Analysis II
(Introduction to Fourier Analysis)

Kazuaki TAIRA

December, 2006

Contents

1 Fourier's Method **2**

2 The Riesz–Schauder theory for compact operators **5**
- 2.1 Compact operators 5
- 2.2 Spectral analysis of compact operators 6

3 Concrete Examples of Fourier's Method **8**
- 3.1 Formulation of the Problem 8
- 3.2 Uniqueness Theorem for the Initial-Boundary Value Problem for the Heat Equation 9
- 3.3 Separation of Variables Method 11
- 3.4 Case of a Rectangle 15
- 3.5 Case of a Disk 17

1 Fourier's Method

Jean-Baptiste Joseph Fourier (1768—1830) "Théorie analytique de la chaleur"

Let Ω be a bounded domain in Euclidean space $\mathbf{R}^n$ with boundary $\partial\Omega$. The heat conduction phenomenon can be described by the following initial boundary value problem for the heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \Omega \times (0, \infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{1.1}$$

Here

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \ldots + \frac{\partial^2}{\partial {x_n}^2}$$

is the usual Laplacian, $u_0(x)$ is an initial thermal distribution and $u(x,t)$ expresses a thermal distribution of position x at time t. The boundary condition is the Dirichlet boundary condition (isothermal condition).

We shall find a non-trivial solution $u(x,t)$ of the initial-boundary value problem (1.1) in the separation of variables form

$$u(x,t) = T(t)X(x).$$

Here we assume that $T(t)$ and $X(x)$ are both non-trivial functions.

Step 1: First, it follows from the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

that

$$T'(t)X(x) = T(t)\Delta X(x).$$

Hence we have formally the formula

$$\frac{T'(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = -\lambda.$$

However, the left-hand side depends only on the variable t, while the right-hand side depends only on the variable x. In particular, the both sides do not depend on t and x, that is, they are a constant, say $-\lambda$.

Step 2: Secondly, since we have the formula

$$\frac{T'(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = -\lambda,$$

we obtain that the initial-boundary value problem (1.1) is equivalent to the following:

$$T'(t) + \lambda T(t) = 0,$$

$$
\begin{cases} \Delta X + \lambda X = 0 & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}
$$

The next claim asserts that $\lambda > 0$:

Claim 1.1. *If $X(x)$ is a non-trivial function, then $\lambda > 0$:*

Proof. Indeed, since the function $X(x)$ satisfies the Dirichlet boundary condition

$$
X = 0 \quad \text{on } \partial\Omega, \tag{1.3}
$$

we have, by integration by parts (Green's formula),

$$
\begin{aligned}
\lambda \int_\Omega X(x)^2\, dx &= -\int_\Omega \Delta X(x) \cdot X(x)\, dx \\
&= -\int_\Omega \left(\frac{\partial^2 X}{\partial {x_1}^2} + \frac{\partial^2 X}{\partial {x_2}^2} + \ldots + \frac{\partial^2 X}{\partial {x_n}^2} \right) \cdot X(x)\, dx \\
&= \sum_{i=1}^n \int_\Omega \left(\frac{\partial X}{\partial x_i} \right)^2 dx \\
&\geq 0.
\end{aligned}
$$

This proves that

$$
\lambda \geq 0.
$$

If $\lambda = 0$, then it follows that

$$
\int_\Omega \left(\frac{\partial X}{\partial x_i} \right)^2 dx = 0, \quad 1 \leq i \leq n,
$$

so that $X(x)$ is a constant. However, we have, by the Dirichlet condition (1.3),

$$
X(x) \equiv 0 \quad \text{in } \Omega.
$$

This implies that $X(x) \equiv 0$ in Ω.

Summing up, we have proved that $\lambda > 0$, provided that $X(x)$ is a non-trivial function. □

Step 3: Now we shall find a non-trivial solution $u(x,t)$ of the initial-boundary value problem (1.1) in the separation of variables form

$$
u(x,t) = Ae^{-\lambda t} X(x), \quad \lambda > 0,
$$

where A and λ are constants to be chosen later on.

Step 4: The next completeness theorem of eigenfunctions is one of the fundamental results in the theory of partial differential equations (see Theorem 2.6):

Theorem 1.1. *The Dirichlet eigenvalue problem*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a countable number of solutions

$$\{\lambda_j, u_j\}_{j=1}^{\infty}$$

Moreover, the countable family of eigenfunctions

$$\{u_j\}_{j=1}^{\infty}$$

forms a complete orthonormal system *in the Hilbert space* $L^2(\Omega)$ *and*

$$\begin{aligned} &0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots, \\ &u_1 > 0 \quad \text{in } \Omega. \end{aligned}$$

Therefore, the Dirichlet eigenvalue problem

$$\begin{cases} \Delta X_j + \lambda_j X_j = 0 & \text{in } \Omega, \\ X_j = 0 & \text{on } \partial\Omega \end{cases}$$

has a countable family $\{\lambda_j, X_j\}_{j=1}^{\infty}$ of solutions.

Step 5: We remark that the functions

$$u_j(x,t) = A_j\, e^{-\lambda_j t}\, X_j(x), \quad j = 1, 2, \dots$$

satisfy the heat equation and the Dirichlet boundary condition

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \Delta\right) u_j = 0 & \text{in } \Omega \times (0, \infty), \\ u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by the *principle of superposition* we obtain that the series of functions

$$u(x,t) = \sum_{j=1}^{\infty} A_j\, e^{-\lambda_j t}\, X_j(x), \quad A_j \in \mathbf{C},$$

is a candidate of solution of the problem (1.1) except for the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

Step 6: Finally, if the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega$$

is satisfied, then the constants A_j satisfy the expansion formula

$$u_0(x) = \sum_{j=1}^{\infty} A_j\, X_j(x).$$

In this way, we are reduced to the following problem:

Problem 1.1. *Can we express the initial function* $u_0(x)$ *as a* Fourier series *of the eigenfunctions* $X_j(x)$ *of the Dirichlet eigenvalue problem* (1.2) *?*

In fact, by virtue of Theorem 1.1 we can expand the function $u_0(x)$ as follows:

$$u_0(x) = \sum_{j=1}^{\infty} A_j\, X_j(x), \quad A_j = \int_{\Omega} u_0(y)\, X_j(y)\, dy.$$

Step 7: Summing up, we have proved that a non-trivial solution $u(x,t)$ of the initial-boundary value problem (1.1) can be expressed as follows:

$$u(x,t) = \sum_{j=1}^{\infty} A_j\, e^{-\lambda_j t}\, X_j(x),$$
$$A_j = \int_{\Omega} u_0(y)\, X_j(y)\, dy.$$

2 The Riesz–Schauder theory for compact operators

This section is devoted to the Riesz–Schauder theory for compact operators (Theorem 2.5).

2.1 Compact operators

A linear operator T on X into Y is said to be *compact* or *completely continuous* if it maps every bounded subset of X onto a relatively compact subset of Y, that is, if the closure of the range $T(B)$ is compact in Y for every bounded subset B of X. This is equivalent to saying that, for every bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ has a subsequence which converges in Y.

We list some facts which follow at once:

(1) Every compact operator is bounded.

(2) Every bounded linear operator with finite dimensional range is compact.

(3) No isomorphism between infinite dimensional spaces is compact.

(4) The product of a compact operator with a bounded operator is compact.

The next theorem states that if Y is a Banach space, then the compact operators on X into Y form a closed subspace of the space $L(X,Y)$ of bounded linear operators on X into Y:

Theorem 2.1. *Let* X *be a normed linear space and* Y *a Banach space. If* $\{T_n\}$ *is a sequence of compact linear operators which converges to an operator* T *in the space* $L(X,Y)$ *with the uniform topology, then* T *is compact.*

The next theorem states that the compactness of operators is inherited by the transposes:

Theorem 2.2. *Let X and Y be normed linear spaces. If T is a compact linear operator on X into Y, then its transpose T' is a compact linear operator on the dual space Y' into the dual space X'.*

2.2 Spectral analysis of compact operators

Now we state the most interesting results on compact linear operators, which are essentially due to F. Riesz in the Hilbert space setting. The results are extended to Banach spaces by Schauder.

Theorem 2.3. *Let X be a Banach space and T a compact linear operator on X into itself. Set*

$$S = I - T.$$

Then we have the following assertions:

(i) The null space $N(S)$ of S is finite dimensional and the range $R(S)$ of S is closed in X.

(ii) The null space $N(S')$ of the transpose S' is finite dimensional and the range $R(S')$ of S' is closed in X'.

(iii) $\dim N(S) = \dim N(S')$.

The next result is an extension of the theory of linear mappings in finite dimensional linear spaces:

Theorem 2.4 (the Fredholm alternative). *Let T be a compact linear operator on a Banach space X into itself. If $S = I - T$ is either one-to-one or onto, then it is an isomorphism of X onto itself.*

Let T be a bounded linear operator on X into itself. The *resolvent set* of T, denoted $\rho(T)$, is defined to be the set of scalars $\lambda \in \mathbf{K}$ such that $\lambda I - T$ is an isomorphism of X onto itself. In this case, the inverse $(\lambda I - T)^{-1}$ is called the *resolvent* of T. The complement of $\rho(T)$, that is, the set of scalars $\lambda \in \mathbf{K}$ such that $\lambda I - T$ is not an isomorphism of X onto itself is called the *spectrum* of T, and is denoted by $\sigma(T)$.

The set $\sigma_p(T)$ of scalars $\lambda \in \mathbf{K}$ such that $\lambda I - T$ is not one-to-one forms a subset of $\sigma(T)$, and is called the *point spectrum* of T. A scalar $\lambda \in \mathbf{K}$ belongs to $\sigma_p(T)$ if and only if there exists a non-zero element $x \in X$ such that $Tx = \lambda x$. In this case, λ is called an *eigenvalue* of T and x an *eigenvector* of T corresponding to λ. Also the null space $N(\lambda I - T)$ of $\lambda I - T$ is called the *eigenspace* of T corresponding to λ, and the dimension of $N(\lambda I - T)$ is called the *multiplicity* of λ.

By using C. Neumann's series, we find that the resolvent set $\rho(T)$ is open in $\mathbf{K}$ and that

$$\{\lambda \in \mathbf{K} : |\lambda| > \|T\|\} \subset \rho(T).$$

Hence the spectrum $\sigma(T) = \mathbf{K} \setminus \rho(T)$ is closed and bounded in $\mathbf{K}$.

If T is a compact operator and λ is a non-zero element of $\sigma(T)$, by applying Banach's closed graph theorem to the operator $\lambda^{-1}T$ we obtain that $\lambda I - T$ is not one-to-one, that is, $\lambda \in \sigma_p(T)$. Also note that if X is infinite dimensional, then T is not an isomorphism of X onto itself; hence $0 \in \sigma_p(T)$. Therefore the scalar field $\mathbf{K}$ can be decomposed as follows:

$$\mathbf{K} = (\sigma_p(T) \cup \{0\}) \cup \rho(T).$$

We can say rather more about the spectrum $\sigma(T)$. In fact, we have the following (see [12, Chapter X, Section 5, Theorem 3]):

Theorem 2.5 (Riesz–Schauder). *Let T be a compact linear operator on a Banach space X into itself. Then we have the following assertions:*

(i) The spectrum $\sigma(T)$ of T is either a finite set or a countable set accumulating only at the zero 0; and every non-zero element of $\sigma(T)$ is an eigenvalue of T.

(ii) $\dim N(\lambda I - T) = \dim N(\lambda I - T') < \infty$ *for all $\lambda \neq 0$.*

(iii) Let $\lambda \neq 0$. The equation

$$(\lambda I - T)x = y$$

has a solution if and only if y is orthogonal to the space $N(\lambda I - T')$, that is, $T'f = \lambda f$ implies that $\langle y, f\rangle = 0$. Here $\langle \cdot, \cdot \rangle$ denotes the duality between X and X'. Similarly, the equation

$$(\lambda I - T')f = g$$

has a solution if and only if g is orthogonal to the space $N(\lambda I - T)$, that is, $Tx = \lambda x$ implies that $\langle x, g\rangle = 0$. Moreover, the operator $\lambda I - T$ is onto if and only if it is one-to-one.

Furthermore, Theorem 2.5 tells us that if T is compact, then the non-zero eigenvalues of T is a countable set accumulating only at the zero 0; hence we can order them in a sequence $\{\lambda_j\}$ such that

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_j| \geq |\lambda_{j+1}| \geq \ldots \longrightarrow 0,$$

where each λ_j is repeated according to its multiplicity. For each λ_j, we let

$$\begin{aligned} V_{\lambda_j} = &\text{the eigenspace } N(\lambda_j I - T) \text{ of } T \\ &\text{corresponding to the eigenvalue } \lambda_j. \end{aligned}$$

The eigenspaces V_{λ_j} are mutually orthogonal. In fact, if $x \in V_{\lambda_j}$ and $y \in V_{\lambda_j}$, then we have the formula

$$\lambda_j(x, y) = (Tx, y) = (x, Ty) = \lambda_j(x, y),$$

so that $(x, y) = 0$ if $\lambda_j \neq \lambda_j$.

Therefore, we can choose an orthonormal basis of V_{λ_j} and combine these into an orthonormal set $\{x_j\}$ of eigenvectors of T such that

$$Tx_j = \lambda_j x_j.$$

The next theorem states that there is an orthonormal basis of X such that the operator T diagonalizes with respect to this basis:

$$T \sim \begin{pmatrix} \lambda_1 & 0 & \dots & \dots \\ 0 & \ddots & & \\ \vdots & & \lambda_j & \\ \vdots & & & \ddots \\ & & & \end{pmatrix}.$$

Theorem 2.6 (Hilbert–Schmidt). *Let T be a self-adjoint, compact linear operator on a Hilbert space X into itself. Then we have, for all $x \in X$,*

$$Tx = \sum_{j=1}^{\infty} \lambda_j (x, x_j) x_j = s - \lim_{n \to \infty} \sum_{j=1}^{n} \lambda_j (x, x_j) x_j.$$

In particular, if T is one-to-one, then we have the expansion formula

$$x = \sum_{j=1}^{\infty} (x, x_j) x_j = s - \lim_{n \to \infty} \sum_{j=1}^{n} (x, x_j) x_j,$$

that is, the family $\{x_j\}$ of eigenvectors is a complete orthonormal system *of X.*

3 Concrete Examples of Fourier's Method

We consider a metal bar of length ℓ with end points $x = 0$ and $x = \ell$. We study the temperature distribution of this metal bar.

3.1 Formulation of the Problem

Let $u(x, t)$ be the temperature of the position x of the metal bar at time t. Then it is known that $u(x, t)$ obeys the heat conduction equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{3.1}$$

Here the constant c depends on the mass density of the material and the specific heat capacity. The partial differential equation (3.1) is called the *heat conduction equation.*

Physically, if an initial temperature distribution of the metal bar is given at time $t = 0$ and if a temperature distribution of the end points is prescribed, then the temperature distribution of the metal bar is uniquely determined at any time t.

Mathematically, the temperature distribution of a metal bar can be formulated as follows:

(i) An initial temperature distribution of the metal bar at time $t = 0$ implies that, for a given function $\varphi(x)$ defined on $[0, \ell]$, the solution $u(x, t)$ satisfies the condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq \ell. \tag{3.2}$$

(ii) A temperature distribution of the end points is described by three conditions:

$$u(0, t) = u(\ell, t) = 0 \quad \text{(isothermal case)} \tag{3.3}$$

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(\ell, t) = 0 \quad \text{(adiabatic case)}, \tag{3.4}$$

$$u(0, t) = 0, \ \frac{\partial u}{\partial x}(t, \ell) + hu(\ell, t) = 0 \quad \text{(mixed case)}. \tag{3.5}$$

Here h is a positive constant.

The condition (3.2) is called an *initial condition* and the conditions (3.3), (3.4) and (3.5) are called *boundary conditions*.

In this section we consider only the isothermal case. Namely, we study the following problem:

Problem 3.1. *For a given function $\varphi(x)$ defined on $[0, \ell]$, find a function $u(x, t)$ defined in $(0, \infty) \times [0, \ell]$ such that*

$$\begin{cases} \dfrac{\partial u}{\partial t} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0, \infty) \times [0, \ell], \\ u(x, 0) = \varphi(x) & \text{in } [0, \ell], \\ u(0, t) = u(\ell, t) = 0 & \text{for } t > 0. \end{cases} \tag{3.6}$$

3.2 Uniqueness Theorem for the Initial-Boundary Value Problem for the Heat Equation

In this subsection we consider the following initial-boundary value problem for the heat equation: For a given function $\varphi(x)$ defined on the closed interval $J = [0, \ell]$, find a function $u(x, t)$ in $J \times [0, \infty)$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for all } 0 < x < \ell \text{ and } t > 0, \\ u(0, t) = u(\ell, t) = 0 & \text{for all } t > 0, \\ u(x, 0) = \varphi(x) & \text{for all } 0 < x < \ell. \end{cases} \tag{3.7}$$

First, we prove the following *uniqueness theorem* for the initial-boundary value problem (3.7):

Theorem 3.1. *Assume that the function $\varphi(x)$ is continuous on the closed interval $J = [0, \ell]$. Then a solution $u(x, t)$ of problem (3.7) is unique.*

Proof. If $u_1(x,t)$ and $u_2(x,t)$ are two solutions of the initial-boundary value problem (3.7), we let

$$u(x,t) := u_1(x,t) - u_2(x,t).$$

Then we have the formulas

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for all } 0 < x < \ell \text{ and } t > 0, \\ u(0,t) = u(\ell,t) = 0 & \text{for all } t > 0, \\ u(x,0) = 0 & \text{for all } 0 < x < \ell. \end{cases} \tag{3.8}$$

Now we consider the *total heat energy*

$$E(t) = \int_0^\ell u(x,t)^2 \, dx.$$

By integration by parts, it follows from formulas (3.8) that

$$\begin{aligned} \frac{dE}{dt} &= 2 \int_0^\ell \frac{\partial u}{\partial t}(x,t) \cdot u(x,t) \, dx \\ &= 2 \int_0^\ell \frac{\partial^2 u}{\partial x^2}(x,t) \cdot u(x,t) \, dx \\ &= 2 \left[\frac{\partial u}{\partial x}(x,t) \cdot u(x,t) \right]_{x=0}^{x=\ell} - 2 \int_0^l \left(\frac{\partial u}{\partial x}(x,t) \right)^2 dx \\ &= -2 \int_0^\ell \left(\frac{\partial u}{\partial x}(x,t) \right)^2 dx \\ &\leq 0. \end{aligned}$$

This proves that the total heat energy $E(t)$ is a non-increasing function of t. However, we have the formula

$$E(0) = \int_0^\ell u(x,0)^2 \, dx = 0,$$

since $u(x,0) = 0$ for all $0 < x < \ell$.

Summing up, we have proved that

$$E(t) = \int_0^\ell u(x,t)^2 \, dx \equiv 0 \quad \text{for all } t \geq 0,$$

so that

$$u_1(x,t) - u_2(x,t) = u(x,t) \equiv 0 \quad \text{for all } (x,t) \in (0,\ell) \times [0,\infty).$$

The proof of Theorem 3.1 is complete. □

By virtue of Theorem 3.1, we have only to find a candidate of solutions of the heat equation (3.7) with the initial condition and the Dirichlet boundary condition. In doing so, we make use of the separation of variables method.

3.3 Separation of Variables Method

We shall find a non-trivial solution $u(x,t)$ of the initial-boundary value problem (3.1), (3.2) and (3.3) in the separation of variables form

$$u(x,t) = X(x)T(t). \tag{3.9}$$

Here we assume that $T(t)$ and $X(x)$ are both non-trivial functions. By substituting formula (3.9) into equation (3.1), we obtain that

$$X(x)T'(t) = c^2 X''(x)T(t).$$

Namely, we have the formulas

$$\frac{1}{c^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

However, the left-hand side depends only on the variable t, while the right-hand side depends only on the variable x. In particular, the both sides do not depend on t and x, that is, they are a constant, say $-\lambda$. Thus we have the formulas

$$X''(x) + \lambda X(x) = 0. \tag{3.10}$$

and

$$T'(t) + c^2\lambda T(t) = 0. \tag{3.11}$$

Moreover, since the Dirichlet boundary condition (3.3) holds true, it follows that the function $X(x)$ should satisfy the boundary condition

$$X(0) = X(\ell) = 0. \tag{3.12}$$

Namely, the function $X(x)$ is a non-trivial solution of the homogeneous Dirichlet boundary value problem (3.10) and (3.12).

(I) The case where $\lambda < 0$: The Dirichlet boundary value problem (3.10) and (3.12) has only the trivial solution $X(x) \equiv 0$. Indeed, a general solution of equation (3.10) is given by the formula

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}.$$

It follows from the boundary condition (3.12) that

$$c_1 = c_2 = 0,$$

so that

$$X(x) \equiv 0 \quad \text{in } \Omega.$$

This implies that $u(x,t) = T(t)X(x) \equiv 0$.

(II) The case where $\lambda = 0$: The Dirichlet boundary value problem (3.10), (3.12) has only the trivial solution $X(x) \equiv 0$. Indeed, a general solution of equation (3.10) is given by the formula

$$X(x) = c_1 + c_2 x.$$

It follows from the boundary condition (3.12) that $X(x) \equiv 0$. This implies that $u(x,t) = T(t)X(x) \equiv 0$.

(III) The case where $\lambda > 0$: A general solution of equation (3.10) is given by the formula

$$X(x) = D_1 \cos \sqrt{\lambda} x + D_2 \sin \sqrt{\lambda} x.$$

It follows from the boundary condition (3.12) that

$$\begin{cases} 0 = X(0) = D_1, \\ 0 = X(\ell) = D_2 \sin \sqrt{\lambda} \ell. \end{cases}$$

However, since $X(x) \not\equiv 0$, it follows that $D_2 \neq 0$. Hence we have the condition

$$\sin \sqrt{\lambda} \ell = 0.$$

This implies that

$$\sqrt{\lambda} = \frac{n\pi}{\ell}, \quad n = 1, 2, 3, \ldots,$$

provided that the $X(x)$ are non-trivial functions.

Therefore, if the Dirichlet boundary value problem (3.10) and (3.12) has non-trivial solutions, then the constant λ should satisfy the conditions

$$\lambda = \lambda_n = \left(\frac{n\pi}{\ell} \right)^2, \quad n = 1, 2, 3, \ldots. \tag{3.13}$$

The corresponding solution is given by the formula

$$X_n(x) = D_n \sin \frac{n\pi}{\ell} x, \quad n = 1, 2, 3, \ldots. \tag{3.14}$$

Here D_n is an arbitrary constant to be chosen later on.

Summing up, we have proved that the Dirichlet boundary value problem (3.10), (3.12) admits non-trivial solutions (3.14) if the parameter λ satisfies the condition (3.13). The number λ_n is called an eigenvalue and the function $X_n(x)$ is called an eigenfunction corresponding the eigenvalue λ_n, respectively.

A solution $T_n(t)$ of equation (3.11) corresponding $\lambda = \lambda_n$ is equal to the following:

$$T_n(t) = c_n e^{-c^2 \lambda_n t}. \tag{3.15}$$

Here c_n is an arbitrary constant to be chosen later on.

Therefore, by combining the formulas (3.14) and (3.15) we obtain from the formula (3.9) that

$$u_n(x,t) = X_n(x) T_n(t) = A_n e^{-c^2 \lambda_n t} \sin \frac{n\pi}{\ell} x.$$

Here A_n is an arbitrary constant to be chosen later on.

We remark that the functions $u_n(x,t)$ are solutions of the heat conduction equation (3.1) and satisfy the Dirichlet boundary condition (3.3). However, they do not satisfy the initial condition (3.2).

Now we make use of the *principle of superposition* in order to treat the initial condition (3.2). Namely, we consider a series of functions $u_n(x,t)$ of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-c^2\lambda_n t} \sin \frac{n\pi}{\ell} x. \tag{3.16}$$

If the function $u(x,t)$ satisfy the initial condition (3.2), then it follows that the constants A_n satisfy the expansion formula

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{\ell} x, \quad 0 \leq x \leq \ell. \tag{3.17}$$

If the function $u(x,t)$, given by the series (3.16), is partially differentiable with respect to t and twice partially differentiable with respect to x in the region $0 < x < \ell,\ t > 0$, then we obtain that the function $u(x,t)$ is a solution of the initial-boundary value problem for the heat equation.

In this way, we are reduced to the following problem:

Problem 3.2. *Given a function* $\varphi(x)$ *defined on the interval* $[0,1]$*, can we expand it as a trigonometric series of the form* (3.17) *?*

By the theory of Fourier series (see Section 2), we know that, under some conditions on $\varphi(x)$, the coefficients A_n of formula (3.16) are given by the formulas

$$A_n = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \sin \frac{n\pi}{\ell} x dx, \quad n = 1, 2, 3, \dots. \tag{3.18}$$

Assuming this fact for the moment, the function $u(x,t)$ defined by formulas (3.16) and (3.18) is a solution of the heat conduction equation (3.1) which satisfies the initial condition (3.2) and the Dirichlet boundary condition (3.3).

If the mass density and the specific heat capacity of the metal bar are not constant and are functions of position x, then we have the following differential equation of Sturm–Liouville type

$$r(x)\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] + q(x)u \tag{3.19}$$

A general boundary condition is given as follows:

$$\begin{cases} \alpha u(a,t) + \alpha' u'(a,t) = 0 & (t > 0), \\ \beta u(b,t) + \beta' u'(b,t) = 0 & (t > 0). \end{cases} \tag{3.20}$$

where

$$\alpha^2 + \alpha'^2 \neq 0\,,\ \beta^2 + \beta'^2 \neq 0$$

We remark that the boundary condition (3.3) is a special case of the general boundary condition (3.20).

In this way, if we try to find a solution $u(x,t)$ of the differential equation (3.19) with initial condition (3.2) and boundary condition (3.20), then we are reduced to the study of the following eigenvalue problem:

Problem 3.3. *Find parameters λ and non-trivial functions $y(x)$ of the eigenvalue problem*

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0, \tag{3.21}$$

$$\alpha y(a) + \alpha' y'(a) = 0, \quad \beta y(b) + \beta' y'(b) = 0. \tag{3.22}$$

Moreover, characterize the analytic properties of eigenvalues λ_n and eigenfunctions $y_n(x)$.

For example, we consider the case $p(x) \equiv 1, q(x) \equiv 0$ and $r(x) \equiv 1$ in the equation (3.21)

$$y'' + \lambda y = 0.$$

If we impose the Dirichlet boundary condition (isothermal condition)

$$y(a) = y(b) = 0,$$

then we obtain the following eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{n\pi}{b-a}\right)^2; \quad y_n(x) = \sin\sqrt{\lambda_n}(x-a),\ n = 1, 2, \ldots.$$

If we impose the Neumann boundary condition (adiabatic condition)

$$y'(a) = y'(b) = 0$$

then we obtain the following eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{n\pi}{b-a}\right)^2; \quad y_n(x) = \cos\sqrt{\lambda_n}(x-a),\ n = 0, 1, 2, \ldots.$$

Summary 3.2. *For simplicity, we let $a > 0, b = 0$. The results in this subsection are summarized as follows:*

Conditions	Eigenvalues	Eigenfunctions	Comments
Dirichlet case	$\frac{m^2\pi^2}{a^2}$	$\sqrt{\frac{2}{a}}\sin\frac{m\pi}{a}x$	$m \geq 1$, Odd-extension
Neumann case	$\frac{m^2\pi^2}{a^2}$	$\sqrt{\frac{2}{a}}\cos\frac{m\pi}{a}x$	$m \geq 0$, Even-extension

If we consider a mixed type boundary condition defined by the formula

$$y(a) = y'(b) = 0,$$

then we find that the eigenvalues and eigenfunctions are given as follows:

$$\lambda_n = \left[\frac{\left(n + \frac{1}{2}\right)\pi}{b-a}\right]^2, \qquad y_n(x) = \sin\sqrt{\lambda_n}(x-a),\ n = 0, 1, 2, \ldots.$$

We remark that the eigenvalues λ_n here are different from the Dirichlet case and the Neumann case.

3.4 Case of a Rectangle

We calculate the eigenvalues and eigenfunctions of the Dirichlet eigenvalue problem for the Laplacian in a rectangle:

Example 3.1. *Let $\Omega = (0, a) \times (0, b)$ be a rectangle in $\mathbf{R}^2$. We consider the following initial-boundary value problem for the heat equation:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \Omega \times (0, \infty), \\ u(x, y, 0) = u_0(x, y) & \text{in } \Omega, \\ u(x, y, t) = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{3.23}$$

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian, $u_0(x, y)$ is an initial thermal distribution and $u(x, y, t)$ expresses a thermal distribution of position (x, y) at time t. The boundary condition is the Dirichlet boundary condition (isothermal condition).

We shall find a non-trivial solution $u(x, y, t)$ of the initial-boundary value problem (3.23) in the separation of variables form

$$u(x, y, t) = T(t)X(x, y) = T(t)U(x)V(y) \tag{3.24}$$

Here we assume that $T(t)$, $U(x)$ and $V(y)$ are non-trivial functions.

(1) First, by substituting formula (3.24) into the heat conduction equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

we obtain that

$$T'(t)X(x, y) = T(t)\Delta X(x, y).$$

Namely, we have the equations

$$\frac{T'(t)}{T(t)} = \frac{\Delta X(x, y)}{X(x, y)} = \frac{U''(x)V(y) + U(x)V''(y)}{U(x)V(y)} = -\lambda.$$

However, the left-hand side depends only on the variable t, while the right-hand side depends only on the variables x and y. In particular, the both sides do not depend on t, x and y, that is, they are a constant, say $-\lambda$.

$$\frac{T'(t)}{T(t)} = \frac{U''(x)}{U(x)} + \frac{V''(y)}{V(y)} = -\lambda.$$

Moreover, we have the formula

$$\frac{U''(x)}{U(x)} = -\lambda - \frac{V''(y)}{V(y)} = -\alpha.$$

However, the left-hand side depends only on the variable x, while the right-hand side depends only on the variable y. In particular, the both sides do not depend on x and y, that is, they are a constant, say $-\alpha$. Namely, it follows that

$$\frac{U''(x)}{U(x)} = -\lambda - \frac{V''(y)}{V(y)} = -\alpha,$$

or equivalently,

$$\frac{U''(x)}{U(x)} = -\alpha,$$
$$\frac{V''(y)}{V(y)} = -\beta.$$

Here α and β are constants such that

$$\alpha + \beta = \lambda.$$

Summing up, we have proved that the initial-boundary value problem (1.1) is equivalent to the following three problems:

$$T'(t) + \lambda T(t) = 0,$$

and

$$\begin{cases} U''(x) + \alpha U(x) = 0 \quad \text{in } (0, a), \\ U(0) = U(a) = 0 \end{cases} \tag{3.25}$$

$$\begin{cases} V''(y) + \beta V(y) = 0 \quad \text{in } (0, b), \\ V(0) = V(b) = 0 \end{cases} \tag{3.26}$$

(2) The Dirichlet eigenvalue problems (3.25) and (3.26) can be solved respectively as follows:

$$\alpha_m = \frac{m^2\pi^2}{a^2}, \quad m = 1, 2, \ldots,$$
$$U_m(x) = \sqrt{\frac{2}{a}} \sin \frac{m\pi}{a} x, \quad m = 1, 2, \ldots,$$

and

$$\beta_n = \frac{n^2\pi^2}{a^2}, \quad n = 1, 2, \ldots,$$
$$V_n(y) = \sqrt{\frac{2}{b}} \sin \frac{n\pi}{b} y, \quad n = 1, 2, \ldots,$$

provided that $U(x)$ and $V(y)$ are non-trivial functions. Here we remark that

$$\lambda_{m,n} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}, \quad m = 1, 2, \ldots, \; n = 1, 2, \ldots.$$

Summary 3.3. *The results in this subsection are summarized as follows:*

Conditions	Eigenvalues	Eigenfunctions	Comments
Dirichlet case	$\dfrac{m^2\pi^2}{a^2} + \dfrac{n^2\pi^2}{b^2}$	$\dfrac{2}{\sqrt{ab}} \sin \dfrac{m\pi}{a} x \sin \dfrac{n\pi}{b} y$	Odd-extension

Therefore, the unique solution $u(x, y, t)$ of problem (3.23) can be expressed as follows:

$$u(x, y, t) = \sum_{m,n=1}^{\infty} C_{m,n}\, e^{-\lambda_{m,n} t}\, U_m(x) V_n(y),$$

$$C_{m,n} = \int \int_{\Omega} u_0(x, y)\, U_m(x) V_n(y)\, dx\, dy.$$

3.5 Case of a Disk

We calculate the eigenvalues and eigenfunctions of the Dirichlet eigenvalue problem for the Laplacian in a *disk*:

Example 3.2. *Let* $\Omega = \{x^2 + y^2 \leq a^2\}$ *be a disk in* $\mathbf{R}^2$. *We consider the Dirichlet eigenvalue problem for the Laplacian*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u(x, y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

The Laplacian Δ is expressed in terms of polar coordinates as follows:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Hence the Laplace equation

$$\Delta u + \lambda u = 0$$

is equivalent to the following:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\lambda u. \tag{3.28}$$

We construct a solution $u(r, \theta) = u(x, y)$ of the equation (3.28) in the separation of variables form

$$u(r, \theta) = U(r) V(\theta). \tag{3.29}$$

Here we assume that $U(r)$ and $V(\theta)$ are both non-trivial functions.

By substituting the formula (3.29) into the equation (3.28), we obtain that

$$\frac{r^2 U''(r) + r U'(r) + r^2 \lambda U(r)}{U(r)} = -\frac{V''(\theta)}{V(\theta)} = \mu,$$

However, the left-hand side depends only on the variable r, while the right-hand side depends only on the variable θ. In particular, the both sides do not depend on r and θ, that is, they are a constant, say μ. In this way we are reduced to the following two eigenvalue problems:

$$\begin{cases} V''(\theta) = -\mu V(\theta), \\ V(0) = V(2\pi), \end{cases} \tag{3.30}$$

and

$$U''(r) + \frac{1}{r}U'(r) + \left(\lambda - \frac{\mu}{r^2}\right) U(r) = 0, \tag{3.31}$$

$$U(a) = 0. \tag{3.32}$$

First, the eigenvalues μ_n and eigenfunctions $V_n(\theta)$ of the eigenvalue problem (3.30) are given as follows:

$$\mu_n = n^2, \quad n = 0, 1, 2, \ldots,$$

and

$$\begin{cases} V_n(\theta) = \cos n\theta, \quad n = 0, 1, 2, \ldots, \\ V_n(\theta) = \sin n\theta, \quad n = 1, 2, \ldots, \end{cases}$$

provided that the $V(\theta)$ are non-trivial functions.

For these eigenvalues $\mu_n = n^2$, we solve the boundary value problem (3.31) and (3.32). If we let

$$s = \sqrt{\lambda} r,$$

$$J(s) = J(\sqrt{\lambda} r) = U(r)$$

in equation (3.31), then we have the *Bessel differential equation*

$$J''(s) + \frac{1}{s}J'(s) + \left(1 - \frac{n^2}{s^2}\right) J(s) = 0. \tag{3.33}$$

It is well known that the solution $J_n(s)$ of the differential equation (3.33), which is smooth near the origin 0, is given by the following power series:

$$J_n(s) = \left(\frac{s}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{s}{2}\right)^{2k} \tag{3.34}$$

The function $J_n(s)$ is called the n-th *Bessel function*. Moreover, the zeros $\nu_{n,m}$ of $J_n(s)$ are distributed on the positive real axis diverging to $+\infty$:

$$\nu_{n,m}, \quad m = 1, 2, \ldots$$

In this case, the boundary condition (3.32) is equivalent to the condition

$$J_n(\sqrt{\lambda} a) = 0.$$

Therefore, we can characterize the eigenvalues λ of the Dirichlet eigenvalue problem (3.27) as follows:

$$\lambda_{n,m} = \frac{\nu_{n,m}^2}{a^2}, \quad n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots$$

The eigenfunctions corresponding to the zeros $\lambda_{n,m}$ are given as follows:

$$\begin{cases} J_n(\nu_{n,m} r/a) \cos n\theta, & n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots \\ J_n(\nu_{n,m} r/a) \sin n\theta, & n = 1, 2, \ldots, \quad m = 1, 2, \ldots \end{cases} \tag{3.35}$$

Summary 3.4. *The results in this subsection are summarized as follows:*

Conditions	Eigenvalues	Eigenfunctions	Comments
Dirichlet case	$\frac{\nu_{n,m}^2}{a^2}$	$J_n(\nu_{n,m} r/a) \cos n\theta$, $J_n(\nu_{n,m} r/a) \sin n\theta$	Bessel functions

Therefore, the unique solution $u(r, \theta, t) = u(x, y, t)$ of problem (3.23) can be expressed as follows:

$$\begin{aligned} u(r, \theta, t) = &\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{n,m}\, e^{-\lambda_{n,m} t}\, J_n(\sqrt{\lambda_{n,m}} r) \cos n\theta \\ &+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} D_{n,m}\, e^{-\lambda_{n,m} t}\, J_n(\sqrt{\lambda_{n,m}} r) \sin n\theta, \\ C_{n,m} = &\int_0^a \int_0^{2\pi} u_0(r, \theta)\, J_n(\sqrt{\lambda_{n,m}} r) \cos n\theta\, r\, dr\, d\theta, \\ D_{n,m} = &\int_0^a \int_0^{2\pi} u_0(r, \theta)\, J_n(\sqrt{\lambda_{n,m}} r) \sin n\theta\, r\, dr\, d\theta. \end{aligned}$$

References

[1] Farlow, S.J. (1982). *Partial differential equations for scientists and engineers* (John Wiley & Sons, New York).

[2] Folland, G.B. (1995). *Introduction to partial differential equations*, second edition (Princeton University Press, Princeton, New Jersey).

[3] Friedman, A. (1970). *Foundations of modern analysis* (Holt, Rinehart and Winston Inc., New York Montreal London).

[4] Gilbarg, D. and Trudinger, N.S. (1998). *Elliptic partial differential equations of second order*, 1998 edition (Springer-Verlag, New York Berlin Heidelberg Tokyo).

[5] McLean, W. (2000). *Strongly elliptic systems and boundary integral equations* (Cambridge University Press, Cambridge).

[6] Mizohata, S. (1973). *The theory of partial differential equations* (Cambridge University Press, London, New York).

[7] Taira, K. (2014). *Semigroups, boundary value problems and Markov processes*, second edition, Springer Monographs in Mathematics (Springer-Verlag, Berlin Heidelberg New York).

[8] Taira, K. (2016). *Analytic semigroups and semilinear initial boundary value problems*, second edition, London Mathematical Society Lecture Note Series, No. 434 (Cambridge University Press, London New York).

[9] Vladimirov, V.S. (1971). *Equations of mathematical physics* (Marcel Dekker, New York).

[10] Watson, G.N. (1944). *A treatise on the theory of Bessel functions*, second edition (Cambridge University Press, Cambridge).

[11] Wloka, J. (1987). *Partial differential equations* (Cambridge University Press, Cambridge).

[12] Yosida, K. (1980). *Functional analysis*, reprint of the sixth (1980) edition, Classics in Mathematics (Springer-Verlag, Berlin Heidelberg New York).