Higher-Dimensional General Jacobi Identities I

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Abstract

It was shown by the author [International Journal of Theoretical Physics 36 (1997), 1099-1131] in synthetic differential geometry that what is called the general Jacobi identity obtaining in microcubes underlies the Jacobi identity of vector fields. It is well known in the theory of Lie algebras that a plethora of higher-dimensional generalizations of the Jacobi identity hold, though it is usually established not as a direct derivation from the axioms of Lie algebras but by making an appeal to the so-called Poincaré-Birkhoff-Witt theorem. The general Jacobi identity was rediscovered by Kirill Mackenzie in the second decade of this century [Geometric Methods in Physics, 357-366, Birkhäuser/Springer 2013]. The principal objective in this paper is to investigate a four-dimensional generalization of the general Jacobi identity in detail. In a subsequent paper we will propose a uniform method for establishing a bevy of higher-dimensional generalizations of the Jacobi identity under a single umbrella.

1 Introduction

It is known in synthetic differential geometry (cf. [2] and [4]) that vector fields on a microlinear space M forms a Lie algebra, for which the following antisymmetry holds:

$$[X_1, X_2] + [X_2, X_1] = 0 \tag{1}$$

It was shown in [3] that a bit deeper theorem in the following underlies the above identity.

Theorem 1 *Let M be a microlinear space. Given microsquares $\gamma_{12}, \gamma_{21} : D^2 \to M$ with $\gamma_{12} \mid D(2) = \gamma_{21} \mid D(2)$, we have*

$$\left(\gamma_{12} \overset{\cdot}{-} \gamma_{21}\right) + \left(\gamma_{21} \overset{\cdot}{-} \gamma_{12}\right) = 0 \tag{2}$$

Now we consider the famous Jacobi identity.

$$[X_1,[X_2,X_3]]+[X_2,[X_3,X_1]]+[X_3,[X_1,X_2]]=0 \tag{3}$$

It claims that the sum of $[X_1,[X_2,X_3]]$'s with the three cyclic permutations of $\{1,2,3\}$ applied vanishes. We note in passing that the three permutations of $\{1,2,3\}$ are no other than the three even permutations of $\{1,2,3\}$. It has been demonstrated in [6], [7], [8] and [10] that the following deeper theorem underlies the above identity.

Theorem 2 *(General Jacobi Identity) Let M be a microlinear space. Given microcubes $\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321}:D^3\to M$ with*

$$\begin{aligned}
\gamma_{123}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_2d_3=0\right\}&=\gamma_{132}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_2d_3=0\right\}\\
\gamma_{231}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_2d_3=0\right\}&=\gamma_{321}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_2d_3=0\right\}\\
\gamma_{231}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_1d_3=0\right\}&=\gamma_{213}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_1d_3=0\right\}\\
\gamma_{312}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_1d_3=0\right\}&=\gamma_{132}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_1d_3=0\right\}\\
\gamma_{312}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_1d_2=0\right\}&=\gamma_{321}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_1d_2=0\right\}\\
\gamma_{123}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_1d_2=0\right\}&=\gamma_{213}\mid\left\{(d_1,d_2,d_3)\in D^3\mid d_1d_2=0\right\}
\end{aligned}$$

we have

$$\begin{aligned}
&\left(\left(\gamma_{123}\underset{1}{\dot{-}}\gamma_{132}\right)\dot{-}\left(\gamma_{231}\underset{1}{\dot{-}}\gamma_{321}\right)\right)+\\
&\left(\left(\gamma_{231}\underset{2}{\dot{-}}\gamma_{213}\right)\dot{-}\left(\gamma_{312}\underset{2}{\dot{-}}\gamma_{132}\right)\right)+\\
&\left(\left(\gamma_{312}\underset{3}{\dot{-}}\gamma_{321}\right)\dot{-}\left(\gamma_{123}\underset{3}{\dot{-}}\gamma_{213}\right)\right)\\
&=0
\end{aligned} \tag{4}$$

The general Jacobi identity was rediscovered by Kirill Mackenzie [5] in a somewhat different context. We add that the general Jacobi identity plays a fundamental role in a combinatorial or geometric proof of Jacobi-like identities in so-called Frölicher-Nijenhuis calculus (cf. [9]).

Now we consider the following four-dimensional analogue of the Jacobi identity.

$$\begin{aligned}
&[X_1,[X_2,[X_3,X_4]]]+[X_1,[X_3,[X_4,X_2]]]+[X_1,[X_4,[X_2,X_3]]]+\\
&[X_2,[X_1,[X_4,X_3]]]+[X_2,[X_3,[X_1,X_4]]]+[X_2,[X_4,[X_3,X_1]]]+\\
&[X_3,[X_1,[X_2,X_4]]]+[X_3,[X_2,[X_4,X_1]]]+[X_3,[X_4,[X_1,X_2]]]+\\
&[X_4,[X_1,[X_3,X_2]]]+[X_4,[X_2,[X_1,X_3]]]+[X_4,[X_3,[X_2,X_1]]]\\
&=0
\end{aligned} \tag{5}$$

It claims that the sum of $[X_1,[X_2,[X_3,X_4]]]$'s with the twelve even permutations of $\{1,2,3,4\}$ applied vanishes.

The principal objective in this paper is to establish a four-dimensional version of the general Jacobi identity underpinning the above identity (5). In a subsequent paper we will discuss a slew of higher-dimensional general Jacobi identities underlying the higher-dimensional Jacobi identities discussed in [1] and [12] (the former called them *generalized Jacobi identities*) from a coherent standpoint. For a good introduction to generalized Jacobi identities, the reader is referred to Chapter 8 of [11]. We know well that various higher-dimensional Jacobi identities are logical consequences of the three-dimensional Jacobi identity, but we guess that higher-dimensional general Jacobi identities are by no means logical consequences of the three-dimensional general Jacobi identity. We assume the reader to be familiar with [4] up to Chapter 3.

2 Strong Differences

First we introduce the notion of a simplicial small object after [6], though in a somewhat generalized form.

Notation 3 *(Simplicial small objects) Let n be a natural number. Given a subset $\mathfrak{p}$ of*

$$\{(i,j)\in\mathbb{N}\times\mathbb{N}\mid 1\leq i\leq n, 1\leq j\leq n, i\neq j\}$$

and a subset ξ of

$$\{i\in\mathbb{N}\mid 1\leq i\leq n\}$$

$D^n\{\mathfrak{p},\xi\}$ denotes the set

$$\{(d_1,...,d_n):D^n\mid d_id_j=0 \textit{ for any } (i,j)\in\mathfrak{p},\ d_i=0 \textit{ for any } i\in\xi\}$$

which is surely a small object. By way of example, we have

$$D(2)=D^2\{(1,2)\}$$
$$D(3)=D^3\{(1,2),(1,3),(2,3)\}$$

and $D^3\{1,3\}$ can be identified with D via the canonical isomorphism

$$d\in D\mapsto(0,d,0)\in D^3\{1,3\}$$

The notion of strong difference in synthetic differential geometry is based upon the following lemma.

Lemma 4 *(cf. the first Lemma in §3.4 of [4]) The diagram*

$$\begin{array}{ccccc} & & D^3\{(1,3),(2,3)\} & & \\ & \nearrow & & \nwarrow & \\ D^2 & & & & D^2 \\ & \nwarrow & & \nearrow & \\ & & D^2\{(1,2)\} & & \end{array} \qquad (6)$$

with the lower two arrows being the canonical injections and the upper two arrows being

$$j_1^2 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, d_1 d_2) \in D^3 \{(1,3), (2,3)\}$$
$$j_2^2 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0) \in D^3 \{(1,3), (2,3)\}$$

from left to right is a quasi-colimit diagram.

Corollary 5 *Let M be a microlinear space with two microsquares $\gamma_1, \gamma_2 : D^2 \to M$ abiding by*

$$\gamma_1 \mid D^2 \{(1,2)\} = \gamma_2 \mid D^2 \{(1,2)\}$$

Then there exists a unique mapping

$$\mathfrak{n}^2_{(\gamma_1,\gamma_2)} : D^3 \{(1,3), (2,3)\} \to M$$

such that $\mathfrak{n}^2_{(\gamma_1,\gamma_2)} \circ j_1^2 = \gamma_1$ and $\mathfrak{n}^2_{(\gamma_1,\gamma_2)} \circ j_2^2 = \gamma_2$.

Notation 6 *In the above notation in Corollary 5 we write $\gamma_1 \dot{-} \gamma_2$ for the mapping*

$$d \in D \mapsto \mathfrak{n}^2_{(\gamma_1,\gamma_2)} (0, 0, d)$$

The notion of strong difference can easily be relativized.

Lemma 7 *Let n be a natural number. The diagram*

$$\begin{array}{ccccc} & & D^{n+3} \{(n+1, n+3), (n+2, n+3)\} & & \\ & \nearrow & & \nwarrow & \\ D^{n+2} & & & & D^{n+2} \\ & \nwarrow & & \nearrow & \\ & & D^{n+2} \{(n+1, n+2)\} & & \end{array} \qquad (7)$$

with the lower two arrows being the canonical injections and the upper two arrows being

$$j_1^{n+2} : (d_1, ..., d_n, d_{n+1}, d_{n+2}) \in D^{n+2} \mapsto (d_1, ..., d_n, d_{n+1}, d_{n+2}, d_{n+1} d_{n+2}) \in D^{n+3} \{(n+1, n+3), (n+2, n+3$$
$$j_2^{n+2} : (d_1, ..., d_n, d_{n+1}, d_{n+2}) \in D^{n+2} \mapsto (d_1, ..., d_n, d_{n+1}, d_{n+2}, 0) \in D^{n+3} \{(n+1, n+3), (n+2, n+3)\}$$

from left to right is a quasi-colimit diagram.

Corollary 8 *Let n be a natural number. Let M be a microlinear space with two mappings $\gamma_1, \gamma_2 : D^{n+2} \to M$ abiding by*

$$\gamma_1 \mid D^{n+2} \{(n+1, n+2)\} = \gamma_2 \mid D^{n+2} \{(n+1, n+2)\}$$

Then there exists a unique mapping

$$\mathfrak{n}^{n+2}_{(\gamma_1,\gamma_2)} : D^{n+3} \{(n+1, n+3), (n+2, n+3)\} \to M$$

such that $\mathfrak{n}^{n+2}_{(\gamma_1,\gamma_2)} \circ j_1^{n+2} = \gamma_1$ and $\mathfrak{n}^{n+2}_{(\gamma_1,\gamma_2)} \circ j_2^3 = \gamma_2$.

Notation 9 *Let n be a natural number. Let M be a microlinear space. Given $\gamma : D^n \to M$ and a permutation σ of $\{1, ..., n\}$, we write γ^σ for the mapping*

$$(d_1, ..., d_n) \in D^n \mapsto \gamma \left(d_{\sigma^{-1}(1)}, ..., d_{\sigma^{-1}(n)} \right) \in M$$

Notation 10 *Let n be a natural number. Let M be a microlinear space.*

1. *Given $\gamma_1, \gamma_2 : D^{n+2} \to M$ with*

   $$\gamma_1 \mid D^{n+2} \{(n+1, n+2)\} = \gamma_2 \mid D^{n+2} \{(n+1, n+2)\}$$

   we write $\gamma_1 \underset{1...n}{\dot{-}} \gamma_2$ for the mapping

   $$(d_1, ..., d_n, d_{n+1}) \in D^{n+1} \mapsto \mathfrak{n}^{n+2}_{(\gamma_1, \gamma_2)} (d_1, ..., d_n, 0, 0, d_{n+1}) \in M$$

2. *Given a permutation σ of $\{1, ..., n, n+1, n+2\}$ and $\gamma_1, \gamma_2 : D^{n+2} \to M$ with*

   $$\gamma_1 \mid D^{n+2} \{(\sigma (n+1), \sigma (n+2))\} = \gamma_2 \mid D^{n+2} \{(\sigma (n+1), \sigma (n+2))\}$$

   we write $\gamma_1 \underset{\sigma(1)...\sigma(n)}{\dot{-}} \gamma_2$ for $(\gamma_1)^\sigma \underset{1...n}{\dot{-}} (\gamma_2)^\sigma$.

The following result is well known.

Lemma 11 *(cf. Proposition 6 in §2.2 of [4]) The diagram*

$$\begin{array}{ccccc} & & D^2 \{(1,2)\} & & \\ & \nearrow & & \nwarrow & \\ D^2 \{1\} & & & & D^2 \{2\} \\ & \nwarrow & & \nearrow & \\ & & 1 & & \end{array} \qquad (8)$$

with the four arrows being the canonical injections is a quasi-colimit diagram.

Corollary 12 *Let M be a microlinear space. Given $\gamma_1, \gamma_2 : D^2 \to M$,*

$$\gamma_1 \mid D^2 \{(1,2)\} = \gamma_2 \mid D^2 \{(1,2)\}$$

obtains iff both

$$\gamma_1 \mid D^2 \{1\} = \gamma_2 \mid D^2 \{1\}$$

and

$$\gamma_1 \mid D^2 \{2\} = \gamma_2 \mid D^2 \{2\}$$

obtain.

It can readily be relativized and generalized.

Lemma 13 *The diagram*

$$
\begin{array}{ccccc}
 & & D^{n+m_1+m_2}\left\{ \begin{array}{c} (n+i,n+m_1+j) \mid \\ 1\leq i\leq m_1, 1\leq j\leq m_2 \end{array} \right\} & & \\
 & \nearrow & & \nwarrow & \\
D^{n+m_1+m_2}\left\{ \begin{array}{c} n+1,..., \\ n+m_1 \end{array} \right\} & & & & D^{n+m_1+m_2}\left\{ \begin{array}{c} n+m_1+1,... \\ n+m_1+m_2 \end{array} \right. \\
 & \nwarrow & & \nearrow & \\
 & & D^n & &
\end{array}
\tag{9}
$$

with the four arrows being the canonical injections is a quasi-colimit diagram.

Corollary 14 *Let M be a microlinear space. Given $\gamma_1, \gamma_2 : D^{n+m_1+m_2} \rightarrow M$,*

$$
\begin{aligned}
&\gamma_1 \mid D^{n+m_1+m_2}\left\{(n+i,n+m_1+j) \mid 1\leq i\leq m_1, 1\leq j\leq m_2\right\} \\
&\quad = \gamma_2 \mid D^{n+m_1+m_2}\left\{(n+i,n+m_1+j) \mid 1\leq i\leq m_1, 1\leq j\leq m_2\right\}
\end{aligned}
$$

obtains iff both

$$
\gamma_1 \mid D^{n+m_1+m_2}\left\{n+1,...,n+m_1\right\} = \gamma_2 \mid D^{n+m_1+m_2}\left\{n+1,...,n+m_1\right\}
$$

and

$$
\gamma_1 \mid D^{n+m_1+m_2}\left\{n+m_1+1,...,n+m_1+m_2\right\} = \gamma_2 \mid D^{n+m_1+m_2}\left\{n+m_1+1,...,n+m_1+m_2\right\}
$$

obtain.

Proposition 15 *Let M be a microlinear space. Then we have the following two statements:*

1. *Given*

$$
\gamma_1 : D^4 \rightarrow M, \gamma_2 : D^4 \rightarrow M, \gamma_3 : D^4 \rightarrow M, \gamma_4 : D^4 \rightarrow M
$$

if it holds that

$$
\begin{aligned}
\gamma_1 \mid D^4\left\{(3,4)\right\} &= \gamma_2 \mid D^4\left\{(3,4)\right\} \\
\gamma_3 \mid D^4\left\{(3,4)\right\} &= \gamma_4 \mid D^4\left\{(3,4)\right\} \\
\gamma_1 \mid D^4\left\{(2,3),(2,4)\right\} &= \gamma_3 \mid D^4\left\{(2,3),(2,4)\right\} \qquad (10) \\
\gamma_2 \mid D^4\left\{(2,3),(2,4)\right\} &= \gamma_4 \mid D^4\left\{(2,3),(2,4)\right\} \qquad (11)
\end{aligned}
$$

then all of

$$
\begin{gathered}
\gamma_1 \underset{12}{\overset{\cdot}{-}} \gamma_2 \\
\gamma_3 \underset{12}{\overset{\cdot}{-}} \gamma_4 \\
\left(\gamma_1 \underset{12}{\overset{\cdot}{-}} \gamma_2\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_3 \underset{12}{\overset{\cdot}{-}} \gamma_4\right)
\end{gathered}
$$

are well defined.

2. Given

$$\gamma_1 : D^4 \to M, \gamma_2 : D^4 \to M, \gamma_3 : D^4 \to M, \gamma_4 : D^4 \to M,$$
$$\gamma_5 : D^4 \to M, \gamma_6 : D^4 \to M, \gamma_7 : D^4 \to M, \gamma_8 : D^4 \to M$$

if it holds that

$$\gamma_1 \mid D^4\{(3,4)\} = \gamma_2 \mid D^4\{(3,4)\}$$
$$\gamma_3 \mid D^4\{(3,4)\} = \gamma_4 \mid D^4\{(3,4)\}$$
$$\gamma_5 \mid D^4\{(3,4)\} = \gamma_6 \mid D^4\{(3,4)\}$$
$$\gamma_7 \mid D^4\{(3,4)\} = \gamma_8 \mid D^4\{(3,4)\}$$
$$\gamma_1 \mid D^4\{(2,3),(2,4)\} = \gamma_3 \mid D^4\{(2,3),(2,4)\}$$
$$\gamma_2 \mid D^4\{(2,3),(2,4)\} = \gamma_4 \mid D^4\{(2,3),(2,4)\}$$
$$\gamma_5 \mid D^4\{(2,3),(2,4)\} = \gamma_7 \mid D^4\{(2,3),(2,4)\}$$
$$\gamma_6 \mid D^4\{(2,3),(2,4)\} = \gamma_8 \mid D^4\{(2,3),(2,4)\}$$
$$\gamma_1 \mid D^4\{(1,2),(1,3),(1,4)\} = \gamma_5 \mid D^4\{(1,2),(1,3),(1,4)\} \tag{12}$$
$$\gamma_2 \mid D^4\{(1,2),(1,3),(1,4)\} = \gamma_6 \mid D^4\{(1,2),(1,3),(1,4)\} \tag{13}$$
$$\gamma_3 \mid D^4\{(1,2),(1,3),(1,4)\} = \gamma_7 \mid D^4\{(1,2),(1,3),(1,4)\} \tag{14}$$
$$\gamma_4 \mid D^4\{(1,2),(1,3),(1,4)\} = \gamma_8 \mid D^4\{(1,2),(1,3),(1,4)\} \tag{15}$$

then all of

$$\gamma_1 \underset{12}{\dot{-}} \gamma_2$$
$$\gamma_3 \underset{12}{\dot{-}} \gamma_4$$
$$\gamma_5 \underset{12}{\dot{-}} \gamma_6$$
$$\gamma_7 \underset{12}{\dot{-}} \gamma_8$$
$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)$$
$$\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)$$
$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \dot{-} \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right)$$

are well defined.

Proof. We deal with the above two statements in order.

1. For the first statement, we have to show that

$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\{(2,3)\} = \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\{(2,3)\}$$

which is, by dint of Corollary 14, tantamout to showing that

$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\{2\} = \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\{2\} \tag{16}$$

$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\{3\} = \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\{3\} \tag{17}$$

because of the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^3\{(2,3)\} & & \\ & \nearrow & & \nwarrow & \\ D^3\{2\} & & & & D^3\{3\} \\ & \nwarrow & & \nearrow & \\ & & D^3\{2,3\} & & \end{array}$$

with the four arrows being the canonical injections (Lemma 13 with $n = m_1 = m_2 = 1$). Due to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^4\{(2,3),(2,4)\} & & \\ & \nearrow & & \nwarrow & \\ D^4\{2\} & & & & D^4\{3,4\} \\ & \nwarrow & & \nearrow & \\ & & D^4\{2,3,4\} & & \end{array}$$

with the four arrows being the canonical injections (Lemma 13 with $n = m_1 = 1$ and $m_2 = 2$), the condition (10) is equivalent to the conditions

$$\gamma_1 \mid D^4\{2\} = \gamma_3 \mid D^4\{2\} \tag{18}$$

$$\gamma_1 \mid D^4\{3,4\} = \gamma_3 \mid D^4\{3,4\} \tag{19}$$

while the condition (11) is equivalent to the conditions

$$\gamma_2 \mid D^4\{2\} = \gamma_4 \mid D^4\{2\} \tag{20}$$

$$\gamma_2 \mid D^4\{3,4\} = \gamma_4 \mid D^4\{3,4\} \tag{21}$$

In order to show that (16) obtains, we note that the quasi-colimit diagram in (7) with $n = 2$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^5\{2,(3,5),(4,5)\} & & \\ & \nearrow & & \nwarrow & \\ D^4\{2\} & & & & D^4\{2\} \\ & \nwarrow & & \nearrow & \\ & & D^4\{2,(3,4)\} & & \end{array}$$

so that the conditions (18) and (20) imply (16). It is easy to see that

$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\{3\}\right) \circ i_2^3 = \left(\gamma_1 \mid D^4\{3,4\}\right) \circ i_2^4 = \left(\gamma_2 \mid D^4\{3,4\}\right) \circ i_2^4$$

$$\left(\left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\{3\}\right) \circ i_2^3 = \left(\gamma_3 \mid D^4\{3,4\}\right) \circ i_2^4 = \left(\gamma_4 \mid D^4\{3,4\}\right) \circ i_2^4$$

obtain with

$$\begin{aligned} i_2^3 &: (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0) \in D^3 \\ i_2^4 &: (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0, 0) \in D^4 \end{aligned}$$

so that (19) and (21) imply (17).

2. For the second statement, we have to show that

$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \mid D^2\{(1,2)\} = \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right) \mid D^2\{(1,2)\}$$

which is tantamout to showing that

$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \mid D^2\{1\} = \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right) \mid D^2\{1\} \tag{22}$$

$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \mid D^2\{2\} = \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right) \mid D^2\{2\} \tag{23}$$

because of the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^2\{(1,2)\} & & \\ & \nearrow & & \nwarrow & \\ D^2\{1\} & & & & D^2\{2\} \\ & \nwarrow & & \nearrow & \\ & & D^2\{1,2\} & & \end{array}$$

with the four arrows being the canonical injections (Lemma 11). Due to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^4\{(1,2),(1,3),(1,4)\} & & \\ & \nearrow & & \nwarrow & \\ D^4\{1\} & & & & D^4\{2,3,4\} \\ & \nwarrow & & \nearrow & \\ & & D^4\{1,2,3,4\} & & \end{array}$$

with the four arrows being the canonical injections (Lemma 13 with $n = 0$, $m_1 = 1$ and $m_2 = 3$), we have

- the condition (12) is equivalent to the conditions

$$\gamma_1 \mid D^4\{1\} = \gamma_5 \mid D^4\{1\} \tag{24}$$
$$\gamma_1 \mid D^4\{2,3,4\} = \gamma_5 \mid D^4\{2,3,4\} \tag{25}$$

- the condition (13) is equivalent to the conditions

$$\gamma_2 \mid D^4\{1\} = \gamma_6 \mid D^4\{1\} \tag{26}$$
$$\gamma_2 \mid D^4\{2,3,4\} = \gamma_6 \mid D^4\{2,3,4\} \tag{27}$$

- the condition (14) is equivalent to the conditions

$$\gamma_3 \mid D^4\{1\} = \gamma_7 \mid D^4\{1\} \tag{28}$$
$$\gamma_3 \mid D^4\{2,3,4\} = \gamma_7 \mid D^4\{2,3,4\} \tag{29}$$

- the condition (15) is equivalent to the conditions

$$\gamma_4 \mid D^4\{1\} = \gamma_8 \mid D^4\{1\} \tag{30}$$
$$\gamma_4 \mid D^4\{2,3,4\} = \gamma_8 \mid D^4\{2,3,4\} \tag{31}$$

In order to show that (22) obtains, we note first that the quasi-colimit diagram in 7 with $n = 2$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^5\{1,(3,5),(4,5)\} & & \\ & \nearrow & & \nwarrow & \\ D^4\{1\} & & & & D^4\{1\} \\ & \nwarrow & & \nearrow & \\ & & D^4\{1,(3,4)\} & & \end{array}$$

so that we have

- the conditions (24) and (26) imply

$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\{1\} = \left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \mid D^3\{1\} \tag{32}$$

- the conditions (28) and (30) imply

$$\left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\{1\} = \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right) \mid D^3\{1\} \tag{33}$$

We note also that the quasi-colimit diagram in (7) with $n = 1$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^4\{1,(2,4),(3,4)\} & & \\ & \nearrow & & \nwarrow & \\ D^3\{1\} & & & & D^3\{1\} \\ & \nwarrow & & \nearrow & \\ & & D^3\{1,(2,3)\} & & \end{array}$$

so that the conditions (32) and (**??**) imply the condition (22). It is easy to see that

$$\begin{aligned}
&\left(\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \mid D^2\{2\}\right) \circ i^2 \\
&= \left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\{2,3\}\right) \circ i^3 = \left(\left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\{2,3\}\right) \circ i^3 \\
&= \left(\gamma_1 \mid D^4\{2,3,4\}\right) \circ i^4 = \left(\gamma_2 \mid D^4\{2,3,4\}\right) \circ i^4 \\
&= \left(\gamma_3 \mid D^4\{2,3,4\}\right) \circ i^4 = \left(\gamma_4 \mid D^4\{2,3,4\}\right) \circ i^4
\end{aligned}$$

and

$$\left(\left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right) \mid D^2\{2\}\right) \circ i^2$$
$$= \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \mid D^3\{2,3\}\right) \circ i^3 = \left(\left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right) \mid D^3\{2,3\}\right) \circ i^3$$
$$= \left(\gamma_5 \mid D^4\{2,3,4\}\right) \circ i^4 = \left(\gamma_6 \mid D^4\{2,3,4\}\right) \circ i^4$$
$$= \left(\gamma_7 \mid D^4\{2,3,4\}\right) \circ i^4 = \left(\gamma_8 \mid D^4\{2,3,4\}\right) \circ i^4$$

obtain with

$$i^2 : d \in D \mapsto (d,0) \in D^2\{2\}$$
$$i^3 : d \in D \mapsto (d,0,0) \in D^3\{2,3\}$$
$$i^4 : d \in D \mapsto (d,0,0,0) \in D^4\{2,3,4\}$$

so that (25), (27), (29) and (31) imply (23).

■

Notation 16 *Let M be a microlinear space.*

1. *We denote by $\mathfrak{X}(M)$ the totality of vector fields on M. It forms a Lie algebra. We take the third viewpoint of a vector field in the essentially equivalent three discussed in §3.2 of [4]. Namely, a vector field X on M is a mapping $d \in D \mapsto X_d \in M^M$ with $X_0 = \mathrm{id}_M$.*

2. *Given $X, ..., X_n \in \mathfrak{X}(M)$, we denote by $X_n * ... * X_1$ the mapping $(d_1, ..., d_n) \in D^n \mapsto (X_n)_{d_n} \circ ... \circ (X_1)_{d_1} \in M^M$.*

3. *We recall (cf. Proposition 8 in §3.4 of [4]) that*

$$[X_1, X_2] = X_2 * X_1 \dot{-} (X_1 * X_2)^{\sigma_{21}}$$

with

$$\sigma_{21} = \begin{pmatrix} 12 \\ 21 \end{pmatrix}$$

We recall (cf. Prposition 2.7 of [6]) that

$$[X_1, [X_2, X_3]]$$
$$= \left(X_3 * X_2 * X_1 \underset{1}{\dot{-}} (X_2 * X_3 * X_1)^{\sigma_{132}}\right) \dot{-} \left((X_1 * X_3 * X_2)^{\sigma_{231}} \underset{1}{\dot{-}} (X_1 * X_2 * X_3)^{\sigma_{321}}\right)$$

with

$$\sigma_{132} = \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \sigma_{231} = \begin{pmatrix} 123 \\ 312 \end{pmatrix}, \sigma_{321} = \begin{pmatrix} 123 \\ 321 \end{pmatrix}$$

We note that

$$
\begin{aligned}
&[X_1,[X_2,[X_3,X_4]]] \\
&= \left(\begin{array}{c} \left(X_4 * X_3 * X_2 * X_1 \underset{12}{\dot{-}} (X_3 * X_4 * X_2 * X_1)^{\sigma_{1243}} \right) \underset{1}{\dot{-}} \\ \left((X_2 * X_4 * X_3 * X_1)^{\sigma_{1342}} \underset{12}{\dot{-}} (X_2 * X_3 * X_4 * X_1)^{\sigma_{1432}} \right) \end{array} \right) \dot{-} \\
&\left(\begin{array}{c} \left((X_1 * X_4 * X_3 * X_2)^{\sigma_{2341}} \underset{12}{\dot{-}} (X_1 * X_3 * X_4 * X_2)^{\sigma_{2431}} \right) \underset{1}{\dot{-}} \\ \left((X_1 * X_2 * X_4 * X_3)^{\sigma_{3421}} \underset{12}{\dot{-}} (X_1 * X_2 * X_3 * X_4)^{\sigma_{4321}} \right) \end{array} \right)
\end{aligned}
$$

with

$$
\sigma_{1243} = \begin{pmatrix} 1234 \\ 1243 \end{pmatrix}, \sigma_{1342} = \begin{pmatrix} 1234 \\ 1423 \end{pmatrix}, \sigma_{1432} = \begin{pmatrix} 1234 \\ 1432 \end{pmatrix}, \sigma_{2341} = \begin{pmatrix} 1234 \\ 4123 \end{pmatrix},
$$

$$
\sigma_{2431} = \begin{pmatrix} 1234 \\ 4132 \end{pmatrix}, \sigma_{3421} = \begin{pmatrix} 1234 \\ 4312 \end{pmatrix}, \sigma_{4321} = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}
$$

3 A Four-Dimensional General Jacobi Identity

Theorem 17 *The diagram consisting of objects*

$P, Q^{1234}, Q^{1243}, Q^{1324}, Q^{1342}, Q^{1423}, Q^{1432}, Q^{2134}, Q^{2143}, Q^{2314}, Q^{2341}, Q^{2413}, Q^{2431},$
$Q^{3124}, Q^{3142}, Q^{3214}, Q^{3241}, Q^{3412}, Q^{3421}, Q^{4123}, Q^{4132}, Q^{4213}, Q^{4231}, Q^{4312}, Q^{4321},$
$R_{12}^{1234,1243}, R_{12}^{1342,1432}, R_{12}^{2341,2431}, R_{12}^{3421,4321}, R_{12}^{2134,2143}, R_{12}^{3412,4312}, R_{13}^{1324,1342}, R_{13}^{1243,1423}$
$R_{13}^{3241,3421}, R_{13}^{2431,4231}, R_{13}^{3124,3142}, R_{13}^{2413,4213}, R_{14}^{1423,1432}, R_{14}^{1234,1324}, R_{14}^{4231,4321}, R_{14}^{2341,3241},$
$R_{14}^{4123,4132}, R_{14}^{2314,3214}, R_{23}^{2314,2341}, R_{23}^{2143,2413}, R_{23}^{3142,3412}, R_{23}^{1432,4132}, R_{23}^{3214,3241}, R_{23}^{1423,4123},$
$R_{24}^{2413,2431}, R_{24}^{2134,2314}, R_{24}^{4132,4312}, R_{24}^{1342,3142}, R_{24}^{4213,4231}, R_{24}^{1324,3124}, R_{34}^{3412,3421}, R_{34}^{3124,3214},$
$R_{34}^{4123,4213}, R_{34}^{1243,2143}, R_{34}^{4312,4321}, R_{34}^{1234,2134}$

with

$$P = D^{53}\left\{\begin{array}{c}(1,5),(2,5),(1,6),(3,6),(1,7),(4,7),(2,8),(3,8),(2,9),(4,9),\\(3,10),(4,10),(5,6),(5,7),(5,8),(5,9),(6,7),(6,8),(6,10),(7,9),\\(7,10),(8,9),(8,10),(9,10),(1,i_{11,15}),(2,i_{11,15}),(3,i_{11,15}),\\(5,i_{11,15}),(6,i_{11,15}),(7,i_{11,15}),(8,i_{11,15}),(9,i_{11,15}),(10,i_{11,15}),\\(1,i_{16,20}),(2,i_{16,20}),(4,i_{16,20}),(5,i_{16,20}),(6,i_{16,20}),(7,i_{16,20}),\\(8,i_{16,20}),(9,i_{16,20}),(10,i_{16,20}),(1,i_{21,25})(3,i_{21,25}),(4,i_{21,25}),\\(5,i_{21,25}),(6,i_{21,25}),(7,i_{21,25}),(8,i_{21,25}),(9,i_{21,25}),(10,i_{21,25}),\\(2,i_{26,30}),(3,i_{26,30}),(4,i_{26,30}),(5,i_{26,30}),(6,i_{26,30}),(7,i_{26,30}),\\(8,i_{26,30}),(9,i_{26,30}),(10,i_{26,30}),\left(i_{11,15},i'_{11,15}\right),(i_{11,15},i_{16,20}),\\(i_{11,15},i_{21,25}),(i_{11,15},i_{26,30}),\left(i_{16,20},i'_{16,20}\right),(i_{16,20},i_{21,25}),\\(i_{16,20},i_{26,30}),\left(i_{21,25},i'_{21,25}\right),(i_{21,25},i_{26,30}),\left(i_{26,30},i'_{26,30}\right),\\(1,i_{31,53}),(2,i_{31,53}),(3,i_{31,53}),(4,i_{31,53}),(5,i_{31,53}),(6,i_{31,53}),\\(7,i_{31,53}),(8,i_{31,53}),(9,i_{31,53}),(10,i_{31,53}),(i_{11,15},i_{31,53}),\\(i_{16,20},i_{31,53}),(i_{21,25},i_{31,53}),(i_{26,30},i_{31,53}),\left(i_{31,53},i'_{31,53}\right)\\ \mid 1 i_{11,15},i'_{11,15},i_{16,20},i'_{16,20},i_{21,25},i'_{21,25},i_{26,30},i'_{26,30},i_{31,53},i'_{31,53}\in\mathbb{N},\\ 1\le i_{11,15}\le 15, 11\le i'_{11,15}\le 15, 16\le i_{16,20}\le 20,\\ 16\le i'_{16,20}\le 20, 21\le i_{21,25}\le 25, 21\le i'_{21,25}\le 25,\\ 26\le i_{26,30}\le 30, 26\le i'_{26,30}\le 30, 31\le i_{31,53}\le 53,\\ 31\le i'_{31,53}\le 53\end{array}\right\}$$

$$\begin{array}{l}Q^{1234}=Q^{1243}=Q^{1324}=Q^{1342}=Q^{1423}=Q^{1432}=\\Q^{2134}=Q^{2143}=Q^{2314}=Q^{2341}=Q^{2413}=Q^{2431}=\\Q^{3124}=Q^{3142}=Q^{3214}=Q^{3241}=Q^{3412}=Q^{3421}=\\Q^{4123}=Q^{4132}=Q^{4213}=Q^{4231}=Q^{4312}=Q^{4321}=D^4\end{array}$$

$$\begin{array}{l}R_{12}^{1234,1243}=R_{12}^{1342,1432}=R_{12}^{2341,2431}=R_{12}^{3421,4321}=\\R_{12}^{2134,2143}=R_{12}^{3412,4312}=D^4\{(3,4)\}\end{array}$$

$$\begin{array}{l}R_{13}^{1324,1342}=R_{13}^{1243,1423}=R_{13}^{3241,3421}=R_{13}^{2431,4231}=\\R_{13}^{3124,3142}=R_{13}^{2413,4213}=D^4\{(2,4)\}\end{array}$$

$$\begin{array}{l}R_{14}^{1423,1432}=R_{14}^{1234,1324}=R_{14}^{4231,4321}=R_{14}^{2341,3241}=\\R_{14}^{4123,4132}=R_{14}^{2314,3214}=D^4\{(2,3)\}\end{array}$$

$$\begin{array}{l}R_{23}^{2314,2341}=R_{23}^{2143,2413}=R_{23}^{3142,3412}=R_{23}^{1432,4132}=\\R_{23}^{3214,3241}=R_{23}^{1423,4123}=D^4\{(1,4)\}\end{array}$$

$$\begin{array}{l}R_{24}^{2413,2431}=R_{24}^{2134,2314}=R_{24}^{4132,4312}=R_{24}^{1342,3142}=\\R_{24}^{4213,4231}=R_{24}^{1324,3124}=D^4\{(1,3)\}\end{array}$$

$$R_{34}^{3412,3421} = R_{34}^{3124,3214} = R_{34}^{4123,4213} = R_{34}^{1243,2143} =$$
$$R_{34}^{4312,4321} = R_{34}^{1234,2134} = D^4 \{(1,2)\}$$

and of morphisms

$$f_{1234} : Q^{1234} \to P, f_{1243} : Q^{1243} \to P, f_{1324} : Q^{1324} \to P, f_{1342} : Q^{1342} \to P,$$
$$f_{1423} : Q^{1423} \to P, f_{1432} : Q^{1432} \to P, f_{2134} : Q^{2134} \to P, f_{2143} : Q^{2143} \to P,$$
$$f_{2314} : Q^{2314} \to P, f_{2341} : Q^{2341} \to P, f_{2413} : Q^{2413} \to P, f_{2431} : Q^{2431} \to P,$$
$$f_{3124} : Q^{3124} \to P, f_{3142} : Q^{3142} \to P, f_{3214} : Q^{3214} \to P, f_{3241} : Q^{3241} \to P,$$
$$f_{3412} : Q^{3412} \to P, f_{3421} : Q^{3421} \to P, f_{4123} : Q^{4123} \to P, f_{4132} : Q^{4132} \to P,$$
$$f_{4213} : Q^{4213} \to P, f_{4231} : Q^{4231} \to P, f_{4312} : Q^{4312} \to P, f_{4321} : Q^{4321} \to P,$$

$$g_{12}^{1234,1243} : R_{12}^{1234,1243} \to Q^{1234}, h_{12}^{1234,1243} : R_{12}^{1234,1243} \to Q^{1243},$$
$$g_{12}^{1342,1432} : R_{12}^{1342,1432} \to Q^{1342}, h_{12}^{1342,1432} : R_{12}^{1342,1432} \to Q^{1432},$$
$$g_{12}^{2341,2431} : R_{12}^{2341,2431} \to Q^{2341}, h_{12}^{2341,2431} : R_{12}^{2341,2431} \to Q^{2431},$$
$$g_{12}^{3421,4321} : R_{12}^{3421,4321} \to Q^{3421}, h_{12}^{3421,4321} : R_{12}^{3421,4321} \to Q^{4321},$$
$$g_{12}^{2134,2143} : R_{12}^{2134,2143} \to Q^{2134}, h_{12}^{2134,2143} : R_{12}^{2134,2143} \to Q^{2143},$$
$$g_{12}^{3412,4312} : R_{12}^{3412,4312} \to Q^{3412}, h_{12}^{3412,4312} : R_{12}^{3412,4312} \to Q^{4312},$$

$$g_{13}^{1324,1342} : R_{13}^{1324,1342} \to Q^{1324}, h_{13}^{1324,1342} : R_{13}^{1324,1342} \to Q^{1342},$$
$$g_{13}^{1243,1423} : R_{13}^{1243,1423} \to Q^{1243}, h_{13}^{1243,1423} : R_{13}^{1243,1423} \to Q^{1423},$$
$$g_{13}^{3241,3421} : R_{13}^{3241,3421} \to Q^{3241}, h_{13}^{3241,3421} : R_{13}^{3241,3421} \to Q^{3421},$$
$$g_{13}^{2431,4231} : R_{13}^{2431,4231} \to Q^{2431}, h_{13}^{2431,4231} : R_{13}^{2431,4231} \to Q^{4231},$$
$$g_{13}^{3124,3142} : R_{13}^{3124,3142} \to Q^{3124}, h_{13}^{3124,3142} : R_{13}^{3124,3142} \to Q^{3142},$$
$$g_{13}^{2413,4213} : R_{13}^{2413,4213} \to Q^{2413}, h_{13}^{2413,4213} : R_{13}^{2413,4213} \to Q^{4213},$$

$$g_{14}^{1423,1432} : R_{14}^{1423,1432} \to Q^{1423}, h_{14}^{1423,1432} : R_{14}^{1423,1432} \to Q^{1432},$$
$$g_{14}^{1234,1324} : R_{14}^{1234,1324} \to Q^{1234}, h_{14}^{1234,1324} : R_{14}^{1234,1324} \to Q^{1324},$$
$$g_{14}^{4231,4321} : R_{14}^{4231,4321} \to Q^{4231}, h_{14}^{4231,4321} : R_{14}^{4231,4321} \to Q^{4321},$$
$$g_{14}^{2341,3241} : R_{14}^{2341,3241} \to Q^{2341}, h_{14}^{2341,3241} : R_{14}^{2341,3241} \to Q^{3241},$$
$$g_{14}^{4123,4132} : R_{14}^{4123,4132} \to Q^{4123}, h_{14}^{4123,4132} : R_{14}^{4123,4132} \to Q^{4132},$$
$$g_{14}^{2314,3214} : R_{14}^{2314,3214} \to Q^{2314}, h_{14}^{2314,3214} : R_{14}^{2314,3214} \to Q^{3214},$$

$$g_{23}^{2314,2341}:R_{23}^{2314,2341}\to Q^{2314},h_{23}^{2314,2341}:R_{23}^{2314,2341}\to Q^{2341},$$
$$g_{23}^{2143,2413}:R_{23}^{2143,2413}\to Q^{2143},h_{23}^{2143,2413}:R_{23}^{2143,2413}\to Q^{2413},$$
$$g_{23}^{3142,3412}:R_{23}^{3142,3412}\to Q^{3142},h_{23}^{3142,3412}:R_{23}^{3142,3412}\to Q^{3412},$$
$$g_{23}^{1432,4132}:R_{23}^{1432,4132}\to Q^{1432},h_{23}^{1432,4132}:R_{23}^{1432,4132}\to Q^{4132},$$
$$g_{23}^{3214,3241}:R_{23}^{3214,3241}\to Q^{3214},h_{23}^{3214,3241}:R_{23}^{3214,3241}\to Q^{3241},$$
$$g_{23}^{1423,4123}:R_{23}^{1423,4123}\to Q^{1423},h_{23}^{1423,4123}:R_{23}^{1423,4123}\to Q^{4123},$$

$$g_{24}^{2413,2431}:R_{24}^{2413,2431}\to Q^{2413},h_{24}^{2413,2431}:R_{24}^{2413,2431}\to Q^{2431},$$
$$g_{24}^{2134,2314}:R_{24}^{2134,2314}\to Q^{2134},h_{24}^{2134,2314}:R_{24}^{2134,2314}\to Q^{2314},$$
$$g_{24}^{4132,4312}:R_{24}^{4132,4312}\to Q^{4132},h_{24}^{4132,4312}:R_{24}^{4132,4312}\to Q^{4312},$$
$$g_{24}^{1342,3142}:R_{24}^{1342,3142}\to Q^{1342},h_{24}^{1342,3142}:R_{24}^{1342,3142}\to Q^{3142},$$
$$g_{24}^{4213,4231}:R_{24}^{4213,4231}\to Q^{4213},h_{24}^{4213,4231}:R_{24}^{4213,4231}\to Q^{4231},$$
$$g_{24}^{1324,3124}:R_{24}^{1324,3124}\to Q^{1324},h_{24}^{1324,3124}:R_{24}^{1324,3124}\to Q^{3124},$$

$$g_{34}^{3412,3421}:R_{34}^{3412,3421}\to Q^{3412},h_{34}^{3412,3421}:R_{34}^{3412,3421}\to Q^{3421},$$
$$g_{34}^{3124,3214}:R_{34}^{3124,3214}\to Q^{3124},h_{34}^{3124,3214}:R_{34}^{3124,3214}\to Q^{3214},$$
$$g_{34}^{4123,4213}:R_{34}^{4123,4213}\to Q^{4123},h_{34}^{4123,4213}:R_{34}^{4123,4213}\to Q^{4213},$$
$$g_{34}^{1243,2143}:R_{34}^{1243,2143}\to Q^{1243},h_{34}^{1243,2143}:R_{34}^{1243,2143}\to Q^{2143},$$
$$g_{34}^{4312,4321}:R_{34}^{4312,4321}\to Q^{4312},h_{34}^{4312,4321}:R_{34}^{4312,4321}\to Q^{4321},$$
$$g_{34}^{1234,2134}:R_{34}^{1234,2134}\to Q^{1234},h_{34}^{1234,2134}:R_{34}^{1234,2134}\to Q^{2134}$$

with

$$f_{1234}\left(d_1,d_2,d_3,d_4\right) = \left(d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{53}{0}\right)$$

$$f_{1243}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{9}{0},\underset{10}{d_3d_4},\underset{11}{0},...,\underset{20}{0},d_1d_3d_4,\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{30}{0}, \\ d_1d_2d_3d_4,\underset{32}{0},...,\underset{53}{0} \end{array}\right)$$

$$f_{1324}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,\underset{9}{0},\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{31}{0}, \\ d_1d_2d_3d_4,\underset{33}{0},...,\underset{53}{0} \end{array}\right)$$

$$f_{1342}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{27}{0}, \\ d_2d_3d_4,\underset{29}{0},...,\underset{32}{0},d_1d_2d_3d_4,\underset{34}{0},...,\underset{53}{0} \end{pmatrix}$$

$$f_{1423}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{20}{0},d_1d_3d_4, \\ \underset{22}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{33}{0},d_1d_2d_3d_4,\underset{35}{0},...,\underset{53}{0} \end{pmatrix}$$

$$f_{1432}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{16}{0},...,\underset{20}{0}, \\ d_1d_3d_4,\underset{22}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{34}{0},d_1d_2d_3d_4,\underset{36}{0},...,\underset{53}{0} \end{pmatrix}$$

$$f_{2134}(d_1,d_2,d_3,d_4) = \left(d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{35}{0},d_1d_2d_3d_4,\underset{37}{0},...,\underset{53}{0}\right)$$

$$f_{2143}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{20}{0},d_1d_3d_4, \\ \underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{36}{0},d_1d_2d_3d_4,\underset{38}{0},...,\underset{53}{0} \end{pmatrix}$$

$$f_{2314}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,\underset{7}{0},...,\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{21}{0},d_1d_3d_4, \\ \underset{23}{0},...,\underset{37}{0},d_1d_2d_3d_4,\underset{39}{0},...,\underset{53}{0} \end{pmatrix}$$

$$f_{2341}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},...,\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0}, \\ d_1d_3d_4,\underset{24}{0},...,\underset{38}{0},d_1d_2d_3d_4,\underset{40}{0},...,\underset{53}{0} \end{pmatrix}$$

$$f_{2413}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{23}{0}, \\ d_1d_3d_4,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{39}{0},d_1d_2d_3d_4,\underset{41}{0},...,\underset{53}{0} \end{pmatrix}$$

$$f_{2431}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4, \\ \underset{19}{0},...,\underset{24}{0},d_1d_3d_4,d_2d_3d_4,\underset{27}{0},...,\underset{40}{0},d_1d_2d_3d_4,\underset{42}{0},...,\underset{53}{0} \end{pmatrix}$$

$$f_{3124}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{21}{0},d_1d_3d_4,\underset{23}{0},...,\underset{26}{0}, \\ d_2d_3d_4,\underset{28}{0},...,\underset{41}{0},d_1d_2d_3d_4,\underset{43}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{3142}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{21}{0},d_1d_3d_4, \\ \underset{23}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{42}{0},d_1d_2d_3d_4,\underset{44}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{3214}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{21}{0},d_1d_3d_4, \\ \underset{23}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{43}{0},d_1d_2d_3d_4,\underset{45}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{3241}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0}, \\ d_1d_3d_4,\underset{24}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{44}{0},d_1d_2d_3d_4,\underset{46}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{3412}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0},d_1d_2d_4, \\ \underset{20}{0},...,\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{45}{0},d_1d_2d_3d_4,\underset{47}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{3421}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0},d_1d_2d_4, \\ \underset{21}{0},\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{46}{0},d_1d_2d_3d_4,\underset{48}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{4123}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,\underset{7}{0},\underset{7}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0},d_1d_3d_4, \\ \underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{47}{0},d_1d_2d_3d_4,\underset{49}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{4132}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0}, \\ d_1d_3d_4,\underset{25}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{48}{0},d_1d_2d_3d_4,\underset{50}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{4213}\left(d_1,d_2,d_3,d_4\right) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{19}{0},d_1d_2d_4, \\ \underset{21}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{49}{0},d_1d_2d_3d_4,\underset{51}{0},...,\underset{53}{0} \end{array} \right)$$

$$f_{4231}\left(d_1,d_2,d_3,d_4\right) = \begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{19}{0}, \\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{50}{0},d_1d_2d_3d_4,\underset{52}{0},\underset{53}{0} \end{pmatrix}$$

$$f_{4312}\left(d_1,d_2,d_3,d_4\right) = \begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0}, \\ d_1d_2d_4,\underset{20}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{51}{0},d_1d_2d_3d_4,\underset{53}{0} \end{pmatrix}$$

$$f_{4321}\left(d_1,d_2,d_3,d_4\right) = \begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0}, \\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{52}{0},d_1d_2d_3d_4 \end{pmatrix}$$

$$g_{12}^{1234,1243} = g_{12}^{1342,1432} = g_{12}^{2341,2431} = g_{12}^{3421,4321} = g_{12}^{2134,2143} = g_{12}^{3412,4312} = h_{12}^{1234,1243} = h_{12}^{1342,1432} = h_{12}^{2341,2431} = h_{12}^{3421,4321} = h_{12}^{2134,2143} = h_{12}^{3412,4312}$$

$$g_{12}^{1234,1243}\left(d_1,d_2,d_3,d_4\right) = h_{12}^{1234,1243}\left(d_1,d_2,d_3,d_4\right) = \left(d_1,d_2,d_3,d_4\right)$$

$$g_{13}^{1324,1342} = g_{13}^{1243,1423} = g_{13}^{3241,3421} = g_{13}^{2431,4231} = g_{13}^{3124,3142} = g_{13}^{2413,4213} = h_{13}^{1324,1342} = h_{13}^{1243,1423} = h_{13}^{3241,3421} = h_{13}^{2431,4231} = h_{13}^{3124,3142} = h_{13}^{2413,4213}$$

$$g_{13}^{1324,1342}\left(d_1,d_2,d_3,d_4\right) = h_{13}^{1324,1342}\left(d_1,d_2,d_3,d_4\right) = \left(d_1,d_2,d_3,d_4\right)$$

$$g_{14}^{1423,1432} = g_{14}^{1234,1324} = g_{14}^{4231,4321} = g_{14}^{2341,3241} = g_{14}^{4123,4132} = g_{14}^{2314,3214} = h_{14}^{1423,1432} = h_{14}^{1234,1324} = h_{14}^{4231,4321} = h_{14}^{2341,3241} = h_{14}^{4123,4132} = h_{14}^{2314,3214}$$

$$g_{14}^{1423,1432}\left(d_1,d_2,d_3,d_4\right) = h_{14}^{1423,1432}\left(d_1,d_2,d_3,d_4\right) = \left(d_1,d_2,d_3,d_4\right)$$

$$g_{23}^{2314,2341} = g_{23}^{2143,2413} = g_{23}^{3142,3412} = g_{23}^{1432,4132} = g_{23}^{3214,3241} = g_{23}^{1423,4123} = h_{23}^{2314,2341} = h_{23}^{2143,2413} = h_{23}^{3142,3412} = h_{23}^{1432,4132} = h_{23}^{3214,3241} = h_{23}^{1423,4123}$$

$$g_{23}^{2314,2341}\left(d_1,d_2,d_3,d_4\right) = h_{23}^{2314,2341}\left(d_1,d_2,d_3,d_4\right) = \left(d_1,d_2,d_3,d_4\right)$$

$$g_{24}^{2413,2431} = g_{24}^{2134,2314} = g_{24}^{4132,4312} = g_{24}^{1342,3142} = g_{24}^{4213,4231} = g_{24}^{1324,3124} = h_{24}^{2413,2431} = h_{24}^{2134,2314} = h_{24}^{4132,4312} = h_{24}^{1342,3142} = h_{24}^{4213,4231} = h_{24}^{1324,3124}$$

$$g_{24}^{2413,2431}\left(d_1,d_2,d_3,d_4\right) = h_{24}^{2413,2431}\left(d_1,d_2,d_3,d_4\right) = \left(d_1,d_2,d_3,d_4\right)$$

$$g_{34}^{3412,3421} = g_{34}^{3124,3214} = g_{34}^{4123,4213} = g_{34}^{1243,2143} = g_{34}^{4312,4321} = g_{34}^{1234,2134} = h_{34}^{3412,3421} = h_{34}^{3124,3214} = h_{34}^{4123,4213} = h_{34}^{1243,2143} = h_{34}^{4312,4321} = h_{34}^{1234,2134}$$

$$g_{34}^{3412,3421}\left(d_1,d_2,d_3,d_4\right) = h_{34}^{3412,3421}\left(d_1,d_2,d_3,d_4\right) = \left(d_1,d_2,d_3,d_4\right)$$

is a quasi-colimit diagram.

Proof. Let $\theta^{1234}:Q^{1234}\rightarrow\mathbb{R}$, $\theta^{1243}:Q^{1243}\rightarrow\mathbb{R}$, $\theta^{1324}:Q^{1324}\rightarrow\mathbb{R}$, $\theta^{1342}:Q^{1342}\rightarrow\mathbb{R}$, $\theta^{1423}:Q^{1423}\rightarrow\mathbb{R}$, $\theta^{1432}:Q^{1432}\rightarrow\mathbb{R}$, $\theta^{2134}:Q^{2134}\rightarrow\mathbb{R}$, $\theta^{2143}:Q^{2143}\rightarrow\mathbb{R}$, $\theta^{2314}:Q^{2314}\rightarrow\mathbb{R}$, $\theta^{2341}:Q^{2341}\rightarrow\mathbb{R}$, $\theta^{2413}:Q^{2413}\rightarrow\mathbb{R}$, $\theta^{2431}:Q^{2431}\rightarrow\mathbb{R}$, $\theta^{3124}:Q^{3124}\rightarrow\mathbb{R}$, $\theta^{3142}:Q^{3142}\rightarrow\mathbb{R}$, $\theta^{3214}:Q^{3214}\rightarrow\mathbb{R}$, $\theta^{3241}:Q^{3241}\rightarrow\mathbb{R}$, $\theta^{3412}:Q^{3412}\rightarrow\mathbb{R}$, $\theta^{3421}:Q^{3421}\rightarrow\mathbb{R}$, $\theta^{4123}:Q^{4123}\rightarrow\mathbb{R}$, $\theta^{4132}:Q^{4132}\rightarrow\mathbb{R}$, $\theta^{4213}:Q^{4213}\rightarrow\mathbb{R}$, $\theta^{4231}:Q^{4231}\rightarrow\mathbb{R}$, $\theta^{4312}:Q^{4312}\rightarrow\mathbb{R}$ and $\theta^{4321}:Q^{4321}\rightarrow\mathbb{R}$ be mappings, which are to be of the following forms by dint of the general Kock-Lawvere axiom (cf. §2.1.3 of [4]):

$$\theta^{1234}\left(d_1,d_2,d_3,d_4\right)$$
$$=a^{1234}+a_1^{1234}d_1+a_2^{1234}d_2+a_3^{1234}d_3+a_4^{1234}d_1+a_{12}^{1234}d_1d_2+a_{13}^{1234}d_1d_3+a_{14}^{1234}d_1d_4+$$
$$a_{23}^{1234}d_2d_3+a_{24}^{1234}d_2d_4+a_{34}^{1234}d_3d_4+a_{123}^{1234}d_1d_2d_3+a_{124}^{1234}d_1d_2d_4+a_{134}^{1234}d_1d_3d_4+$$
$$a_{234}^{1234}d_2d_3d_4+a_{1234}^{1234}d_1d_2d_3d_4$$

$$\theta^{1243}\left(d_1,d_2,d_3,d_4\right)$$
$$=a^{1243}+a_1^{1243}d_1+a_2^{1243}d_2+a_3^{1243}d_3+a_4^{1243}d_1+a_{12}^{1243}d_1d_2+a_{13}^{1243}d_1d_3+a_{14}^{1243}d_1d_4+$$
$$a_{23}^{1243}d_2d_3+a_{24}^{1243}d_2d_4+a_{34}^{1243}d_3d_4+a_{123}^{1243}d_1d_2d_3+a_{124}^{1243}d_1d_2d_4+a_{134}^{1243}d_1d_3d_4+$$
$$a_{234}^{1243}d_2d_3d_4+a_{1234}^{1243}d_1d_2d_3d_4$$

$$\theta^{1324}\left(d_1,d_2,d_3,d_4\right)$$
$$=a^{1324}+a_1^{1324}d_1+a_2^{1324}d_2+a_3^{1324}d_3+a_4^{1324}d_1+a_{12}^{1324}d_1d_2+a_{13}^{1324}d_1d_3+a_{14}^{1324}d_1d_4+$$
$$a_{23}^{1324}d_2d_3+a_{24}^{1324}d_2d_4+a_{34}^{1324}d_3d_4+a_{123}^{1324}d_1d_2d_3+a_{124}^{1324}d_1d_2d_4+a_{134}^{1324}d_1d_3d_4+$$
$$a_{234}^{1324}d_2d_3d_4+a_{1234}^{1324}d_1d_2d_3d_4$$

$$\theta^{1342}\left(d_1,d_2,d_3,d_4\right)$$
$$=a^{1342}+a_1^{1342}d_1+a_2^{1342}d_2+a_3^{1342}d_3+a_4^{1342}d_1+a_{12}^{1342}d_1d_2+a_{13}^{1342}d_1d_3+a_{14}^{1342}d_1d_4+$$
$$a_{23}^{1342}d_2d_3+a_{24}^{1342}d_2d_4+a_{34}^{1342}d_3d_4+a_{123}^{1342}d_1d_2d_3+a_{124}^{1342}d_1d_2d_4+a_{134}^{1342}d_1d_3d_4+$$
$$a_{234}^{1342}d_2d_3d_4+a_{1234}^{1342}d_1d_2d_3d_4$$

$$\theta^{1423}\left(d_1,d_2,d_3,d_4\right)$$
$$=a^{1423}+a_1^{1423}d_1+a_2^{1423}d_2+a_3^{1423}d_3+a_4^{1423}d_1+a_{12}^{1423}d_1d_2+a_{13}^{1423}d_1d_3+a_{14}^{1423}d_1d_4+$$
$$a_{23}^{1423}d_2d_3+a_{24}^{1423}d_2d_4+a_{34}^{1423}d_3d_4+a_{123}^{1423}d_1d_2d_3+a_{124}^{1423}d_1d_2d_4+a_{134}^{1423}d_1d_3d_4+$$
$$a_{234}^{1423}d_2d_3d_4+a_{1234}^{1423}d_1d_2d_3d_4$$

$$\theta^{1432}\left(d_1,d_2,d_3,d_4\right)$$
$$=a^{1432}+a_1^{1432}d_1+a_2^{1432}d_2+a_3^{1432}d_3+a_4^{1432}d_1+a_{12}^{1432}d_1d_2+a_{13}^{1432}d_1d_3+a_{14}^{1432}d_1d_4+$$
$$a_{23}^{1432}d_2d_3+a_{24}^{1432}d_2d_4+a_{34}^{1432}d_3d_4+a_{123}^{1432}d_1d_2d_3+a_{124}^{1432}d_1d_2d_4+a_{134}^{1432}d_1d_3d_4+$$
$$a_{234}^{1432}d_2d_3d_4+a_{1234}^{1432}d_1d_2d_3d_4$$

$\theta^{2134}(d_1,d_2,d_3,d_4)$
$= a^{2134} + a_1^{2134}d_1 + a_2^{2134}d_2 + a_3^{2134}d_3 + a_4^{2134}d_1 + a_{12}^{2134}d_1d_2 + a_{13}^{2134}d_1d_3 + a_{14}^{2134}d_1d_4 +$
$a_{23}^{2134}d_2d_3 + a_{24}^{2134}d_2d_4 + a_{34}^{2134}d_3d_4 + a_{123}^{2134}d_1d_2d_3 + a_{124}^{2134}d_1d_2d_4 + a_{134}^{2134}d_1d_3d_4 +$
$a_{234}^{2134}d_2d_3d_4 + a_{1234}^{2134}d_1d_2d_3d_4$

$\theta^{2143}(d_1,d_2,d_3,d_4)$
$= a^{2143} + a_1^{2143}d_1 + a_2^{2143}d_2 + a_3^{2143}d_3 + a_4^{2143}d_1 + a_{12}^{2143}d_1d_2 + a_{13}^{2143}d_1d_3 + a_{14}^{2143}d_1d_4 +$
$a_{23}^{2143}d_2d_3 + a_{24}^{2143}d_2d_4 + a_{34}^{2143}d_3d_4 + a_{123}^{2143}d_1d_2d_3 + a_{124}^{2143}d_1d_2d_4 + a_{134}^{2143}d_1d_3d_4 +$
$a_{234}^{2143}d_2d_3d_4 + a_{1234}^{2143}d_1d_2d_3d_4$

$\theta^{2314}(d_1,d_2,d_3,d_4)$
$= a^{2314} + a_1^{2314}d_1 + a_2^{2314}d_2 + a_3^{2314}d_3 + a_4^{2314}d_1 + a_{12}^{2314}d_1d_2 + a_{13}^{2314}d_1d_3 + a_{14}^{2314}d_1d_4 +$
$a_{23}^{2314}d_2d_3 + a_{24}^{2314}d_2d_4 + a_{34}^{2314}d_3d_4 + a_{123}^{2314}d_1d_2d_3 + a_{124}^{2314}d_1d_2d_4 + a_{134}^{2314}d_1d_3d_4 +$
$a_{234}^{2314}d_2d_3d_4 + a_{1234}^{2314}d_1d_2d_3d_4$

$\theta^{2341}(d_1,d_2,d_3,d_4)$
$= a^{2341} + a_1^{2341}d_1 + a_2^{2341}d_2 + a_3^{2341}d_3 + a_4^{2341}d_1 + a_{12}^{2341}d_1d_2 + a_{13}^{2341}d_1d_3 + a_{14}^{2341}d_1d_4 +$
$a_{23}^{2341}d_2d_3 + a_{24}^{2341}d_2d_4 + a_{34}^{2341}d_3d_4 + a_{123}^{2341}d_1d_2d_3 + a_{124}^{2341}d_1d_2d_4 + a_{134}^{2341}d_1d_3d_4 +$
$a_{234}^{2341}d_2d_3d_4 + a_{1234}^{2341}d_1d_2d_3d_4$

$\theta^{2413}(d_1,d_2,d_3,d_4)$
$= a^{2413} + a_1^{2413}d_1 + a_2^{2413}d_2 + a_3^{2413}d_3 + a_4^{2413}d_1 + a_{12}^{2413}d_1d_2 + a_{13}^{2413}d_1d_3 + a_{14}^{2413}d_1d_4 +$
$a_{23}^{2413}d_2d_3 + a_{24}^{2413}d_2d_4 + a_{34}^{2413}d_3d_4 + a_{123}^{2413}d_1d_2d_3 + a_{124}^{2413}d_1d_2d_4 + a_{134}^{2413}d_1d_3d_4 +$
$a_{234}^{2413}d_2d_3d_4 + a_{1234}^{2413}d_1d_2d_3d_4$

$\theta^{2431}(d_1,d_2,d_3,d_4)$
$= a^{2431} + a_1^{2431}d_1 + a_2^{2431}d_2 + a_3^{2431}d_3 + a_4^{2431}d_1 + a_{12}^{2431}d_1d_2 + a_{13}^{2431}d_1d_3 + a_{14}^{2431}d_1d_4 +$
$a_{23}^{2431}d_2d_3 + a_{24}^{2431}d_2d_4 + a_{34}^{2431}d_3d_4 + a_{123}^{2431}d_1d_2d_3 + a_{124}^{2431}d_1d_2d_4 + a_{134}^{2431}d_1d_3d_4 +$
$a_{234}^{2431}d_2d_3d_4 + a_{1234}^{2431}d_1d_2d_3d_4$

$\theta^{3124}(d_1,d_2,d_3,d_4)$
$= a^{3124} + a_1^{3124}d_1 + a_2^{3124}d_2 + a_3^{3124}d_3 + a_4^{3124}d_1 + a_{12}^{3124}d_1d_2 + a_{13}^{3124}d_1d_3 + a_{14}^{3124}d_1d_4 +$
$a_{23}^{3124}d_2d_3 + a_{24}^{3124}d_2d_4 + a_{34}^{3124}d_3d_4 + a_{123}^{3124}d_1d_2d_3 + a_{124}^{3124}d_1d_2d_4 + a_{134}^{3124}d_1d_3d_4 +$
$a_{234}^{3124}d_2d_3d_4 + a_{1234}^{3124}d_1d_2d_3d_4$

$\theta^{3142}(d_1, d_2, d_3, d_4)$
$= a^{3142} + a_1^{3142}d_1 + a_2^{3142}d_2 + a_3^{3142}d_3 + a_4^{3142}d_1 + a_{12}^{3142}d_1d_2 + a_{13}^{3142}d_1d_3 + a_{14}^{3142}d_1d_4 +$
$a_{23}^{3142}d_2d_3 + a_{24}^{3142}d_2d_4 + a_{34}^{3142}d_3d_4 + a_{123}^{3142}d_1d_2d_3 + a_{124}^{3142}d_1d_2d_4 + a_{134}^{3142}d_1d_3d_4 +$
$a_{234}^{3142}d_2d_3d_4 + a_{1234}^{3142}d_1d_2d_3d_4$

$\theta^{3214}(d_1, d_2, d_3, d_4)$
$= a^{3214} + a_1^{3214}d_1 + a_2^{3214}d_2 + a_3^{3214}d_3 + a_4^{3214}d_1 + a_{12}^{3214}d_1d_2 + a_{13}^{3214}d_1d_3 + a_{14}^{3214}d_1d_4 +$
$a_{23}^{3214}d_2d_3 + a_{24}^{3214}d_2d_4 + a_{34}^{3214}d_3d_4 + a_{123}^{3214}d_1d_2d_3 + a_{124}^{3214}d_1d_2d_4 + a_{134}^{3214}d_1d_3d_4 +$
$a_{234}^{3214}d_2d_3d_4 + a_{1234}^{3214}d_1d_2d_3d_4$

$\theta^{3241}(d_1, d_2, d_3, d_4)$
$= a^{3241} + a_1^{3241}d_1 + a_2^{3241}d_2 + a_3^{3241}d_3 + a_4^{3241}d_1 + a_{12}^{3241}d_1d_2 + a_{13}^{3241}d_1d_3 + a_{14}^{3241}d_1d_4 +$
$a_{23}^{3241}d_2d_3 + a_{24}^{3241}d_2d_4 + a_{34}^{3241}d_3d_4 + a_{123}^{3241}d_1d_2d_3 + a_{124}^{3241}d_1d_2d_4 + a_{134}^{3241}d_1d_3d_4 +$
$a_{234}^{3241}d_2d_3d_4 + a_{1234}^{3241}d_1d_2d_3d_4$

$\theta^{3412}(d_1, d_2, d_3, d_4)$
$= a^{3412} + a_1^{3412}d_1 + a_2^{3412}d_2 + a_3^{3412}d_3 + a_4^{3412}d_1 + a_{12}^{3412}d_1d_2 + a_{13}^{3412}d_1d_3 + a_{14}^{3412}d_1d_4 +$
$a_{23}^{3412}d_2d_3 + a_{24}^{3412}d_2d_4 + a_{34}^{3412}d_3d_4 + a_{123}^{3412}d_1d_2d_3 + a_{124}^{3412}d_1d_2d_4 + a_{134}^{3412}d_1d_3d_4 +$
$a_{234}^{3412}d_2d_3d_4 + a_{1234}^{3412}d_1d_2d_3d_4$

$\theta^{3421}(d_1, d_2, d_3, d_4)$
$= a^{3421} + a_1^{3421}d_1 + a_2^{3421}d_2 + a_3^{3421}d_3 + a_4^{3421}d_1 + a_{12}^{3421}d_1d_2 + a_{13}^{3421}d_1d_3 + a_{14}^{3421}d_1d_4 +$
$a_{23}^{3421}d_2d_3 + a_{24}^{3421}d_2d_4 + a_{34}^{3421}d_3d_4 + a_{123}^{3421}d_1d_2d_3 + a_{124}^{3421}d_1d_2d_4 + a_{134}^{3421}d_1d_3d_4 +$
$a_{234}^{3421}d_2d_3d_4 + a_{1234}^{3421}d_1d_2d_3d_4$

$\theta^{4123}(d_1, d_2, d_3, d_4)$
$= a^{4123} + a_1^{4123}d_1 + a_2^{4123}d_2 + a_3^{4123}d_3 + a_4^{4123}d_1 + a_{12}^{4123}d_1d_2 + a_{13}^{4123}d_1d_3 + a_{14}^{4123}d_1d_4 +$
$a_{23}^{4123}d_2d_3 + a_{24}^{4123}d_2d_4 + a_{34}^{4123}d_3d_4 + a_{123}^{4123}d_1d_2d_3 + a_{124}^{4123}d_1d_2d_4 + a_{134}^{4123}d_1d_3d_4 +$
$a_{234}^{4123}d_2d_3d_4 + a_{1234}^{4123}d_1d_2d_3d_4$

$\theta^{4132}(d_1, d_2, d_3, d_4)$
$= a^{4132} + a_1^{4132}d_1 + a_2^{4132}d_2 + a_3^{4132}d_3 + a_4^{4132}d_1 + a_{12}^{4132}d_1d_2 + a_{13}^{4132}d_1d_3 + a_{14}^{4132}d_1d_4 +$
$a_{23}^{4132}d_2d_3 + a_{24}^{4132}d_2d_4 + a_{34}^{4132}d_3d_4 + a_{123}^{4132}d_1d_2d_3 + a_{124}^{4132}d_1d_2d_4 + a_{134}^{4132}d_1d_3d_4 +$
$a_{234}^{4132}d_2d_3d_4 + a_{1234}^{4132}d_1d_2d_3d_4$

$\theta^{4213}(d_1, d_2, d_3, d_4)$
$= a^{4213} + a_1^{4213}d_1 + a_2^{4213}d_2 + a_3^{4213}d_3 + a_4^{4213}d_1 + a_{12}^{4213}d_1d_2 + a_{13}^{4213}d_1d_3 + a_{14}^{4213}d_1d_4 +$
$a_{23}^{4213}d_2d_3 + a_{24}^{4213}d_2d_4 + a_{34}^{4213}d_3d_4 + a_{123}^{4213}d_1d_2d_3 + a_{124}^{4213}d_1d_2d_4 + a_{134}^{4213}d_1d_3d_4 +$
$a_{234}^{4213}d_2d_3d_4 + a_{1234}^{4213}d_1d_2d_3d_4$

$\theta^{4231}(d_1, d_2, d_3, d_4)$
$= a^{4231} + a_1^{4231}d_1 + a_2^{4231}d_2 + a_3^{4231}d_3 + a_4^{4231}d_1 + a_{12}^{4231}d_1d_2 + a_{13}^{4231}d_1d_3 + a_{14}^{4231}d_1d_4 +$
$a_{23}^{4231}d_2d_3 + a_{24}^{4231}d_2d_4 + a_{34}^{4231}d_3d_4 + a_{123}^{4231}d_1d_2d_3 + a_{124}^{4231}d_1d_2d_4 + a_{134}^{4231}d_1d_3d_4 +$
$a_{234}^{4231}d_2d_3d_4 + a_{1234}^{4231}d_1d_2d_3d_4$

$\theta^{4312}(d_1, d_2, d_3, d_4)$
$= a^{4312} + a_1^{4312}d_1 + a_2^{4312}d_2 + a_3^{4312}d_3 + a_4^{4312}d_1 + a_{12}^{4312}d_1d_2 + a_{13}^{4312}d_1d_3 + a_{14}^{4312}d_1d_4 +$
$a_{23}^{4312}d_2d_3 + a_{24}^{4312}d_2d_4 + a_{34}^{4312}d_3d_4 + a_{123}^{4312}d_1d_2d_3 + a_{124}^{4312}d_1d_2d_4 + a_{134}^{4312}d_1d_3d_4 +$
$a_{234}^{4312}d_2d_3d_4 + a_{1234}^{4312}d_1d_2d_3d_4$

$\theta^{4321}(d_1, d_2, d_3, d_4)$
$= a^{4321} + a_1^{4321}d_1 + a_2^{4321}d_2 + a_3^{4321}d_3 + a_4^{4321}d_1 + a_{12}^{4321}d_1d_2 + a_{13}^{4321}d_1d_3 + a_{14}^{4321}d_1d_4 +$
$a_{23}^{4321}d_2d_3 + a_{24}^{4321}d_2d_4 + a_{34}^{4321}d_3d_4 + a_{123}^{4321}d_1d_2d_3 + a_{124}^{4321}d_1d_2d_4 + a_{134}^{4321}d_1d_3d_4 +$
$a_{234}^{4321}d_2d_3d_4 + a_{1234}^{4321}d_1d_2d_3d_4$

The conditions

$$\theta^{1234}\circ g_{12}^{1234,1243}=\theta^{1243}\circ h_{12}^{1234,1243},\theta^{1342}\circ g_{12}^{1342,1432}=\theta^{1432}\circ h_{12}^{1342,1432},$$
$$\theta^{2341}\circ g_{12}^{2341,2431}=\theta^{2431}\circ h_{12}^{2341,2431},\theta^{3421}\circ g_{12}^{3421,4321}=\theta^{4321}\circ h_{12}^{3421,4321},$$
$$\theta^{2134}\circ g_{12}^{2134,2143}=\theta^{2143}\circ h_{12}^{2134,2143},\theta^{3412}\circ g_{12}^{3412,4312}=\theta^{4312}\circ h_{12}^{3412,4312},$$
$$\theta^{1324}\circ g_{13}^{1324,1342}=\theta^{1342}\circ h_{13}^{1324,1342},\theta^{1243}\circ g_{13}^{1243,1423}=\theta^{1423}\circ h_{13}^{1243,1423},$$
$$\theta^{3241}\circ g_{13}^{3241,3421}=\theta^{3421}\circ h_{13}^{3241,3421},\theta^{2431}\circ g_{13}^{2431,4231}=\theta^{4231}\circ h_{13}^{2431,4231},$$
$$\theta^{3124}\circ g_{13}^{3124,3142}=\theta^{3142}\circ h_{13}^{3124,3142},\theta^{2413}\circ g_{13}^{2413,4213}=\theta^{4213}\circ h_{13}^{2413,4213},$$
$$\theta^{1423}\circ g_{14}^{1423,1432}=\theta^{1432}\circ h_{14}^{1423,1432},\theta^{1234}\circ g_{14}^{1234,1324}=\theta^{1324}\circ h_{14}^{1234,1324},$$
$$\theta^{4231}\circ g_{14}^{4231,4321}=\theta^{4321}\circ h_{14}^{4231,4321},\theta^{2341}\circ g_{14}^{2341,3241}=\theta^{3241}\circ h_{14}^{2341,3241},$$
$$\theta^{4123}\circ g_{14}^{4123,4132}=\theta^{4132}\circ h_{14}^{4123,4132},\theta^{2314}\circ g_{14}^{2314,3214}=\theta^{3214}\circ h_{14}^{2314,3214},$$
$$\theta^{2314}\circ g_{23}^{2314,2341}=\theta^{2341}\circ h_{23}^{2314,2341},\theta^{2143}\circ g_{23}^{2143,2413}=\theta^{2413}\circ h_{23}^{2143,2413},$$
$$\theta^{3142}\circ g_{23}^{3142,3412}=\theta^{3412}\circ h_{23}^{3142,3412},\theta^{1432}\circ g_{23}^{1432,4132}=\theta^{4132}\circ h_{23}^{1432,4132},$$
$$\theta^{3214}\circ g_{23}^{3214,3241}=\theta^{3241}\circ h_{23}^{3214,3241},\theta^{1423}\circ g_{23}^{1423,4123}=\theta^{4123}\circ h_{23}^{1423,4123},$$
$$\theta^{2413}\circ g_{24}^{2413,2431}=\theta^{2431}\circ h_{24}^{2413,2431},\theta^{2134}\circ g_{24}^{2134,2314}=\theta^{2314}\circ h_{24}^{2134,2314},$$
$$\theta^{4132}\circ g_{24}^{4132,4312}=\theta^{4312}\circ h_{24}^{4132,4312},\theta^{1342}\circ g_{24}^{1342,3142}=\theta^{3142}\circ h_{24}^{1342,3142},$$
$$\theta^{4213}\circ g_{24}^{4213,4231}=\theta^{4231}\circ h_{24}^{4213,4231},\theta^{1324}\circ g_{24}^{1324,3124}=\theta^{3124}\circ h_{24}^{1324,3124},$$
$$\theta^{3412}\circ g_{34}^{3412,3421}=\theta^{3421}\circ h_{34}^{3412,3421},\theta^{3124}\circ g_{34}^{3124,3214}=\theta^{3214}\circ h_{34}^{3124,3214},$$
$$\theta^{4123}\circ g_{34}^{4123,4213}=\theta^{4213}\circ h_{34}^{4123,4213},\theta^{1243}\circ g_{34}^{1243,2143}=\theta^{2143}\circ h_{34}^{1243,2143},$$
$$\theta^{4312}\circ g_{34}^{4312,4321}=\theta^{4321}\circ h_{34}^{4312,4321},\theta^{1234}\circ g_{34}^{1234,2134}=\theta^{2134}\circ h_{34}^{1234,2134}$$

are tantamount to the following conditions in terms of coefficients of the polynomials

$$a^{1342}=a^{1432},a_1^{1342}=a_1^{1432},a_2^{1342}=a_2^{1432},a_3^{1342}=a_3^{1432},a_4^{1342}=a_4^{1432},$$
$$a_{12}^{1342}=a_{12}^{1432},a_{13}^{1342}=a_{13}^{1432},a_{14}^{1342}=a_{14}^{1432},a_{23}^{1342}=a_{23}^{1432},a_{24}^{1342}=a_{24}^{1432},$$
$$a_{123}^{1342}=a_{123}^{1432},a_{124}^{1342}=a_{124}^{1432}$$

$$a^{2341}=a^{2431},a_1^{2341}=a_1^{2431},a_2^{2341}=a_2^{2431},a_3^{2341}=a_3^{2431},a_4^{2341}=a_4^{2431},$$
$$a_{12}^{2341}=a_{12}^{2431},a_{13}^{2341}=a_{13}^{2431},a_{14}^{2341}=a_{14}^{2431},a_{23}^{2341}=a_{23}^{2431},a_{24}^{2341}=a_{24}^{2431},$$
$$a_{123}^{2341}=a_{123}^{2431},a_{124}^{2341}=a_{124}^{2431}$$

$$a^{3421}=a^{4321},a_1^{3421}=a_1^{4321},a_2^{3421}=a_2^{4321},a_3^{3421}=a_3^{4321},a_4^{3421}=a_4^{4321},$$
$$a_{12}^{3421}=a_{12}^{4321},a_{13}^{3421}=a_{13}^{4321},a_{14}^{3421}=a_{14}^{4321},a_{23}^{3421}=a_{23}^{4321},a_{24}^{3421}=a_{24}^{4321},$$
$$a_{123}^{3421}=a_{123}^{4321},a_{124}^{3421}=a_{124}^{4321}$$

$a^{2341} = a^{2314}, a_1^{2341} = a_1^{2314}, a_2^{2341} = a_2^{2314}, a_3^{2341} = a_3^{2314}, a_4^{2341} = a_4^{2314},$
$a_{12}^{2341} = a_{12}^{2314}, a_{13}^{2341} = a_{13}^{2314}, a_{14}^{2341} = a_{14}^{2314}, a_{24}^{2341} = a_{24}^{2314}, a_{34}^{2341} = a_{34}^{2314},$
$a_{124}^{2341} = a_{124}^{2314}, a_{134}^{2341} = a_{134}^{2314}$

$a^{2413} = a^{2143}, a_1^{2413} = a_1^{2143}, a_2^{2413} = a_2^{2143}, a_3^{2413} = a_3^{2143}, a_4^{2413} = a_4^{2143},$
$a_{12}^{2413} = a_{12}^{2143}, a_{13}^{2413} = a_{13}^{2143}, a_{14}^{2413} = a_{14}^{2143}, a_{24}^{2413} = a_{24}^{2143}, a_{34}^{2413} = a_{34}^{2143},$
$a_{124}^{2413} = a_{124}^{2143}, a_{134}^{2413} = a_{134}^{2143}$

$a^{3412} = a^{3142}, a_1^{3412} = a_1^{3142}, a_2^{3412} = a_2^{3142}, a_3^{3412} = a_3^{3142}, a_4^{3412} = a_4^{3142},$
$a_{12}^{3412} = a_{12}^{3142}, a_{13}^{3412} = a_{13}^{3142}, a_{14}^{3412} = a_{14}^{3142}, a_{24}^{3412} = a_{24}^{3142}, a_{34}^{3412} = a_{34}^{3142},$
$a_{124}^{3412} = a_{124}^{3142}, a_{134}^{3412} = a_{134}^{3142}$

$a^{4132} = a^{1432}, a_1^{4132} = a_1^{1432}, a_2^{4132} = a_2^{1432}, a_3^{4132} = a_3^{1432}, a_4^{4132} = a_4^{1432},$
$a_{12}^{4132} = a_{12}^{1432}, a_{13}^{4132} = a_{13}^{1432}, a_{14}^{4132} = a_{14}^{1432}, a_{24}^{4132} = a_{24}^{1432}, a_{34}^{4132} = a_{34}^{1432},$
$a_{124}^{4132} = a_{124}^{1432}, a_{134}^{4132} = a_{134}^{1432}$

$a^{3412} = a^{3421}, a_1^{3412} = a_1^{3421}, a_2^{3412} = a_2^{3421}, a_3^{3412} = a_3^{3421}, a_4^{3412} = a_4^{3421},$
$a_{12}^{3412} = a_{12}^{3421}, a_{13}^{3412} = a_{13}^{3421}, a_{14}^{3412} = a_{14}^{3421}, a_{23}^{3412} = a_{23}^{3421}, a_{24}^{3412} = a_{24}^{3421},$
$a_{123}^{3412} = a_{123}^{3421}, a_{124}^{3412} = a_{124}^{3421}$

$a^{3124} = a^{3214}, a_1^{3124} = a_1^{3214}, a_2^{3124} = a_2^{3214}, a_3^{3124} = a_3^{3214}, a_4^{3124} = a_4^{3214},$
$a_{12}^{3124} = a_{12}^{3214}, a_{13}^{3124} = a_{13}^{3214}, a_{14}^{3124} = a_{14}^{3214}, a_{23}^{3124} = a_{23}^{3214}, a_{24}^{3124} = a_{24}^{3214},$
$a_{123}^{3124} = a_{123}^{3214}, a_{124}^{3124} = a_{124}^{3214}$

$a^{4123} = a^{4213}, a_1^{4123} = a_1^{4213}, a_2^{4123} = a_2^{4213}, a_3^{4123} = a_3^{4213}, a_4^{4123} = a_4^{4213},$
$a_{12}^{4123} = a_{12}^{4213}, a_{13}^{4123} = a_{13}^{4213}, a_{14}^{4123} = a_{14}^{4213}, a_{23}^{4123} = a_{23}^{4213}, a_{24}^{4123} = a_{24}^{4213},$
$a_{123}^{4123} = a_{123}^{4213}, a_{124}^{4123} = a_{124}^{4213}$

$a^{1243} = a^{2143}, a_1^{1243} = a_1^{2143}, a_2^{1243} = a_2^{2143}, a_3^{1243} = a_3^{2143}, a_4^{1243} = a_4^{2143},$
$a_{12}^{1243} = a_{12}^{2143}, a_{13}^{1243} = a_{13}^{2143}, a_{14}^{1243} = a_{14}^{2143}, a_{23}^{1243} = a_{23}^{2143}, a_{24}^{1243} = a_{24}^{2143},$
$a_{123}^{1243} = a_{123}^{2143}, a_{124}^{1243} = a_{124}^{2143}$

$a^{4123} = a^{4132}, a_1^{4123} = a_1^{4132}, a_2^{4123} = a_2^{4132}, a_3^{4123} = a_3^{4132}, a_4^{4123} = a_4^{4132},$
$a_{12}^{4123} = a_{12}^{4132}, a_{13}^{4123} = a_{13}^{4132}, a_{23}^{4123} = a_{23}^{4132}, a_{24}^{4123} = a_{24}^{4132}, a_{34}^{4123} = a_{34}^{4132},$
$a_{123}^{4123} = a_{123}^{4132}, a_{234}^{4123} = a_{234}^{4132}$

$$a^{4231} = a^{4321}, a_1^{4231} = a_1^{4321}, a_2^{4231} = a_2^{4321}, a_3^{4231} = a_3^{4321}, a_4^{4231} = a_4^{4321},$$
$$a_{12}^{4231} = a_{12}^{4321}, a_{13}^{4231} = a_{13}^{4321}, a_{23}^{4231} = a_{23}^{4321}, a_{24}^{4231} = a_{24}^{4321}, a_{34}^{4231} = a_{34}^{4321},$$
$$a_{123}^{4231} = a_{123}^{4321}, a_{234}^{4231} = a_{234}^{4321}$$

$$a^{1234} = a^{1324}, a_1^{1234} = a_1^{1324}, a_2^{1234} = a_2^{1324}, a_3^{1234} = a_3^{1324}, a_4^{1234} = a_4^{1324},$$
$$a_{12}^{1234} = a_{12}^{1324}, a_{13}^{1234} = a_{13}^{1324}, a_{23}^{1234} = a_{23}^{1324}, a_{24}^{1234} = a_{24}^{1324}, a_{34}^{1234} = a_{34}^{1324},$$
$$a_{123}^{1234} = a_{123}^{1324}, a_{234}^{1234} = a_{234}^{1324}$$

$$a^{2314} = a^{3214}, a_1^{2314} = a_1^{3214}, a_2^{2314} = a_2^{3214}, a_3^{2314} = a_3^{3214}, a_4^{2314} = a_4^{3214},$$
$$a_{12}^{2314} = a_{12}^{3214}, a_{13}^{2314} = a_{13}^{3214}, a_{23}^{2314} = a_{23}^{3214}, a_{24}^{2314} = a_{24}^{3214}, a_{34}^{2314} = a_{34}^{3214},$$
$$a_{123}^{2314} = a_{123}^{3214}, a_{234}^{2314} = a_{234}^{3214}$$

which can succintly be summarized as

$$a^{1234} = a^{1243} = a^{1324} = a^{1342} = a^{1423} = a^{1432} =$$
$$a^{2134} = a^{2143} = a^{2314} = a^{2341} = a^{2413} = a^{2431} =$$
$$a^{3124} = a^{3142} = a^{3214} = a^{3241} = a^{3412} = a^{3421} =$$
$$a^{4123} = a^{4132} = a^{4213} = a^{4231} = a^{4312} = a^{4321}$$

$$a_1^{1234} = a_1^{1243} = a_1^{1324} = a_1^{1342} = a_1^{1423} = a_1^{1432} =$$
$$a_1^{2134} = a_1^{2143} = a_1^{2314} = a_1^{2341} = a_1^{2413} = a_1^{2431} =$$
$$a_1^{3124} = a_1^{3142} = a_1^{3214} = a_1^{3241} = a_1^{3412} = a_1^{3421} =$$
$$a_1^{4123} = a_1^{4132} = a_1^{4213} = a_1^{4231} = a_1^{4312} = a_1^{4321}$$

$$a_2^{1234} = a_2^{1243} = a_2^{1324} = a_2^{1342} = a_2^{1423} = a_2^{1432} =$$
$$a_2^{2134} = a_2^{2143} = a_2^{2314} = a_2^{2341} = a_2^{2413} = a_2^{2431} =$$
$$a_2^{3124} = a_2^{3142} = a_2^{3214} = a_2^{3241} = a_2^{3412} = a_2^{3421} =$$
$$a_2^{4123} = a_2^{4132} = a_2^{4213} = a_2^{4231} = a_2^{4312} = a_2^{4321}$$

$$a_3^{1234} = a_3^{1243} = a_3^{1324} = a_3^{1342} = a_3^{1423} = a_3^{1432} =$$
$$a_3^{2134} = a_3^{2143} = a_3^{2314} = a_3^{2341} = a_3^{2413} = a_3^{2431} =$$
$$a_3^{3124} = a_3^{3142} = a_3^{3214} = a_3^{3241} = a_3^{3412} = a_3^{3421} =$$
$$a_3^{4123} = a_3^{4132} = a_3^{4213} = a_3^{4231} = a_3^{4312} = a_3^{4321}$$

$$a_4^{1234} = a_4^{1243} = a_4^{1324} = a_4^{1342} = a_4^{1423} = a_4^{1432} =$$
$$a_4^{2134} = a_4^{2143} = a_4^{2314} = a_4^{2341} = a_4^{2413} = a_4^{2431} =$$
$$a_4^{3124} = a_4^{3142} = a_4^{3214} = a_4^{3241} = a_4^{3412} = a_4^{3421} =$$
$$a_4^{4123} = a_4^{4132} = a_4^{4213} = a_4^{4231} = a_4^{4312} = a_4^{4321}$$

$$a_{12}^{1234} = a_{12}^{1243} = a_{12}^{1324} = a_{12}^{1342} = a_{12}^{1423} = a_{12}^{1432} = a_{12}^{3124} = a_{12}^{3142} = a_{12}^{3412} = a_{12}^{4123} = a_{12}^{4132} = a_{12}^{4312}$$

$$a_{12}^{2134} = a_{12}^{2143} = a_{12}^{2314} = a_{12}^{2341} = a_{12}^{2413} = a_{12}^{2431} = a_{12}^{3214} = a_{12}^{3241} = a_{12}^{3421} = a_{12}^{4213} = a_{12}^{4231} = a_{12}^{4321}$$

$$a_{13}^{1342} = a_{13}^{1324} = a_{13}^{1432} = a_{13}^{1423} = a_{13}^{1234} = a_{13}^{1243} = a_{13}^{4132} = a_{13}^{4123} = a_{13}^{4213} = a_{13}^{2134} = a_{13}^{2143} = a_{13}^{2413}$$

$$a_{13}^{3142} = a_{13}^{3124} = a_{13}^{3412} = a_{13}^{3421} = a_{13}^{3214} = a_{13}^{3241} = a_{13}^{4312} = a_{13}^{4321} = a_{13}^{4231} = a_{13}^{2314} = a_{13}^{2341} = a_{13}^{2431}$$

$$a_{14}^{1423} = a_{14}^{1432} = a_{14}^{1243} = a_{14}^{1234} = a_{14}^{1342} = a_{14}^{1324} = a_{14}^{2143} = a_{14}^{2134} = a_{14}^{2314} = a_{14}^{3142} = a_{14}^{3124} = a_{14}^{3214}$$

$$a_{14}^{4123} = a_{14}^{4132} = a_{14}^{4213} = a_{14}^{4231} = a_{14}^{4312} = a_{14}^{4321} = a_{14}^{2413} = a_{14}^{2431} = a_{14}^{2341} = a_{14}^{3412} = a_{14}^{3421} = a_{14}^{3241}$$

$$a_{23}^{2314} = a_{23}^{2341} = a_{23}^{2134} = a_{23}^{2143} = a_{23}^{2431} = a_{23}^{2413} = a_{23}^{1234} = a_{23}^{1243} = a_{23}^{1423} = a_{23}^{4231} = a_{23}^{4213} = a_{23}^{4123}$$

$$a_{23}^{3214} = a_{23}^{3241} = a_{23}^{3124} = a_{23}^{3142} = a_{23}^{3421} = a_{23}^{3412} = a_{23}^{1324} = a_{23}^{1342} = a_{23}^{1432} = a_{23}^{4321} = a_{23}^{4312} = a_{23}^{4132}$$

$$a_{24}^{2413} = a_{24}^{2431} = a_{24}^{2143} = a_{24}^{2134} = a_{24}^{2341} = a_{24}^{2314} = a_{24}^{1243} = a_{24}^{1234} = a_{24}^{1324} = a_{24}^{3241} = a_{24}^{3214} = a_{24}^{3124}$$

$$a_{24}^{4213} = a_{24}^{4231} = a_{24}^{4123} = a_{24}^{4132} = a_{24}^{4321} = a_{24}^{4312} = a_{24}^{1423} = a_{24}^{1432} = a_{24}^{1342} = a_{24}^{3421} = a_{24}^{3412} = a_{24}^{3142}$$

$$a_{34}^{3412} = a_{34}^{3421} = a_{34}^{3142} = a_{34}^{3124} = a_{34}^{3241} = a_{34}^{3214} = a_{34}^{1342} = a_{34}^{1324} = a_{34}^{1234} = a_{34}^{2341} = a_{34}^{2314} = a_{34}^{2134}$$

$$a_{34}^{4312} = a_{34}^{4321} = a_{34}^{4132} = a_{34}^{4123} = a_{34}^{4231} = a_{34}^{4213} = a_{34}^{1432} = a_{34}^{1423} = a_{34}^{1243} = a_{34}^{2431} = a_{34}^{2413} = a_{34}^{2143}$$

$$a_{123}^{1234} = a_{123}^{1243} = a_{123}^{1423} = a_{123}^{4123}$$
$$a_{123}^{1324} = a_{123}^{1342} = a_{123}^{1432} = a_{123}^{4132}$$
$$a_{123}^{2134} = a_{123}^{2143} = a_{123}^{2413} = a_{123}^{4213}$$
$$a_{123}^{2314} = a_{123}^{2341} = a_{123}^{2431} = a_{123}^{4231}$$
$$a_{123}^{3124} = a_{123}^{3142} = a_{123}^{3412} = a_{123}^{4312}$$
$$a_{123}^{3214} = a_{123}^{3241} = a_{123}^{3421} = a_{123}^{4321}$$
$$a_{124}^{1243} = a_{124}^{1234} = a_{124}^{1324} = a_{124}^{3124}$$
$$a_{124}^{1423} = a_{124}^{1432} = a_{124}^{1342} = a_{124}^{3142}$$
$$a_{124}^{2143} = a_{124}^{2134} = a_{124}^{2314} = a_{124}^{3214}$$
$$a_{124}^{2413} = a_{124}^{2431} = a_{124}^{2341} = a_{124}^{3241}$$
$$a_{124}^{4123} = a_{124}^{4132} = a_{124}^{4312} = a_{124}^{3412}$$
$$a_{124}^{4213} = a_{124}^{4231} = a_{124}^{4321} = a_{124}^{3421}$$
$$a_{134}^{1342} = a_{134}^{1324} = a_{134}^{1234} = a_{134}^{2134}$$
$$a_{134}^{1432} = a_{134}^{1423} = a_{134}^{1243} = a_{134}^{2143}$$
$$a_{134}^{3142} = a_{134}^{3124} = a_{134}^{3214} = a_{134}^{2314}$$
$$a_{134}^{3412} = a_{134}^{3421} = a_{134}^{3241} = a_{134}^{2341}$$
$$a_{134}^{4132} = a_{134}^{4123} = a_{134}^{4213} = a_{134}^{2413}$$
$$a_{134}^{4312} = a_{134}^{4321} = a_{134}^{4231} = a_{134}^{2431}$$
$$a_{234}^{2341} = a_{234}^{2314} = a_{234}^{2134} = a_{234}^{1234}$$
$$a_{234}^{2431} = a_{234}^{2413} = a_{234}^{2143} = a_{234}^{1243}$$
$$a_{234}^{3241} = a_{234}^{3214} = a_{234}^{3124} = a_{234}^{1324}$$
$$a_{234}^{3421} = a_{234}^{3412} = a_{234}^{3142} = a_{234}^{1342}$$
$$a_{234}^{4231} = a_{234}^{4213} = a_{234}^{4123} = a_{234}^{1423}$$
$$a_{234}^{4321} = a_{234}^{4312} = a_{234}^{4132} = a_{234}^{1432}$$

Therefore there exists a unique mapping $\theta : P \to \mathbb{R}$ such that

$$\theta^{1234} = \theta \circ f_{1234}, \theta^{1243} = \theta \circ f_{1243}, \theta^{1324} = \theta \circ f_{1324}, \theta^{1342} = \theta \circ f_{1342}, \theta^{1423} = \theta \circ f_{1423}, \theta^{1432} = \theta \circ f_{1432},$$
$$\theta^{2134} = \theta \circ f_{2134}, \theta^{2143} = \theta \circ f_{2143}, \theta^{2314} = \theta \circ f_{2314}, \theta^{2341} = \theta \circ f_{2341}, \theta^{2413} = \theta \circ f_{2413}, \theta^{2431} = \theta \circ f_{2431},$$
$$\theta^{3124} = \theta \circ f_{3124}, \theta^{3142} = \theta \circ f_{3142}, \theta^{3214} = \theta \circ f_{3214}, \theta^{3241} = \theta \circ f_{3241}, \theta^{3412} = \theta \circ f_{3412}, \theta^{3421} = \theta \circ f_{3421},$$
$$\theta^{4123} = \theta \circ f_{4123}, \theta^{4132} = \theta \circ f_{4132}, \theta^{4213} = \theta \circ f_{4213}, \theta^{4231} = \theta \circ f_{4231}, \theta^{4312} = \theta \circ f_{4312}, \theta^{4321} = \theta \circ f_{4321}$$

The proof is now complete. ■

Remark 18 *For our convenience we display the positions* $5-30$ *of* $f_{ijkl}(d_1, d_2, d_3, d_4)$ *as tables as follows:*

	$5/d_1d_2$	$6/d_1d_3$	$7/d_1d_4$	$8/d_2d_3$	$9/d_2d_4$	$10/d_3d_4$
1234	0	0	0	0	0	0
1243	0	0	0	0	0	d_3d_4
1324	0	0	0	d_2d_3	0	0
1342	0	0	0	d_2d_3	d_2d_4	0
1423	0	0	0	0	d_2d_4	d_3d_4
1432	0	0	0	d_2d_3	d_2d_4	d_3d_4
2134	d_1d_2	0	0	0	0	0
2143	d_1d_2	0	0	0	0	d_3d_4
2314	d_1d_2	d_1d_3	0	0	0	0
2341	d_1d_2	d_1d_3	d_1d_4	0	0	0
2413	d_1d_2	0	d_1d_4	0	0	d_3d_4
2431	d_1d_2	d_1d_3	d_1d_4	0	0	d_3d_4
3124	0	d_1d_3	0	d_2d_3	0	0
3142	0	d_1d_3	0	d_2d_3	d_2d_4	0
3214	d_1d_2	d_1d_3	0	d_2d_3	0	0
3241	d_1d_2	d_1d_3	d_1d_4	d_2d_3	0	0
3412	0	d_1d_3	d_1d_4	d_2d_3	d_2d_4	0
3421	d_1d_2	d_1d_3	d_1d_4	d_2d_3	d_2d_4	0
4123	0	0	d_1d_4	0	d_2d_4	d_3d_4
4132	0	0	d_1d_4	d_2d_3	d_2d_4	d_3d_4
4213	d_1d_2	0	d_1d_4	0	d_2d_4	d_3d_4
4231	d_1d_2	d_1d_3	d_1d_4	0	d_2d_4	d_3d_4
4312	0	d_1d_3	d_1d_4	d_2d_3	d_2d_4	d_3d_4
4321	d_1d_2	d_1d_3	d_1d_4	d_2d_3	d_2d_4	d_3d_4

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	$11-15/d_1d_2d_3$	$16-20/d_1d_2d_4$	$21-25/d_1d_3d_4$	$26-30/d_2d_3d_4$
1234				
1243			21	26
1324	11			27
1342	11	16		28
1423		16	21	29
1432	11	16	21	30
2134	12	17		
2143	12	17	21	26
2314	13	17	22	
2341	13	18	23	
2413	12	18	24	26
2431	13	18	25	26
3124	14		22	27
3142	14	16	22	28
3214	15	17	22	27
3241	15	18	23	27
3412	14	19	23	28
3421	15	20	23	28
4123		19	24	29
4132	11	19	24	30
4213	12	20	24	29
4231	13	20	25	29
4312	14	19	25	30
4321	15	20	25	30

(35)

Corollary 19 *Let M be a microlinear space with mappings*

$$\begin{aligned}
&\gamma_{1234}:Q^{1234}\rightarrow M,\gamma_{1243}:Q^{1243}\rightarrow M,\gamma_{1324}:Q^{1324}\rightarrow M,\\
&\gamma_{1342}:Q^{1342}\rightarrow M,\gamma_{1423}:Q^{1423}\rightarrow M,\gamma_{1432}:Q^{1432}\rightarrow M,\\
&\gamma_{2134}:Q^{2134}\rightarrow M,\gamma_{2143}:Q^{2143}\rightarrow M,\gamma_{2314}:Q^{2314}\rightarrow M,\\
&\gamma_{2341}:Q^{2341}\rightarrow M,\gamma_{2413}:Q^{2413}\rightarrow M,\gamma_{2431}:Q^{2431}\rightarrow M,\\
&\gamma_{3124}:Q^{3124}\rightarrow M,\gamma_{3142}:Q^{3142}\rightarrow M,\gamma_{3214}:Q^{3214}\rightarrow M,\\
&\gamma_{3241}:Q^{3241}\rightarrow M,\gamma_{3412}:Q^{3412}\rightarrow M,\gamma_{3421}:Q^{3421}\rightarrow M,\\
&\gamma_{4123}:Q^{4123}\rightarrow M,\gamma_{4132}:Q^{4132}\rightarrow M,\gamma_{4213}:Q^{4213}\rightarrow M,\\
&\gamma_{4231}:Q^{4231}\rightarrow M,\gamma_{4312}:Q^{4312}\rightarrow M,\gamma_{4321}:Q^{4321}\rightarrow M
\end{aligned}$$

abiding by

$$\begin{aligned}
&\gamma_{1234} \circ g_{12}^{1234,1243} = \gamma_{1243} \circ h_{12}^{1234,1243}, \gamma_{1342} \circ g_{12}^{1342,1432} = \gamma_{1432} \circ h_{12}^{1342,1432},\\
&\gamma_{2341} \circ g_{12}^{2341,2431} = \gamma_{2431} \circ h_{12}^{2341,2431}, \gamma_{3421} \circ g_{12}^{3421,4321} = \gamma_{4321} \circ h_{12}^{3421,4321},\\
&\gamma_{2134} \circ g_{12}^{2134,2143} = \gamma_{2143} \circ h_{12}^{2134,2143}, \gamma_{3412} \circ g_{12}^{3412,4312} = \gamma_{4312} \circ h_{12}^{3412,4312},\\
&\gamma_{1324} \circ g_{13}^{1324,1342} = \gamma_{1342} \circ h_{13}^{1324,1342}, \gamma_{1243} \circ g_{13}^{1243,1423} = \gamma_{1423} \circ h_{13}^{1243,1423},\\
&\gamma_{3241} \circ g_{13}^{3241,3421} = \gamma_{3421} \circ h_{13}^{3241,3421}, \gamma_{2431} \circ g_{13}^{2431,4231} = \gamma_{4231} \circ h_{13}^{2431,4231},\\
&\gamma_{3124} \circ g_{13}^{3124,3142} = \gamma_{3142} \circ h_{13}^{3124,3142}, \gamma_{2413} \circ g_{13}^{2413,4213} = \gamma_{4213} \circ h_{13}^{2413,4213},\\
&\gamma_{1423} \circ g_{14}^{1423,1432} = \gamma_{1432} \circ h_{14}^{1423,1432}, \gamma_{1234} \circ g_{14}^{1234,1324} = \gamma_{1324} \circ h_{14}^{1234,1324},\\
&\gamma_{4231} \circ g_{14}^{4231,4321} = \gamma_{4321} \circ h_{14}^{4231,4321}, \gamma_{2341} \circ g_{14}^{2341,3241} = \gamma_{3241} \circ h_{14}^{2341,3241},\\
&\gamma_{4123} \circ g_{14}^{4123,4132} = \gamma_{4132} \circ h_{14}^{4123,4132}, \gamma_{2314} \circ g_{14}^{2314,3214} = \gamma_{3214} \circ h_{14}^{2314,3214},\\
&\gamma_{2314} \circ g_{23}^{2314,2341} = \gamma_{2341} \circ h_{23}^{2314,2341}, \gamma_{2143} \circ g_{23}^{2143,2413} = \gamma_{2413} \circ h_{23}^{2143,2413},\\
&\gamma_{3142} \circ g_{23}^{3142,3412} = \gamma_{3412} \circ h_{23}^{3142,3412}, \gamma_{1432} \circ g_{23}^{1432,4132} = \gamma_{4132} \circ h_{23}^{1432,4132},\\
&\gamma_{3214} \circ g_{23}^{3214,3241} = \gamma_{3241} \circ h_{23}^{3214,3241}, \gamma_{1423} \circ g_{23}^{1423,4123} = \gamma_{4123} \circ h_{23}^{1423,4123},\\
&\gamma_{2413} \circ g_{24}^{2413,2431} = \gamma_{2431} \circ h_{24}^{2413,2431}, \gamma_{2134} \circ g_{24}^{2134,2314} = \gamma_{2314} \circ h_{24}^{2134,2314},\\
&\gamma_{4132} \circ g_{24}^{4132,4312} = \gamma_{4312} \circ h_{24}^{4132,4312}, \gamma_{1342} \circ g_{24}^{1342,3142} = \gamma_{3142} \circ h_{24}^{1342,3142},\\
&\gamma_{4213} \circ g_{24}^{4213,4231} = \gamma_{4231} \circ h_{24}^{4213,4231}, \gamma_{1324} \circ g_{24}^{1324,3124} = \gamma_{3124} \circ h_{24}^{1324,3124},\\
&\gamma_{3412} \circ g_{34}^{3412,3421} = \gamma_{3421} \circ h_{34}^{3412,3421}, \gamma_{3124} \circ g_{34}^{3124,3214} = \gamma_{3214} \circ h_{34}^{3124,3214},\\
&\gamma_{4123} \circ g_{34}^{4123,4213} = \gamma_{4213} \circ h_{34}^{4123,4213}, \gamma_{1243} \circ g_{34}^{1243,2143} = \gamma_{2143} \circ h_{34}^{1243,2143},\\
&\gamma_{4312} \circ g_{34}^{4312,4321} = \gamma_{4321} \circ h_{34}^{4312,4321}, \gamma_{1234} \circ g_{34}^{1234,2134} = \gamma_{2134} \circ h_{34}^{1234,2134}
\end{aligned}$$

Then there exists a unique mapping

$$\mathfrak{m} : P \to M$$

such that

$$\begin{aligned}
&\mathfrak{m} \circ f_{1234} = \gamma_{1234}, \mathfrak{m} \circ f_{1243} = \gamma_{1243}, \mathfrak{m} \circ f_{1324} = \gamma_{1324}, \mathfrak{m} \circ f_{1342} = \gamma_{1342}, \mathfrak{m} \circ f_{1423} = \gamma_{1423}, \mathfrak{m} \circ f_{1432} = \gamma_{1432},\\
&\mathfrak{m} \circ f_{2134} = \gamma_{2134}, \mathfrak{m} \circ f_{2143} = \gamma_{2143}, \mathfrak{m} \circ f_{2314} = \gamma_{2314}, \mathfrak{m} \circ f_{2341} = \gamma_{2341}, \mathfrak{m} \circ f_{2413} = \gamma_{2413}, \mathfrak{m} \circ f_{2431} = \gamma_{2431},\\
&\mathfrak{m} \circ f_{3124} = \gamma_{3124}, \mathfrak{m} \circ f_{3142} = \gamma_{3142}, \mathfrak{m} \circ f_{3214} = \gamma_{3214}, \mathfrak{m} \circ f_{3241} = \gamma_{3241}, \mathfrak{m} \circ f_{3412} = \gamma_{3412}, \mathfrak{m} \circ f_{3421} = \gamma_{3421},\\
&\mathfrak{m} \circ f_{4123} = \gamma_{4123}, \mathfrak{m} \circ f_{4132} = \gamma_{4132}, \mathfrak{m} \circ f_{4213} = \gamma_{4213}, \mathfrak{m} \circ f_{4231} = \gamma_{4231}, \mathfrak{m} \circ f_{4312} = \gamma_{4312}, \mathfrak{m} \circ f_{4321} = \gamma_{4321}
\end{aligned}$$

Theorem 20 *Let M be a microlinear space. Let*

$$\begin{array}{c}
\gamma_{1234}, \gamma_{1243}, \gamma_{1324}, \gamma_{1342}, \gamma_{1423}, \gamma_{1432}, \gamma_{2134}, \gamma_{2143}, \gamma_{2314}, \gamma_{2341}, \gamma_{2413}, \gamma_{2431},\\
\gamma_{3124}, \gamma_{3142}, \gamma_{3214}, \gamma_{3241}, \gamma_{3124}, \gamma_{3142}, \gamma_{4123}, \gamma_{4132}, \gamma_{4213}, \gamma_{4231}, \gamma_{4312}, \gamma_{4321}
\end{array} : D^4 \to M$$

with

$$\gamma_{1234} \mid D^4\{(3,4)\} = \gamma_{1243} \mid D^4\{(3,4)\}, \gamma_{1342} \mid D^4\{(3,4)\} = \gamma_{1432} \mid D^4\{(3,4)\},$$
$$\gamma_{2341} \mid D^4\{(3,4)\} = \gamma_{2431} \mid D^4\{(3,4)\}, \gamma_{3421} \mid D^4\{(3,4)\} = \gamma_{4321} \mid D^4\{(3,4)\},$$
$$\gamma_{2143} \mid D^4\{(3,4)\} = \gamma_{2134} \mid D^4\{(3,4)\}, \gamma_{4312} \mid D^4\{(3,4)\} = \gamma_{3412} \mid D^4\{(3,4)\},$$
$$\gamma_{1342} \mid D^4\{(2,4)\} = \gamma_{1324} \mid D^4\{(2,4)\}, \gamma_{1423} \mid D^4\{(2,4)\} = \gamma_{1243} \mid D^4\{(2,4)\},$$
$$\gamma_{3421} \mid D^4\{(2,4)\} = \gamma_{3241} \mid D^4\{(2,4)\}, \gamma_{4231} \mid D^4\{(2,4)\} = \gamma_{2431} \mid D^4\{(2,4)\},$$
$$\gamma_{3124} \mid D^4\{(2,4)\} = \gamma_{3142} \mid D^4\{(2,4)\}, \gamma_{2413} \mid D^4\{(2,4)\} = \gamma_{4213} \mid D^4\{(2,4)\},$$
$$\gamma_{1423} \mid D^4\{(2,3)\} = \gamma_{1432} \mid D^4\{(2,3)\}, \gamma_{1234} \mid D^4\{(2,3)\} = \gamma_{1324} \mid D^4\{(2,3)\},$$
$$\gamma_{4231} \mid D^4\{(2,3)\} = \gamma_{4321} \mid D^4\{(2,3)\}, \gamma_{2341} \mid D^4\{(2,3)\} = \gamma_{3241} \mid D^4\{(2,3)\},$$
$$\gamma_{4132} \mid D^4\{(2,3)\} = \gamma_{4123} \mid D^4\{(2,3)\}, \gamma_{3214} \mid D^4\{(2,3)\} = \gamma_{2314} \mid D^4\{(2,3)\},$$
$$\gamma_{2314} \mid D^4\{(1,4)\} = \gamma_{2341} \mid D^4\{(1,4)\}, \gamma_{2143} \mid D^4\{(1,4)\} = \gamma_{2413} \mid D^4\{(1,4)\},$$
$$\gamma_{3142} \mid D^4\{(1,4)\} = \gamma_{3412} \mid D^4\{(1,4)\}, \gamma_{1432} \mid D^4\{(1,4)\} = \gamma_{4132} \mid D^4\{(1,4)\},$$
$$\gamma_{3241} \mid D^4\{(1,4)\} = \gamma_{3214} \mid D^4\{(1,4)\}, \gamma_{4123} \mid D^4\{(1,4)\} = \gamma_{1423} \mid D^4\{(1,4)\},$$
$$\gamma_{2431} \mid D^4\{(1,3)\} = \gamma_{2413} \mid D^4\{(1,3)\}, \gamma_{2314} \mid D^4\{(1,3)\} = \gamma_{2134} \mid D^4\{(1,3)\},$$
$$\gamma_{4312} \mid D^4\{(1,3)\} = \gamma_{4132} \mid D^4\{(1,3)\}, \gamma_{3142} \mid D^4\{(1,3)\} = \gamma_{1342} \mid D^4\{(1,3)\},$$
$$\gamma_{4213} \mid D^4\{(1,3)\} = \gamma_{4231} \mid D^4\{(1,3)\}, \gamma_{1324} \mid D^4\{(1,3)\} = \gamma_{3124} \mid D^4\{(1,3)\},$$
$$\gamma_{3412} \mid D^4\{(1,2)\} = \gamma_{3421} \mid D^4\{(1,2)\}, \gamma_{3124} \mid D^4\{(1,2)\} = \gamma_{3214} \mid D^4\{(1,2)\},$$
$$\gamma_{4123} \mid D^4\{(1,2)\} = \gamma_{4213} \mid D^4\{(1,2)\}, \gamma_{1243} \mid D^4\{(1,2)\} = \gamma_{2143} \mid D^4\{(1,2)\},$$
$$\gamma_{4321} \mid D^4\{(1,2)\} = \gamma_{4312} \mid D^4\{(1,2)\}, \gamma_{2134} \mid D^4\{(1,2)\} = \gamma_{1234} \mid D^4\{(1,2)\}$$

Then we have

$$\begin{aligned}
&\left(\left(\left(\gamma_{1234} \overset{\cdot}{\underset{12}{-}} \gamma_{1243}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1342} \overset{\cdot}{\underset{12}{-}} \gamma_{1432}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{2341} \overset{\cdot}{\underset{12}{-}} \gamma_{2431}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3421} \overset{\cdot}{\underset{12}{-}} \gamma_{4321}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{1342} \overset{\cdot}{\underset{13}{-}} \gamma_{1324}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1423} \overset{\cdot}{\underset{13}{-}} \gamma_{1243}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{3421} \overset{\cdot}{\underset{13}{-}} \gamma_{3241}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4231} \overset{\cdot}{\underset{13}{-}} \gamma_{2431}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{1423} \overset{\cdot}{\underset{14}{-}} \gamma_{1432}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1234} \overset{\cdot}{\underset{14}{-}} \gamma_{1324}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{4231} \overset{\cdot}{\underset{14}{-}} \gamma_{4321}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2341} \overset{\cdot}{\underset{14}{-}} \gamma_{3241}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{2143} \overset{\cdot}{\underset{21}{-}} \gamma_{2134}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2431} \overset{\cdot}{\underset{21}{-}} \gamma_{2341}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{1432} \overset{\cdot}{\underset{21}{-}} \gamma_{1342}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4312} \overset{\cdot}{\underset{21}{-}} \gamma_{3412}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{2314} \overset{\cdot}{\underset{23}{-}} \gamma_{2341}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2143} \overset{\cdot}{\underset{23}{-}} \gamma_{2413}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{3142} \overset{\cdot}{\underset{23}{-}} \gamma_{3412}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1432} \overset{\cdot}{\underset{23}{-}} \gamma_{4132}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{2431} \overset{\cdot}{\underset{24}{-}} \gamma_{2413}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2314} \overset{\cdot}{\underset{24}{-}} \gamma_{2134}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{4312} \overset{\cdot}{\underset{24}{-}} \gamma_{4132}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3142} \overset{\cdot}{\underset{24}{-}} \gamma_{1342}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{3124} \overset{\cdot}{\underset{31}{-}} \gamma_{3142}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3241} \overset{\cdot}{\underset{31}{-}} \gamma_{3421}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{1243} \overset{\cdot}{\underset{31}{-}} \gamma_{1423}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2413} \overset{\cdot}{\underset{31}{-}} \gamma_{4213}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{3241} \overset{\cdot}{\underset{32}{-}} \gamma_{3214}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3412} \overset{\cdot}{\underset{32}{-}} \gamma_{3142}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{2413} \overset{\cdot}{\underset{32}{-}} \gamma_{2143}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4123} \overset{\cdot}{\underset{32}{-}} \gamma_{1423}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{3412} \overset{\cdot}{\underset{34}{-}} \gamma_{3421}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3124} \overset{\cdot}{\underset{34}{-}} \gamma_{3214}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{4123} \overset{\cdot}{\underset{34}{-}} \gamma_{4213}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1243} \overset{\cdot}{\underset{34}{-}} \gamma_{2143}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{4132} \overset{\cdot}{\underset{41}{-}} \gamma_{4123}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4321} \overset{\cdot}{\underset{41}{-}} \gamma_{4231}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{1324} \overset{\cdot}{\underset{41}{-}} \gamma_{1234}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3214} \overset{\cdot}{\underset{41}{-}} \gamma_{2314}\right)\right)\right) + \\
&\left(\left(\left(\gamma_{4213} \overset{\cdot}{\underset{42}{-}} \gamma_{4231}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4132} \overset{\cdot}{\underset{42}{-}} \gamma_{4312}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{2134} \overset{\cdot}{\underset{42}{-}} \gamma_{2314}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1324} \overset{\cdot}{\underset{42}{-}} \gamma_{3124}\right)\right)\right) + \\
&\left(\left(\gamma_{4321} \overset{\cdot}{\underset{43}{-}} \gamma_{4312}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4213} \overset{\cdot}{\underset{43}{-}} \gamma_{4123}\right)\right) \overset{\cdot}{-} \left(\left(\gamma_{3214} \overset{\cdot}{\underset{43}{-}} \gamma_{3124}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2134} \overset{\cdot}{\underset{43}{-}} \gamma_{1234}\right)\right) \\
&= 0 \qquad (36)
\end{aligned}$$

Proof. The proof is divided into thirteen steps.

1. Since

$$\begin{aligned}
&\gamma_{1234}\left(d_1, d_2, d_3, d_4\right) \\
&= \mathfrak{m}\left(d_1, d_2, d_3, d_4, \underset{5}{0}, \ldots, \underset{53}{0}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{1243}\left(d_1, d_2, d_3, d_4\right) \\
&= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \ldots, \underset{9}{0}, \underset{10}{d_3 d_4}, \underset{11}{0}, \ldots, \underset{20}{0}, d_1 d_3 d_4, \underset{22}{0}, \ldots, \underset{25}{0}, d_2 d_3 d_4, \underset{27}{0}, \ldots, \underset{30}{0}, \\ d_1 d_2 d_3 d_4, \underset{32}{0}, \ldots, \underset{53}{0} \end{array}\right)
\end{aligned}$$

we have

$$\left(\gamma_{1234} \underset{12}{\dot{-}} \gamma_{1243}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(d_1, d_2, \underset{3}{0}, ..., \underset{9}{0}, \underset{10}{-d_3}, \underset{11}{0}, ..., \underset{20}{0}, -d_1d_3, \underset{22}{0}, ..., \underset{25}{0}, -d_2d_3, \underset{27}{0}, ..., \underset{30}{0}, -d_1d_2d_3, \underset{32}{0}, ..., \underset{53}{0}\right) \tag{37}$$

For we have

$$\mathfrak{n}^4_{(\gamma_{1234},\gamma_{1243})}(d_1, d_2, d_3, d_4, d_5)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{9}{0}, \underset{10}{d_3d_4 - d_5}, \underset{11}{0}, ..., \underset{20}{0}, d_1d_3d_4 - d_1d_5, \underset{22}{0}, ..., \underset{25}{0}, \\ d_2d_3d_4 - d_2d_5, \underset{27}{0}, ..., \underset{30}{0}, d_1d_2d_3d_4 - d_1d_2d_5, \underset{32}{0}, ..., \underset{53}{0} \end{array}\right)$$

Since

$$\gamma_{1342}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{27}{0}, \\ d_2d_3d_4, 0, 0, 0, 0, d_1d_2d_3d_4, \underset{34}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{1432}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{20}{0}, \\ d_1d_3d_4, \underset{22}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{34}{0}, d_1d_2d_3d_4, \underset{36}{0}, ..., \underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{1342} \underset{12}{\dot{-}} \gamma_{1432}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, \underset{3}{0}, ..., \underset{9}{0}, -d_3, \underset{11}{0}, ..., \underset{20}{0}, -d_1d_3, \underset{22}{0}, ..., \underset{27}{0}, d_2d_3, \underset{29}{0}, -d_2d_3, \underset{31}{0}, \underset{32}{0}, d_1d_2d_3, \underset{34}{0}, \\ -d_1d_2d_3, \underset{36}{0}, ..., \underset{53}{0} \end{array}\right) \tag{38}$$

For we have

$$\mathfrak{n}^4_{(\gamma_{1342},\gamma_{1432})}(d_1, d_2, d_3, d_4, d_5)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{9}{0}, \underset{10}{d_3d_4 - d_5}, \underset{11}{0}, ..., \underset{20}{0}, d_1d_3d_4 - d_1d_5, \underset{22}{0}, ..., \underset{27}{0}, \\ d_2d_5, \underset{29}{0}, d_2d_3d_4 - d_2d_5, \underset{31}{0}, \underset{32}{0}, d_1d_2d_5, \underset{34}{0}, d_1d_2d_3d_4 - d_1d_2d_5, \underset{36}{0}, ..., \underset{53}{0} \end{array}\right)$$

For the sake of the completeness of our proof, we have provided the reason why (37) and (38) are derivable, but we will omit such reasoning from now on. (37) and (38) imply that

$$\left(\left(\gamma_{1234}\underset{12}{\dot{-}}\gamma_{1243}\right)\underset{1}{\dot{-}}\left(\gamma_{1342}\underset{12}{\dot{-}}\gamma_{1432}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\left(d_1,\underset{2}{0},...,\underset{25}{0},-d_2,\underset{27}{0},-d_2,\underset{29}{0},d_2,-d_1d_2,\underset{32}{0},-d_1d_2,\underset{34}{0},d_1d_2,\underset{36}{0},...,\underset{53}{0}\right) \tag{39}$$

Since

$$\gamma_{2341}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},...,\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0},\\ d_1d_3d_4,\underset{24}{0},...,\underset{38}{0},d_1d_2d_3d_4,\underset{40}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{2431}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\\ \underset{19}{0},...,\underset{24}{0},d_1d_3d_4,d_2d_3d_4,\underset{27}{0},...,\underset{40}{0},d_1d_2d_3d_4,\underset{42}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{2341}\underset{12}{\dot{-}}\gamma_{2431}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,\underset{3}{0},...,\underset{9}{0},-d_3,\underset{11}{0},...,\underset{22}{0},d_1d_3,0,-d_1d_3,-d_2d_3,\\ \underset{27}{0},...,\underset{38}{0},d_1d_2d_3,0,-d_1d_2d_3,\underset{42}{0},...,\underset{53}{0} \end{array}\right) \tag{40}$$

Since

$$\gamma_{3421}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0},d_1d_2d_4,\\ \underset{21}{0},\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{46}{0},d_1d_2d_3d_4,\underset{48}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4321}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0},\\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{52}{0},d_1d_2d_3d_4 \end{array}\right)$$

we have

$$\left(\gamma_{3421}\underset{12}{\dot{-}}\gamma_{4321}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,\underset{3}{0},...,\underset{9}{0},-d_3,\underset{11}{0},...,\underset{22}{0},d_1d_3,\underset{24}{0},-d_1d_3,\underset{26}{0},\underset{27}{0},d_2d_3,\underset{29}{0},-d_2d_3,\\ \underset{31}{0},...,\underset{46}{0},d_1d_2d_3,\underset{48}{0},...,\underset{52}{0},-d_1d_2d_3 \end{array}\right) \tag{41}$$

(40) and (41) imply that

$$\begin{aligned}&\left(\left(\gamma_{2341}\overset{\cdot}{\underset{12}{-}}\gamma_{2431}\right)\overset{\cdot}{\underset{1}{-}}\left(\gamma_{3421}\overset{\cdot}{\underset{12}{-}}\gamma_{4321}\right)\right)(d_1,d_2)\\ &=\mathfrak{m}\left(\begin{array}{c} d_1,\underset{2}{0},...,\underset{25}{0},-d_2,\underset{27}{0},-d_2,\underset{29}{0},d_2,\underset{31}{0},...,\underset{38}{0},d_1d_2,\underset{40}{0},\\ -d_1d_2,\underset{42}{0},...,\underset{46}{0},-d_1d_2,\underset{48}{0},...,\underset{52}{0},d_1d_2\end{array}\right)\end{aligned}\tag{42}$$

(39) and (42) imply that

$$\begin{aligned}&\left(\begin{array}{c}\left(\left(\gamma_{1234}\overset{\cdot}{\underset{12}{-}}\gamma_{1243}\right)\overset{\cdot}{\underset{1}{-}}\left(\gamma_{1342}\overset{\cdot}{\underset{12}{-}}\gamma_{1432}\right)\right)\overset{\cdot}{-}\\ \left(\left(\gamma_{2341}\overset{\cdot}{\underset{12}{-}}\gamma_{2431}\right)\overset{\cdot}{\underset{1}{-}}\left(\gamma_{3421}\overset{\cdot}{\underset{12}{-}}\gamma_{4321}\right)\right)\end{array}\right)(d)\\ &=\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},-d,\underset{32}{0},-d,\underset{34}{0},d,\underset{36}{0},...,\underset{38}{0},-d,0,d,\underset{42}{0},...,\underset{46}{0},d,\underset{48}{0},...,\underset{52}{0},-d\right)\end{aligned}\tag{43}$$

2. Since

$$\begin{aligned}&\gamma_{1342}(d_1,d_2,d_3,d_4)\\ &=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{27}{0},\\ d_2d_3d_4,0,0,0,0,d_1d_2d_3d_4,\underset{34}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

and

$$\begin{aligned}&\gamma_{1324}(d_1,d_2,d_3,d_4)\\ &=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,\underset{9}{0},\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{31}{0},\\ d_1d_2d_3d_4,\underset{33}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

we have

$$\begin{aligned}&\left(\gamma_{1342}\overset{\cdot}{\underset{13}{-}}\gamma_{1324}\right)(d_1,d_2,d_3)\\ &=\mathfrak{m}\left(\begin{array}{c} d_1,0,d_2,\underset{4}{0},...,\underset{8}{0},d_3,\underset{10}{0},...,\underset{15}{0},d_1d_3,\underset{17}{0},...,\underset{26}{0},-d_2d_3,d_2d_3,\underset{29}{0},...,\underset{31}{0},\\ -d_1d_2d_3,d_1d_2d_3,\underset{34}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}\tag{44}$$

Since

$$\begin{aligned}&\gamma_{1423}(d_1,d_2,d_3,d_4)\\ &=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{20}{0},d_1d_3d_4,\\ \underset{22}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{33}{0},d_1d_2d_3d_4,\underset{35}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

and

$$\gamma_{1243}\left(d_1,d_2,d_3,d_4\right) = \begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{9}{0},\underset{10}{d_3d_4},\underset{11}{0},...,\underset{20}{0},d_1d_3d_4,\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{30}{0}, \\ d_1d_2d_3d_4,\underset{32}{0},...,\underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{1423}\underset{13}{\dot{-}}\gamma_{1243}\right)\left(d_1,d_2,d_3\right) = \mathfrak{m}\begin{pmatrix} d_1,0,d_2,\underset{4}{0},...,\underset{8}{0},d_3,\underset{10}{0},...,\underset{15}{0},d_1d_3,\underset{17}{0},...,\underset{25}{0},-d_2d_3,\underset{27}{0},\underset{28}{0},d_2d_3,\underset{30}{0}, \\ -d_1d_2d_3,\underset{32}{0},\underset{33}{0},d_1d_2d_3,\underset{35}{0},...,\underset{53}{0} \end{pmatrix} \tag{45}$$

(44) and (45) imply that

$$\left(\left(\gamma_{1324}\underset{13}{\dot{-}}\gamma_{1342}\right)\underset{1}{\dot{-}}\left(\gamma_{1243}\underset{13}{\dot{-}}\gamma_{1423}\right)\right)\left(d_1,d_2\right) = \mathfrak{m}\left(d_1,\underset{2}{0},...,\underset{25}{0},d_2,-d_2,d_2,-d_2,\underset{30}{0},d_1d_2,-d_1d_2,d_1d_2,-d_1d_2,\underset{35}{0},...,\underset{53}{0}\right) \tag{46}$$

Since

$$\gamma_{3421}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0},d_1d_2d_4, \\ 0,0,d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{46}{0},d_1d_2d_3d_4,\underset{48}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{3241}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0}, \\ d_1d_3d_4,\underset{24}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{44}{0},d_1d_2d_3d_4,\underset{46}{0},...,\underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{3421}\underset{13}{\dot{-}}\gamma_{3241}\right)\left(d_1,d_2,d_3\right) = \mathfrak{m}\begin{pmatrix} d_1,0,d_2,\underset{4}{0},...,\underset{8}{0},d_3,\underset{10}{0},...,\underset{17}{0},-d_1d_3,\underset{19}{0},d_1d_3,\underset{21}{0},...,\underset{26}{0},-d_2d_3,d_2d_3, \\ \underset{29}{0},...,\underset{44}{0},-d_1d_2d_3,\underset{46}{0},d_1d_2d_3,\underset{48}{0},...,\underset{53}{0} \end{pmatrix} \tag{47}$$

Since

$$\gamma_{4231}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{19}{0}, \\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{50}{0},d_1d_2d_3d_4,0,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{2431}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},0,d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4, \\ \underset{19}{0},...,\underset{24}{0},d_1d_3d_4,d_2d_3d_4,\underset{27}{0},...,\underset{40}{0},d_1d_2d_3d_4,\underset{42}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{4231}\underset{13}{\dot{-}}\gamma_{2431}\right)\left(d_1,d_2,d_3\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,\underset{2}{0},d_2,\underset{4}{0},...,\underset{8}{0},d_3,\underset{10}{0},...,\underset{17}{0},-d_1d_3,\underset{19}{0},d_1d_3,\underset{21}{0},...,\underset{25}{0},-d_2d_3,\underset{27}{0},\underset{28}{0},d_2d_3, \\ \underset{30}{0},...,\underset{40}{0},-d_1d_2d_3,\underset{42}{0},...,\underset{50}{0},d_1d_2d_3,\underset{52}{0},\underset{53}{0} \end{array}\right) \tag{48}$$

(47) and (48) imply that

$$\left(\left(\gamma_{3421}\underset{13}{\dot{-}}\gamma_{3241}\right)\underset{1}{\dot{-}}\left(\gamma_{4231}\underset{13}{\dot{-}}\gamma_{2431}\right)\right)\left(d_1,d_2\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,\underset{2}{0},...,\underset{25}{0},d_2,-d_2,d_2,-d_2,\underset{30}{0},...,\underset{40}{0},d_1d_2,\underset{42}{0},...,\underset{44}{0}, \\ -d_1d_2,\underset{46}{0},d_1d_2,\underset{48}{0},...,\underset{50}{0},-d_1d_2,\underset{52}{0},\underset{53}{0} \end{array}\right) \tag{49}$$

(46) and (49) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{1324}\underset{13}{\dot{-}}\gamma_{1342}\right)\underset{1}{\dot{-}}\left(\gamma_{1243}\underset{13}{\dot{-}}\gamma_{1423}\right)\right)\dot{-} \\ \left(\left(\gamma_{3241}\underset{13}{\dot{-}}\gamma_{3421}\right)\underset{1}{\dot{-}}\left(\gamma_{2431}\underset{13}{\dot{-}}\gamma_{4231}\right)\right) \end{array}\right)(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},d,-d,d,-d,\underset{35}{0},...,\underset{40}{0},-d,\underset{42}{0},...\underset{44}{0},d,\underset{46}{0},-d,\underset{48}{0},...,\underset{50}{0},d,\underset{52}{0},\underset{53}{0}\right) \tag{50}$$

3. Since

$$\gamma_{1423}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{20}{0},d_1d_3d_4, \\ \underset{22}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{33}{0},d_1d_2d_3d_4,\underset{35}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{1432}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{20}{0}, \\ d_1d_3d_4,\underset{22}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{34}{0},d_1d_2d_3d_4,\underset{36}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{1423} \underset{14}{\dot{-}} \gamma_{1432}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, ..., \underset{7}{0}, -d_3, \underset{9}{0}, \underset{10}{0}, -d_1d_3, \underset{12}{0}, ..., \underset{28}{0}, d_2d_3, -d_2d_3, \\ \underset{31}{0}, ..., \underset{33}{0}, d_1d_2d_3, -d_1d_2d_3, \underset{36}{0}, ..., \underset{53}{0} \end{array}\right) \tag{51}$$

Since

$$\gamma_{1234}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{53}{0}\right)$$

and

$$\gamma_{1324}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2d_3, \underset{9}{0}, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{30}{0}, \\ 0, d_1d_2d_3d_4, \underset{33}{0}, ..., \underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{1234} \underset{14}{\dot{-}} \gamma_{1324}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, ..., \underset{7}{0}, -d_3, \underset{9}{0}, \underset{10}{0}, -d_1d_3, \underset{12}{0}, ..., \underset{26}{0}, -d_2d_3, \\ \underset{28}{0}, ..., \underset{31}{0}, -d_1d_2d_3, \underset{33}{0}, ..., \underset{53}{0} \end{array}\right) \tag{52}$$

(51) and (52) imply that

$$\left(\left(\gamma_{1423} \underset{14}{\dot{-}} \gamma_{1432}\right) \underset{1}{\dot{-}} \left(\gamma_{1234} \underset{14}{\dot{-}} \gamma_{1324}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(d_1, \underset{2}{0}, ..., \underset{26}{0}, d_2, \underset{28}{0}, d_2, -d_2, \underset{31}{0}, d_1d_2, \underset{33}{0}, d_1d_2, -d_1d_2, \underset{36}{0}, ..., \underset{53}{0}\right) \tag{53}$$

Since

$$\gamma_{4231}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, d_2d_4, d_3d_4, \underset{11}{0}, \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{50}{0}, d_1d_2d_3d_4, \underset{52}{0}, \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4321}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{52}{0}, d_1d_2d_3d_4 \end{array}\right)$$

we have

$$\left(\gamma_{4231} \underset{14}{\dot{-}} \gamma_{4321}\right)(d_1,d_2,d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, ..., \underset{7}{0}, -d_3, \underset{9}{0}, ..., \underset{12}{0}, d_1d_3, \underset{14}{0}, -d_1d_3, \underset{16}{0}, ..., \underset{28}{0}, d_2d_3, -d_2d_3, \\ \underset{31}{0}, ..., \underset{50}{0}, d_1d_2d_3, \underset{52}{0}, -d_1d_2d_3 \end{array}\right) \tag{54}$$

Since

$$\gamma_{2341}(d_1,d_2,d_3,d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, ..., \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, ..., \underset{22}{0}, \\ d_1d_3d_4, \underset{24}{0}, ..., \underset{38}{0}, d_1d_2d_3d_4, \underset{40}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3241}(d_1,d_2,d_3,d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, \underset{9}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, ..., \underset{22}{0}, \\ d_1d_3d_4, \underset{24}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{44}{0}, d_1d_2d_3d_4, \underset{46}{0}, ..., \underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{2341} \underset{14}{\dot{-}} \gamma_{3241}\right)(d_1,d_2,d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, ..., \underset{7}{0}, -d_3, \underset{9}{0}, ..., \underset{12}{0}, d_1d_3, \underset{14}{0}, -d_1d_3, \underset{16}{0}, ..., \underset{26}{0}, -d_2d_3, \\ \underset{28}{0}, ..., \underset{38}{0}, d_1d_2d_3, \underset{40}{0}, ..., \underset{44}{0}, -d_1d_2d_3, \underset{46}{0}, ..., \underset{53}{0} \end{array}\right) \tag{55}$$

(54) and (55) imply that

$$\left(\left(\gamma_{4231} \underset{14}{\dot{-}} \gamma_{4321}\right) \underset{1}{\dot{-}} \left(\gamma_{2341} \underset{14}{\dot{-}} \gamma_{3241}\right)\right)(d_1,d_2)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, ..., \underset{26}{0}, d_2, \underset{28}{0}, d_2, -d_2, \underset{31}{0}, ..., \underset{38}{0}, -d_1d_2, \\ \underset{40}{0}, ..., \underset{44}{0}, d_1d_2, \underset{46}{0}, ..., \underset{50}{0}, d_1d_2, \underset{52}{0}, -d_1d_2 \end{array}\right) \tag{56}$$

(53) and (56) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{1423} \underset{14}{\dot{-}} \gamma_{1432}\right) \underset{1}{\dot{-}} \left(\gamma_{1234} \underset{14}{\dot{-}} \gamma_{1324}\right)\right) \dot{-} \\ \left(\left(\gamma_{4231} \underset{14}{\dot{-}} \gamma_{4321}\right) \underset{1}{\dot{-}} \left(\gamma_{2341} \underset{14}{\dot{-}} \gamma_{3241}\right)\right) \end{array}\right)(d)$$
$$= \mathfrak{m}\left(\underset{1}{0}, ..., \underset{31}{0}, d, \underset{33}{0}, d, -d, \underset{36}{0}, ..., \underset{38}{0}, d, \underset{40}{0}, ..., \underset{44}{0}, -d, \underset{46}{0}, ..., \underset{50}{0}, -d, \underset{52}{0}, d\right) \tag{57}$$

4. Since

$$\begin{aligned}
&\gamma_{2143}\left(d_1,d_2,d_3,d_4\right)\\
&=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{20}{0},d_1d_3d_4,\\ \underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{36}{0},d_1d_2d_3d_4,\underset{38}{0},...,\underset{53}{0}\end{array}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{2134}\left(d_1,d_2,d_3,d_4\right)\\
&=\mathfrak{m}\left(d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{35}{0},d_1d_2d_3d_4,\underset{37}{0},...,\underset{53}{0}\right)
\end{aligned}$$

we have

$$\begin{aligned}
&\left(\gamma_{2143}\underset{21}{\dot{-}}\gamma_{2134}\right)\left(d_1,d_2,d_3\right)\\
&=\mathfrak{m}\left(\begin{array}{c} d_2,d_1,\underset{3}{0},\underset{4}{0},d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3,\underset{11}{0},...,\underset{20}{0},d_2d_3,\underset{22}{0},...,\underset{25}{0},d_1d_3,\\ \underset{27}{0},...,\underset{35}{0},-d_1d_2d_3,d_1d_2d_3,\underset{38}{0},...,\underset{53}{0}\end{array}\right) \qquad (58)
\end{aligned}$$

Since

$$\begin{aligned}
&\gamma_{2431}\left(d_1,d_2,d_3,d_4\right)\\
&=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\\ \underset{19}{0},...,\underset{24}{0},d_1d_3d_4,d_2d_3d_4,\underset{27}{0},...,\underset{40}{0},d_1d_2d_3d_4,\underset{42}{0},...,\underset{53}{0}\end{array}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{2341}\left(d_1,d_2,d_3,d_4\right)\\
&=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},...,\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0},\\ d_1d_3d_4,\underset{24}{0},...,\underset{38}{0},d_1d_2d_3d_4,\underset{40}{0},...,\underset{53}{0}\end{array}\right)
\end{aligned}$$

we have

$$\begin{aligned}
&\left(\gamma_{2431}\underset{21}{\dot{-}}\gamma_{2341}\right)\left(d_1,d_2,d_3\right)\\
&=\mathfrak{m}\left(\begin{array}{c} d_2,d_1,\underset{3}{0},\underset{4}{0},d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3,\underset{11}{0},...,\underset{22}{0},-d_2d_3,\underset{24}{0},d_2d_3,d_1d_3,\\ \underset{27}{0},...,\underset{38}{0},-d_1d_2d_3,\underset{40}{0},d_1d_2d_3,\underset{42}{0},...,\underset{53}{0}\end{array}\right) \qquad (59)
\end{aligned}$$

(58) and (59) imply that

$$\begin{aligned}
&\left(\left(\gamma_{2143}\underset{21}{\dot{-}}\gamma_{2134}\right)\underset{1}{\dot{-}}\left(\gamma_{2431}\underset{21}{\dot{-}}\gamma_{2341}\right)\right)\left(d_1,d_2\right)\\
&=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_1,\underset{3}{0},...,\underset{20}{0},d_2,\underset{22}{0},d_2,\underset{24}{0},-d_2,\underset{26}{0},...,\underset{35}{0},\\ -d_1d_2,d_1d_2,\underset{38}{0},d_1d_2,\underset{40}{0},-d_1d_2,\underset{42}{0},...,\underset{53}{0}\end{array}\right) \qquad (60)
\end{aligned}$$

Since

$\gamma_{1432}(d_1, d_2, d_3, d_4)$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{20}{0}, \\ d_1d_3d_4, \underset{22}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{34}{0}, d_1d_2d_3d_4, \underset{36}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$\gamma_{1342}(d_1, d_2, d_3, d_4)$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{27}{0}, \\ d_2d_3d_4, 0, 0, 0, 0, d_1d_2d_3d_4, \underset{34}{0}, ..., \underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{1432} \underset{21}{\dot{-}} \gamma_{1342}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_2, d_1, \underset{3}{0}, \underset{4}{0}, d_1d_2, \underset{6}{0}, ..., \underset{9}{0}, d_3, \underset{11}{0}, ..., \underset{20}{0}, d_1d_3, \underset{22}{0}, ..., \underset{27}{0}, -d_2d_3, \underset{29}{0}, d_2d_3, \\ \underset{31}{0}, \underset{32}{0}, -d_1d_2d_3, \underset{34}{0}, d_1d_2d_3, \underset{36}{0}, ..., \underset{53}{0} \end{pmatrix} \tag{61}$$

Since

$\gamma_{4312}(d_1, d_2, d_3, d_4)$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, \\ d_1d_2d_4, \underset{20}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{51}{0}, d_1d_2d_3d_4, \underset{53}{0} \end{pmatrix}$$

and

$\gamma_{3412}(d_1, d_2, d_3, d_4)$

$$= \begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, d_1d_2d_4, \\ \underset{20}{0}, ..., \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{45}{0}, d_1d_2d_3d_4, \underset{47}{0}, ..., \underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{4312} \underset{21}{\dot{-}} \gamma_{3412}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_2, d_1, \underset{3}{0}, \underset{4}{0}, d_1d_2, \underset{6}{0}, ..., \underset{9}{0}, d_3, \underset{11}{0}, ..., \underset{20}{0}, \underset{21}{0}, \underset{22}{0}, -d_2d_3, \underset{24}{0}, d_2d_3, \underset{26}{0}, \underset{27}{0}, \\ -d_1d_3, \underset{29}{0}, d_1d_3, \underset{31}{0}, ..., \underset{45}{0}, -d_1d_2d_3, \underset{47}{0}, ..., \underset{51}{0}, d_1d_2d_3, \underset{53}{0} \end{pmatrix} \tag{62}$$

(61) and (62) imply that

$$\left(\left(\gamma_{1432} \underset{21}{\dot{-}} \gamma_{1342}\right) \underset{1}{\dot{-}} \left(\gamma_{4312} \underset{21}{\dot{-}} \gamma_{3412}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, d_1, \underset{3}{0}, ..., \underset{20}{0}, d_2, \underset{22}{0}, d_2, \underset{24}{0}, -d_2, \underset{26}{0}, ..., \underset{32}{0}, -d_1d_2, \underset{34}{0}, d_1d_2, \\ \underset{36}{0}, ..., \underset{45}{0}, d_1d_2, \underset{47}{0}, ..., \underset{51}{0}, -d_1d_2, \underset{53}{0} \end{pmatrix} \tag{63}$$

(60) and (63) imply that

$$\begin{aligned} & \left(\begin{array}{c} \left(\left(\gamma_{2143} \underset{21}{\dot{-}} \gamma_{2134}\right) \underset{1}{\dot{-}} \left(\gamma_{2431} \underset{21}{\dot{-}} \gamma_{2341}\right)\right) \dot{-} \\ \left(\left(\gamma_{1432} \underset{21}{\dot{-}} \gamma_{1342}\right) \underset{1}{\dot{-}} \left(\gamma_{4312} \underset{21}{\dot{-}} \gamma_{3412}\right)\right) \end{array} \right)(d) \\ & = \mathfrak{m}\left(\underset{1}{0}, ..., \underset{32}{0}, d, \underset{34}{0}, -d, -d, d, \underset{38}{0}, d, \underset{40}{0}, -d, \underset{42}{0}, ..., \underset{45}{0}, -d, \underset{47}{0}, ..., \underset{51}{0}, d, \underset{53}{0}\right) \end{aligned} \tag{64}$$

5. Since

$$\begin{aligned} & \gamma_{2314}\left(d_1, d_2, d_3, d_4\right) \\ & = \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, \underset{7}{0}, ..., \underset{12}{0}, d_1 d_2 d_3, 0, 0, 0, d_1 d_2 d_4, \underset{18}{0}, ..., \underset{21}{0}, d_1 d_3 d_4, \\ \underset{23}{0}, ..., \underset{37}{0}, d_1 d_2 d_3 d_4, \underset{39}{0}, ..., \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2341}\left(d_1, d_2, d_3, d_4\right) \\ & = \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, \underset{8}{0}, ..., \underset{12}{0}, d_1 d_2 d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1 d_2 d_4, \underset{19}{0}, ..., \underset{22}{0}, \\ d_1 d_3 d_4, \underset{24}{0}, ..., \underset{38}{0}, d_1 d_2 d_3 d_4, \underset{40}{0}, ..., \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2314} \underset{23}{\dot{-}} \gamma_{2341}\right)\left(d_1, d_2, d_3\right) \\ & = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, d_2, \underset{4}{0}, ..., \underset{6}{0}, -d_3, \underset{8}{0}, ..., \underset{16}{0}, d_1 d_3, -d_1 d_3, \underset{19}{0}, ..., \underset{21}{0}, d_2 d_3, -d_2 d_3, \\ \underset{24}{0}, ..., \underset{37}{0}, d_1 d_2 d_3, -d_1 d_2 d_3, \underset{40}{0}, ..., \underset{53}{0} \end{array} \right) \end{aligned} \tag{65}$$

Since

$$\begin{aligned} & \gamma_{2143}\left(d_1, d_2, d_3, d_4\right) \\ & = \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, \underset{6}{0}, ..., \underset{9}{0}, d_3 d_4, 0, d_1 d_2 d_3, \underset{13}{0}, ..., \underset{16}{0}, d_1 d_2 d_4, 0, 0, 0, d_1 d_3 d_4, \\ \underset{22}{0}, ..., \underset{25}{0}, d_2 d_3 d_4, \underset{27}{0}, ..., \underset{36}{0}, d_1 d_2 d_3 d_4, \underset{38}{0}, ..., \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2413}\left(d_1, d_2, d_3, d_4\right) \\ & = \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, \underset{6}{0}, d_1 d_4, \underset{8}{0}, \underset{9}{0}, d_3 d_4, \underset{11}{0}, d_1 d_2 d_3, \underset{13}{0}, ..., \underset{17}{0}, d_1 d_2 d_4, \underset{19}{0}, ..., \underset{23}{0}, \\ d_1 d_3 d_4, \underset{25}{0}, d_2 d_3 d_4, \underset{27}{0}, ..., \underset{39}{0}, d_1 d_2 d_3 d_4, \underset{41}{0}, ..., \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\left(\gamma_{2143} \underset{23}{\overset{\cdot}{-}} \gamma_{2413}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, d_2, \underset{4}{0}, ..., \underset{6}{0}, -d_3, \underset{8}{0}, ..., \underset{16}{0}, d_1d_3, -d_1d_3, \underset{19}{0}, \underset{20}{0}, d_2d_3, \underset{22}{0}, \underset{23}{0}, \\ -d_2d_3, \underset{25}{0}, ..., \underset{36}{0}, d_1d_2d_3, \underset{38}{0}, \underset{39}{0}, -d_1d_2d_3, \underset{41}{0}, ..., \underset{53}{0} \end{array}\right) \tag{66}$$

(65) and (66) imply that

$$\left(\left(\gamma_{2314} \underset{23}{\overset{\cdot}{-}} \gamma_{2341}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{2143} \underset{23}{\overset{\cdot}{-}} \gamma_{2413}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(\underset{1}{0}, d_1, \underset{3}{0}, ..., \underset{20}{0}, -d_2, d_2, -d_2, d_2, \underset{25}{0}, ..., \underset{36}{0}, -d_1d_2, d_1d_2, -d_1d_2, d_1d_2, \underset{41}{0}, ..., \underset{53}{0}\right) \tag{67}$$

Since

$$\gamma_{3142}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, 0, d_1d_2d_4, \underset{17}{0}, ..., \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{42}{0}, d_1d_2d_3d_4, \underset{44}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3412}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, d_1d_2d_4, \\ \underset{20}{0}, ..., \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{45}{0}, d_1d_2d_3d_4, \underset{47}{0}, ..., \underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{3142} \underset{23}{\overset{\cdot}{-}} \gamma_{3412}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, d_2, \underset{4}{0}, ..., \underset{6}{0}, -d_3, \underset{8}{0}, ..., \underset{15}{0}, d_1d_3, \underset{17}{0}, \underset{18}{0}, -d_1d_3, \underset{20}{0}, \underset{21}{0}, d_2d_3, -d_2d_3, \\ \underset{24}{0}, ..., \underset{42}{0}, d_1d_2d_3, \underset{44}{0}, \underset{45}{0}, -d_1d_2d_3, \underset{47}{0}, ..., \underset{53}{0} \end{array}\right) \tag{68}$$

Since

$$\gamma_{1432}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{20}{0}, \\ d_1d_3d_4, \underset{22}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{34}{0}, d_1d_2d_3d_4, \underset{36}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4132}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, d_1d_4, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, ..., \underset{18}{0}, d_1d_2d_4, \underset{20}{0}, ..., \underset{23}{0}, \\ d_1d_3d_4, \underset{25}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{48}{0}, d_1d_2d_3d_4, \underset{50}{0}, ..., \underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{1432} \underset{23}{\overset{\cdot}{-}} \gamma_{4132}\right)(d_1, d_2, d_3) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, d_2, \underset{4}{0}, ..., \underset{6}{0}, -d_3, \underset{8}{0}, ..., \underset{15}{0}, d_1 d_3, \underset{17}{0}, \underset{18}{0}, -d_1 d_3, \underset{20}{0}, d_2 d_3, \underset{22}{0}, \underset{23}{0}, \\ -d_2 d_3, \underset{25}{0}, ..., \underset{34}{0}, d_1 d_2 d_3, \underset{36}{0}, ..., \underset{48}{0}, -d_1 d_2 d_3, \underset{50}{0}, ..., \underset{53}{0} \end{array}\right) \tag{69}$$

(68) and (69) imply that

$$\left(\left(\gamma_{3142} \underset{23}{\overset{\cdot}{-}} \gamma_{3412}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{1432} \underset{23}{\overset{\cdot}{-}} \gamma_{4132}\right)\right)(d_1, d_2) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, \underset{3}{0}, ..., \underset{20}{0}, -d_2, d_2, -d_2, d_2, \underset{25}{0}, ..., \underset{34}{0}, -d_1 d_2, \underset{36}{0}, ..., \underset{42}{0}, \\ d_1 d_2, \underset{44}{0}, \underset{45}{0}, -d_1 d_2, \underset{47}{0}, \underset{48}{0}, d_1 d_2, \underset{50}{0}, ..., \underset{53}{0} \end{array}\right) \tag{70}$$

(67) and (70) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{2314} \underset{23}{\overset{\cdot}{-}} \gamma_{2341}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{2143} \underset{23}{\overset{\cdot}{-}} \gamma_{2413}\right)\right) \overset{\cdot}{-} \\ \left(\left(\gamma_{3142} \underset{23}{\overset{\cdot}{-}} \gamma_{3412}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{1432} \underset{23}{\overset{\cdot}{-}} \gamma_{4132}\right)\right) \end{array}\right)(d) = \mathfrak{m}\left(\underset{1}{0}, ..., \underset{34}{0}, d, \underset{36}{0}, -d, d, -d, d, \underset{41}{0}, \underset{42}{0}, -d, \underset{44}{0}, \underset{45}{0}, d, \underset{47}{0}, \underset{48}{0}, -d, \underset{50}{0}, ..., \underset{53}{0}\right) \tag{71}$$

6. Since

$$\gamma_{2431}(d_1, d_2, d_3, d_4) = \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, \underset{8}{0}, \underset{9}{0}, d_3 d_4, \underset{11}{0}, \underset{12}{0}, d_1 d_2 d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1 d_2 d_4, \\ \underset{19}{0}, ..., \underset{24}{0}, d_1 d_3 d_4, d_2 d_3 d_4, \underset{27}{0}, ..., \underset{40}{0}, d_1 d_2 d_3 d_4, \underset{42}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{2413}(d_1, d_2, d_3, d_4) = \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, \underset{6}{0}, d_1 d_4, \underset{8}{0}, \underset{9}{0}, d_3 d_4, \underset{11}{0}, d_1 d_2 d_3, \underset{13}{0}, ..., \underset{17}{0}, d_1 d_2 d_4, \underset{19}{0}, ..., \underset{23}{0}, \\ d_1 d_3 d_4, 0, d_2 d_3 d_4, \underset{27}{0}, ..., \underset{39}{0}, d_1 d_2 d_3 d_4, \underset{41}{0}, ..., \underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{2431} \underset{24}{\overset{\cdot}{-}} \gamma_{2413}\right)(d_1, d_2, d_3) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, \underset{3}{0}, d_2, \underset{5}{0}, d_3, \underset{7}{0}, ..., \underset{11}{0}, -d_1 d_3, d_1 d_3, \underset{14}{0}, ..., \underset{23}{0}, \\ -d_2 d_3, d_2 d_3, \underset{26}{0}, ..., \underset{39}{0}, -d_1 d_2 d_3, d_1 d_2 d_3, \underset{42}{0}, ..., \underset{53}{0} \end{array}\right) \tag{72}$$

Since

$$\gamma_{2314}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,\underset{7}{0},...,\underset{12}{0},d_1d_2d_3,0,0,0,d_1d_2d_4,\underset{18}{0},...,\underset{21}{0},d_1d_3d_4,\\ \underset{23}{0},...,\underset{37}{0},d_1d_2d_3d_4,\underset{39}{0},...,\underset{53}{0}\end{pmatrix}$$

and

$$\gamma_{2134}\left(d_1,d_2,d_3,d_4\right)$$
$$=\left(d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{35}{0},d_1d_2d_3d_4,\underset{37}{0},...,\underset{53}{0}\right)$$

we have

$$\left(\gamma_{2314}\underset{24}{\dot{-}}\gamma_{2134}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\begin{pmatrix} \underset{1}{0},d_1,\underset{3}{0},d_2,\underset{5}{0},d_3,\underset{7}{0},...,\underset{11}{0},-d_1d_3,d_1d_3,\underset{14}{0},...,\underset{21}{0},d_2d_3,\\ \underset{23}{0},...,\underset{35}{0},-d_1d_2d_3,\underset{37}{0},d_1d_2d_3,\underset{39}{0},...,\underset{53}{0}\end{pmatrix} \tag{73}$$

(72) and (73) imply that

$$\left(\left(\gamma_{2431}\underset{24}{\dot{-}}\gamma_{2413}\right)\underset{1}{\dot{-}}\left(\gamma_{2314}\underset{24}{\dot{-}}\gamma_{2134}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\left(\underset{1}{0},d_1,\underset{3}{0},...,\underset{21}{0},-d_2,\underset{23}{0},-d_2,d_2,\underset{26}{0},...,\underset{35}{0},d_1d_2,\underset{37}{0},-d_1d_2,\underset{39}{0},-d_1d_2,d_1d_2,\underset{42}{0},...,\underset{53}{0}\right) \tag{74}$$

Since

$$\gamma_{4312}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0},\\ d_1d_2d_4,\underset{20}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{51}{0},d_1d_2d_3d_4,\underset{53}{0}\end{pmatrix}$$

and

$$\gamma_{4132}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0},\\ d_1d_3d_4,\underset{25}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{48}{0},d_1d_2d_3d_4,\underset{50}{0},...,\underset{53}{0}\end{pmatrix}$$

we have

$$\left(\gamma_{4312}\underset{24}{\dot{-}}\gamma_{4132}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\begin{pmatrix} \underset{1}{0},d_1,\underset{3}{0},d_2,\underset{5}{0},d_3,\underset{7}{0},...,\underset{10}{0},-d_1d_3,\underset{12}{0},\underset{13}{0},d_1d_3,\underset{15}{0},...,\underset{23}{0},\\ -d_2d_3,d_2d_3,\underset{26}{0},...,\underset{48}{0},-d_1d_2d_3,\underset{50}{0},\underset{51}{0},d_1d_2d_3,\underset{53}{0}\end{pmatrix} \tag{75}$$

Since

$$\gamma_{3142}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{17}{0},...,\underset{21}{0},d_1d_3d_4, \\ \underset{23}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{42}{0},d_1d_2d_3d_4,\underset{44}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{1342}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{27}{0}, \\ d_2d_3d_4,0,0,0,0,d_1d_2d_3d_4,\underset{34}{0},...,\underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{3142}\overset{\cdot}{\underset{24}{-}}\gamma_{1342}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\begin{pmatrix} \underset{1}{0},d_1,\underset{3}{0},d_2,\underset{5}{0},d_3,\underset{7}{0},...,\underset{10}{0},-d_1d_3,\underset{12}{0},\underset{13}{0},d_1d_3,\underset{15}{0},...,\underset{21}{0},d_2d_3, \\ \underset{23}{0},...,\underset{32}{0},-d_1d_2d_3,\underset{34}{0},...,\underset{42}{0},d_1d_2d_3,\underset{44}{0},...,\underset{53}{0} \end{pmatrix} \tag{76}$$

(75) and (76) imply that

$$\left(\left(\gamma_{4312}\overset{\cdot}{\underset{24}{-}}\gamma_{4132}\right)\overset{\cdot}{\underset{1}{-}}\left(\gamma_{3142}\overset{\cdot}{\underset{24}{-}}\gamma_{1342}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\begin{pmatrix} \underset{1}{0},d_1,\underset{3}{0},...,\underset{21}{0},-d_2,\underset{23}{0},-d_2,d_2,\underset{26}{0},...,\underset{32}{0},d_1d_2,\underset{34}{0},...,\underset{42}{0}, \\ -d_1d_2,\underset{44}{0},...,\underset{48}{0},-d_1d_2,\underset{50}{0},\underset{51}{0},d_1d_2,\underset{53}{0} \end{pmatrix} \tag{77}$$

(74) and (77) imply that

$$\begin{pmatrix} \left(\left(\gamma_{2431}\overset{\cdot}{\underset{24}{-}}\gamma_{2413}\right)\overset{\cdot}{\underset{1}{-}}\left(\gamma_{2314}\overset{\cdot}{\underset{24}{-}}\gamma_{2134}\right)\right)\overset{\cdot}{-} \\ \left(\left(\gamma_{4312}\overset{\cdot}{\underset{24}{-}}\gamma_{4132}\right)\overset{\cdot}{\underset{1}{-}}\left(\gamma_{3142}\overset{\cdot}{\underset{24}{-}}\gamma_{1342}\right)\right) \end{pmatrix}(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{32}{0},-d,\underset{34}{0},\underset{35}{0},d,\underset{37}{0},-d,\underset{39}{0},-d,d,\underset{42}{0},d,\underset{44}{0},...,\underset{48}{0},d,\underset{50}{0},\underset{51}{0},-d,\underset{53}{0}\right) \tag{78}$$

7. Since

$$\gamma_{3124}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{21}{0},d_1d_3d_4,\underset{23}{0},...,\underset{26}{0}, \\ d_2d_3d_4,\underset{28}{0},...,\underset{41}{0},d_1d_2d_3d_4,\underset{43}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{3142}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{17}{0},...,\underset{21}{0},\\ d_1d_3d_4,\underset{23}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{42}{0},d_1d_2d_3d_4,\underset{44}{0},...,\underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{3124}\underset{31}{\dot{-}}\gamma_{3142}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\begin{pmatrix} d_2,\underset{2}{0},d_1,\underset{4}{0},...,\underset{8}{0},-d_3,\underset{10}{0},...,\underset{15}{0},-d_2d_3,\underset{17}{0},...,\underset{26}{0},d_1d_3,-d_1d_3,\\ \underset{29}{0},...,\underset{41}{0},d_1d_2d_3,-d_1d_2d_3,\underset{44}{0},...,\underset{53}{0} \end{pmatrix} \tag{79}$$

Since

$$\gamma_{3241}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0},\\ d_1d_3d_4,\underset{24}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{44}{0},d_1d_2d_3d_4,\underset{46}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{3421}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0},d_1d_2d_4,\\ \underset{21}{0},0,d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{46}{0},d_1d_2d_3d_4,\underset{48}{0},...,\underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{3241}\underset{31}{\dot{-}}\gamma_{3421}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\begin{pmatrix} d_2,\underset{2}{0},d_1,\underset{4}{0},...,\underset{8}{0},-d_3,\underset{10}{0},...,\underset{17}{0},d_2d_3,\underset{19}{0},-d_2d_3,\underset{21}{0},...,\underset{26}{0},\\ d_1d_3,-d_1d_3,\underset{29}{0},...,\underset{44}{0},d_1d_2d_3,\underset{46}{0},-d_1d_2d_3,\underset{48}{0},...,\underset{53}{0} \end{pmatrix} \tag{80}$$

(79) and (80) imply that

$$\left(\left(\gamma_{3124}\underset{31}{\dot{-}}\gamma_{3142}\right)\underset{1}{\dot{-}}\left(\gamma_{3241}\underset{31}{\dot{-}}\gamma_{3421}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\begin{pmatrix} \underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{15}{0},-d_2,\underset{17}{0},-d_2,\underset{19}{0},d_2,\underset{21}{0},...,\underset{41}{0},\\ d_1d_2,-d_1d_2,\underset{44}{0},-d_1d_2,\underset{46}{0},d_1d_2,\underset{48}{0},...,\underset{53}{0} \end{pmatrix} \tag{81}$$

Since

$$\gamma_{1243}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{9}{0},\underset{10}{d_3d_4},\underset{11}{0},...,\underset{20}{0},d_1d_3d_4,\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{30}{0},\\ d_1d_2d_3d_4,\underset{32}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{1423}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{20}{0},d_1d_3d_4, \\ \underset{22}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{33}{0},d_1d_2d_3d_4,\underset{35}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{1243}\underset{31}{\dot{-}}\gamma_{1423}\right)\left(d_1,d_2,d_3\right) = \mathfrak{m}\left(\begin{array}{c} d_2,\underset{2}{0},d_1,\underset{4}{0},...,\underset{8}{0},-d_3,\underset{10}{0},...,\underset{15}{0},-d_2d_3,\underset{17}{0},...,\underset{25}{0},d_1d_3, \\ \underset{27}{0},\underset{28}{0},-d_1d_3,\underset{30}{0},d_1d_2d_3,\underset{32}{0},\underset{33}{0},-d_1d_2d_3,\underset{35}{0},...,\underset{53}{0} \end{array}\right) \tag{82}$$

Since

$$\gamma_{2413}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{23}{0}, \\ d_1d_3d_4,0,d_2d_3d_4,\underset{27}{0},...,\underset{39}{0},d_1d_2d_3d_4,\underset{41}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4213}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{19}{0},d_1d_2d_4, \\ \underset{21}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{49}{0},d_1d_2d_3d_4,\underset{51}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{2413}\underset{31}{\dot{-}}\gamma_{4213}\right)\left(d_1,d_2,d_3\right) = \mathfrak{m}\left(\begin{array}{c} d_2,\underset{2}{0},d_1,\underset{4}{0},...,\underset{8}{0},-d_3,\underset{10}{0},...,\underset{17}{0},d_2d_3,\underset{19}{0},-d_2d_3,\underset{21}{0},...,\underset{25}{0},d_1d_3, \\ \underset{27}{0},\underset{28}{0},-d_1d_3,\underset{30}{0},...,\underset{39}{0},d_1d_2d_3,\underset{41}{0},...,\underset{49}{0},-d_1d_2d_3,\underset{51}{0},...,\underset{53}{0} \end{array}\right) \tag{83}$$

(82) and (83) imply that

$$\left(\left(\gamma_{1243}\underset{31}{\dot{-}}\gamma_{1423}\right)\underset{1}{\dot{-}}\left(\gamma_{2413}\underset{31}{\dot{-}}\gamma_{4213}\right)\right)\left(d_1,d_2\right) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{15}{0},-d_2,\underset{17}{0},-d_2,\underset{19}{0},d_2,\underset{21}{0},...,\underset{30}{0},d_1d_2,\underset{32}{0},\underset{33}{0},-d_1d_2, \\ \underset{35}{0},...,\underset{39}{0},-d_1d_2,\underset{41}{0},...,\underset{49}{0},d_1d_2,\underset{51}{0},...,\underset{53}{0} \end{array}\right) \tag{84}$$

(81) and (84) imply that

$$\left(\begin{array}{c}\left(\left(\gamma_{3124}\underset{31}{\dot{-}}\gamma_{3142}\right)\underset{1}{\dot{-}}\left(\gamma_{3241}\underset{31}{\dot{-}}\gamma_{3421}\right)\right)\dot{-}\\ \left(\left(\gamma_{1243}\underset{31}{\dot{-}}\gamma_{1423}\right)\underset{1}{\dot{-}}\left(\gamma_{2413}\underset{31}{\dot{-}}\gamma_{4213}\right)\right)\end{array}\right)(d)$$

$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},-d,\underset{32}{0},\underset{33}{0},d,\underset{35}{0},...,\underset{39}{0},d,\underset{41}{0},d,-d,\underset{44}{0},-d,\underset{46}{0},d,\underset{48}{0},\underset{49}{0},-d,\underset{51}{0},...,\underset{53}{0}\right) \tag{85}$$

8. Since

$$\gamma_{3241}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0},\\ d_1d_3d_4,\underset{24}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{44}{0},d_1d_2d_3d_4,\underset{46}{0},...,\underset{53}{0}\end{array}\right)$$

and

$$\gamma_{3214}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,0,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{18}{0},...,\underset{21}{0},d_1d_3d_4,\\ \underset{23}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{43}{0},d_1d_2d_3d_4,\underset{45}{0},...,\underset{53}{0}\end{array}\right)$$

we have

$$\left(\gamma_{3241}\underset{32}{\dot{-}}\gamma_{3214}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},d_2,d_1,\underset{4}{0},...,\underset{6}{0},d_3,\underset{8}{0},...,\underset{16}{0},-d_2d_3,d_2d_3,\underset{19}{0},...,\underset{21}{0},-d_1d_3,d_1d_3,\\ \underset{24}{0},...,\underset{43}{0},-d_1d_2d_3,d_1d_2d_3,\underset{46}{0},...,\underset{53}{0}\end{array}\right) \tag{86}$$

Since

$$\gamma_{3412}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0},d_1d_2d_4,\\ \underset{20}{0},...,\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{45}{0},d_1d_2d_3d_4,\underset{47}{0},...,\underset{53}{0}\end{array}\right)$$

and

$$\gamma_{3142}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{17}{0},...,\underset{21}{0},d_1d_3d_4,\\ \underset{23}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{42}{0},d_1d_2d_3d_4,\underset{44}{0},...,\underset{53}{0}\end{array}\right)$$

we have

$$\left(\gamma_{3412}\underset{32}{\dot{-}}\gamma_{3142}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},d_2,d_1,\underset{4}{0},...,\underset{6}{0},d_3,\underset{8}{0},...,\underset{15}{0},-d_2d_3,\underset{17}{0},\underset{18}{0},d_2d_3,\underset{20}{0},\underset{21}{0},\\ -d_1d_3,d_1d_3,\underset{24}{0},...,\underset{42}{0},-d_1d_2d_3,\underset{44}{0},\underset{45}{0},d_1d_2d_3,\underset{47}{0},...,\underset{53}{0}\end{array}\right) \tag{87}$$

(86) and (87) imply that

$$\begin{aligned}&\left(\left(\gamma_{3241}\overset{\cdot}{\underset{32}{-}}\gamma_{3214}\right)\overset{\cdot}{\underset{1}{-}}\left(\gamma_{3412}\overset{\cdot}{\underset{32}{-}}\gamma_{3142}\right)\right)(d_1,d_2)\\&=\mathfrak{m}\left(\underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{15}{0},d_2,-d_2,d_2,-d_2,\underset{20}{0},...,\underset{42}{0},d_1d_2,-d_1d_2,d_1d_2,-d_1d_2,\underset{47}{0},...,\underset{53}{0}\right)\end{aligned}\tag{88}$$

Since

$$\begin{aligned}&\gamma_{2413}(d_1,d_2,d_3,d_4)\\&=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{23}{0},\\d_1d_3d_4,0,d_2d_3d_4,\underset{27}{0},...,\underset{39}{0},d_1d_2d_3d_4,\underset{41}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

and

$$\begin{aligned}&\gamma_{2143}(d_1,d_2,d_3,d_4)\\&=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3d_4,0,d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,0,0,0,d_1d_3d_4,\\\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{36}{0},d_1d_2d_3d_4,\underset{38}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

we have

$$\begin{aligned}&\left(\gamma_{2413}\overset{\cdot}{\underset{32}{-}}\gamma_{2143}\right)(d_1,d_2,d_3)\\&=\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},d_2,d_1,\underset{4}{0},...,\underset{6}{0},d_3,\underset{8}{0},...,\underset{16}{0},-d_2d_3,d_2d_3,\underset{19}{0},\underset{20}{0},-d_1d_3,\\\underset{22}{0},\underset{23}{0},d_1d_3,\underset{25}{0},...,\underset{36}{0},-d_1d_2d_3,\underset{38}{0},\underset{39}{0},d_1d_2d_3,\underset{41}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}\tag{89}$$

Since

$$\begin{aligned}&\gamma_{4123}(d_1,d_2,d_3,d_4)\\&=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0},d_1d_3d_4,\\\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{47}{0},d_1d_2d_3d_4,\underset{49}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

and

$$\begin{aligned}&\gamma_{1423}(d_1,d_2,d_3,d_4)\\&=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{20}{0},d_1d_3d_4,\\\underset{22}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{33}{0},d_1d_2d_3d_4,\underset{35}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

we have

$$\begin{aligned}&\left(\gamma_{4123}\overset{\cdot}{\underset{32}{-}}\gamma_{1423}\right)(d_1,d_2,d_3)\\&=\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},d_2,d_1,\underset{4}{0},...,\underset{6}{0},d_3,\underset{8}{0},...,\underset{15}{0},-d_2d_3,\underset{17}{0},\underset{18}{0},d_2d_3,\underset{20}{0},-d_1d_3,\\\underset{22}{0},\underset{23}{0},d_1d_3,\underset{25}{0},...,\underset{33}{0},-d_1d_2d_3,\underset{35}{0},...,\underset{47}{0},d_1d_2d_3,\underset{49}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}\tag{90}$$

(**??**) and (90) imply that

$$\left(\left(\gamma_{2413}\underset{32}{\dot{-}}\gamma_{2143}\right)\underset{1}{\dot{-}}\left(\gamma_{4123}\underset{32}{\dot{-}}\gamma_{1423}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{15}{0},d_2,-d_2,d_2,-d_2,\underset{20}{0},...,\underset{33}{0},d_1d_2,\underset{35}{0},\underset{36}{0},-d_1d_2,\\ \underset{38}{0},\underset{39}{0},d_1d_2,\underset{41}{0},...,\underset{47}{0},-d_1d_2,\underset{49}{0},...,\underset{53}{0}\end{array}\right) \tag{91}$$

(88) and (91) imply that

$$\left(\begin{array}{c}\left(\left(\gamma_{3241}\underset{32}{\dot{-}}\gamma_{3214}\right)\underset{1}{\dot{-}}\left(\gamma_{3412}\underset{32}{\dot{-}}\gamma_{3142}\right)\right)\dot{-}\\ \left(\left(\gamma_{2413}\underset{32}{\dot{-}}\gamma_{2143}\right)\underset{1}{\dot{-}}\left(\gamma_{4123}\underset{32}{\dot{-}}\gamma_{1423}\right)\right)\end{array}\right)(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{33}{0},-d,\underset{35}{0},\underset{36}{0},d,\underset{38}{0},\underset{39}{0},-d,\underset{41}{0},\underset{42}{0},d,-d,d,-d,\underset{47}{0},d,\underset{49}{0},...,\underset{53}{0}\right) \tag{92}$$

9. Since

$$\gamma_{3412}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0},d_1d_2d_4,\\ \underset{20}{0},...,\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{45}{0},d_1d_2d_3d_4,\underset{47}{0},...,\underset{53}{0}\end{array}\right)$$

and

$$\gamma_{3421}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0},d_1d_2d_4,\\ \underset{21}{0},\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{46}{0},d_1d_2d_3d_4,\underset{48}{0},...,\underset{53}{0}\end{array}\right)$$

we have

$$\left(\gamma_{3412}\underset{34}{\dot{-}}\gamma_{3421}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},\underset{2}{0},d_1,d_2,-d_3,\underset{6}{0},...,\underset{13}{0},d_1d_3,-d_1d_3,\underset{16}{0},...,\underset{18}{0},d_2d_3,-d_2d_3,\\ \underset{21}{0},...,\underset{45}{0},d_1d_2d_3,-d_1d_2d_3,\underset{48}{0},...,\underset{53}{0}\end{array}\right) \tag{93}$$

Since

$$\gamma_{3124}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{21}{0},d_1d_3d_4,\underset{23}{0},...,\underset{26}{0},\\ d_2d_3d_4,\underset{28}{0},...,\underset{41}{0},d_1d_2d_3d_4,\underset{43}{0},...,\underset{53}{0}\end{array}\right)$$

and

$$\gamma_{3214}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{18}{0},...,\underset{21}{0},d_1d_3d_4, \\ \underset{23}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{43}{0},d_1d_2d_3d_4,\underset{45}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{3124}\underset{34}{\dot{-}}\gamma_{3214}\right)\left(d_1,d_2,d_3\right) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,d_2,-d_3,\underset{6}{0},...,\underset{13}{0},d_1d_3,-d_1d_3,\underset{16}{0},-d_2d_3,\underset{18}{0},...,\underset{41}{0}, \\ d_1d_2d_3,\underset{43}{0},-d_1d_2d_3,\underset{45}{0},...,\underset{53}{0} \end{array}\right) \quad (94)$$

(93) and (94) imply that

$$\left(\left(\gamma_{3412}\underset{34}{\dot{-}}\gamma_{3421}\right)\underset{1}{\dot{-}}\left(\gamma_{3124}\underset{34}{\dot{-}}\gamma_{3214}\right)\right)\left(d_1,d_2\right) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{16}{0},d_2,\underset{18}{0},d_2,-d_2,\underset{21}{0},...\underset{41}{0},-d_1d_2,\underset{43}{0},d_1d_2, \\ \underset{45}{0},d_1d_2,-d_1d_2,\underset{48}{0},...,\underset{53}{0} \end{array}\right) \quad (95)$$

Since

$$\gamma_{4123}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0},d_1d_3d_4, \\ \underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{47}{0},d_1d_2d_3d_4,\underset{49}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4213}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,0,d_1d_4,0,d_2d_4,d_3d_4,0,d_1d_2d_3,\underset{13}{0},...,\underset{19}{0},d_1d_2d_4, \\ \underset{21}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{49}{0},d_1d_2d_3d_4,\underset{51}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{4123}\underset{34}{\dot{-}}\gamma_{4213}\right)\left(d_1,d_2,d_3\right) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,d_2,-d_3,\underset{6}{0},...,\underset{11}{0},-d_1d_3,\underset{13}{0},...,\underset{18}{0},d_2d_3,-d_2d_3, \\ \underset{21}{0},...,\underset{47}{0},d_1d_2d_3,\underset{49}{0},-d_1d_2d_3,\underset{51}{0},...,\underset{53}{0} \end{array}\right) \quad (96)$$

Since

$$\gamma_{1243}\left(d_1,d_2,d_3,d_4\right) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{9}{0},\underset{10}{d_3d_4},\underset{11}{0},...,\underset{20}{0},d_1d_3d_4,\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{30}{0}, \\ d_1d_2d_3d_4,\underset{32}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{2143}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3d_4,0,d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{20}{0},d_1d_3d_4, \\ \underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{36}{0},d_1d_2d_3d_4,\underset{38}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{1243}\underset{34}{\dot{-}}\gamma_{2143}\right)(d_1,d_2,d_3) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,d_2,-d_3,\underset{6}{0},...,\underset{11}{0},-d_1d_3,\underset{13}{0},...,\underset{16}{0},-d_2d_3, \\ \underset{18}{0},...,\underset{30}{0},d_1d_2d_3,\underset{32}{0},...,\underset{36}{0},-d_1d_2d_3,\underset{38}{0},...,\underset{53}{0} \end{array}\right) \tag{97}$$

(96) and (97) imply that

$$\left(\left(\gamma_{4123}\underset{34}{\dot{-}}\gamma_{4213}\right)\underset{1}{\dot{-}}\left(\gamma_{1243}\underset{34}{\dot{-}}\gamma_{2143}\right)\right)(d_1,d_2) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{16}{0},d_2,\underset{18}{0},d_2,-d_2,\underset{21}{0},...,\underset{30}{0},-d_1d_2,\underset{32}{0},...,\underset{36}{0}, \\ d_1d_2,\underset{38}{0},...,\underset{47}{0},d_1d_2,\underset{49}{0},-d_1d_2,\underset{51}{0},...,\underset{53}{0} \end{array}\right) \tag{98}$$

(95) and (98) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{3412}\underset{34}{\dot{-}}\gamma_{3421}\right)\underset{1}{\dot{-}}\left(\gamma_{3124}\underset{34}{\dot{-}}\gamma_{3214}\right)\right)\dot{-} \\ \left(\left(\gamma_{4123}\underset{34}{\dot{-}}\gamma_{4213}\right)\underset{1}{\dot{-}}\left(\gamma_{1243}\underset{34}{\dot{-}}\gamma_{2143}\right)\right) \end{array}\right)(d) = \mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},d,\underset{32}{0},...,\underset{36}{0},-d,\underset{38}{0},...,\underset{41}{0},-d,\underset{43}{0},d,\underset{45}{0},d,-d,-d,\underset{49}{0},d,\underset{51}{0},...,\underset{53}{0}\right) \tag{99}$$

10. Since

$$\gamma_{4132}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0}, \\ d_1d_3d_4,\underset{25}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{48}{0},d_1d_2d_3d_4,\underset{50}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4123}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0},d_1d_3d_4, \\ \underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{47}{0},d_1d_2d_3d_4,\underset{49}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{4132} \underset{41}{\dot{-}} \gamma_{4123}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{7}{0}, d_3, \underset{9}{0}, ..., \underset{10}{0}, d_2d_3, \underset{12}{0}, ..., \underset{28}{0}, -d_1d_3, d_1d_3, \\ \underset{31}{0}, ..., \underset{47}{0}, -d_1d_2d_3, d_1d_2d_3, \underset{50}{0}, ..., \underset{53}{0} \end{pmatrix} \tag{100}$$

Since

$$\gamma_{4321}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{52}{0}, d_1d_2d_3d_4 \end{pmatrix}$$

and

$$\gamma_{4231}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, 0, d_2d_4, d_3d_4, 0, 0, d_1d_2d_3, \underset{14}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{50}{0}, d_1d_2d_3d_4, 0, \underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{4321} \underset{41}{\dot{-}} \gamma_{4231}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{7}{0}, d_3, \underset{9}{0}, ..., \underset{12}{0}, -d_2d_3, 0, d_2d_3, \underset{16}{0}, ..., \underset{28}{0}, \\ -d_1d_3, d_1d_3, \underset{31}{0}, ..., \underset{50}{0}, -d_1d_2d_3, \underset{52}{0}, d_1d_2d_3 \end{pmatrix} \tag{101}$$

(100) and (**??**) imply that

$$\left(\left(\gamma_{4132} \underset{41}{\dot{-}} \gamma_{4123}\right) \underset{1}{\dot{-}} \left(\gamma_{4321} \underset{41}{\dot{-}} \gamma_{4231}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(\underset{1}{0}, ..., \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{10}{0}, d_2, \underset{12}{0}, d_2, \underset{14}{0}, -d_2, \underset{16}{0}, ..., \underset{47}{0}, -d_1d_2, d_1d_2, \underset{50}{0}, d_1d_2, \underset{52}{0}, -d_1d_2\right) \tag{102}$$

Since

$$\gamma_{1324}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, \underset{7}{0}, d_2d_3, \underset{9}{0}, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{31}{0}, \\ d_1d_2d_3d_4, \underset{33}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{1234}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{53}{0}\right)$$

we have

$$\left(\gamma_{1324} \underset{41}{\dot{-}} \gamma_{1234}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{7}{0}, d_3, \underset{9}{0}, \underset{10}{0}, d_2d_3, \underset{12}{0}, ..., \underset{26}{0}, d_1d_3, \underset{28}{0}, ..., \underset{31}{0}, d_1d_2d_3, \underset{33}{0}, ..., \underset{53}{0}\right) \tag{103}$$

Since

$$\gamma_{3214}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, \underset{7}{0}, d_2d_3, \underset{9}{0}, ..., \underset{14}{0}, d_1d_2d_3, 0, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{43}{0}, d_1d_2d_3d_4, \underset{45}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{2314}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, \underset{7}{0}, ..., \underset{12}{0}, d_1d_2d_3, 0, 0, 0, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, ..., \underset{37}{0}, d_1d_2d_3d_4, \underset{39}{0}, ..., \underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{3214} \underset{41}{\dot{-}} \gamma_{2314}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{7}{0}, d_3, \underset{9}{0}, ..., \underset{12}{0}, -d_2d_3, \underset{14}{0}, d_2d_3, \underset{16}{0}, ..., \underset{26}{0}, d_1d_3, \\ \underset{28}{0}, ..., \underset{37}{0}, -d_1d_2d_3, \underset{39}{0}, ..., \underset{43}{0}, d_1d_2d_3, \underset{45}{0}, ..., \underset{53}{0} \end{array}\right) \tag{104}$$

(103) and (104) imply that

$$\left(\left(\gamma_{1324} \underset{41}{\dot{-}} \gamma_{1234}\right) \underset{1}{\dot{-}} \left(\gamma_{3214} \underset{41}{\dot{-}} \gamma_{2314}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, ..., \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{10}{0}, d_2, \underset{12}{0}, d_2, \underset{14}{0}, -d_2, \underset{16}{0}, ..., \underset{31}{0}, d_1d_2, \\ \underset{33}{0}, ..., \underset{37}{0}, d_1d_2, \underset{39}{0}, ..., \underset{43}{0}, -d_1d_2, \underset{45}{0}, ..., \underset{53}{0} \end{array}\right) \tag{105}$$

(102) and (105) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{4132} \underset{41}{\dot{-}} \gamma_{4123}\right) \underset{1}{\dot{-}} \left(\gamma_{4321} \underset{41}{\dot{-}} \gamma_{4231}\right)\right) \dot{-} \\ \left(\left(\gamma_{1324} \underset{41}{\dot{-}} \gamma_{1234}\right) \underset{1}{\dot{-}} \left(\gamma_{3214} \underset{41}{\dot{-}} \gamma_{2314}\right)\right) \end{array}\right)(d)$$
$$= \mathfrak{m}\left(\underset{1}{0}, ..., \underset{31}{0}, -d, \underset{33}{0}, ..., \underset{37}{0}, -d, \underset{39}{0}, ..., \underset{43}{0}, d, \underset{45}{0}, ..., \underset{47}{0}, -d, d, \underset{50}{0}, d, \underset{52}{0}, \underset{53}{-d}\right) \tag{106}$$

11. Since

$\gamma_{4213}(d_1,d_2,d_3,d_4)$

$$= \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{19}{0},d_1d_2d_4, \\ \underset{21}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{49}{0},d_1d_2d_3d_4,\underset{51}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$\gamma_{4231}(d_1,d_2,d_3,d_4)$

$$= \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{19}{0}, \\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{50}{0},d_1d_2d_3d_4,\underset{52}{0},\underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{4213}\underset{42}{\dot{-}}\gamma_{4231}\right)(d_1,d_2,d_3)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0},d_2,\underset{3}{0},d_1,\underset{5}{0},-d_3,\underset{7}{0},...,\underset{11}{0},d_2d_3,-d_2d_3,\underset{14}{0},...,\underset{23}{0}, \\ d_1d_3,-d_1d_3,\underset{26}{0},...,\underset{49}{0},d_1d_2d_3,\underset{51}{-d_1d_2d_3},0,0 \end{pmatrix} \tag{107}$$

Since

$\gamma_{4132}(d_1,d_2,d_3,d_4)$

$$= \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0}, \\ d_1d_3d_4,\underset{25}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{48}{0},d_1d_2d_3d_4,\underset{50}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$\gamma_{4312}(d_1,d_2,d_3,d_4)$

$$= \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,0,d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0}, \\ d_1d_2d_4,\underset{20}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{51}{0},d_1d_2d_3d_4,\underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{4132}\underset{42}{\dot{-}}\gamma_{4312}\right)(d_1,d_2,d_3)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0},d_2,\underset{3}{0},d_1,\underset{5}{0},-d_3,\underset{7}{0},...,\underset{10}{0},d_2d_3,\underset{12}{0},\underset{13}{0},-d_2d_3,\underset{15}{0},...,\underset{23}{0},d_1d_3,-d_1d_3, \\ \underset{26}{0},...,\underset{48}{0},d_1d_2d_3,\underset{50}{0},\underset{51}{0},-d_1d_2d_3,\underset{53}{0} \end{pmatrix} \tag{108}$$

(107) and (108) imply that

$$\left(\left(\gamma_{4213}\underset{42}{\dot{-}}\gamma_{4231}\right)\underset{1}{\dot{-}}\left(\gamma_{4132}\underset{42}{\dot{-}}\gamma_{4312}\right)\right)(d_1,d_2)$$
$$= \mathfrak{m}\left(\underset{1}{0},...,\underset{3}{0},d_1,\underset{5}{0},...,\underset{10}{0},-d_2,d_2,-d_2,d_2,\underset{15}{0},...,\underset{48}{0},-d_1d_2,d_1d_2,-d_1d_2,d_1d_2,\underset{53}{0}\right) \tag{109}$$

Since

$$\gamma_{2134}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{35}{0},d_1d_2d_3d_4,\underset{37}{0},...,\underset{53}{0}\right)$$

and

$$\gamma_{2314}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,\underset{7}{0},...,\underset{12}{0},d_1d_2d_3,0,0,0,d_1d_2d_4,\underset{18}{0},...,\underset{21}{0},d_1d_3d_4, \\ \underset{23}{0},...,\underset{37}{0},d_1d_2d_3d_4,\underset{39}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{2134}\underset{42}{\dot{-}}\gamma_{2314}\right)\left(d_1,d_2,d_3\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_2,\underset{3}{0},d_1,\underset{5}{0},-d_3,\underset{7}{0},...,\underset{11}{0},d_2d_3,-d_2d_3,\underset{14}{0},...,\underset{21}{0},-d_1d_3, \\ \underset{23}{0},...,\underset{35}{0},d_1d_2d_3,0,-d_1d_2d_3,\underset{39}{0},...,\underset{53}{0} \end{array}\right) \quad (110)$$

Since

$$\gamma_{1324}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},\underset{7}{0},d_2d_3,\underset{9}{0},\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{26}{0},d_2d_3d_4,0,0,\underset{30}{0}, \\ 0,d_1d_2d_3d_4,\underset{33}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3124}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{21}{0},d_1d_3d_4,\underset{23}{0},...,\underset{26}{0}, \\ d_2d_3d_4,\underset{28}{0},...,\underset{41}{0},d_1d_2d_3d_4,\underset{43}{0},...,\underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{1324}\underset{42}{\dot{-}}\gamma_{3124}\right)\left(d_1,d_2,d_3\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_2,\underset{3}{0},d_1,\underset{5}{0},-d_3,\underset{7}{0},...,\underset{10}{0},d_2d_3,0,0,-d_2d_3,\underset{15}{0},...,\underset{21}{0}, \\ -d_1d_3,\underset{23}{0},...,\underset{31}{0},d_1d_2d_3,\underset{33}{0},...,\underset{41}{0},-d_1d_2d_3,\underset{43}{0},...,\underset{53}{0} \end{array}\right) \quad (111)$$

(110) and (111) imply that

$$\left(\left(\gamma_{2134}\underset{42}{\dot{-}}\gamma_{2314}\right)\underset{1}{\dot{-}}\left(\gamma_{1324}\underset{42}{\dot{-}}\gamma_{3124}\right)\right)\left(d_1,d_2\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},...,\underset{3}{0},d_1,\underset{5}{0},...,\underset{10}{0},-d_2,d_2,-d_2,d_2,\underset{15}{0},...,\underset{31}{0},-d_1d_2,\underset{33}{0},...,\underset{35}{0}, \\ d_1d_2,0,-d_1d_2,\underset{39}{0},...,\underset{41}{0},d_1d_2,\underset{43}{0},...,\underset{53}{0} \end{array}\right) \quad (112)$$

(109) and (112) imply that

$$\begin{pmatrix} \left(\left(\gamma_{4213} \underset{42}{\dot{-}} \gamma_{4231}\right) \underset{1}{\dot{-}} \left(\gamma_{4132} \underset{42}{\dot{-}} \gamma_{4312}\right)\right) \dot{-} \\ \left(\left(\gamma_{2134} \underset{42}{\dot{-}} \gamma_{2314}\right) \underset{1}{\dot{-}} \left(\gamma_{1324} \underset{42}{\dot{-}} \gamma_{3124}\right)\right) \end{pmatrix} (d)$$
$$= \mathfrak{m}\left(\underset{1}{0}, ..., \underset{31}{0}, d, \underset{33}{0}, ..., \underset{35}{0}, -d, 0, d, \underset{39}{0}, ..., \underset{41}{0}, -d, \underset{43}{0}, ..., \underset{48}{0}, -d, d, -d, d, \underset{53}{0}\right) \tag{113}$$

12. Since

$$\gamma_{4321}\left(d_1, d_2, d_3, d_4\right)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{52}{0}, d_1d_2d_3d_4 \end{pmatrix}$$

and

$$\gamma_{4312}\left(d_1, d_2, d_3, d_4\right)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, \\ d_1d_2d_4, \underset{20}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{51}{0}, d_1d_2d_3d_4, \underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{4321} \underset{43}{\dot{-}} \gamma_{4312}\right)\left(d_1, d_2, d_3\right)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, \underset{2}{0}, d_2, d_1, d_3, \underset{6}{0}, ..., \underset{13}{0}, -d_2d_3, d_2d_3, \underset{16}{0}, ..., \underset{18}{0}, -d_1d_3, d_1d_3, \\ \underset{21}{0}, ..., \underset{51}{0}, -d_1d_2d_3, d_1d_2d_3 \end{pmatrix} \tag{114}$$

Since

$$\gamma_{4213}\left(d_1, d_2, d_3, d_4\right)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, d_1d_4, \underset{8}{0}, d_2d_4, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{19}{0}, d_1d_2d_4, \\ \underset{21}{0}, ..., \underset{23}{0}, d_1d_3d_4, \underset{25}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{49}{0}, d_1d_2d_3d_4, \underset{51}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{4123}\left(d_1, d_2, d_3, d_4\right)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, 0, 0, d_1d_4, 0, 0, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{18}{0}, d_1d_2d_4, \underset{20}{0}, ..., \underset{23}{0}, d_1d_3d_4, \\ \underset{25}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{47}{0}, d_1d_2d_3d_4, \underset{49}{0}, ..., \underset{53}{0} \end{pmatrix}$$

we have

$$\left(\gamma_{4213} \underset{43}{\dot{-}} \gamma_{4123}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, \underset{2}{0}, d_2, d_1, d_3, \underset{6}{0}, ..., \underset{11}{0}, d_2d_3, \underset{13}{0}, ..., \underset{18}{0}, -d_1d_3, d_1d_3, \underset{21}{0}, ..., \underset{47}{0}, -d_1d_2d_3, \underset{49}{0}, \\ d_1d_2d_3, \underset{51}{0}, ..., \underset{53}{0} \end{array}\right) \tag{115}$$

(114) and (115) imply that

$$\left(\left(\gamma_{4321} \underset{43}{\dot{-}} \gamma_{4312}\right) \underset{1}{\dot{-}} \left(\gamma_{4213} \underset{43}{\dot{-}} \gamma_{4123}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(\underset{1}{0}, ..., \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{11}{0}, -d_2, 0, -d_2, d_2, \underset{16}{0}, ..., \underset{47}{0}, d_1d_2, \underset{49}{0}, -d_1d_2, \underset{51}{0}, -d_1d_2, d_1d_2\right) \tag{116}$$

Since

$$\gamma_{3214}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, 0, d_2d_3, \underset{9}{0}, ..., \underset{14}{0}, d_1d_2d_3, 0, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{43}{0}, d_1d_2d_3d_4, \underset{45}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3124}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, \underset{7}{0}, d_2d_3, \underset{9}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{21}{0}, d_1d_3d_4, \underset{23}{0}, ..., \underset{26}{0}, \\ d_2d_3d_4, \underset{28}{0}, ..., \underset{41}{0}, d_1d_2d_3d_4, \underset{43}{0}, ..., \underset{53}{0} \end{array}\right)$$

we have

$$\left(\gamma_{3214} \underset{43}{\dot{-}} \gamma_{3124}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, \underset{2}{0}, d_2, d_1, d_3, \underset{6}{0}, ..., \underset{13}{0}, -d_2d_3, d_2d_3, 0, d_1d_3, \\ \underset{18}{0}, ..., \underset{41}{0}, -d_1d_2d_3, 0, d_1d_2d_3, \underset{45}{0}, ..., \underset{53}{0} \end{array}\right) \tag{117}$$

Since

$$\gamma_{2134}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, ..., \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{16}{0}, d_1d_2d_4, \underset{18}{0}, ..., \underset{35}{0}, d_1d_2d_3d_4, \underset{37}{0}, ..., \underset{53}{0}\right)$$

and

$$\gamma_{1234}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{53}{0}\right)$$

we have

$$\left(\gamma_{2134}\underset{43}{\dot{-}}\gamma_{1234}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\underset{1}{0},\underset{2}{0},d_2,d_1,d_3,\underset{6}{0},...,\underset{11}{0},d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_3,\underset{18}{0},...,\underset{35}{0},d_1d_2d_3,\underset{37}{0},...,\underset{53}{0}\right) \tag{118}$$

(117) and (118) imply that

$$\left(\left(\gamma_{3214}\underset{43}{\dot{-}}\gamma_{3124}\right)\underset{1}{\dot{-}}\left(\gamma_{2134}\underset{43}{\dot{-}}\gamma_{1234}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},...,\underset{3}{0},d_1,\underset{5}{0},...,\underset{11}{0},-d_2,0,-d_2,d_2,\underset{16}{0},...,\underset{35}{0},-d_1d_2,\\ \underset{37}{0},...,\underset{41}{0},-d_1d_2,\underset{43}{0},d_1d_2,\underset{45}{0},...,\underset{53}{0}\end{array}\right) \tag{119}$$

(116) and (119) imply that

$$\left(\begin{array}{c}\left(\left(\gamma_{4321}\underset{43}{\dot{-}}\gamma_{4312}\right)\underset{1}{\dot{-}}\left(\gamma_{4213}\underset{43}{\dot{-}}\gamma_{4123}\right)\right)\dot{-}\\ \left(\left(\gamma_{3214}\underset{43}{\dot{-}}\gamma_{3124}\right)\underset{1}{\dot{-}}\left(\gamma_{2134}\underset{43}{\dot{-}}\gamma_{1234}\right)\right)\end{array}\right)(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{35}{0},d,\underset{37}{0},...,\underset{41}{0},d,0,-d,\underset{45}{0},...,\underset{47}{0},d,\underset{49}{0},-d,\underset{51}{0},-d,d\right) \tag{120}$$

13. By (**??**), (**??**), (**??**), (**??**), (71), (78), (85), (92), (99), (106), (113) and

(120), we have

the left-hand side of the equation (36)

$$
\begin{aligned}
&= \Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},-d,\underset{32}{0},-d,\underset{34}{0},d,\underset{36}{0},...,\underset{38}{0},-d,0,d,\underset{42}{0},...,\underset{46}{0},d,\underset{48}{0},...,\underset{52}{0},-d\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},d,-d,d,-d,\underset{35}{0},...,\underset{40}{0},-d,\underset{42}{0},...\underset{44}{0},d,\underset{46}{0},-d,\underset{48}{0},...,\underset{50}{0},d,\underset{52}{0},\underset{53}{0}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},d,\underset{33}{0},d,-d,\underset{36}{0},...,\underset{38}{0},d,\underset{40}{0},...,\underset{44}{0},-d,\underset{46}{0},...,\underset{50}{0},-d,\underset{52}{0},d\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{32}{0},d,\underset{34}{0},-d,-d,d,\underset{38}{0},d,\underset{40}{0},-d,\underset{42}{0},...,\underset{45}{0},-d,\underset{47}{0},...,\underset{51}{0},d,\underset{53}{0}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{34}{0},d,\underset{36}{0},-d,d,-d,d,\underset{41}{0},\underset{42}{0},-d,\underset{44}{0},\underset{45}{0},d,\underset{47}{0},\underset{48}{0},-d,\underset{50}{0},...,\underset{53}{0}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{32}{0},-d,\underset{34}{0},\underset{35}{0},d,\underset{37}{0},-d,\underset{39}{0},-d,d,\underset{42}{0},d,\underset{44}{0},...,\underset{48}{0},d,\underset{50}{0},\underset{51}{0},-d,\underset{53}{0}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},-d,\underset{32}{0},\underset{33}{0},d,\underset{35}{0},...,\underset{39}{0},d,\underset{41}{0},d,-d,\underset{44}{0},-d,\underset{46}{0},d,\underset{48}{0},\underset{49}{0},-d,\underset{51}{0},...,\underset{53}{0}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},-d,\underset{32}{0},\underset{33}{0},d,\underset{35}{0},...,\underset{39}{0},d,\underset{41}{0},d,-d,\underset{44}{0},-d,\underset{46}{0},d,\underset{48}{0},\underset{49}{0},-d,\underset{51}{0},...,\underset{53}{0}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{33}{0},-d,\underset{35}{0},\underset{36}{0},d,\underset{38}{0},\underset{39}{0},-d,\underset{41}{0},\underset{42}{0},d,-d,d,-d,\underset{47}{0},d,\underset{49}{0},...,\underset{53}{0}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},d,\underset{32}{0},...,\underset{36}{0},-d,\underset{38}{0},...,\underset{41}{0},-d,\underset{43}{0},d,\underset{45}{0},d,-d,-d,\underset{49}{0},d,\underset{51}{0},...,\underset{53}{0}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},-d,\underset{33}{0},...,\underset{37}{0},-d,\underset{39}{0},...,\underset{43}{0},d,\underset{45}{0},...,\underset{47}{0},-d,d,\underset{50}{0},d,0,\underset{53}{-d}\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},d,\underset{33}{0},...,\underset{35}{0},-d,0,d,\underset{39}{0},...,\underset{41}{0},-d,\underset{43}{0},...,\underset{48}{0},-d,d,-d,d,0\right)\Big)+\\
&\Big(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{35}{0},d,\underset{37}{0},...,\underset{41}{0},d,0,-d,\underset{45}{0},...,\underset{47}{0},d,0,-d,0,-d,d\right)\Big)\\
&= \lambda d.\mathfrak{m}\begin{pmatrix}
\underset{1}{0},...,\underset{30}{0},-d+\underset{31}{d}-d+d,-d+\underset{32}{d}-d+d,-d+\underset{33}{d}+d-d,\\
-d+\underset{34}{d}+d-d,d-\underset{35}{d}-d+d,-d+\underset{36}{d}-d+d,\\
d-\underset{37}{d}+d-d,d-\underset{38}{d}-d+d,-d+\underset{39}{d}+d-d,\\
d-\underset{40}{d}+d-d,d-\underset{41}{d}-d+d,d-\underset{42}{d}-d+d,\\
-d+\underset{43}{d}-d+d,-d+\underset{44}{d}+d-d,d-\underset{45}{d}-d+d,\\
-d+\underset{46}{d}-d+d,d-\underset{47}{d}+d-d,d-\underset{48}{d}-d+d,\\
-d+\underset{49}{d}+d-d,-d+\underset{50}{d}+d-d,d-\underset{51}{d}+d-d,\\
d-\underset{52}{d}+d-d,-d+\underset{53}{d}-d+d
\end{pmatrix}\\
&= \lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{53}{0}\right) \qquad (121)
\end{aligned}
$$

■

Remark 21 *For our convenience we display the positions* $31-53$ *in (43), (50),* *(57), (64), (71), (78), (85), (92), (99), (106), (113) and (120) as a table:*

	1/12	2/13	3/14	4/21	5/23	6/24	7/31	8/32	9/34	10/41	11/42	12/43
31/1243	$-d$	d					$-d$		d			
32/1324		$-d$	d							$-d$	d	
33/1342	$-d$	d		d		$-d$						
34/1423		$-d$	d				d	$-d$				
35/1432	d		$-d$	$-d$	d							
36/2134				$-d$		d					$-d$	d
37/2143				d	$-d$			d	$-d$			
38/2314					d	$-d$				$-d$	d	
39/2341	$-d$		d	d	$-d$							
40/2413					d	$-d$	d	$-d$				
41/2431	d	$-d$		$-d$		d						
42/3124							d		$-d$		$-d$	d
43/3142					$-d$	d	$-d$	d				
44/3214								$-d$	d	d		$-d$
45/3241		d	$-d$				$-d$	d				
46/3412				$-d$	d			$-d$	d			
47/3421	d	$-d$					d		$-d$			
48/4123								d	$-d$	$-d$		d
49/4132					$-d$	d				d	$-d$	
50/4213							$-d$		d		d	$-d$
51/4231		d	$-d$							d	$-d$	
52/4312				d		$-d$					d	$-d$
53/4321	$-d$		d							$-d$		d

(122)

Corollary 22 *Let M be a microlinear space with*

$$X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$$

Then we have

$$\begin{aligned}&[X_1,[X_2,[X_3,X_4]]]+[X_1,[X_3,[X_4,X_2]]]+[X_1,[X_4,[X_2,X_3]]]+\\&[X_2,[X_1,[X_4,X_3]]]+[X_2,[X_3,[X_1,X_4]]]+[X_2,[X_4,[X_3,X_1]]]+\\&[X_3,[X_1,[X_2,X_4]]]+[X_3,[X_2,[X_4,X_1]]]+[X_3,[X_4,[X_1,X_2]]]+\\&[X_4,[X_1,[X_3,X_2]]]+[X_4,[X_2,[X_1,X_3]]]+[X_4,[X_3,[X_2,X_1]]]\\&=0\end{aligned}$$

Proof. Let

$$\begin{aligned}
&\gamma_{1234} = X_4 * X_3 * X_2 * X_1, \gamma_{1243} = (X_3 * X_4 * X_2 * X_1)^{\sigma_{1243}},\\
&\gamma_{1324} = (X_4 * X_2 * X_3 * X_1)^{\sigma_{1324}}, \gamma_{1342} = (X_2 * X_4 * X_3 * X_1)^{\sigma_{1342}},\\
&\gamma_{1423} = (X_3 * X_2 * X_4 * X_1)^{\sigma_{1423}}, \gamma_{1432} = (X_2 * X_3 * X_4 * X_1)^{\sigma_{1432}},\\
&\gamma_{2134} = (X_4 * X_3 * X_1 * X_2)^{\sigma_{2134}}, \gamma_{2143} = (X_3 * X_4 * X_1 * X_2)^{\sigma_{2143}},\\
&\gamma_{2314} = (X_4 * X_1 * X_3 * X_2)^{\sigma_{2314}}, \gamma_{2341} = (X_1 * X_4 * X_3 * X_2)^{\sigma_{2341}},\\
&\gamma_{2413} = (X_3 * X_1 * X_4 * X_2)^{\sigma_{2413}}, \gamma_{2431} = (X_1 * X_3 * X_4 * X_2)^{\sigma_{2431}},\\
&\gamma_{3124} = (X_4 * X_2 * X_1 * X_3)^{\sigma_{3124}}, \gamma_{3142} = (X_2 * X_4 * X_1 * X_3)^{\sigma_{3142}},\\
&\gamma_{3214} = (X_4 * X_1 * X_2 * X_3)^{\sigma_{3214}}, \gamma_{3241} = (X_1 * X_4 * X_2 * X_3)^{\sigma_{3241}},\\
&\gamma_{3412} = (X_2 * X_1 * X_4 * X_3)^{\sigma_{3412}}, \gamma_{3421} = (X_1 * X_2 * X_4 * X_3)^{\sigma_{3421}},\\
&\gamma_{4123} = (X_3 * X_2 * X_1 * X_4)^{\sigma_{4123}}, \gamma_{4132} = (X_2 * X_3 * X_1 * X_4)^{\sigma_{4132}},\\
&\gamma_{4213} = (X_3 * X_1 * X_2 * X_4)^{\sigma_{4213}}, \gamma_{4231} = (X_1 * X_3 * X_2 * X_4)^{\sigma_{4231}},\\
&\gamma_{4312} = (X_2 * X_1 * X_3 * X_4)^{\sigma_{4312}}, \gamma_{4321} = (X_1 * X_2 * X_3 * X_4)^{\sigma_{4321}}
\end{aligned}$$

with

$$\begin{aligned}
&\sigma_{1243} = \begin{pmatrix} 1234 \\ 1243 \end{pmatrix}, \sigma_{1324} = \begin{pmatrix} 1234 \\ 1324 \end{pmatrix}, \sigma_{1342} = \begin{pmatrix} 1234 \\ 1423 \end{pmatrix}, \sigma_{1423} = \begin{pmatrix} 1234 \\ 1342 \end{pmatrix}, \sigma_{1432} = \begin{pmatrix} 1234 \\ 1432 \end{pmatrix},\\
&\sigma_{2134} = \begin{pmatrix} 1234 \\ 2134 \end{pmatrix}, \sigma_{2143} = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, \sigma_{2314} = \begin{pmatrix} 1234 \\ 3124 \end{pmatrix}, \sigma_{2341} = \begin{pmatrix} 1234 \\ 4123 \end{pmatrix}, \sigma_{2413} = \begin{pmatrix} 1234 \\ 3142 \end{pmatrix},\\
&\sigma_{2431} = \begin{pmatrix} 1234 \\ 4132 \end{pmatrix}, \sigma_{3124} = \begin{pmatrix} 1234 \\ 2314 \end{pmatrix}, \sigma_{3142} = \begin{pmatrix} 1234 \\ 2413 \end{pmatrix}, \sigma_{3214} = \begin{pmatrix} 1234 \\ 3214 \end{pmatrix}, \sigma_{3241} = \begin{pmatrix} 1234 \\ 4213 \end{pmatrix},\\
&\sigma_{3412} = \begin{pmatrix} 1234 \\ 3412 \end{pmatrix}, \sigma_{3421} = \begin{pmatrix} 1234 \\ 4312 \end{pmatrix}, \sigma_{4123} = \begin{pmatrix} 1234 \\ 2341 \end{pmatrix}, \sigma_{4132} = \begin{pmatrix} 1234 \\ 2431 \end{pmatrix}, \sigma_{4213} = \begin{pmatrix} 1234 \\ 3241 \end{pmatrix},\\
&\sigma_{4231} = \begin{pmatrix} 1234 \\ 4231 \end{pmatrix}, \sigma_{4312} = \begin{pmatrix} 1234 \\ 3421 \end{pmatrix}, \sigma_{4321} = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}
\end{aligned}$$

Then it is easy to see that

$$[X_1, [X_2, [X_3, X_4]]]$$
$$= \left(\left(\gamma_{1234} \underset{12}{\dot{-}} \gamma_{1243}\right) \underset{1}{\dot{-}} \left(\gamma_{1342} \underset{12}{\dot{-}} \gamma_{1432}\right)\right) \dot{-} \left(\left(\gamma_{2341} \underset{12}{\dot{-}} \gamma_{2431}\right) \underset{1}{\dot{-}} \left(\gamma_{3421} \underset{12}{\dot{-}} \gamma_{4321}\right)\right)$$

$$[X_1, [X_3, [X_4, X_2]]]$$
$$= \left(\left(\gamma_{1342} \underset{13}{\dot{-}} \gamma_{1324}\right) \underset{1}{\dot{-}} \left(\gamma_{1423} \underset{13}{\dot{-}} \gamma_{1243}\right)\right) \dot{-} \left(\left(\gamma_{3421} \underset{13}{\dot{-}} \gamma_{3241}\right) \underset{1}{\dot{-}} \left(\gamma_{4231} \underset{13}{\dot{-}} \gamma_{2431}\right)\right)$$

$$[X_1, [X_4, [X_2, X_3]]]$$
$$= \left(\left(\gamma_{1423} \underset{14}{\dot{-}} \gamma_{1432}\right) \underset{1}{\dot{-}} \left(\gamma_{1234} \underset{14}{\dot{-}} \gamma_{1324}\right)\right) \dot{-} \left(\left(\gamma_{4231} \underset{14}{\dot{-}} \gamma_{4321}\right) \underset{1}{\dot{-}} \left(\gamma_{2341} \underset{14}{\dot{-}} \gamma_{3241}\right)\right)$$

$$[X_2,[X_1,[X_4,X_3]]]$$
$$= \left(\left(\gamma_{2143} \overset{\cdot}{\underset{21}{-}} \gamma_{2134} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2431} \overset{\cdot}{\underset{21}{-}} \gamma_{2341} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{1432} \overset{\cdot}{\underset{21}{-}} \gamma_{1342} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4312} \overset{\cdot}{\underset{21}{-}} \gamma_{3412} \right) \right)$$

$$[X_2,[X_3,[X_1,X_4]]]$$
$$= \left(\left(\gamma_{2314} \overset{\cdot}{\underset{23}{-}} \gamma_{2341} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2143} \overset{\cdot}{\underset{23}{-}} \gamma_{2413} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{3142} \overset{\cdot}{\underset{23}{-}} \gamma_{3412} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1432} \overset{\cdot}{\underset{23}{-}} \gamma_{4132} \right) \right)$$

$$[X_2,[X_4,[X_3,X_1]]]$$
$$= \left(\left(\gamma_{2431} \overset{\cdot}{\underset{24}{-}} \gamma_{2413} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2314} \overset{\cdot}{\underset{24}{-}} \gamma_{2134} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{4312} \overset{\cdot}{\underset{24}{-}} \gamma_{4132} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3142} \overset{\cdot}{\underset{24}{-}} \gamma_{1342} \right) \right)$$

$$[X_3,[X_1,[X_2,X_4]]]$$
$$= \left(\left(\gamma_{3124} \overset{\cdot}{\underset{31}{-}} \gamma_{3142} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3241} \overset{\cdot}{\underset{31}{-}} \gamma_{3421} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{1243} \overset{\cdot}{\underset{31}{-}} \gamma_{1423} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2413} \overset{\cdot}{\underset{31}{-}} \gamma_{4213} \right) \right)$$

$$[X_3,[X_2,[X_4,X_1]]]$$
$$= \left(\left(\gamma_{3241} \overset{\cdot}{\underset{32}{-}} \gamma_{3214} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3412} \overset{\cdot}{\underset{32}{-}} \gamma_{3142} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{2413} \overset{\cdot}{\underset{32}{-}} \gamma_{2143} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4123} \overset{\cdot}{\underset{32}{-}} \gamma_{1423} \right) \right)$$

$$[X_3,[X_4,[X_1,X_2]]]$$
$$= \left(\left(\gamma_{3412} \overset{\cdot}{\underset{34}{-}} \gamma_{3421} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3124} \overset{\cdot}{\underset{34}{-}} \gamma_{3214} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{4123} \overset{\cdot}{\underset{34}{-}} \gamma_{4213} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1243} \overset{\cdot}{\underset{34}{-}} \gamma_{2143} \right) \right)$$

$$[X_4,[X_1,[X_3,X_2]]]$$
$$= \left(\left(\gamma_{4132} \overset{\cdot}{\underset{41}{-}} \gamma_{4123} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4321} \overset{\cdot}{\underset{41}{-}} \gamma_{4231} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{1324} \overset{\cdot}{\underset{41}{-}} \gamma_{1234} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3214} \overset{\cdot}{\underset{41}{-}} \gamma_{2314} \right) \right)$$

$$[X_4,[X_2,[X_1,X_3]]]$$
$$= \left(\left(\gamma_{4213} \overset{\cdot}{\underset{42}{-}} \gamma_{4231} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4132} \overset{\cdot}{\underset{42}{-}} \gamma_{4312} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{2134} \overset{\cdot}{\underset{42}{-}} \gamma_{2314} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1324} \overset{\cdot}{\underset{42}{-}} \gamma_{3124} \right) \right)$$

$$[X_4,[X_3,[X_2,X_1]]]$$
$$= \left(\left(\gamma_{4321} \overset{\cdot}{\underset{43}{-}} \gamma_{4312} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4213} \overset{\cdot}{\underset{43}{-}} \gamma_{4123} \right) \right) \overset{\cdot}{-} \left(\left(\gamma_{3214} \overset{\cdot}{\underset{43}{-}} \gamma_{3124} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2134} \overset{\cdot}{\underset{43}{-}} \gamma_{1234} \right) \right)$$

■

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