Topological Methods in Semilinear Elliptic Boundary Value Problems

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Purpose

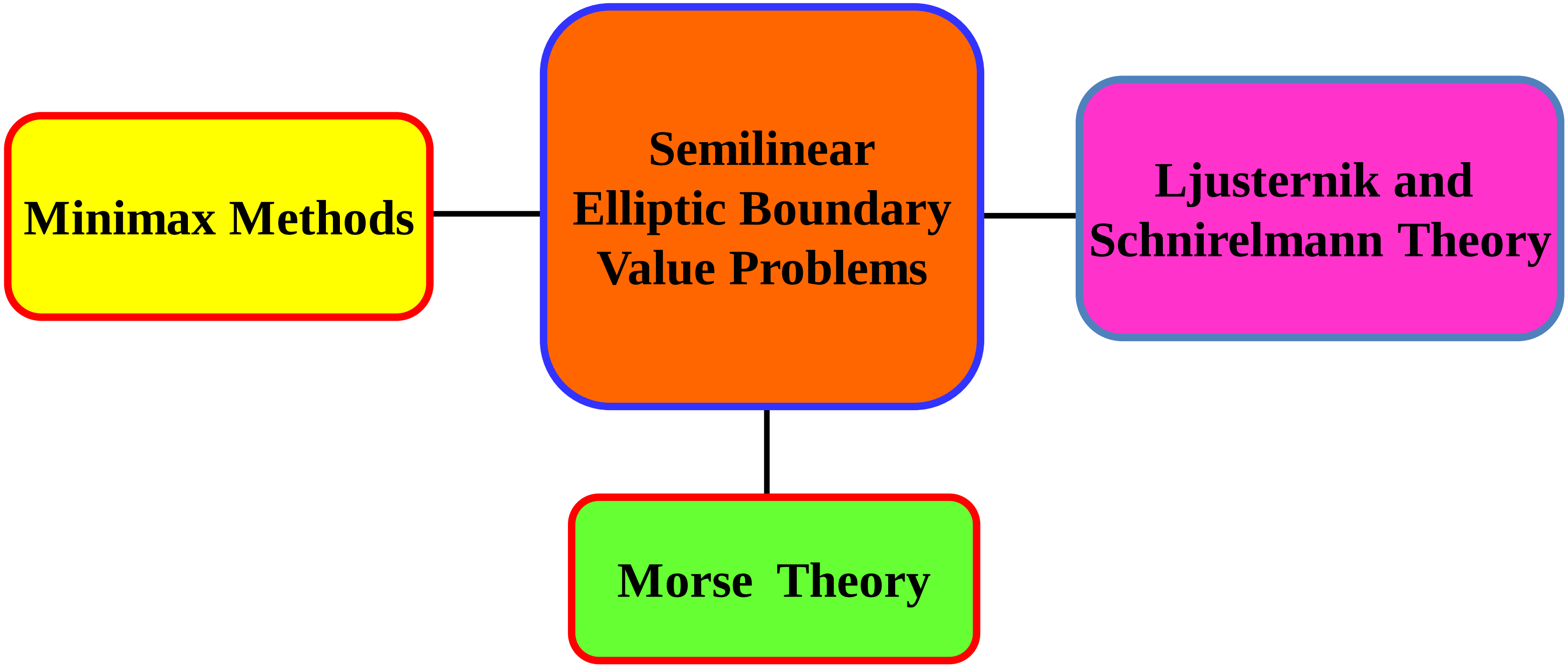
- **The purpose of this talk is to study a class of semilinear degenerate elliptic boundary value problems in the framework of Sobolev spaces which include as particular cases the Dirichlet and Robin problems.**
- **The approach here is based on the following:**

(1) Minimax Method

(2) Morse theory

(3) Ljusternik-Schnirelmann theory.

Bird’s-Eye View


Minimax Methods
Semilinear Elliptic Boundary Value Problems
Ljusternik and Schnirelmann Theory
Morse Theory

References

References (Monographs)

- **Ambrosetti and Prodi: A Primer of Nonlinear Analysis, Cambridge University Press, 1993**
- **K.C. Chang: Methods in Nonlinear Analysis, Springer-Verlag, 2005**
- **Ambrosetti and Malchiodi: Nonlinear Analysis and Semilinear Elliptic Problems, , Cambridge University Press, 2007**

References (Papers)

- **Ambrosetti and Lupo: On a class of nonlinear Dirichlet problems with multiple solutions, Nonlinear Analysis, 8 (1984), 1145-1150**
- **Thews: Multiple solutions for elliptic boundary value problems with odd nonlinearities, Math. Z. 163 (1978), 163-175**
- **Amann and Zehnder: Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations, Ann. Scuola Norm. Sup. Pisa 7 (1980), 539-603**

References (Papers)

- **Berger and Podolak:** On the solutions of a nonlinear Dirichlet problem, Indiana Univ. Math. J. 24 (1975), 837-846
- **Ambrosetti and Mancini:** Sharp nonuniqueness results for some nonlinear problems, Nonlinear Analysis, 3 (1979), 635-645
- **Amann and Hess:** A multiplicity result for a class of elliptic boundary value problems, Proc. Roy. Soc. Edinburgh 84A (1979), 145-151

My Works

My Works

- **Taira: Degenerate Elliptic Boundary Value Problems with Asymmetric Nonlinearity, Journal of the Mathematical Society of Japan, 62 (2010), 431-465**
- **Taira: Semilinear Degenerate Elliptic Boundary Value Problems at Resonance, Annali dell'Universit`a di Ferrara, 56 (2010), 369-392**

My Works

- **Taira : Multiple Solutions of Semilinear Degenerate Elliptic Boundary Value Problems, Mathematische Nachrichten, 284 (2011), 105-123**
- **Taira: Multiple Solutions of Semilinear Degenerate Elliptic Boundary Value Problems II, Mathematische Nachrichten, 284 (2011), 1554-1556**

My Works

- **Taira : Degenerate Elliptic Boundary Value Problems with Asymptotically Linear Nonlinearity, Rendiconti del Circolo Matematico di Palermo, 60 (2011), 283-308**
- **Taira: Multiple Solutions of Semilinear Elliptic Problems with Degenerate Boundary Conditions, Mediterranean Journal of Mathematics, 10 (2013), 731-752**

My Works

- **Taira: Semilinear Degenerate Elliptic Boundary Value Problems via Critical Point Theory, Tsukuba Journal of Mathematics, 36 (2012), 311-365**
- **Taira: Semilinear Degenerate Elliptic Boundary Value Problems via Morse Theory, Journal of the Mathematical Society of Japan, 67 (2015), 339-382**

My Works

- **Taira: Semilinear Degenerate Elliptic Boundary Value Problems via Critical Point Theory, Tsukuba Journal of Mathematics, 36 (2012), 311-365**
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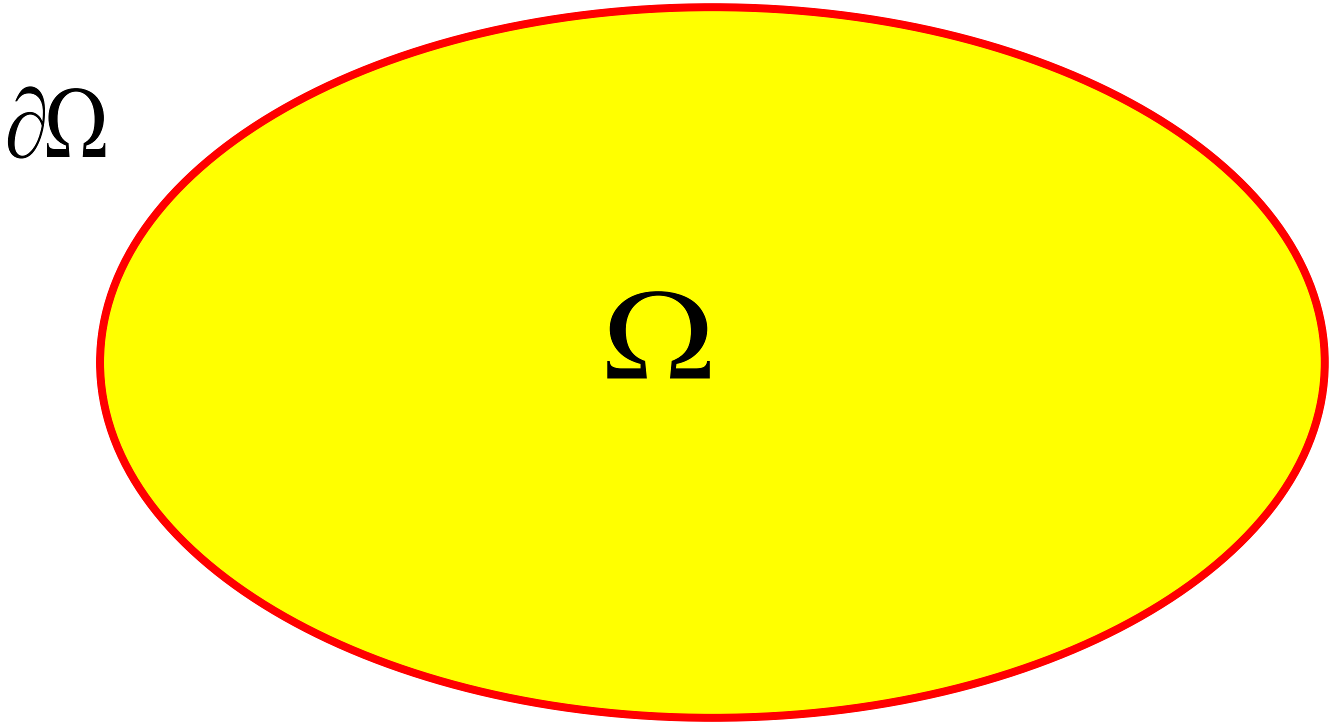
Typical Example

Bounded Domain

$\mathbf{R}^N, \quad N \geq 2$



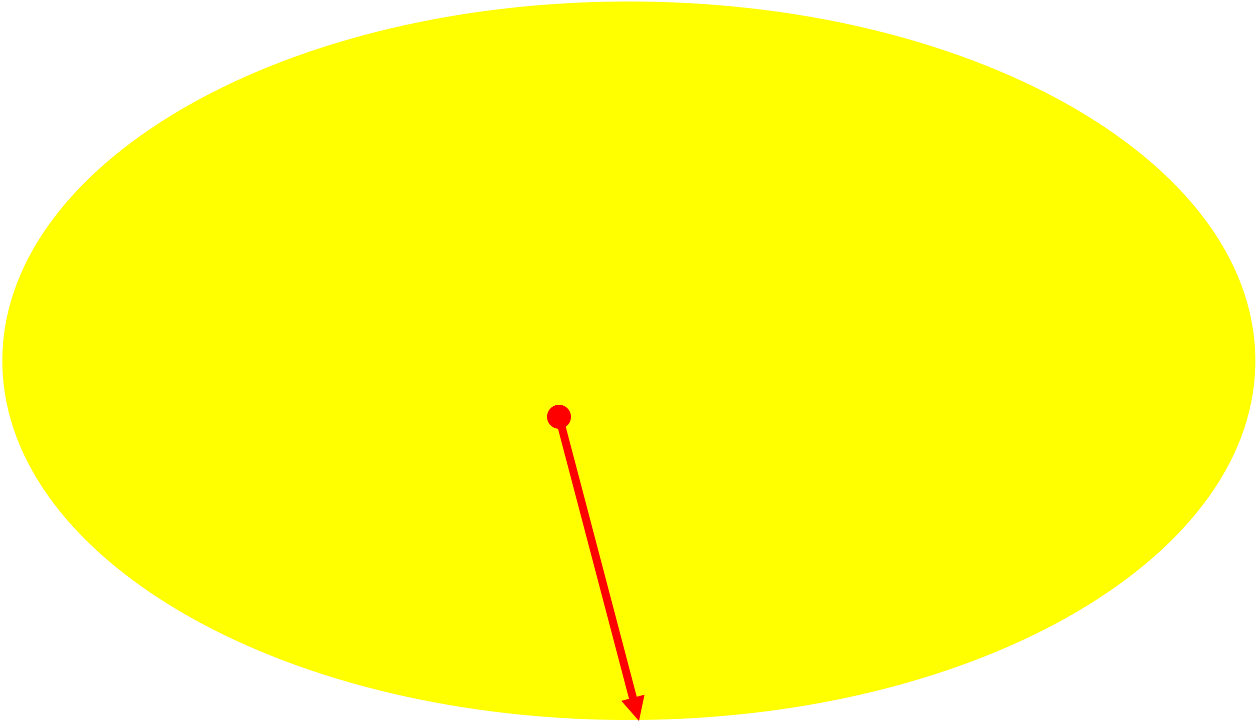
Typical Example

$$\begin{cases} -\Delta u = f(u) \text{ in } \Omega, \\ Bu(x') = a(x')\dfrac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0 \text{ on } \partial\Omega. \end{cases}$$

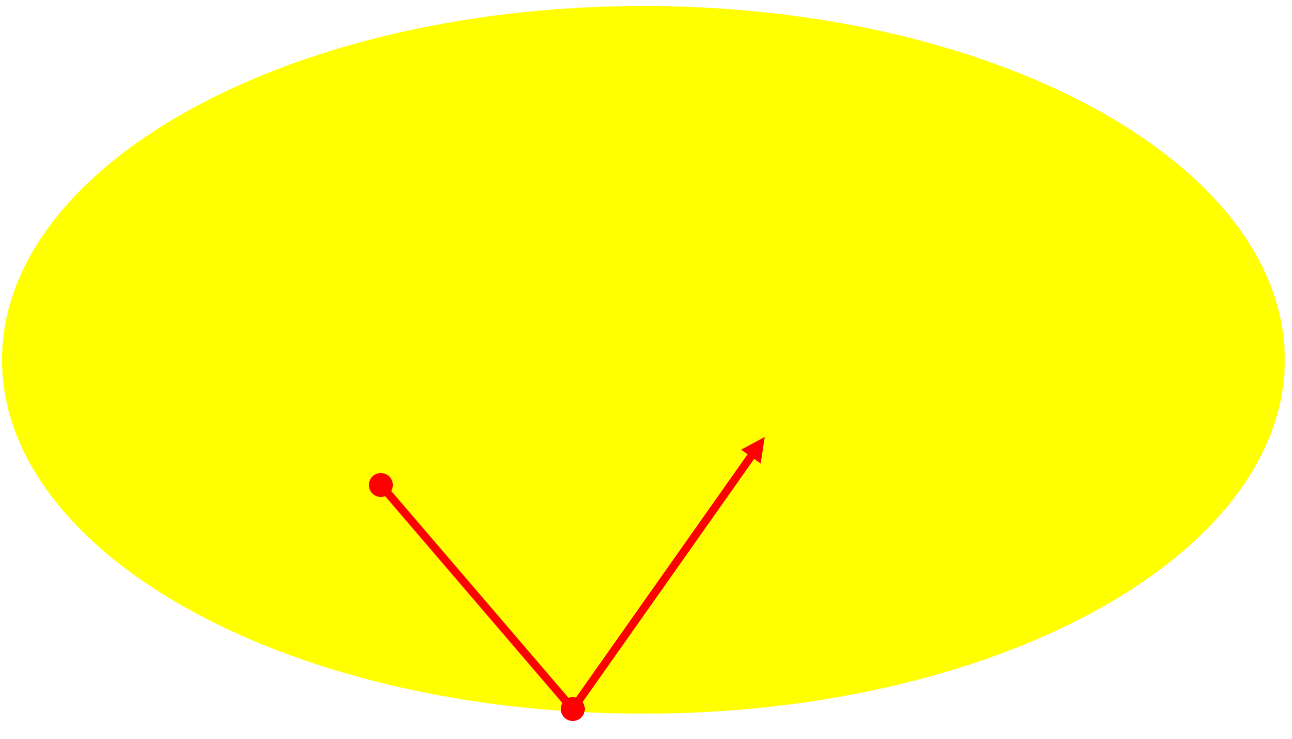
(H.1) $0 \le a(x') \le 1$ on $\partial\Omega$.

(H.2) $\boxed{a(x') \not\equiv 1 \text{ on } \partial\Omega.}$

Absorption Phenomenon (Dirichlet Condition)



Reflection Phenomenon
(Neumann Condition)



Difficult Point

Degeneracy
of
a Pseudo-Differential Operator

Associate Pseudo-Differential Operator (Fredholm Boundary Operator)

$$T = BP = a(x')\sqrt{-\Lambda} + \left(1 - a(x')\right)$$

$$\Lambda = \text{Laplace-Beltrami Operator}$$

$$\boldsymbol{\sigma}(T)(x', \xi') = a(x')\left|\xi'\right| + 1 - a(x')$$

$$\boxed{0 \le a(x') \le 1 \text{ on } \partial\Omega.}$$

Non-Degenerate (Elliptic) Case

Dirichlet Case: $a\left(x'\right) \equiv 0 \text{ on } \partial\Omega$

$$T = BP = a(x')\sqrt{-\Lambda} + I = I$$

Robin Case: $a\left(x'\right) > 0 \text{ on } \partial\Omega$

$$T = BP = a(x')\left(\sqrt{-\Lambda} + \frac{1 - a(x')}{a(x')} I\right)$$

Formulation of a Problem

Elliptic Differential Operator

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u$$

(1) $a^{ij}(x) \in C^{\infty}(\overline{\Omega}),\ a^{ij}(x) = a^{ji}(x)$

for all $x \in \overline{\Omega}$ and $\exists a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i \xi_j \geq a_0 |\xi|^2, \ \forall x \in \overline{\Omega}, \forall \xi \in \mathbf{R}^N.$$

(2) $c(x) \in C^{\infty}(\overline{\Omega})$ and $c(x) \geq 0$ on $\overline{\Omega}$.

Degenerate Robin Condition

$$Bu(x') = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 \text{ on } \partial\Omega.$$

(1) $a(x') \in C^\infty(\partial\Omega)$ and $a(x') \geq 0$ on $\partial\Omega$.

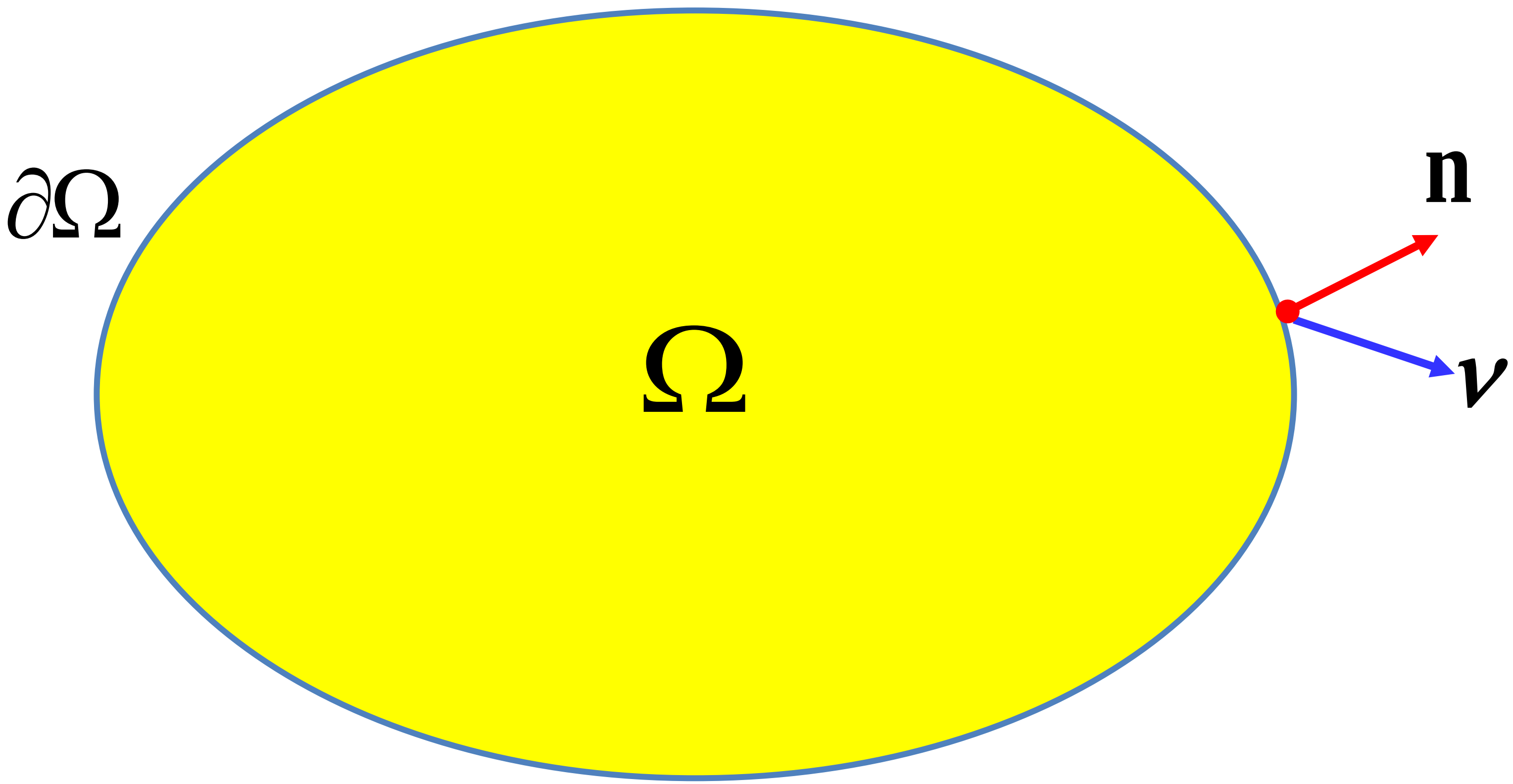
(2) $b(x') \in C^\infty(\partial\Omega)$ and $b(x') \geq 0$ on $\partial\Omega$.

Conormal Derivative (1)

$$\frac{\partial}{\partial \nu} = \sum_{i=1}^{N} \nu_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} a^{ij}(x') n_j \right) \frac{\partial}{\partial x_i}.$$

$\mathbf{n} = (n_1, n_2, \cdots, n_N)$ is the unit exterior normal.

Conormal Derivative (2)



Degenerate Boundary Conditions

$$Bu(x') = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 \text{ on } \partial\Omega.$$

(H.1) $a(x') + b(x') > 0$ on $\partial\Omega$.

(H.2) $b(x') \not\equiv 0$ on $\partial\Omega$.

Semilinear Degenerate Elliptic Boundary Value Problems

For a given function $f(t)$,

find a function $u(x)$ in Ω such that

$$\begin{cases} Au = f(u) \text{ in } \Omega, \\ Bu = 0 \text{ on } \partial\Omega. \end{cases}$$

Nonlinearities

1. **Superlinear Nonlinearity**
2. **Odd Nonlinearity**

Example of Superlinear Nonlinearity

$$f(s) = \begin{cases} s^{p} & s \geq 0 \\ s|s|^{q-1} & s < 0 \end{cases}$$

Here

$p > 1, \quad q > 1$

Example of Odd Nonlinearity

$$f(s) = s|s|^{p-1}$$

Here

$$p > 1$$

Main Results

Superlinear Case

Nonlinearity Conditions (1)

$$f(s) = \lambda s - g(s), \qquad \lambda \in \mathbf{R}$$

Nonlinearity Conditions (2)

(A) $g \in C^1(\mathbf{R})$, $\boxed{g(0) = g'(0) = 0}$.

(B) The limits $g'(\pm\infty)$ satisfies the conditions

$$\boxed{g'(\pm\infty) = \lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty}$$

Example 1

$$g(s) = \begin{cases} s^{p} & s \geq 0 \\ s|s|^{q-1} & s < 0 \end{cases}$$

Here

$p > 1, \quad q > 1$

Remarks on Nonlinearity Conditions

(F.1) $f \in C^1(\mathbf{R}), \quad f(0) = f'(0) = 0$

(F.2) $\exists c_1 > 0$ such that

$$|f(t)| \le c_1(1 + |t|^p), \; 1 < p < \frac{n+2}{n-2}$$

(F.3) $0 < \exists \theta < 1/2, \; \exists c_2 > 0$ such that

$$0 < F(t) = \int_0^t f(s)\,ds \le \theta t\, f(t), \quad |t| \ge c_2$$

Linear Operator $\mathfrak{A}$

We define a linear operator

$$\mathfrak{A} : L^2(\Omega) \to L^2(\Omega)$$

as follows:

(a) The domain $D(\mathfrak{A})$ is the set

$$D(\mathfrak{A}) = \{u \in H^2(\Omega) = W^{2,2}(\Omega) : Bu = 0\}.$$

(b) $\mathfrak{A}u = Au, \ \forall u \in D(\mathfrak{A})$.

$\Longrightarrow$

$\mathfrak{A}$ is a **positive definite**, self-adjoint operator

Spectral Properties of $\mathfrak{A}$

(1) The first eigenvalue λ_1 is **positive** and **algebraically simple**.

(2) The corresponding eigenfunction $\phi_1(x)$ may be chosen **strictly positve** in Ω:

$$\boxed{\begin{array}{l} \mathfrak{A}\phi_1 = \lambda_1 \phi_1, \\ \phi_1(x) > 0 \text{ in } \Omega \end{array}}$$

(3) No other eigenvalues $\lambda_j,\ j \geq 2,$ have positive eigenfunctions.

Remark (Neumann Case)

$b(x') \equiv 1$ on $\partial\Omega$ (**Neumann**)

$\Rightarrow$

$\lambda_1 = 0$

References

- **Taira: Boundary Value Problems and Markov Processes, Second Edition, Lecture Notes in Mathematics, Springer-Verlag, 2009**
- **Taira: Degenerate Elliptic Eigenvalue Problems with Indefinite Weights, Mediterranean Journal of Mathematics, 5 (2008), 133-162**

The Case

$$\lambda > \lambda_1$$

Existence Theorem 1

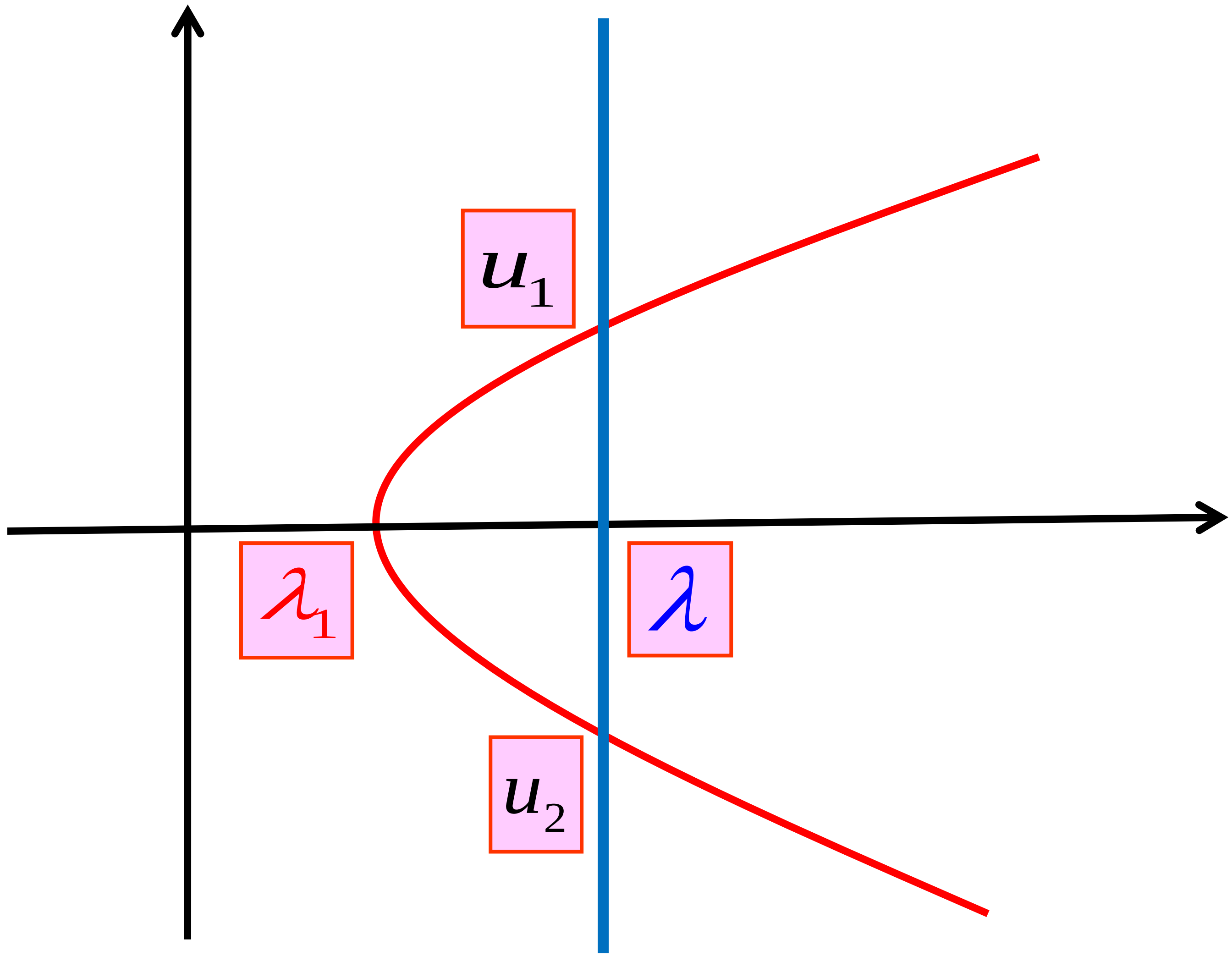
The semilinear problem

$$\begin{cases} Au = \lambda u - g(u) \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega \end{cases}$$

has at least **two** **non - trivial** solutions

$u_1 > 0, u_2 < 0$ for each $\boxed{\lambda > \lambda_1}$.

Outline of $f(s) = \lambda s - g(s)$



Comment 1

$$f'(s) = \lambda - g'(s):$$

$$f'(\infty) = -\infty < \lambda_1 < \lambda = f'(0)$$

$f'(s)$ crosses at least one eigenvalue λ_1 of $\mathfrak{A}$ if $|s|$ goes from 0 to ∞.

The Case

$$\lambda > \lambda_2$$

Existence Theorem 2

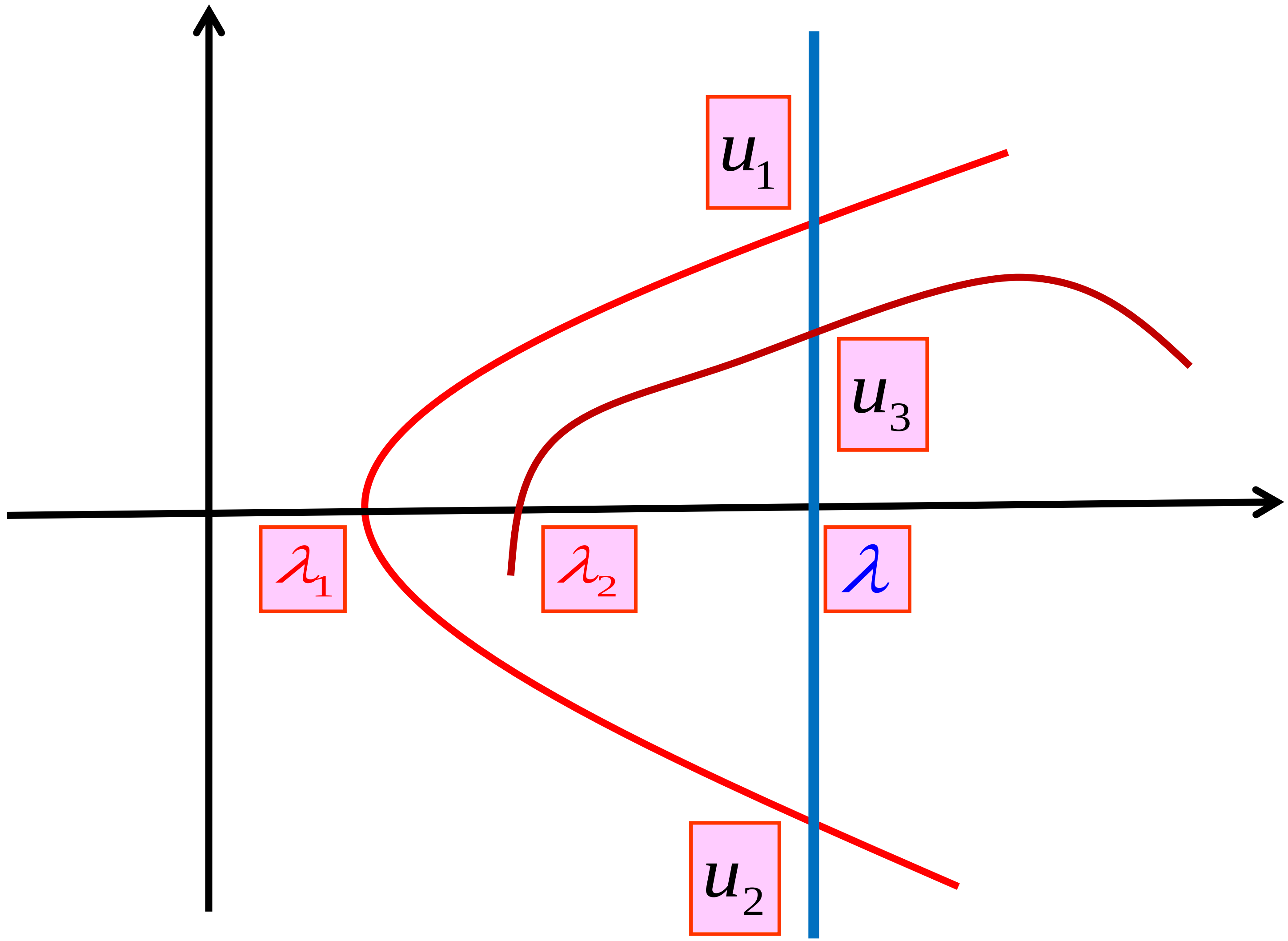
The semilinear problem

$$\begin{cases} Au = \lambda u - g(u) \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega \end{cases}$$

has at least **three** **non-trivial** solutions

for each $\boxed{\lambda > \lambda_2}$.

Outline of $f(s) = \lambda s - g(s)$



Comment 2

$$f'(s) = \lambda - g'(s):$$

$$f'(\infty) = -\infty < \lambda_1 < \lambda_2 < \lambda = f'(0)$$

$f'(s)$ crosses at least two eigenvalues λ_1, λ_2 of $\mathfrak{A}$ if $|s|$ goes from 0 to ∞.

Odd Nonlinearity Case

Nonlinearity Conditions (3)

(A) $g \in C^1(\mathbf{R})$, $\boxed{g(0) = g'(0) = 0}$.

(B) The limits $g'(\pm\infty)$ satisfies the conditions

$$g'(\pm\infty) = \lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty$$

(C) $\boxed{g(-s) = -g(s), \quad \forall s \in \mathbf{R}.}$

Example 2

$$g(s) = s|s|^{p-1}$$

Here

$$p > 1$$

The Case

$$\lambda > \lambda_k$$

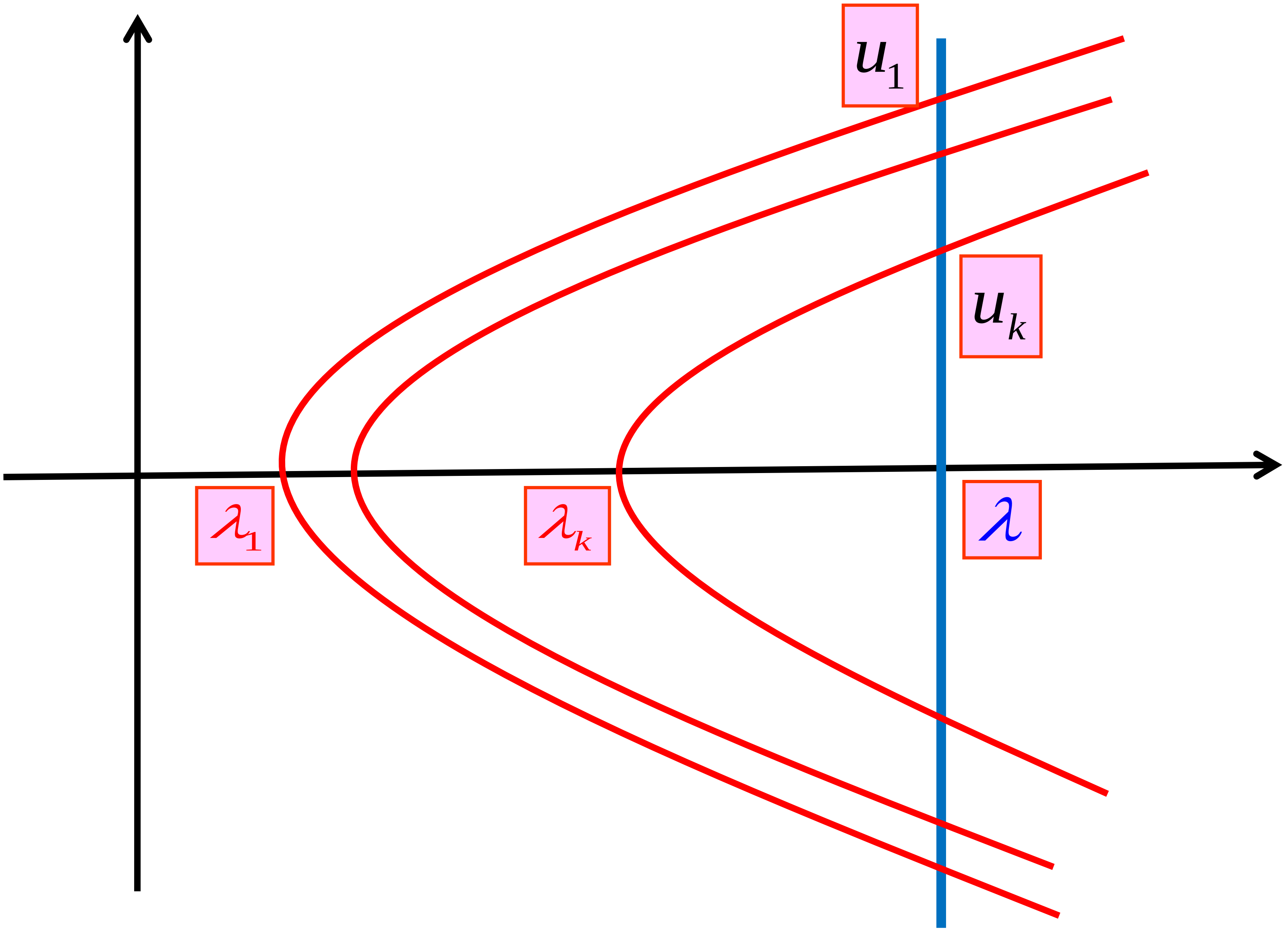
Existence Theorem 3

The semilinear problem

$$\begin{cases} Au = \lambda u - g(u) \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega \end{cases}$$

has at least k**-pairs** of **non - trivial** solutions for each $\boxed{\lambda > \lambda_k}$.

Outline of $f(s) = \lambda s - g(s)$



Comment 3

$$f'(s) = \lambda - g'(s):$$

$$f'(\infty) = -\infty < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k < \lambda = f'(0)$$

$f'(s)$ crosses at least k eigenvalues $\lambda_1, \cdots, \lambda_k$ of $\mathfrak{A}$ if $|s|$ goes from 0 to ∞.

Further Results

Asymptotically Linear Case

References

- **Amann:** **Saddle points and multiple solutions of differential equations, Math. Z. 169 (1979), 127-166**
- **Thews:** **A resduction method for some nonlinear Dirichlet, J. Nonlinear Analysis 3 (1979), 795-813**
- **Amann and Zehnder:** **Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations, Ann. Scuola Norm. Sup. Pisa 7 (1980), 539-603**

Semilinear Degenerate Elliptic Boundary Value Problems

For a given function $f(t)$,

find a function $u(x)$ in Ω such that

$$\begin{cases} Au = f(u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Nonlinearity Conditions A

(A) $f \in C^1(\mathbf{R})$.

(B) The limit $f'(\infty)$ is not an eigenvalue of $\mathfrak{A}$.

Example A

$$f(s) = \frac{\lambda_1 + \lambda_2}{2} s + \frac{1}{1+s^2}$$

$$f'(\infty) = \frac{\lambda_1 + \lambda_2}{2}$$

Existence Theorem A

The semilinear problem

$$\begin{cases} Au = f(u) \text{ in } \Omega, \\ Bu = 0 \text{ on } \partial\Omega \end{cases}$$

has at least **one** solution.

Nonlinearity Conditions B

(A) $f \in C^1(\mathbf{R})$, $\boxed{f(0) = 0}$

(B) The limit $f'(\infty)$ is not an eigenvalue of $\mathfrak{A}$.

(C) $\exists \lambda_j$ of $\mathfrak{A}$ such that

$$\boxed{f'(0) < \lambda_j < f'(\infty)}$$

or $\boxed{f'(\infty) < \lambda_j < f'(0)}$

Example B

$$f(s) = \frac{\lambda_1}{2} s + \frac{\lambda_2}{2} \frac{s}{1+s^2}$$

$$f'(\infty) = \frac{\lambda_1}{2}$$

$$f'(0) = \frac{\lambda_1 + \lambda_2}{2}$$

Existence Theorem B

The semilinear problem

$$\begin{cases} Au = f(u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

has at least **two solutions** -

one trivial solution and

one non-trivial solution.

Minimax Methods

Gradient

Let H be a Hilbert space and $f \in C^1(H, \mathbf{R})$.

The Frechet derivative $df(u)$ can be expressed as follows:

$$df(u)(v) = \left(\nabla f(u), v\right)_H, \quad \forall v \in H$$

$\nabla f(u) \in H$: the **gradient** of f at u

Palais-Smale Condition

Palais-Smale Condition

Let H be a Hilbert space and $f \in C^1(H, \mathbf{R})$.

(1) f satisfies $(\mathrm{PS})_c$ condition if

$$\{x_j\} \subset H, f(x_j) \to c, \nabla f(x_j) \to 0$$

$$\Rightarrow \exists \{x_{j'}\} \text{ is convergent}$$

(2) f satisfies (PS) condition

if it satisfies $(\mathrm{PS})_c$ for every $c \in \mathbf{R}$.

Minimizing Method for Minimum Points

Let H be a Hilbert space and $f \in C^1(H, \mathbf{R})$.

(1) $f(x)$ is **bounded from below**

(2) $f(x)$ satisfies $(\mathrm{PS})_c$ condition with

$$c = \inf_{x \in H} f(x)$$

$$\Rightarrow$$

$\exists x^* \in H$ such that

$$f(x^*) = c = \inf_{x \in H} f(x)$$

$$\nabla f(x^*) = 0$$

Ekeland Variational Principle

Let (X, d) be a complete metric space

$f : X \to \mathbf{R} \cup \{+\infty\}$

(1) $f(x)$ is **bounded from below**

and lower semi-continuous.

(2) $\exists \varepsilon > 0$, $\exists x_{\varepsilon} \in X$ such that

$$f(x_{\varepsilon}) < \inf_{x \in X} f(x) + \varepsilon.$$

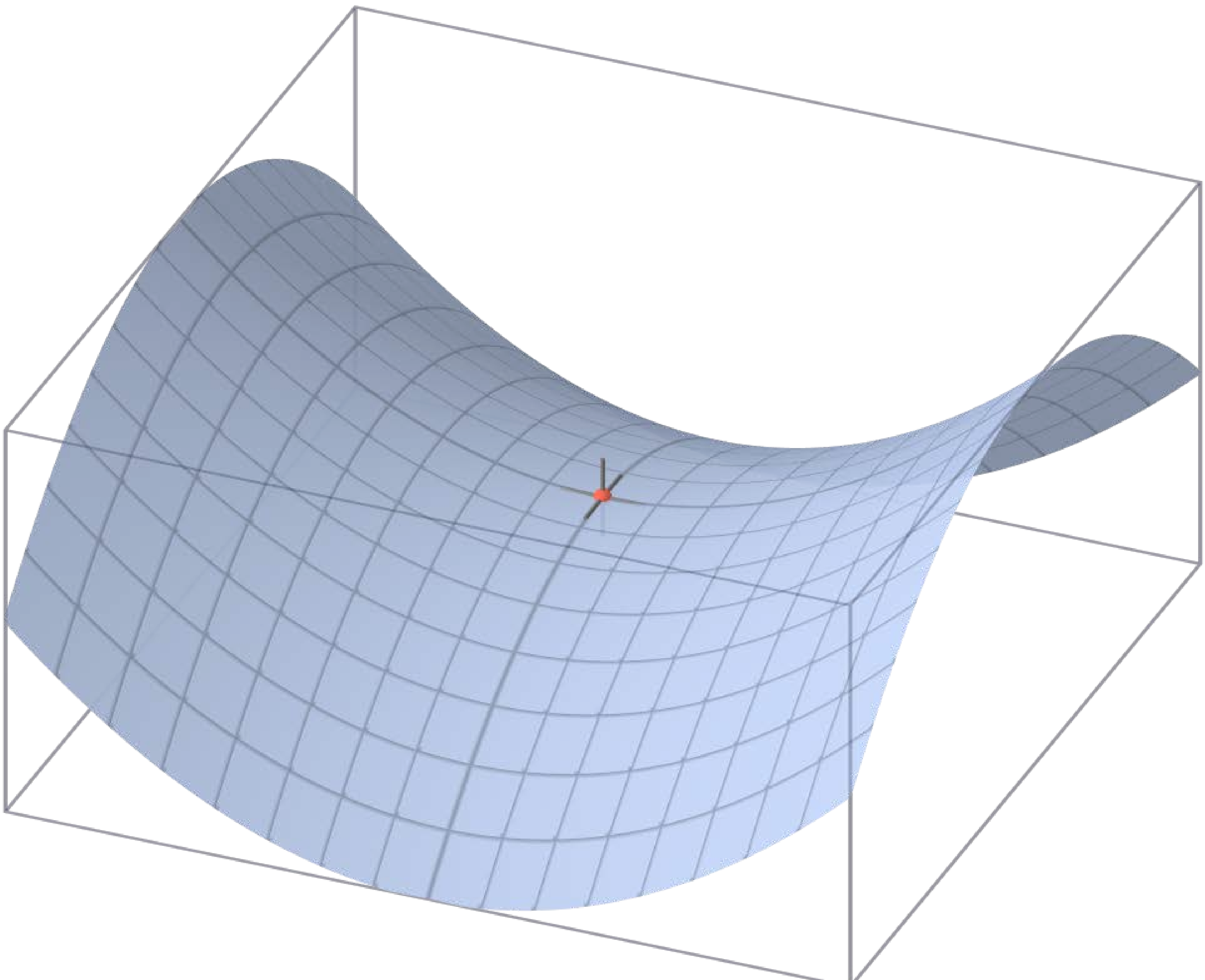
$\Longrightarrow \exists y_{\varepsilon} \in X$ such that

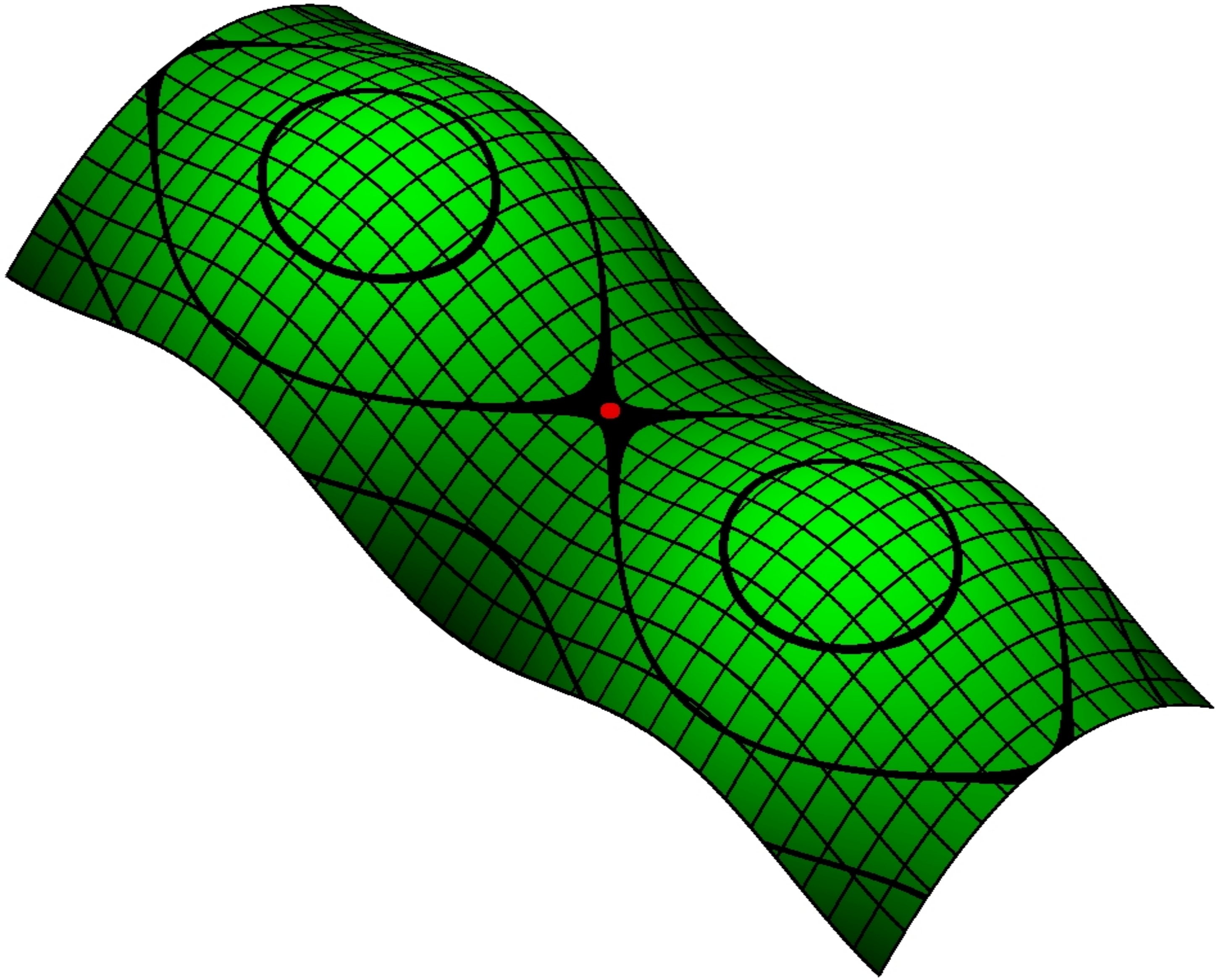
(a) $f(y_{\varepsilon}) \leq f(x_{\varepsilon})$.

(b) $d(x_{\varepsilon}, y_{\varepsilon}) \leq 1$.

(c) $f(x) > f(y_{\varepsilon}) - \varepsilon d(y_{\varepsilon}, x), \quad \forall x \neq y_{\varepsilon}$.

Existence of Saddle Points





Existence of Saddle Points

Nonlinearity	Approach	Studied by
•Asymptotically Linear Case	•Linking Theorem	Rabinowitz (1978)
•Superlinear Case	•Morse Theory	Ambrosetti and Lupo (1984)
•Odd Nonlinearity Case	•Ljusternik and Schnirelmann Theory	Castro and Lazer (1979)

Morse Theory on Hilbert Spaces

Marston Morse

Marston Morse

- Marston Morse (1892-1977) American Mathematician

References (Papers)

- **Palais:** **Morse theory on Hilbert manifolds, Topology 2 (1963), 299-340**
- **Palais and Smale:** **A generalized Morse theory, Bull. Amer. Math. Soc. 70 (1964), 165-172**
- **Smale:** **An infinite-dimensional version of Sard's theorem, Amer. J. Math. 87 (1965), 861-866**
- **Marino and Prodi:** **Metodi perturbativi nella teoria di Morse, Boll. Un. Mat. Ital. 11 (1975), 1-32**

Regular and Critical Points

Let H be a Hilbert space and $f \in C^1(H, \mathbf{R})$

(1) u is called a **regular point** of f

if $\nabla f(u) \neq 0$

(2) u is called a **critical point** of f

if $\nabla f(u) = 0$

Hessian

Let H be a Hilbert space and $f \in C^2(H, \mathbf{R})$.

The Frechet derivative $D^2 f(u)$ of $\nabla f(u)$

can be expressed as follows:

$$d^2 f(u)(v,w) = \left(D^2 f(u)v, w\right)_H, \forall v, w \in H$$

$D^2 f(u) : H \to H$: the **Hessian** of f at u

Non-Degeneracy of Critical Points

Let H be a Hilbert space and $f \in C^2(H, \mathbf{R})$

A critical point u of f is called

non-degenerate if the Hessian

$D^2 f(u) : H \to H$

has a **bounded inverse**.

Splitting Theorem

Morse's Lemma (1)

M : compact finite dimensional manifold

$f \in C^2(M, \mathbf{R})$

p : **non-degenerate, critical point** of f

Morse's Lemma (2)

Then:

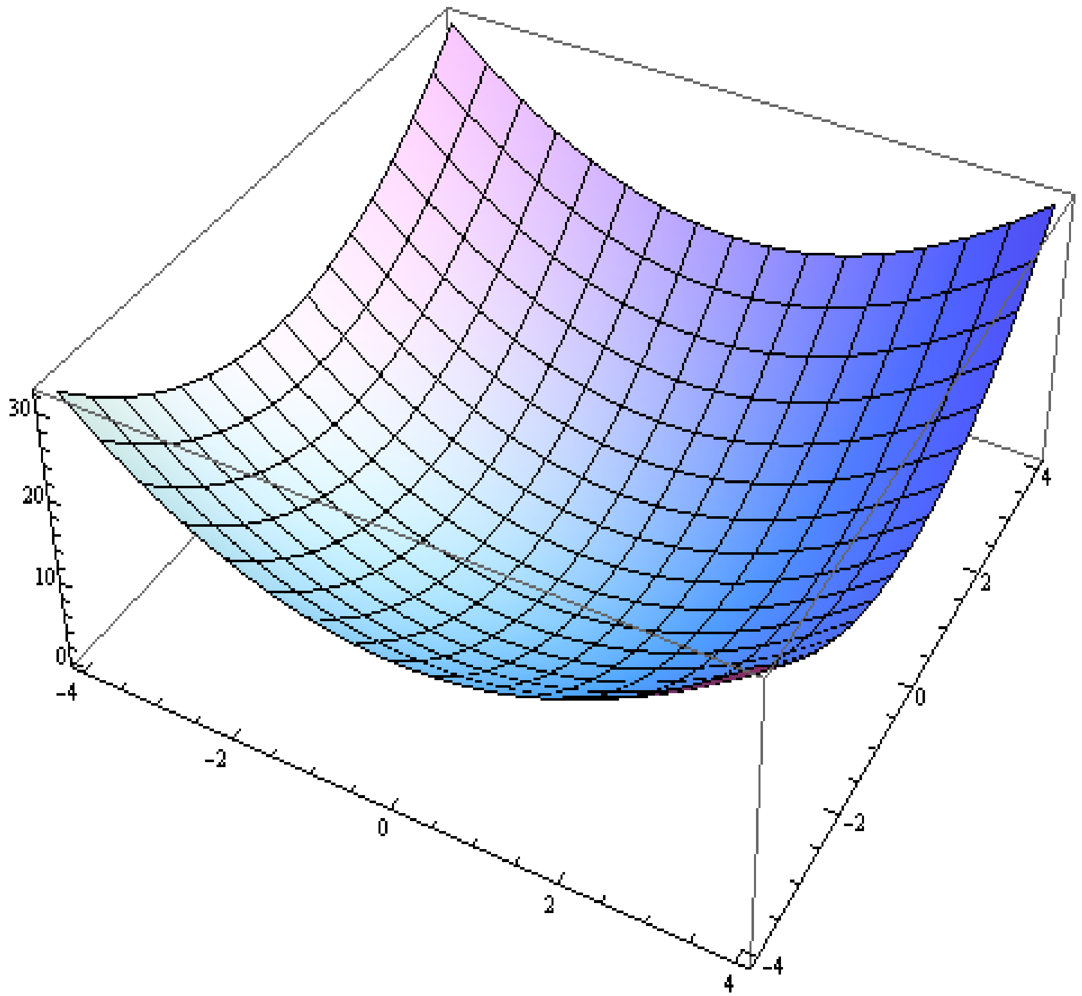
$\exists(y_1, y_2, \cdots, y_n)$ a local coordinate system near p such that

$$f(y) = f(p) - y_1^2 - y_2^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + y_{\lambda+2}^2 + \cdots + y_n^2$$

Here

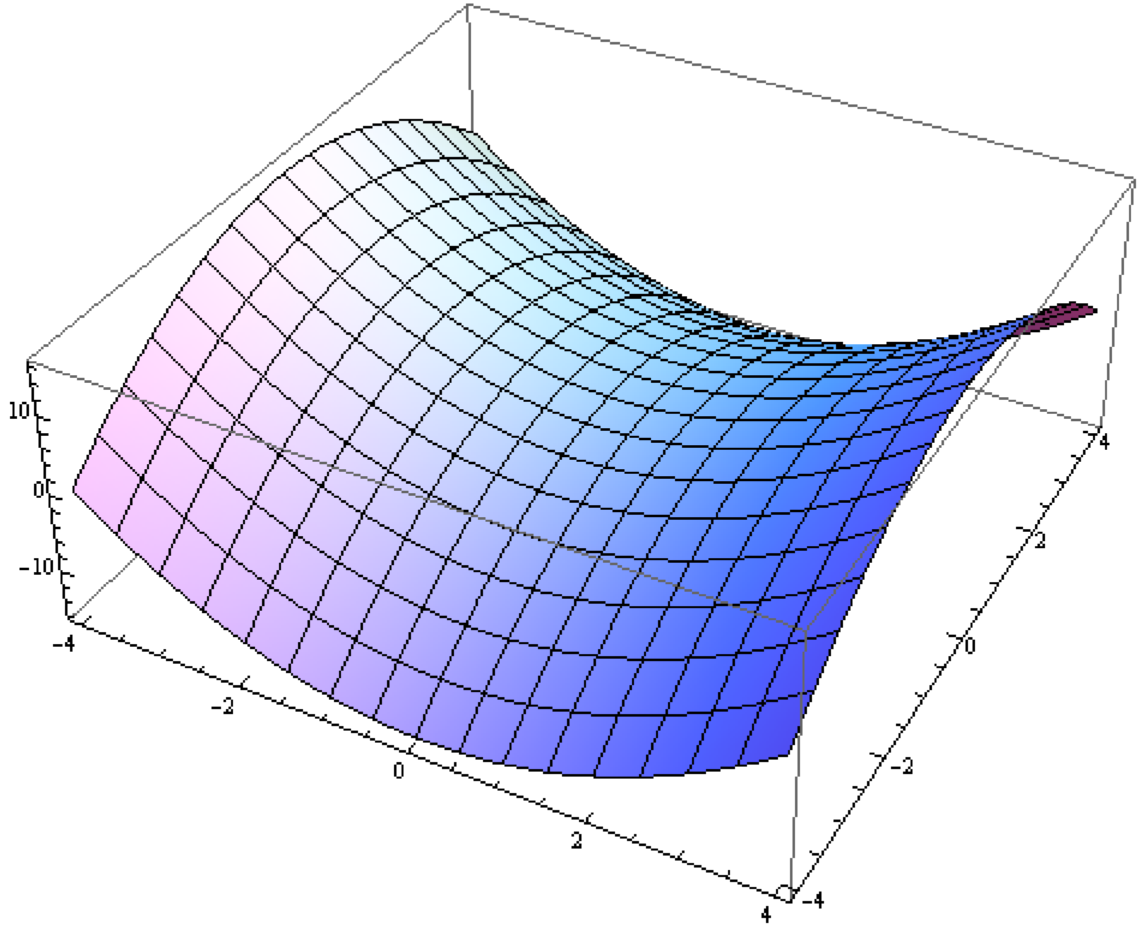
λ is the **Morse index** of f at p

$$f(x,y) = x^2 + y^2$$ (minimal point)



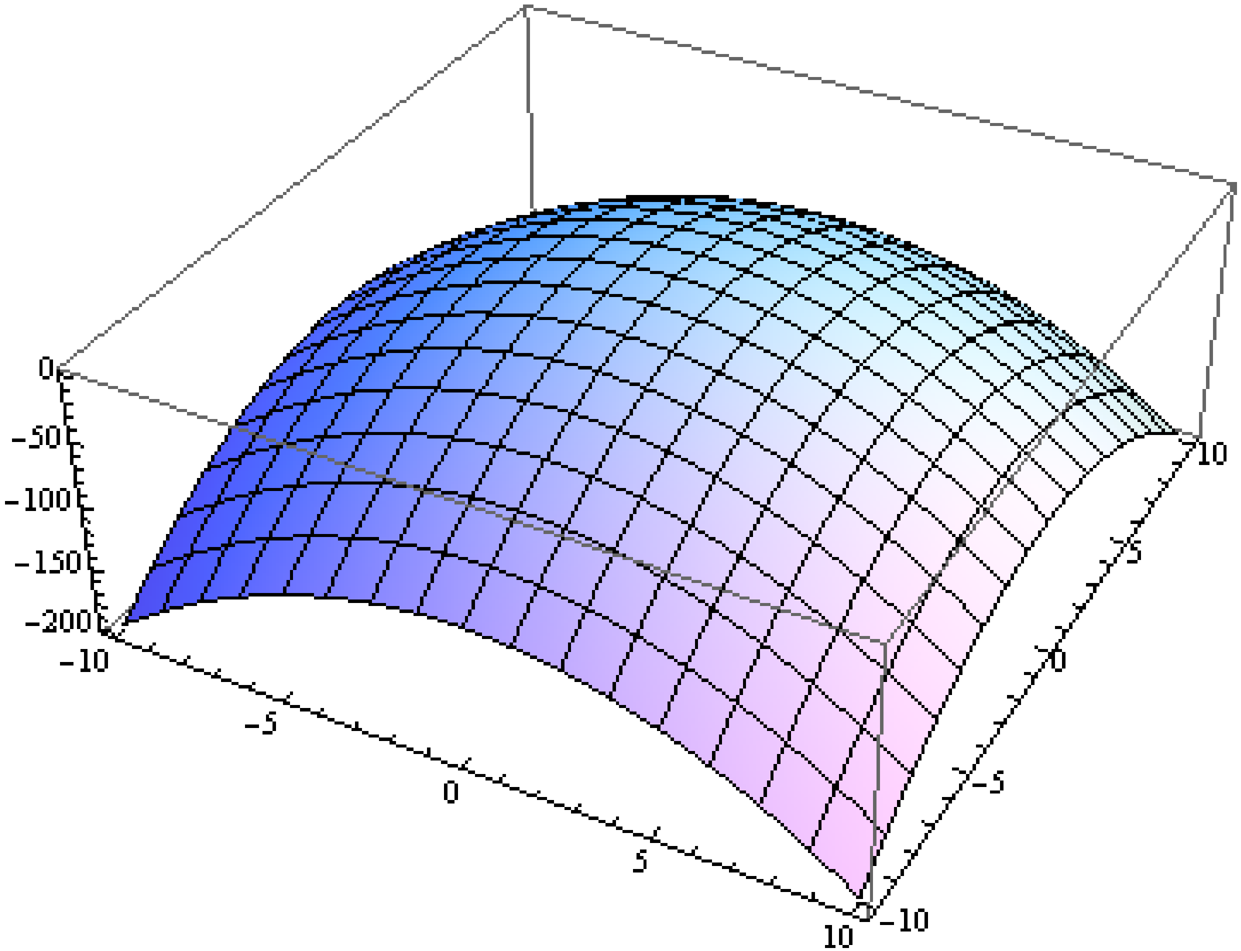
0 : Morse Index

$g(x, y) = x^2 - y^2$ （saddle point）



1 : Morse Index

$h(x, y) = -x^2 - y^2$ **(maximal point)**



2 : Morse Index

Critical Groups and Morse Indices

$C_*(f,0)$	Morse Index
$G \oplus 0 \oplus 0$	**0**
$0 \oplus G \oplus 0$	**1**
$0 \oplus 0 \oplus G$	**2**

Splitting Theorem (1)

H : Hilbert space

U : convex neighborhood of 0 in H

$f \in C^2(U, \mathbf{R})$

Assume that:

(1) 0 is the **only critical point** of f

(2) $\boxed{A = D^2 f(0)}$ is **Fredholm** with

$N = \mathrm{Ker}\, A$

Splitting Theorem (2)

Then:

(1) $\exists B \subset U$ an open ball about 0 in H

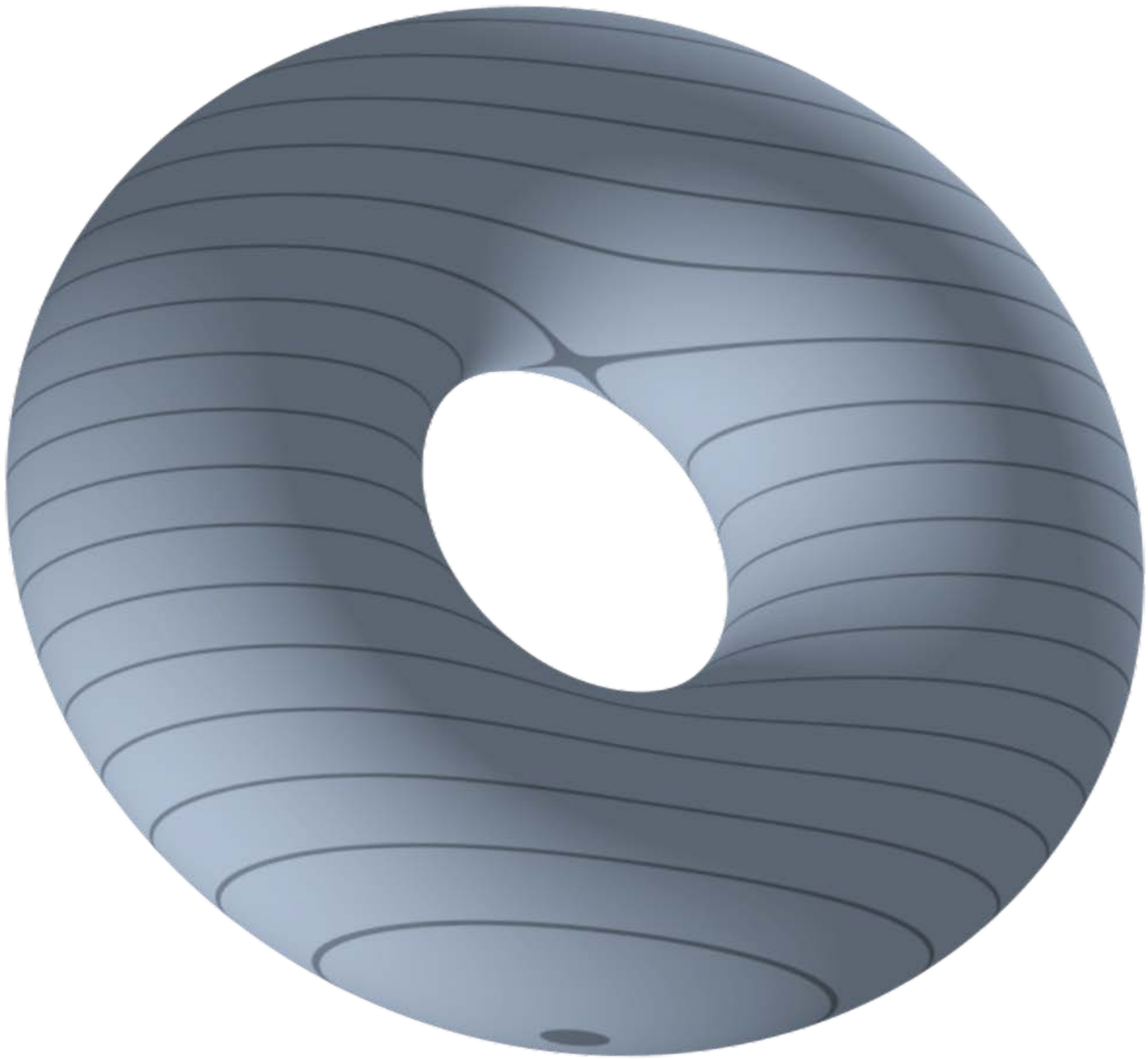
(2) $\exists \varphi : B \to B$ **homeomorphism**

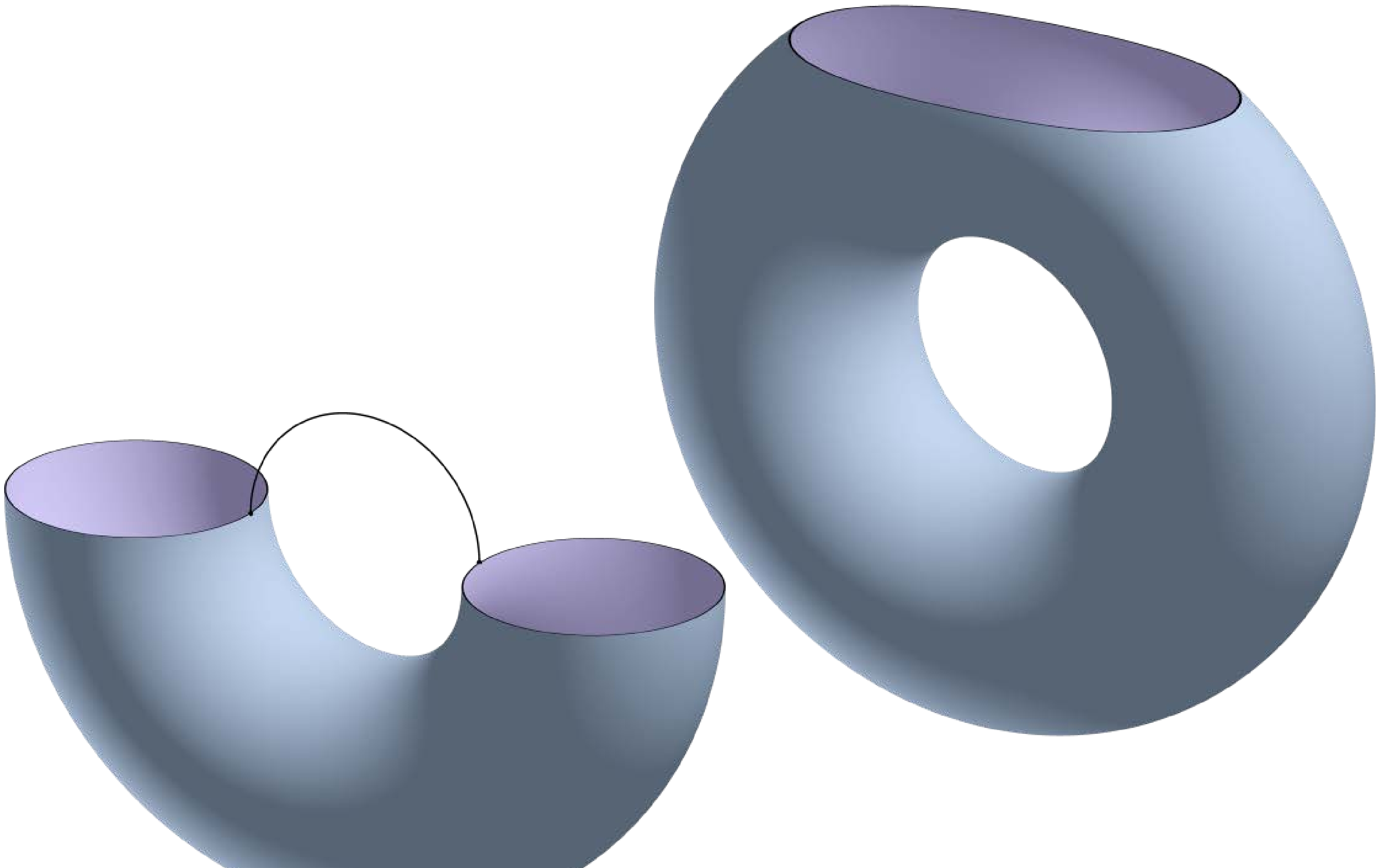
(3) $\exists h : B \cap N \to N^{\perp}$ a C^1 map

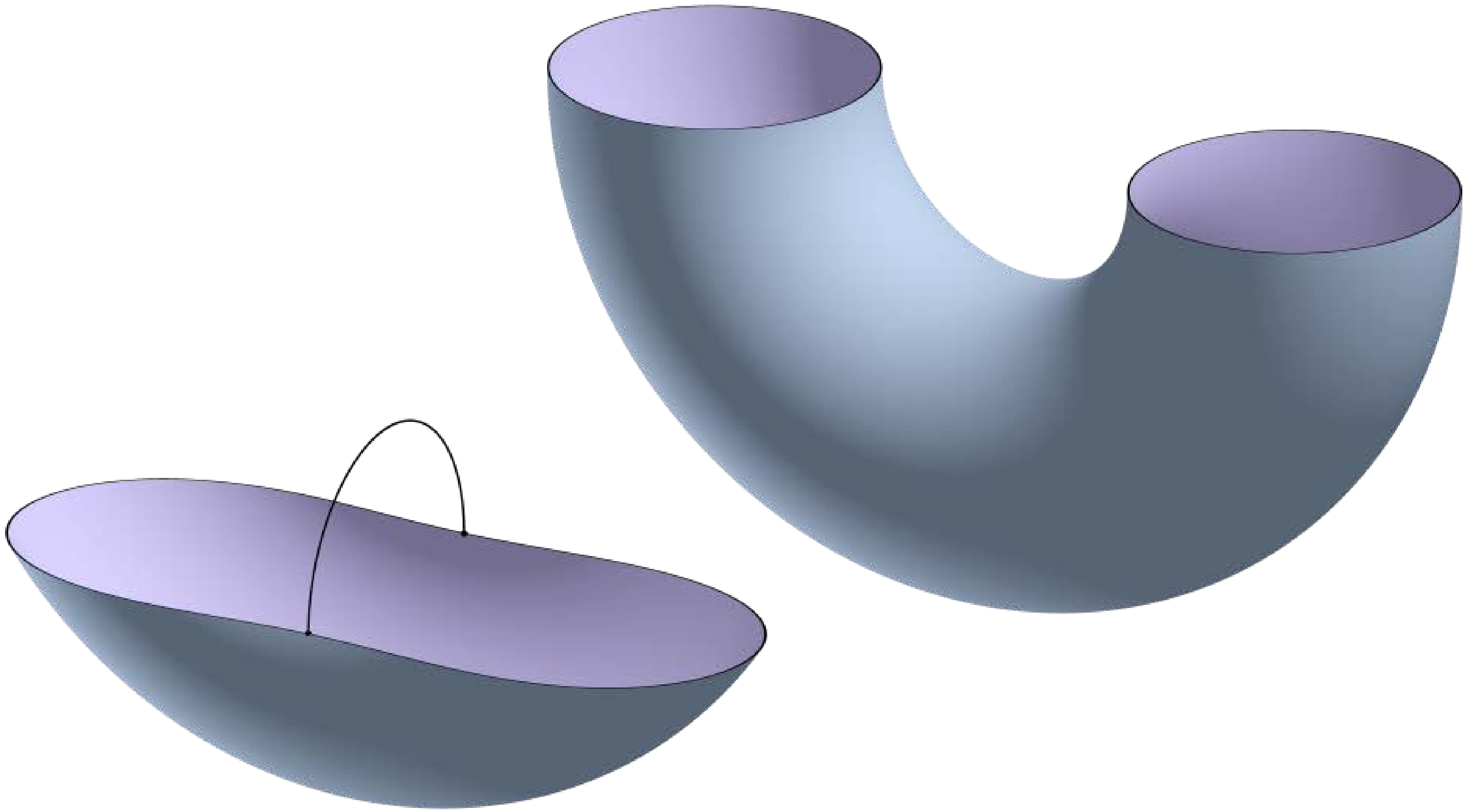
such that

$$(f \circ \varphi)(y+\xi) = \frac{1}{2}(A\xi, \xi)_H + f(y+h(y))$$

$\forall y \in B \cap N, \ \xi \in B \cap N^{\perp}$







Relative Homology Groups

Betti Numbers

G : Abelian group

(X, Y) : pair of topological spaces with $Y \subset X$

$H_*(X, Y; G)$: **relative singular homology group**

$$\beta_q(X, Y) = \text{rank}\, H_q(X, Y; G), \quad q = 0, 1, 2, \cdots$$

the q-th **Betti number** of (X, Y)

Euler-Poincare Characteristic

$$\beta_q(X,Y) = \text{rank } H_q(X,Y;G), \quad q = 0,1,2,\cdots$$

$$\chi(X,Y) = \sum_{q=0}^{\infty} (-1)^q \beta_q(X,Y)$$

the **Euler - Poincare characteristic** of (X,Y)

Non-Trivial Interval Theorem

Non-Trivial Interval Theorem (1)

H : Hilbert space

$f \in C^1(H, \mathbf{R})$

$$K = \{x \in H : \nabla f(x) = 0\}$$

$$f^a = f^{-1}((-\infty, a]) = \{x \in H : f(x) \le a\}$$

Non-Trivial Interval Theorem (2)

$$H_q(f^b, f^a; G) \neq 0, \quad a < b$$

$$\Rightarrow$$

$$f^{-1}\left([a,b]\right) \cap K \neq \varnothing$$

Non-Critical Interval Theorem

Deformation Retract (1)

X : topological space

A continuous map $\eta : X \times [0,1] \to X$

is called a **deformation** of X if

$\eta(\cdot, 0) =$ identity on X

Deformation Retract (2)

(X, Y) : pair of topological spaces with $Y \subset X$

(1) A continuous map $r : X \to Y$ is called

a **deformation retract** if

$r \circ i =$ identity on Y

$i \circ r \simeq$ identity on X

(2) Y is called a **deformation retraction** of X

Strong Deformation Retract

A deformation retract $r : X \to Y$ is called a **strong deformation retract** if $\exists$a **deformation** $\eta : X \times [0,1] \to X$ such that

$\eta(\cdot, t) = \text{identity on } Y$ for all $t \in [0,1]$

$\eta(\cdot, 1) = i \circ r$ on X

Excision Property

Y is a **strong deformation retraction** of X

$\Rightarrow$

$$H_q(X,Y;G) = 0, \quad q = 0,1,2,\cdots$$

$\Rightarrow$

$$H_q(X;G) = H_q(Y;G), \quad q = 0,1,2,\cdots$$

Non-Critical Interval Theorem (1)

Let H be a Hilbert space and $f \in C^1(H, \mathbf{R})$

(1) f satisfies $(\mathrm{PS})_c$ condition for $\forall c \in [a, b]$

(2) Let K be the set of all **critical points** of f

$$f^{-1}([a,b]) \cap K = \varnothing$$

$\Rightarrow$

f^a is a **strong deformation retraction** of f^b

Non-Critical Interval Theorem (2)

f^a is a **strong deformation retraction** of f^b

$\Rightarrow$

$$\boxed{H_q(f^b, f^a; G) = 0, \quad q = 0, 1, 2, \cdots}$$

$\Rightarrow$

$$\boxed{H_q(f^b; G) = H_q(f^a; G), \quad q = 0, 1, 2, \cdots}$$

Critical Groups

Critical Group (1)

H : Hilbert space

$f \in C^1(H, \mathbf{R})$

z : **isolated, critical point** of f

U a neighborhood of z such that

$U \cap K = \{z\}$

Critical Group (2)

$$C_q(f,z) = H_q\left(f^c \cap U, \left(f^c \setminus \{z\}\right) \cap U; G\right)$$

$$c = f(z)$$

$$f^c = f^{-1}((-\infty, c]) = \{x \in H : f(x) \le c\}$$

$C_q(f,z)$ is called a **critical group** of f at z

Critical Groups and Morse Indices

$C_*(f,0)$	**Morse Index**
$G \oplus 0 \oplus 0$	**0**
$0 \oplus G \oplus 0$	**1**
$0 \oplus 0 \oplus G$	**2**

Example 1 (Minimum Point)

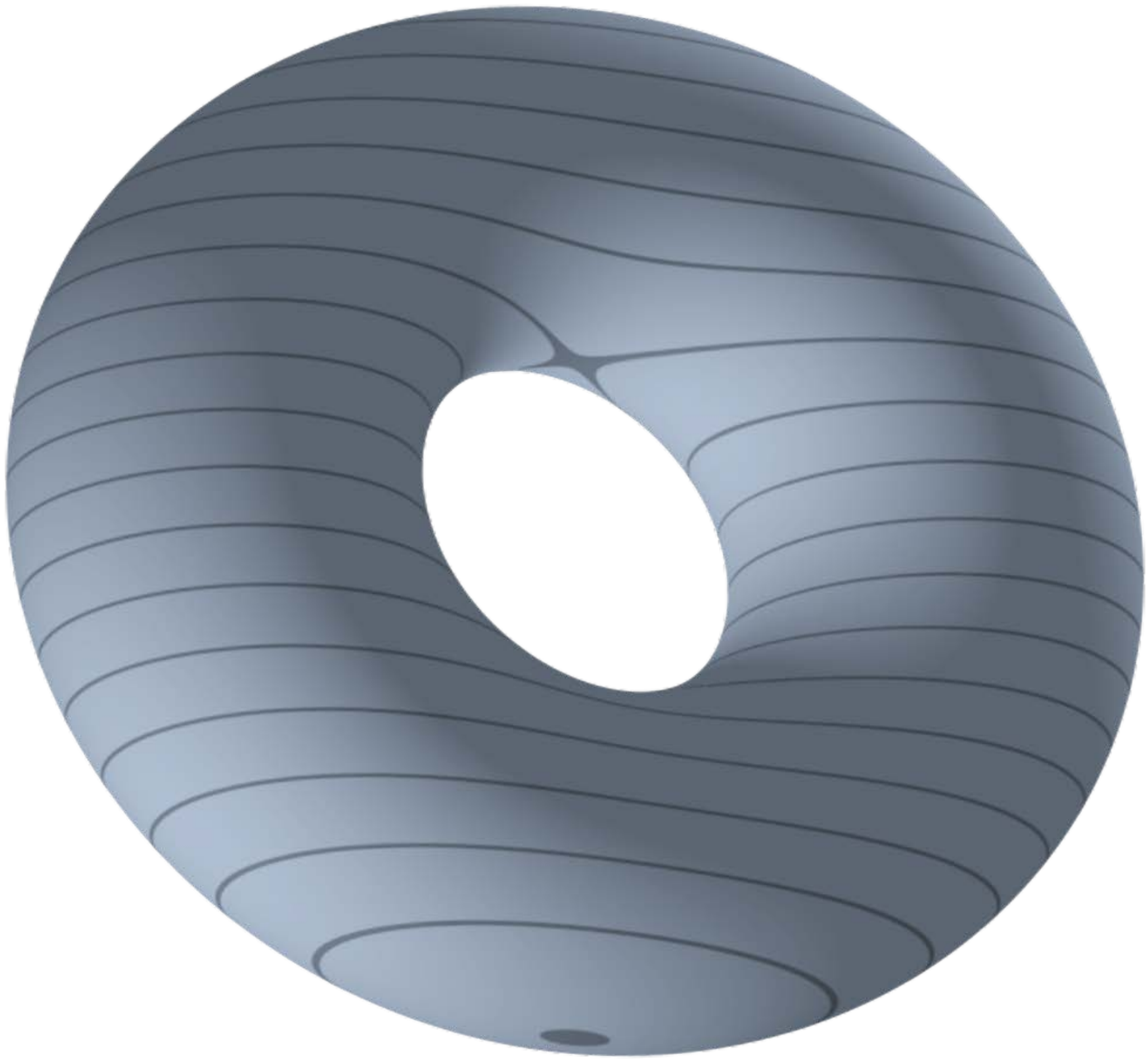
H : Hilbert space

$f \in C^1(H, \mathbf{R})$

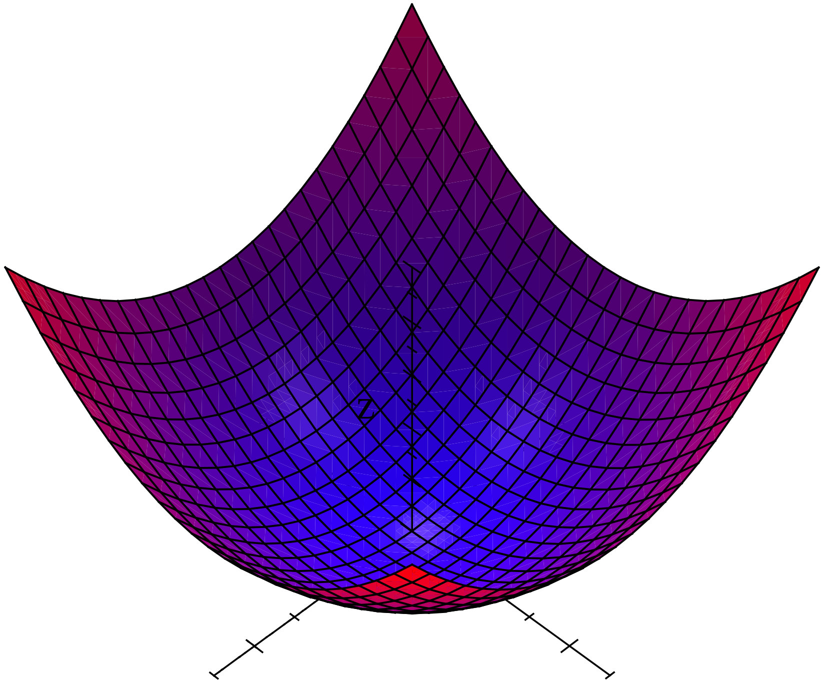
z : **isolated, local minimum** of f

$$C_q(f, z) = H_q\left(f^c \cap U, \left(f^c \setminus \{z\}\right) \cap U; G\right)$$

$$= H_q\left(\{z\}; G\right) = \begin{cases} G & \text{if } q = 0 \\ 0 & \text{if } q \geq 1 \end{cases}$$

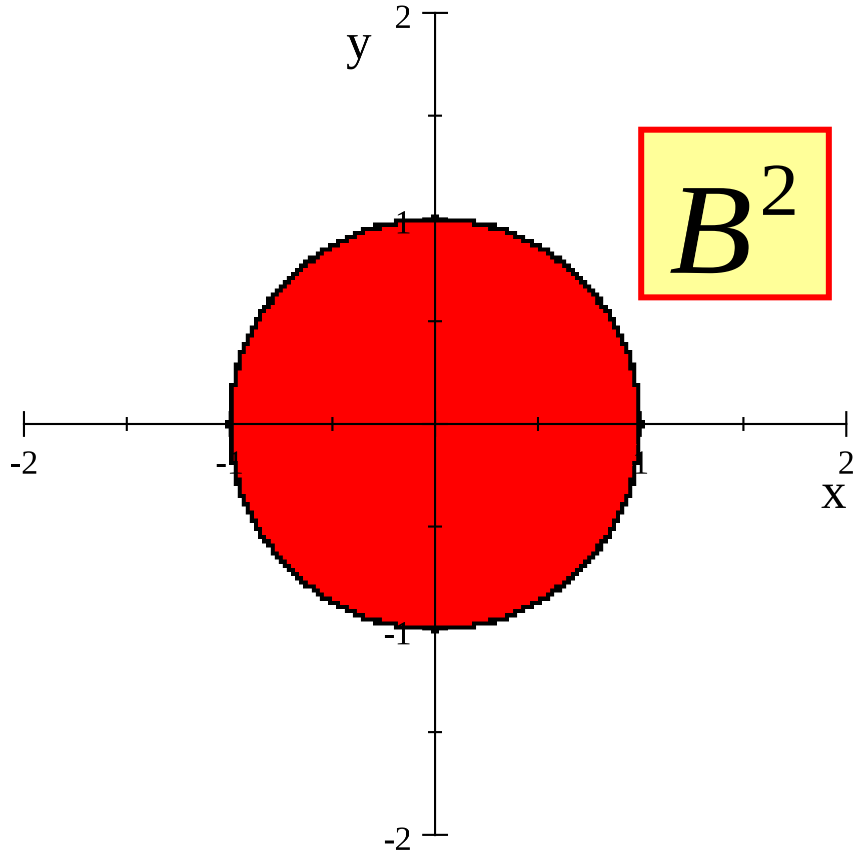


Minimum Point



$$f(x, y) = x^2 + y^2$$

$$f^{1} = \{(x, y) \in \mathbf{R}^2 : f(x, y) \le 1\}$$



Critical Groups and Homology Groups

$$(1)\ C_*(f,0) = H_*\left(\{0\};G\right) = G \oplus 0 \oplus 0$$

$$(2)\ H_*\left(f^1, f^{-1};G\right) = H_*\left(B^2;G\right) = G \oplus 0 \oplus 0$$

Here

$$f^a = \left\{(x,y) \in \mathbf{R}^2 : f(x,y) \le a\right\}$$

0 : Morse Index

Example 2 (Maximum Point)

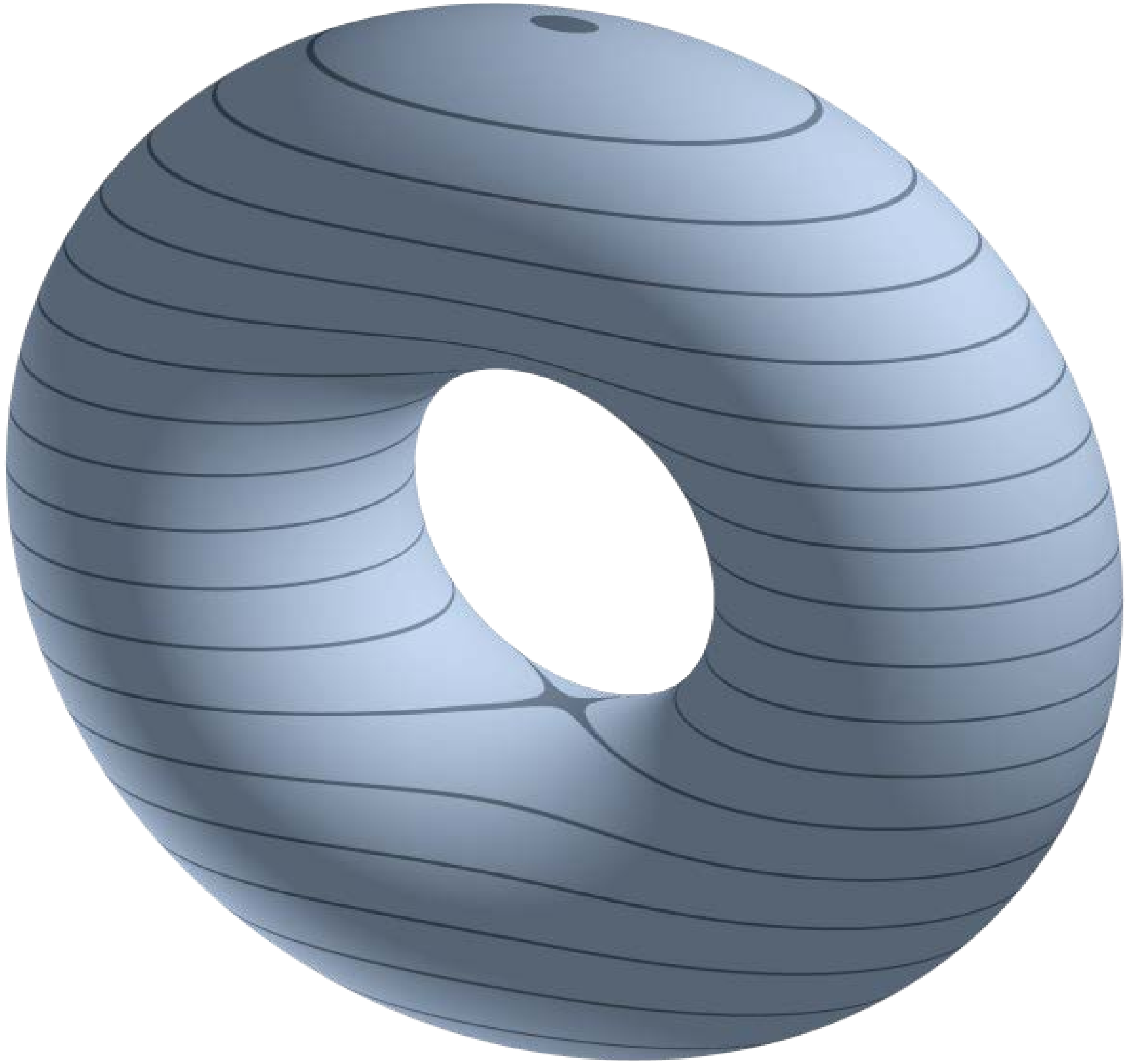
H : Hilbert space

$f \in C^1(H, \mathbf{R})$

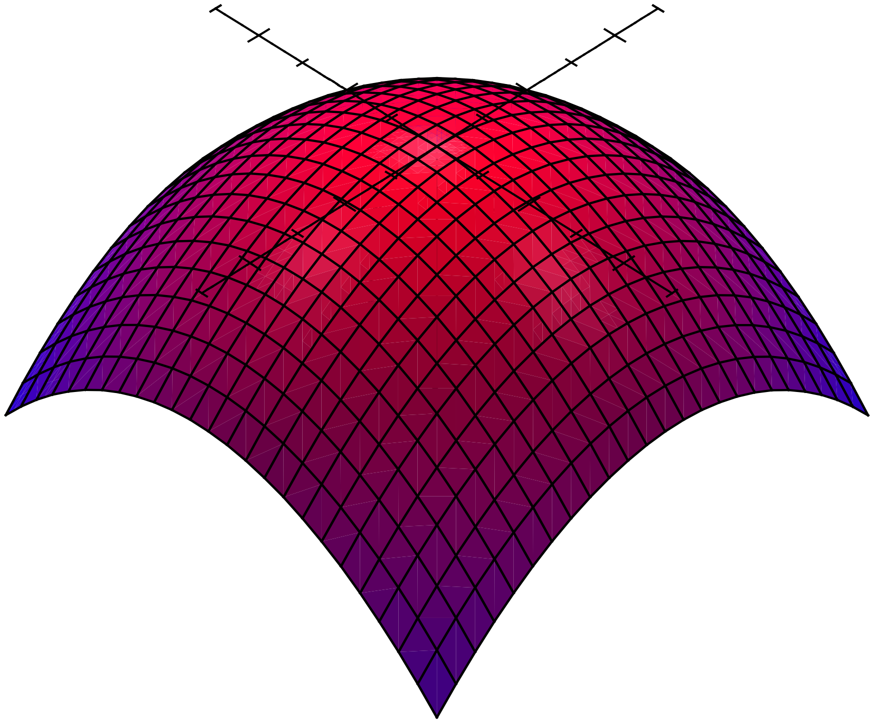
z : **isolated, local maximum** of f

$$C_q(f, z) = H_q\left(f^c \cap U, \left(f^c \setminus \{z\}\right) \cap U; G\right)$$

$$= H_q\left(B^j, S^{j-1}; G\right) = \begin{cases} G & \text{if } q = j \\ 0 & \text{if } q \neq j \end{cases}$$

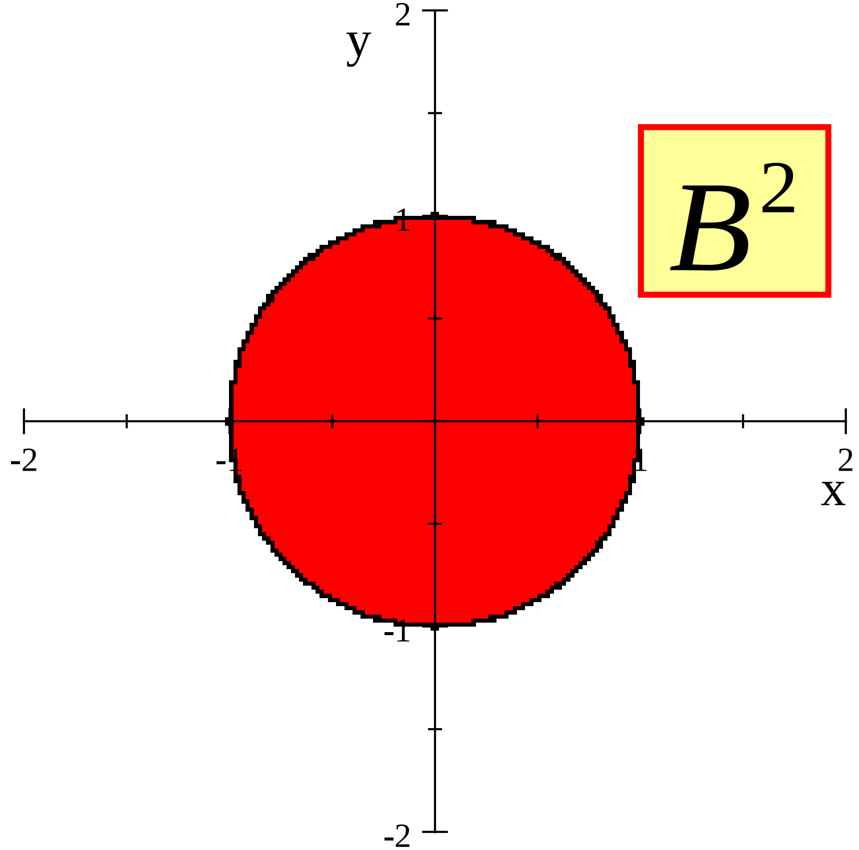


Maximum Point



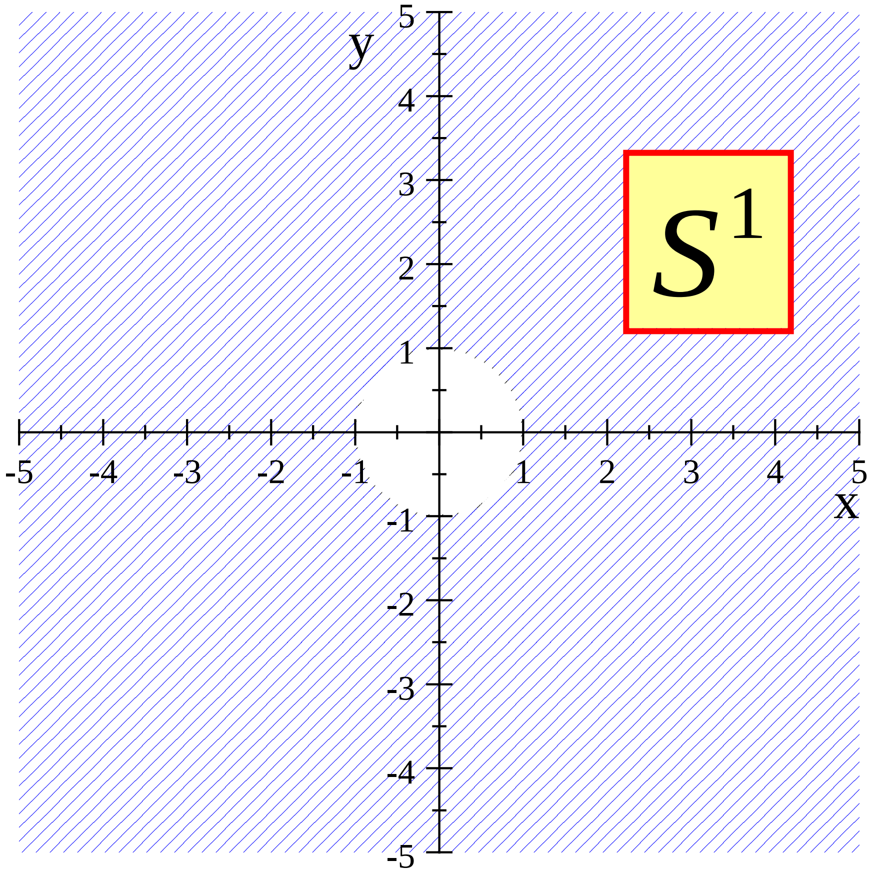
$$g(x, y) = -x^2 - y^2$$

$$g^1 = \{(x, y) \in \mathbf{R}^2 : g(x, y) \le 1\}$$



B^2

$$g^{-1} = \left\{(x, y) \in \mathbf{R}^2 : g(x, y) \leq -1\right\}$$



Critical Groups and Homology Groups

(1) $C_*(g,0) = H_*\left(B^2, S^1; G\right) = 0 \oplus 0 \oplus G$

(2) $H_*\left(g^1, g^{-1}; G\right) = H_*\left(B^2, S^1; G\right) = 0 \oplus 0 \oplus G$

Here

$$g^a = \left\{(x,y) \in \mathbf{R}^2 : g(x,y) \le a\right\}$$

2 : Morse Index

Example 3 (Saddle Point)

H : Hilbert space

$f \in C^2(H, \mathbf{R})$

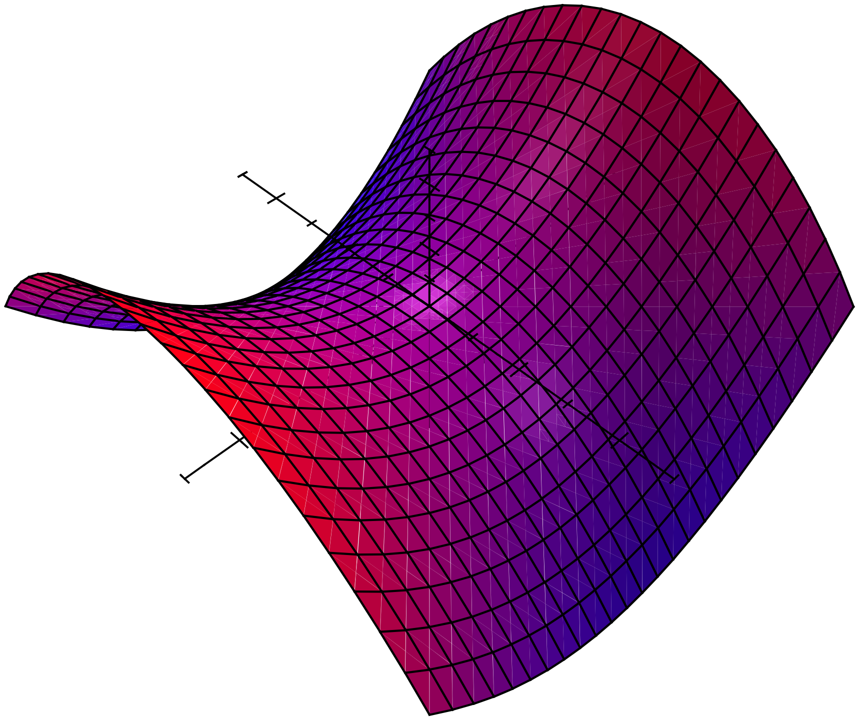
z : **non-degenerate**, critical point of f

with Morse index j

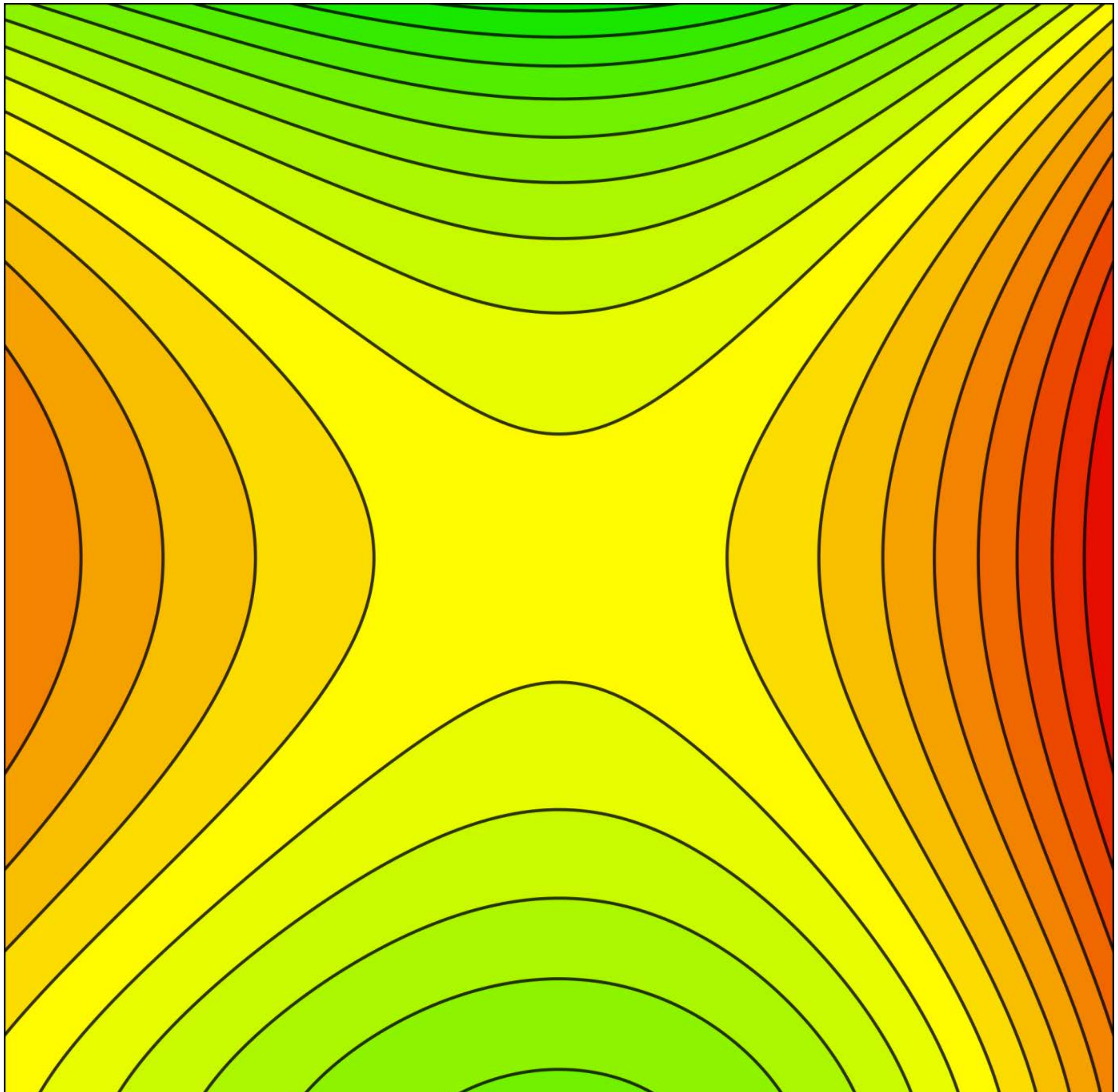
$$C_q(f,z) = H_q\left(B^j, S^{j-1}; G\right)$$

$$= \begin{cases} G & \text{if } q = j \\ 0 & \text{if } q \neq j \end{cases}$$

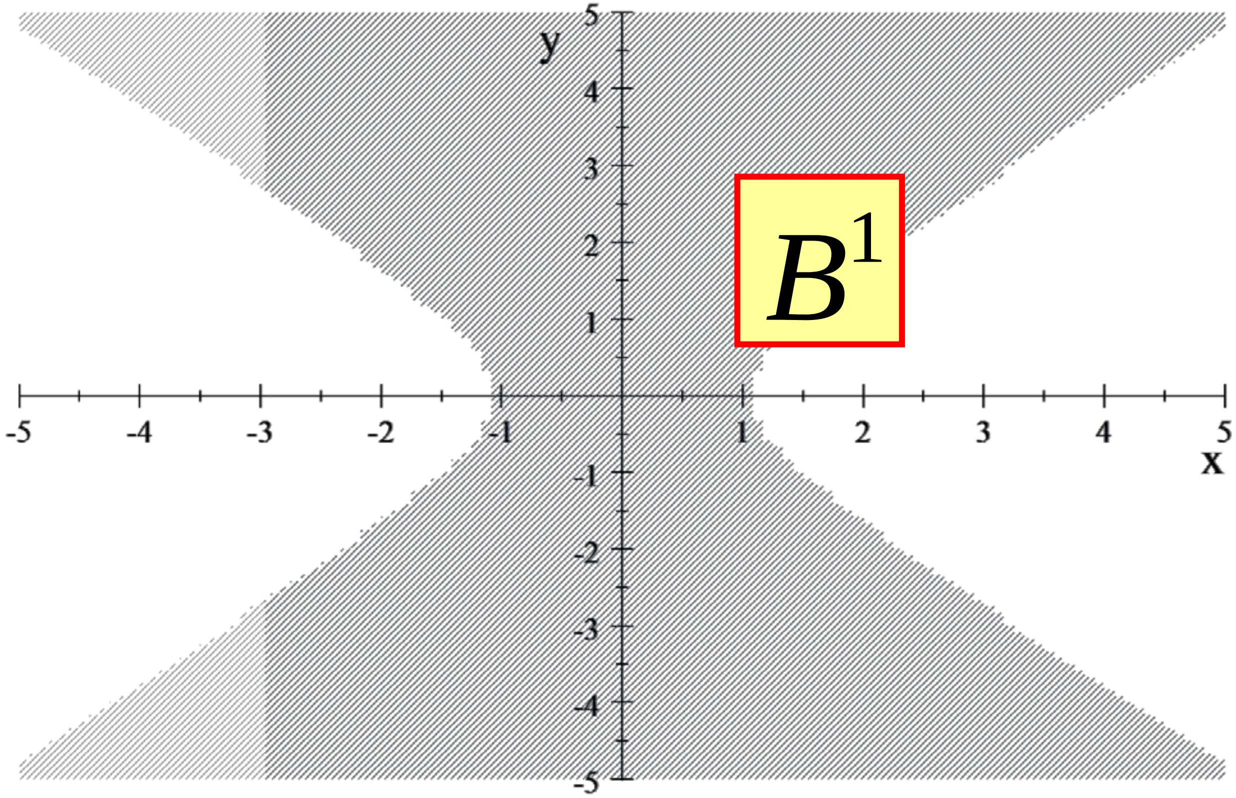
Saddle Point



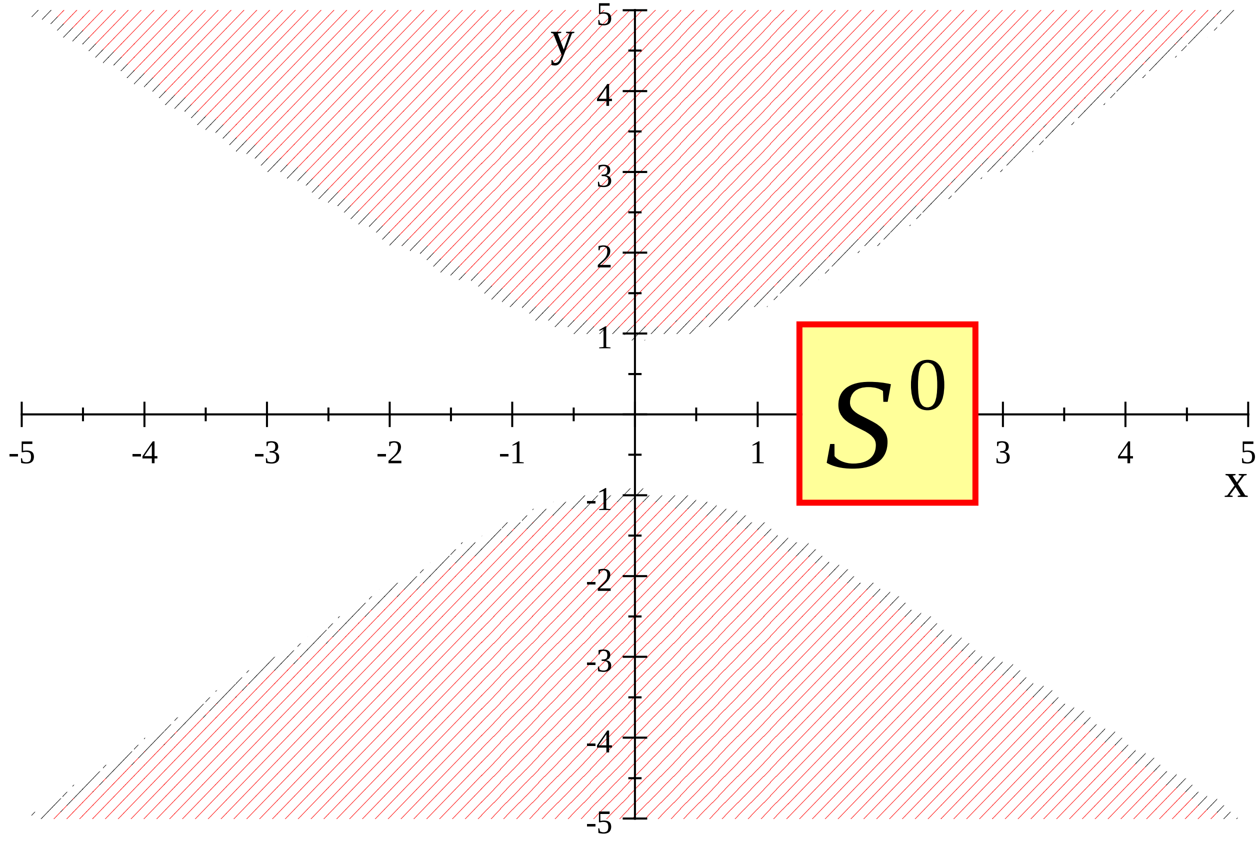
$$h(x, y) = x^2 - y^2$$



$$h^1 = \left\{ (x, y) \in \mathbf{R}^2 : h(x, y) \le 1 \right\}$$



$$h^{-1} = \left\{(x, y) \in \mathbf{R}^2 : h(x, y) \leq -1\right\}$$



Critical Groups and Homology Groups

$$(1)\ C_*(h,0) = H_*\left(B^1, S^0; G\right) = 0 \oplus G \oplus 0$$

$$(2)\ H_*\left(h^1, h^{-1}; G\right) = H_*\left(B^1, S^0; G\right) = 0 \oplus G \oplus 0$$

Here

$$h^a = \left\{(x,y) \in \mathbf{R}^2 : h(x,y) \le a\right\}$$

1: Morse Index

Minimum Point

0 : Morse Index

$$C_*(f, 0) = G \oplus 0 \oplus 0$$

Saddle Point

1: Morse Index

$$C_*(h,0) = 0 \oplus G \oplus 0$$

Maximum Point

2 : Morse Index

$$C_*(g,0) = 0 \oplus 0 \oplus G$$

Splitting Theorem

$f \in C^2(H, \mathbf{R})$

0 : **non-degenerate**, critical point of f

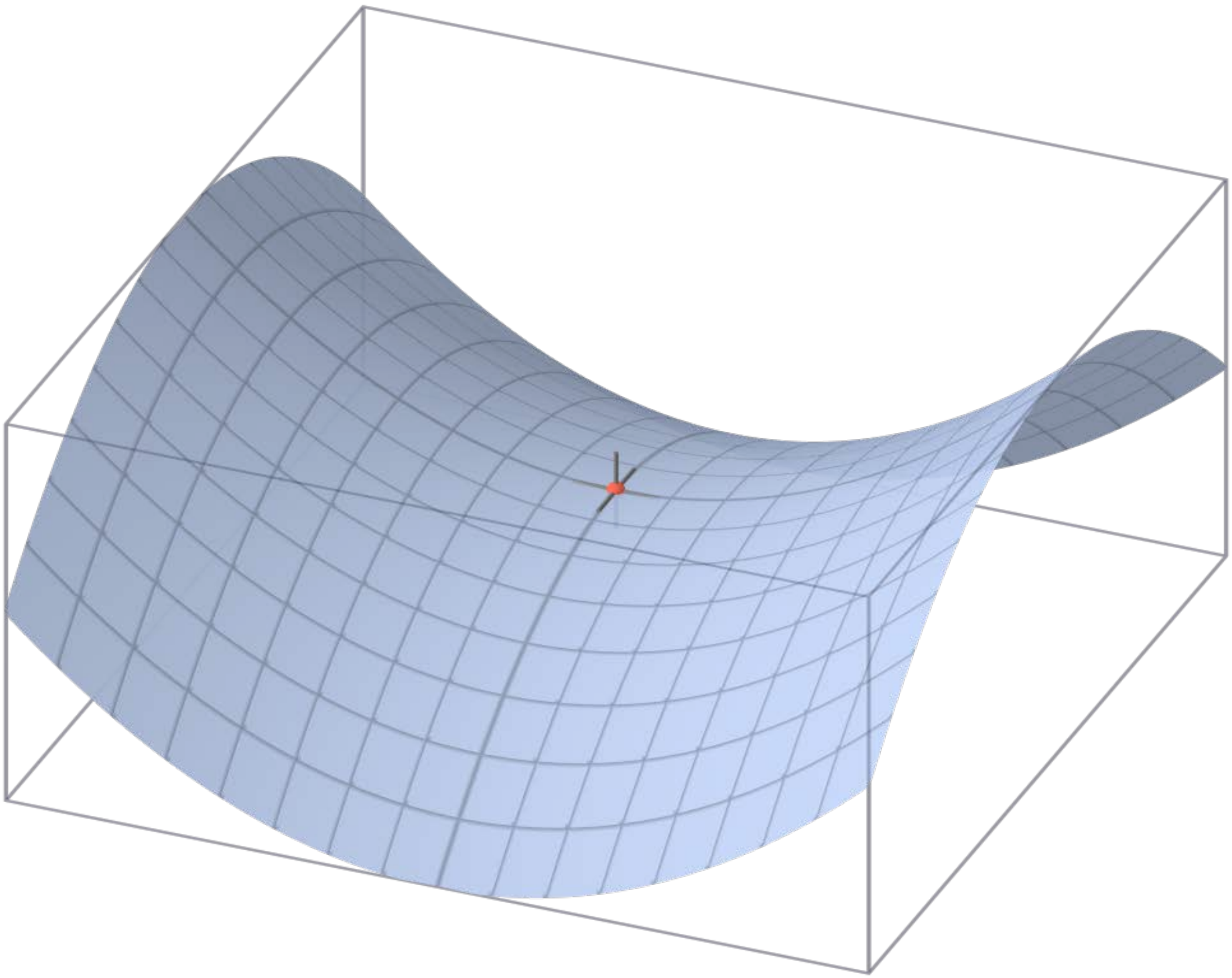
with Morse index j

$\Rightarrow$

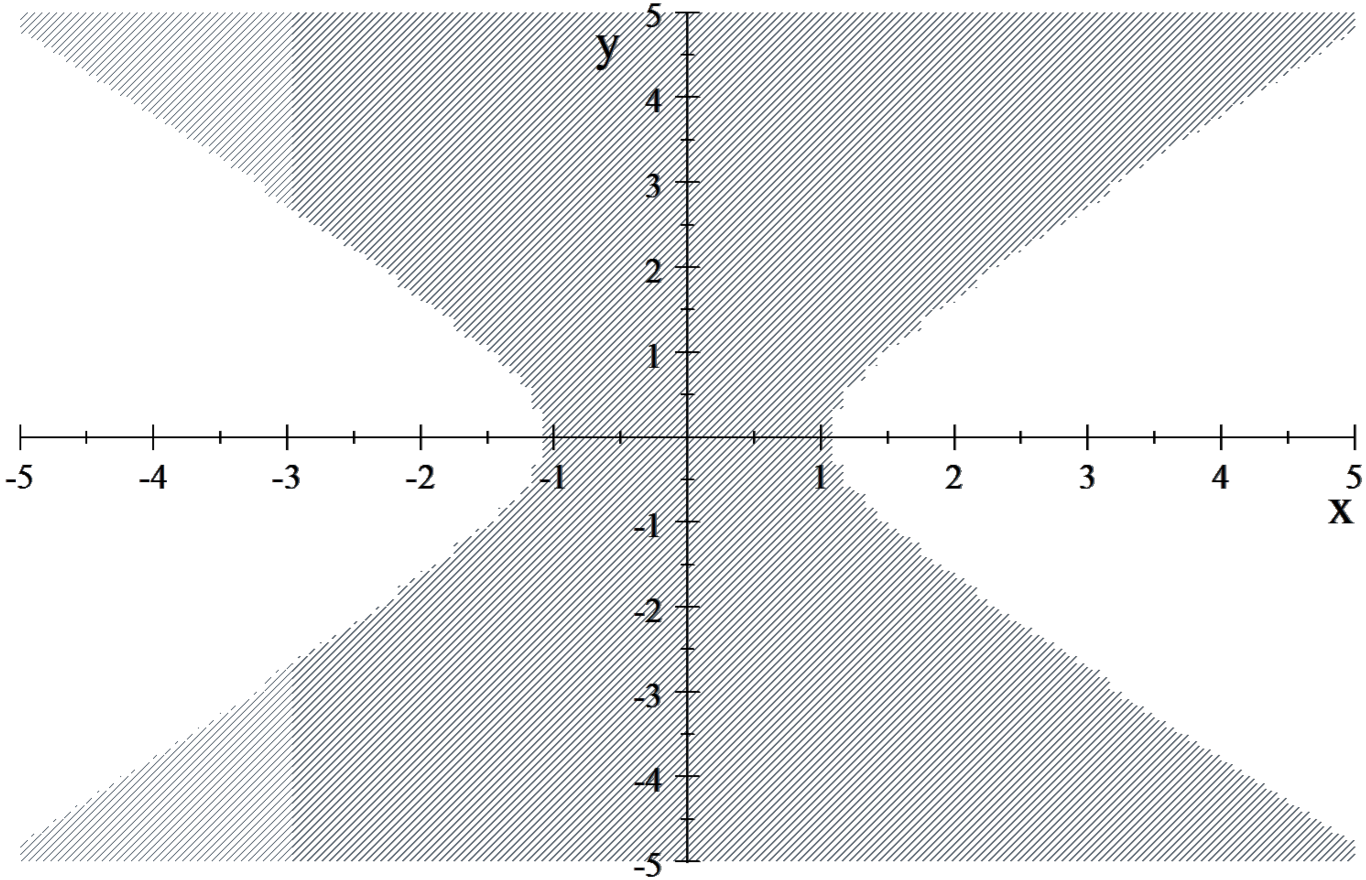
$H = H_+ \oplus H_-, \quad \dim H_- = j$

$$f(x) = \frac{1}{2}\|x_+\|_H^2 - \frac{1}{2}\|x_-\|_H^2$$

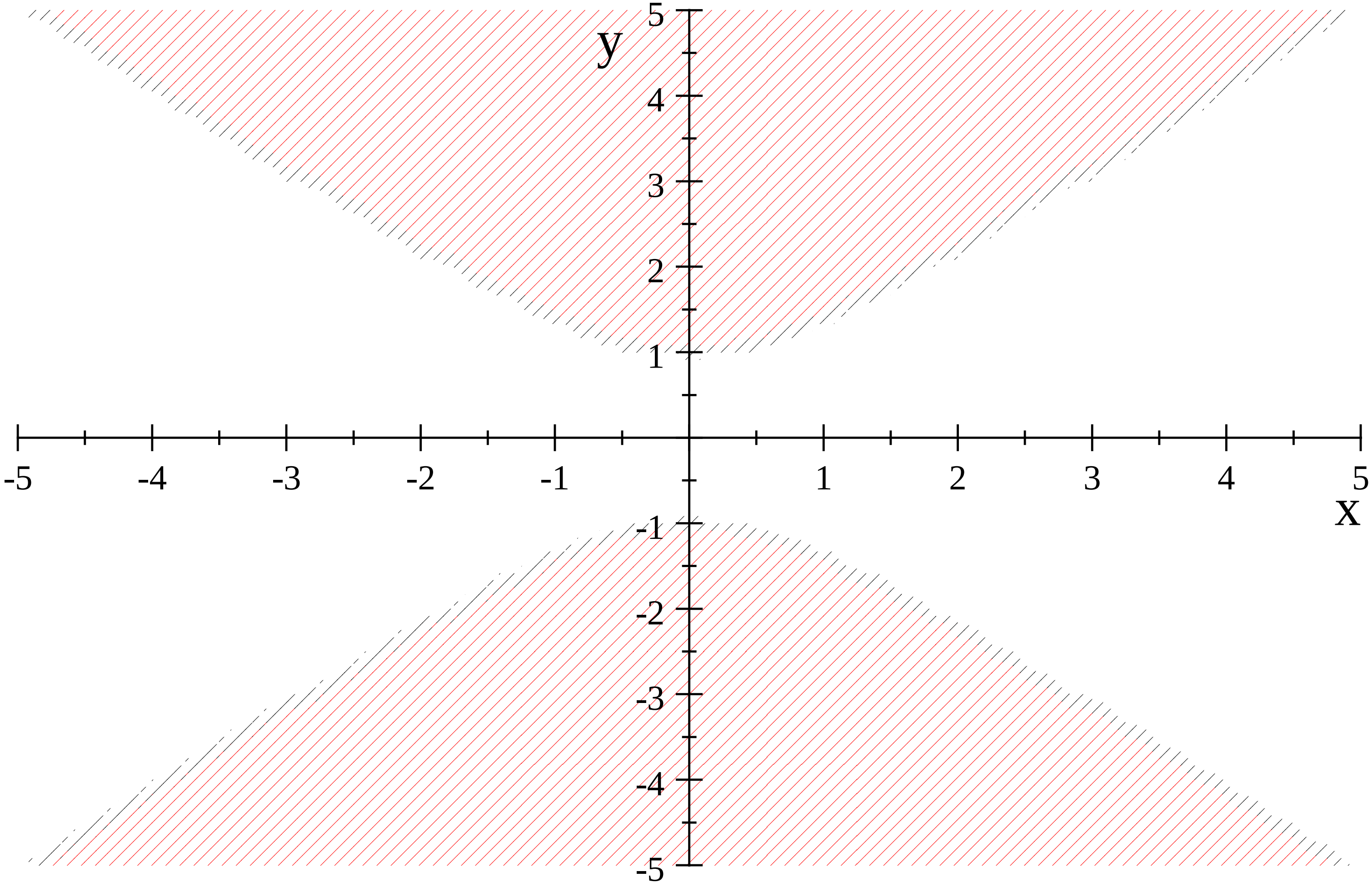
$$f(x) = \frac{1}{2}\|x_+\|_H^2 - \frac{1}{2}\|x_-\|_H^2$$



B^j



$$S^{j-1}$$



Critical Groups and Morse Indices

0 : **non - degenerate,** critical point of f

with Morse index j

$\Rightarrow$

$$C_q(f,z) = H_q\left(B^j, S^{j-1}; G\right) = \begin{cases} G & \text{if } q = j \\ 0 & \text{if } q \neq j \end{cases}$$

Relative Homology Groups

$$(1)\ H_*\left(B^1, S^0; G\right) = 0 \oplus G \quad (j = 1)$$

$$(2)\ H_*\left(B^2, S^1; G\right) = 0 \oplus 0 \oplus G \quad (j = 2)$$

$$(3)\ H_q\left(B^j, S^{j-1}; G\right) \cong H_{q-1}\left(S^{j-1}; G\right)$$

$$= \begin{cases} G & \text{if } q = j \\ 0 & \text{if } q \neq j \end{cases} \qquad (j \geq 3)$$

Morse Type Numbers

Morse Type Number (1)

H a Hilbert space

$f \in C^1(H, \mathbf{R})$

isolated, critical values of f :

$$\ldots < c_{-2} < c_{-1} < c_0 < c_1 < c_2 < \ldots$$

isolated, critical points of f :

$$f^{-1}(c_i) \cap K = \left\{ z_1^i, z_2^i, \ldots, z_{m_i}^i \right\}$$

Morse Type Number (2)

(a,b) a **pair of regular values** of f with $a<b$

$$M_q(a,b)=\sum_{a<c_i<b}\operatorname{rank}H_q(f^{c_i+\varepsilon_i},f^{c_i-\varepsilon_i};G)$$

$$0<\varepsilon_i<\min\{c_{i+1}-c_i,c_i-c_{i-1}\}$$

Morse type number of f on (a,b)

Morse Type Numbers and Critical Groups

$f \in C^2(H, \mathbf{R})$

f satisfies **(PS) condition**

(a, b) a **pair of regular values** of f

$$M_q(a,b) = \sum_{a<c_i<b} \sum_{j=1}^{m_i} \operatorname{rank} C_q(f, z_j^i)$$

$$f^{-1}(c_i) \cap K = \{z_1^i, z_2^i, \ldots, z_{m_i}^i\}$$

Morse Type Numbers and Critical Points

$f \in C^2(H, \mathbf{R})$

f satisfies (PS) condition

$M_q(a,b) =$ the number of critical points of f in (a,b) with **Morse index** q

Morse Inequalities

Fundamental Assumptions

H : Hilbert space

$f \in C^2(H, \mathbf{R})$

f satisfies **(PS) condition**

All critical points of f are **non-degenerate**

Betti Numbers

(a,b) a **pair of regular values** of f with $a<b$

$$\beta_q(a,b) = \operatorname{rank} H_q(f^b, f^a; G)$$

Betti number of (f^b, f^a)

$$f^a = f^{-1}\big((-\infty, a]\big) = \{x \in H : f(x) \le a\}$$

Morse Inequalities (1)

$f \in C^2(H, \mathbf{R})$

(1) f satisfies **(PS) condition**

(2) f is **bounded from below**

(3) f has isolated **local minima**

(4) f has **non-degenerate,** critical points of **positive Morse index**

Morse Inequalities (2)

$\beta_k(b) = \text{rank}\, H_k(f^b; G)$ (Betti number)

$C_0(b)$ = the number of isolated **local minima**

$C_m(b)$ = the number of **non-degenerate**, critical points of **Morse index** m in f^b

Morse Inequalities (3)

$$\beta_0(b) \le C_0(b)$$

$$\boxed{\beta_1(b) - \beta_0(b) \le C_1(b) - C_0(b)}$$

$$\sum_{m=0}^{k} (-1)^{k-m} \beta_m(b) \le \sum_{m=0}^{k} (-1)^{k-m} C_m(b)$$

Four-Solution Theorem for Saddle Points

Four-Solution Theorem (1)

$f \in C^2(H, \mathbf{R})$

(1) f satisfies **(PS) condition**

(2) f is **bounded from below**

(3) 0 is a **non-degenerate** critical point with **Morse index** $\boxed{q_0 \geq 2}$

(4) f has **two** **local minima** u_1, u_2

Four-Solution Theorem (2)

f has at least **another non - zero** critical point u_3

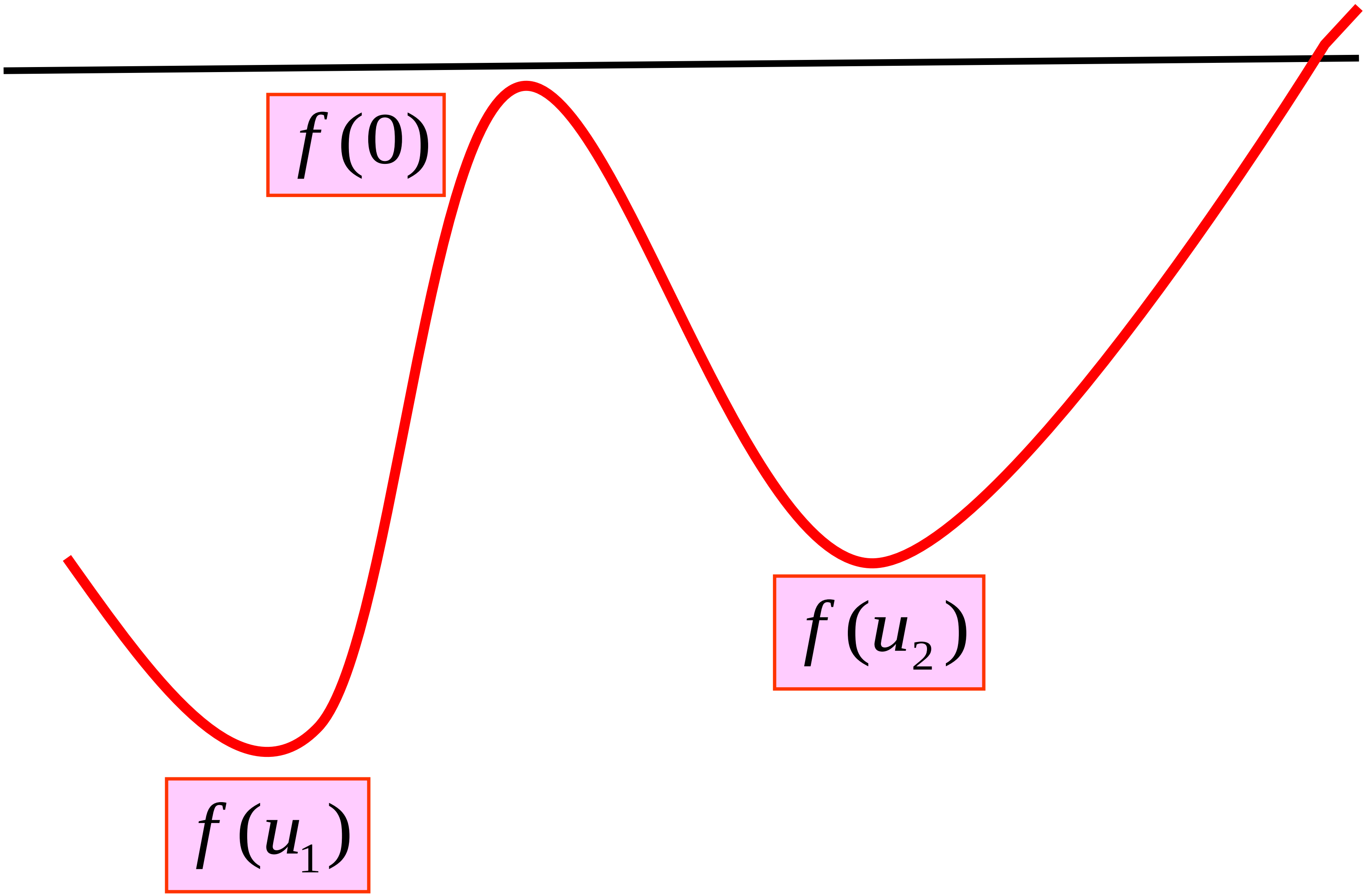
Proof (1)

Reduction to Absurdity

f has only **three** **critical points** $u_1, u_2, 0$

$b > \max\{f(u_1), f(u_2), f(0)\}$

$$f^b = \{ x \in H : f(x) \leq b \}$$



Proof (2)

$$C_q(b) = \begin{cases} 2 & \text{if } q = 0 \\ 0 & \text{if } q \geq 1, q \neq q_0 \\ 1 & \text{if } q = q_0 \geq 2 \end{cases}$$

$$\Rightarrow$$

$$C_0(b) = 2, \quad C_1(b) = 0$$

Proof (3)

$$\beta_q(b) = \operatorname{rank} H_q(f^b; G)$$

$$= \operatorname{rank} H_q(H; G) = \begin{cases} 1 & \text{if } q = 0 \\ 0 & \text{if } q \geq 1 \end{cases}$$

$$\Rightarrow$$

$$\boxed{\beta_0(b) = 1, \quad \beta_1(b) = 0}$$

Proof (4)

$$\beta_1(b) - \beta_0(b) = -1$$
$$\leq C_1(b) - C_0(b) = -2$$

Contradiction!

Ljusternik-Schnirelmann Theory on Hilbert Spaces

References (Papers)

- **Schwartz:** **Generalizing the Lusternik-Schnirelman theory of critical points, Comm. Pure Appl. Math. 17 (1964), 307-315**
- **Palais:** **Lusternik-Schnirelman theory on Banach manifolds, Topology 5 (1966), 115-132**
- **Clark:** **A variant of Lusternik-Schnirelman theory, Indiana Univ. Math. J. 22 (1972), 65-74**

Krasnoselskii Genus

Symmetric Sets and Odd Maps

H : real Hilbert space

(1) A subset $A \subset H$ is said to be **symmetric** with respect to 0 if

$$u \in A \Rightarrow -u \in A$$

(2) A map $f : A \to \mathbf{R}^n$ is said to be **odd** if

$$f(-x) = -f(x), \quad \forall x \in A$$

Krasnoselskii Genus (1)

$\gamma(A)$ = the least integer n such that

$\exists \phi \in C\left(A, R^n \setminus \{0\}\right)$ **odd map**

Krasnoselskii Genus (2)

$\gamma(A)$ = the least integer n such that

$\exists \psi \in C\left(H, R^{n}\right)$ **odd map**

$\psi(x) \neq 0, \forall x \in A$

Tietze's Extension Theorem

Let X be a metric space and A a closed subset.

Let L be a locally convex topological

linear space and $f: A \to L$ a continuous map.

Then there exists a **continuous extension map**

$$\boxed{F: X \to L}$$

of f.

Fundamental Properties

(1) $A \subset B \Rightarrow \gamma(A) \leq \gamma(B)$ **(Monotonicity)**

(2) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$ **(Subadditivity)**

(3) $p \neq 0, [p] = \{p, -p\} \Rightarrow \gamma([p]) = 1$ **(Normality)**

(4) $\boxed{\gamma(A) = m \Rightarrow \#(\gamma(A)) \geq m}$

(5) $\gamma(S^n) = n + 1$

Borsuk-Ulam Theorem

Let Ω be a symmetric, bounded open subset of $\mathbf{R}^n$ including the origin, with boundary $\partial\Omega$. Let $g:\partial\Omega \to \mathbf{R}^m$ be a continuous and **odd map** for $\boxed{m < n}$.

Then there exists a point $x_0 \in \partial\Omega$ such that

$$\boxed{g(x_0) = 0}$$

Multiplicity Theorems

Multiplicity Theorem 1

H : real Hilbert space

$f \in C^1(H, \mathbf{R}),\ f(-x) = f(x), \forall x \in H$

$$c_n(f) = \inf_{\gamma(A) \geq n} \sup_{x \in A} f(x), \quad n = 1, 2, \cdots$$

(1) $c = c_{k+1}(f) = \cdots = c_{k+m}(f) < \infty$

(2) f satisfies (PS) condition

$\Longrightarrow$

$$\gamma(K_c) \geq m$$

$K_c = \{x \in H : f(x) = c, \nabla f(x) = 0\}$

Multiplicity Theorem 2-1 (Analytic Version)

Let H be a Hilbert space, $f \in C^1(H, \mathbf{R})$

and $a < b$

(1) $f(0) > b$ and

$$\boxed{f(-x) = f(x), \forall x \in H}$$

(2) f satisfies **(PS) condition**

Multiplicity Theorem 2-2 (Analytic Version)

Assume the following:

(i) $\exists E \subset H$, $\boxed{\dim E = m}$, $\exists \rho > 0$ such that

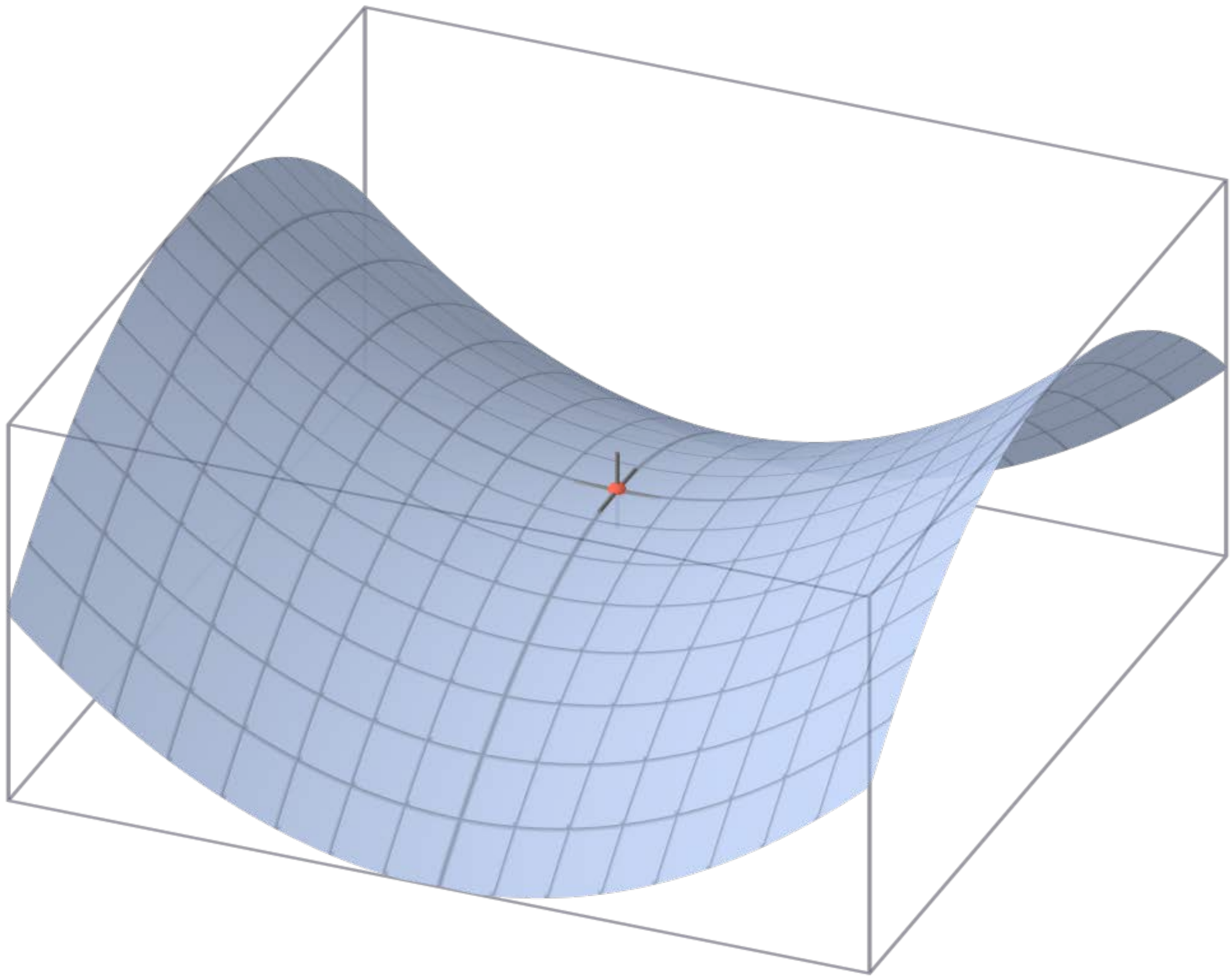
$$\boxed{\sup_{x \in E \cap S_\rho(0)} f(x) \le b}$$

(ii) $\exists F \subset H$, $\boxed{\dim F = j}$ such that

$$\boxed{\inf_{x \in F^\perp} f(x) > a}$$

(iii) $m > j$

Then $f(x)$ has at least $(m - j)$ pairs of distinct critical points.



Variational Theory

L^2 Approach

Linear Boundary Value Problems

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u$$

$$Bu(x') = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 \text{ on } \partial\Omega.$$

Linear Operator $\mathfrak{A}$

We define a linear operator

$$\mathfrak{A} : L^2(\Omega) \to L^2(\Omega)$$

as follows:

(a) The domain $D(\mathfrak{A})$ is the set

$$D(\mathfrak{A}) = \{u \in H^2(\Omega) = W^{2,2}(\Omega) : Bu = 0\}.$$

(b) $\mathfrak{A}u = -\Delta u, \ \forall u \in D(\mathfrak{A})$.

$\Rightarrow$

$\mathfrak{A}$ is a **positive definite**, self-adjoint operator

Spectral Properties of $\mathfrak{A}$

(1) The first eigenvalue λ_1 is positive and **algebraically simple**.

(2) The corresponding eigenfunction $\phi_1(x)$ may be chosen **strictly positve** in Ω:

$$\boxed{\begin{array}{l}\mathfrak{A}\phi_1 = \lambda_1\phi_1, \\ \phi_1(x) > 0 \text{ in } \Omega\end{array}}$$

(3) No other eigenvalues $\lambda_j, \; j \geq 2,$ have positive eigenfunctions.

Infinite Dimensional Version
of
Perron-Frobenius Theorem

Ordered Vector Space

V **is an ordered vector space**

$\overset{\text{def}}{\Longleftrightarrow}$

(i) $(V, \leq)$ **is an ordered set.**

(ii) V **is a real vector space.**

(iii) **The ordering** $\leq$ **is linear :**

(a) $x, y \in V, x \leq y \Rightarrow x + z \leq y + z, \forall z \in V.$

(b) $x, y \in V, x \leq y \Rightarrow \alpha x \leq \alpha y, \forall \alpha \geq 0.$

Ordered Banach Space

E **is an ordered Banach space**

$\overset{\text{def}}{\Leftrightarrow}$

(i) E **is a Banach space.**

(ii) $(E,\leq)$ **is an ordered vector space.**

(iii) $P:=\{x\in E: x\geq 0\}$, **positive cone, is closed.**

(a) $x,y\in P\Rightarrow \alpha x+\beta y\in P,\ \forall\alpha,\beta\geq 0.$

(b) $P\cap(-P)=\{0\}.$

Krein-Rutman Theorem (1)

Let (E, P) be an ordered Banach space with non-empty interior, and assume that $K : E \to E$ is strongly positive and compact.

$$K\left(P \setminus \{0\}\right) \subset \mathrm{Int}\left(P\right)$$

Krein-Rutman Theorem (2)

(i) $r = \lim_{n\to\infty} \|K^n\|^{1/n} > 0$, **(spectral radius)**

r **is a unique eigenvalue of** K

having a positive eigenfunction.

r **is algebraically simple.**

(ii) r **is also an algebraically simple**

eigenvalue of the adjoint $K^* : E^* \to E^*$

with a positive eigenfunction.

Difficult Point

Typical Example

$$\begin{cases} -\Delta u = g \quad \text{in } \Omega, \\ Bu(x') = a(x')\dfrac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))u = 0 \text{ on } \partial\Omega. \end{cases}$$

(H.1) $0 \leq a(x') \leq 1$ on $\partial\Omega$.

(H.2) $\boxed{a(x') \not\equiv 1 \text{ on } \partial\Omega.}$

Construction of Green Operator

Reduction to the Boundary

Reduction to the Boundary (1)

Consider the boundary value problem

$$\begin{cases} -\Delta u = g \quad \text{in } \Omega, \\ a(x')\dfrac{\partial u}{\partial \mathbf{n}} + \left(1 - a(x')\right)u = 0 \quad \text{on } \partial\Omega \end{cases}$$

Reduction to the Boundary (2)

Solve the Robin problem

$$\begin{cases} -\Delta v = g \in H^{s-2,p}(\Omega), \\ \dfrac{\partial v}{\partial \mathbf{n}} + \left(1 - a(x')\right) v = 0 \end{cases}$$

We let

$$v := \exists! Gg \in W^{s,p}\left(\Omega\right)$$

(H.2) $a(x') \not\equiv 1 \ on \ \partial\Omega.$

Reduction to the Boundary (3)

Let

$$w := u - v = u - Gg$$

Reduction to the Boundary (4)

$$B(Gg)$$

$$= a(x')\frac{\partial v}{\partial \mathbf{n}} + \left(1 - a(x')\right)v$$

$$= \left(1 - a(x')\right)^2 v$$

Gain of One Derivative

$$Bw = Bu - Bv$$

$$= -B(Gg)$$

$$= -\left(1 - a(x')\right)^2 v$$

Reduction to the Boundary (5)

Then

$$\begin{cases} -\Delta u = g \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} -\Delta w = -\Delta u + \Delta v = 0 \ \text{ in } \Omega, \\ Bw = -\left(1 - a(x')\right)^2 v \ \text{ on } \partial\Omega \end{cases}$$

Reduction to the Boundary (7)

$$-\Delta w = 0 \quad \text{in } \Omega$$

$$\Leftrightarrow$$

$$w = P\psi$$ **(Poisson operator)**

Reduction to the Boundary (8)

$$\begin{cases} -\Delta u = g \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega \end{cases}$$

$$\Longleftrightarrow$$

$$(BP)\psi = Bw$$

$$= -\left(1 - a(x')\right)^2 v \text{ on } \partial\Omega$$

Fredholm Boundary Operator

$$(BP)\psi = a(x')\frac{\partial}{\partial \mathbf{n}}\left(P\psi\right) + \left(1 - a(x')\right)\psi$$

Degeneracy
of
a Pseudo-Differential Operator

Associate Pseudo-Differential Operator (Fredholm Boundary Operator)

$$T = BP = a(x')\sqrt{-\Lambda} + \left(1 - a(x')\right)$$

$$\boldsymbol{\sigma}(T)(x', \xi') = a(x')\left|\xi'\right| + 1 - a(x')$$

$$\boxed{0 \leq a(x') \leq 1 \text{ on } \partial\Omega.}$$

Criteria for Parametrices

Criteria for Parametrices (1)

Let $A = p(x, D)$ be a properly supported pseudo-differential operator in the class $L^m_{1,0}(\Omega)$

Criteria for Parametrices (2)

Assume that:

$$\left| D_\xi^\alpha D_x^\beta p(x,\xi) \right| \le \exists C_{K,\alpha,\beta} \left| p(x,\xi) \right| \left(1+|\xi|\right)^{-|\alpha|+(1/2)|\beta|}$$

$$\left| p(x,\xi)^{-1} \right| \le \exists C_K, \ \forall x \in K \subset \Omega, \ \forall \ |\xi| \ge C_K.$$

Criteria for Parametrices (3)

$$\Longrightarrow$$

$$\exists B \in L^{\mathbf{0}}_{1,\mathbf{1/2}}(\Omega) \quad \text{such} \quad \text{that}$$

$$AB \equiv I \mod L^{-\infty}(\Omega),$$

$$BA \equiv I \mod L^{-\infty}(\Omega).$$

Existence
of
a Parametrix

Loss of One Derivative

$$T = a(x')\sqrt{-\Lambda} + 1 - a(x') \in L^{1}_{1,0}(\partial\Omega)$$

$$\Rightarrow$$

$$\exists S \in L^{0}_{1,1/2}(\partial\Omega) \text{ such that}$$

$$TS \equiv ST \equiv I \mod L^{-\infty}(\partial\Omega)$$

Elementary Lemma

$$f(x) \in C^2(\mathbf{R}),$$

$$f(x) \geq 0 \text{ on } \mathbf{R},$$

$$\sup_{x \in \mathbf{R}} |f''(x)| \leq \exists c$$

$$\Rightarrow$$

$$|f'(x)| \leq \sqrt{2c}\left(f(x)\right)^{1/2} \text{ on } \mathbf{R}.$$

Infinite Dimensional Version
of
Perron-Frobenius Theorem

Ordered Vector Space

V **is an ordered vector space**

$\overset{\text{def}}{\Longleftrightarrow}$

(i) $(V, \leq)$ **is an ordered set.**

(ii) V **is a real vector space.**

(iii) **The ordering** $\leq$ **is linear :**

(a) $x, y \in V, x \leq y \Rightarrow x + z \leq y + z, \forall z \in V.$

(b) $x, y \in V, x \leq y \Rightarrow \alpha x \leq \alpha y, \forall \alpha \geq 0.$

Ordered Banach Space

E **is an ordered Banach space**

$\overset{\text{def}}{\Leftrightarrow}$

(i) E **is a Banach space.**

(ii) $(E, \leq)$ **is an ordered vector space.**

(iii) $P := \{x \in E : x \geq 0\}$, **positive cone, is closed.**

(a) $x, y \in P \Rightarrow \alpha x + \beta y \in P,\ \forall \alpha, \beta \geq 0.$

(b) $P \cap (-P) = \{0\}.$

Example

$$Y = C(\overline{\Omega}),$$

$$u \le v \overset{\text{def}}{\Longleftrightarrow} u(x) \le v(x),\ \forall x \in \overline{\Omega}$$

$$\Rightarrow$$

$$P_Y = \left\{ u \in C(\overline{\Omega}) : u \ge 0 \ \textbf{on}\ \overline{\Omega} \right\},$$

$$\mathrm{Int}(P_Y) = \left\{ u \in C(\overline{\Omega}) : u > 0 \ \textbf{on}\ \overline{\Omega} \right\}$$

Strong Positivity

$$Ku(x) = \int_{\Omega} k(x,y)u(y)dy$$

$$k(x,y) > 0$$

$$\Leftrightarrow$$

$u(x) \geq 0 \;\Rightarrow\; Ku(x) > 0$ **strongly positive**

$$K\left(P \setminus \{0\}\right) \subset \mathrm{Int}(P)$$

Krein-Rutman Theorem (1)

Let (E, P) be an ordered Banach space with non-empty interior, and assume that $K : E \to E$ is strongly positive and compact.

$$K\left(P \setminus \{0\}\right) \subset \mathrm{Int}\left(P\right)$$

Krein-Rutman Theorem (2)

(i) $r = \lim_{n\to\infty} \|K^n\|^{1/n} > 0$, **(spectral radius)**

r **is a unique eigenvalue of** K

having a positive eigenfunction.

r **is algebraically simple.**

(ii) r **is also an algebraically simple**

eigenvalue of the adjoint $K^* : E^* \to E^*$

with a positive eigenfunction.

Reduction to the Boundary

Existence
of
a Parametrix

Associate Pseudo-Differential Operator (Fredholm Boundary Operator)

$$T = BP = a(x')\sqrt{-\Lambda} + b(x')$$

$$\boldsymbol{\sigma}(T)(x', \xi') = a(x')\left|\xi'\right| + b(x')$$

$$\Lambda = \text{Laplace-Beltrami Operator}$$

$$T = a(x')\sqrt{-\Lambda} + b(x') \in L^{1}_{1,0}(\partial\Omega)$$

$$\boxed{a(x') + b(x') > 0 \text{ on } \partial\Omega}$$

$\Rightarrow$

$\exists S \in L^{0}_{1,1/2}(\partial\Omega)$ such that

$TS \equiv ST \equiv I \mod L^{-\infty}(\partial\Omega)$

Besov-Space Boundedness Theorem

Besov-Space Boundedness Theorem

Every properly supported operator

$A \in L^{m}_{1,\delta}(\Omega), \quad 0 \le \delta < 1,$

extends to a continuous linear operator

$A : H^{s,p}_{\mathrm{loc}}(\Omega) \to H^{s-m,p}_{\mathrm{loc}}(\Omega), \forall s \in \mathbf{R}, 1 < \forall p < \infty$

and to a continuous linear operator

$A : B^{s,p}_{\mathrm{loc}}(\Omega) \to B^{s-m,p}_{\mathrm{loc}}(\Omega), \forall s \in \mathbf{R}, 1 \le \forall p \le \infty$

Hilbert Space

Fractional Power of $\mathfrak{A}$

$C = \sqrt{\mathfrak{A}}$, **square root of** $\mathfrak{A}$

$$Cu = -\frac{1}{\pi}\int_0^\infty s^{-1/2}(sI + \mathfrak{A})^{-1}\mathfrak{A}u\,ds$$

$$u \in D(\mathfrak{A})$$

Function Space (1)

$C = \sqrt{\mathfrak{A}}$, **square root** of $\mathfrak{A}$

H = the domain $D(C)$ with the inner product $(u,v)_H = (Cu, Cv)$.

Function Space (2)

H = the **completion** of the domain $D(\mathfrak{A})$

with respect to the **inner product**

$$(u,v)_H = \sum_{i,j=1}^{N} a^{ij}(x) \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_j} dx$$

$$+ \int_\Omega c(x) u \cdot v dx + \int_{\{a \neq 0\}} \frac{b(x')}{a(x')} u \cdot v \, d\sigma$$

$$
\begin{array}{ccc}
L^2(\Omega) & \xrightarrow{\;C^{-1}\;} & D(C) = H \\
\uparrow & & \uparrow \\
D(C) & \xrightarrow{\;C^{-1}\;} & D(C^2) = D(\mathfrak{A}) \\
\uparrow & & \uparrow \\
D(\mathfrak{A}) & \xrightarrow{\;C^{-1}\;} & D(\mathfrak{A}^{3/2})
\end{array}
$$

Function Space (3)

$$D(\mathfrak{A}) \subset D(C) = H \subset H^1(\Omega) = W^{1,2}(\Omega)$$

Function Spaces (4)

Observe the following fact (due to D. Fujiwara):

$$H = \begin{cases} H_0^1(\Omega) \text{ if } a(x') \equiv 0 \text{ on } \partial\Omega \text{ (\textbf{Dirichlet}),} \\ H^1(\Omega) \text{ if } a(x') \equiv 1 \text{ on } \partial\Omega \text{ (\textbf{Robin}).} \end{cases}$$

Weak Solutions

Semilinear Elliptic Boundary Value Problems

For a given function $p(t)$,

find a function $u(x)$ in Ω such that

$$\begin{cases} Au = p(u) \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Definition of a Weak Solution

$u \in H$ is a **weak solution**

$\Leftrightarrow$

$$(u,v)_H - \int_\Omega p(u) v \, dx$$

$$= \sum_{i,j=1}^{N} a^{ij}(x) \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_j} dx + \int_\Omega c(x) u \cdot v dx$$

$$- \int_\Omega p(u) v \, dx + \int_{\{a \neq 0\}} \frac{b(x')}{a(x')} u \cdot v \, d\sigma$$

$$= 0 \quad \forall v \in H$$

Superlinear Case

Semilinear Elliptic Boundary Value Problems

For a given function $g(t)$,

find a function $u(x)$ in Ω such that

$$\begin{cases} -\Delta u = \lambda u - g(u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Nonlinearity Conditions (1)

$$f(t) = \lambda t - g(t), \quad \lambda \in \mathbf{R}$$

Nonlinearity Conditions (2)

(A) $g \in C^1(\mathbf{R})$, $\boxed{g(0) = g'(0) = 0}$.

(B) The limits $g'(\pm\infty)$ satisfies the conditions

$$\boxed{g'(\pm\infty) = \lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty}$$

The Case

$$\lambda > \lambda_1$$

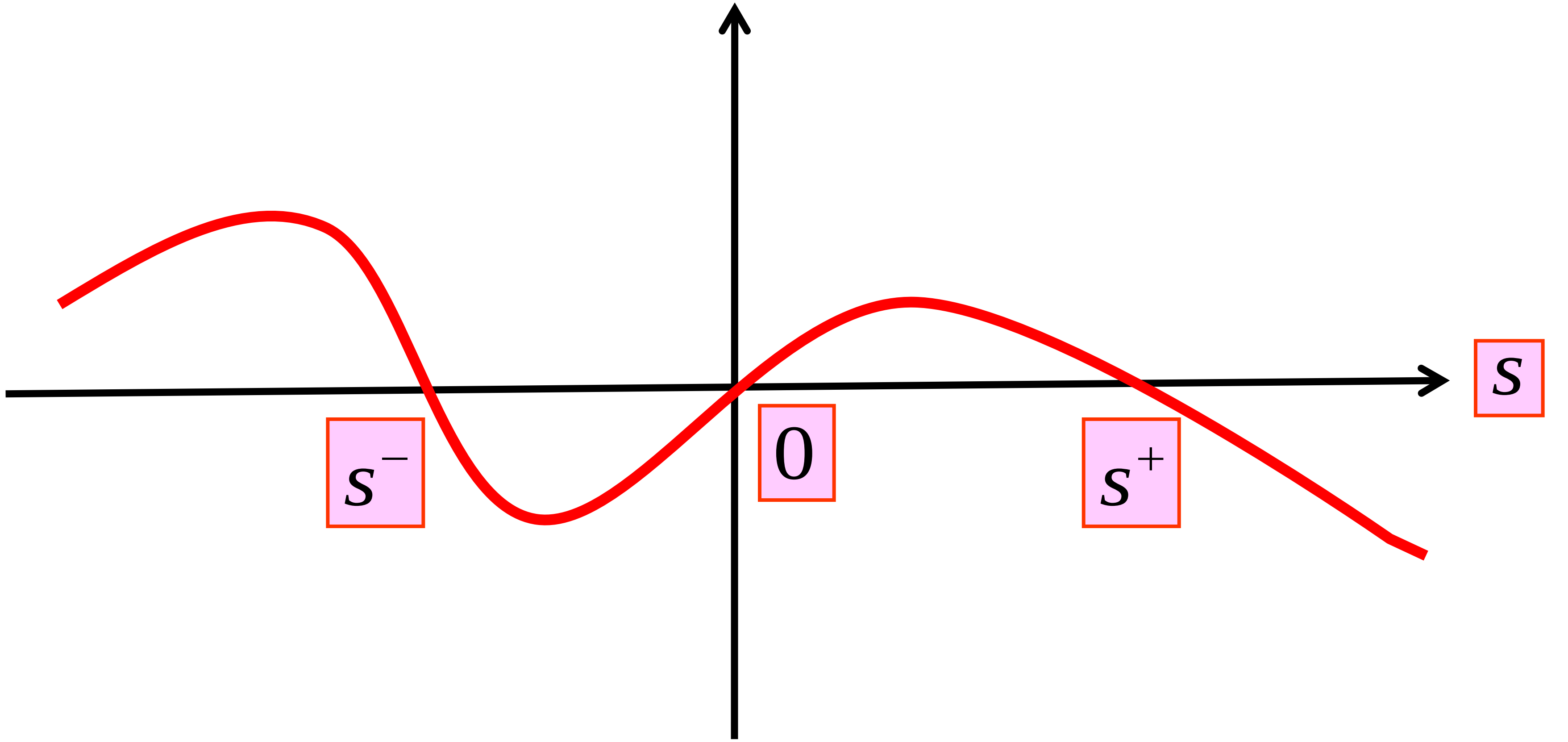
Truncation
of
a Non-linear Term

Nonlinearity Conditions (3)

$$\exists s^- < 0 < \exists s^+ \text{ such that}$$

$$\lambda s^+ - g(s^+) \le 0 \le \lambda s^- - g(s^-)$$

Outline of $p(s)$



Example

$$g(s) = s|s|^{p-1}, \quad p > 1$$

$$\Rightarrow$$

$$s^{\pm} = \pm\lambda^{\frac{1}{p-1}}$$

Nonlinearity Conditions (4)

$$p(s) = \begin{cases} < 0, & s > s^{+} \\ \lambda s - g(s), & s^{-} \leq s \leq s^{+} \\ > 0, & s < s^{-} \end{cases}$$

$$|p(s)| \leq L \quad \text{on } \mathbf{R}$$

$$|p'(s)| \leq L \quad \text{on } \mathbf{R}$$

Modified Semilinear Problem

For a given function $p(t)$,

find a function $u(x)$ in Ω such that

$$\begin{cases} Au = p(u) \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Energy Functionals

Energy Functional

$$F(u) = \frac{1}{2}(u,u)_H - \int_{\Omega} P(u(x))\,dx$$

$$P(s) = \int_0^s p(t)\,dt$$

Gradient

$$\left(\nabla F(u), v\right)_H = \left(u, v\right)_H - \int_\Omega p(u) v \, dx$$

$$= \left(u, v\right)_H - \left(\mathfrak{A}^{-1}(p(u)), v\right)_H$$

$$= \left(u - \mathfrak{A}^{-1}(p(u)), v\right)_H$$

$$\Leftrightarrow$$

$$\boxed{\nabla F(u) = u - \mathfrak{A}^{-1}(p(u))}$$

Critical Points (Euler-Lagrange)

$$\boxed{\nabla F(u^*) = 0}$$

$$\Leftrightarrow$$

$$u^* = \mathfrak{A}^{-1}(p(u^*))$$

$$\Leftrightarrow$$

$$\boxed{\mathfrak{A}u^* = p(u^*)}$$

Critical Points and Weak Solutions

u^* is a **critical point** of F

$\Leftrightarrow$

u^* is a **weak solution**

of the problem

$$\begin{cases} Au^* = p(u^*) \ \text{ in } \Omega, \\ Bu^* = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Weak Solutions
imply
Classical Solutions

Regularity Theorem (1)

$$\begin{cases} u \in L^p(\Omega) \\ Au \in W^{s-2,p}(\Omega) \\ Bu = 0 \end{cases}$$

$$\Rightarrow u \in W^{s,p}(\Omega), \forall s > 1/p + 1$$

Regularity Theorem (2)

$$\begin{cases} u \in L^p(\Omega) \\ Au \in C^\alpha(\overline{\Omega}) \\ Bu = 0 \end{cases}$$

$$\Longrightarrow u \in C^{2+\alpha}(\overline{\Omega})$$

Reduction to the Boundary

Associate Pseudo-Differential Operator (Fredholm Boundary Operator)

$$T = BP = a(x')\sqrt{-\Lambda} + b(x')$$

$$\boldsymbol{\sigma}(T)(x', \xi') = a(x')\left|\xi'\right| + b(x')$$

$$\Lambda = \text{Laplace-Beltrami Operator}$$

$$T = a(x')\sqrt{-\Lambda} + b(x') \in L^1_{1,0}(\partial\Omega)$$

$$\boxed{a(x') + b(x') > 0 \text{ on } \partial\Omega}$$

$$\Rightarrow$$

$\exists S \in L^0_{1,1/2}(\partial\Omega)$ such that

$TS \equiv ST \equiv I \mod L^{-\infty}(\partial\Omega)$

Besov-Space Boundedness Theorem

Besov-Space Boundedness Theorem

Every properly supported operator

$A \in L^m_{1,\delta}(\Omega), \quad \boxed{0 \leq \delta < 1},$

extends to a continuous linear operator

$A : H^{s,p}_{\mathrm{loc}}(\Omega) \to H^{s-m,p}_{\mathrm{loc}}(\Omega), \ \forall s \in \mathbf{R}, 1 < \forall p < \infty$

and to a continuous linear operator

$A : B^{s,p}_{\mathrm{loc}}(\Omega) \to B^{s-m,p}_{\mathrm{loc}}(\Omega), \ \forall s \in \mathbf{R}, 1 \leq \forall p \leq \infty$

Sobolev Imbedding Theorem

Sobolev Imbedding Theorems

$$W^{2,q}(\Omega) \subset L^r(\Omega) \begin{cases} \forall r \geq 1 \quad \text{if } 2q \geq n \\ r = q^* = \dfrac{nq}{n-2q} \quad \text{if } 2q < n \end{cases}$$

Bootstrap Argument

Maximum Principle

Maximum Principle

$$\begin{cases} Au = p(u) \ \text{in}\, \Omega, \\ Bu = 0 \ \ \text{on}\, \partial\Omega \end{cases}$$

$$\Longrightarrow$$

$$\boxed{s^- \le u(x) \le s^+ \quad \text{in}\ \Omega}$$

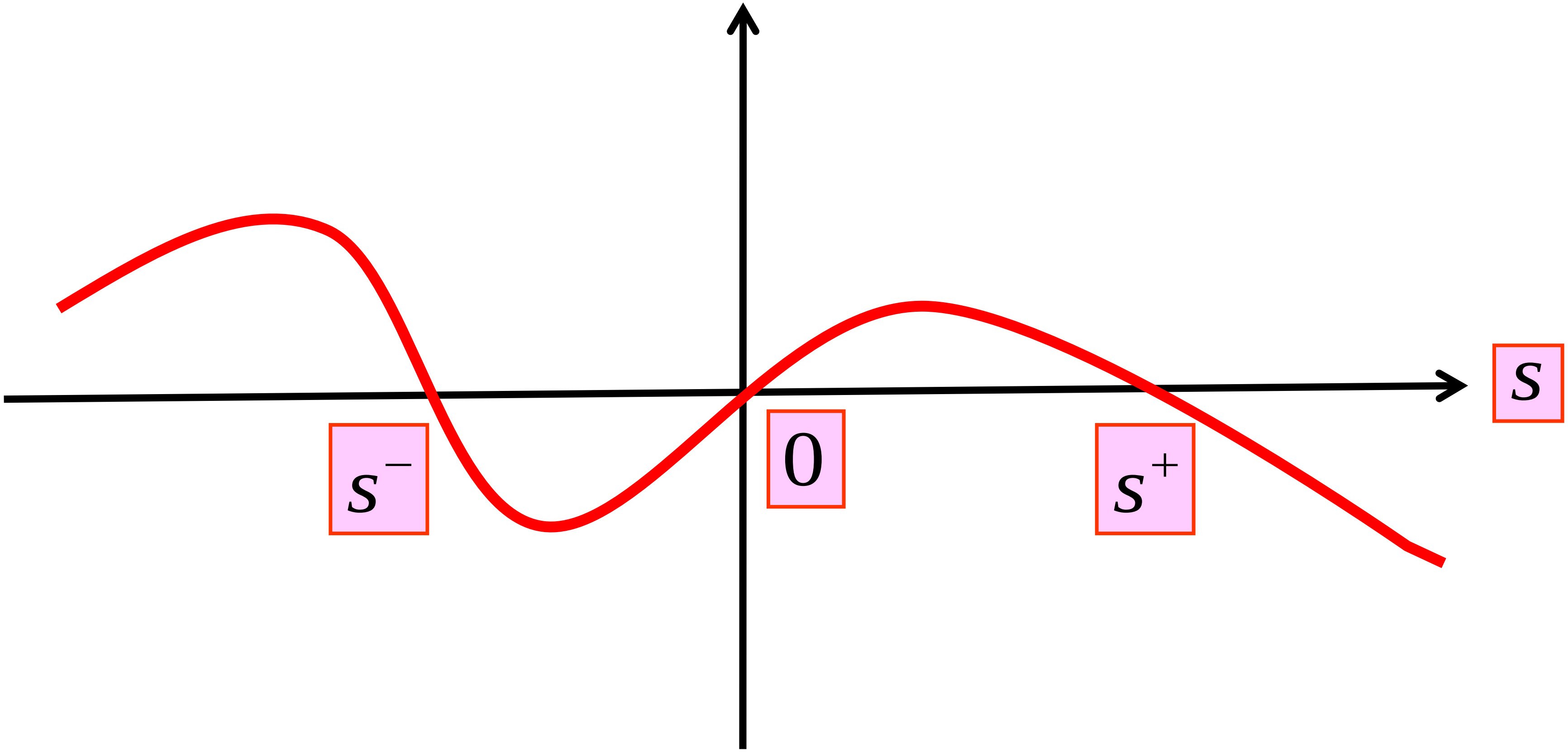
$$\Longrightarrow$$

$$p(u) = \lambda u - g(u) = f(u)$$

Truncation

$$p(s) = \begin{cases} < 0, & s > s^{+} \\ \lambda s - g(s), & s^{-} \le s \le s^{+} \\ > 0, & s < s^{-} \end{cases}$$

Outline of $p(s)$



Original Semilinear Problem

$$\begin{cases} Au = \lambda u - g(u) \quad \text{in } \Omega, \\ Bu = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Existence
of
Critical Points

Minimizing Method

$F \in C^1(H, \mathbf{R})$.

(1) $F(u)$ is **bounded from below**

(2) $F(u)$ satisfies **$(\mathbf{PS})_{\mathbf{c}}$ condition** with

$$c = \inf_{u \in H} F(u)$$

$\Rightarrow$

$\exists u^* \in H$ such that

$$F(u^*) = c = \inf_{u \in H} F(u)$$

$$\nabla F(u^*) = 0$$

Lower Bound for Energy Functional

$$F(u) = \frac{1}{2}(u,u)_H - \int_\Omega P(u(x))\,dx$$

$$\geq -\frac{L^2 |\Omega|}{2\lambda_1}, \quad \forall u \in H$$

$$|p(s)| \leq L \text{ on } \mathbf{R}$$

$$|p'(s)| \leq L \text{ on } \mathbf{R}$$

Remark (Neumann Case)

$a(x') \equiv 1$ on $\partial\Omega$ (**Neumann**)

$\Rightarrow$

$$\boxed{\lambda_1 = 0}$$

Palais–Smale Condition

Palais-Smale Condition (1)

$$\{u_j\} \subset H \text{ such that}$$

$$F(u_j) \to c \quad \text{in } \mathbf{R}$$

$$\nabla F(u_j) \to 0 \quad \text{in } H$$

Palais-Smale Condition (2)

$$u_j \to \exists u \quad \text{in } L^q(\Omega)$$

$$1 \leq \forall q < 2^* = \frac{2N}{N-2}$$

$$D(C) = H \subset H^1(\Omega) = W^{1,2}(\Omega)$$

Palais-Smale Condition (3)

$$\left|(u_j - u, v)_H\right| \le \left\|\nabla F(u_j)\right\|_H \cdot \|v\|_H$$
$$+\exists C_2 \left\|p(u_j) - p(u)\right\|_{L^{2N/(N+2)}(\Omega)} \cdot \|v\|_H$$
$$\forall v \in H$$

Palais-Smale Condition (4)

Riesz Representation Theorem

$$\Rightarrow$$

$$\left\| u_j - u \right\|_H$$

$$\leq \left\| \nabla F(u_j) \right\|_H + C_2 \left\| p(u_j) - p(u) \right\|_{L^{2N/(N+2)}(\Omega)}$$

Nemytskii Operator

Nemytskii Operator

$$N : u(x) \mapsto p(u(x))$$

$$u_j \to u \quad \text{in } L^q(\Omega)$$

$$1 \leq \forall q < 2^* = \frac{2N}{N-2}$$

Continuity of Nemytskii Operator (1)

(i) $s \mapsto f(x,s)$ is **continuous**

for almost every $x \in \Omega$

(ii) $x \mapsto f(x,s)$ is **measurable** for all $s \in \mathbf{R}$

(iii) $\boxed{\left|f(x,s)\right| \le \exists a + \exists b \left|s\right|^{q/p}, \quad \forall (x,s) \in \Omega \times \mathbf{R}}$

$\Rightarrow$

$F : u \in L^q(\Omega) \mapsto f(x,u(x)) \in L^p(\Omega)$

continuous

Continuity of Nemytskii Operator (2)

$$N : u \in L^q(\Omega) \mapsto P(u) \in L^{2N/(N+2)}(\Omega)$$

continuous

$$1 \leq \forall q < 2^* = \frac{2N}{N-2}$$

Palais-Smale Condition (5)

$$\left\| N(u_j) - N(u) \right\|_{L^{2N/(N+2)}(\Omega)} \to 0$$

$$\nabla F(u_j) \to 0 \quad \text{in } H$$

Palais-Smale Condition (6)

$$\left\| u_j - u \right\|_H \leq \left\| \nabla F(u_j) \right\|_H + C_2 \left\| N(u_j) - N(u) \right\|_{L^{2N/(N+2)}(\Omega)} \to 0$$

Existence Theorem (1)

The semilinear problem

$$\begin{cases} Au = \lambda u - g(u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

has at least **one** **non-trivial** solution for each $\lambda > \lambda_1$.

Another Truncation of a Non-linear Term

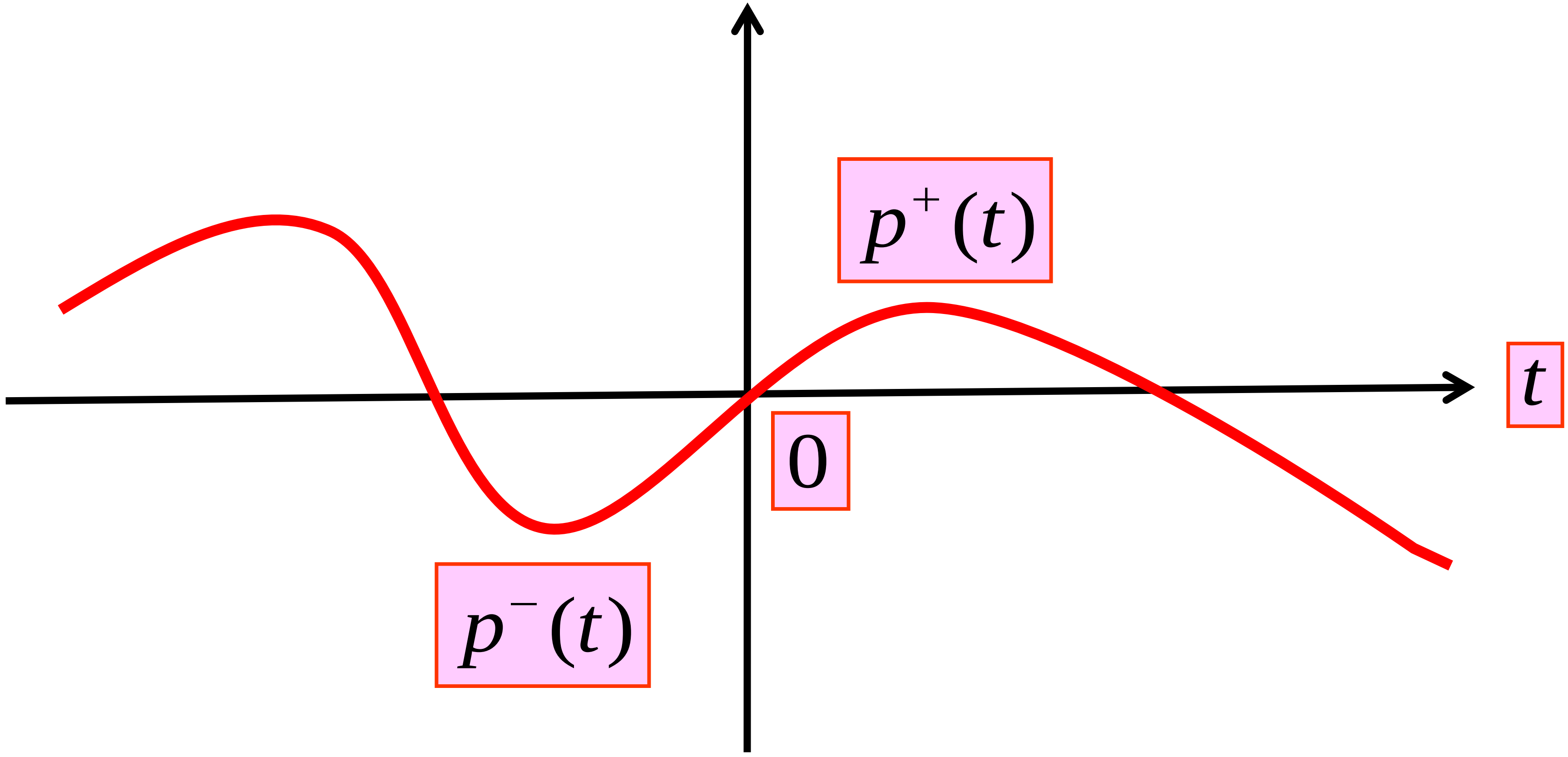
Another Truncation

$$p^{+}(t) = \max\{p(t), 0\}$$

$$p^{-}(t) = p(t) - p^{+}(t)$$

$p^{\pm}(t)$: **Lipschitz Continuous**

Outline of $p^{\pm}(t)$



Modified Semilinear Problems

For given functions $p^{\pm}(t)$,

find a function $u(x)$ in Ω such that

$$\begin{cases} Au = p^{\pm}(u) \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega. \end{cases}$$

New Energy Functionals

$$F^{\pm}(u) = \frac{1}{2}(u,u)_H - \int_{\Omega} P^{\pm}(u(x))\,dx$$

$$P^{\pm}(s) = \int_0^s p^{\pm}(t)\,dt$$

Existence Theorem (2)

The semilinear problem

$$\begin{cases} Au = p^{+}(u) \text{ in } \Omega, \\ Bu = 0 \text{ on } \partial\Omega \end{cases}$$

has at least **one non-trivial** solution u_1 for each $\lambda > \lambda_1$.

Maximum Principle

$$\begin{cases} Au_1 = p^+(u_1) \geq 0 \text{ in } \Omega, \\ Bu_1 = 0 \text{ on } \partial\Omega \end{cases}$$

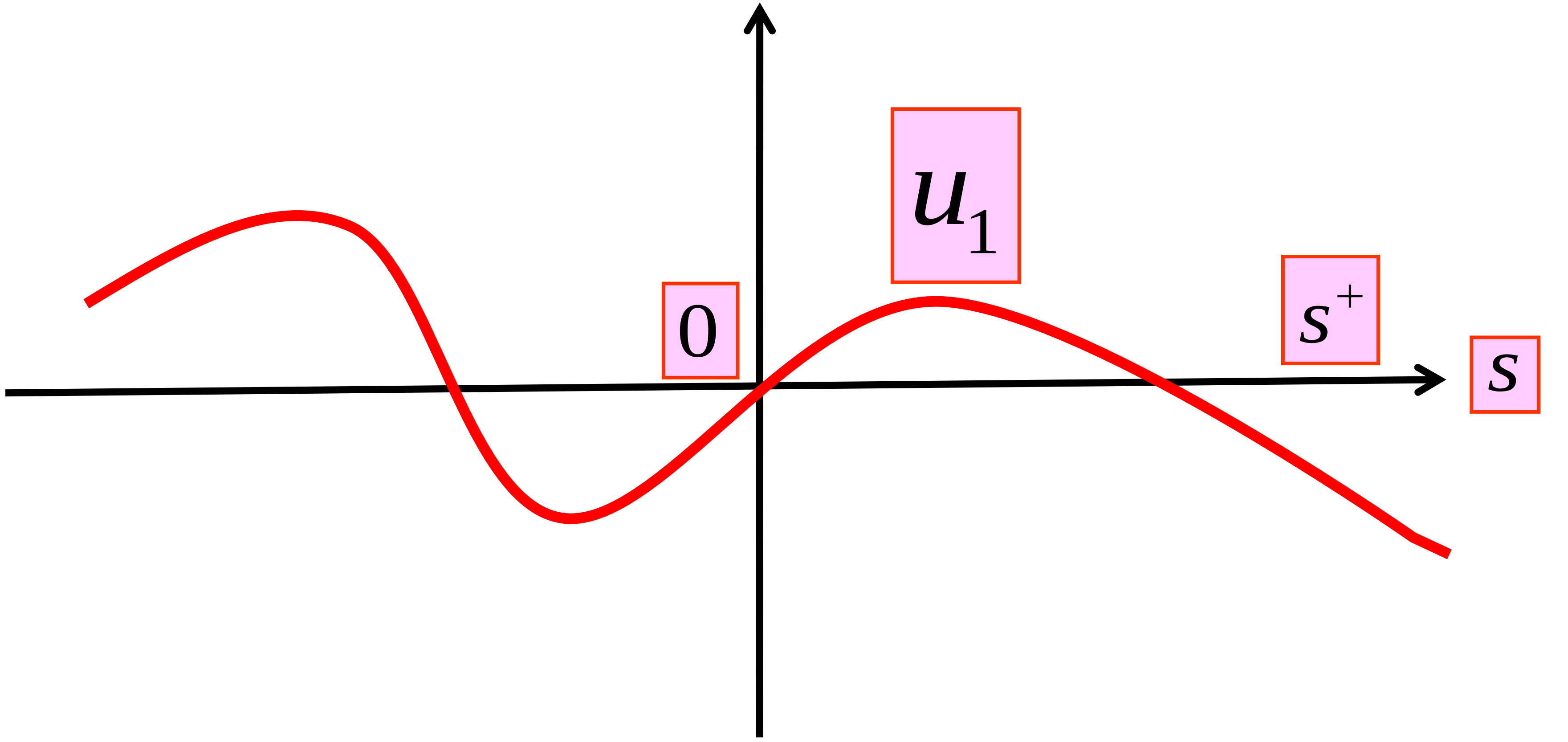
$$\Longrightarrow$$

$$\boxed{0 \leq u_1(x) \leq s^+ \quad \text{in } \Omega}$$

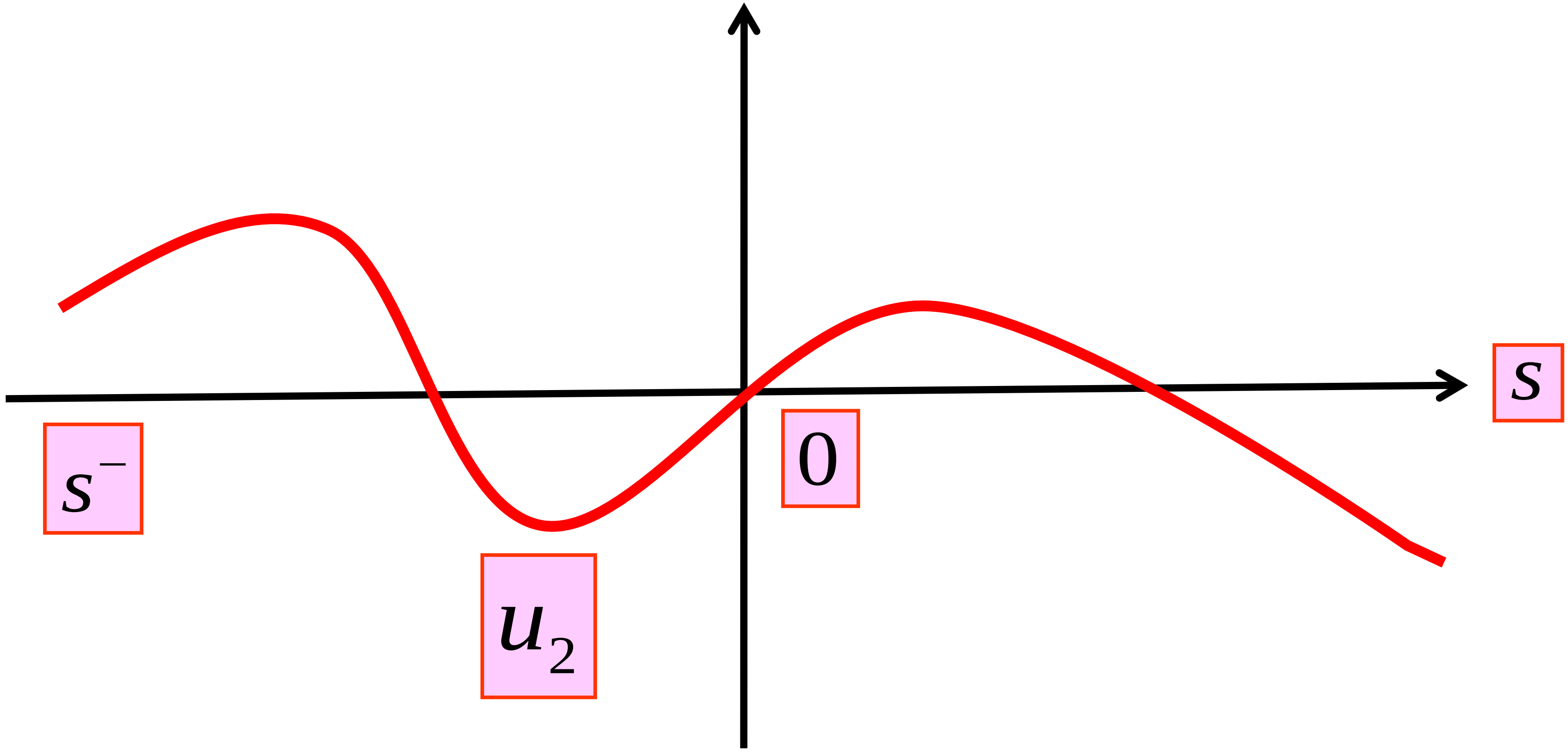
$$\Longrightarrow$$

$$p^+(u_1) = \lambda u_1 - g(u_1) = f(u_1)$$

Outline of $p(s)$



Outline of $p(s)$



Existence Theorem (3)

The semilinear problem

$$\begin{cases} Au = \lambda u - g(u) \ \text{in}\, \Omega, \\ Bu = 0 \ \text{on}\, \partial\Omega \end{cases}$$

has at least **one** **positive** solution

$u_1 > 0$ for each $\boxed{\lambda > \lambda_1}$.

Existence Theorem (4)

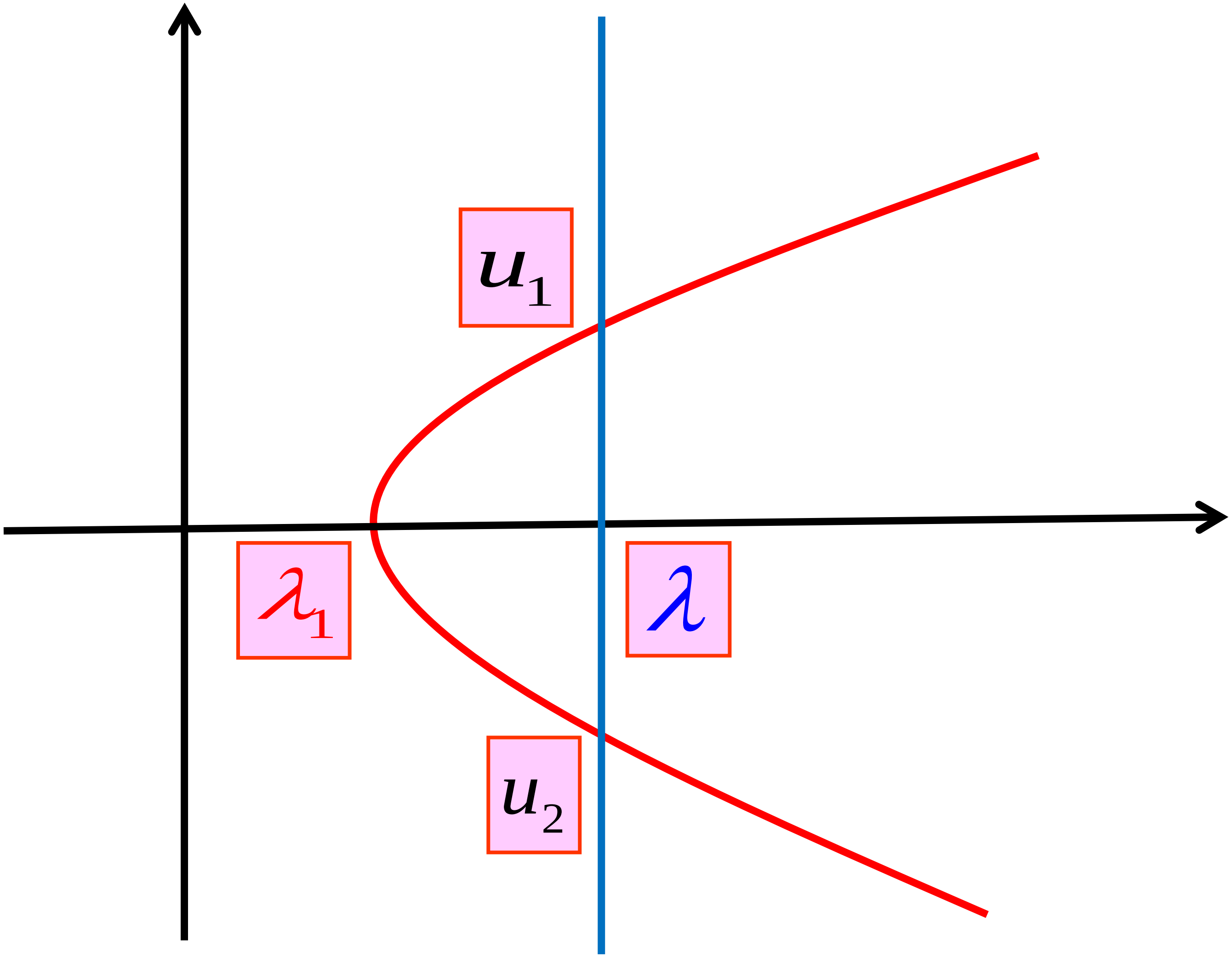
The semilinear problem

$$\begin{cases} Au = \lambda u - g(u) \text{ in } \Omega, \\ Bu = 0 \text{ on } \partial\Omega \end{cases}$$

has at least **one** **negative** solution

$u_2 < 0$ for each $\boxed{\lambda > \lambda_1}$.

Outline of $f(s) = \lambda s - g(s)$



The Case

$$\lambda > \lambda_2 > \lambda_1$$

Lyapunov-Schmidt Procedure

$$|p(s)| \le L \quad \text{on } \mathbf{R}$$

$$|p'(s)| \le L \quad \text{on } \mathbf{R}$$

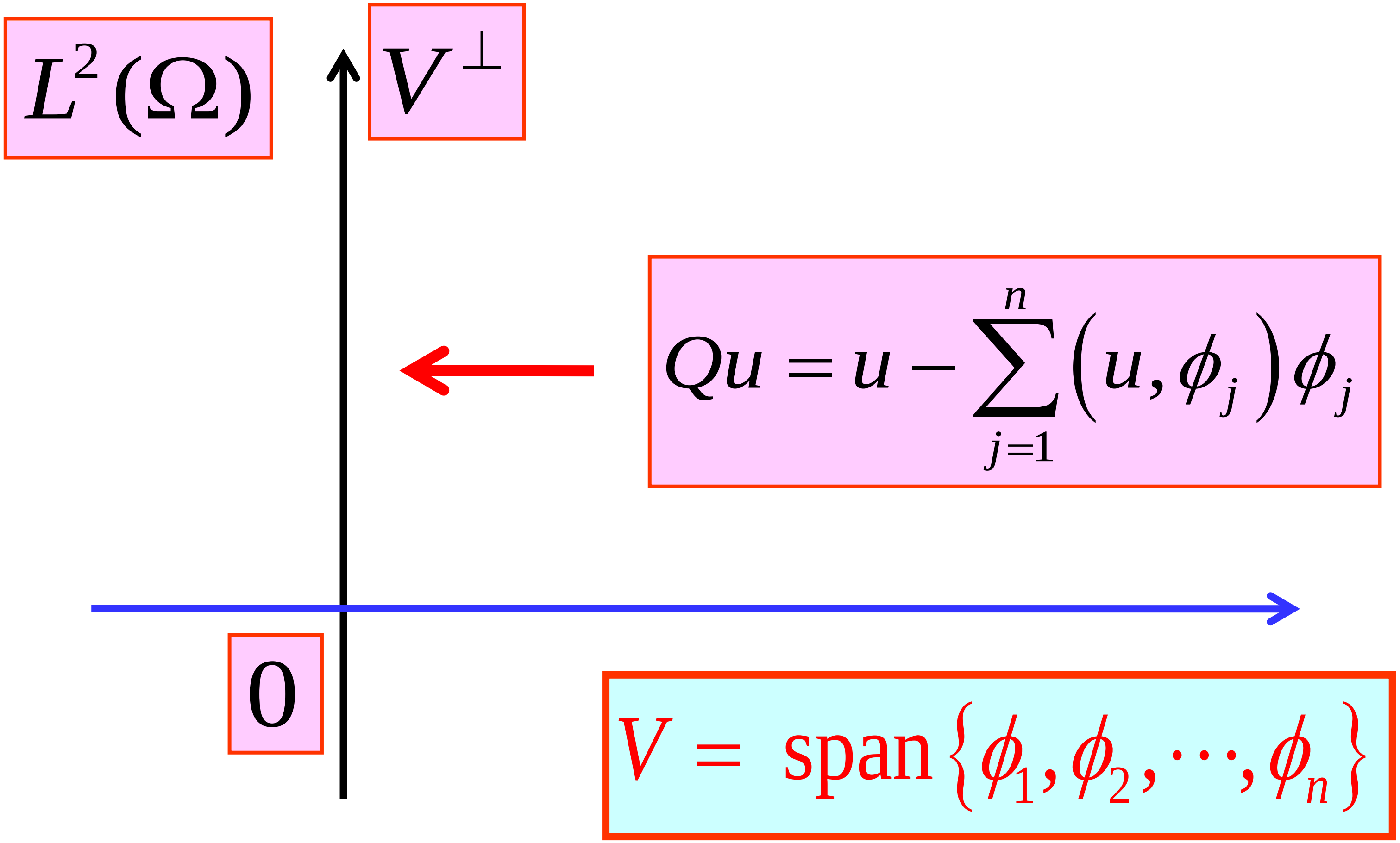
$$p'(s) < \exists \lambda_n, \quad \forall s \in \mathbf{R}$$

$$\mathrm{V} = \operatorname{span}\{\phi_1, \phi_2, \cdots, \phi_n\}$$

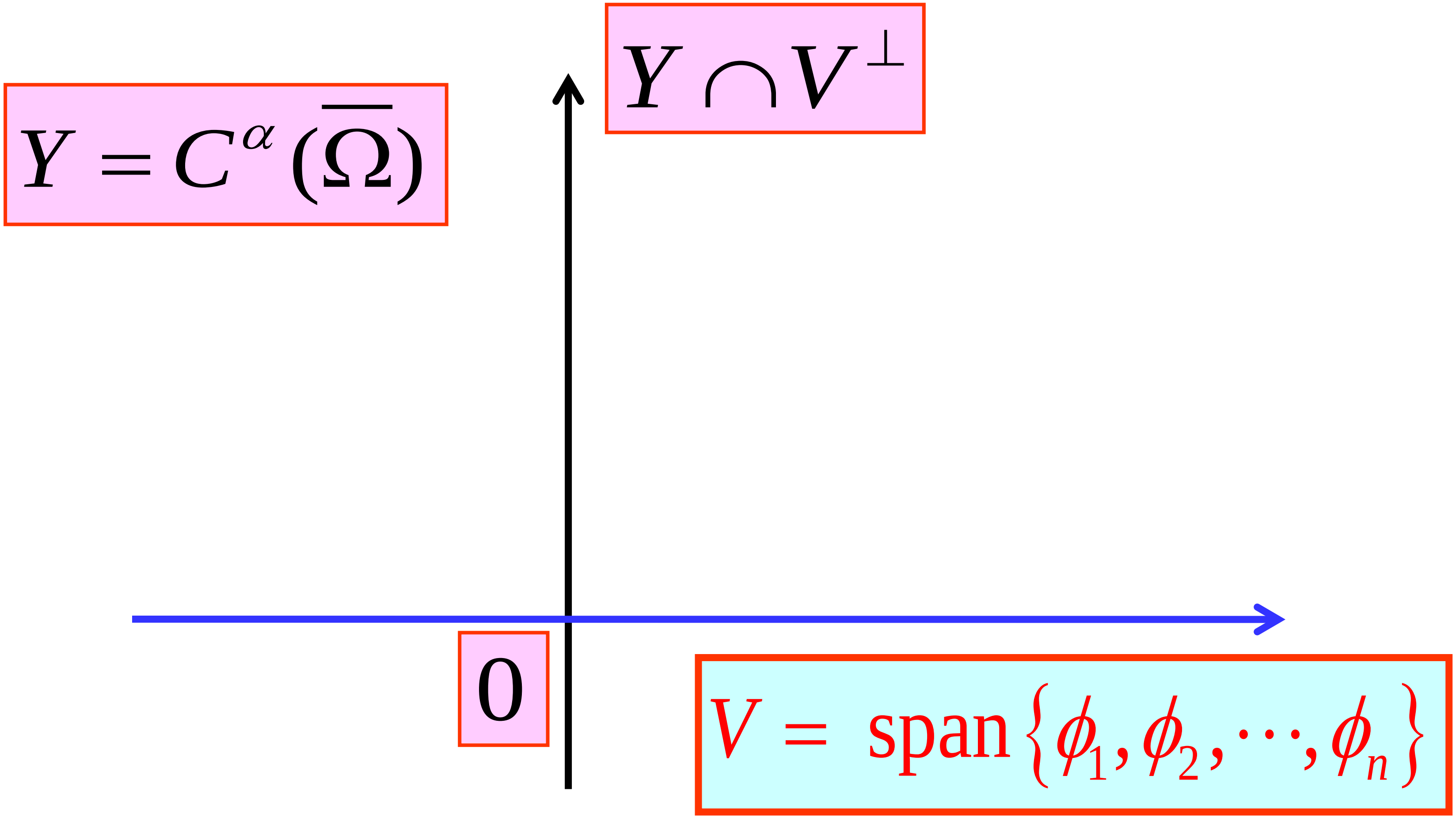
$$\dim \mathrm{V} = n$$

Orthogonal Decomposition

Orthogonal Decomposition (1)



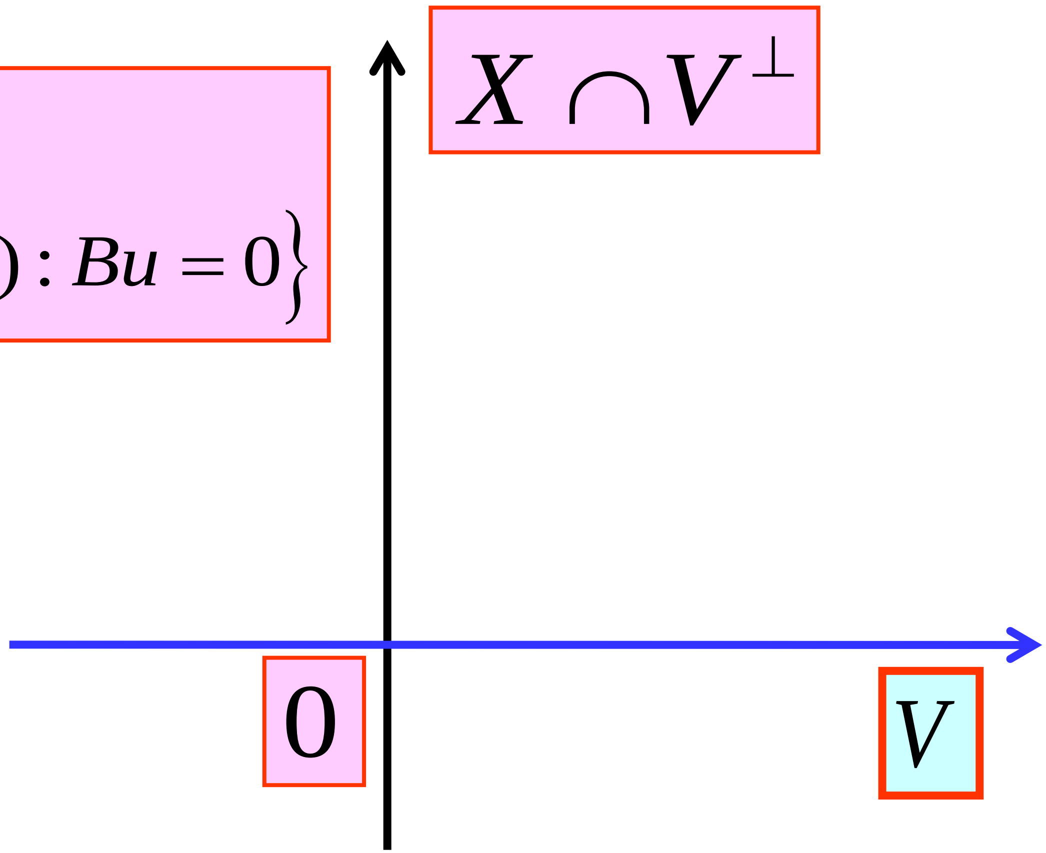
Orthogonal Decomposition (2)



Orthogonal Decomposition (3)

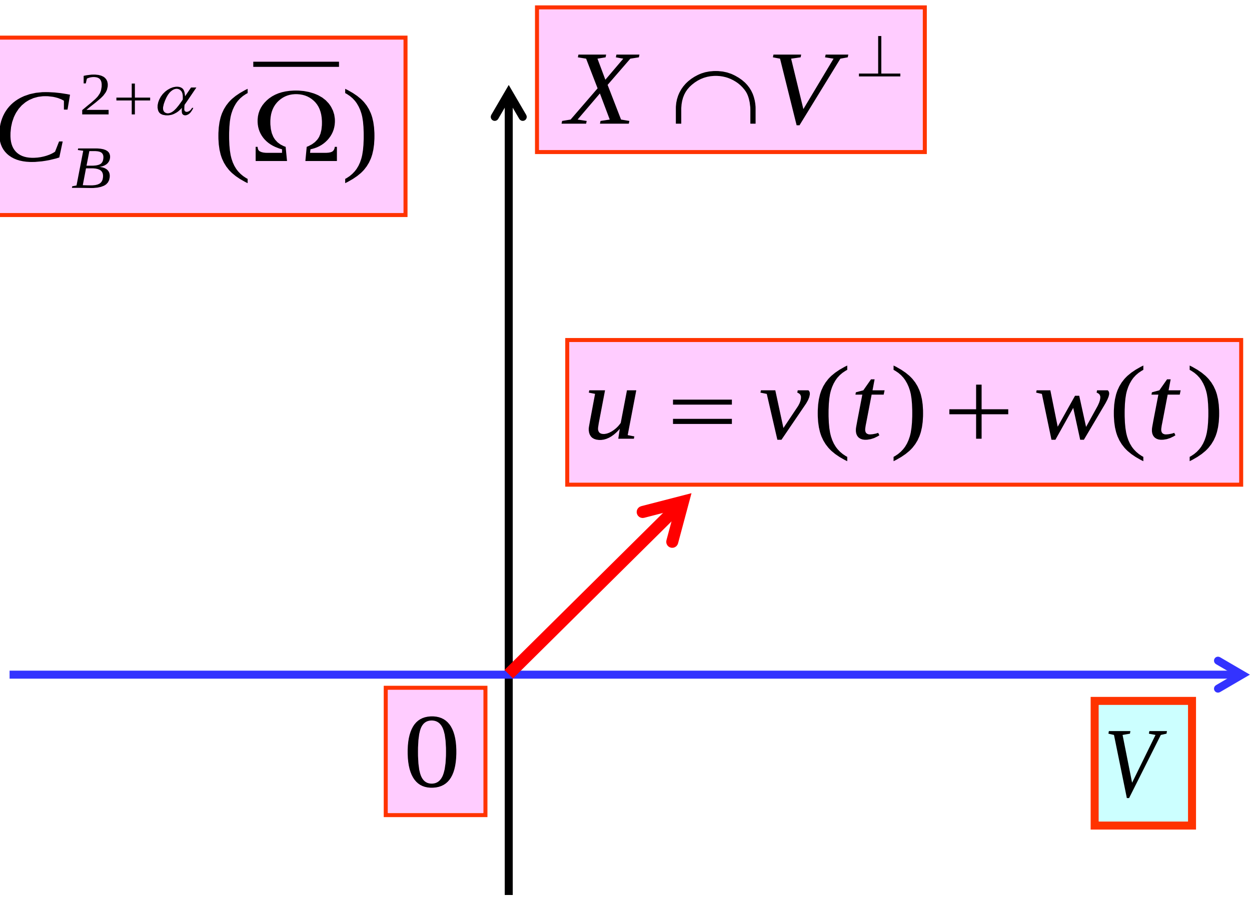
$$X = C_B^{2+\alpha}(\overline{\Omega})$$

$$= \left\{ u \in C_B^{2+\alpha}(\overline{\Omega}) : Bu = 0 \right\}$$

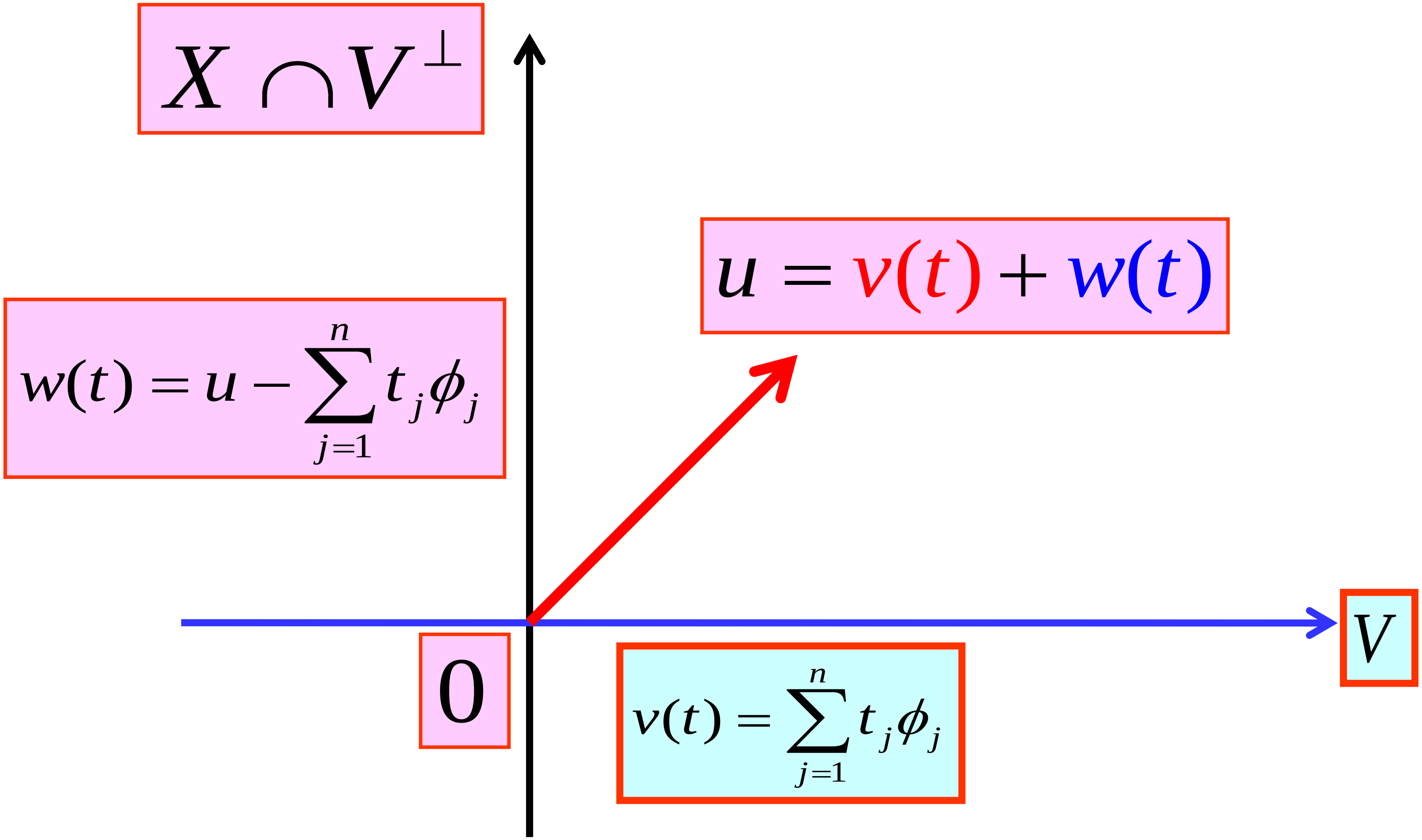


Orthogonal Decomposition (4)

$$X = C_B^{2+\alpha}(\overline{\Omega})$$



Orthogonal Decomposition (5)



$$
\boxed{\begin{aligned} Au &= p(u) \ \text{ in } \Omega, \\ Bu &= 0 \ \text{ on } \partial\Omega \end{aligned}}
$$

$$\Leftrightarrow$$

$$
\boxed{\begin{aligned} & u = v(t) + w(t) \\ & \begin{cases} Aw(t) = Q\Big(p(v(t)+w(t))\Big) \\ Av(t) = (I-Q)\Big(p(v(t)+w(t))\Big) \end{cases} \end{aligned}}
$$

Lyapunov-Schmidt Procedure

$$\begin{cases} Au = p(u) = h \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega \end{cases}$$

$$\Leftrightarrow$$

$$Aw(t) = Q\big(p(v(t)+w(t))\big), \ \ w(t) \in X \cap V^{\perp}$$

$$\int_{\Omega} p(v(t)+w(t))\phi_j(x)\,dx = \lambda_j t_j, \ \ 1 \le j \le n$$

Reduction to an Operator Equation

Infinite-dimensional Equation (1)

$$Aw(t) = Q\Big(p(v(t) + w(t))\Big), \;\; w(t) \in X \cap V^{\perp}$$

$$\Phi : \mathbf{R}^n \times (X \cap V^{\perp}) \rightarrow Y \cap V^{\perp}$$

$$\boxed{(t, w) \mapsto Aw - Q\Big(p(v(t) + w)\Big)}$$

Here

$$v(t) = \sum_{j=1}^{n} t_j \phi_j \in \mathbf{V}$$

$$t = (t_1, t_2, \cdots, t_n) \in \mathbf{R}^n$$

Global Inversion Theorem

M **arcwise connected** metric space

N **simply connected** metric space

$F : M \to N$

(1) **proper**

(2) **locally invertible on all of** M

$\Rightarrow F : M \to N$ **homeomorphism**

Infinite-dimensional Equation (2)

$$\Phi : \mathbf{R}^n \times (X \cap V^\perp) \to Y \cap V^\perp$$

$$\Phi(t, w) = Aw - Q\big(p(v(t) + w)\big) = Qh$$

$$\Leftrightarrow$$

$$\exists! w(t) \in X \cap V^\perp \quad \text{such that}$$

$$Aw(t) - Q\big(p(v(t) + w(t))\big) = Qh$$

Infinite-dimensional Equation (3)

$$h = 0$$

$$\Rightarrow$$

$\exists!\, w(t) \in X \cap V^{\perp}$ such that

$$\boxed{Aw(t) = Q\left(p(v(t) + w(t))\right)}$$

Finite-dimensional Equation

$$\boxed{Av(t) = (I - Q)\left(p(v(t) + w(t))\right)}$$

$$\Leftrightarrow$$

$$\boxed{\lambda_j t_j = \int_\Omega p\left(\sum_{k=1}^{n} t_k \phi_k + w(t)\right)\phi_j(x)dx}$$

$$(1 \le j \le n)$$

Morse Theory

Energy Functional

$$\Psi(t) = \frac{1}{2}\left(w(t), w(t) \right)_H + \frac{1}{2}\sum_{j=1}^{N} \lambda_j t_j^2 - \int_\Omega P\left(v(t) + w(t) \right) dx$$

Gradient

$$\frac{\partial \Psi}{\partial t_j} = \lambda_j t_j - \int_\Omega p\left(v(t) + w(t)\right) \phi_j(x) dx$$

$$(1 \leq j \leq n)$$

Critical Points

$$\boxed{\nabla\Psi(t)=0}$$

$$\Leftrightarrow$$

$$\boxed{\lambda_j t_j = \int_\Omega p\left(v(t)+w(t)\right)\phi_j(x)dx}$$

$$\Leftrightarrow$$

$$\boxed{Av(t)=(I-Q)\left(p(v(t)+w(t))\right)}$$

Hessian (1)

$$\frac{\partial^2 \Psi}{\partial t_i \partial t_j}(0) = \left(\lambda_j - \lambda\right)\delta_{ij}$$

$$(1 \le i, j \le n)$$

$$\lambda > \lambda_2 > \lambda_1$$

Non-Degenerate Case

Non-Degenerate Case

$$\lambda > \lambda_2 > \lambda_1$$

$$\lambda \neq \lambda_k, \quad \forall k \geq 3$$

Hessian (2)

$$\left(\frac{\partial^2 \Psi}{\partial t_i \partial t_j}(0) \right)$$

$$= \begin{pmatrix} \lambda_1 - \lambda & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 & 0 \\ 0 & 0 & \lambda_k - \lambda & 0 \\ 0 & 0 & 0 & \lambda_N - \lambda \end{pmatrix}$$

Hessian (3)

0 is a **non-degenerate** critical point with **Morse index** $\boxed{q_0 \geq 2}$

Four-Solution Theorem (1)

$\Psi \in C^2(\mathbf{R}^n, \mathbf{R})$

(1) $\Psi(t)$ satisfies **(PS) condition**

(2) $\Psi(t)$ is **bounded from below**

(3) 0 is a **non-degenerate** critical point

with **Morse index** $\boxed{q_0 \geq 2}$

(4) $\Psi(t)$ has **two** **local minima** t_1, t_2

Four-Solution Theorem (2)

$\Psi(t)$ has at least **another non-zero** critical point t_3

Non-Degenerate Case

The semilinear problem

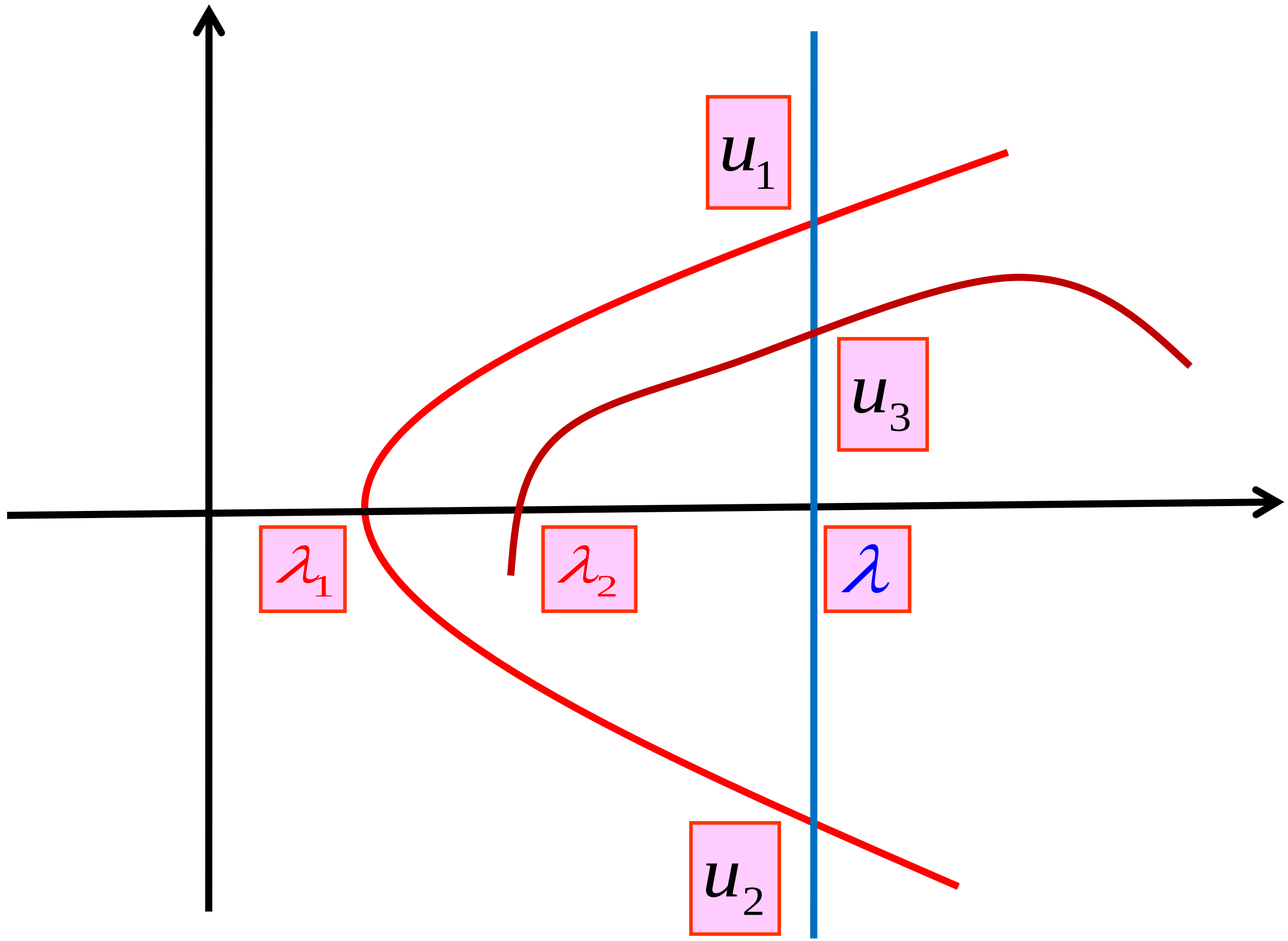
$$\begin{cases} Au = \lambda u - g(u) \ \text{ in } \Omega, \\ Bu = 0 \ \text{ on } \partial\Omega \end{cases}$$

has at least **three** **non-trivial** solutions

u_1, u_2, u_3 for each $\boxed{\lambda > \lambda_2, \lambda \neq \lambda_k, \forall k \geq 3}$

$$u_1 = v(t_1) + w(t_1)$$
$$u_2 = v(t_2) + w(t_2)$$
$$u_3 = v(t_3) + w(t_3)$$

Outline of $f(s) = \lambda s - g(s)$



Degenerate Case

Degenerate Case

$$\lambda > \lambda_2 > \lambda_1$$

$$\lambda = \lambda_k, \quad \exists k \geq 3$$

Hessian (4)

$$\left(\frac{\partial^2 \Psi}{\partial t_i \partial t_j}(0) \right) = \begin{pmatrix} \lambda_1 - \lambda & 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \lambda_N - \lambda \end{pmatrix}$$

Proof (1)

Reduction to Absurdity

$\Psi(t)$ has only **three** **critical points** $t_1, t_2, 0$

$b > \max\left\{\Psi(t_1), \Psi(t_2), \Psi(0)\right\}$

Resolution
of
Critical Points

Resolution of Critical Points (1)

$f \in C^2(H, \mathbf{R})$

(1) f satisfies **(PS) condition**

(2) x_0 is an isolated (**degenerate**) critical point

Resolution of Critical Points (2)

$\forall \varepsilon > 0,\ \exists \mathrm{g} \in C^2(H, \mathbf{R})$ such that

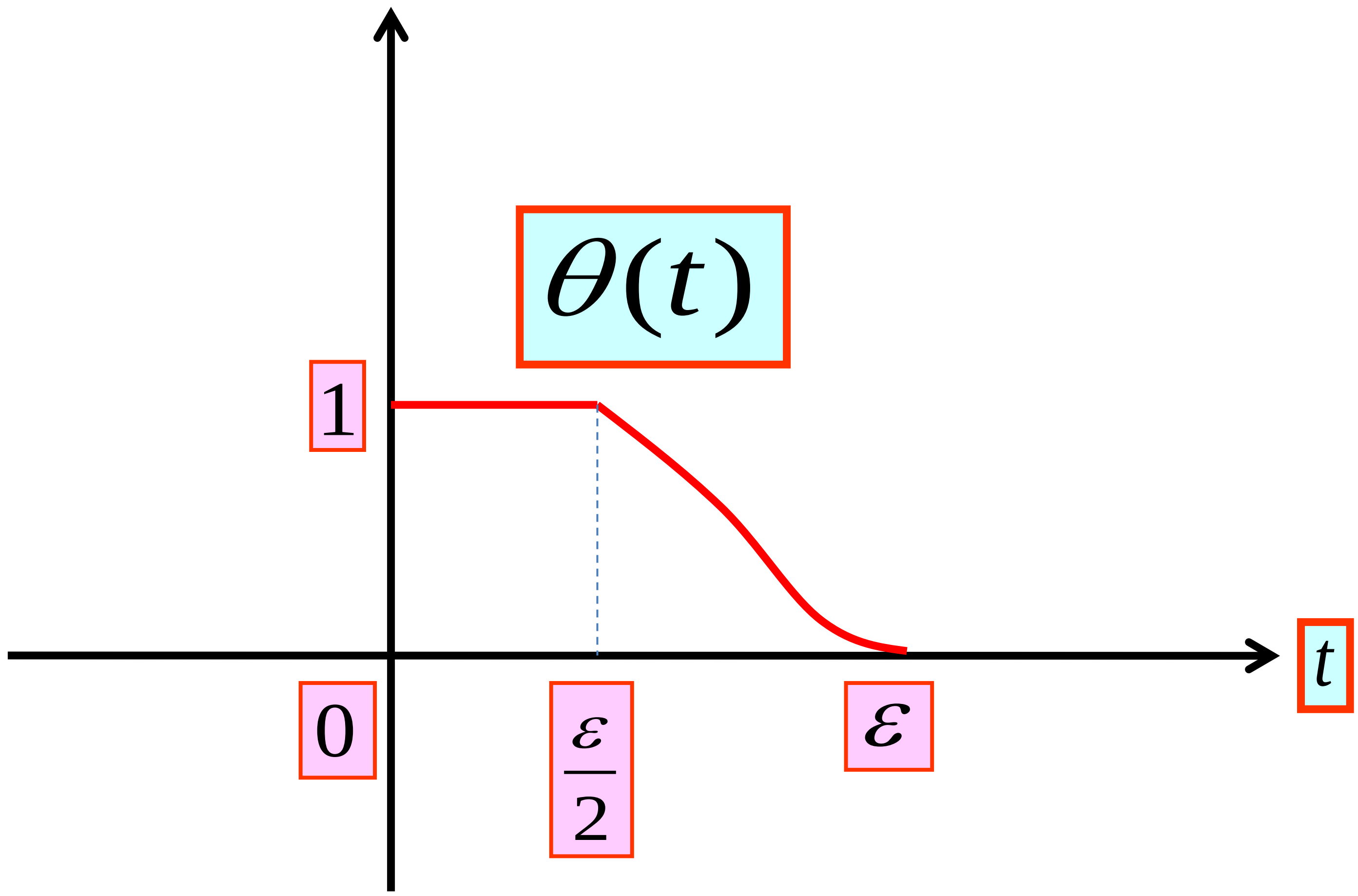
(a) g satisfies **(PS) condition**

(b) $g(x) = f(x)$ for $\|x - x_0\| \geq \varepsilon$

(c) g has a **finite number** of **non-degenerate** critical points in $\|x - x_0\| < \varepsilon$

(d) $\boxed{\|D^2 g(x) - D^2 f(x)\| < \varepsilon}$

Proof (1)



Proof (2)

$$x_0 = 0$$

$$g(x)$$

$$= f(x) - \theta\left(\|x\|\right)(x, y)$$

$$= f(x) - \theta\left(\sqrt{x_1^2 + x_2^2 + \cdots x_n^2}\right)\sum_{j=1}^{n} x_j y_j$$

Proof (3)

$$\nabla g(x) = 0, \|x\| \le \frac{\varepsilon}{2}$$

$D^2 g(x)$: **singular**

$\Leftrightarrow$

$$\nabla f(x) = y, \|x\| \le \frac{\varepsilon}{2}$$

$D^2 f(x)$: **singular**

$\Leftrightarrow$

y : **critical value of** ∇f in $\|x\| \le \frac{\varepsilon}{2}$

Proof (4)

Sard's Lemma

$\Rightarrow$

$g(x)$ has a **finite number** of **non-degenerate, critical points** in the open ball $\|x\| < \varepsilon$

Energy Functional

$$\Psi(t) = \frac{1}{2}\left(w(t), w(t)\right)_H + \frac{1}{2}\sum_{j=1}^{N} \lambda_j t_j^2 - \int_\Omega P\left(v(t) + w(t)\right) dx$$

$\Psi \in C^2(\mathbf{R}^n, \mathbf{R})$

(1) $\Psi(t)$ satisfies **(PS) condition**

(2) $\Psi(t)$ is **bounded from below**

(3) 0 is a **degenerate** critical point

with **Morse index** $\boxed{q_0 \geq 2}$

(4) $\Psi(t)$ has **two** **local minima** t_1, t_2

$\exists \tilde{\Psi} \in C^2(\mathbf{R}^n, \mathbf{R})$ such that:

(1) $\tilde{\Psi}(t)$ satisfies **(PS) condition**

(2) $\tilde{\Psi}(t)$ has only **two critical points** t_1, t_2

in the closed set $\{|t| \geq \varepsilon\}$

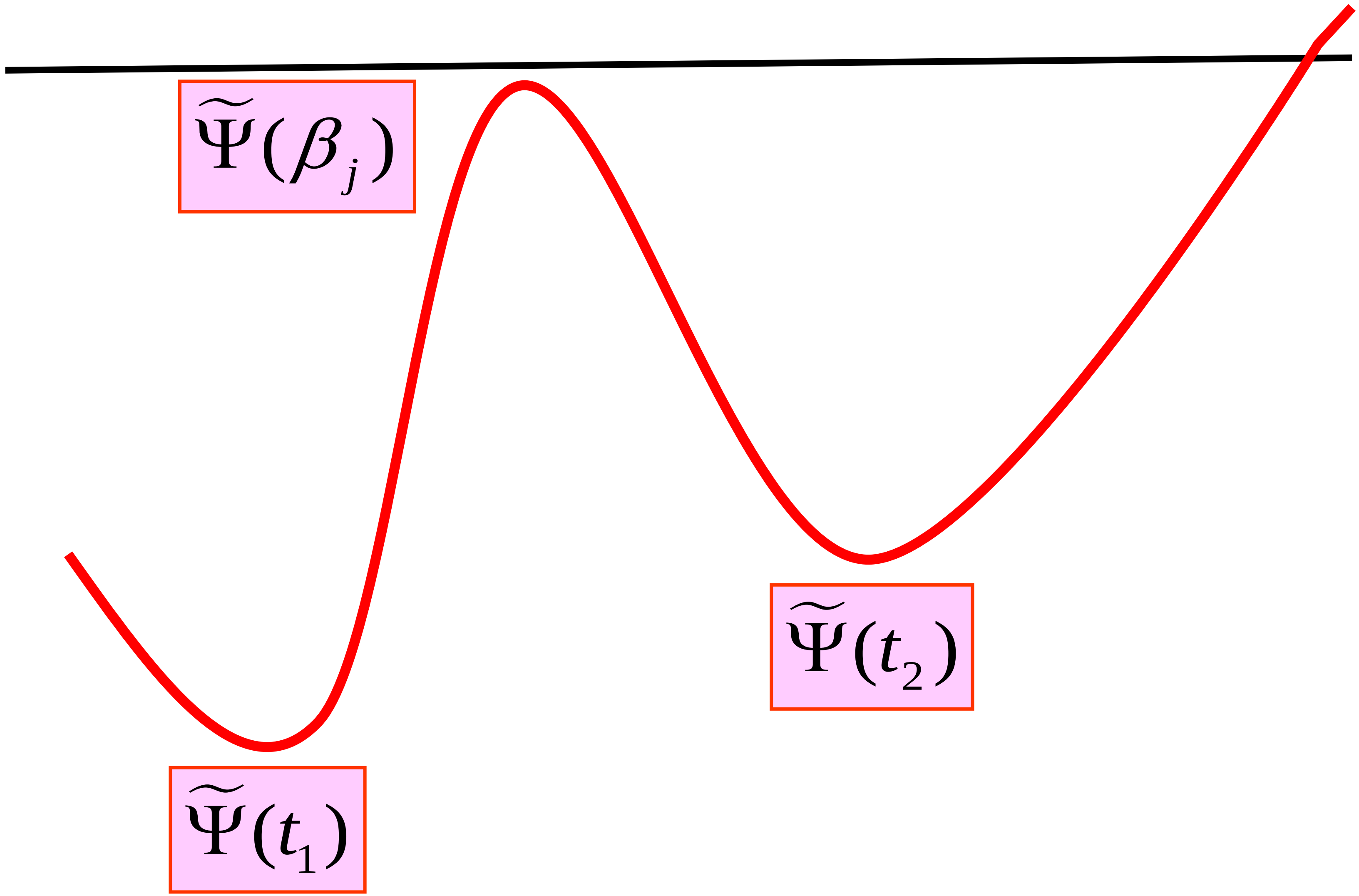
(3) $\tilde{\Psi}(t)$ has only a finite number of

non-degenerate critical points

$\beta_1, \beta_2, \cdots \beta_\ell$ with **Morse index** $\boxed{\geq 2}$

in the open set $\{|t| < \varepsilon\}$

$$\widetilde{\Psi}^{b} = \left\{ t \in \mathbf{R}^{n} : \widetilde{\Psi}(t) \le b \right\}$$



Proof (5)

$$\beta_1(b) - \beta_0(b) = -1$$

$$\leq C_1(b) - C_0(b) = -2$$

Contradiction!

Proof (6)

Reduction to Absurdity

$\Psi(t)$ has only **three critical points** $t_1, t_2, 0$

$b > \max\{\Psi(t_1), \Psi(t_2), \Psi(0)\}$

⇓

$\Psi(t)$ has at least **another non-zero** critical point t_3

Degenerate Case

The semilinear problem

$$\begin{cases} Au = \lambda u - g(u) \ \text{in}\, \Omega, \\ Bu = 0 \ \text{on}\, \partial\Omega \end{cases}$$

has at least **three** **non-trivial** solutions

u_1, u_2, u_3 for each $\boxed{\lambda > \lambda_2}$

$$u_1 = v(t_1) + w(t_1)$$
$$u_2 = v(t_2) + w(t_2)$$
$$u_3 = v(t_3) + w(t_3)$$

Odd Nonlinear Case

Ljusternik-Schnirelmann Theory

Odd Nonlinearity Conditions

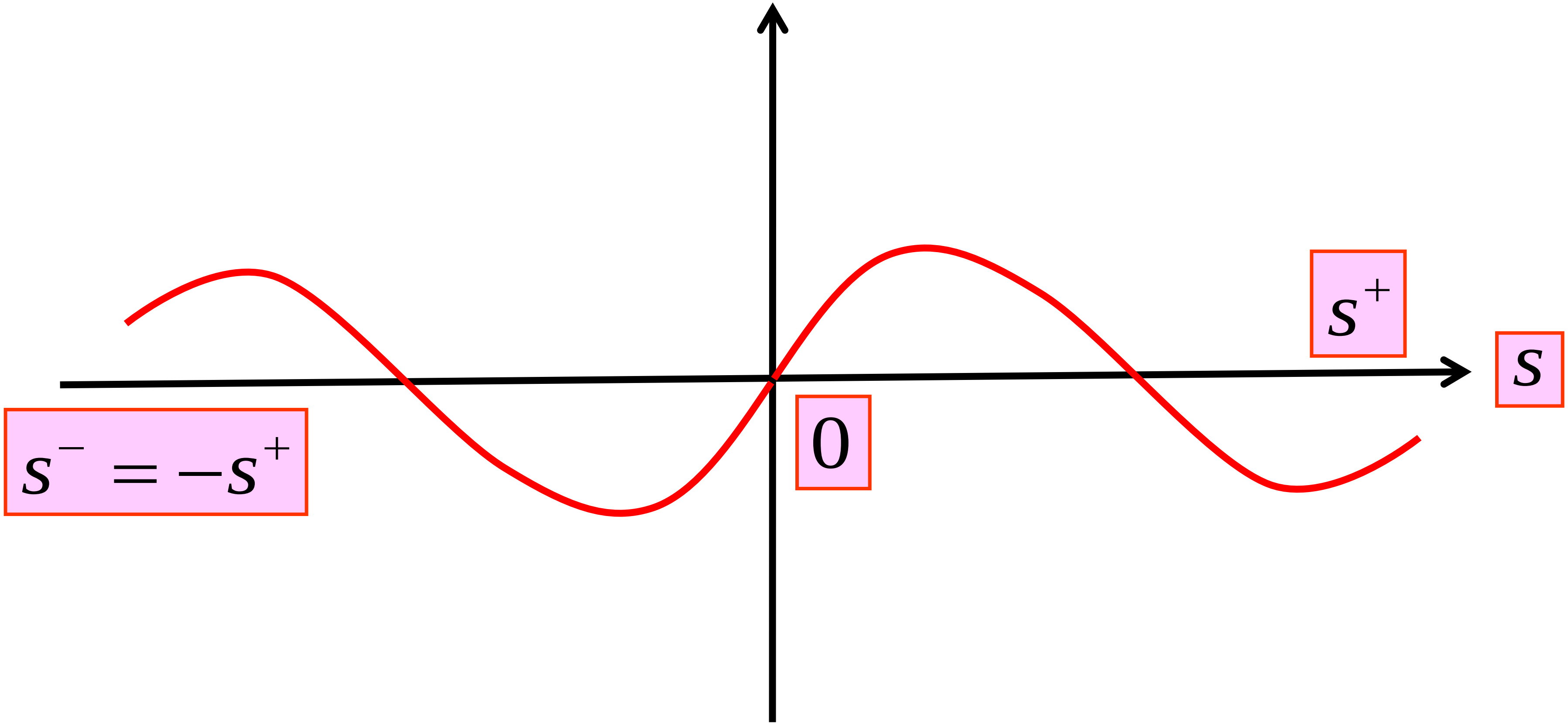
(A) $g \in C^1(\mathbf{R}), \quad \boxed{g(0) = g'(0) = 0}.$

(B) The limits $g'(\pm\infty)$ satisfies the conditions

$$g'(\pm\infty) = \lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty$$

(C) $\boxed{g(-s) = -g(s), \quad \forall s \in \mathbf{R}.}$

Outline of $p(s)$



Energy Functional

$$F(u) = \frac{1}{2}(u,u)_H - \int_\Omega P\big(u(x)\big)\,dx$$

$$P(s) = \int_0^s p(t)\,dt$$

Lower Bound for Energy Functional

$$F(u) = \frac{1}{2}(u,u)_H - \int_\Omega P(u(x))\,dx$$

$$= \frac{1}{2}\|u\|_H^2 - \int_\Omega \left(\int_0^{u(x)} p(t)\,dt \right) dx$$

$$\geq -\frac{L^2|\Omega|}{2\lambda_1}, \quad \forall u \in H$$

Multiplicity Theorem (1)

$F \in C^1(H, \mathbf{R})$

(1) $F(-u) = F(u), \quad \forall u \in H$

(2) F satisfies **(PS) condition**

Multiplicity Theorem (2)

Assume the following:

(i) $\dim V = k, \ \exists \rho > 0$ such that

$$\boxed{\sup_{u \in V \cap S_\rho(0)} F(u) \le \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_k}\right)\rho^2}$$

(ii) $F(u)$ is **bounded from below**

Then $F(u)$ has at least k - *pairs* of **distinct critical points**.

Behavior of Energy Functional (1)

$$F(u) = \frac{1}{2}\|u\|_H^2 - \int_\Omega P\big(u(x)\big)\,dx$$

$$= \frac{1}{2}\|u\|_H^2 - \int_\Omega \int_0^{u(x)} \big(\lambda t - g(t)\big)\,dt\,dx$$

$$= \frac{1}{2}\|u\|_H^2 - \frac{\lambda}{2}\int_\Omega u(x)^2\,dx + \int_\Omega \int_0^{u(x)} g(t)\,dt\,dx$$

Finite-Dimensional Linear Space

$$\lambda > \lambda_k$$

$$\Rightarrow$$

$$V = \operatorname{span}\{\phi_1, \phi_2, \cdots, \phi_k\}$$

$$\dim V = k$$

Behavior of Energy Functional (2)

All norms on the **finite-dimensional space** V are **equivalent.**

Nonlinearity Conditions

(A) $g \in C^1(\mathbf{R})$, $\boxed{g(0) = g'(0) = 0}$.

(B) The limits $g'(\pm\infty)$ satisfies the conditions

$$g'(\pm\infty) = \lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty$$

(C) $\boxed{g(-s) = -g(s), \quad \forall s \in \mathbf{R}.}$

Behavior of Energy Functional (3)

$$g(t) = o(t) \quad \text{as } t \to 0$$

$$\Rightarrow$$

$$\int_\Omega \int_0^{u(x)} g(t)\,dt\,dx = o(\rho^2)$$

as $u \in V$ and $\|u\|_H = \rho \to 0$

Behavior of Energy Functional (4)

$$F(u) = \frac{1}{2}\|u\|_H^2 - \int_\Omega P(u(x))\,dx$$

$$\le \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\rho^2 + o\left(\rho^2\right)$$

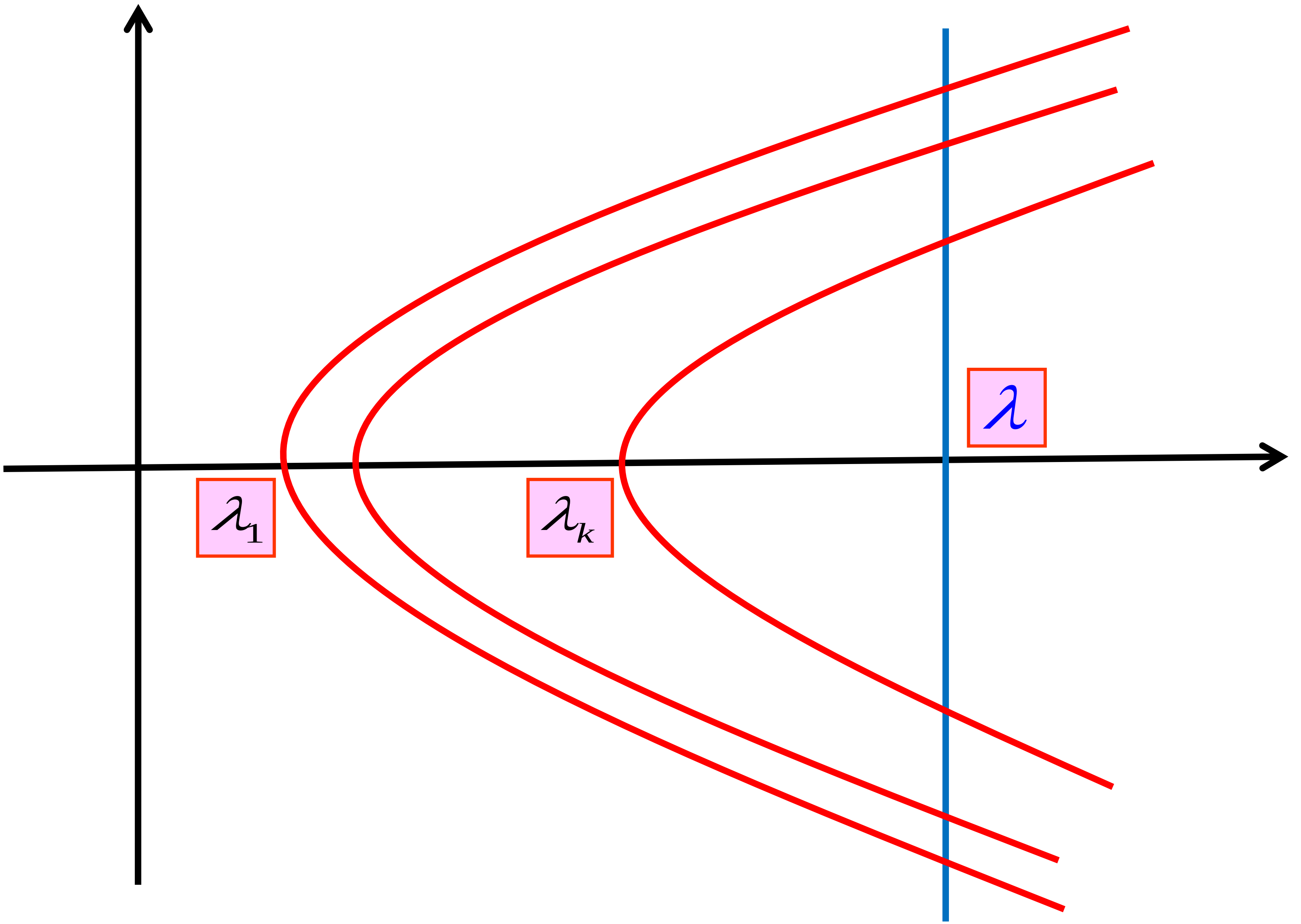
as $u \in V$ and $\|u\|_H = \rho \to 0$

Behavior of Energy Functional (5)

$$\dim V = k, \ \exists \rho > 0 \text{ such that}$$

$$\sup_{u \in V \cap S_\rho(0)} F(u) \le \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_k}\right)\rho^2$$

Outline of $f(s) = \lambda s - g(s)$



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