let Q_ϵ denote the space obtained from the disjoint union of two copies of the completion $\bar{R}_\epsilon$ of R_ϵ glued along each segment $1 \times [-\epsilon, \epsilon]$ of the boundaries:

$$Q_\epsilon = \bar{R}_\epsilon \amalg_{1\times[-\epsilon,\epsilon]} \bar{R}_\epsilon.$$

Let $r : Q_\epsilon \to Q_\epsilon$ be the reflection induced from $(x, y) \to (x, -y)$. Let

$$M_\epsilon = (\mathbb{S}^1(1) \times Q_\epsilon)/(z, p) \sim (-z, r(p)).$$

As $\epsilon \to 0$, M_ϵ inradius collapses to $\mathbb{S}^1(1/2) \times [0, 2]$. Let

$$\pi_\epsilon : M_\epsilon \to \mathbb{S}^1(1/2) \times [0, 2]$$

be the projection induced by $[z, (x, y)] \to (z, x)$. Then both $\pi_\epsilon^{-1}(\mathbb{S}^1(1/2) \times [0, 1)$ and $\pi_\epsilon^{-1}(\mathbb{S}^1(1/2) \times (1, 2])$ are solid Klein bottles and their I-fiber structures are isomorphic to the twisted singular I-bundle models around respective boundary of $\mathbb{S}^1(1/2) \times [0, 2]$.

Theorem 4.1.4. *Let $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to an $(n-1)$-dimensional Alexandrov space N. Then there is a singular I-bundle:*

$$I \to M_i \stackrel{\pi}{\to} N.$$

More precisely,

1. *If N has no boundary, then M_i is homeomorphic to a product $N \times I$ or a twisted product $N \tilde{\times} I$;*

2. *If N has non-empty boundary, each component $\partial_\alpha N$ of ∂N has a neighborhood V such that $\pi^{-1}(V)$ is homeomorphic to either $\mathcal{U}_1(\partial_\alpha N)$ or $\mathcal{U}_2(\partial_\alpha N)$ as I-fibered spaces;*

3. *If $\pi^{-1}(V)$ is homeomorphic to $\mathcal{U}_1(\partial_\alpha N)$ for some component $\partial_\alpha N$, then M_i is homeomorphic to*
$$D(N) \times [-1, 1]/(x, t) \sim (r(x), -t),$$
where r is the canonical reflection of the double $D(N)$.

Recall that
$$Y = C_0 \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f},$$
where $\tilde{f} = (f, -\mathrm{id})$, C_0 and Y are the noncollapsing limit of $(\partial M_i)^{\mathrm{int}}$ and $\tilde{M}_i$ respectively. Therefore both C_0 is a smoothable space in the sense of [Ka02].

Let $F \subset C_0$ denote the fixed point set of the isometry $f : C_0 \to C_0$. By Proposition 1.2.6 and Theorem 1.2.4 , $\eta_0(F)$ is an extremal subset of N and it has a topological stratification.

Lemma 4.1.5. *$\eta_0(F)$ coincides with ∂N if f is not the identity.*

We postpone the proof of Lemma 4.1.5 for a moment.

Proof of Theorem 4.1.4. (1) By Lemma 4.1.5, if N has no boundary, F is empty, and therefore either $N = N_1$ or $N = N_2$. If $N = N_1$, then $C_0 = N$ and Y is homeomorphic to $N \times I$. If $N = N_2$, then $N = C_0/f$ has no boundary, and Y is homeomorphic to either $N \times I$ or $C_0 \times [-1,1]/(x,t) \sim (f(x), -t)$ which is a twisted I bundle over N.

(2) Suppose N has non-empty boundary. Note that

$$N_1 = \eta_0(F).$$

By Proposition 1.2.7, each component $\partial_\alpha N$ of ∂N has a collar neighborhood V_α. Let $\varphi : V_\alpha \to \partial_\alpha N \times [0,1)$ be a homeomorphism. Let $\pi : Y \to N$ be the projection. By the I-fiber structure of Y, $\pi^{-1}(\varphi^{-1}(x \times [0,1))$ is canonically homeomorphic to R_{t_0}. In particular $\pi^{-1}(V_\alpha)$ is an R_{t_0}-bundle over $\partial_\alpha N$. If this bundle is trivial, $\pi^{-1}(V_\alpha)$ is isomorphic to the product singular I-bundle structure $\mathcal{U}_1(\partial_\alpha N) = \partial_\alpha N \times R_{t_0}$.

Suppose that this bundle is nontrivial, and let P_α be the boundary of $\pi^{-1}(\varphi^{-1}(\partial_\alpha N \times \{1/2\}))$, which is a double covering of ∂N_α. Let $\Phi = (\varphi, r)$, and $\rho : P_\alpha \to \partial_\alpha N$ the projection.

Lemma 4.1.6. *$\pi^{-1}(V_\alpha)$ is isomorphic to the twisted singular I-bundle structure $\mathcal{U}_2(\partial_\alpha N) = (P_\alpha \times R_{t_0})/\Phi$.*

Proof. Note that

$$\mathcal{U}_2(\partial_\alpha N) := (P_\alpha \times R_{t_0})/(p,x,y) \sim (\varphi(p), x, -y),$$
$$\pi^{-1}(V_\alpha) = \pi^{-1}\varphi^{-1}(\partial_\alpha N \times [0,1).$$

And for each $(p,x) \in P_\alpha \times [0,1)$, $\{p, \varphi(p)\}$ can be identified with with the boundary of the I-fiber $I_{\rho(p),x} := \pi^{-1}\varphi^{-1}(\rho(p) \times \{x\})$. We define a map

$$\Psi : \mathcal{U}_2(\partial_\alpha N) \to \pi^{-1}(V_\alpha)$$

as follows: let $\Psi(p,x,y)$, $-t_0 \le y \le t_0$, be the arc on the fiber $I_{\rho(p),x}$ from p to $\varphi(p)$. Clearly, $\Psi : \mathcal{U}_2(\partial_\alpha N) \to \pi^{-1}(V_\alpha)$ gives an isomorphism between I fibered spaces. □

(3) Put int$N := N \setminus \partial N$ for simplicity.

Assertion 4.1.7. *There is an isometric imbedding $g : N \to C_0$ such that $\eta_0 \circ g = 1_N$.*

Proof. Set $F_\alpha := \eta_0^{-1}(\partial_\alpha N)$. From the assumption, we may assume that F_α is two-sided in the sense that the complement of F_α in some connected neighborhood of it is disconnected. Thus there is a connected neighborhood V_α of $\partial_\alpha N$ in int$N$ for which there is an isometric imbedding $g_\alpha : V_\alpha \to C_0 \setminus F$ such that $\eta_0 \circ g_\alpha = 1_{V_\alpha}$.

Let W be the maximal connected open subset of int$N$ for which there is an isometric imbedding $g_0 : W \to C_0 \setminus F$ such that $\eta_0 \circ g_0 = 1_W$ and $g_0(W) \supset g_\alpha(V_\alpha)$. We only have to show that $W = \mathrm{int}N$. Otherwise, there is a point $x \in \partial W \cap \mathrm{int}N$. Take a connected neighborhood W_x of x in int$N$ such that $\eta_0^{-1}(W_x)$ is a disjoint union of open sets U_1 and U_2 such that $\eta_0 : U_i \to W_x$ is an isometry for $i = 1, 2$. Obviously one of U_i, say U_1, meets $g_0(W)$ and the other does not. We extend g_0 to $g_1 : W \cup W_x \to C_0 \setminus F$ by requiring $g_1|_{W_x} = \eta_0^{-1} : W_x \to U_1$. Since g_1 is an isometric imbedding, this is a contradiction to the maximality of W.

Thus we have an isometric imbedding $g_0 : \mathrm{int}N \to C_0 \setminus F$. Since int$N$ is convex and η_0 is 1-Lipschitz, g_0 preserves the distance. It follows that g_0 extends to an isometric imbedding $g : N \to C_0$ which preserves distance. □

Assertion 4.1.7 shows that every component of F is two-sided. It follows that $C_0 = D(N)$, and that f is the reflection of the double $D(N)$. This completes the proof of Theorem 4.1.4 □

Proof of Lemma 4.1.5. Obviously $\partial N \subset \eta_0(F)$. Suppose that $\eta_0(F) \cap (\mathrm{int}N)$ is not empty.

Sublemma 4.1.8. $\dim(\eta_0(F) \cap \mathrm{int}N) \le m - 2$*, where* $m := \dim N$*.*

Proof. If
$$\dim(\eta_0(F) \cap \mathrm{int}N) = m - 1,$$
then the top-dimensional strata S of $\eta_0(F) \cap \mathrm{int}N$ is a topological $(m-1)$-manifold, and therefore it meets the m-dimensional strata of N because $N^{\mathrm{sing}} \cap \mathrm{int}N$ has codimension ≥ 2 (Theorem 1.2.1). Take $p \in \eta_0^{-1}(S)$. It is now easy to see that f is the reflection with respect to $\eta_0^{-1}(S)$ in a small neighborhood of p. It follows that S is a subset of ∂N, contradiction to the hypothesis. □

Take a point $x = \eta_0(p) \in \eta_0(F) \cap \mathrm{int}N$, and consider the directional derivative $f_* : \Sigma_p(C_0) \to \Sigma_p(C_0)$ of f at p which is also an isometric involution with fixed point set
$$F_* := \Sigma_p(F)$$
By Corollary 1.2.5 and Sublemma 4.1.8, $\dim F_* \le m - 3$ while $\dim \Sigma_p(C_0) = m - 1$. Repeating this we have a finite sequence of directional derivatives of f, $f_* \ldots$, each of which is an isometric involution:
$$f_{*k} : \Sigma_{*k}(C_0) \to \Sigma_{*k}(C_0),$$
where $\Sigma_{*k}(C_0)$ denotes a k iterated space of directions,
$$\Sigma_{*k}(C_0) = \Sigma_{\xi_{k-1}}(\cdots(\Sigma_{\xi_1}(\Sigma_p(C_0)))\cdots),$$

and ξ_i is taken from the fixed point set of the iterated directional derivatives:

$$\xi_1 \in \Sigma_p(F),\ \xi_2 \in \Sigma_{\xi_1}(F_*), \dots, \xi_k \in \Sigma_{\xi_{i-1}}(F_{*(k-1)}),$$

and F_{*i} denotes the fixed point set of $f_{*i} : \Sigma_{*i}(C_0) \to \Sigma_{*i}(C_0)$ which coincides with $F_{*i} = \Sigma_{\xi_{i-1}}(F_{*(i-1)})$.

Note that the iterated space of directions $\Sigma_{*k}(C_0)$ has dimension $m-k$, and the iterated fixed point set $F_{*k} \subset \Sigma_{*k}(C_0)$ has dimension $\le m-k-2$. It follows that for some $k \le m-2$, F_{*k} becomes a finite set. It follows that for any $\xi_{k+1} \in F_{*k}$,

$$f_{*(k+1)} : \Sigma_{\xi_{k+1}}(\Sigma_{*k}(C_0)) \to \Sigma_{\xi_{k+1}}(\Sigma_{*k}(C_0))$$

has no fixed points. Put

$$D := C_0 \times_{\tilde{\phi}} [-t_0, t_0],$$

and let $\tilde{f}$ be an isometric involution on D defined by $\tilde{f} = (f, -\mathrm{id})$. From Theorem 3.3.4,

$$Y = D/\tilde{f}.$$

Let $x = \eta_0(p)$, $p = (p, 0)$, $\xi_i \in \Sigma_{\xi_{i-1}}(F_{*(i-1)})$, $1 \le i \le k+1$, be as above. Note that

$$\Sigma_x(Y) = \Sigma_p(D)/\tilde{f}_*,\ \Sigma_x(X) = \Sigma_p(C_0)/f_*.$$

Let $\zeta_1 \in \Sigma_x(\eta_0(F)) \subset \Sigma_x(X) \subset \Sigma_x(Y)$ be the element corresponding to $\xi_1 \in \Sigma_p(F) \subset \Sigma_p(C_0) \subset \Sigma_p(D)$. Note that

$$\Sigma_p(D) = \{\xi_\pm\} * \Sigma_p(C_0)$$

and $\tilde{f}_* = (f_*, -\mathrm{id})$ interchanges ξ_+ and ξ_- and preserves $\Sigma_p(C_0)$. Next consider

$$\Sigma_{\zeta_1}(\Sigma_x(Y)) = \Sigma_{\xi_1}(\Sigma_p(D))/\tilde{f}_{**},$$

where $\tilde{f}_{**}$ denotes the directional derivative of f_* at ζ_1. Note that $\Sigma_{\xi_1}(\Sigma_p(D))$ is still isometric to $\{\xi_\pm\} * \Sigma_{\xi_1}(\Sigma_p(C_0))$ and $\tilde{f}_{**} = (f_{**}, -\mathrm{id})$ interchanges ξ_+ and ξ_- and preserves $\Sigma_{\xi_1}(\Sigma_p(C_0))$. Similarly and finally we consider

$$\Sigma_{\zeta_{k+1}}(\Sigma_{*k}(Y)) = \Sigma_{\xi_{k+1}}(\Sigma_{*k}(D))/\tilde{f}_{*k+1}, \tag{4.1}$$

where $\zeta_{k+1} \in \Sigma_{*k}(Y)$ is the element corresponding to $\xi_{k+1} \in \Sigma_{*k}(D)$, and $\tilde{f}_{*k+1} = (f_{*k+1}, -\mathrm{id})$ freely acts on $\Sigma_{\xi_{k+1}}(\Sigma_{*k}(D))$. Recall that

$$\ell := \dim \Sigma_{\xi_{k+1}}(\Sigma_{*k}(D)) = m - k \ge 2.$$

Note that the iterated spaces of directions $\Sigma(Y \setminus C_{t_0})$ must be all homeomorphic to spheres (Theorem 1.2.10). However (4.1) shows that $\Sigma_{\zeta_{k+1}}(\Sigma_{*k}(Y))$ is homeomorphic to a quotient $\mathbb{S}^\ell/\mathbb{Z}_2$ for $\ell \ge 2$ by a free $\mathbb{Z}_2$-action, which is a contradiction. This completes the proof of Lemma 4.1.5. □

Remark 4.1.9. Though $Y \setminus C_{t_0}$ is not smoothable, since Kapovich's proof [Ka02] is local in nature, it also applies to the interior of Y.

4.2 Inradius collapses to almost regular spaces

Next we consider the case where M_i inradius collapses to an almost regular Alexandrov space N. The idea of using an equivariant fibration-capping theorem in [Ya02] was inspired by a recent work [MY:pre].

First we recall this theorem. Let X be a k-dimensional complete Alexandrov space with curvature $\geq \kappa$ possibly non-empty boundary. We denote by $D(X)$ the gluing space of X and it copy along their isometric boundaries, which is also an Alexandrov space with curvature $\geq \kappa$. (see [Pr94]).

A (k, δ)-strainer $\{(a_i, b_i)\}$ of $D(X)$ at $p \in X$ is called *admissible* if $a_i \in X$, $b_j \in X$ for every $1 \leq i \leq k$, $1 \leq j \leq k-1$ (clearly, $b_k \in X^*$ if $p \in \partial X$ for instance). Let $R^D_\delta(X)$ denote the set of points of X at which there are admissible (k, δ)-strainers. It has the structure of a Lipschitz k-manifold with boundary. Note that every point of $R^D_\delta(X) \cap \partial X$ has a small neighborhood in X almost isometric to an open subset of the half space $\mathbb{R}^k_+$ for small δ.

If Y is a closed domain of $R^D_\delta(X)$, then the δ_D-*strain radius* of Y is defined as the infimum of positive numbers ℓ such that there exists an admissible (k, δ)-strainer of length $\geq \ell$ at every point in Y, denoted by δ_D-str.rad(Y).

For a small $\nu > 0$, we put

$$Y_\nu := \{x \in Y \mid d(\partial X, x) \geq \nu\}.$$

We use the following special notations:

$$\partial_0 Y_\nu := Y_\nu \cap \{d_{\partial X} = \nu\}, \quad \mathrm{int}_0 Y_\nu := Y_\nu - \partial_0 Y_\nu.$$

Let M^n be another n-dimensional complete Alexandrov space with curvature $\geq \kappa$ having no boundary. Let $R_\delta(M)$ denote the set of all (n, δ)-strained points of M.

A surjective map $f : M \to X$ is called an ϵ-*almost Lipschitz submersion* if

1. it is an ϵ-approximation;

2. for every $p, q \in M$
$$\left| \frac{d(f(p), f(q))}{d(p,q)} - \sin \theta_{p,q} \right| < \epsilon,$$

where $\theta_{p,q}$ denotes the infimum of $\angle qpx$ when x runs over $f^{-1}(f(p))$.

Now let a Lie group G act on M^n and X as isometries. Let

$$d_{e.GH}((M, G), (X, G))$$

denote the equivariant Gromov-Hausdorff distance as defined in Section 1.1. We need to assume the following on the existence of slice for G-orbits:

Assumption 4.2.1. For each $p \in X$, there is a *slice* L_p at p. Namely $U_p := GL_p$ provides a G-invariant tubular neighborhood of Gp which is G-isomorphic to $G \times_{G_p} L_p$.

Obviously Assumption 4.2.1 is automatically satisfied if G is discrete. By [HS:pre], Assumption 4.2.1 also holds true if G is compact.

Theorem 4.2.2 (Equivariant Fibration-Capping Theorem([Ya02], Thm 18.9))**.** *Let X and G be as above such that X/G is compact. Given k and $\mu > 0$ there exist positive numbers $\delta = \delta_k$, $\epsilon_{X,G}(\mu)$ and $\nu = \nu_{X,G}(\mu)$ satisfying the following : Suppose $X = R^D_\delta(X)$ and δ_D-str.rad$(X) > \mu$. Suppose $M = R_{\delta_n}(M)$ and $d_{eGH}((M,G),(X,G)) < \epsilon$ for some $\epsilon \le \epsilon_{X,G}(\mu)$. Then there exists a G-invariant decomposition*

$$M = M_{\rm int} \cup M_{\rm cap}$$

of M into two closed domains glued along their boundaries, and a G-equivariant Lipschitz map $f : M \to X_\nu$ such that

1. *$M_{\rm int}$ is the closure of $f^{-1}(\mathrm{int}_0 X_\nu)$, and $M_{\rm cap} = f^{-1}(\partial_0 X_\nu)$;*
2. *the restrictions $f|_{M_{\rm int}} : M_{\rm int} \to X_\nu$ and $f|_{M_{\rm cap}} : M_{\rm cap} \to \partial_0 X_\nu$ are*
   - *(a) locally trivial fiber bundles;*
   - *(b) $\tau(\delta, \nu, \epsilon/\nu)$-Lipschitz submersions.*

Here, $\tau(\epsilon_1, \ldots, \epsilon_k)$ denotes a function depending on a priori constants and ϵ_i satisfying

$$\lim_{\epsilon_i \to 0} \tau(\epsilon_1, \ldots, \epsilon_k) = 0.$$

Remark 4.2.3. If X has no boundary, then X_ν is replaced by X, $M_{\rm cap} = \emptyset$ and $M = N$ in the statement above.

We go back to the situation of Theorem 0.2.3. Assume that M_i inradius collapses to an almost regular Alexandrov space N. Let us consider the double and the partial double of $\tilde{M}_i$ and Y respectively

$$D(\tilde{M}_i) := \tilde{M}_i \amalg_{\partial \tilde{M}_i} \tilde{M}_i, \quad W := Y \amalg_{C_{t_0}} Y.$$

From Perelman's result [Pr94], both $D(\tilde{M}_i)$ and W are Alexandrov space. Note that both $D(M_i)$ and W admit canonical isometric $\mathbb{Z}_2$ actions by the reflections.

Lemma 4.2.4. *$(D(\tilde{M}_i), \mathbb{Z}_2)$ converges to $(W, \mathbb{Z}_2)$ with respect to the equivariant Gromov-Hausdorff convergence.*

Proof. Let $\psi_i : \tilde{M}_i \to Y$ and $\varphi_i : Y \to \tilde{M}_i$ be ε_i-approximations, where $\varepsilon_i \searrow 0$, and $\psi_i^* : \tilde{M}_i^* \to Y^*$ and $\varphi_i^* : Y^* \to \tilde{M}_i^*$ the copy of ψ_i and φ_i respectively. We define two maps $\bar{\psi}_i : D(\tilde{M}_i) \to W$ and $\bar{\psi}_i : W \to D(\tilde{M}_i)$ as follows,

$$\bar{\psi}_i(x) = \begin{cases} \psi_i(x) & \text{if } \ x \in \tilde{M}_i \\ \psi_i^*(x) & \text{if } \ x \in \tilde{M}_i^*, \end{cases} \bar{\varphi}_i(x) = \begin{cases} \varphi_i(x) & \text{if } \ x \in Y \\ \varphi_i^*(x) & \text{if } \ x \in Y^*. \end{cases}$$

We claim that $\bar{\psi}_i$ is a ε_i''-approximation, where $\varepsilon_i'' \to 0$. Clearly,

$$B_{\varepsilon_i}(\bar{\psi}_i(D(\tilde{M}_i))) = W.$$

Next, we shall prove that $||x_i y_i|_{D(\tilde{M}_i)} - |\bar{\psi}_i(x_i)\bar{\psi}_i(y_i)|_W| < \varepsilon_i''$ for any two points x_i, y_i in $D(\tilde{M}_i)$, where $\varepsilon_i'' \to 0$. Let $\gamma_i : [0,1] \to D(\tilde{M}_i)$ be a shortest path joining x_i and y_i. It suffices to consider the following two cases.

Case 1. x_i, $y_i \in \tilde{M}_i$.

Without loss of generality, we assume $\gamma_i \subset \tilde{M}_i($ or $\tilde{M}_i^*)$. Clearly, $|x_i y_i|_{D(\tilde{M}_i)} = |x_i y_i|_{\tilde{M}_i}$ and both $\bar{\psi}_i(x_i)$ and $\bar{\psi}_i(y_i)$ in Y in this case. Thus $\bar{\psi}_i(x_i) = \psi_i(x_i)$ and $\bar{\psi}_i(y_i) = \psi(y_i)$ and them can be joined by a shortest path completely contained in Y. Hence $|\bar{\psi}_i(x_i)\bar{\psi}_i(y_i)|_W = |\psi_i(x_i)\psi_i(y_i)|_Y$. Therefore,

$$||x_i y_i|_{D(\tilde{M}_i)} - |\bar{\psi}_i(x_i)\bar{\psi}_i(y_i)|_W| = ||x_i y_i|_{\tilde{M}_i} - |\psi_i(x_i)\psi_i(y_i)|_Y| < \varepsilon_i.$$

Case 2. $x_i \in \tilde{M}_i$ and $y_i \in \tilde{M}_i^*$.

Clearly, there exists a point $c_i \in (0,1)$ such that $z_i := \gamma_i(c_i) \in \partial\tilde{M}_i$. By triangle inequality, we have

$$\begin{aligned} &|\bar{\psi}_i(x_i)\bar{\psi}_i(y_i)|_W - |x_i y_i|_{D(\tilde{M}_i)} \\ &\le |\psi_i(x_i)\psi_i(z_i)|_Y + |\psi_i^*(z_i)\psi_i^*(y_i)|_{Y^*} - (|x_i z_i|_{\tilde{M}_i} + |z_i y_i|_{\tilde{M}_i^*}) \\ &\le |\psi_i(x_i)\psi_i(z_i)|_Y - |x_i z_i|_{\tilde{M}_i}) + (|\psi_i^*(z_i)\psi_i^*(y_i)|_{Y^*} - |z_i y_i|_{\tilde{M}_i^*}) \\ &< 2\varepsilon_i. \end{aligned}$$

Let γ_i' be a shortest path in Z joining $\bar{\psi}_i(x_i) \in Y$ and $\bar{\psi}_i(y_i) \in Y^*$, $\xi_i \in \gamma_i' \cap C_{t_0} \subset \partial Y$. Suppose $z_i' \in \tilde{M}_i$ such that

$$|\xi_i \psi_i(z_i')|_Y < \varepsilon_i. \tag{4.2}$$

Hence

$$|\psi_i(z_i')C_{t_0}|_Y < \varepsilon_i \tag{4.3}$$

Such z_i' also satisfies

$$|z_i'\partial\tilde{M}_i|_{\tilde{M}_i} \to 0. \tag{4.4}$$

Otherwise, there exists a subsequence of M_i such that $|z_i'\partial\tilde{M}_i|_{\tilde{M}_i} \geq 2\varepsilon_0$ for some fix positive number ε_0. Thus $|z_i'\partial M_i^{\text{ext}}| \leq t_0 - \varepsilon_0$. Let $z' := \lim_{i\to\infty}\psi_i(z_i')$. Recall that t_0 is the height of C_i. Hence

$$|z'X|_Y \leq t_0 - \varepsilon_0,$$

since $\lim_{i\to\infty}\partial M_i^{\text{ext}} = X$ by Proposition 2.1.9. Thus by Lemma 2.1.2, we have

$$|z'C_{t_0}| \geq \varepsilon_0,$$

which contradicts to (4.3). Hence the inequality (4.4) holds.

Let $\eta : D(\tilde{M}_i) \to D(\tilde{M}_i)$ be the reflection, $z_i'' := \eta(z_i') \in \tilde{M}_i^*$. Clearly,

$$|\xi_i\psi_i^*(z_i'')|_Y = |\xi_i\psi_i(z_i')|_Y < \varepsilon_i \text{ and } \varepsilon_i' := |z_i'z_i''|_{D(\tilde{M}_i)} = 2|z_i'\partial\tilde{M}_i|_{\tilde{M}_i} \to 0,$$

by the symmetry, (4.2) and (4.4). Then

$$\begin{aligned}
&|\bar{\psi}_i(x_i)\bar{\psi}_i(y_i)|_W - |x_iy_i|_{D(\tilde{M}_i)} \\
&\geq |\psi_i(x_i)\xi|_Y + |\xi\psi_i(y_i)|_{Y^*} - |x_iz_i'|_{\tilde{M}_i} - |z_i'y_i|_{\tilde{M}_i^*} \\
&\geq (|\psi_i(x_i)\psi_i(z_i')|_Y - |x_iz_i'|_{\tilde{M}_i}) - |\xi\psi_i(z_i')|_Y - |z_i'z_i''|_{D(\tilde{M}_i)} \\
&+ (|\psi_i^*(z_i'')\psi_i^*(y_i)|_{Y^*} - |z_i''y_i|_{\tilde{M}_i^*}) - |\xi\psi_i^*(z_i'')|_{Y^*} \\
&> -4\varepsilon_i - \varepsilon_i'
\end{aligned}$$

Hence, $\bar{\psi}_i$ is an ε_i''-approximation, where $\varepsilon_i'' = 4\varepsilon_i + \varepsilon_i' \to 0$ as $i \to \infty$. Clearly, $\bar{\psi}_i$ is $\mathbb{Z}_2$-equivariant. It can be proved in a similar way that $\bar{\varphi}_i$ is an equivariant ε_i'''-approximation, where $\varepsilon_i''' \to 0$ as $i \to \infty$. □

Proof of Theorem 0.2.3. By Lemma 4.2.4, for any $\varepsilon > 0$, if i large,

$$d_{eGH}((D(\tilde{M}_i), \mathbb{Z}_2), (W, \mathbb{Z}_2)) < \varepsilon.$$

By Theorem 3.3.4, Y is almost regular with almost regular boundary. Hence, $W = R_\delta^D(W)$ and δ_D-str.rad$(W)) > \mu$ for some $\mu > 0$. Thus by Theorem 4.2.2 and its remark, there exists a $\mathbb{Z}_2$-equivariant capping fibration

$$\tilde{f}_i : D(\tilde{M}_i) \to W_\nu,$$

where

$$W_\nu = \{x \in W \mid d(x, \partial W) \geq \nu\}.$$

Notice that the image W_ν is homeomorphic to W because of the form of Y. Obviously, $\tilde{f}_i$ induces a map $f_i : \tilde{M}_i \to Y$. By the remark after Corollary 3.3.6, $\eta_0 : C_0 \to X$ is either an isometry or a locally isometric double covering.

Case (a). If $\eta_0 : C_0 \to X$ is a double covering, then $C_{t_0} = \partial Y$. Hence W has no boundary. Thus in this case, $f_i : \tilde{M}_i \to Y$ is a fiber bundle with fiber F_i which are closed almost nonngetively curved manifolds. Since Y is an I-bundle over N by Theorem 3.3.4, $\tilde{M}_i$ and hence M_i is an $F_i \times I$-bundle over N.

Case (b). If $\eta_0 : C_0 \to X$ is an isometry, then Y is isometric to $N \times_\phi [0, t_0]$, and therefore ∂Y consists of $\eta(C_0) = X$ and $\eta(C_{t_0})$. Thus ∂W consists two copies of $\eta_0(C_0)$. Therefore by Theorem 4.2.2, there exists a $\mathbb{Z}_2$-invariant decomposition

$$D(\tilde{M}_i) = (D(\tilde{M}_i))_{\rm int} \cup (D(\tilde{M}_i))_{\rm cap}, \tag{4.5}$$

of $D(\tilde{M}_i)$ into two closed domains glued along their boundaries such that

1. $(D(\tilde{M}_i))_{\rm int}$ is the closure of $\tilde{f}_i^{-1}({\rm int}_0 W_\nu)$, and $(D(\tilde{M}_i))_{\rm cap} = \tilde{f}_i^{-1}(\partial_0 W_\nu)$;
2. $\tilde{f}_i|_{(D(\tilde{M}_i))_{\rm int}} : (D(\tilde{M}_i))_{\rm int} \to W_\nu$, $\tilde{f}_i|_{(D(\tilde{M}_i))_{\rm cap}} : (D(\tilde{M}_i))_{\rm cap} \to \partial_0 W_\nu$ are locally trivial fiber bundles,

where

$$\partial_0 W_\nu := \{x \in W \mid d(x, \partial W) = \nu\}, \quad {\rm int}_0 W_\nu := W_\nu \setminus \partial_0 W_\nu.$$

Since (4.5) is $\mathbb{Z}_2$-invariant, it induces a decomposition

$$\tilde{M}_i = (\tilde{M}_i)_{\rm int} \cup (\tilde{M}_i)_{\rm cap}.$$

Since $\tilde{f}_i$ is $\mathbb{Z}_2$-equivariant, these fibrations induce fibrations

$$F_i \longrightarrow (\tilde{M}_i)_{\rm int} \longrightarrow Y_\nu,$$
$${\rm Cap}_i \longrightarrow (\tilde{M}_i)_{\rm cap} \longrightarrow \partial_0 Y_\nu.$$

From construction, $\partial{\rm Cap}_i$ is homeomorphic to F_i. Note that every cylindrical geodesic in the warped cylinder $C_i \subset \tilde{M}_i$ is almost perpendicular to the fibers ([Ya91], [Ya96]). This implies that $(\tilde{M}_i)_{\rm int}$ is homeomorphic to $\partial(\tilde{M}_i)_{\rm int} \times [0,1]$, and therefore $\tilde{M}_i$ and hence M_i is homeomorphic to $(\tilde{M}_i)_{\rm cap}$. Noting $\partial_0 Y_\nu$ is homeomorphic to N, we obtain a fiber bundle

$${\rm Cap}_i \longrightarrow M_i \longrightarrow N.$$

This completes the proof. □

Chapter 5

The case of unbounded diameters

In this chapter we provide the proof of Theorem 0.2.5. All results in this chapter are due to a joint work with Prof. Takao Yamaguchi [YZ15].

Let $\mathcal{M}(n,\kappa,\lambda)$ denote the set of all isometry classes of n-dimensional complete Riemannian manifolds M satisfying

$$K_M \geq \kappa, \ |\Pi_{\partial M}| \leq \kappa.$$

Let

$$\tilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$$

denote the set of all $(\tilde{M}, M, p)$ with $M \in \mathcal{M}(n,\kappa,\lambda)$ and $p \in \partial M$.

We denote by

$$\partial_0\mathcal{M}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$$

the set of all pointed Gromov-Hausdorff limit spaces (Y, X, x) of sequences $(\tilde{M}_i, M_i, p_i)$ in $\tilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ with $\mathrm{inrad}(M_i) \to 0$. From now, $(\tilde{M}_i, M_i, p_i)$ and $(\tilde{M}, M, p)$ are always assumed to be elements in $\tilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$.

We first remark

Lemma 5.0.5. *Let $(\tilde{M}_i, M_i, p_i)$ converges to (Y, X, x) in $\partial_0\mathcal{M}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ with $\mathrm{inrad}(M_i)$ converges to zero. Then all the structure results for the limit spaces in Section 3.1 and Lemma 3.2.1 still holds for (Y, X).*

Proof. This is because all the argument there are local. □

However in the proof of Corollary 3.2.2, the compactness of C_0 is essentially used. To improve this proof for the noncompact C_0, we need to establish the following result, which is a weaker form of Lemma 3.3.2.

Proposition 5.0.6. *For every $x, y \in X$ and $p \in C_0$ with $\eta_0(p) = x$, there exists a point $q \in C_0$ such that $\eta_0(q) = y$ and $|p, q| = |x, y|_{X^{\mathrm{int}}}$.*

Proof. Let $c : [0, \ell] \to X$ be a unit speed minimal geodesic joining x to y, and let $v := \dot{c}(0) \in \Sigma_x(X)$. By Lemmas 3.1.2 and 3.1.3, $\angle(\xi_+, v) = \pi/2$. Let $\xi \in \Sigma_x(Y)$ be an element satisfying (3.1) or (3.5) together with $\angle(\xi_+, \xi) = \pi/4$. We may assume that there is a geodesic γ from x in the direction ξ. Consider $\tilde{\gamma} := \eta^{-1}(\gamma)$ and $\tilde{\sigma} := \tilde{\pi}(\tilde{\gamma})$. Let $\sigma := \pi(\gamma) = \eta_0(\tilde{\sigma})$, and σ define a direction $\hat{v}$. As was shown in the proof of Lemma 3.1.2, $\hat{v}$ also satisfies the same equation (3.1) as v. This shows that $\hat{v} = v$ and σ is infinitesimally minimizing because it has a definite direction at x. Therefore for every $\epsilon > 0$, setting $x_1 := \sigma(t_1)$ for small enough $t_1 > 0$, we have

$$|y, x_1| \le |x, y| - (1 - \epsilon)L(\sigma_1),$$

where $\sigma_1 = \sigma|_{[0,t_1]}$. We repeat this argument for a minimal geodesic $c_1 : [0, \ell_1] \to X$ joining x_1 to y, and finally we have an infinite sequence of points $\{x_i\}_{i=1}^{\infty}$ and Lipschitz curves $\{\sigma_i\}_{i-1}^{\infty}$ and $\{\tilde{\sigma}_i\}_{i-1}^{\infty}$ joining x_{i-1} to x_i and $\tilde{x}_{i-1}$ to $\tilde{x}_i$ respectively such that

1. $\eta_0(\tilde{\sigma}_i) = \sigma_i$;
2. $|y, x_k| \le |x, y| - (1 - \epsilon) \sum_{i=1}^{k} L(\sigma_i)$ for each $1 \le k < \infty$;
3. $\lim x_k = y$.

Let the curves σ_ϵ and $\tilde{\sigma}_\epsilon$ be defined by the union of those σ_i and $\tilde{\sigma}_i$ respectively. It follows that σ_ϵ is an almost minimizing curve joining x to y. Passing to a subsequence we may assume that σ_ϵ and $\tilde{\sigma}_\epsilon$ converge to curves σ_∞ and $\tilde{\sigma}_\infty$ respectively satisfying $\eta_0(\tilde{\sigma}_\infty) = \sigma_\infty$. Note that both σ_∞ and $\tilde{\sigma}_\infty$ are minimizing since η_0 is 1-Lipschitz and preserving length by Proposition 3.1.5. Thus we have a required point q as the endpoint of $\tilde{\sigma}_\infty$ different from p. □

By Proposition 5.0.6, we can improve the proof of Corollary 3.2.2 without assuming the compactness. Thus Corollary 3.2.2 holds for the present case, too. We also see that all the results in Section 3 still holds true except Corollary 3.2.3. In particular we have

Theorem 5.0.7. *Let a sequence of pointed complete Riemannian manifolds (M_i, p_i) in $\mathcal{M}(n, \kappa, \lambda)$ inradius collapse to a pointed length space (N, q) with respect to the pointed Gromov-Hausdorff convergence. Then N is an Alexandrov space with curvature $\ge c(\kappa, \lambda)$, where $c(\kappa, \lambda)$ is a constant depending only on κ and λ.*

To have Corollary 3.2.3 in the case when Y is noncompact is the main purpose of the rest of this section.

We introduce a more refined version of the pointed Gromov-Hausdorff convergence. Let $\iota_{\partial M} : (\partial M)^{\rm int} \to: (\partial M)^{\rm ext}$ be the canonical map, where $(\partial M)^{\rm ext}$ is equipped with the exterior metric in M. Let $\omega_M : M \to \partial M$ be a nearest point map (compare Proposition 2.1.9).

Definition 5.0.8. For $(\tilde{M}, M, p) \in \tilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ and $(Y, X, x) \in \partial_0\mathcal{M}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ with

$$Y = X \bigcup_{\eta_0} C_0 \times_\phi [0, t_0], \tag{5.1}$$

we define the pointed Gromov-Hausdorff distance

$$d_{pGH}((\tilde{M}, M, p), (Y, X, x))$$

as the infimum of those $\delta > 0$ such that

1. there exists a componentwise δ-approximation

$$\psi : B^{\tilde{M}}(p, 1/\delta) \cap (\partial M)^{\mathrm{int}} \to B^Y(x, 1/\delta) \cap C_0^{int};$$

2. the map $\varphi : B^{M^{\mathrm{int}}}(p, 1/\delta) \to B^{X^{\mathrm{int}}}(x, 1/\delta)$ defined by

$$\varphi = \eta_0 \circ \psi \circ \iota_{\partial M}^{-1} \circ \omega_M$$

   is a δ-approximation;

3. the map $\varPhi : B^{\tilde{M}}(p, 1/\delta) \to B^Y(x, 1/\delta)$ defined by

$$\varPhi(q) = \begin{cases} \varphi(q), & q \in B^{\tilde{M}}(p, 1/\delta) \cap M \\ (\eta_0 \circ \psi \circ \iota_{\partial M}^{-1}(q_1), t), & q = (q_1, t) \in B^{\tilde{M}}(p, 1/\delta) \cap \partial M \times [0, t_0]. \end{cases}$$

   is a δ-approximation.

This definition is justified by the following lemma.

Lemma 5.0.9. *Let*

$$(\tilde{M}_i, M_i, p_i) \in \tilde{\mathcal{M}}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$$

converge to (Y, X, x) *in* $\partial_0\mathcal{M}\mathcal{M}(n,\kappa,\lambda)_{\mathrm{pt}}$ *with respect to the pointed Gromov-Hausdorff topology. Then there exists a componentwise* δ_i*-approximation*

$$\psi_i : B^{\tilde{M}_i}(p_i, 1/\delta_i) \cap (\partial M_i)^{\mathrm{int}} \to B^Y(x, 1/\delta_i) \cap C_0$$

with $\lim_{i\to\infty} \delta_i = 0$ *such that the maps*

$$\varphi_i : B^{M_i^{\mathrm{int}}}(p_i, 1/\delta_i) \to B^{X^{\mathrm{int}}}(x, 1/\delta_i),$$
$$\varPhi_i : B^{\tilde{M}_i}(p_i, 1/\delta_i) \to B^Y(x, 1/\delta_i)$$

defined as in Definition 5.0.8 via ψ_i *are* δ_i'*-approximations with* $\lim_{i\to\infty} \delta_i' = 0$.

Proof. Let

$$\lambda_i : B^{\tilde{M}_i}(p_i, 1/\epsilon_i) \to B^Y(x, 1/\epsilon_i)$$

be an ϵ_i-approximation with $\lim_{i\to\infty} \epsilon_i = 0$. When it is restricted to the boundary, it provides a componentwise ϵ_i-approximation

$$\lambda_i' : B^{\tilde{M}_i}(p_i, 1/\epsilon_i) \cap \partial M_i \to B^Y(x, 1/\epsilon_i) \cap C_{t_0}.$$

Since $\partial \tilde{M}_i$ and C_{t_0} are totally geodesic and $\phi(t_0)$-homeomorc to $(\partial M_i)^{\mathrm{int}}$ and C_0 respectively, λ_i' gives a component-wise $\epsilon_i/\phi(t_0)$-approximation

$$\psi_i : B^{\tilde{M}_i}(p_i, 1/\epsilon_i) \cap (\partial M_i)^{\mathrm{int}} \to B^Y(x, 1/\epsilon_i) \cap C_0.$$

Now the conclusion follows in a way similar to Proposition 2.1.9. □

Lemma 5.0.10. *For each $\delta > 0$ there exists a positive number $\epsilon = \epsilon(\delta)$ such that if (M, p) in $\mathcal{M}(n, \kappa, \lambda)$ satisfies* $\mathrm{inrad}(M) < \epsilon$*, then*

$$d_{pGH}((\tilde{M}, M, p), (Y, X, x)) < \delta,$$

for some (Y, X, x) contained in $\partial_0 \mathcal{MM}(n, \kappa, \lambda)$.

Proof. Lemma 5.0.10 follows from Lemma 5.0.9 and the precompactness of $\tilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\mathrm{pt}}$ combined with a contradiction argument. □

If $(Y, X, x) \in \partial_0 \mathcal{MM}(n, \kappa, \lambda)$ satisfies the conclusion of Lemma 5.0.10, we call it a *δ-limit* of $(\tilde{M}, M, p)$, which is also denoted by $\mathcal{Y}(M, p)$ for simplicity:

$$\mathcal{Y}(M, p) = (Y, X, x).$$

Definition 5.0.11. Let $(Y, X, x) \in \partial_0 \mathcal{MM}(n, \kappa, \lambda)$ and $y \in X$. We call y a *single point* (resp. *double point*) if $\#\eta_0^{-1}(y) = 1$ (resp. $\#\eta_0^{-1}(y) = 2$). We say that (Y, X, x) is single (resp. double) if every element of X is single (resp double). If (Y, X, x) neither single nor double, it is called *mixed*. We also say that (Y, X, x) is single (resp. double) in scale R if every element of $X \cap B^Y(x, R)$ is single (resp. double). If (Y, X, x) is neither single nor double in scale R, it is called mixed in scale R.

From now on, to prove Theorem 0.2.5, we analyze the local structure of ∂M about the connectedness when $\mathrm{inrad}(M) < \epsilon$. By Lemma 5.0.10, for any $p \in M$, there exists a δ-limit $\mathcal{Y}(M, p) = (Y, X, x)$ together with

1. a δ-approximation $\psi : (\partial M)^{\mathrm{int}} \cap B^M(p, R) \to C_0(p, R)$, where $C_0(p, R)$ is a closed domain in C_0^{int};
2. a δ-approximation $\varphi := \eta_0 \circ \psi \circ \iota_{\partial M}^{-1} \circ \omega_M : B^M(p, R) \to B^{X^{\mathrm{int}}}(x, R)$.

We shall use those δ-approximations in the proofs of Lemmas 5.0.12, 5.0.13 and 5.0.14 below.

Lemma 5.0.12. *For any $R > 0$ there exists $\delta_0 < 1/R$ satisfying the following: For every $0 < \delta \le \delta_0$, let $\epsilon = \epsilon(\delta) > 0$ be as in Lemma 5.0.10. Then for M in $\mathcal{M}(n, \kappa, \lambda)$ with $\mathrm{inrad}(M) < \epsilon$, if some δ-limit $\mathcal{Y}(M, p)$ is single in scale R for some $p \in M$, then every $p_1, p_2 \in \partial M \cap B^{\tilde{M}}(p, R)$ can be joined by a curve in ∂M of length $\le |p_1, p_2|_M + 2\delta$.*

Proof. Let $(Y, X, x) := \mathcal{Y}(M, p)$, and ψ, φ be δ-approximations as above. Put $x_i := \varphi(p_i) \in X$, $i = 1, 2$. Take $\tilde{x}_i \in C_0$ such that $\eta_0(\tilde{x}_i) = x_i$. Lemma 3.2.6 shows $|\tilde{x}_1, \tilde{x}_2| = |x_1, x_2|$. Since ψ is a δ-approximation, we have

$$|p_1, p_2|_{\partial M} < |\tilde{x}_1, \tilde{x}_2| + \delta = |x_1, x_2| + \delta < |p_1, p_2|_M + 2\delta.$$

$\square$

Lemma 5.0.13. *For any $R > 0$ there exists $\delta_0 < 1/R$ satisfying the following: For every $0 < \delta \le \delta_0$, let $\epsilon = \epsilon(\delta) > 0$ be as in Lemma 5.0.10. Then for M in $\mathcal{M}(n, \kappa, \lambda)$ with $\mathrm{inrad}(M) < \epsilon$, if a δ-limit $\mathcal{Y}(M, p)$ is double in scale R for some $p \in M$, then there exists a point $p' \in M$ satisfying*

1. *$|p, p'|_M < \delta$;*
2. *every $q \in \partial M \cap B^{\tilde{M}}(p, R)$ can be joined to p or p' by a curve in ∂M of length $\le |p, q|_M + 3\delta$.*

Proof. Let $(Y, X, x) := \mathcal{Y}(M, p)$, and ψ, φ be δ-approximations as above. Set $x := \varphi(p)$, $y := \varphi(q)$. Since (Y, X, x) is double in scale R, we can put $\{\tilde{x}_1, \tilde{x}_2\} := \eta_0^{-1}(x)$ and $\{\tilde{y}_1, \tilde{y}_2\} := \eta_0^{-1}(y)$. Let $\gamma : [0, 1] \to X$ be a minimal geodesic joining x to y. From Lemma 3.3.2, there are lifts $\tilde{\gamma}_i : [0, 1] \to C_0$ of γ starting from $\tilde{x}_i$, where we may assume $\tilde{\gamma}(1) = \tilde{y}_i$ and $\tilde{x}_1 = \psi(p)$. Since ψ is a δ-approximation, if $\psi(q) = \tilde{y}_1$, then

$$|p, q|_{(\partial M)^{\mathrm{int}}} < |\tilde{x}_1, \tilde{y}_1| + \delta = |x, y| + \delta < |p, q|_{\partial M^{\mathrm{ext}}} + 2\delta$$

Similarly, if $\psi(q) = \tilde{y}_2$, then putting $p' := \psi^{-1}(\tilde{x}_2)$, we have $|p', q|_{(\partial M)^{\mathrm{int}}} < |p, q|_M + 3\delta$. This completes the proof. $\square$

Lemma 5.0.14. *For any $R > 0$ there exists $\delta_0 < 1/R$ satisfying the following: For every $0 < \delta \le \delta_0$, let $\epsilon = \epsilon(\delta) > 0$ be as in Lemma 5.0.10. Then for M in $\mathcal{M}(n, \kappa, \lambda)$ with $\mathrm{inrad}(M) < \epsilon$, if a δ-limit $\mathcal{Y}(M, p)$ is mixed in scale R for some $p \in M$, then there exists a point $p_0 \in \partial M \cap B^{\tilde{M}}(p, R)$ such that every point q in $\partial M \cap B^{\tilde{M}}(p, R)$ can be joined to p_0 by a minimal geodesic in ∂M of length $|p_0, q|_M + 2\delta$.*

Proof. Let $(Y, X, x) := \mathcal{Y}(M, p)$, and ψ, φ be δ-approximations as above. Let $x_0 \in X$ be a single point with $|x, x_0| \le R$, and take $\tilde{x}_0 \in C_0$ and $p_0 \in \partial M$ such that $\eta_0(\tilde{x}_0) = x_0$ and $|\psi(p_0), \tilde{x}_0| < \delta$ Let $\gamma : [0, 1] \to X$ be a minimal geodesic from x_0 to $\varphi(q)$. Since $\tilde{x}_0 \in C_0^1$, there is a unique minimal geodesic $\tilde{\gamma} : [0, 1] \to C_0$ from $\tilde{x}_0$ to $\psi(q)$ with $\eta_0 \circ \tilde{\gamma} = \gamma$ (see Lemma 3.3.2). Since ψ is a δ-approximation, we have

$$\begin{aligned}|p_0, q| &< |\tilde{x}_0, \psi(q)| + \delta = |x_0, \varphi(q)|_{X^{\rm int}} + \delta \\ &\le |\varphi(q), \varphi(p_0)|_{X^{\rm int}} + |\varphi(p_0), x_0|_{X^{\rm int}} + \delta \\ &\le |p_0, q|_M + 2\delta.\end{aligned}$$

□

Lemma 5.0.15. *For any $R > 0$ and $\delta < 1/R$, there exists $\epsilon > 0$ satisfying the following: If M in $\mathcal{M}(n, \kappa, \lambda)$ with* $\mathrm{inrad}(M) < \epsilon$ *has disconnected boundary ∂M, then every δ-limit $\mathcal{Y}(M, p)$ is double in scale R for every $p \in M$.*

Proof. Suppose that some limit $\mathcal{Y}(M, p) = (Y, X, x)$ is single or mixed in scale R. First note that by Lemmas 5.0.12 and 5.0.14, every points q_1, q_2 in $\partial M \cap B^{\tilde{M}}(p, R)$ can be joined by a curve in ∂M. Take a point $p_\alpha \in \partial M$ contained in a component different from that containing p. Let $c : [0, \ell] \to M$ be a unit speed minimal geodesic in M from p to $p_\alpha \in \partial$. For each i with $1 \le i \le [\ell/R]$, take $p_i \in \partial M$ with $|p_i, c(iR)|_M < \epsilon$. By applying Lemmas 5.0.12, 5.0.14 and 5.0.13 to p_i together with a standard monodoromy argument, we see that p_α can be joined to p in ∂M, which is a contradiction. □

We are now ready to prove Theprem 0.2.5.

Proof of Thoerem 0.2.5. (1) Suppose that ∂M is disconnected. By Lemma 5.0.15, every δ-limit $\mathcal{Y}(M, p)$ is double in scale R for every $p \in M$. Take p_α and p_β from distinct components of ∂M. For every $p \in \partial M$, let $c : [0, \ell] \to M$ be a unit speed curve in M from p_α to p_β through p. For each i with $1 \le i \le [\ell/R]$, take $p_i \in \partial M$ with $|p_i, c(iR)|_M < \epsilon$. By applying Lemma 5.0.13 to each p_i together with a standard monodoromy argument, we see that p can be joined to p_α or p_β by a curve in ∂M. Therefore we conclude that the number of boundary components of M is at most two.

(2) Suppose that ∂M has two components. By Lemma 5.0.13, any δ-limit $\mathcal{Y}(M, p) = (Y, X, x)$ is double in scale R for every $p \in \partial M$. Therefore for any $x \in X$, there are distinct $y_1 \ne y_2 \in C_{t_0}$ with $|y_i, x| = t_0$. Take $q_i \in \partial \tilde{M}$, $i = 1, 2$, which are δ-close to y_i in the Gromov-Hausdorff distance. From Lemma 5.0.13, q_1 and q_2 must belong to distinct components of $\partial \tilde{M}$, which implies $|q_1, q_2| \ge 2t_0$, and hence $|y_1, y_2| = 2t_0$. Let W be a component of $\partial \tilde{M}$, and consider the distance function d_W from W. The above observation shows that for every $0 < \epsilon_0 < \pi$, d_W is ϵ_0-regular on a neighborhood of M in

$\tilde{M}$ if $\delta = \delta(\epsilon_0, t_0) > 0$ is taken small enough. This means that for any $p \in M$, there exists a point $q \in \partial\tilde{M}$ such that

$$\tilde{\angle}Wpq > \pi - \epsilon_0.$$

This makes it possible to define locally defined gradient-like vector fields for d_W on neighborhoods of the points of M. Then by gluing those local gradient-like vector fields, we get a globally defined gradient-like vector field V on $\tilde{M}$ whose support is contained in a neighborhood of M. It is now straightforward to obtain a diffeomorphism between $\tilde{M}$ and $W \times [0, 1]$ by means of integral curves of V. $\square$

Chapter 6

Typical inradius collapses

In this chapter, we adopt the same notations as Chapter 2. For a convergent sequence $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ with respect to Gromov-Hausdorff distance, we will show that if the its limit space is topological or metrical nice, then M_i inradius collapses. This result is a joint work with Prof. Takao Yamaguchi [YZ15].

Note that, Wong, cf. Proposition 2 and Lemma 3 in [Wo10], also studied the conditions for manifolds inradius collapse. However, his conditions are very strong. The reason for him to study inradius collapse is proving his fibration theorem for codimention one collapse, cf. Theorem 3 in [Wo10], by Theorem 0.1.6.

We always assume $m = \dim Y = \lim_{GH} \tilde{M}_i$ as before. For convenience, we assume there exists a $c_0 > 0$ such that $\mathrm{inrad}(M_i) > c_0$ for all i. Then we immediately have

$$\dim X = m.$$

Let X_0 be the limit space of $\partial M_i^{\mathrm{ext}}$. Since we don't assume M_i inradius collapses, X_0 is not equal to X in general. Let $\eta_0 : C_0 \to X_0$ be the limit map of the inclusion map $\iota_i : \partial M_i^{\mathrm{int}} \to \partial M_i^{\mathrm{ext}}$. It is also easy to see

Lemma 6.0.16. *X_0 coincides with the topological boundary ∂X of X in Y.*

As before we have a description of Y:

Lemma 6.0.17. *There is a canonical surjective 1-Lipschitz map $\eta_0 : C_0 \to X_0$ such that Y is isometric to the length space*

$$X \bigcup_{\eta_0} C_0 \times_\phi [0, t_0],$$

where $(x, 0) \in C_0 \times 0$ is identified with $\eta_0(x) \in X_0$ for each $x \in C_0$.

Lemmas 3.1.1, 3.1.2, 3.1.3, Propositions 3.1.5 and 3.1.7 still hold if one replaces X by X_0 in this generality. Especially we have the following results. The proofs are similar, and hence omitted.

Lemma 6.0.18. *For every $x \in X_0$, we have*

1. $\#\eta_0^{-1}(x) \leq 2$;
2. *for $x \in X_0$, suppose $\#\eta_0^{-1}(x) = 2$, and let $\gamma_\pm$ be the two shortest geodesics from x to C_{t_0}, and let $\xi_\pm \in \Sigma_x(Y)$ be the directions of $\gamma_\pm$ respectively. Then $\Sigma_x(Y)$ is isometric to the spherical suspension $\{\xi_\pm\} * \Sigma_x(X_0)$.*

As an application of Lemma 6.0.18, we have the following result, which gives a sufficient condition for inradius collapse.

Proposition 6.0.19. *Let M_i in $\mathcal{M}(n, \kappa, \lambda, d)$ converge to a compact length space N with respect to the Gromov-Hausdorff distance, and suppose that N is a closed topological manifold or a closed Alexandrov space. Then* $\mathrm{inrad}(M_i)$ *converges to zero.*

Proof. We assume that N is a closed Alexandrov space with curvature bounded below. The case when N is a closed topological manifold is similar. Suppose that Proposition 6.0.19 does not hold. Let $r_i := \mathrm{inrad}(M_i)$, and take a point $p_i \in M_i$ and $q_i \in \partial M_i$ such that $|p_, q_i| = r_i$. Passing to a subsequence, we may assume that $(B(p_i, r_i), q_i)$ converges to a metric ball $(B(x_0, r), y_0)$ in X under the convergence $\tilde{M}_i \to Y$, where $r > 0$.

Take a minimal geodesics γ and γ^+ from y_0 to x_0 and C_{t_0} respectively. Note that

$$\angle(\gamma, \gamma^+) = \pi \tag{6.1}$$

We claim that $\#\eta_0^{-1}(y) = 1$ for every point of X_0 near y_0. Otherwise we have a sequence $y_i \in X_0$ converging to y_0 with $\#\eta_0^{-1}(y_i) = 2$. Take two minimal geodesics $\gamma_i^\pm$ from y_i to C_{t_0}. For every $\delta > 0$ take $s_0 > 0$ such that

$$\tilde{\angle}\gamma(s_0)y_0\gamma_+(s_0) > \pi - \delta.$$

Since $\gamma_i^\pm$ converges to γ^+, we obtain

$$\angle\gamma(s_0)y_i\gamma_i^\pm(s_0) \geq \tilde{\angle}\gamma(s_0)y_i\gamma_i^\pm(s_0) > \pi - 2\delta,$$

for large enough i. This implies that $\angle(\gamma_i^+, \gamma_i^-) < 2\delta$ contradicting to Lemma 6.0.18.

Let $p_0 \in C_0$ with $\eta_0(p_0) = y_0$, and set $a := \gamma^+(t_0)$. From the previous consideration, it is possible to take neighborhoods U_0 of p_0 in C_0, V_0 of y_0 in X_0 respectively in such a way that $\eta_0 : U_0 \to V_0$ is a homeomorphism. From (6.1), we may assume that the distance function d_a from a is regular on U_0. Choose a neighborhood $U_1 \subset U_0$ homeomorphic to $\mathbb{R}^{m-1}$. Perelman's fibration theorem 1.2.9 now implies that a small neighborhood of any point $y \in \eta_0(U_1)$ in Y is homeomorphic to $\mathbb{R}^m$. On the other hand, since X is bi-Lipschitz homeomorphic to a closed Alexandrov space and since $\dim \eta_0(U_1) = m - 1$, one can take a point $y \in \eta_0(U_1)$ having a neighborhood in X homeomorphic to $\mathbb{R}^m$. Since y is a boundary point of X in Y, this contradicts to the domain invariance theorem in $\mathbb{R}^m$. This completes the proof. ∎

Bibliography

[AB98] Alexander, S. B.; Bishop, R. L. *Thin Riemannian manifolds with boundary.* Math. Ann. **311** no. 1, 55-70, 1998.

[BBI] Burago,Y.; Burago, D.; Ivanov, S: *A course in metric geometry.* Graduate Studies in Mathematics 2001; Volume: 33.

[BGP] Burago, Y., Gromov, M. and Perelman, G.: *A. D. Alexandrov spaces with curvature bounded below.* Uspekhi Mat. Nauk. **42**: 2, 3-51, 1992.

[CC96] Cheeger, J.; Colding, T. H. *Lower bounds on Ricci curvature and the almost rigidity of warped products.* Ann. of Math. (2) **144** (1996), no. 1, 189-237.

[CC97] Cheeger, J; Colding, T:*On the structure of spaces with Ricci curvature bounded below. I.* J. Differential Geom. 46 (1997), no. 3, 406-480.

[CG86] Cheeger, J.; Gromov, M.: *Collapsing Riemannian manifolds while keeping their curvature bounded. I.* J. Differential Geom. **23** (1986), no. 3, 309-346.

[CG90] Cheeger, J.; Gromov, M.:*Collapsing Riemannian manifolds while keeping their curvature bounded. II.* J. Differential Geom. **32** (1990), no. 1, 269-298.

[CFG92] Cheeger, J; Fukaya, K; Gromov, M *Nilpotent structures and invariant metrics on collapsed manifolds.* J. Amer. Math. Soc. **5** (1992), no. 2, 327-372.

[Co97] Colding, T. H. : *Ricci curvature and volume convergence.* Ann. of Math. (2) **145** (1997), no. 3, 477-501.

[Fu86] Fukaya, K.: *Collapsing Riemannian manifolds to ones of lower dimensions.* J. Differential Geom. **25** (1987), no. 1, 139-156.

[FY92] Fukaya, K.; Yamaguchi, T.:*The fundamental groups of almost non-negatively curved manifolds.* Ann. of Math. (2) **136** (1992), no. 2, 253-333.

[GW88] Greene, R. E.; Wu, H. :*Lipschitz convergence of Riemannian manifolds.* Pacific J. Math. **131** (1988), no. 1, 119-141.

[G:AFM] Gromov, M.: *Almost flat manifolds.* J. Differential Geom. **13** (1978), no. 2, 231-241.

[G'bk] Gromov, M.: *Structures métriques pour les variétés riemanniennes.* Edited by J. Lafontaine and P. Pansu. Textes Mathmatiques, 1. CEDIC, Paris, 1981. iv+152 pp.

[GP88] Grove, K.; Petersen, P. *Bounding Homotopy Types by Geometry.* Ann. of Math. (2) **128** (1988), no. 1, 195-206.

[GP91] Grove, K.; Petersen, P. *Manifolds near the boundary of existence.* J. Differential Geom. **33** (1991), no. 2, 379-394.

[GPW] Grove, K.; Petersen, P.; J, Wu.*Geometric finiteness theorems via controlled topology.* Invent. math. **99**, (1990), 205-213.

[HS:pre] Harvey, J.; Searle, C.: *Orientation and symmetries of Alexandrov spaces with applications in positive curvature.* arXiv:1209.1366

[Ka02] Kapovitch, V., *Regularity of limits of noncollapsing sequences of manifolds.* Geom. Funct. Anal. **12** (2002), no. 1, 121C137.

[Ka07] Kapovitch, V., *Perelman's stability theorem.* Surveys in differential geometry. Vol. XI, 103-136, Surv. Differ. Geom., 11, Int. Press, Somerville, MA, 2007.

[KPT10] Kapovitch, V; Petrunin, A; Tuschmann, W: *Nilpotency, almost nonnegative curvature, and the gradient flow on Alexandrov spaces.*Ann. of Math. (2) **171** (2010), no. 1, 3430-373.

[KW11] Kapovitch, V; Wilking, B: *Structure of fundamental groups of manifolds with Ricci curvature bounded below.* preprint, 2011

[Kat85] Katsuda, A: *Gromov's convergence theorem and its application.* Nagoya Math. J. **100** (1985), 11-48.

[Kod90] Kodani, S.: *Convergence theorem for Riemannian manifolds with boundary.* Compositio Math. **75** (1990), no. 2, 171-192.

[Kos02] Kosovskiĭ, N. N.: *Gluing of Riemannian manifolds of curvature* $\geq \kappa$. Algebra i Analiz **14**: 3 (2002), 140-157.

[MY15] Mitsuishi, A.; Yamaguchi, T.: *Collapsing three-dimensional closed Alexandrov spaces with a lower curvature bound.* Trans. Amer. Math. Soc. **367** (2015), no. 4, 2339-2410.

[MY:pre] Mitsuishi, A.; Yamaguchi, T.: *Collapsing three-dimensional Alexandrov spaces with bound*. in preparation.

[OS94] Otsu,Y.; Shioya,T.: *The Riemannian structure of Alexandrov spaces.* J. Differential Geom. **39** (1994), no. 3, 629–658

[Pera] Raquel, P.: *Convergence of Manifolds and Metric Spaces with Boundary.* preprint, 2015.

[Pr94] Perelman, G.:*Alexandrov spaces with curvature bounded below II.* preprint, 1994.

[Pr94'] Perelman, G.: *Elements of Morse theory on Alexandrov spaces.* St. Petersburg Math. J. **5** (1994) 207–214.

[PP93] Perelman, G.; Petrunin, A.: *Extremal subsets in Aleksandrov spaces and the generalized Liberman theorem.* Algebra i Analiz **5** (1993), no. 1, 242–256; translation in St. Petersburg Math. J. **5** (1994), no. 1, 215C227.

[PP:pre] Perelman, G and Petrunin, A. :*Quasigeodesics and gradient curves in Alexandrov spaces,* preprint.

[Pe02] Perelman, G:*Ricci flow with surgery on three-manifolds*, preprint, 2002.

[Pe03] Perelman, G:*The entropy formula for the Ricci flow and its geometric applications*, 2003.

[Peters87] Peters, S.:*Convergence of Riemannian manifolds.* Compositio Math. **62** (1987), no. 1, 3-16.

[Pet97] Petrunin, A.: *Applications of quasigeodesics and gradient curves. Comparison geometry* (Berkeley, CA, 1993), 203-219, Math. Sci. Res. Inst. Publ., **30**, Cambridge Univ. Press, Cambridge, 1997.

[Pet98] Petrunin, A: *Parallel Transportation for Alexandrov space with Curvature Bounded Below*. Geom. Funct. Anal. **8** (1998), no. 1, 123-148.

[Pet07] Petrunin, A.: *Semiconcave functions in Alexandrov's geometry.* Surveys in Comparison Geometry, 2007.

[Ruh82] Ruh, E.: *Almost flat manifolds.* J. Differential Geom. **17** (1982), no. 1, 1-14.

[Sie72] Siebenmann, L. C.: *Deformation of homeomorphisms on stratified sets. I, II.* Comment. Math. Helv. **47** (1972), 123-136; ibid. **47** (1972), 137-163.

[SY00] Shioya, T.; Yamaguchi, T.: *Collapsing three-manifolds under a lower curvature bound*. J. Differential Geom. **56** (2000), no. 1, 1-66.

[SY05] Shioya, T; Yamaguchi, T: *Volume collapsed three-manifolds with a lower curvature bound.* Math. Ann. **333** (2005), no. 1, 131-155.

[Wo06] Wong, J.: *Collapsing manifolds with boundary.* PhD thesis, University of Illinois at urbana-Champaign, 1-81, 2006

[Wo08] Wong, J.: *An extension procedure for manifolds with boundary.* Pacific J. **235**: 173-199. 2008

[Wo10] Wong, J.: *Collapsing manifolds with boundary.* Geom Dedicata. **149**: 291-334. 2010

[Ya91] Yamaguchi, T.: *Collapsing and pinching under a lower curvature bound.* Ann. Math. **133**(2) 317-357, 1991

[Ya96] Yamaguchi, T.: *A convergence theorem in the geometry of Alexandrov spaces.* Actes de la Table Ronde de Géométrie Différentielle, Luminy 1992. Sémin. Congr., vol. 1, 601-642, Soc. Math. France, Paris (1996)

[Ya02] Yamaguchi, T.: *Collapsing 4-manifolds with a lower curvature bound*, 1-133, preprint, 2002.

[YZ15] Yamaguchi, T., Zhang, Z.: *Inradius collapsed manifolds*, preprint, 2015