# On some two phase problem for compressible and compressible viscous fluid flow separated by sharp interface

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**Abstract**

In this paper, we prove a local in time unique existence theorem for some two phase problem of compressible and compressible barotropic viscous fluid flow without surface tension in the $L_p$ in time and the $L_q$ in space framework with $2 < p < \infty$ and $N < q < \infty$ under the assumption that the initial domain is a uniform $W_q^{2-1/q}$ domain in $\mathbb{R}^N (N \geq 2)$. After transforming a unknown time dependent domain to the initial domain by the Lagrangian transformation, we solve problem by contraction mapping principle with the maximal $L_p$-$L_q$ regularity of the generalized Stokes operator for the compressible viscous fluid flow with free boundary condition. The key step of our method is to prove the existence of $\mathcal{R}$-bounded solution operator to resolvent problem corresponding to linearized problem. The $\mathcal{R}$-boundedness combined with Weis's operator valued Fourier multiplier theorem implies the generation of analytic semigroup and the maximal $L_p$-$L_q$ regularity theorem.

## 1 Introduction

In this paper, we consider some two phase problem for compressible and compressible viscous fluid flow without surface tension. Our problem is formulated in the following: let $\Omega_\pm$ be regions in $\mathbb{R}^N$ $(N \geq 2)$ occupied by compressible barotropic viscous fluids. We assume that the boundaries of $\Omega_\pm$ consist of three parts $\Gamma$ and $\Gamma_\pm$ with $\partial\Omega_\pm = \Gamma \cup \Gamma_\pm$, $\Gamma_\pm \cap \Gamma = \emptyset$ and $\Gamma_+ \cap \Gamma_- = \emptyset$. Set $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$. Let $\Omega_{\pm,t}$, $\Gamma_{-,t}$ and $\Gamma_t$ be time evolutions of $\Omega_\pm$, $\Gamma_-$ and $\Gamma$ respectively. We assume that $\Omega_{+,t} \cap \Omega_{-,t} = \emptyset$ $(t \geq 0)$.

Our problem is described by the following system:

$$\begin{cases} \partial_t \rho_\pm + \mathrm{div}\,(\rho_\pm \vec{v}_\pm) = 0 & \text{in } \Omega_{\pm,t}, \\ \rho_\pm(\partial_t \vec{v}_\pm + (\vec{v}_\pm \cdot \nabla)\vec{v}_\pm) - \mathrm{Div}\, S_\pm(\vec{v}_\pm) + \nabla P_\pm(\rho_\pm) = 0 & \text{in } \Omega_{\pm,t}, \\ (S_+(\vec{v}_+) - P_+(\rho_+)I)\vec{n}_t|_{\Gamma_t+0} & \\ \qquad -(S_-(\vec{v}_-) - P_-(\rho_-)I)\vec{n}_t|_{\Gamma_t-0} = -\pi_0 \vec{n}_t|_{\Gamma_t}, & \\ \vec{v}_+|_{\Gamma_t+0} = \vec{v}_-|_{\Gamma_t-0}, & \\ (S_-(\vec{v}_-) - P_-(\rho_-)I)\vec{n}_{-,t}|_{\Gamma_{-,t}} = -P_-(\rho_{0,-})\vec{n}_{-,t}, & \\ v_+|_{\Gamma_+} = 0 & \end{cases} \tag{1.1}$$

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for $0 < t < T$, subject to the initial condition $(\rho_\pm, \vec{v}_\pm)|_{t=0} = (\rho_{0,\pm} + \theta_{0,\pm}, \vec{v}_{0,\pm})$. Here $\rho_\pm = \rho_\pm(x,t)$ and $\vec{v}_\pm = \vec{v}_\pm(x,t) = (v_{\pm,1}(x,t), \ldots, v_{\pm,N}(x,t))$ are unknown mass density and unknown velocity field respectively. $\rho_{0,\pm}$ are a positive constant describing reference mass density of $\Omega_\pm$ and $(\theta_{0,\pm}, \vec{v}_{0,\pm})$ is the given initial data. $P_\pm(\rho)$ are pressure functions which are $C^\infty$-functions defined on $\rho > 0$ and satisfy $P'_\pm(\rho) > 0$. $\vec{n}_t$ and $\vec{n}_{-,t}$ are the unit outward normal to $\Gamma_t$ and $\Gamma_{-,t}$, respectively. Let $S_\pm(\vec{v}_\pm) = 2\mu_1^\pm D(\vec{v}_\pm) + \mu_2^\pm(\mathrm{div}\,\vec{v}_\pm)I$, where $D(u) = (\nabla u + {}^T\nabla u)/2$ is the $N \times N$ matrix called the Cauchy deformation tensor [1], $I$ denotes the $N \times N$ identity matrix and $\mu_1^\pm$ and $\mu_2^\pm$ are viscosity coefficients with $\mu_1^\pm > 0$ and $\mu_1^\pm + \mu_2^\pm > 0$. For $N \times N$ matrix function $M = (M_{ij})$, the $i$-th component of Div $M$ is defined by $\sum_{j=1}^N \partial_j M_{ij}$. Moreover $f|_{\Gamma_t \pm 0}$ and $\pi_0$ mean that $f|_{\Gamma_t \pm 0} = \lim_{x \to x_0, x \in \Omega_{\pm,t}} f(x,t)$ $(x_0 \in \Gamma_t)$ and $\pi_0 = P_+(\rho_{0,+}) - P_-(\rho_{0,-})$ and $T$ is a positive number describing time.

The kinematic condition for $\Gamma_t$ and $\Gamma_{-,t}$ is satisfied, namely they give

$$\Gamma_t = \{x = \vec{x}(\xi, t) \mid \xi \in \Gamma\}, \qquad \Gamma_{-,t} = \{x = \vec{x}(\xi, t) \mid \xi \in \Gamma_-\}, \tag{1.2}$$

1.3

[1] ${}^T M$ denotes the transposed $M$.

where $x = \vec{x}(\xi, t)$ is the solution to the Cauchy problem

$$\frac{dx}{dt} = \vec{v}(x,t), \qquad x|_{t=0} = \xi, \qquad \vec{v} = \begin{cases} \vec{v}_+ & \text{in } \Omega_{+,t}, \\ \vec{v}_- & \text{in } \Omega_{-,t}. \end{cases}$$

This fact means that the interface $\Gamma_t$ and free surface $\Gamma_{-,t}$ consist of the same fluid particles, which do not leave them and are not incident of them from inside $\Omega_{+,t} \cup \Omega_{-,t}$ for $t > 0$. It is clear that $\Omega_{\pm,t} = \{x = x(\xi,t) \mid \xi \in \Omega_\pm\}$.

A free boundary problem for a viscous compressible barotrophic fluid has been studied by some mathematicians. For the results for one phase problem, local in time unique existence of solutions to the free boundary problem without surface tension in the multi-dimensional case was proved by Secchi and Valli [9] in $L_2$ framework and by Tani [14] in the Hölder space, respectively. Later on, the same problem with surface tension was studied by Solonnikov and Tani [12] in the $L_2$ framework and Dennisova and Solonnikov [2], [3] in Hölder spaces.

For two phase problem of compressible and incompressible viscous fluids, Denisova [1] first showed a local in time existence theorem with surface tension on $\Gamma_t$ under the assumption that $\mu_1^+ < \mu_1^-$ and $\mu_1^+ + \mu_2^+ < \mu_1^+/R_\infty$ with some positive constant $R_\infty$ and $\Gamma_{-,t}$ and $\Gamma_+$ are empty sets. Recently Kubo, Shibata and Soga [8] considered corresponding resolvent problem and showed the existence of its $\mathcal{R}$-bounded solution operator, which implies maximal $L_p$-$L_q$ regularity theorem for linearlized problem and the local in time existence theorem for two phase problem.

On the other hand, for two phase problem of compressible and compressible viscous fluid, Tani [14],[15] studied a local in time existence theorem under the natural condition in Hölder space framework. As far as we know, there are no literatures concerning the $L_p$ approach to the two phase problem of compressible and compressible viscous fluid. In this paper, we shall consider the two phase problem of compressible and compressible fluid in $L_p$-$L_q$ framework and prove the local in time existence theorem of our problem in a similar way as Enomoto, Below and Shibata [5] and Kubo, Shibata and Soga[8]. As we shall explain later, after transforming a unknown time dependent domain to the initial domain by the Lagrangian transformation, we solve our problem by contraction mapping principle with maximal $L_p$-$L_q$ regularity theorem for the generalized Stokes operator for the compressible viscous fluid flow with free boundary condition. Maximal $L_p$-$L_q$ regularity theorem follows from the $\mathcal{R}$-boundedness of solution operator to the generalized resolvent equation corresponding to our liearized problem with the help of Weis's operator valued Fourier multiplier theorem. Therefore our goal of this paper is to prove the existence of $\mathcal{R}$-bounded solution operator.

We shall now go back to our approach. As we mentioned, we transfer $\Omega_{\pm,t}$ to some fixed domain. Our problem can be written as an initial boundary value problem in the given domain $\Omega_\pm$ if we transfer the Euler coordinates $x \in \Omega_{\pm,t}$ to Lagrange coordinates $\xi \in \Omega_\pm$. If velocity field $\vec{u}_\pm(\xi,t)$ defined on $\Omega_\pm$ is known as functions of the Lagrange coordinates $\xi \in \Omega_\pm$, then this connection can be written in the form

$$x = \xi + \int_0^t \vec{u}_\pm(\xi,s)ds \equiv X_{\vec{u}_\pm}(\xi,t), \tag{1.3}$$

where $\vec{u}_\pm(\xi,t) = \vec{v}_\pm(X_{\vec{u}_\pm}(\xi,t),t)$ are the velocity vector fields defined on $\Omega_\pm$ known as functions of the Lagrange coordinates $\xi \in \Omega_\pm$. Let $A_\pm$ be the Jacobi matrix of the transformation $x = X_{\vec{u}_\pm}(\xi,t)$ with element $a_{ij}^\pm = \delta_{ij} + \int_0^t (\partial_{\xi_j} u_{\pm,i})(\xi,s)ds$. There exists a small number $\sigma$ such that $A_\pm$ are invertible, that is $\det A_\pm \neq 0$ whenever

$$\max_{i,j=1,\dots,N} \left\| \int_0^t (\partial_{\xi_j} u_{\pm,i})(\cdot,s)ds \right\|_{L_\infty(\Omega_\pm)} < \sigma \qquad (0 < t < T). \tag{1.4}$$

In this case, we have $\nabla_x = A_\pm^{-1}\nabla_\xi = (I + V_0(\int_0^t \nabla \vec{u}_\pm(\xi,s)ds))\nabla_\xi$ with $\nabla_x = {}^T(\partial_{x_1},\dots,\partial_{x_N})$ and $\nabla_\xi = {}^T(\partial_{\xi_1},\dots,\partial_{\xi_N})$, where $V_0(w_\pm)$ is a $N \times N$ matrix of $C^\infty$ functions with respect to $w_\pm = (w_{\pm,ij})$, $w_{\pm,ij} = \int_0^t (\partial_{\xi_j} u_{\pm,i})(\cdot,s)ds$, defined on $|w_\pm| < 2\sigma$ and $V_0(0) = 0$. For the unit outer normal vector $\vec{n}$ and $\vec{n}_-$ to $\Gamma$ and $\Gamma_-$, by (1.2), we see that the relation between $(\vec{n}_t, \vec{n}_{-,t})$ and $(\vec{n}, \vec{n}_-)$ is given by

$$\vec{n}_t = \frac{A_-^{-1}\vec{n}}{|A_-^{-1}\vec{n}|}, \qquad \vec{n}_{-,t} = \frac{A_-^{-1}\vec{n}_-}{|A_-^{-1}\vec{n}_-|}. \tag{1.5}$$

Since $\partial_t(\rho(X_{\vec{u}_\pm}(\xi,t),t)\det A_\pm) = (\partial_t\rho + \mathrm{div}\,(\rho\vec{v}_\pm))\det A_\pm = 0$, we see

$$\rho(X_{\vec{u}_\pm}(\xi,t),t) = (\rho_{0,\pm}+\theta_{0,\pm})(\det A_\pm)^{-1} \tag{1.6}$$

and

$$\mathrm{div}\,_x\vec{w} = (\det A_\pm)^{-1}\mathrm{div}\,_\xi({}^T(\mathrm{cof}A_\pm)\hat{w}) \tag{1.7}$$

with $\hat{w}(\xi,t) = \vec{w}(X_{\vec{u}_\pm}(\xi,t),t)$, where cof $A$ denotes the cofactor matrix of $A$. Setting $\rho(X_{\vec{u}_\pm}(\xi,t),t) = \rho_{0,\pm}+\theta_{0,\pm}(\xi)+\theta_\pm(\xi,t)$ and using (1.5)-(1.7), we write dynamical system (1.1) in the Lagrangian coordinate introduced by (1.3) as follows:

$$\begin{cases} \partial_t\theta_\pm + (\rho_{0,\pm}+\theta_{0,\pm})\mathrm{div}\,\vec{u}_\pm = F_\pm & \text{in } \Omega_\pm, \\ (\rho_{0,\pm}+\theta_{0,\pm})\partial_t\vec{u}_\pm - \mathrm{Div}\,S_\pm(\vec{u}_\pm) + \nabla(P'_\pm(\rho_{0,\pm}+\theta_{0,\pm})\theta_\pm) = \vec{g}_\pm + \vec{G}_\pm & \text{in } \Omega_\pm, \\ (S_+(\vec{u}_+) - P'_+(\rho_{0,+}+\theta_{0,+})\theta_+ I)\vec{n}|_{\Gamma+0} & \\ \qquad\qquad -(S_-(\vec{u}_-) - P'_-(\rho_{0,-}+\theta_{0,-})\theta_- I)\vec{n}|_{\Gamma-0} = \vec{h}+\vec{H}, & \\ \vec{u}_+|_{\Gamma+0} = \vec{u}_-|_{\Gamma-0}, & \\ (S_-(\vec{u}_-) - P'_-(\rho_{0,-}+\theta_{0,-})\theta_- I)\vec{n}_-|_{\Gamma_-} = \vec{h}_- + \vec{H}_-, & \\ \vec{u}_+|_{\Gamma_+} = 0, & \end{cases} \tag{P}$$

for $0<t<T$, subject to the initial condition $(\theta_\pm,\vec{u}_\pm)|_{t=0} = (0,\vec{v}_{0,\pm})$. Here

$$\begin{aligned} \vec{g}_\pm &= -P'_\pm(\rho_{0,\pm}+\theta_{0,\pm})\nabla\theta_{0,\pm}, \\ \vec{h} &= [P_+(\rho_{0,+}+\theta_{0,+}) - P_+(\rho_{0,+}) - \{P_-(\rho_{0,-}+\theta_{0,-}) - P_-(\rho_{0,-})\}]\vec{n}, \\ \vec{h}_- &= (P_-(\rho_{0,-}+\theta_{0,-}) - P_-(\rho_{0,-}))\vec{n}_- \end{aligned}$$

and $F_\pm(\theta_\pm,\vec{u}_\pm)$, $\vec{G}_\pm(\theta_\pm,\vec{u}_\pm)$, $\vec{H}(\theta_\pm,\vec{u}_\pm)$ and $\vec{H}_-(\theta_-,\vec{u}_-)$ are nonlinear functions with respect to $\theta_\pm,\vec{u}_\pm$ and $w_\pm = \int_0^t \nabla\vec{u}_\pm(\xi,s)ds$ of the form:

$$\begin{aligned} &F_\pm(\theta_\pm,\vec{u}_\pm) \\ &= -\theta_\pm\mathrm{div}\,\vec{u}_\pm - (\rho_{0,\pm}+\theta_{0,\pm}+\theta_\pm)V_0(w_\pm)\mathrm{div}\,\vec{u}_\pm, \\ &\vec{G}_\pm(\theta_\pm,\vec{u}_\pm) \\ &= -\theta_\pm\partial_t\vec{u}_\pm + \mathrm{Div}\,\{\mu_1^\pm V_D(w_\pm)\nabla\vec{u}_\pm + \mu_2^\pm V_{\mathrm{div}}\,(w_\pm)(\nabla\vec{u}_\pm)\} \\ &\quad + V_{\mathrm{Div}}\,(w_\pm)\mathrm{Div}\,\{\mu_1^\pm(D(\vec{u}_\pm) + V_D(w_\pm)\nabla\vec{u}) + \mu_2^\pm(\mathrm{div}\,\vec{u}_\pm + V_{\mathrm{div}}\,(w_\pm)\nabla\vec{u}_\pm)\} \\ &\qquad + V_0(w_\pm)P'_\pm(\rho_{0,\pm}+\theta_{0,\pm}+\theta_\pm)\nabla(\theta_{0,\pm}+\theta_\pm) \\ &\qquad\quad + \nabla\int_0^1 P'_\pm(\rho_{0,\pm}+\theta_{0,\pm}+\tau\theta_\pm)(1-\tau)d\tau\theta_\pm^2, \\ &\vec{H}(\theta_\pm,\vec{u}_\pm) \\ &= -[\mu_1^+ V_D(w_+)\nabla\vec{u}_+ + \mu_2^+(V_{\mathrm{Div}}\,(w_+)\nabla\vec{u}_+)I]\vec{n} \\ &\quad + [\mu_1^+ V_D(w_-)\nabla\vec{u}_- + \mu_2^-(V_{\mathrm{Div}}\,(w_-)\nabla\vec{u}_-)I]\vec{n} \\ &\quad - V_0(w_+)[\mu_1^+(D(\vec{u}_+) + V_D(w_+)\nabla\vec{u}_+) + \mu_2^+(\mathrm{div}\,\vec{u}_+ + V_{\mathrm{div}}\,(w_+)(\nabla\vec{u}_+)I]\vec{n} \\ &\qquad + V_0(w_-)[\mu_1^-(D(\vec{u}_-) + V_D(w_-)\nabla\vec{u}_-) + \mu_2^-(\mathrm{div}\,\vec{u}_- + V_{\mathrm{div}}\,(w_-)(\nabla\vec{u}_-)I]\vec{n} \\ &\qquad\quad - \int_0^1 P''_+(\rho_{0,+}+\theta_{0,+}+\tau\theta_+)(1-\tau)d\tau\theta_+^2\vec{n} \\ &\qquad\qquad - \int_0^1 P''_-(\rho_{0,-}+\theta_{0,-}+\tau\theta_-)(1-\tau)d\tau\theta_-^2\vec{n} \\ &\qquad\qquad + V_0(w_+)[P_+(\rho_{0,+}+\theta_{0,+}+\theta_+) - P_+(\rho_{0,+})]\vec{n} \\ &\qquad\qquad\quad - V_0(w_-)[P_-(\rho_{0,-}+\theta_{0,-}+\theta_-) - P_-(\rho_{0,-})]\vec{n}, \\ &\vec{H}_-(\theta_-,\vec{u}_-) \end{aligned}$$

$$= -\,(\mu_1^- V_D(w_-)\nabla \vec{u}_- + \mu_2^- V_{\rm div}\,(w_-)({\rm div}\,\vec{u}_-)I)\vec{n}_- + \left(\int_0^1 P_-''(\rho_{0,-}+\theta_{0,-}+\tau\theta_-)(1-\tau)d\tau\theta_-^2\right)\vec{n}_-, \tag{1.8}$$

form_FGH

where $V_{\rm div}\,(w_\pm)$,$V_D(w_\pm)$ and $V_{\rm Div}\,(w_\pm)$ are some matrices of $C^\infty$ functions with respect to $w_\pm$ defined on $|w_\pm| < 2\sigma$, which satisfy conditions $V_{\rm div}\,(0) = V_D(0) = V_{\rm Div}\,(0) = 0$ and relations: ${\rm div}\,\vec{v}_\pm = {\rm div}\,\vec{u}_\pm + V_{\rm div}\,(w_\pm)\nabla\vec{u}_\pm$, $D(\vec{v}_\pm) = D(\vec{u}_\pm) + V_D(w_\pm)\nabla\vec{u}_\pm$, ${\rm Div}\,K_\pm = {\rm div}\,\widehat{K_\pm} + V_{\rm Div}\,(w_\pm)\nabla\widehat{K_\pm}$ with $\widehat{K_\pm} = K_\pm(X_{\vec{u}_\pm}(\xi,t),t)$.

To state our theorem on the local well-posedness of problem (P), we introduce some functional spaces and the definition of uniform $W_r^{2-1/r}$ domain. For any domain $D$ and $1 \le q \le \infty$, $L_q(D)$ and $W_q^m(D)$ denote the usual Lebesgue space and Sobolev space. We set $W_q^0(D) = L_q(D)$. For any Banach space $X$ and $1 \le p \le \infty$, $L_p((a,b),X)$ and $W_p^m((a,b),X)$ denote the usual Lebesgue space and Sobolev space of $X$-valued functions defined on an interval $(a,b)$. For $0 < \theta < 1$ and $\ell = 1, 2$, $B^{\ell\theta}_{q,p}(D)$ denotes the real interpolation space defined by $B^{\ell\theta}_{q,p}(D) = (L_q(D), W_q^\ell(D))_{\theta,p}$ with real interpolation functor $(\cdot,\cdot)_{\theta,p}$. We set $W_q^{\ell\theta} = B^{\ell\theta}_{q,q}$. $\|\cdot\|_{L_q(D)}, \|\cdot\|_{W_q^m(D)}, \|\cdot\|_{L_p((a,b),X)}, \|\cdot\|_{W_q^m((a,b),X)}$ and $\|\cdot\|_{B^{\ell\theta}_{q,p}(D)}$ denote their norms. For any functional space $Y$ with norm $\|\cdot\|_Y$, we set $Y^d = \{f = (f_1,\ldots,f_d) \mid f_i \in Y (i = 1,\ldots,d)\}$. For the simplicity of notations we use $\|\cdot\|_Y$ as its norm instead of $\|\cdot\|_{Y^d}$.

**Definition 1.1** (uniform $W_r^{2-1/r}$ domain)**.** Let $1 < r < \infty$ and let $\Omega$ be a domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. We say that $\Omega$ is a uniform $W_r^{2-1/r}$ domain, if there exist positive constants $\alpha, \beta$ and $K$ such that for any $x_0 = (x_{01},\ldots,x_{0N}) \in \partial\Omega$ there exist a coordinate number $j$, a $W_r^{2-1/r}$ function $h(\check{x})$ ($\check{x} = (x_1,\ldots,\check{x}_j,\ldots,x_N)$) defined on $B'_\alpha(\check{x}_0)$ with $\check{x}_0 = (x_{01},\ldots,\check{x}_{0j},\ldots,x_{0N})$ and $\|h\|_{W_r^{2-1/r}(B'_\alpha(\check{x}_0))} \le K$ such that

$$\begin{aligned}\Omega \cap B_\beta(x_0) &= \{x \in \mathbb{R}^N \mid x_j > h(\check{x})\ (\check{x} \in B'_\alpha(\check{x}_0))\} \cap B_\beta(x_0),\\ \partial\Omega \cap B_\beta(x_0) &= \{x \in \mathbb{R}^N \mid x_j = h(\check{x})\ (\check{x} \in B'_\alpha(\check{x}_0))\} \cap B_\beta(x_0).\end{aligned} \tag{1.9}$$

def-domain

Here $(x_1,\ldots,\check{x}_j,\ldots,x_N) = (x_1,\ldots,x_{j-1},x_j,\ldots,x_N)$, $B'_\alpha(\check{x}_0) = \{\check{x} \in \mathbb{R}^{N-1} \mid |\check{x} - \check{x}_0| < \alpha\}$ and $B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\}$.

The following theorem is concerned with the local well-posedness of problem (P).

th_LS **Theorem 1.2.** *Let $N \ge 2$, $2 < p < \infty$, $N < q < \infty$ and $R > 0$. Let $\rho_{0,\pm}$ be positive constants describing the reference mass density and let $P_\pm(\rho)$ be $C^\infty$ functions defined on $(\rho_{0,\pm}/2, 2\rho_{0,\pm})$ such that $0 \le P'_\pm(\rho_\pm) \le \rho_{1,\pm}$ with certain positive constants $\rho_{1,\pm}$ for any $\rho_\pm \in (\rho_{0,\pm}/2, 2\rho_{0,\pm})$. Let $\Omega_\pm$ be a uniform $W_q^{2-1/q}$ domains in $\mathbb{R}^N$. Let $\theta_{0,\pm} \in W_q^1(\Omega)$ and $\vec{v}_{0,\pm} \in B^{2(1-1/p)}_{q,p}(\Omega)^N$ be initial data for (P) with $\|\theta_{0,\pm}\|_{W_q^1(\Omega)} + \|\vec{v}_{0,\pm}\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le R$, which satisfy the compatibility condition:*

$$\begin{aligned}&[S_+(\vec{v}_{0,+}) - P_+(\rho_{0,+}+\theta_{0,+})I]\vec{n}|_{\Gamma+0} - [S_-(\vec{v}_{0,-}) - P_-(\rho_{0,-}+\theta_{0,-})I]\vec{n}|_{\Gamma-0} = -\pi_0\vec{n}|_\Gamma,\\ &\vec{v}_{0,+}|_{\Gamma+0} = \vec{v}_{0,-}|_{\Gamma-0},\ \ \vec{v}_{0,+}|_{\Gamma_+} = 0,\\ &[S_-(\vec{v}_{0,-}) - P_-(\rho_{0,-}+\theta_{0,-})I]\vec{n}_-|_{\Gamma_-} = -P_-(\rho_{0,-})\vec{n}_-\end{aligned} \tag{1.10}$$

compatibility

*and the range condition: $\|\theta_{0,\pm}\|_{L_\infty(\Omega_\pm)} \le \rho_{0,\pm}/2$, so that*

$$\frac{1}{2}\rho_{0,\pm} \le \rho_{0,\pm} + \theta_{0,\pm} \le \frac{3}{2}\rho_{0,\pm}. \tag{1.11}$$

range

*Then there exists a $T > 0$ depending on $R$ such that problem (P) admits a unique solution $(\theta_\pm, \vec{u}_\pm)$ with*

$$\theta_\pm \in W_p^1((0,T), W_q^1(\Omega_\pm)), \qquad \vec{u}_\pm \in W_p^1((0,T), L_q(\Omega_\pm)^N) \cap L_p((0,T), W_q^2(\Omega_\pm)^N)$$

*satisfying (1.4) and the estimate:*

$$\|\theta_\pm\|_{W_p^1((0,T),W_q^1(\Omega_\pm))} + \|\vec{u}_\pm\|_{L_p((0,T),W_q^2(\Omega_\pm))} + \|\partial_t\vec{u}_\pm\|_{L_p((0,T),L_q(\Omega_\pm))} \le C_R$$

*with some constant $C_R$ depending on $R, \rho_{0,\pm}$, $\rho_{1,\pm}$, $\mu_1^\pm$, $\mu_2^\pm$, $p$ and $q$.*

Using the argument due to Ströhmer [13], we can show the injectivity of the map $x = X_{\vec{u}_\pm}(\xi, t)$, so that we have the following the local well-posedness theorem for (1.1).

**Theorem 1.3.** *Let $2 < p < \infty$, $N < q < \infty$ and $R > 0$. Assume that $\Omega_\pm$ are uniform $W^{2-1/q}_q$ domains. Let $\theta_{0,\pm} \in W^1_q(\Omega_\pm)$ and $\vec{v}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^N$ be initial data for (1.1), which satisfy compatibility condition (1.10), range condition (1.11) and $\|\theta_{0,\pm}\|_{W^1_q(\Omega)} + \|\vec{v}_{0,\pm}\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \le R$. Then there exists a $T > 0$ depending on $R$ such that (1.1) with kinematic condition (1.2) admits a unique solution $(\rho_\pm, \vec{v}_\pm)$ with*

$$
\begin{aligned}
&\rho_\pm \in W^1_p((0,T), L_q(\Omega_{\pm,t})) \cap L_p((0,T), W^1_q(\Omega_{\pm,t})),\\
&\vec{v}_\pm \in W^1_p((0,T), L_q(\Omega_{\pm,t})^N) \cap L_p((0,T), W^2_q(\Omega_{\pm,t})^N)
\end{aligned}
$$

Finally we introduce more symbols and functional spaces used throughout this paper. $\mathbb{N}$ and $\mathbb{C}$ denote the sets of all natural numbers and complex numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $1 < q < \infty$, let $q' = q/(q-1)$. For any multi-index $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial^\kappa_x = \partial^{\kappa_1}_1 \ldots \partial^{\kappa_N}_N$ with $x = (x_1, \ldots, x_N)$ and $\partial_j = \partial/\partial x_j$. For the differentations of a scalar function $f$ and $N$-vector $\vec{g} = (g_1, \ldots, g_N)$, we use the following symbols:

$$
\begin{aligned}
&\nabla f = (\partial_1 f, \ldots, \partial_N f), && \nabla^2 f = (\partial^\kappa_x f \mid |\kappa| = 2),\\
&\nabla \vec{g} = (\partial_i g_j \mid i, j = 1, \ldots, N), && \nabla^2 \vec{g} = (\partial_i \partial_j g_k \mid i, j, k = 1, \ldots, N).
\end{aligned}
$$

We set $W^{m,\ell}_q(\Omega) = \{(f, \vec{g}) \mid f \in W^m_q(\Omega), \vec{g} \in W^\ell_q(\Omega)^N\}$. For Banach space $X$, we set

$$
\begin{aligned}
W^m_{p,\gamma}(\mathbb{R}, X) &= \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R}, X) \mid e^{-\gamma t}\partial^j_t f(t) \in L_p(\mathbb{R}, X),\ (j = 0, 1, \ldots, m)\},\\
W^m_{p,\gamma,0}(\mathbb{R}, X) &= \{f(t) \in W^m_{p,\gamma} \mid f(t) = 0\ (t < 0)\}
\end{aligned}
$$

with $\partial^0_t f(t) = f(t)$ and set $L_{p,\gamma}(\mathbb{R}, X) = W^0_{p,\gamma}(\mathbb{R}, X)$, $L_{p,\gamma,0}(\mathbb{R}, X) = W^0_{p,\gamma,0}(\mathbb{R}, X)$.

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all $X$-valued $C^\infty$-functions having compact support and the Schwartz space of rapidly decreasing $X$-valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$
T_M \varphi = \mathcal{F}^{-1}[M\mathcal{F}[\varphi]] \qquad (\mathcal{F}[\varphi] \in \mathcal{D}(\mathbb{R}, X)). \tag{1.12}
$$

Here $\mathcal{F}_x$ and $\mathcal{F}^{-1}_\xi$ denote the Fourier transform and its inversion defined by

$$
\mathcal{F}_x[u](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x)\,dx, \qquad \mathcal{F}^{-1}_\xi[v](\xi) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} v(\xi)\,d\xi,
$$

respectively. Let $\mathcal{F}_{x'}$ and $\mathcal{F}^{-1}_{\xi'}$ denote the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{N-1})$ and its inversion defined by

$$
\begin{aligned}
&\mathcal{F}_{x'}[u](\xi', x_N) = \widehat{u}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} u(x', x_N)\,dx',\\
&\mathcal{F}^{-1}_{\xi'}[v(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}}\int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} v(\xi', x_N)\,d\xi',
\end{aligned} \tag{1.13}
$$

respectively. Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and its inversion, which are defined by

$$
\begin{aligned}
\mathcal{L}[f](\lambda) &= \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\,dt = \mathcal{F}_t[e^{-\gamma t} f(t)](\tau),\\
\mathcal{L}^{-1}[g](t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\tau)\,d\tau = e^{\gamma t}\mathcal{F}^{-1}_\tau[g(\tau)](t)
\end{aligned}
$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$, respectively. Given $s \in \mathbb{R}$ and $X$-valued function $f(t)$, we set

$$
\Lambda^s_\gamma f(t) = \mathcal{L}^{-1}[\lambda^s \mathcal{L}[f](\lambda)](t).
$$

The Bessel potential space of $X$-valued functions of order $s>0$ are defined by

$$
\begin{aligned}
H^s_{p,\gamma}(\mathbb{R},X) &= \{f\in L_p(\mathbb{R},X)\mid e^{-\gamma' t}\Lambda^s_{\gamma'}f(t)\in L_p(\mathbb{R},X) \text{ for any } \gamma'\geq\gamma\},\\
H^s_{p,\gamma,0}(\mathbb{R},X) &= \{f\in H^s_{p,\gamma}(\mathbb{R},X)\mid f(t)=0\ (t<0)\}.
\end{aligned}
$$

For any two Banach spaces $X$ and $Y$, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from $X$ to $Y$. $\mathrm{Hol}(U,X)$ denotes the set of all $X$-valued holomorphic functions defined on $U$. The letter $C$ denotes generic constants and the constant $C_{a,b,\ldots}$ depends on $a$, $b$, $\ldots$. The values of constants $C$ and $C_{a,b,\ldots}$ may change from line to line.

Following the argument due to Enomoto, Below and Shibata [5], we can prove Theorem 1.2 by contraction mapping principle with the help of the maximal $L_p$-$L_q$ results, Theorems 2.1 and 2.2 (see [5] for detail). Thus, this paper consists of the following four sections. In Sect. 2, we present the maximal $L_p$-$L_q$ regularity theorem (Theorems 2.1 and 2.2) and the theorem concerning the existence of $\mathcal{R}$-bounded solution operator for linearized problem (Theorem 2.6). As was seen in Enomoto, Below and Shibata [5], the maximal $L_p$-$L_q$ regularity theorem is direct consequence of Theorem 2.6 concerning the generalized resolvent problem for the linearized equations with the help of Weis' operator valued Fourier multiplier theorem, so that the main part of this paper is to show Theorem 2.6. In Sect 3, we consider the generalized resolvent problem for the linearized problem in the half-space and we show the existence of its $\mathcal{R}$-bounded solution operator. In Sect 4, following the argumentation due to Enomoto, Below and Shibata [5], we show Theorem 2.6.

## 2 Main theorem for linear problem

In order to prove Theorem 1.2, we use the contraction mapping principle based on the maximal $L_p$-$L_q$ regularity for solutions to the linearized problem

$$
\begin{cases}
\partial_t\theta_\pm+\gamma_1^\pm\mathrm{div}\,\vec{u}_\pm=f_\pm & \text{in } \Omega_\pm\times(0,T),\\
\gamma_0^\pm\partial_t\vec{u}_\pm-\mathrm{Div}\,S_\pm(\vec{u}_\pm)+\nabla(\gamma_2^\pm\theta_\pm)=\vec{g}_\pm & \text{in } \Omega_\pm\times(0,T),\\
(S_+(\vec{u}_+)-\gamma_2^+\theta_+I)\vec{n}|_{\Gamma+0}-(S_-(\vec{u}_-)-\gamma_2^-\theta_-I)\vec{n}|_{\Gamma-0}=\vec{h}, & \\
\vec{u}_+|_{\Gamma+0}=\vec{u}_-|_{\Gamma-0}, & \\
(S_-(\vec{u}_-)-\gamma_2^-\theta_-I)\vec{n}_-|_{\Gamma_-}=\vec{h}_-, & \\
\vec{u}_+|_{\Gamma_+}=0, &
\end{cases}
\tag{LP}
$$

subject to the initial condition $(\theta_\pm,\vec{u}_\pm)|_{t=0}=(\theta_{0,\pm},\vec{u}_{0,\pm})$. Here $\gamma_i^\pm=\gamma_i^\pm(x)$ $(i=0,1,2)$ are uniformly continuous functions defined on $\overline{\Omega_\pm}$ such that

$$
\frac{1}{2}\rho_{0,\pm}\leq\gamma_0^\pm(x)\leq 2\rho_{0,\pm},\qquad 0\leq\gamma_k^\pm(x)\leq\rho_{2,\pm}\ (x\in\overline{\Omega_\pm}),\qquad \|\nabla\gamma_\ell^\pm\|_{L_r(\Omega_\pm)}\leq\rho_{2,\pm} \tag{2.1}
$$

for $k=1,2$ and $\ell=0,1,2$ with some positive constants $\rho_{2,\pm}$ and $N<r<\infty$.

The following two theorems are maximal $L_p$-$L_q$ regularity theorem for linear problem (LP). First theorem is the maximal $L_p$-$L_q$ regularity theorem for (LP) with $f_\pm=0$, $\vec{g}_\pm=\vec{h}=\vec{h}_-=\vec{0}$.

**Theorem 2.1.** *Let $1<p,q<\infty$ and $N<r<\infty$. Assume that $r\geq\max(q,q')$ and that $\Omega_\pm$ are uniformly $W_r^{2-1/r}$ domains. Then there exists a positive number $\lambda_1$ such that the following assertion is valid: for any initial data $\theta_{0,\pm}\in W^1_q(\Omega_\pm)$ and $\vec{u}_{0,\pm}\in B^{2(1-1/p)}_{q,p}(\Omega_\pm)$ satisfying the compatibility conditions:*

$$
\begin{aligned}
&(S_+(\vec{u}_{0,+})-\gamma_2^+\theta_{0,+}I)\vec{n}|_{\Gamma+0}-(S_-(\vec{u}_{0,-})-\gamma_2^-\theta_{0,-}I)\vec{n}|_{\Gamma-0}=0,\\
&(S_-(\vec{u}_{0,-})-\gamma_2^-\theta_{0,-}I)\vec{n}_-|_{\Gamma_-}=0,\ \ \vec{u}_{0,+}|_{\Gamma+0}=\vec{u}_{0,-}|_{\Gamma-0},\ \ \vec{u}_{0,+}|_{\Gamma_+}=0,
\end{aligned}
$$

*problem (LP) admits a unique solution $(\theta_\pm,\vec{u}_\pm)$ with*

$$
\begin{aligned}
&\theta_\pm\in W^1_{p,\lambda_1}((0,\infty),W^1_q(\Omega_\pm)),\\
&\vec{u}_\pm\in L_{p,\lambda_1}((0,\infty),W^2_q(\Omega_\pm)^N)\cap W^1_{p,\lambda_1}((0,\infty),L_q(\Omega_\pm)^N)
\end{aligned}
$$

*possesing the estimate:*

$$\begin{aligned}
&\sum_{\ell=+,-} \|e^{-\gamma t}(\partial_t \theta_\ell, \gamma \theta_\ell)\|_{L_p((0,\infty),W^1_q(\Omega_\ell))} \\
&+ \sum_{\ell=+,-} \Big( \|e^{-\gamma t}(\partial_t \vec{u}_\ell, \gamma \vec{u}_\ell)\|_{L_p((0,\infty),L_q(\Omega_\ell))} + \|e^{-\gamma t} \vec{u}_\ell\|_{L_p((0,\infty),W^2_q(\Omega_\ell))} \Big) \\
&\le C \sum_{\ell=+,-} \Big( \|\theta_{0,\ell}\|_{W^1_q(\Omega_\ell)} + \|\vec{u}_{0,\ell}\|_{B^{2(1-1/p)}_{q,p}(\Omega_\ell)} \Big)
\end{aligned}$$

*for $\gamma \ge \lambda_1$ with some constant $C$. Here $\lambda_1$ and $C$ depend on $\mu_1^\pm$, $\mu_2^\pm$, $q$, $r$, $\Omega_\pm$, $N$, $\rho_{0,\pm}$ and $\rho_{2,\pm}$.*

Second theorem is the maximal $L_p$-$L_q$ regularity theorem for (LP) with $\theta_{0,\pm} = 0$ and $\vec{u}_{0,\pm} = \vec{0}$.

th_MR2 **Theorem 2.2.** *Let $1 < p, q < \infty$ and $N < r < \infty$. Assume that $r \ge \max(q, q')$ and that $\Omega_\pm$ are uniformly $W^{2-1/r}_r$ domains. Then there exists a positive number $\lambda_2$ such that the following assertion is valid: for $(f_\pm, \vec{g}_\pm) \in L_{p,\lambda_2,0}(\mathbb{R}, W^{1,0}_q(\Omega))$ and $\vec{h}, \vec{h_-} \in L_{p,\lambda_2,0}(\mathbb{R}, W^1_q(\Omega)^N) \cap H^{1/2}_{p,\lambda_2,0}(\mathbb{R}, L_q(\Omega)^N)$, problem (LP) admits a unique solution $(\theta_\pm, \vec{u}_\pm)$ with*

$$\theta_\pm \in W^1_{p,\lambda_2}(\mathbb{R}, W^1_q(\Omega_\pm)), \qquad \vec{u}_\pm \in L_{p,\lambda_2}(\mathbb{R}, W^2_q(\Omega_\pm)^N) \cap W^1_{p,\lambda_2}(\mathbb{R}, L_q(\Omega_\pm)^N)$$

*possesing the estimate:*

$$\begin{aligned}
&\sum_{\ell=+,-} \|e^{-\gamma t}(\partial_t \theta_\ell, \gamma \theta_\ell)\|_{L_p(\mathbb{R},W^1_q(\Omega_\ell))} + \|e^{-\gamma t}(\partial_t \vec{u}_\ell, \gamma \vec{u}_\ell, \Lambda^{1/2}_\gamma \nabla \vec{u}_\ell, \nabla^2 \vec{u}_\ell)\|_{L_p(\mathbb{R},L_q(\Omega_\ell))} \\
&\le C \sum_{\ell=+,-} \|e^{-\gamma t}(f_\ell, \vec{g}_\ell)\|_{L_p(\mathbb{R},W^{1,0}_q(\Omega_\ell))} \\
&\quad + C \sum_{\ell=+,-} \|e^{-\gamma t}(\Lambda^{1/2}_\gamma \vec{h}, \nabla \vec{h}, \Lambda^{1/2}_\gamma \vec{h_-}, \nabla \vec{h_-})\|_{L_p(\mathbb{R},L_q(\Omega_\ell))}
\end{aligned}$$

*for $\gamma \ge \lambda_2$ with some constant $C$. Here $\lambda_2$ and $C$ depend on $\mu_1^\pm$, $\mu_2^\pm$, $q$, $r$, $\Omega_\pm$, $N$, $\rho_{0,\pm}$ and $\rho_{2,\pm}$.*

In order to prove our main results for (LP) (Theorem 2.1 and Theorem 2.2), we introduce the definition of $\mathcal{R}$-bounded operator family and operator valued Fourier multiplier theorem due to Weis [16]. The definition of $\mathcal{R}$-boundedness which is the key word in our method is the following.

def_R **Definition 2.3.** Let $X$ and $Y$ be Banach spaces. A family of operator $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{x_j\}_{j=1}^n \subset X$ and sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$ there holds the inequality:

$$\left\{ \int_0^1 \| \sum_{j=1}^n r_j(u) T_j x_j \|_Y^p \, du \right\}^{1/p} \le C \left\{ \int_0^1 \| \sum_{j=1}^n r_j(u) x_j \|_X^p \, du \right\}^{1/p}.$$

The smallest such $C$ is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

The following theorem is given by Weis [16]

th_Weis **Theorem 2.4.** *Let $X$ and $Y$ be two UMD spaces and $1 < p < \infty$. Let $M$ be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)} \left( \left\{ \left( \tau \frac{d}{d\tau} \right)^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\} \right\} \right) \le \kappa < \infty \ (\ell = 0, 1)$$

*with some constant $\kappa$. Then the operator $T_M$ defined by (1.12) may uniquely be extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover denoting this extension by $T_M$, we have*

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \le C\kappa$$

*for some positive constant $C$ depending on $p$, $X$ and $Y$.*

The following lemma concerning the $\mathcal{R}$-boundedness of the summation and composition of operator is known (see Denk, Hieber and Prüß[4]).

**Lemma 2.5.** (1) *Let $X$ and $Y$ be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$. Then $\mathcal{T}+\mathcal{S}=\{T+S \mid T\in\mathcal{T}, S\in\mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X,Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}+\mathcal{S})\le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})+\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let $X$, $Y$ and $Z$ be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then $\mathcal{S}\mathcal{T}=\{ST \mid T\in\mathcal{T}, S\in\mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X,Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T})\le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

In order to prove the maximal $L_p$-$L_q$ regularity theorem with the help of Theorem 2.4, we need the $\mathcal{R}$-boundedness for solution operator to the following generalized resolvent problem:

$$\begin{cases} \lambda\theta_\pm+\gamma_1^\pm \operatorname{div}\vec{u}_\pm=f_\pm & \text{in } \Omega_\pm,\\ \gamma_0^\pm\lambda\vec{u}_\pm-\operatorname{Div}S_\pm(\vec{u}_\pm)+\nabla(\gamma_2^\pm\theta_\pm)=\vec{g}_\pm & \text{in } \Omega_\pm,\\ (S_+(\vec{u}_+)-\gamma_2^+\theta_+I)\vec{n}|_{\Gamma+0}-(S_-(\vec{u}_-)-\gamma_2^-\theta_-I)\vec{n}|_{\Gamma-0}=\vec{h}, & \\ \vec{u}_+|_{\Gamma+0}-\vec{u}_-|_{\Gamma-0}=\vec{k}, & \\ (S_-(\vec{u}_-)-\gamma_2^-\theta_-I)\vec{n}_-|_{\Gamma_-}=\vec{h}_-, & \\ \vec{u}_+|_{\Gamma_+}=0, & \end{cases} \tag{RP}$$

The resolvent parameter $\lambda$ varies in $\Lambda_{\varepsilon,\lambda_0}=\Sigma_{\varepsilon,\lambda_0}\cap K_\varepsilon$,where

$$\begin{aligned} &\Sigma_\varepsilon=\{\lambda\in\mathbb{C}\backslash\{0\}\mid |\arg\lambda|\le\pi-\varepsilon\}, \qquad \Sigma_{\varepsilon,\lambda_0}=\{\lambda\in\Sigma_\varepsilon \mid |\lambda|\ge\lambda_0\},\\ &K_\varepsilon=\left\{\lambda\in\mathbb{C}\mid (\mathrm{Re}\lambda+\gamma_m+\varepsilon)^2+(\mathrm{Im}\lambda)^2\ge(\gamma_m+\varepsilon)^2\right\} \end{aligned} \tag{2.2}$$

with $\gamma_m=\max\left(\sup_{x\in\Omega_+}\frac{\gamma_1^+\gamma_2^+}{\mu_1^++\mu_2^+},\ \sup_{x\in\Omega_-}\frac{\gamma_1^-\gamma_2^-}{\mu_1^-+\mu_2^-}\right)$.

We can show the existence of the $\mathcal{R}$-bounded solution operator to (RP) as follows:

**Theorem 2.6.** *Let $1<q<\infty$, $0<\varepsilon<\pi/2$ and $N<r<\infty$. Assume that $r>\max(q,q')$. Let $\Omega_\pm$ be are uniform $W_r^{2-1/r}$ domains. Let $\Lambda_{\varepsilon,\lambda_0}$ be the set defined in (2.2). Set*

$$\begin{aligned} X_q=\{&(f_+,f_-,\vec{g}_+,\vec{g}_-,\vec{h},\vec{h}_-,\vec{k})\mid\\ &f_\pm\in W_q^1(\Omega_\pm),\ \ \vec{g}_\pm\in L_q(\Omega_\pm)^N,\ \ \vec{h},\vec{h}_-\in W_q^1(\Omega)^N,\ \ \vec{k}\in W_q^2(\Omega)^N\},\\ \mathcal{X}_q=\{&(F_{0+},F_{0-},F_{1+},F_{1-},F_2,F_3,F_{2-},F_{3-},F_4,F_5,F_6)\mid\\ &F_{0\pm}\in W_q^1(\Omega_\pm),F_{1\pm}\in L_q(\Omega_\pm)^N,\\ &F_2,F_{2-},F_5\in L_q(\Omega)^{N^2},\ \ F_3,F_{3-},F_6\in L_q(\Omega)^N,\ \ F_4\in L_q(\Omega)^{N^3}\}. \end{aligned}$$

*Then, there exist operator families*

$$\mathcal{P}_\pm(\lambda)\in\mathrm{Hol}(\Lambda_{\varepsilon,\lambda_0},\mathcal{L}(\mathcal{X}_q,W_q^1(\Omega_\pm))),\quad \vec{\mathcal{U}}_\pm(\lambda)\in\mathrm{Hol}(\Lambda_{\varepsilon,\lambda_0},\mathcal{L}(\mathcal{X}_q,W_q^2(\Omega_\pm)^N))$$

*such that for any $(f_+,f_-,\vec{g}_+,\vec{g}_-,\vec{h},\vec{h}_-,\vec{k})\in X_q$ and $\lambda\in\Lambda_{\varepsilon,\lambda_0}$,*

$$\rho_\pm=\mathcal{P}_\pm(\lambda)(f_+,f_-,\vec{g}_+,\vec{g}_-,\nabla\vec{h},\lambda^{1/2}\vec{h},\nabla\vec{h}_-,\lambda^{1/2}\vec{h}_-,\nabla^2\vec{k},\lambda^{1/2}\nabla\vec{k},\lambda\vec{k}), \tag{2.3}$$

$$\vec{u}_\pm=\vec{\mathcal{U}}_\pm(\lambda)(f_+,f_-,\vec{g}_+,\vec{g}_-,\nabla\vec{h},\lambda^{1/2}\vec{h},\nabla\vec{h}_-,\lambda^{1/2}\vec{h}_-,\nabla^2\vec{k},\lambda^{1/2}\nabla\vec{k},\lambda\vec{k}) \tag{2.4}$$

*solve problem (RP) uniquely. Moreover, there exists a constant $C$ depending on $\varepsilon$, $\lambda_0$, $q$ and $N$ such that*

$$\begin{aligned} &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q,W_q^1(\mathbb{R}_+^N))}(\{(\tau\partial_\tau)^\ell\{(\lambda,\gamma)\mathcal{P}_\pm(\lambda)\}\mid\lambda\in\Lambda_{\varepsilon,\lambda_0}\})\le C\quad(\ell=0,1),\\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q,L_q(\mathbb{R}_\pm^N)^{N^3+N^2+2N})}(\{(\tau\partial_\tau)^\ell(G_\lambda\vec{\mathcal{U}}_\pm(\lambda))\mid\lambda\in\Lambda_{\varepsilon,\lambda_0}\})\le C\quad(\ell=0,1), \end{aligned}$$

*where $G_\lambda u=(\lambda u,\gamma u,\lambda^{1/2}\nabla u,\nabla^2u)$ and $\lambda=\gamma+i\tau$.*

Let $\mathcal{B}$ be the linear operator defined by

$$
\begin{aligned}
&\mathcal{B}(\theta_+,\theta_-,\vec{u}_+,\vec{u}_-)\\
&=(-\gamma_1^+\operatorname{div}\vec{u}_+,-\gamma_1^-\operatorname{div}\vec{u}_-,\\
&\qquad(\gamma_0^+)^{-1}\mathrm{Div}\,S_+(\vec{u}_+)-(\gamma_0^+)^{-1}\nabla(\gamma_2^+\theta_+),(\gamma_0^-)^{-1}\mathrm{Div}\,S_-(\vec{u}_-)-(\gamma_0^-)^{-1}\nabla(\gamma_2^-\theta_-))
\end{aligned}
$$

for $(\theta_+,\theta_-,\vec{u}_+,\vec{u}_-)\in\mathcal{D}_q(\mathcal{B})$, where

$$
\begin{aligned}
\mathcal{D}_q(\mathcal{B})=\{(\theta_+,\theta_-,\vec{u}_+,\vec{u}_-)\mid&\\
&\theta_\pm\in W^1_q(\Omega_\pm),\vec{u}_\pm\in W^2_q(\Omega_\pm)^N,\\
&(S_+(\vec{u}_+)-\gamma_2^+\theta_+I)\vec{n}|_{\Gamma+0}-(S_-(\vec{u}_-)-\gamma_2^-\theta_-I)\vec{n}|_{\Gamma-0}=0,\\
&(S_-(\vec{u}_-)-\gamma_2^-\theta_-I)\vec{n}_-|_{\Gamma_-}=0,\ \vec{u}_+|_{\Gamma+0}=\vec{u}_-|_{\Gamma-0},\ \vec{u}_+|_{\Gamma_+}=0\}.
\end{aligned}
$$

Since Definition 2.3 with $n=1$ implies that the boundedness of the operator family $\mathcal{T}$, it follows from Theorem 2.6 that $\Lambda_{\varepsilon,\lambda_0}$ is contained in the resolvent set of $\mathcal{B}$ and for any $\lambda\in\Lambda_{\varepsilon,\lambda_0}$ and $(f_+,f_-,\vec{g}_+,\vec{g}_-,\vec{h},\vec{h}_-,\vec{k})\in\mathcal{X}_q$, $(\rho_\pm,\vec{u}_\pm)$ given by (2.3) and (2.4) satisfies the estimate:

$$
\begin{aligned}
&\sum_{\ell=+,-}\Big(|\lambda|\|\rho_\ell\|_{W^1_q(\Omega_\ell)}+\|(\lambda\vec{u}_\ell,\lambda^{1/2}\nabla\vec{u}_\ell,\nabla^2\vec{u}_\ell)\|_{L_q(\Omega_\ell)}\Big)\\
&\le C\sum_{\ell=+,-}\Big(\|(f_\ell,\vec{g}_\ell)\|_{W^{1,0}_q(\Omega_\ell)}+\|(\lambda^{1/2}\vec{h},\nabla\vec{h},\lambda^{1/2}\vec{h}_-,\nabla\vec{h}_-,\nabla^2\vec{k},\lambda^{1/2}\nabla\vec{k},\lambda\vec{k})\|_{L_q(\Omega)}\Big)
\end{aligned}
\tag{2.5}
$$

resolvent_est

for $1<q<\infty$. By (2.5) with $\vec{h}=\vec{h}_-=\vec{0}$, we have the following theorem:

th_semigroup

**Theorem 2.7.** *Let $1<q<\infty$ and $N<r<\infty$. Assume that $r\ge\max(q,q')$. Let $\Omega_\pm$ be uniform $W^{2-1/r}_r$ domains. Then the operator $\mathcal{B}$ generates an analytic semigroup $\{T(t)\}_{t\ge0}$ on $W^{1,0}_{\pm,q}(\Omega)$, where $W^{1,0}_{\pm,q}(\Omega)=\{(\theta_+,\theta_-,\vec{u}_+,\vec{u}_-)\mid(\theta_\pm,\vec{u}_\pm)\in W^{1,0}_q(\Omega)\}$. Moreover there exists constant $\lambda_3>0$ and $M>0$ such that for any $\mathbf{F}=(f_+,f_-,\vec{g}_+,\vec{g}_-)\in W^{1,0}_{\pm,q}(\Omega)$, $(\rho_+(t),\rho_-(t),\vec{u}_+(t),\vec{u}_-(t))=T(t)\mathbf{F}$ satisfies the following estimate:*

$$
\begin{aligned}
&\sum_{\ell=+,-}\Big(\|\rho_\ell\|_{W^1_q(\Omega_\ell)}+\|\vec{u}_\ell\|_{L_q(\Omega_\ell)}+t^{1/2}\|\nabla\vec{u}_\ell\|_{L_q(\Omega_\ell)}+t\|\nabla^2\vec{u}_\ell\|_{L_q(\Omega_\ell)}\Big)\\
&\le Me^{\lambda_3 t}\|F\|_{W^{1,0}_{\pm,q}(\Omega)}
\end{aligned}
$$

*for $t>0$. Here $\lambda_3$ and $M$ depend solely on $\mu_1^\pm$, $\mu_2^\pm$, $q$, $r$, $\Omega_\pm$, $N$, $\lambda_0$, $\rho_{0,\pm}$ and $\rho_{2,\pm}$.*

Following the argument due to Enomoto, Below and Shibata [5], we can obtain Theorem 2.1 and Theorem 2.2 as direct consequence of Theorem 2.6 with the help of Theorem 2.4. Moreover we can prove Theorem 1.2 by contraction mapping principle with Theorem 2.1 and 2.2 (see [5] for detail). Thus we omit the proof of Theorem 1.2, Theorem 2.1 and Theorem 2.2 and show the proof of only Theorem 2.6 in this paper.

In order to show Theorem 2.6, substituting the relation $\theta_\pm=\lambda^{-1}(f_\pm-\gamma_1^\pm\operatorname{div}\vec{u}_\pm)$ given by the first equation in (RP) to the second, third and fifth equations in (RP), we have

$$
\begin{cases}
\gamma_0^\pm\lambda\vec{u}_\pm-\mathrm{Div}\,S_\pm(\vec{u}_\pm)-\lambda^{-1}\nabla\left(\gamma_1^\pm\gamma_2^\pm\operatorname{div}\vec{u}_\pm\right)=\vec{g'}_\pm & \text{in }\Omega_\pm,\\
(S_+(\vec{u}_+)+\gamma_1^+\gamma_2^+\lambda^{-1}(\operatorname{div}\vec{u}_+)I)\vec{n}\,|_{\Gamma+0}\\
\qquad-(S_-(\vec{u}_-)+\gamma_1^-\gamma_2^-\lambda^{-1}(\operatorname{div}\vec{u}_-)I)\vec{n}\,|_{\Gamma-0}=\vec{h'},\\
\vec{u}_+|_{\Gamma+0}-\vec{u}_-|_{\Gamma-0}=0,\\
(S_-(\vec{u}_-)+\gamma_1^-\gamma_2^-\lambda^{-1}(\operatorname{div}\vec{u}_-)I)\vec{n}_-|_{\Gamma_-}=\vec{h'}_-,\\
\vec{u}_+|_{\Gamma_+}=\vec{0},
\end{cases}
\tag{2.6}
$$

2.8

where $\vec{g'}_\pm=\vec{g}_\pm-\lambda^{-1}\nabla f_\pm$, $\vec{h'}=\vec{h}+\gamma_1^+\gamma_2^+\lambda^{-1}(f_+I)\vec{n}|_{\Gamma+0}-\gamma_1^-\gamma_2^-\lambda^{-1}(f_-I)\vec{n}|_{\Gamma-0}$ and $\vec{h'}_-=\vec{h}_-+\gamma_1^-\gamma_2^-\lambda^{-1}(f_-I)\vec{n}_-|_{\Gamma_-}$. We shall solve (2.6) by constructing of parametrix in the similar way to Enomoto, Below and Shibata [5]. To this end, we need the $\mathcal{R}$-bounded solution operator of the following four problems:

$$\begin{cases} \gamma_0^{\pm}\lambda\vec{u}_{\pm} - \mathrm{Div}\, S_{\pm}(\vec{u}_{\pm}) - \lambda^{-1}\nabla(\gamma_1^{\pm}\gamma_2^{\pm}\mathrm{div}\,\vec{u}_{\pm}) = \vec{g}_{\pm} \quad \text{in } \Omega_{\pm}, \\ (S_+(\vec{u}_+) + \gamma_1^+\gamma_2^+\lambda^{-1}(\mathrm{div}\,\vec{u}_+)I)\vec{n}\,|_{\Gamma+0} \\ -(S_-(\vec{u}_-) + \gamma_1^-\gamma_2^-\lambda^{-1}(\mathrm{div}\,\vec{u}_-)I)\vec{n}\,|_{\Gamma-0} = \vec{h}, \\ \vec{u}_+|_{\Gamma+0} - \vec{u}_-|_{\Gamma-0} = 0, \end{cases} \tag{RP1}$$

$$\begin{cases} \gamma_0^{\pm}\lambda\vec{u}_{+} - \mathrm{Div}\, S_{\pm}(\vec{u}_{+}) - \lambda^{-1}\nabla(\gamma_1^{+}\gamma_2^{+}\mathrm{div}\,\vec{u}_{+}) = \vec{g}_{+} \quad \text{in } \Omega_{+}, \\ \vec{u}_+\,|_{\Gamma+} = 0, \end{cases} \tag{RP2}$$

$$\begin{cases} \gamma_0^{\pm}\lambda\vec{u}_{-} - \mathrm{Div}\, S_{-}(\vec{u}_{-}) - \lambda^{-1}\nabla(\gamma_1^{-}\gamma_2^{-}\mathrm{div}\,\vec{u}_{-}) = \vec{g}_{-} \quad \text{in } \Omega_{-}, \\ (S_-(\vec{u}_-) + \gamma_1^-\gamma_2^-\lambda^{-1}(\mathrm{div}\,\vec{u}_-)I)\vec{n}_-\,|_{\Gamma-} = \vec{h}_-, \end{cases} \tag{RP3}$$

and

$$\gamma_0^{\pm}\lambda\vec{u}_{-} - \mathrm{Div}\, S_{-}(\vec{u}_{-}) - \lambda^{-1}\nabla(\gamma_1^{\pm}\gamma_2^{\pm}\mathrm{div}\,\vec{u}_{-}) = \vec{g}_{\pm} \quad \text{in } \mathbb{R}^N. \tag{RP4}$$

Since the existence of $\mathcal{R}$-bounded solution operators to (RP2)-(RP4) is given in [5], it is sufficient to prove the existence of the $\mathcal{R}$-bounded solution operator of (RP1). For this purpose, we shall first prove the existence of $\mathcal{R}$-bounded solution operator to the following generalized resolvent problem in half-spaces:

$$\begin{cases} \tilde{\gamma}_0^{\pm}\lambda\vec{u}_{\pm} - \mathrm{Div}\ \ S_{\pm}(\vec{u}_{\pm}) + \delta\nabla(\mathrm{div}\,\vec{u}_{\pm}) = \vec{g}_{\pm} & \text{in } \mathbb{R}^N_{\pm}, \\ \vec{u}_+|_{x_N=0+} - \vec{u}_-|_{x_N=0-} = \vec{k} & \text{on } \mathbb{R}^N_0, \\ (S_+(\vec{u}_+) - \delta(\mathrm{div}\,\vec{u}_+)I)\vec{n}\,|_{x_N=0+} - (S_-(\vec{u}_-) - \delta(\mathrm{div}\,\vec{u}_-)I)\vec{n}\,|_{x_N=0-} = \vec{h} & \text{on } \mathbb{R}^N_0, \end{cases} \tag{2.7}$$

where $\vec{n} = (0, \ldots, 0, -1)$ is the unit outer normal to $\mathbb{R}^n_-$ and $\tilde{\gamma}_i^{\pm} (i = 0, 1, 2)$ are constants satisfying $\tilde{\gamma}_0^{\pm} \geq \rho_{0,\pm}/2$, $\tilde{\gamma}_1^{\pm}$ and $\tilde{\gamma}_2^{\pm} \geq 0$. $\mathbb{R}^N_+$, $\mathbb{R}^N_-$ and $\mathbb{R}^N_0$ denote the upper half-space, the lower half-space and their boundary defined by $\mathbb{R}^N_{\pm} = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \pm x_N > 0\}$ and $\mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}$. Let $\delta$ and $\lambda$ satisfy one of the following conditions:

(C1) $\delta = \tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}/\lambda$ and $\lambda \in \Sigma_{\varepsilon,\lambda_0} \cap K_{\varepsilon}$,

(C2) $\delta \in \Sigma_{\varepsilon}$ with $\mathrm{Re}\,\delta < 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$ and $\mathrm{Re}\,\lambda \geq |\mathrm{Re}\,\delta/\mathrm{Im}\,\delta||\mathrm{Im}\,\lambda|$,

(C3) $\delta \in \Sigma_{\varepsilon}$ with $\mathrm{Re}\,\delta \geq 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$ and $\mathrm{Re}\,\lambda \geq \lambda_0|\mathrm{Im}\,\lambda|$

for $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. We denote $\Gamma_{\varepsilon,\lambda_0}$ by

$$\Gamma_{\varepsilon,\lambda_0} = \begin{cases} \Sigma_{\varepsilon,\lambda_0} \cap K_{\varepsilon} & \text{in case of (C1)}, \\ \{\lambda \in \mathbb{C} \mid |\lambda| \geq \lambda_0,\ \mathrm{Re}\,\lambda \geq |\mathrm{Re}\,\delta/\mathrm{Im}\,\delta||\mathrm{Im}\,\lambda|\} & \text{in case of (C2)}, \\ \{\lambda \in \mathbb{C} \mid |\lambda| \geq \lambda_0\ \mathrm{Re}\,\lambda \geq \lambda_0|\mathrm{Im}\,\lambda|\} & \text{in case of (C3)}. \end{cases} \tag{2.8}$$

The case (C1) is used to prove the existence of $\mathcal{R}$-bounded solution operator to (2.7) and the cases (C2) and (C3) are used for some homotopic argument in proving the exponential stability of analytic semigroup in a bounded domain ([6] and [11] in the nonslip condition cases for detail.) In case (C1), $|\delta| = \tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}/|\lambda| \leq \tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}\lambda_0^{-1}$ and in cases of (C2) and (C3), we assume that $|\delta| \leq \delta_0$ for some $\delta_0 > 0$. Therefore we see

$$|\delta| \leq \max(\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}\lambda_0^{-1}, \delta_0). \tag{2.9}$$

Then we obtain the following theorem on existence of $\mathcal{R}$-bounded solution operators to (2.7):

**Theorem 2.8.** *Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. Set*

$$\begin{aligned} Y_q &= \{(\vec{g}_+, \vec{g}_-, \vec{h}, \vec{k}) \mid \vec{g}_{\pm} \in L_q(\mathbb{R}^N_{\pm})^N,\ \ \vec{h} \in W^1_q(\mathbb{R}^N)^N,\ \ \vec{k} \in W^2_q(\mathbb{R}^N)^N\}, \\ \mathcal{Y}_q &= \{(F_{1+}, F_{1-}, F_2, F_3, F_4, F_5, F_6) \mid \\ &\quad F_{1\pm} \in L_q(\mathbb{R}^N_{\pm})^N,\ \ F_2, F_5 \in L_q(\mathbb{R}^N)^{N^2},\ \ F_3, F_6 \in L_q(\mathbb{R}^N)^N,\ \ F_4 \in L_q(\mathbb{R}^N)^{N^3}\}. \end{aligned}$$

*Then, there exist operator family* $\vec{\mathcal{U}}_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0})$ *such that*

$$\vec{u}_\pm = \vec{\mathcal{U}}_\pm(\lambda)(\vec{g}_+, \vec{g}_-, \nabla\vec{h}, \lambda^{1/2}\vec{h}, \nabla^2\vec{k}, \lambda^{1/2}\nabla\vec{k}, \lambda\vec{k})$$

*solve problem* (2.7) *uniquely for any* $(\vec{g}_+, \vec{g}_-, \vec{h}, \vec{k}) \in Y_q$ *and* $\lambda \in \Gamma_{\varepsilon,\lambda_0}$. *Moreover, there exists a constant* $C$ *depending on* $\varepsilon$, $\lambda_0$, $q$ *and* $N$ *such that*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q, L_q(\mathbb{R}^N_\pm)^{N^3+N^2+2N})}(\{(\tau\partial_\tau)^\ell (G_\lambda \vec{\mathcal{U}}_\pm(\lambda)) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le C \quad (\ell = 0, 1), \tag{2.10}$$

*where* $G_\lambda u = (\lambda u, \gamma u, \lambda^{1/2}\nabla u, \nabla^2 u)$ *and* $\lambda = \gamma + i\tau$.

Following the argument in [5], by Theorem 2.8 and a change of variables, we can prove Theorem 2.6. In section 4, we shall describe the proof of Theorem 2.6 for detail.

# 3 $\mathcal{R}$-bounded solution operators for model problem

## 3.1 model problem and its solution formula

In order to prove Theorem 2.8, we first reduce problem (2.7) to the following problem:

$$\begin{cases} \tilde\gamma_0^\pm \lambda \vec{v}_\pm - \mathrm{Div}\ S_\pm(\vec{v}_\pm) + \delta\nabla(\mathrm{div}\,\vec{v}_\pm) = \vec{0} & \text{in } \mathbb{R}^N_\pm, \\ \vec{v}_+|_{x_N=0+} - \vec{v}_-|_{x_N=0-} = \vec{k} & \text{on } \mathbb{R}^N_0, \\ (S_+(\vec{v}_+) - \delta(\mathrm{div}\,\vec{v}_+)I)\vec{n}\,|_{x_N=0+} - (S_-(\vec{v}_-) - \delta(\mathrm{div}\,\vec{v}_-)I)\vec{n}\,|_{x_N=0-} = \vec{h} & \text{on } \mathbb{R}^N_0, \end{cases} \tag{3.1}$$

Given function $f_\pm$ defined on $\mathbb{R}^N_\pm$, $f^e_\pm$ and $f^o_\pm$ denote their even extension and odd extension to $\mathbb{R}^N$, respectively, that is

$$f^e_+(x) = \begin{cases} f_+(x) & x_N > 0, \\ f_+(x', -x_N) & x_N < 0 \end{cases}, \qquad f^o_+(x) = \begin{cases} f_+(x) & x_N > 0, \\ -f_+(x', -x_N) & x_N < 0 \end{cases},$$

$$f^e_-(x) = \begin{cases} f_-(x', -x_N) & x_N > 0, \\ f_-(x) & x_N < 0 \end{cases}, \qquad f^o_-(x) = \begin{cases} -f_-(x', -x_N) & x_N > 0, \\ f_-(x) & x_N < 0. \end{cases}$$

Let $\vec{g}_\pm = (g_{\pm,1}, \dots, g_{\pm,N}) \in L_q(\mathbb{R}^N_\pm)^N$ and set $\vec{G}_\pm = (g^e_{\pm,1}, \dots, g^e_{\pm,N-1}, g^o_{\pm,N})$. Let $\mathcal{S}_0(\lambda)$ be the solution operator to $\tilde\gamma_0\lambda\vec{u} - \mathrm{Div}\,S(\vec{u}) - \delta\nabla(\mathrm{div}\,\vec{u}) = \vec{g}$ in $\mathbb{R}^N$ and let $\mathcal{S}^1_\pm(\lambda)$ be an operator defined by $\mathcal{S}^1_\pm(\lambda)\vec{g}_\pm = \mathcal{S}_0(\lambda)\vec{G}_\pm$. Götz and Shibata [7] showed that $\vec{U}_\pm = S^1_\pm(\lambda)\vec{g}_\pm$ satisfies the equation

$$\begin{cases} \tilde\gamma_0^\pm \lambda \vec{U}_\pm - \mathrm{Div}\,S(\vec{U}_\pm) - \delta\nabla(\mathrm{div}\,\vec{U}_\pm) = \vec{g}_\pm & \text{in } \mathbb{R}^N_\pm, \\ \partial_N U_{N,\pm} = 0 & \text{on } \mathbb{R}^N_0 \end{cases}$$

with $\vec{U}_\pm = (U_{\pm,1}, \dots, U_{\pm,N})$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell G_\lambda S^1_\pm(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le C \ (\ell = 0, 1) \tag{3.2}$$

with $\lambda = \gamma + i\tau$ and some constant $C$, where $G_\lambda u = (\lambda u, \gamma u, \lambda^{1/2}\nabla u, \nabla^2 u)$. Set $\vec{u}_\pm = \vec{U}_\pm + \vec{v}_\pm$ in the equation (2.7), and then $\vec{v}_\pm$ satisfies the equation (3.1), replacing $\vec{k}$ by $\vec{k} - \vec{U}_+|_{x_N=0+} + \vec{U}_-|_{x_N=0-}$, $h_j$ by $h_j - \mu_1^+(D_N U_{+,j} + D_j U_{+,N}) + \mu_1^-(D_N U_{-,j} + D_j U_{-,N})$ and $h_N$ by $h_N - (\mu_2^+ + \delta)\sum_{j=1}^{N-1}\partial_j U_{+,j} + (\mu_2^- + \delta)\sum_{j=1}^{N-1}\partial_j U_{-,j}$. Thus it is sufficient to consider the problem (3.1).

In the second step, applying the partial Fourier transform defined by (1.13), we derive a solution formula of the problem (3.1). We consider the following generalized resolvent problem:

$$\tilde\gamma_0^\pm \lambda \vec{v}_\pm - \mathrm{Div}\left[2\mu_1^\pm D(\vec{v}_\pm) + \left(\mu_2^\pm + \delta\right)(\mathrm{div}\,\vec{v}_\pm)I\right] = 0 \quad \text{in } \mathbb{R}^N_\pm, \tag{3.3}$$

$$v_{+,J} - v_{-,J} = k_J \quad \text{on } \mathbb{R}^N_0, \tag{3.4}$$

$$\mu_1^+\left(D_N v_{+,j} + D_j v_{+,N}\right) - \mu_1^-\left(D_N v_{-,j} + D_j v_{-,N}\right) = h_j \quad \text{on } \mathbb{R}^N_0, \tag{3.5}$$

$$2\mu_1^+ D_N v_{+,N} + \left(\mu_2^+ + \delta\right)\mathrm{div}\,\vec{v}_+ - (2\mu_1^- D_N v_{-,N} + \left(\mu_2^- + \delta\right)\mathrm{div}\,\vec{v}_-) = h_N \quad \text{on } \mathbb{R}^N_0. \tag{3.6}$$

Here and hereafter, $j$ and $J$ run from 1 through $N-1$ and $N$.

Applying the divergence to (3.3), we have $\left[\tilde{\gamma}_0^\pm\lambda - \left(2\mu_1^\pm + \mu_2^\pm + \delta\right)\Delta\right] \operatorname{div}\vec{v}_\pm = 0$, which implies that

$$\left[\tilde{\gamma}_0^\pm\lambda - \left(2\mu_1^\pm + \mu_2^\pm + \delta\right)\Delta\right](\tilde{\gamma}_0^\pm\lambda - \mu_1^\pm\Delta)\vec{v}_\pm = 0. \tag{3.7}$$

Applying the partial Fourier transform to (3.3)-(3.6) and (3.7), we obtain

$$\begin{cases} \tilde{\gamma}_0^\pm\lambda\widehat{v_{+,j}} - \mu_1^+\left[(D_N^2 - |\xi'|^2)\widehat{v_{+,j}} + i\xi_j\widehat{\operatorname{div}\vec{v}_+}\right] - \left(\mu_2^+ + \delta\right) i\xi_j\widehat{\operatorname{div}\vec{v}_+} = 0, \\ \tilde{\gamma}_0^\pm\lambda\widehat{v_{+,N}} - \mu_1^+\left[(D_N^2 - |\xi'|^2)\widehat{v_{+,N}} + D_N\widehat{\operatorname{div}\vec{v}_+}\right] - \left(\mu_2^+ + \delta\right) D_N\widehat{\operatorname{div}\vec{v}_+} = 0, \\ \tilde{\gamma}_0^\pm\lambda\widehat{v_{-,j}} - \mu_1^-\left[(D_N^2 - |\xi'|^2)\widehat{v_{-,j}} + i\xi_j\widehat{\operatorname{div}\vec{v}_+}\right] - \left(\mu_2^- + \delta\right) i\xi_j\widehat{\operatorname{div}\vec{v}_-} = 0, \\ \tilde{\gamma}_0^\pm\lambda\widehat{v_{-,N}} - \mu_1^-\left[(D_N^2 - |\xi'|^2)\widehat{v_{-,N}} + D_N\widehat{\operatorname{div}\vec{v}_-}\right] - \left(\mu_2^- + \delta\right) D_N\widehat{\operatorname{div}\vec{v}_-} = 0 \end{cases} \tag{3.8}$$

and

$$\left[\tilde{\gamma}_0^\pm\lambda + \left(2\mu_1^\pm + \mu_2^\pm + \delta\right)(|\xi'|^2 - D_N^2)\right]\left[\tilde{\gamma}_0^\pm\lambda + \mu_1^\pm(|\xi'|^2 - D_N^2)\right]\widehat{v_{\pm,J}} = 0. \tag{3.9}$$

By (3.9), we see that the characteristic roots of (3.8) are

$$A_\pm = \sqrt{\left(2\mu_1^\pm + \mu_2^\pm + \delta\right)^{-1}\tilde{\gamma}_0^\pm\lambda + A^2}, \qquad B_\pm = \sqrt{\left(\mu_1^\pm\right)^{-1}\tilde{\gamma}_0^\pm\lambda + A^2}, \qquad A = |\xi'|.$$

By using $B_\pm$, we rewrite (3.8) as follows:

$$\begin{cases} \mu_1^+(B_+^2 - D_N^2)\widehat{v_{+,j}} - \left(\mu_1^+ + \mu_2^+ + \delta\right) i\xi_j\widehat{\operatorname{div}\vec{v}_+} = 0, \\ \mu_1^+(B_+^2 - D_N^2)\widehat{v_{+,N}} - \left(\mu_1^+ + \mu_2^+ + \delta\right) D_N\widehat{\operatorname{div}\vec{v}_+} = 0, \\ \mu_1^-(B_-^2 - D_N^2)\widehat{v_{-,j}} - \left(\mu_1^- + \mu_2^- + \delta\right) i\xi_j\widehat{\operatorname{div}\vec{v}_-} = 0, \\ \mu_1^-(B_-^2 - D_N^2)\widehat{v_{-,N}} - \left(\mu_1^- + \mu_2^- + \delta\right) D_N\widehat{\operatorname{div}\vec{v}_-} = 0. \end{cases} \tag{3.10}$$

From now, we shall find the solution $\widehat{v_{\pm,J}}$ to (3.8) of the forms:

$$\widehat{v_{+,J}} = \alpha_J^+(e^{-B_+x_N} - e^{-A_+x_N}) + \beta_J^+e^{-B_+x_N}, \tag{3.11}$$

$$\widehat{v_{-,J}} = \alpha_J^-(e^{B_-x_N} - e^{A_-x_N}) + \beta_J^-e^{B_-x_N}. \tag{3.12}$$

We see that $(B_\pm^2 - D_N^2)\widehat{v_{\pm,J}} = (A_\pm^2 - B_\pm^2)\alpha_J^\pm e^{\mp A_\pm x_N}$ and

$$\begin{aligned} \widehat{\operatorname{div}\vec{v}_+} &= (i\xi'\cdot\alpha'_+ + i\xi'\cdot\beta'_+ - B_+(\alpha_N^+ + \beta_N^+))e^{-B_+x_N} + (A_+\alpha_N^+ - i\xi'\cdot\alpha'_+)e^{-A_+x_N}, \\ \widehat{\operatorname{div}\vec{v}_-} &= (i\xi'\cdot\alpha'_- + i\xi'\cdot\beta'_- + B_-(\alpha_N^- + \beta_N^-))e^{B_-x_N} - (A_-\alpha_N^- + i\xi'\cdot\alpha'_-)e^{A_-x_N}, \end{aligned} \tag{3.13}$$

where $\alpha'_\pm = (\alpha_1^\pm, \ldots, \alpha_{N-1}^\pm)$, $\beta'_\pm = (\beta_1^\pm, \ldots, \beta_{N-1}^\pm)$.

Substituting (3.13) into (3.10) and equating the coefficients of $e^{\mp B_\pm x_N}$ and $e^{\mp A_\pm x_N}$, we have

$$\begin{cases} i\xi'\cdot\alpha'_+ + i\xi'\cdot\beta'_+ - B_+(\alpha_N^+ + \beta_N^+) = 0, \\ i\xi'\cdot\alpha'_- + i\xi'\cdot\beta'_- + B_-(\alpha_N^- + \beta_N^-) = 0, \\ \mu_1^+(A_+^2 - B_+^2)\alpha_j^+ - \left(\mu_1^+ + \mu_2^+ + \delta\right) i\xi_j(A_+\alpha_N^+ - i\xi'\cdot\alpha'_+) = 0, \\ \mu_1^+(A_+^2 - B_+^2)\alpha_N^+ + \left(\mu_1^+ + \mu_2^+ + \delta\right) A_+(A_+\alpha_N^+ - i\xi'\cdot\alpha'_+) = 0, \\ \mu_1^-(A_-^2 - B_-^2)\alpha_j^- + \left(\mu_1^- + \mu_2^- + \delta\right) i\xi_j(A_-\alpha_N^- + i\xi'\cdot\alpha'_-) = 0, \\ \mu_1^-(A_-^2 - B_-^2)\alpha_N^- + \left(\mu_1^- + \mu_2^- + \delta\right) A_-(A_-\alpha_N^+ + i\xi'\cdot\alpha'_-) = 0. \end{cases} \tag{3.14}$$

Since $\mu_1^+(A_+^2 - B_+^2) + \left(\mu_1^+ + \mu_2^+ + \delta\right)A_+^2 = \left(\mu_1^+ + \mu_2^+ + \delta\right)A^2$, the fourth equation in (3.14) implies that $\alpha_N^+ = A^{-2}A_+i\xi'\cdot\alpha'_+$. By the first equation in (3.14), we have

$$i\xi'\cdot\alpha'_+ = \frac{A^2}{A_+B_+ - A^2}(i\xi'\cdot\beta'_+ - B_+\beta_N^+), \qquad \alpha_N^+ = \frac{A_+}{A_+B_+ - A^2}(i\xi'\cdot\beta'_+ - B_+\beta_N^+). \tag{3.15}$$

Similarly, by the sixth equation and the second equation in (3.14), we obtain

$$i\xi'\cdot\alpha'_- = \frac{A^2}{A_-B_- - A^2}(i\xi'\cdot\beta'_- + B_-\beta_N^-), \qquad \alpha_N^- = \frac{-A_-}{A_-B_- - A^2}(i\xi'\cdot\beta'_- + B_-\beta_N^-). \tag{3.16}$$

Next we consider the boundary condition (3.4)-(3.6). Applying the partial Fourier transform to (3.4)-(3.6), we obtain

$$\widehat{k_J} = \beta_J^+ - \beta_J^-, \tag{3.17}$$

$$\begin{aligned} &\widehat{h_j} \\ &= \mu_1^+\left((A_+ - B_+)\alpha_j^+ - B_+\beta_j^+ + i\xi_j\beta_N^+\right) - \mu_1^-\left((B_- - A_-)\alpha_j^- + B_-\beta_j^- + i\xi_j\beta_N^-\right), \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\widehat{h_N} \\ &= 2\mu_1^+(A_+ - B_+)\alpha_N^+ - 2\mu_1^+B_+\beta_N^+ + \left(\mu_2^+ + \delta\right)\left(i\xi'\cdot\beta'_+ - B_+\beta_N^+ + (A_+ - B_+)\alpha_N^+\right) \\ &- 2\mu_1^-(B_- - A_-)\alpha_N^- - 2\mu_1^-B_-\beta_N^- - \left(\mu_2^- + \delta\right)\left(i\xi'\cdot\beta'_- + B_-\beta_N^- + (B_- - A_-)\alpha_N^-\right). \end{aligned} \tag{3.19}$$

By (3.15) and (3.16), we have

$$\begin{aligned} &i\xi'\cdot\widehat{h'} \\ &= \mu_1^+\left((A_+ - B_+)i\xi'\cdot\alpha'_+ - B_+i\xi'\cdot\beta'_+ - A^2\beta_N^+\right) \\ &\qquad - \mu_1^-\left((B_- - A_-)i\xi'\cdot\alpha'_- + B_-i\xi'\cdot\beta'_- - A^2\beta_N^-\right) \\ &= \mu_1^+\left(\frac{A^2(A_+ - B_+)}{A_+B_+ - A^2}(i\xi'\cdot\beta'_+ - B_+\beta_N^+) - B_+i\xi'\cdot\beta'_+ - A^2\beta_N^+\right) \\ &\qquad\qquad + \mu_1^-\left(\frac{A^2(B_- - A_-)}{A_-B_- - A^2}(i\xi'\cdot\beta'_- + B_-\beta_N^-) + B_-i\xi'\cdot\beta'_- - A^2\beta_N^-\right) \\ &= \mu_1^+\left(\frac{A_+(A^2 - B_+^2)}{A_+B_+ - A^2}i\xi'\cdot\beta'_+ - \frac{A^2(2A_+B_+ - B_+^2 - A^2)}{A_+B_+ - A^2}\beta_N^+\right) \\ &\qquad\qquad - \mu_1^-\left(\frac{A_-(B_-^2 - A^2)}{A_-B_- - A^2}i\xi'\cdot\beta'_- + \frac{A^2(A^2 + B_-^2 - 2A_-B_-)}{A_-B_- - A^2}\beta_N^-\right) \end{aligned}$$

and

$$\begin{aligned} &\widehat{h_N} \\ &= \left(2\mu_1^+ + \mu_2^+ + \delta\right)(A_+ - B_+)\alpha_N^+ - \left(2\mu_1^+ + \mu_2^+ + \delta\right)B_+\beta_N^+ + \left(\mu_2^+ + \delta\right)i\xi'\cdot\beta'_+ \\ &- \left[\left(2\mu_1^- + \mu_2^- + \delta\right)(B_- - A_-)\alpha_N^- + \left(2\mu_1^- + \mu_2^- + \delta\right)B_-\beta_N^- + \left(\mu_2^- + \delta\right)i\xi'\cdot\beta'_-\right] \\ &= \left(2\mu_1^+ + \mu_2^+ + \delta\right)\frac{A_+(A_+ - B_+)}{A_+B_+ - A^2}(i\xi'\cdot\beta'_+ - B_+\beta_N^+) \\ &\quad - \left(2\mu_1^+ + \mu_2^+ + \delta\right)B_+\beta_N^+ + \left(\mu_2^+ + \delta\right)i\xi'\cdot\beta'_+ \\ &\qquad - \left(2\mu_1^- + \mu_2^- + \delta\right)\frac{A_-(A_- - B_-)}{A_-B_- - A^2}(i\xi'\cdot\beta'_- + B_-\beta_N^-) \\ &\qquad\quad - \left(2\mu_1^- + \mu_2^- + \delta\right)B_-\beta_N^- - \left(\mu_2^- + \delta\right)i\xi'\cdot\beta'_- \\ &= \frac{1}{A_+B_+ - A^2}\left[2\mu_1^+(A_+^2 - A_+B_+) + \left(\mu_2^+ + \delta\right)(A_+^2 - A^2)\right]i\xi'\cdot\beta'_+ \\ &\quad - \left(2\mu_1^+ + \mu_2^+ + \delta\right)\frac{A_+^2 - A^2}{A_+B_+ - A^2}B_+\beta_N^+ - \left(2\mu_1^- + \mu_2^- + \delta\right)\frac{A_-^2 - A^2}{A_-B_- - A^2}B_-\beta_N^- \\ &\qquad - \frac{1}{A_-B_- - A^2}\left[2\mu_1^-(A_-^2 - A_-B_-) + \left(\mu_2^- + \delta\right)(A_-^2 - A^2)\right]i\xi'\cdot\beta'_-. \end{aligned}$$

Substituting the relation $\beta_J^+ = \beta_J^- + \widehat{k_J}$ given by (3.17) into the formula of $i\xi' \cdot \widehat{h'}$ and $\widehat{h_N}$, we obtain

$$
\begin{aligned}
&-\mu_1^+ \frac{A_+(A^2 - B_+^2)}{A_+B_+ - A^2} i\xi' \cdot \widehat{k'} + \mu_1^- \frac{A^2(2A_+B_+ - B_+^2 - A^2)}{A_+B_+ - A^2} \widehat{k_N} + i\xi' \cdot \widehat{h'} \\
&= -\left[\mu_1^+ \frac{A_+(B_+^2 - A^2)}{A_+B_+ - A^2} + \mu_1^- \frac{A_-(B_-^2 - A^2)}{A_-B_- - A^2}\right] i\xi' \cdot \beta_-' \\
&\qquad\qquad - A^2 \left[\mu_1^+ \frac{2A_+B_+ - B_+^2 - A^2}{A_+B_+ - A^2} - \mu_1^- \frac{2A_-B_- - A^2 - B_-^2}{A_-B_- - A^2}\right] \beta_N^-
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{A_+B_+ - A^2}\left[2\mu_1^+(A_+^2 - A_+B_+) + \left(\mu_2^+ + \delta\right)(A_+^2 - A^2)\right] i\xi' \cdot \widehat{k'} \\
&\qquad\qquad + \left(2\mu_1^+ + \mu_2^+ + \delta\right) \frac{A_+^2 - A^2}{A_+B_+ - A^2} B_+ \widehat{k_N} + \widehat{h_N} \\
&= \frac{1}{A_+B_+ - A^2}\left[2\mu_1^+(A_+^2 - A_+B_+) + \left(\mu_2^+ + \delta\right)(A_+^2 - A^2)\right] i\xi' \cdot \beta_-' \\
&\quad - \frac{1}{A_-B_- - A^2}\left[2\mu_1^-(A_-^2 - A_-B_-) + \left(\mu_2^- + \delta\right)(A_-^2 - A^2)\right] i\xi' \cdot \beta_-' \\
&\quad - \left(2\mu_1^+ + \mu_2^+ + \delta\right) \frac{A_+^2 - A^2}{A_+B_+ - A^2} B_+ \beta_N^- - \left(2\mu_1^- + \mu_2^- + \delta\right) \frac{A_-^2 - A^2}{A_-B_- - A^2} B_- \beta_N^-.
\end{aligned}
$$

Here setting

$$
\begin{cases}
L_{11}^\pm = -\mu_1^\pm \dfrac{A_\pm(B_\pm^2 - A^2)}{A_\pm B_\pm - A^2}, \\
L_{12}^\pm = \mp\mu_1^\pm A^2 \dfrac{2A_\pm B_\pm - A^2 - B_\pm^2}{A_\pm B_\pm - A^2}, \\
L_{21}^\pm = \dfrac{\pm 1}{A_\pm B_\pm - A^2}\left[2\mu_1^\pm A_\pm(A_\pm - B_\pm) + \left(\mu_2^\pm + \delta\right)(A_\pm^2 - A^2)\right], \\
L_{22}^\pm = -\left(2\mu_1^\pm + \mu_2^\pm + \delta\right) \dfrac{A_\pm^2 - A^2}{A_\pm B_\pm - A^2} B_\pm
\end{cases}
\tag{3.20}
$$

and $L_{ij} = L_{ij}^+ + L_{ij}^-$, $L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$, we obtain

$$
L \begin{pmatrix} i\xi' \cdot \beta_-' \\ \beta_N^- \end{pmatrix} = \begin{pmatrix} i\xi' \cdot \widehat{h'} - L_{11}^+ i\xi' \cdot \widehat{k'} - L_{12}^+ \widehat{k_N} \\ \widehat{h_N} - L_{21}^+ i\xi' \cdot \widehat{k'} - L_{22}^+ \widehat{k_N} \end{pmatrix}. \tag{3.21}
$$

If $\det L \neq 0$, the inverse of $L$ exists and we see

$$
\begin{pmatrix} i\xi' \cdot \beta_-' \\ \beta_N^- \end{pmatrix} = \frac{1}{\det L} \begin{pmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{pmatrix} \begin{pmatrix} i\xi' \cdot \widehat{h'} - L_{11}^+ i\xi' \cdot \widehat{k'} - L_{12}^+ \widehat{k_N} \\ \widehat{h_N} - L_{21}^+ i\xi' \cdot \widehat{k'} - L_{22}^+ \widehat{k_N} \end{pmatrix}. \tag{3.22}
$$

In this section, we assume $\det L \neq 0$ and continue to obtain the solution formula. We shall prove $\det L \neq 0$ when $\lambda \in \Gamma_{\varepsilon,\lambda_0}$ in next section. By (3.22), we obtain

$$
\begin{aligned}
i\xi' \cdot \beta_-' = \frac{1}{\det L} &\Big[ i\xi' \cdot \widehat{h'}(L_{22}^+ + L_{22}^-) - \widehat{h_N}(L_{12}^+ + L_{12}^-) \\
&- i\xi' \cdot \widehat{k'}(L_{22}^+ L_{11}^+ + L_{22}^- L_{11}^+ - L_{12}^+ L_{21}^+ - L_{12}^- L_{21}^+) - \widehat{k_N}(L_{22}^- L_{12}^+ - L_{12}^- L_{22}^+) \Big], \\
\beta_{-,N} = \frac{1}{\det L} &\Big[ -i\xi' \cdot \widehat{h'}(L_{21}^+ + L_{21}^-) + \widehat{h_N}(L_{11}^+ + L_{11}^-) \\
&+ i\xi' \cdot \widehat{k'}(L_{21}^- L_{11}^+ - L_{11}^- L_{21}^+) + \widehat{k_N}(L_{21}^+ L_{12}^+ + L_{21}^- L_{12}^+ - L_{11}^+ L_{22}^+ - L_{11}^- L_{22}^+) \Big].
\end{aligned}
$$

Setting

$$\begin{aligned}
M^h_{1,j} &= -\frac{i\xi_j}{A}(L^+_{22} + L^-_{22})A,\\
M^h_{1,N} &= L^+_{12} + L^-_{12},\\
M^k_{1,j} &= -\frac{i\xi_j}{A}(L^+_{22}L^+_{11} + L^-_{22}L^+_{11} - L^+_{12}L^+_{21} - L^-_{12}L^+_{21})A,\\
M^k_{1,N} &= -(L^-_{22}L^+_{12} - L^-_{12}L^+_{22})
\end{aligned} \tag{3.23}$$

formM1

and

$$\begin{aligned}
M^h_{2,j} &= \frac{i\xi_j}{A}(L^+_{21} + L^-_{21})A,\\
M^h_{2,N} &= -(L^+_{11} + L^-_{11}),\\
M^k_{2,j} &= \frac{i\xi_j}{A}(L^-_{21}L^+_{11} - L^-_{11}L^+_{21})A,\\
M^k_{2,N} &= L^+_{21}L^+_{12} + L^-_{21}L^+_{12} - L^+_{11}L^+_{22} - L^-_{11}L^+_{22}),
\end{aligned} \tag{3.24}$$

formM2

we see

$$\begin{aligned}
i\xi' \cdot \beta'_- &= \frac{1}{\det L}\sum_{J=1}^N (M^h_{1,J}\widehat{h_J}(0) + M^k_{1,J}\widehat{k_J}(0)),\\
\beta_{-,N} &= \frac{1}{\det L}\sum_{J=1}^N (M^h_{2,J}\widehat{h_J}(0) + M^k_{2,J}\widehat{k_J}(0)).
\end{aligned}$$

Moreover we have

$$\begin{aligned}
\alpha_{\pm,j} &= \frac{-i\xi_j}{A_\pm B_\pm - A^2}\frac{1}{\det L}\sum_{J=1}^N \Big((M^h_{1,J} \mp B_\pm M^h_{2,J})\widehat{h_J}(0) + (M^k_{1,J} \mp B_\pm M^k_{2,J})\widehat{k_J}(0)\Big),\\
\alpha_{\pm,N} &= \frac{-A_\pm}{A_\pm B_\pm - A^2}\frac{1}{\det L}\sum_{J=1}^N \Big((M^h_{1,J} - B_\pm M^h_{2,J})\widehat{h_J}(0) + (\pm M^k_{1,J} - B_\pm M^k_{2,J})\widehat{k_J}(0)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{-,j}\\
&= \frac{1}{\mu_1^+ B_+ + \mu_1^- B_-}\\
&\Bigg(\frac{-i\xi_j\mu_1^+(A_+ - B_+)}{A_+B_+ - A^2}\frac{1}{\det L}\sum_{J=1}^N \Big((M^h_{1,J} - B_+ M^h_{2,J})\widehat{h_J}(0) + (M^k_{1,J} - B_+ M^k_{2,J})\widehat{k_J}(0)\Big)\\
&- \frac{i\xi_j\mu_1^-(A_- - B_-)}{A_-B_- - A^2}\frac{1}{\det L}\sum_{J=1}^N \Big((M^h_{1,J} + B_- M^h_{2,J})\widehat{h_J}(0) + (M^k_{1,J} + B_- M^k_{2,J})\widehat{k_J}(0)\Big)\\
&+ \frac{i\xi_j(\mu_1^+ - \mu_1^-)}{\det L}\sum_{J=1}^N (M^h_{2,J}\widehat{h_J}(0) + M^k_{2,J}\widehat{k_J}(0)) + \mu_1^+(i\xi_j\widehat{k_N}(0) - B_+\widehat{k_j}(0)) + \widehat{h_j}(0)\Bigg).
\end{aligned}$$

Summing up, we obtain the following representation formula of solutions:

$$\begin{aligned}
\widehat{v_{\pm,J}} &= B^2_\pm M_\pm(x_N)\sum_{\ell=1}^N (Q^{J,1}_{\pm,\ell}\widehat{h_\ell}(0) + R^{J,1}_{\pm,\ell}\widehat{k_\ell}(0))\\
&\quad + B_\pm e^{\mp B_\pm x_N}\sum_{\ell=1}^N (Q^{J,2}_{\pm,\ell}\widehat{h_\ell}(0) + R^{J,2}_{\pm,\ell}\widehat{k_\ell}(0))
\end{aligned}$$

$$+ B_\pm e^{\mp B_\pm x_N}\sum_{\ell=1}^N (Q^{J,3}_{\pm,\ell}\widehat{h}_\ell(0) + R^{J,3}_{\pm,\ell}\widehat{k}_\ell(0)),$$

where we have set

$$\begin{aligned}
Q^{j,1}_{\pm,\ell} &= \frac{-i\xi_\ell}{A_\pm B_\pm - A^2}\frac{B_\pm - A_\pm}{B_\pm^2\det L}(M^h_{1,\ell} \mp B_\pm M^h_{2,\ell}),\\
R^{j,1}_{\pm,\ell} &= \frac{-i\xi_\ell}{A_\pm B_\pm - A^2}\frac{B_\pm - A_\pm}{B_\pm^2\det L}(M^k_{1,\ell} \mp B_\pm M^k_{2,\ell}),\\
Q^{j,2}_{\pm,\ell} &= \frac{-i\xi_\ell}{A_+ B_+ - A^2}\frac{\mu_1^+(A_+ - B_+)}{\mu_1^+ B_+ + \mu_1^- B_-}\frac{1}{B_\pm \det L}(M^h_{1,\ell} - B_+ M^h_{2,\ell})\\
&\quad + \frac{i\xi_\ell(\mu_1^+ - \mu_1^-)}{\mu_1^+ B_+ + \mu_1^- B_-}\frac{1}{B_\pm\det L}M^h_{2,\ell}\\
&\qquad + \frac{-i\xi_\ell}{A_- B_- - A^2}\frac{\mu_1^-(A_- - B_-)}{\mu_1^+ B_+ + \mu_1^- B_-}\frac{1}{B_\pm\det L}(M^h_{1,\ell} + B_- M^h_{2,\ell}),\\
R^{j,2}_{\pm,\ell} &= \frac{-i\xi_\ell}{A_+ B_+ - A^2}\frac{\mu_1^+(A_+ - B_+)}{\mu_1^+ B_+ + \mu_1^- B_-}\frac{1}{B_\pm \det L}(M^k_{1,\ell} - B_+ M^k_{2,\ell})\\
&\quad + \frac{i\xi_\ell(\mu_1^+ - \mu_1^-)}{\mu_1^+ B_+ + \mu_1^- B_-}\frac{1}{B_\pm\det L}M^k_{2,\ell}\\
&\qquad + \frac{-i\xi_\ell}{A_- B_- - A^2}\frac{\mu_1^-(A_- - B_-)}{\mu_1^+ B_+ + \mu_1^- B_-}\frac{1}{B_\pm\det L}(M^k_{1,\ell} + B_- M^k_{2,\ell}),\\
Q^{j,3}_{\pm,\ell} &= \frac{1}{(\mu_1^+ B_+ + \mu^- B_-)B_\pm}\ (\ell = j), \quad Q^{j,3}_{\pm,\ell} = 0\ (\ell \neq j),\\
R^{j,3}_{\pm,\ell} &= \mp\frac{\mu_1^\pm}{\mu_1^+ B_+ + \mu_1^- B_-}\ (\ell = j), \quad R^{j,3}_{\pm,\ell} = \frac{\mu_1^+ i\xi_j}{(\mu_1^+ B_+ + \mu^- B_-)B_\pm}\ (\ell = N),\\
&\quad R^{j,3}_{\pm,\ell} = 0\ (\ell \neq j, N),
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
Q^{N,1}_{\pm,\ell} &= \frac{-A_\pm}{A_\pm B_\pm - A^2}\frac{B_\pm - A_\pm}{B_\pm^2\det L}(\pm M^h_{1,\ell} - B_\pm M^h_{2,\ell}),\\
R^{N,1}_{\pm,\ell} &= \frac{-A_\pm}{A_\pm B_\pm - A^2}\frac{B_\pm - A_\pm}{B_\pm^2\det L}(\pm M^k_{1,\ell} - B_\pm M^k_{2,\ell}),\\
Q^{N,2}_{\pm,\ell} &= \frac{1}{B_\pm\det L}M^h_{2,\ell}, \quad Q^{N,3}_{\pm,\ell} = 0,\\
R^{N,2}_{\pm,\ell} &= \frac{1}{B_\pm\det L}M^k_{2,\ell}, \quad R^{N,3}_{\pm,\ell} = 1\ (\ell = N), \quad R^{N,3}_{\pm,\ell} = 0\ (\ell \neq N).
\end{aligned} \tag{3.26}$$

formQR

By the Volevich trick:

$$\begin{aligned}
a(x_N)b(0) &= -\int_0^\infty \{a'(x_N + y_N)b(y_N) + a(x_N + y_N)b'(y_N)\}dy_N\\
&= \int_{-\infty}^0 \{a'(x_N + y_N)b(y_N) + a(x_N + y_N)b'(y_N)\}dy_N
\end{aligned}$$

and the identity: $1 = \lambda/\mu_1^\pm B_\pm^2 - \sum_{m=1}^{N-1}(i\xi_m)(i\xi_m)/B_\pm^2$, we define $v_{\pm,J}$ as follows:

$$\begin{aligned}
&v_{\pm,J}\\
&= \mp\int_0^{\pm\infty}\mathcal{F}^{-1}_{\xi'}\left[B_\pm^2 M_\pm(x_N + y_N)\sum_{\ell=1}^N\left(Q^{J,1}_{\pm,\ell}\mathcal{F}_{\xi'}[\partial_N h_\ell(y_N)] + R^{J,1}_{\pm,\ell}\frac{\lambda^{1/2}}{B_\pm^2\mu_1^\pm}\mathcal{F}_{\xi'}[\lambda^{1/2}\partial_N k_\ell(y_N)]\right)\right]dy_N\\
&\quad \pm\int_0^{\pm\infty}\mathcal{F}^{-1}_{\xi'}\left[B_\pm^2 M_\pm(x_N + y_N)\sum_{\ell=1}^N\sum_{m=1}^{N-1}\frac{R^{J,1}_{\pm,\ell}i\xi_m}{B_\pm^2}\mathcal{F}_{\xi'}[\partial_m\partial_N k_\ell]\right]dy_N
\end{aligned}$$

$$\mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^N \left( Q^{J,2}_{\pm,\ell} \mathcal{F}_{\xi'}[\partial_N h_\ell(y_N)] + \frac{R^{J,2}_{\pm,\ell}\lambda^{1/2}}{\mu_1^\pm B_\pm^2} \mathcal{F}_{\xi'}[\lambda^{1/2}\partial_N k_\ell(y_N)] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^N \sum_{m=1}^{N-1} \frac{R^{J,2}_{\pm,\ell} i\xi_m}{B_\pm^2} \mathcal{F}_{\xi'}[\partial_m\partial_N k_\ell(y_N)] \right] dy_N$$
$$\mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^N \left( Q^{J,3}_{\pm,\ell} \mathcal{F}_{\xi'}[\partial_N h_\ell(y_N)] + \frac{R^{J,3}_{\pm,\ell}\lambda^{1/2}}{\mu_1^\pm B_\pm^2} \mathcal{F}_{\xi'}[\lambda^{1/2}\partial_N k_\ell(y_N)] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^N \sum_{m=1}^{N-1} \frac{R^{J,3}_{\pm,\ell} i\xi_m}{B_\pm^2} \mathcal{F}_{\xi'}[\partial_m\partial_N k_\ell(y_N)] \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ (e^{\mp B_\pm(x_N+y_N)} + A_\pm M_\pm(x_N+y_N)) B_\pm \sum_{\ell=1}^N \frac{Q^{J,1}_{\pm,\ell}\lambda^{1/2}}{B_\pm \mu_1^\pm} \mathcal{F}_{\xi'}[\lambda^{1/2} h_\ell(y_N)] \right] dy_N$$
$$\mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ (e^{\mp B_\pm(x_N+y_N)} + A_\pm M_\pm(x_N+y_N)) B_\pm \sum_{\ell=1}^N \left( \sum_{m=1}^{N-1} \frac{Q^{J,1}_{\pm,\ell} i\xi_m}{B_\pm} \mathcal{F}_{\xi'}[\partial_m h_\ell(y_N)] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ (e^{\mp B_\pm(x_N+y_N)} + A_\pm M_\pm(x_N+y_N)) B_\pm \sum_{\ell=1}^N \frac{R^{J,1}_{\pm,\ell}}{B_\pm \mu_1^\pm} \mathcal{F}_{\xi'}[\lambda k_\ell(y_N)] \right] dy_N$$
$$\mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ (e^{\mp B_\pm(x_N+y_N)} + A_\pm M_\pm(x_N+y_N)) B_\pm \sum_{\ell=1}^N \left( \sum_{m=1}^{N-1} \frac{R^{J,1}_{\pm,\ell}}{B_\pm} \mathcal{F}_{\xi'}[\partial_m^2 k_\ell(y_N)] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N-1} \left( \frac{Q^{J,2}_{\pm,\ell}\lambda^{1/2}}{B_\pm \mu_1^\pm} \mathcal{F}_{\xi'}[\lambda^{1/2} h_\ell(y_N)] - \sum_{m=1}^{N-1} \frac{Q^{J,2}_{\pm,\ell} i\xi_m}{B_\pm} \mathcal{F}_{\xi'}[\partial_m h_\ell] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N-1} \left( \frac{R^{J,2}_{\pm,\ell}}{B_\pm \mu_1^\pm} \mathcal{F}_{\xi'}[\lambda k_\ell(y_N)] - \sum_{m=1}^{N-1} \frac{R^{J,2}_{\pm,\ell}}{B_\pm} \mathcal{F}_{\xi'}[\partial_m^2 k_\ell] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N-1} \left( \frac{Q^{J,3}_{\pm,\ell}\lambda^{1/2}}{\mu_1^\pm B_\pm} \mathcal{F}_{\xi'}[\lambda^{1/2} h_\ell(y_N)] - \sum_{m=1}^{N-1} \frac{Q^{J,3}_{\pm,\ell} i\xi_m}{B_\pm} \mathcal{F}_{\xi'}[\partial_m h_\ell] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N-1} \left( \frac{R^{J,3}_{\pm,\ell}}{\mu_1^\pm B_\pm} \mathcal{F}_{\xi'}[\lambda k_\ell(y_N)] - \sum_{m=1}^{N-1} \frac{R^{J,3}_{\pm,\ell}}{B_\pm} \mathcal{F}_{\xi'}[\partial_m^2 k_\ell] \right) \right] dy_N.$$

Let $F_{2\ell}, F_{3j\ell}, F_{4\ell}, F_{5j\ell}$ and $F_{6jm\ell}$ be corresponding variables to $\lambda^{1/2}h_\ell$, $\partial_j h_\ell$, $\lambda k_\ell$, $\lambda^{1/2}\partial_j k_\ell$ and $\partial_j\partial_m k_\ell$ respectively. For $F_2 = (F_{21}, \dots, F_{2N}), F_3 = (F_{3j\ell} \mid j, \ell = 1, \dots, N), F_4 = (F_{41}, \dots, F_{4N}), F_5 = (F_{5j\ell} \mid j, \ell = 1, \dots, N)$ and $F_6 = (F_{6mj\ell} \mid j, m, \ell = 1, \dots, N)$, we define operators $\mathcal{T}_{\pm,J}(\lambda)$ by

$$\mathcal{T}_{\pm,J}(F_2, F_3, F_4, F_5, F_6)$$
$$= \mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm^2 M_\pm(x_N+y_N) \sum_{\ell=1}^N \left( Q^{J,1}_{\pm,\ell} \mathcal{F}_{\xi'}[F_{3N\ell}] + R^{J,1}_{\pm,\ell} \frac{\lambda^{1/2}}{B_\pm^2 \mu_1^\pm} \mathcal{F}_{\xi'}[F_{5N\ell}] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm^2 M_\pm(x_N+y_N) \sum_{\ell=1}^N \sum_{m=1}^{N-1} \frac{R^{J,1}_{\pm,\ell} i\xi_m}{B_\pm^2} \mathcal{F}_{\xi'}[F_{6mN\ell}] \right] dy_N$$
$$\mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^N \left( Q^{J,2}_{\pm,\ell} \mathcal{F}_{\xi'}[F_{3N\ell}] + \frac{R^{J,2}_{\pm,\ell}\lambda^{1/2}}{\mu_1^\pm B_\pm^2} \mathcal{F}_{\xi'}[F_{5N\ell}] \right) \right] dy_N$$
$$\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^N \sum_{m=1}^{N-1} \frac{R^{J,2}_{\pm,\ell} i\xi_m}{B_\pm^2} \mathcal{F}_{\xi'}[F_{6mN\ell}] \right] dy_N$$
$$\mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'} \left[ B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^N \left( Q^{J,3}_{\pm,\ell} \mathcal{F}_{\xi'}[F_{3N\ell}] + \frac{R^{J,3}_{\pm,\ell}\lambda^{1/2}}{\mu_1^\pm B_\pm^2} \mathcal{F}_{\xi'}[F_{5N\ell}] \right) \right] dy_N$$

$$
\begin{aligned}
&\pm \int_0^{\infty} \mathcal{F}^{-1}_{\xi'}\left[B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N}\sum_{m=1}^{N-1} \frac{R^{J,3}_{\pm,\ell} i\xi_m}{B_\pm^2} \mathcal{F}_{\xi'}[F_{6mN\ell}]\right] dy_N \\
&\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'}\left[(e^{\mp B_\pm(x_N+y_N)} + A_\pm M_\pm(x_N+y_N))B_\pm \sum_{\ell=1}^{N} \frac{Q^{J,1}_{\pm,\ell}\lambda^{1/2}}{B_\pm \mu_1^\pm} \mathcal{F}_{\xi'}[F_{2\ell}]\right] dy_N \\
&\mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'}\left[(e^{\mp B_\pm(x_N+y_N)} + A_\pm M_\pm(x_N+y_N))B_\pm \sum_{\ell=1}^{N} \left(\sum_{m=1}^{N-1} \frac{Q^{J,1}_{\pm,\ell} i\xi_m}{B_\pm} \mathcal{F}_{\xi'}[F_{3m\ell}]\right)\right] dy_N \\
&\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'}\left[(e^{\mp B_\pm(x_N+y_N)} + A_\pm M_\pm(x_N+y_N))B_\pm \sum_{\ell=1}^{N} \frac{R^{J,1}_{\pm,\ell}}{B_\pm \mu_1^\pm} \mathcal{F}_{\xi'}[F_{4\ell}]\right] dy_N \\
&\mp \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'}\left[(e^{\mp B_\pm(x_N+y_N)} + A_\pm M_\pm(x_N+y_N))B_\pm \sum_{\ell=1}^{N} \left(\sum_{m=1}^{N-1} \frac{R^{J,1}_{\pm,\ell}}{B_\pm} \mathcal{F}_{\xi'}[F_{6mm\ell}]\right)\right] dy_N \\
&\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'}\left[B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N-1} \left(\frac{Q^{J,2}_{\pm,\ell}\lambda^{1/2}}{B_\pm \mu_1^\pm} \mathcal{F}_{\xi'}[F_{2\ell}] - \sum_{m=1}^{N-1} \frac{Q^{J,2}_{\pm,\ell} i\xi_m}{B_\pm} \mathcal{F}_{\xi'}[F_{3m\ell}]\right)\right] dy_N \\
&\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'}\left[B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N-1} \left(\frac{R^{J,2}_{\pm,\ell}}{B_\pm \mu_1^\pm} \mathcal{F}_{\xi'}[F_{4\ell}] - \sum_{m=1}^{N-1} \frac{R^{J,2}_{\pm,\ell}}{B_\pm} \mathcal{F}_{\xi'}[F_{6mm\ell}]\right)\right] dy_N \\
&\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'}\left[B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N-1} \left(\frac{Q^{J,3}_{\pm,\ell}\lambda^{1/2}}{\mu_1^\pm B_\pm} \mathcal{F}_{\xi'}[F_{2\ell}] - \sum_{m=1}^{N-1} \frac{Q^{J,3}_{\pm,\ell} i\xi_m}{B_\pm} \mathcal{F}_{\xi'}[F_{3m\ell}]\right)\right] dy_N \\
&\pm \int_0^{\pm\infty} \mathcal{F}^{-1}_{\xi'}\left[B_\pm e^{\mp B_\pm(x_N+y_N)} \sum_{\ell=1}^{N-1} \left(\frac{R^{J,3}_{\pm,\ell}}{\mu_1^\pm B_\pm} \mathcal{F}_{\xi'}[F_{4\ell}] - \sum_{m=1}^{N-1} \frac{R^{J,3}_{\pm,\ell}}{B_\pm} \mathcal{F}_{\xi'}[F_{6mm\ell}]\right)\right] dy_N. \qquad (3.27)
\end{aligned}
$$
sol_form

## 3.2 Anaylsis of Lopatinski determinant

In order to analyze Lopatinski determinant, we shall prove the following lemma, which is one of the essential steps in this paper.

Lem5.1 **Lemma 3.1.** *Let $L$ be the matrix defined in (3.20). Then, there exists a positive constant $\omega$ depending on $\mu_1^\pm$, $\mu_2^\pm$, $\varepsilon$, $\lambda_0$ and $\delta_0$ such that*

$$|\det L| \geq \omega(|\lambda|^{1/2} + A)^2 \tag{3.28}$$
lem5.1_1

*for any $\lambda \in \Gamma_{\varepsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\backslash\{0\}$. Moreover, the following inequality holds:*

$$|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell(\det L)^{-1}\}| \leq C_{\kappa'}(|\lambda|^{1/2} + A)^{-2-|\kappa'|} \tag{3.29}$$
lem5.1_2

*for $\ell = 0, 1$ and any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$, $\lambda \in \Gamma_{\varepsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\backslash\{0\}$.*

*Proof.* Since we can prove (3.29) by using (3.28) with Leibniz rule and the Bell formula

$$\partial_{\xi'}^{\kappa'} f(g(\xi')) = \sum_{\ell=1}^{|\kappa'|} f^{(\ell)}(g(\xi')) \sum_{\substack{\kappa'_1+\cdots+\kappa'_\ell=\kappa' \\ |\kappa'_i|\geq 1}} \Gamma^{\kappa'}_{\kappa'_1,\ldots,\kappa'_\ell}(\partial_{\xi'}^{\kappa'_1} g(\xi'))\cdots(\partial_{\xi'}^{\kappa'_\ell} g(\xi')) \tag{3.30}$$
Bell

with $f(t) = 1/t$ and $g(\xi') = \det L$, it is sufficient to prove (3.28).

Before proving (3.28), setting $\widetilde{\delta} = \max(\delta_0, \tilde{\gamma}_1^\pm\tilde{\gamma}_2^\pm\lambda_0^{-1})$, we recall that $\delta \in \Sigma_\varepsilon$, $|\delta| \leq \widetilde{\delta}$ and

$$\left(\sin\frac{\varepsilon}{2}\right)(s\mu_1^\pm + \mu_2^\pm) \leq |s\mu_1^\pm + \mu_2^\pm + \delta| \leq s\mu_1^\pm + \mu_2^\pm + \widetilde{\delta} \tag{3.31}$$
base

with $s = 0, 1, 2$. In order to prove (3.28), we consider the three cases: (i) $R_1|\lambda|^{1/2} \leq A$, (ii) $R_2 A \leq |\lambda|^{1/2}$, (iii) $R_2^{-1}|\lambda|^{1/2} \leq A \leq R_1|\lambda|^{1/2}$ for large $R_1$ and $R_2$.

First we consider the case: $R_1|\lambda|^{1/2} \leq A$ with large $R_1 \geq 1$. In this case, we notice that there exists a very small positive constant $\delta_1$ such that

$$|(s_1\mu_1^{\pm} + s_2\mu_2^{\pm} + \delta)^{-1}\tilde{\gamma}_0^{\pm}\lambda A^{-2}| \leq (\sin(\varepsilon/2))^{-1}(s_1\mu_1^{\pm} + s_2\mu_2^{\pm})^{-1}\tilde{\gamma}_0^{\pm}R_1^{-2} \leq \delta_1$$

for $s_1, s_2 \in \mathbb{R}$. Therefore we have $A_{\pm} = A(1 + O(\delta_1))$ and $B_{\pm} = A(1 + O(\delta_1))$ as small $\delta_1$. Since

$$\frac{1}{A_{\pm}B_{\pm} - A^2} = \frac{(2\mu_1^{\pm} + \mu_2^{\pm} + \delta)\mu_1^{\pm}}{(3\mu_1^{\pm} + \mu_2^{\pm} + \delta)\tilde{\gamma}_0^{\pm}\lambda} P_{\pm}(\lambda, \xi'), \tag{3.32}$$

formP

$$P_{\pm}(\lambda, \xi') = \frac{A_{\pm}B_{\pm} + A^2}{(3\mu_1^{\pm} + \mu_2^{\pm} + \delta)^{-1}\tilde{\gamma}_0^{\pm}\lambda + A}, \tag{3.33}$$

we see that $P_{\pm}(\lambda, \xi') = 2 + O(\delta_1)$ and

$$\frac{1}{A_{\pm}B_{\pm} - A^2} = \frac{(2\mu_1^{\pm} + \mu_2^{\pm} + \delta)\mu_1^{\pm}}{(3\mu_1^{\pm} + \mu_2^{\pm} + \delta)\tilde{\gamma}_0^{\pm}\lambda}(2 + O(\delta_1)).$$

Therefore we can obtain

$$\begin{aligned}
L_{11}^{\pm} &= -\frac{\mu_1^{\pm}(2\mu_1^{\pm} + \mu_2^{\pm} + \delta)}{3\mu_1^{\pm} + \mu_2^{\pm} + \delta}A(2 + O(\delta_1)), \\
L_{12}^{\pm} &= \mp\frac{2(\mu_1^{\pm})^2}{3\mu_1^{\pm} + \mu_2^{\pm} + \delta}A^2(1 + O(\delta_1)), \\
L_{21}^{\pm} &= \mp\frac{2(\mu_1^{\pm})^2}{3\mu_1^{\pm} + \mu_2^{\pm} + \delta}(1 + O(\delta_1)), \\
L_{22}^{\pm} &= -\frac{2(2\mu_1^{\pm} + \mu_2^{\pm} + \delta)(\mu_1^{\pm})}{3\mu_1^{\pm} + \mu_2^{\pm} + \delta}A(1 + O(\delta_1)),
\end{aligned} \tag{3.34}$$

est_Lop_1

which imply that

$$\det L = \left(\mu_1^{+} + \frac{\mu_1^{-}(\mu_1^{-} + \mu_2^{-} + \delta)}{3\mu_1^{-} + \mu_2^{-} + \delta}\right)\left(\frac{\mu_1^{+}(\mu_1^{+} + \mu_2^{+} + \delta)}{3\mu_1^{+} + \mu_2^{+} + \delta} + \mu_1^{-}\right)A^2(4 + O(\delta_1)).$$

Taking the fact: $\mu_1^{\pm} > 0$, $\mu_1^{\pm} + \mu_2^{\pm} > 0$ and (3.31) into account, we see

$$\begin{aligned}
&\left|\mu_1^{\pm} + \frac{\mu_1^{\mp}(\mu_1^{\mp} + \mu_2^{\mp} + \delta)}{3\mu_1^{\mp} + \mu_2^{\mp} + \delta}\right| \\
&= \left|\frac{1}{3\mu_1^{\mp} + \mu_2^{\mp} + \delta}\left(\mu_1^{\pm}(3\mu_1^{\mp} + \mu_2^{\mp}) + \mu_1^{\mp}(\mu_1^{\mp} + \mu_2^{\mp}) + \delta(\mu_1^{\pm} + \mu_1^{\mp})\right)\right| \\
&\geq \left(\sin\frac{\varepsilon}{2}\right)\frac{1}{3\mu_1^{\mp} + \mu_2^{\mp}}\left(\mu_1^{\pm}(3\mu_1^{\mp} + \mu_2^{\mp}) + \mu_1^{\mp}(\mu_1^{\mp} + \mu_2^{\mp})\right) > 0.
\end{aligned}$$

Summing up, we can show that there exists a positive constant $\omega$ such that $|\det L| \geq \omega A^2$.

Secondly we shall consider the case $R_2 A \leq |\lambda|^{1/2}$ for large $R_2$. In this case, we notice that there exists a very small positive constant $\delta_2$ such that $|(s_1\mu_1^{\pm} + s_2\mu_2^{\pm} + \delta)\tilde{\gamma}_0^{\pm}\lambda^{-1}A^2| \leq \delta_2$. Therefore we see that

$$\begin{aligned}
A_{\pm} &= (2\mu_1^{\pm} + \mu_2^{\pm} + \delta)^{-1/2}(\tilde{\gamma}_0^{\pm})^{1/2}\lambda^{1/2}(1 + O(\delta_2)), \\
B_{\pm} &= (\mu_1^{\pm})^{-1/2}(\tilde{\gamma}_0^{\pm})^{1/2}\lambda^{1/2}(1 + O(\delta_2)) \\
P_{\pm}(\lambda, \xi') &= \frac{(3\mu_1^{\pm} + \mu_2^{\pm} + \delta)}{(2\mu_1^{\pm} + \mu_2^{\pm} + \delta)^{1/2}(\mu_1^{\pm})^{1/2}}(1 + O(\delta_2))
\end{aligned}$$

as small $\delta_2$. By these relation, we obtain

$$
\begin{aligned}
L_{11}^{\pm} &= -(\tilde{\gamma}_0^{\pm}\mu_1^{\pm})^{1/2}|\lambda|^{1/2}(1+O(\delta_2)),\\
L_{12}^{\pm} &= \mp\frac{(\mu_1^{\pm})^{1/2}(2\mu_1^{\pm}-\mu_2^{\pm}-\delta)}{2(\mu_1^{\pm})^{1/2}+(2\mu_1^{\pm}+\mu_2^{\pm}+\delta)^{1/2}}A^2(1+O(\delta_2)),\\
L_{21}^{\pm} &= \pm(\mu_1^{\pm})^{1/2}\left(\frac{-2\mu_1^{\pm}(\mu_1^{\pm}+\mu_2^{\pm}+\delta)}{(2\mu_1^{\pm}+\mu_2^{\pm}+\delta)+(\mu_1^{\pm})^{1/2}}+\frac{\mu_2^{\pm}+\delta}{(2\mu_1^{\pm}+\mu_2^{\pm}+\delta)^{1/2}}\right)(1+O(\delta_2)),\\
L_{22}^{\pm} &= -(\tilde{\gamma}_0^{\pm})^{1/2}|\lambda|^{1/2}(2\mu_1^{\pm}+\mu_2^{\pm}+\delta)^{1/2}(\mu_1^{\pm})^{-1/2}(1+O(\delta_2)),
\end{aligned}
\tag{3.35}
$$

est_Lop_2

which imply that

$$
|\det L| \tag{3.36}
$$

$$
=\left|\left((\tilde{\gamma}_0^{+}\mu_1^{+})^{1/2}+(\tilde{\gamma}_0^{-}\mu_1^{-})^{1/2}\right)\left(\frac{2\mu_1^{+}+\mu_2^{+}+\delta}{\tilde{\gamma}_0^{+}\mu_1^{+}}+\frac{2\mu_1^{-}+\mu_2^{-}+\delta}{\tilde{\gamma}_0^{-}\mu_1^{-}}\right)\right||\lambda|(1+O(\delta_2)). \tag{3.37}
$$

Since $\mu_1^{\pm}>0, \mu_1^{\pm}+\mu_2^{\pm}>0$ and (3.31), there exists a positive constant $\omega$ such that $|\det L|\geq\omega|\lambda|$.

Thirdly, we consider the case $R_2^{-1}|\lambda|^{1/2}\leq A\leq R_1|\lambda|^{1/2}$. Set $\widetilde{\lambda}=\lambda/(|\lambda|^{1/2}+A)^2$ and

$$
\widetilde{A}=\frac{A}{|\lambda|^{1/2}+A},\qquad \widetilde{A_{\pm}}=\sqrt{\frac{\tilde{\gamma}_0^{\pm}}{2\mu_1^{\pm}+\mu_2^{\pm}+\delta}\widetilde{\lambda}+\widetilde{A}^2},\qquad \widetilde{B_{\pm}}=\sqrt{\frac{\tilde{\gamma}_0^{\pm}}{\mu_1^{\pm}}\widetilde{\lambda}+\widetilde{A}^2}
$$

and

$$
\begin{aligned}
D(R_1,R_2)=\{(\widetilde{\lambda},\widetilde{A})\mid\\
&(1+R_1)^{-2}\leq|\widetilde{\lambda}|\leq R_2^2(1+R_2)^2,(1+R_2)^{-1}\leq\widetilde{A}\leq R_1(1+R_1)^{-1}\}.
\end{aligned}
$$

We remark $(\widetilde{\lambda},\widetilde{A})\in D(R_1,R_2)$ if $(\lambda,\xi')$ satisfies the condition $R_2^{-1}|\lambda|^{1/2}\leq A\leq R_1|\lambda|^{1/2}$. We also define $\widetilde{L}_{ij}$ by replacing $A_{\pm}, A$ and $B_{\pm}$ by $\widetilde{A_{\pm}},\widetilde{A}$ and $\widetilde{B}$, respectively. And we set $\det\widetilde{L}=\widetilde{L}_{11}\widetilde{L}_{22}-\widetilde{L}_{12}\widetilde{L}_{21}$ and then we have $\det L=(|\lambda|^{1/2}+A)^2\det\widetilde{L}$.

First we shall prove that $\det\widetilde{L}\neq 0$ provided that $(\widetilde{\lambda},\widetilde{A})\in D(R_1,R_2)$, $\widetilde{\lambda}\in\Sigma_\varepsilon$ by contradiction. To this end, we assume that $\det\widetilde{L}=0$, namely $\det L=0$. In this case, in view of (3.21) we may assume that there exists a $\vec{w}_{\pm}(x_N)=(w_{\pm,1}(x_N),\dots,w_{\pm,N}(x_N))\neq\vec{0}$ satisfying (3.8) and (3.17)-(3.19) with $\widehat{h}_J(0)=0$ and $\widehat{k}_J(0)=0$, that is $\vec{w}_{\pm}$ satisfy the following homogeneous equations:

$$
\begin{aligned}
&\tilde{\gamma}_0^{\pm}\lambda w_{\pm,j}-\mu_1^{\pm}\sum_{\ell=1}^{N-1}i\xi_\ell(i\xi_j w_{\pm,\ell}+i\xi_\ell w_{\pm,j})\\
&\qquad -\mu_1^{\pm}D_N(i\xi_j w_{\pm,N}+D_N w_{\pm,j})-(\mu_2^{\pm}+\delta)i\xi_j(i\xi'\cdot w'_{\pm}+D_N w_{\pm,N})=0,
\end{aligned}
\tag{3.38}
$$

w_1

$$
\begin{aligned}
&\tilde{\gamma}_0^{\pm}\lambda w_{\pm,N}-\mu_1^{\pm}\sum_{\ell=1}^{N-1}i\xi_\ell(D_N w_{\pm,\ell}+i\xi_\ell w_{\pm,N})\\
&\qquad -2\mu_1^{\pm}D_N^2 w_{\pm,N}-(\mu_2^{\pm}+\delta)D_N(i\xi'\cdot w'_{\pm}+D_N w_{\pm,N})=0,
\end{aligned}
\tag{3.39}
$$

w_2

$$
\mu_1^{+}(D_N w_{+,j}+i\xi_j w_{+,N})|_{x_N=0}-\mu_1^{-}(D_N w_{-,j}+i\xi_j w_{-,N})|_{x_N=0}=0, \tag{3.40}
$$

w_3

$$
\begin{aligned}
&2\mu_1^{-}D_N w_{+,N}+(\mu_2^{+}+\delta)(i\xi'\cdot w'_{+}+D_N w_{+,N})|_{x_N=0}\\
&\qquad\qquad -(2\mu_1^{-}D_N w_{-,N}+(\mu_2^{-}+\delta)(i\xi'\cdot w'_{-}+D_N w_{-,N}))
\end{aligned}
\tag{3.41}
$$

w_4

Let

$$
(a,b)_{\pm}=\pm\int_0^{\pm\infty}a(x_N)\overline{b(x_N)}dx_N,\qquad \|a\|_{\pm}=\sqrt{(a,a)_{\pm}}.
$$

Multipling (3.38) by $\overline{w_{\pm,j}}$ and (3.39) by $\overline{w_{\pm,N}}$ and by integration by parts, we obtain

$$
\begin{aligned}
&\tilde{\gamma}_0^\pm\lambda\|w_{\pm,j}\|_\pm^2 + \mu_1^\pm\sum_{\ell=1}^{N-1}\big((i\xi_\ell w_{\pm,\ell}, i\xi_j w_{\pm,j})_\pm + \|i\xi_\ell w_{\pm,j}\|_\pm^2\big)\\
&\quad + \mu_1^\pm(i\xi_j w_{\pm,N}, D_N w_{\pm,j})_\pm + \mu_1^\pm\|D_N w_{\pm,j}\|_\pm^2\\
&\qquad + (\mu_2^\pm+\delta)\big((i\xi'\cdot w_\pm', i\xi_j w_{\pm,j})_\pm + (D_N w_{\pm,N}, i\xi_j w_{\pm,j})_\pm\big) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{\gamma}_0^\pm\lambda\|w_{\pm,N}\|_\pm^2 + \mu_1^\pm\sum_{\ell=1}^{N-1}\big((D_N w_{\pm,\ell}, i\xi_\ell w_{\pm,N})_\pm + \|i\xi_\ell w_{\pm,N}\|_\pm^2\big) + 2\mu_1^\pm\|D_N w_{\pm,N}\|_\pm^2\\
&\qquad + (\mu_2^\pm+\delta)\big((i\xi'\cdot w_\pm', D_N w_{\pm,N})_\pm + \|D_N w_{\pm,N}\|_\pm^2\big) = 0,
\end{aligned}
$$

where we set $i\xi'\cdot w_\pm' = \sum_{\ell=1}^{N-1} i\xi_\ell w_{\pm,\ell}$. Summing up, we see

$$
\begin{aligned}
&\tilde{\gamma}_0^\pm\lambda\sum_{j=1}^N\|w_{\pm,j}\|_\pm^2 + \mu_1^\pm\|i\xi'\cdot w_\pm'\|_\pm^2 + \mu_1^\pm\sum_{\ell,j=1}^{N-1}\|i\xi_\ell w_{\pm,j}\|_\pm^2\\
&+ \mu_1^\pm\sum_{j=1}^{N-1}(i\xi_\ell w_{\pm,N}, D_N w_{\pm,j})_\pm + \mu_1^\pm\sum_{j=1}^{N-1}\|D_N w_{\pm,j}\|_\pm^2\\
&\quad + \mu_1^\pm\sum_{\ell=1}^{N-1}\big((D_N w_{\pm,\ell}, i\xi_\ell w_{\pm,N})_\pm + \|i\xi_\ell w_{\pm,N}\|_\pm^2\big) + 2\mu_1^\pm\|D_N w_{\pm,N}\|_\pm^2\\
&\qquad + (\mu_2^\pm+\delta)\|i\xi'\cdot w_\pm'\|_\pm^2 + (\mu_2^\pm+\delta)(i\xi'\cdot w_\pm', D_N w_{\pm,N})_\pm\\
&\qquad\quad + (\mu_2^\pm+\delta)(D_N w_{\pm,N}, i\xi'\cdot w_\pm')_\pm + (\mu_2^\pm+\delta)\|D_N w_{\pm,N}\|_\pm^2 = 0.
\end{aligned}
$$

Since

$$
\begin{aligned}
&(i\xi_\ell w_{\pm,N}, D_N w_{\pm,j})_\pm + \|D_N w_{\pm,j}\|_\pm^2 + (D_N w_{\pm,j}, i\xi_j w_{\pm,N})_\pm + \|i\xi_j w_{\pm,N}\|_\pm^2\\
&= \|i\xi_j w_{\pm,j} + D_N w_{\pm,j}\|_\pm^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\|i\xi'\cdot w_\pm'\|_\pm^2 + (i\xi'\cdot w_\pm', D_N w_{\pm,N})_\pm + (D_N w_{\pm,N}, i\xi'\cdot w_\pm')_\pm + \|D_N w_{\pm,N}\|_\pm^2\\
&= \|i\xi'\cdot w_\pm' + D_N w_{\pm,N}\|_\pm^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\tilde{\gamma}_0^\pm\lambda\sum_{j=1}^N\|w_{\pm,j}\|_\pm^2 + \mu_1^\pm\|i\xi'\cdot w_\pm'\|_\pm^2 + \mu_1^\pm\sum_{\ell,j=1}^{N-1}\|i\xi_\ell w_{\pm,j}\|_\pm^2 + 2\mu_1^\pm\|D_N w_{\pm,N}\|_\pm^2\\
&+ \mu_1^\pm\sum_{j=1}^{N-1}\|i\xi_j w_{\pm,N} + D_N w_{\pm,j}\|_\pm^2 + (\mu_2^\pm+\delta)\|i\xi'\cdot w_\pm' + D_N w_{\pm,N}\|_\pm^2 = 0.
\end{aligned}
\tag{3.42}
$$

w_est

Taking the real part and the imaginary part in (3.42) and using the relation:

$$
\begin{aligned}
\|i\xi'\cdot w_\pm'\|_\pm^2 + \sum_{\ell,j=1}^{N-1}\|i\xi_\ell w_{\pm,j}\|_\pm^2 + 2\|D_N w_{\pm,N}\|_\pm^2 &\geq 2\big(\|i\xi'\cdot w_\pm'\|_\pm^2 + \|D_N w_{\pm,N}\|_\pm^2\big)\\
&\geq \|i\xi'\cdot w_\pm' + D_N w_{\pm,N}\|_\pm^2,
\end{aligned}
\tag{3.43}
$$

w_est2

we have

$$\tilde{\gamma}_0^{\pm}(\mathrm{Im}\lambda)K_1 + (\mathrm{Im}\delta)K_2 = 0, \tag{3.44}$$

$$\tilde{\gamma}_0^{\pm}(\mathrm{Re}\lambda)K_1 + (\mu_1^{\pm} + \mu_2^{\pm} + \mathrm{Re}\ \delta)K_2 + \mu_1^{\pm}\sum_{j=1}^{N-1}\|i\xi_j w_{\pm,j} + D_N w_{\pm,j}\|_{\pm}^2 \le 0, \tag{3.45}$$

where we set $K_1 = \sum_{j=1}^{N}\|w_{\pm,j}\|_{\pm}^2$ and $K_2 = \|i\xi'\cdot w'_{\pm} + D_N w_{\pm,N}\|_{\pm}^2$.

First we consider the case $\delta = 0$. In the case where $\mathrm{Im}\ \lambda \neq 0$ and where $\mathrm{Im}\ \lambda = 0$ and $\mathrm{Re}\ \lambda > 0$, we have $K_1 = 0$, that is $\vec{w}_{\pm} = \vec{0}$. In the case where $\mathrm{Im}\ \lambda = 0$ and $\mathrm{Re}\ \lambda \le 0$, it follows from $\lambda \in \Sigma_\varepsilon$ that $\lambda = 0$. Choosing $\varepsilon' > 0$ as $\mu_1^{\pm} + \mu_2^{\pm} - \varepsilon' > 0$, by (3.42) with $\lambda = \delta = 0$ and (3.43), we have

$$\mu_1^{\pm}\sum_{j=1}^{N-1}\|i\xi_j w_{\pm,N} + D_N w_{\pm,j}\|_{\pm}^2 + \varepsilon'\left(\|i\xi'\cdot w'_{\pm}\|_{\pm}^2 + \sum_{\ell,j=1}^{N-1}\|i\xi_\ell w_{\pm,j}\|_{\pm}^2 + 2\|D_N w_{\pm,N}\|_{\pm}^2\right) \le 0.$$

Therefore we see that $\|D_N w_{\pm,N}\|_{\pm} = 0$ and $\|D_N w_{\pm,j} + i\xi_j w_{\pm,N}\|_{\pm} = 0$. Since $w_{\pm,J}(x_N) \to 0$ as $\pm x_N \to \infty$ for $J = 1,\dots,N$, we obtain $\vec{w}_{\pm} = \vec{0}$. Summing up, in the case $\delta = 0$, we obtain $\vec{w}_{\pm} = \vec{0}$, which contradicts to $\vec{w}_{\pm} \neq \vec{0}$. Thus we have $\det \widetilde{L} \neq 0$ when $\delta = 0$, which implies that

$$c_1 = \inf\{|\det \widetilde{L}| \mid (\widetilde{\lambda}, \widetilde{A}) \in D(R_1, R_2), \widetilde{\lambda} \in \Sigma_\varepsilon, \delta = 0\} > 0.$$

Since $\widetilde{A} = \sqrt{(2\mu_1^{\pm} + \mu_2^{\pm} + \delta)^{-1}\tilde{\gamma}_0^{\pm}\widetilde{\lambda} + \widetilde{A}^2} + O(|\delta|)$, there exists a $\delta_5 > 0$ such that

$$\inf\{|\det \widetilde{L}| \mid (\widetilde{\lambda}, \widetilde{A}) \in D(R_1, R_2), \widetilde{\lambda} \in \Sigma_\varepsilon, |\delta| \le \delta_5\} \ge c_1/2,$$

which implies that $|\det L| \ge \omega(|\lambda|^{1/2} + A)^2$ with some positive number $\omega$ provide that $R_2|\lambda|^{1/2} \le A \le R_1^{-1}|\lambda|^{1/2}$ and $\lambda \in \mathbb{C}$ with $|\delta| \le \delta_5$.

Finally, we consider the case where $\delta_5 \le |\delta| \le \widetilde{\delta}$. First we consider the case (C1), that is $\delta = \tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}/\lambda$. Since $\mathrm{Re}\ \delta = \tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}\mathrm{Re}\lambda/|\lambda|^2$ and $\mathrm{Im}\ \delta = -\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}\mathrm{Im}\lambda/|\lambda|^2$, by (3.44) and (3.45), we have

$$(\mathrm{Im}\lambda)\left(\tilde{\gamma}_0^{\pm}K_1 - \frac{\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}}{|\lambda|^2}K_2\right) = 0,$$

$$\tilde{\gamma}_0^{\pm}(\mathrm{Re}\lambda)K_1 + \left(\mu_1^{\pm} + \mu_2^{\pm} + \frac{\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}(\mathrm{Re}\lambda)}{|\lambda|^2}\right)K_2 \le 0. \tag{3.46}$$

In the case $\mathrm{Im}\lambda = 0$, it follows that $\mathrm{Re}\ \lambda \ge \lambda_0$ from $\lambda \in \Sigma_{\varepsilon,\lambda_0}$. Since $\mu_1^{\pm} + \mu_2^{\pm} > 0$ and $\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm} > 0$, we see $K_1 = K_2 = 0$, which implies $\vec{w}_{\pm} = \vec{0}$. In the case $\mathrm{Im}\ \lambda \neq 0$, by (3.46), we have $\tilde{\gamma}_0^{\pm}K_1 = \tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}K_2/|\lambda|^2$ and

$$\left(\frac{\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}}{|\lambda|^2}(2\mathrm{Re}\ \lambda) + (\mu_1^{\pm} + \mu_2^{\pm})\right)K_2 \le 0.$$

Since

$$2\mathrm{Re}\lambda\frac{\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}}{|\lambda|^2} + \mu_1^{\pm} + \mu_2^{\pm} = \frac{\mu_1^{\pm} + \mu_2^{\pm}}{|\lambda|^2}\left(\left(\mathrm{Re}\lambda + \frac{\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}}{\mu_1^{\pm} + \mu_2^{\pm}}\right)^2 + (\mathrm{Im}\lambda)^2 - \left(\frac{\tilde{\gamma}_1^{\pm}\tilde{\gamma}_2^{\pm}}{\mu_1^{\pm} + \mu_2^{\pm}}\right)\right),$$

the condition $\lambda \in K_{\varepsilon,\lambda_0}$ implies $K_1 = K_2 = 0$, namely $\vec{w}_{\pm} = \vec{0}$ by (3.44), which contradict to $\vec{w}_{\pm} \neq \vec{0}$. Therefore we see that there exists a positive constant $\omega$ such that $|\det \widetilde{L}| \ge \omega$. Therefore we obtain $|\det L| \ge \omega(|\lambda|^{1/2} + A)^2$, which implies Lemma 3.1.

Secondly, we consider the case where $\delta \in \Sigma_\varepsilon$ and $\lambda$ satisfying that $|\delta| \leq \delta_0$, $\mathrm{Re}\ \delta \leq 0, |\lambda| \geq \lambda_0$ and $\mathrm{Re}\ \lambda \geq |\mathrm{Re}\ \delta/\mathrm{Im}\ \delta||\mathrm{Im}\ \lambda|$. The case (C2) is included in this case. We prove that $\det \widetilde{L} \neq 0$ provided that $(\widetilde{\lambda}, \widetilde{A}) \in D(R_1, R_2)$ and $\mathrm{Re}\ \widetilde{\lambda} \geq |\mathrm{Re}\delta/\mathrm{Im}\delta||\mathrm{Im}\widetilde{\lambda}|$ by contradiction. Asuume that $\det \widetilde{L} = 0$, that is $\det L = 0$. Thus by the argument above, we have

$$\tilde{\gamma}_0^\pm(\mathrm{Im}\lambda)K_1 + (\mathrm{Im}\ \delta)K_2 = 0, \qquad \tilde{\gamma}_0^\pm(\mathrm{Re}\lambda)K_1 + \left(\mu_1^\pm + \mu_2^\pm + \mathrm{Re}\delta\right)K_2 \leq 0. \tag{3.47}$$

When $\mathrm{Im}\ \lambda = 0$ and $\mathrm{Im}\ \delta \neq 0$, we have $K_2 = 0$, so that $(\mathrm{Re}\ \lambda)K_1 \leq 0$. Since $\mathrm{Re}\ \lambda = \lambda_0 > 0$, we have $K_1 = 0$. Thus we have $\vec{w}_\pm = \vec{0}$. When $\mathrm{Im}\lambda \neq 0$, $\mathrm{Im}\ \delta \neq 0$ and $(\mathrm{Im}\ \lambda)(\mathrm{Im}\ \delta) > 0$, we have $K_1 = K_2 = 0$. On the other hand, when $\mathrm{Im}\lambda \neq 0$, $\mathrm{Im}\ \delta \neq 0$ and $(\mathrm{Im}\ \lambda)(\mathrm{Im}\ \delta) < 0$, we have $K_1 = |\mathrm{Im}\delta/\mathrm{Im}\lambda|K_2$ by the first formula of (3.47), so that it follows from the second formula of (3.47) that

$$\left[\tilde{\gamma}_0^\pm(\mathrm{Re}\ \lambda)\left|\frac{\mathrm{Im}\ \delta}{\mathrm{Im}\ \lambda}\right| + (\mu_1^\pm + \mu_2^\pm + \mathrm{Re}\ \delta)\right] K_2 \leq 0.$$

Since $\mathrm{Re}\ \delta = -|\mathrm{Re}\ \delta|, \mathrm{Re}\ \lambda \geq |\mathrm{Re}\delta/\mathrm{Im}\delta||\mathrm{Im}\lambda|$, we see $(\mathrm{Re}\ \lambda)\,|\mathrm{Im}\ \delta/\mathrm{Im}\ \lambda| + (\mu_1^\pm + \mu_2^\pm + \mathrm{Re}\ \delta) > 0$, which implies $K_1 = K_2 = 0$. Therefore we obtain $\vec{w}_\pm = \vec{0}$, which contradicts to $\vec{w}_\pm \neq \vec{0}$. Therefore we obtain $\det \widetilde{L} \neq 0$ and we see that there exists a positive constant $\omega$ such that $|\det L| \geq \omega(|\lambda|^{1/2} + A)^2$. In a similar way, we can obtain the case where $\delta \in \Sigma_\varepsilon$, $|\delta| \leq \delta_0$, $\mathrm{Re}\ \delta \geq 0, |\lambda| \geq \lambda_0$ and $\mathrm{Re}\ \lambda \geq \lambda_0|\mathrm{Im}\ \lambda|$. This completes the proof of Lemma 3.1. □

## 3.3 Proof of Theorem 2.8

In this section, we shall show the proof of Theorem 2.8. In order to prove Theorem 2.8, we use the following lemmas which is proven by Götz and Shibata [7].

**Lemma 3.2.** *Let $\Lambda$ be a domain in $\mathbb{C}$ and set $\tilde{\Lambda} = \Lambda \times (\mathbb{R}^{N-1} \setminus \{0\})$. Let $m(\lambda, \xi')$ be multipliers defined on $\tilde{\Lambda}$ such that*

$$|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell m(\lambda, \xi')\}| \leq C_{\kappa'}(|\lambda|^{1/2} + A)^{-2-|\kappa'|}, \tag{3.48}$$

*for any $\kappa' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \tilde{\Lambda}$. Let $K_i^\pm$ $(i = 1, 2)$ be operators defined by*

$$K_1^\pm(\lambda)g = \pm\int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi')B_\pm e^{\mp B_\pm(x_N + y_N)}\hat{g}(\xi', y_N)](x')\,dy_N,$$
$$K_2^\pm(\lambda)g = \pm\int_0^{\pm\infty} \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi')B_\pm^2 M_\pm(x_N + y_N)\hat{g}(\xi', y_N)](x')\,dy_N.$$

*Then, there exists a constant $C$ such that*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_\pm), L_q(\mathbb{R}^N_\pm)^{2+N+N^2})}(\{(\tau\partial_\tau)^\ell G_\lambda K_i^\pm(\lambda) \mid \lambda \in \Lambda\}) \leq C \quad (\ell = 0, 1, \ \ i = 1, 2),$$

*where $G_\lambda$ is an operator defined by $G_\lambda u = (\lambda u, \gamma u, \lambda^{1/2}\nabla u, \nabla^2 u)$.*

Applying Lemma 3.2 to each term in solution formula (3.27), we shall show the existence of $\mathcal{R}$-boundedness solution operator. In order to check that the each multipler in (3.27) satisfy the condition of Lemma 3.2, we use the following elementary property (see Kubo, Shibata and Soga [8] for detail).

**Lemma 3.3.** *Let $0 < \varepsilon < \pi/2$, $\lambda_0 > 0$ and $s \geq 0$.*

(1) *For any $\lambda \in \Sigma_\varepsilon$ and $\alpha, \beta > 0$, we have $|\alpha\lambda + \beta| \geq (\sin(\varepsilon/2))(\alpha|\lambda| + \beta)$.*

(2) *There exists a number $\sigma \in (0, \pi)$ depending on $s, \mu_1^\pm, \mu_2^\pm, \lambda_0$ and $\varepsilon$ such that $(s\mu_1^\pm + \mu_2^\pm + \delta)^{-1}\lambda \in \Sigma_\sigma$ for any $\lambda \in \Gamma_{\varepsilon,\lambda_0}$.*

(3) *There exists constant $\delta_3$ and $\delta_4$ depending on $s, \mu_1^\pm, \mu_2^\pm, \lambda_0$ and $\varepsilon$ such that*

$$\delta_3(|\lambda| + |\xi|^2) \leq |(s\mu_1^\pm + \mu_2^\pm + \delta)^{-1}\lambda + |\xi|^2| \leq \delta_4(|\lambda| + |\xi|^2)$$

*for any $\lambda \in \Gamma_{\varepsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\backslash\{0\}$.*

First we start to estimate $M_1 = A_\pm, B_\pm, (A_\pm + B_\pm)$ and $(A + B_\pm)$. To this end, we use the following estimates which follows from Lemma 3.3

$$c(|\lambda|^{1/2} + A) \le \mathrm{Re} M_1 \le |M_1| \le c'(|\lambda|^{1/2} + A) \tag{3.49}$$

for any $\lambda \in \Gamma_{\varepsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$ with some positive constant $c$ and $c'$ depending on $\mu_1^\pm, \mu_2^\pm, \varepsilon, \lambda_0$ and $\delta_0$. By using (3.49) and Bell's formula (3.30) for the derivatives of the composite function of $f(t)$ and $t = g(\xi')$ and suitable coefficients $\Gamma^{\kappa'}_{\kappa'_1,\dots,\kappa'_\ell}$, we obtain

$$|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell (M_1)^s\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{s-|\kappa'|} \tag{3.50}$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$, $\lambda \in \Gamma_{\varepsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$.

Second, we shall estimate $P_\pm(\lambda, \xi')$ defined in (3.32). By Lemma 3.3, (3.30) and (3.50), we have

$$|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell (A_\pm B_\pm + A^2)\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{2-|\kappa'|}, \tag{3.51}$$

$$|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell (3\mu_1^\mp + \mu_2^\pm + \delta_\lambda^\pm)^{-1}\lambda + A)^{-1}\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-2-|\kappa'|} \tag{3.52}$$

for $\ell = 0, 1$ and any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$, $\lambda \in \Lambda_{\varepsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$, so that by the Leibniz rule we have

$$|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell P_\pm(\lambda, \xi')\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-|\kappa'|} \tag{3.53}$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$, $\lambda \in \Lambda_{\varepsilon,\lambda_0}$ and $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$.

By (3.50) and (3.53) with (3.20), we see

$$\begin{aligned}
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell L_{11}^\pm\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{1-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell L_{12}^\pm\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{2-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell L_{21}^\pm\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell L_{22}^\pm\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{1-|\kappa'|}.
\end{aligned} \tag{3.54}$$

Moreover by (3.23) and (3.24), we have

$$\begin{aligned}
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell M_{1,J}^h\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{2-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell M_{1,J}^k\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{3-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell M_{2,J}^h\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{1-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell M_{2,J}^k\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{2-|\kappa'|}.
\end{aligned} \tag{3.55}$$

By (3.26) with (3.29), (3.53) and (3.55), we have

$$\begin{aligned}
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell Q_{\pm,\tilde\ell}^{J,1}\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-2-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell R_{\pm,\tilde\ell}^{J,1}\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-2-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell Q_{\pm,\tilde\ell}^{J,2}\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-2-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell R_{\pm,\tilde\ell}^{J,2}\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-1-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell Q_{\pm,\tilde\ell}^{J,3}\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-2-|\kappa'|},\\
&|\partial_{\xi'}^{\kappa'}\{(\tau\partial_\tau)^\ell R_{\pm,\tilde\ell}^{J,3}\}| \le C_{\kappa'}(|\lambda|^{1/2} + A)^{-1-|\kappa'|}.
\end{aligned} \tag{3.56}$$

Therefore the following multipliers satisfy the condition (3.48) in Lemma 3.2:

$$\begin{aligned}
&Q_{\pm,\ell}^{J,1},\ \frac{\lambda^{1/2}Q_{\pm,\ell}^{J,1}}{B_\pm},\ \frac{i\xi_m Q_{\pm,\ell}^{J,1}}{B_\pm},\ Q_{\pm,\ell}^{J,2},\ \frac{\lambda^{1/2}Q_{\pm,\ell}^{J,2}}{B_\pm},\ \frac{i\xi_m Q_{\pm,\ell}^{J,2}}{B_\pm},\ Q_{\pm,\ell}^{J,3},\ \frac{\lambda^{1/2}Q_{\pm,\ell}^{J,3}}{B_\pm},\ \frac{i\xi_m Q_{\pm,\ell}^{J,3}}{B_\pm},\\
&\frac{\lambda^{1/2}R_{\pm,\ell}^{J,1}}{B_\pm^2},\ \frac{i\xi_m R_{\pm,\ell}^{J,1}}{B_\pm^2},\ \frac{R_{\pm,\ell}^{J,1}}{B_\pm},\ \frac{\lambda^{1/2}R_{\pm,\ell}^{J,2}}{B_\pm^2},\ \frac{i\xi_m R_{\pm,\ell}^{J,2}}{B_\pm^2},\ \frac{\lambda^{1/2}R_{\pm,\ell}^{J,3}}{B_\pm^2},\ \frac{i\xi_m R_{\pm,\ell}^{J,3}}{B_\pm^2},\ \frac{R_{\pm,\ell}^{J,3}}{B_\pm}.
\end{aligned}$$

Therefore applying Lemma 3.2 to $\mathcal{T}_{\pm,J}(\lambda)$, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Z}(\mathbb{R}^N), L_q(\mathbb{R}^N_\pm)^{2+N+N^2})}(\{(\tau\partial_\tau)^\ell G_\lambda \mathcal{T}_{\pm,J}(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq C \; (\ell = 0, 1), \tag{3.57}$$

where $\mathcal{Z}(\mathbb{R}^N) = \{(F_2, F_3, F_4, F_5, F_6) \mid F_2, F_4 \in L_q(\mathbb{R}^N), F_3, F_5 \in L_q(\mathbb{R}^N)^{N^2}, \; F_6 \in L_q(\mathbb{R}^N)^{N^3}\}$. By Lemma 2.5 with (3.57) and (3.2), we can prove Theorem 2.8. This completes the proof of Theorem 2.8.

# 4 Proof of Theorem 2.6

We shall prove Theorem 2.6. First step is to show the existence of $\mathcal{R}$-bounded solution operator to a bent space problem by using the change of variable and Theorem 2.8.

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be two bijections of $C^1$ class and let $\Phi^{-1}$ be their inverse maps. We set $\nabla\Phi_\pm = \mathcal{A} + \mathcal{B}(x)$ and $\nabla\Phi_\pm = \mathcal{A}_{-1} + \mathcal{B}_{-1}(x)$ and assume that $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthogonal matrices with constant coefficients and $\mathcal{B}(x)$ and $\mathcal{B}_{-1}(x)$ are matrices of function in $W^1_r(\mathbb{R}^N)$ with $N < r < \infty$ satisfying

$$\|(\mathcal{B}, \mathcal{B}_{-1})\|_{L_\infty(\mathbb{R}^N)} \leq M_1, \qquad \|\nabla(\mathcal{B}, \mathcal{B}_{-1})\|_{L_r(\mathbb{R}^N)} \leq M_2. \tag{4.1}$$

Set $\Omega_\pm = \Phi(\mathbb{R}^N_\pm)$ and $\Gamma = \Phi(\mathbb{R}^N_0)$.

Let $\gamma_0^\pm(x)$ and $\gamma_3^\pm(x)$ be real-valued functions defined on $\mathbb{R}^N$ satisfying the following conditions:

$$\frac{1}{2}\rho_{0,\pm} \leq \gamma_0^\pm(x) \leq 2\rho_{0,\pm}, \quad 0 \leq \gamma_3^\pm(x) \leq \rho_{2,\pm}^2, \quad \|\gamma_\ell^\pm - \tilde{\gamma}_\ell^\pm\|_{L_\infty(\Omega_\pm)} \leq M_1, \quad \|\nabla\gamma_\ell^\pm\|_{L_r(\Omega_\pm)} \leq C_{M_2} \tag{4.2}$$

for $\ell = 0, 3$, where $\tilde{\gamma}_\ell^\pm$ $(\ell = 0, 3)$ are some constants with $\rho_{0,\pm}/2 < \tilde{\gamma}_0^\pm < 2\rho_{0,\pm}$ and $0 \leq \tilde{\gamma}_3^\pm \leq \rho_{2,\pm}^2$. In the same way as in [5], we can prove the existence of $\mathcal{R}$-bounded solution operator to (RP1) as follows:

**Theorem 4.1.** *Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. Set*

$$\begin{aligned} Y_q(\Omega_\pm) &= \{(\vec{g}_+, \vec{g}_-, \vec{h}, \vec{k}) \mid \vec{g}_\pm \in L_q(\Omega_\pm)^N, \; \vec{h} \in W^1_q(\Omega)^N, \; \vec{k} \in W^2_q(\Omega)^N\}, \\ \mathcal{Y}_q(\Omega_\pm) &= \{(F_{1+}, F_{1-}, F_2, F_3, F_4, F_5, F_6) \mid \\ &\qquad F_{1\pm} \in L_q(\Omega_\pm)^N, \; F_2, F_5 \in L_q(\Omega)^{N^2}, \; F_3, F_6 \in L_q(\Omega)^N, \; F_4 \in L_q(\Omega)^{N^3}\}. \end{aligned}$$

*Assume that $\mu_1^\pm > 0$, $\mu_1^\pm + \mu_2^\pm > 0$ and $\gamma_0^\pm$ and $\gamma_3^\pm$ satisfy the condition (4.2). Then there exist constant $M_1 \in (0,1)$ and $\lambda_0 \geq 1$, and operator families $\mathcal{A}_\pm(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q(\Omega_\pm), W^2_q(\Omega_\pm)^N)$ such that for any $(\vec{g}_+, \vec{g}_-, \vec{h}, \vec{h}_-) \in Y_q(\Omega_\pm)$ and $\lambda \in \Gamma_{\varepsilon,\lambda_0}$, $\vec{u}_\pm = \mathcal{A}_\pm(\lambda)(\vec{g}, \lambda^{1/2}\vec{h}, \nabla\vec{h}, \lambda^{1/2}\vec{h}_-, \nabla\vec{h}_-)$ are unique solution to (RP1). Moreover $\mathcal{A}_\pm(\lambda)$ possess the estimate*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega_\pm), L_q(\Omega_\pm))}(\{(\tau\partial_\tau)^\ell G_\lambda \mathcal{A}_\pm(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \leq C \qquad (\ell = 0, 1),$$

*where $G_\lambda u = (\lambda u, \gamma u, \lambda^{1/2}\nabla u, \nabla^2 u)$ and $\lambda = \gamma + i\tau$.*

Finally we give the proof of Theorem 2.6. We start with introducing the following proposition concerning some important properties of a uniform $W^{2-1/r}_r$ domain that was proved in Enomoto and Shibata [6]. This proposition will be used to construct a solution operator in $\Omega$.

**Proposition 4.2.** *Let $N < r < \infty$ and let $\Omega_\pm$ be uniform $W^{2-1/r}_r$ domains in $\mathbb{R}^N$. Let $M_1$ be the number given in (4.1). Then, there exist constants $M_2 > 0$, $0 < d^0, d^1_\pm, d^2_\pm < 1$, at most countably many $N$-vector of functions $\Phi^0_j, \Phi^1_{j,\pm} \in W^2_r(\mathbb{R}^N)^N$ and points $x^0_j \in \Gamma$, $x^1_{j,\pm} \in \Gamma_\pm$, $x^2_{j,\pm} \in \Omega_\pm$ such that the following assertions hold.*

(i) *The maps: $\mathbb{R}^N \ni x \mapsto \Phi^0_j(x) \in \mathbb{R}^N$ and $\mathbb{R}^N \ni x \mapsto \Phi^1_{j,\pm}(x) \in \mathbb{R}^N$ $(j \in \mathbb{N})$ are bijective.*

(ii) *The following relations hold: $B_{d^2_\pm}(x^2_{j,\pm}) \subset \Omega_\pm$ $(j \in \mathbb{N})$,*

$$\Omega_\pm = \left(\bigcup_{j=1}^\infty (\Phi^0_j(\mathbb{R}^N_\pm) \cap B_{d^0}(x^0_j))\right) \cup \left(\bigcup_{j=1}^\infty (\Phi^1_{j,\pm}(\mathbb{R}^N_\pm) \cap B_{d^1_\pm}(x^1_{j,\pm}))\right) \cup \left(\bigcup_{j=1}^\infty B_{d^2_\pm}(x^2_{j,\pm})\right)$$

*and*

$$\begin{aligned}
&\Phi^0_j(\mathbb{R}^N_\pm)\cap B_{d^0}(x^0_j)=\Omega_\pm\cap B_{d^0}(x^0_j), && \Phi^0_j(\mathbb{R}^N_0)\cap B_{d^0}(x^0_j)=\Gamma\cap B_{d^0}(x^0_j),\\
&\Phi^1_{j,\pm}(\mathbb{R}^N_\pm)\cap B_{d^1_\pm}(x^1_{j,\pm})=\Omega_\pm\cap B_{d^1_\pm}(x^1_{j,\pm}), && \Phi^1_{j,\pm}(\mathbb{R}^N_0)\cap B_{d^1_\pm}(x^1_{j,\pm})=\Gamma\cap B_{d^1_\pm}(x^1_{j,\pm}) \quad (j\in\mathbb{N}).
\end{aligned}$$

(iii) *There exist $C^\infty$ functions $\zeta^k_{j,\pm}$, and $\tilde\zeta^k_{j,\pm}$ $(k=0,1,2,\ j\in\mathbb{N})$ such that*

$$\begin{gathered}
\zeta^k_{j,\pm},\ \tilde\zeta^k_{j,\pm}\le 1,\ \ \|\zeta^k_{j,\pm}\|_{W^2_\infty(\mathbb{R}^N)},\ \|\tilde\zeta^k_{j,\pm}\|_{W^2_\infty(\mathbb{R}^N)}\le c_0,\ \ \tilde\zeta^k_{j,\pm}=1 \ \text{ on supp } \zeta^k_{j,\pm},\\
\text{supp } \zeta^0_{j,\pm},\ \ \text{supp } \tilde\zeta^0_{j,\pm}\subset B_{d^0}(x^0_j),\ \ \text{supp } \zeta^i_{j,\pm},\ \ \text{supp } \tilde\zeta^i_{j,\pm}\subset B_{d^i_\pm}(x^i_{j,\pm})\ \ (i=1,2)\\
\sum_{k=0}^2\sum_{j=1}^\infty \zeta^k_{j,\pm}=1 \quad \text{on } \overline{\Omega_\pm},\ \ \sum_{j=1}^\infty\zeta^0_{j,\pm}=1 \quad \text{on } \Gamma,\ \ \sum_{j=1}^\infty\zeta^1_{j,\pm}=1 \quad \text{on } \Gamma_\pm.
\end{gathered}$$

*Here $c_0$ is a constant independent of $j\in\mathbb{N}$.*

(iv) $\nabla\Phi^0_j=\mathcal{A}^0_j+\mathcal{B}^0_j$, $\nabla\Phi^1_{j,\pm}=\mathcal{A}^{\pm,1}_j+\mathcal{B}^{\pm,1}_j$, $\nabla(\Phi^0_j)^{-1}=\mathcal{A}^0_{j,-}+\mathcal{B}^0_{j,-}$, $\nabla(\Phi^1_{j,\pm})^{-1}=\mathcal{A}^{\pm,1}_{j,-}+\mathcal{B}^{\pm,1}_{j,-}$, *where $\mathcal{A}^0_j$, $\mathcal{A}^{\pm 1}_j$, $\mathcal{A}^0_{j,-}$, and $\mathcal{A}^{\pm,1}_{j,-}$ are $N\times N$ constant orthogonal matrices and $\mathcal{B}^0_j$, $\mathcal{B}^{\pm,1}_j$ $\mathcal{B}^0_{j,-}$, and $\mathcal{B}^{\pm,1}_{j,-}$ are $N\times N$ matrices of $W^1_r(\mathbb{R}^N)$ functions defined in $\mathbb{R}^N$ satisfying*

$$\|(\mathcal{B}^0_j,\mathcal{B}^{\pm,1}_j,\mathcal{B}^0_{j,-},\mathcal{B}^{\pm,1}_{j,-})\|_{L_\infty(\mathbb{R}^N)}\le M_1,\qquad \|(\nabla\mathcal{B}^0_j,\nabla\mathcal{B}^{\pm,1}_j,\nabla\mathcal{B}^0_{j,-},\nabla\mathcal{B}^{\pm,1}_{j,-})\|_{L_r(\mathbb{R}^N)}\le M_2.$$

(v) *There exists a natural number $L\ge 2$ such that any $L+1$ distinct sets of $\{B_{d^0}(x^0_j)\mid j\in\mathbb{N}\}$ and $\{B_{d^k_\pm}(x^k_{j,\pm})\mid k=1,2,j\in\mathbb{N}\}$ have an empty intersection.*

Since $\gamma^\pm_\ell(x)$ $(\ell=0,1,2)$ are uniform continuous functions defined on $\overline{\Omega_\pm}$ satisfying the conditions in (2.1), choosing $d^i_\pm>0$ smaller if necessary, we may assume that $|\gamma^\pm_\ell(x)-\gamma^\pm_\ell(x^0_j)|\le M_1$ for any $x\in B_{d^0}(x^0_j)$ and $|\gamma^\pm_\ell(x)-\gamma^\pm_\ell(x^k_{j,\pm})|\le M_1$ for any $x\in B_{d^k_\pm}(x^k_{j,\pm})$ $(k=1,2)$. Moreover after choosing $M_2$, $d^0$ and $d^k_\pm$ according to $M_1$ in Proposition 4.2, we choose $M_2$ again so large that $\|\gamma^\pm_\ell\|_{L_r(B_{d^0}(x^0_j))}\le M_2$ and $\|\gamma^\pm_\ell\|_{L_r(B_{d^k_\pm}(x^k_{j,\pm}))}\le M_2$ $(k=1,2)$. Summing up, we may assume that

$$\frac12\rho_{0,\pm}\le\gamma^\pm_0(x)\le 2\rho_{0,\pm},\qquad\qquad 0\le\gamma^\pm_k(x)\le\rho_{2,\pm}\ (k=1,2)$$

in $\Omega_\pm\cap B_{d^0}(x^0_j)$ and $\Omega_\pm\cap B_{d^k_\pm}(x^k_{j,\pm})$ $(k=1,2)$ and

$$\begin{aligned}
&\|\gamma^\pm_\ell-\gamma^\pm_\ell(x^0_j)\|_{L_\infty(\Omega_\pm\cap B_{d^0_\pm}(x^0_j))}\le M_1, && \|\nabla\gamma^\pm_\ell\|_{L_r(\Omega_\pm\cap B_{d^0_{j,\pm}}(x^0_j))}\le M_2,\\
&\|\gamma^\pm_\ell-\gamma^\pm_\ell(x^k_{j,\pm})\|_{L_\infty(\Omega_\pm\cap B_{d^k_\pm}(x^k_{j,\pm}))}\le M_1, && \|\nabla\gamma^\pm_\ell\|_{L_r(\Omega_\pm\cap B_{d^k_{j,\pm}}(x^k_{j,\pm}))}\le M_2 \qquad (\ell=0,1,2).
\end{aligned}$$

In the sequel, we write $B^0_{j,\pm}=B_{d^0}(x^0_j)$, $B^k_{j,\pm}=B_{d^k_\pm}(x^k_{j,\pm})$ $(k=1,2)$, $\mathcal{H}^0_{j,\pm}=\Phi^0_j(\mathbb{R}^N_\pm)$, $\mathcal{H}^1_{j,\pm}=\Phi^1_{j,\pm}(\mathbb{R}^N_\pm)$, $\mathcal{H}^2_j=\Phi^2_{j,\pm}(\mathbb{R}^N)$, $\partial\mathcal{H}^0_j=\Phi^0_j(\mathbb{R}^N_0)$, and $\partial\mathcal{H}^1_{j,\pm}=\Phi^1_{j,\pm}(\mathbb{R}^N_0)$ for simplicity of notations. We define the function $\gamma^{i,\pm}_{j\ell}$ by

$$\gamma^{0,\pm}_{j\ell}(x)=(\gamma^\pm_\ell(x)-\gamma^\pm_\ell(x^0_j))\tilde\zeta^0_{j,\pm}(x)+\gamma^\pm_\ell(x^0_j),\quad \gamma^{k,\pm}_{j\ell}(x)=(\gamma^\pm_\ell(x)-\gamma^\pm_\ell(x^k_{j,\pm}))\tilde\zeta^k_{j,\pm}(x)+\gamma^\pm_\ell(x^k_{j,\pm})$$

for $\ell=0,1,2$, $k=1,2$ and $j\in\mathbb{N}$. We see that

$$\begin{aligned}
&\frac12\rho_{0,\pm}\le\gamma^{i,\pm}_{j0}(x)\le 2\rho_{0,\pm}, && 0\le\rho^{i,\pm}_{jk}(x)\le\rho_{2,\pm} && (x\in\mathcal{H}^i_{j,\pm}),\\
&\|\gamma^i_{j\ell}-\gamma^i_{j\ell}(x^i_j)\|_{L_\infty(\mathcal{H}^i_{j,\pm})}\le M_1, && \|\nabla\gamma^i_{j\ell}\|_{L_r(\mathcal{H}^i_{j,\pm})}\le C_{M_2}
\end{aligned}\tag{4.3}$$

and $\gamma^{i,\pm}_{j\ell}(x) = \gamma^{\pm}_{\ell}(x)$ in supp $\zeta^i_{j,\pm}$ for $\ell, i = 0,1,2$, $k = 1,2$ and $j \in \mathbb{N}$. Let $\vec g_\pm \in L_q(\Omega_\pm)$ and $\vec h, \vec h_- \in W^1_q(\Omega_\pm)$. We consider the equations

$$\begin{cases} \gamma^{0,\pm}_{j0}\lambda \vec u^0_{j,\pm} - \mathrm{Div}\, S_\pm(\vec u^0_{j,\pm}) - \lambda^{-1}\nabla(\gamma^{0,\pm}_{j1}\gamma^{0,\pm}_{j2}\mathrm{div}\, \vec u^0_{j,\pm}) = \tilde\zeta^0_{j,\pm}\vec g_\pm & \text{in } \mathcal{H}^0_{j,\pm}, \\ (S_+(\vec u^0_{j,+}) + \gamma^{0,+}_{j1}\gamma^{0,+}_{j2}\lambda^{-1}(\mathrm{div}\, \vec u^0_{j,+})I)\vec n^0_j\,|_{\partial\mathcal{H}^0_{j,+}} & \\ \qquad -(S_-(\vec u^0_{j,-}) + \gamma^{0,-}_{j1}\gamma^{0,-}_{j2}\lambda^{-1}(\mathrm{div}\, \vec u^0_{j,-})I)\vec n^0_j\,|_{\partial\mathcal{H}^0_{j,-}} = \tilde\zeta^0_{j,\pm}\vec h, & \\ \vec u^0_{j+}|_{\partial\mathcal{H}^0_{j,+}} - \vec u^0_{j-}|_{\partial\mathcal{H}^0_{j,-}} = 0, & \end{cases} \tag{4.4}$$

$$\begin{cases} \gamma^{1,+}_{j0}\lambda \vec u^1_{j,+} - \mathrm{Div}\, S_+(\vec u^1_{j,+}) - \lambda^{-1}\nabla(\gamma^{1,+}_{j1}\gamma^{1,+}_{j2}\mathrm{div}\, \vec u^1_{j,+}) = \tilde\zeta^1_{j,+}\vec g_+ & \text{in } \mathcal{H}^1_{j,+}, \\ \vec u^1_{j,+}\,|_{\partial\mathcal{H}^1_{j,+}} = 0, & \end{cases} \tag{4.5}$$

$$\begin{cases} \gamma^{1,-}_{j0}\lambda \vec u^1_{j,-} - \mathrm{Div}\, S_-(\vec u^1_{j,-}) - \lambda^{-1}\nabla(\gamma^{1,-}_{j1}\gamma^{1,-}_{j2}\mathrm{div}\, \vec u^1_{j,-}) = \tilde\zeta^1_{j,-}\vec g_- & \text{in } \mathcal{H}^1_{j,-}, \\ (S_-(\vec u^1_{j,-}) + \gamma^{1-}_{j1}\gamma^{1-}_{j2}\lambda^{-1}(\mathrm{div}\, \vec u^1_{j,-})I)\vec n^1_j\,|_{\Gamma_-} = \vec h_-, & \end{cases} \tag{4.6}$$

and

$$\gamma^{2,\pm}_{j0}\lambda \vec u^2_{j,\pm} - \mathrm{Div}\, S_\pm(\vec u^2_{j,\pm}) - \lambda^{-1}\nabla(\gamma^{2,\pm}_{j1}\gamma^{2,\pm}_{j2}\mathrm{div}\, \vec u^2_{j,\pm}) = \tilde\zeta^2_{j,\pm}\vec g_\pm \qquad \text{in } \mathcal{H}^2_{j,\pm}. \tag{4.7}$$

Here $\vec n^0_j$ and $\vec n^1_j$ are the unit outward normals to $\partial\mathcal{H}^0_{j,-}$ and $\partial\mathcal{H}^1_{j,-}$. Since $\gamma^{i,\pm}_{j\ell}$ ($\ell = 0,1,2$) satisfy the conditioin (4.3), by Theorem 4.1 and the results due to Enomoto, Below and Shibata [5], there exist operator families $\mathcal{T}^i_{j,\pm}(\lambda)$ with

$$\mathcal{T}^0_{j,\pm}(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}^0_q(\mathcal{H}^0_{j,\pm}), W^2_q(\mathcal{H}^0_{j,\pm})^N)), \quad \mathcal{T}^1_{j,+}(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(L_q(\mathcal{H}^1_{j,+}), W^2_q(\mathcal{H}^1_{j,+})^N)),$$
$$\mathcal{T}^1_{j,-}(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}^1_q(\mathcal{H}^1_{j,-}), W^2_q(\mathcal{H}^0_{j,\pm})^N)), \quad \mathcal{T}^2_{j,\pm}(\lambda) \in \mathrm{Hol}(\Gamma_{\varepsilon,\lambda_0}, \mathcal{L}(L_q(\mathcal{H}^2_{j,\pm}), W^2_q(\mathcal{H}^2_{j,\pm})^N)),$$

where

$$\mathcal{Y}^0_q(\mathcal{H}^0_{j,\pm}) = \{(F_{1+}, F_{1-}, F_2, F_3) \mid F_{1\pm} \in L_q(\mathcal{H}^0_{j,\pm})^N, \ F_2 \in L_q(\mathcal{H}^0_j)^{N^2}, \ F_3 \in L_q(\mathcal{H}^0_j)^N\},$$
$$\mathcal{Y}^1_q(\mathcal{H}^1_{j,-}) = \{(F_{1+}, F_{1-}, F_2, F_3) \mid F_{1\pm} \in L_q(\mathcal{H}^1_{j,-})^N, \ F_2 \in L_q(\mathcal{H}^1_j)^{N^2}, \ F_3 \in L_q(\mathcal{H}^1_j)^N\}.$$

Moreover

$$\vec u^0_{j,\pm} = \mathcal{T}^0_{j,\pm}(\lambda)F_\lambda(\tilde\zeta^0_{j,\pm}\vec g_\pm, \tilde\zeta^0_{j,\pm}\vec h), \qquad \vec u^1_{j,+} = \mathcal{T}^1_{j,+}(\lambda)\tilde\zeta^k_{j,+}\vec g_+,$$
$$\vec u^1_{j,-} = \mathcal{T}^1_{j,-}(\lambda)F_\lambda(\tilde\zeta^0_{j,-}\vec g_\pm, \tilde\zeta^0_{j,-}\vec h_-), \qquad \vec u^k_{2,\pm} = \mathcal{T}^2_{j,\pm}(\lambda)\tilde\zeta^k_{j,\pm}\vec g_\pm,$$

where $F_\lambda(\vec g, \vec h) = (\vec g, \lambda^{1/2}\vec h, \nabla\vec h)$, uniquely solve problem (4.4)-(4.7), respectively and we have their $\mathcal{R}$-boundedness:

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathcal{H}^0_{j,\pm}), L_q(\mathcal{H}^0_{j,\pm}))}(\{(\tau\partial_\tau)^\ell G_\lambda \mathcal{T}^0_{j,\pm}(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le \kappa_2, \qquad (\ell = 0,1),$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathcal{H}^1_{j,+}), L_q(\mathcal{H}^1_{j,+}))}(\{(\tau\partial_\tau)^\ell G_\lambda \mathcal{T}^1_{j,+}(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le \kappa_2, \qquad (\ell = 0,1),$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathcal{H}^1_{j,-}), L_q(\mathcal{H}^1_{j,-}))}(\{(\tau\partial_\tau)^\ell G_\lambda \mathcal{T}^1_{j,-}(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le \kappa_2, \qquad (\ell = 0,1),$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathcal{H}^2_{j,\pm}), L_q(\mathcal{H}^2_{j,\pm}))}(\{(\tau\partial_\tau)^\ell G_\lambda \mathcal{T}^2_{j,\pm}(\lambda) \mid \lambda \in \Gamma_{\varepsilon,\lambda_0}\}) \le \kappa_2, \qquad (\ell = 0,1)$$

with some constant $\kappa_2$ independent of $j \in \mathbb{N}$ and the following resolvent estimates:

$$\|(\lambda\vec u^0_{j,\pm}, \lambda^{1/2}\nabla\vec u^0_{j,\pm}, \nabla^2\vec u^0_{j,\pm})\|_{L_q(\mathcal{H}^0_{j,\pm})} \le \kappa_2\|(\tilde\zeta^0_{j,\pm}\vec g_\pm, \lambda^{1/2}\tilde\zeta^0_{j,\pm}\vec h, \nabla(\tilde\zeta^0_{j,\pm}\vec h)\|_{L_q(\mathcal{H}^0_{j,\pm})},$$
$$\|(\lambda\vec u^1_{j,+}, \lambda^{1/2}\nabla\vec u^1_{j,+}, \nabla^2\vec u^1_{j,+})\|_{L_q(\mathcal{H}^1_{j,+})} \le \kappa_2\|\tilde\zeta^0_{j,+}\vec g_+\|_{L_q(\mathcal{H}^0_{j,\pm})},$$
$$\|(\lambda\vec u^1_{j,-}, \lambda^{1/2}\nabla\vec u^1_{j,-}, \nabla^2\vec u^1_{j,-})\|_{L_q(\mathcal{H}^1_{j,-})} \le \kappa_2\|(\tilde\zeta^1_{j,-}\vec g_-, \lambda^{1/2}\tilde\zeta^1_{j,-}\vec h_-, \nabla(\tilde\zeta^1_{j,-}\vec h_-)\|_{L_q(\mathcal{H}^1_{j,-})},$$
$$\|(\lambda\vec u^2_{j,\pm}, \lambda^{1/2}\nabla\vec u^2_{j,\pm}, \nabla^2\vec u^2_{j,\pm})\|_{L_q(\mathcal{H}^2_{j,\pm})} \le \kappa_2\|\tilde\zeta^2_{j,\pm}\vec g_\pm\|_{L_q(\mathcal{H}^2_{j,\pm})} \tag{4.8}$$

for $j \in \mathbb{N}$. For $\vec{g}_\pm \in L_q(\Omega_\pm)$ and $\vec{h}, \vec{h}_- \in W_q^1(\Omega)$, we consider the equation (2.6). We set the parametrix $\vec{\mathcal{U}}_\pm(\lambda)(\vec{g}_\pm, \vec{h}, \vec{h}_-) = \sum_{i=0}^2 \sum_{j=1}^\infty \zeta_{j,\pm}^i \vec{u}_{j,\pm}^i$. Since $\vec{\mathcal{U}}_\pm(\lambda)(\vec{g}_\pm, \vec{h}, \vec{h}_-) \in W_q^2(\Omega)^N$ by (4.8), inserting $\vec{\mathcal{U}}_\pm(\lambda)(\vec{g}_\pm, \vec{h}, \vec{h}_-)$ into (2.6) and noting the facts that $\vec{n} = \vec{n}_j^0$ on supp $\zeta_{j,\pm}^0 \cap \Gamma$ and $\vec{n}_- = \vec{n}_j^1$ on supp $\zeta_{j,-}^1 \cap \Gamma_-$, we have

$$\begin{cases} \gamma_0^\pm \lambda \vec{v}_\pm - \mathrm{Div}\, S_\pm \vec{v}_\pm - \lambda^{-1}\nabla(\gamma_1^\pm\gamma_2^\pm \mathrm{div}\, \vec{v}_\pm) = \vec{g}_\pm - \vec{\mathcal{V}}_\pm^1(\lambda)(\vec{g}_\pm, \vec{h}, \vec{h}_-) \qquad \text{in } \Omega_\pm \\ (S_+(\vec{v}_+) + \gamma_1^+\gamma_2^+\lambda^{-1}(\mathrm{div}\,\vec{v}_+)I)\vec{n}\,|_{\Gamma+0} - (S_-(\vec{v}_-) + \gamma_1^-\gamma_2^-\lambda^{-1}(\mathrm{div}\,\vec{v}_-)I)\vec{n}\,|_{\Gamma-0} = \vec{h} - \vec{\mathcal{V}}^2(\lambda)(\vec{g}_\pm, \vec{h}), \\ \vec{v}_+|_{\Gamma+0} - \vec{v}_-|_{\Gamma-0} = 0, \\ (S_-(\vec{v}_-) + \gamma_1^-\gamma_2^-\lambda^{-1}(\mathrm{div}\,\vec{v}_-)I)\vec{n}_-|_{\Gamma_-} = \vec{h}'_- - \vec{\mathcal{V}}^3(\lambda)(\vec{g}_\pm, \vec{h}_-), \\ \vec{v}_+|_{\Gamma_+} = \vec{0} \end{cases}$$

with $\vec{v}_\pm = \vec{\mathcal{U}}_\pm(\lambda)(\vec{g}_\pm, \vec{h}, \vec{h}_-)$, where

$$\begin{aligned} \vec{\mathcal{V}}_\pm^1(\lambda)(\vec{g}_\pm, \vec{h}, \vec{h}_-) &= \sum_{i=0}^2 \sum_{j=1}^\infty \Big( [\mathrm{Div}\, S_\pm(\zeta_{j,\pm}^i \vec{u}_{j,\pm}^i) - \zeta_{j,\pm}^i \mathrm{Div}\, S_\pm(\vec{u}_{j,\pm}^i)] \\ &\qquad - \lambda^{-1}[\nabla(\gamma_{j1}^{i,\pm}\gamma_{j2}^{i,\pm}\zeta_{j,\pm}^i \mathrm{div}\, \vec{u}_{j,\pm}^i) - \zeta_{j,\pm}^i \nabla(\gamma_{j1}^{i,\pm}\gamma_{j2}^{i,\pm}\mathrm{div}\,\vec{u}_{j,\pm}^i)]\Big), \\ \vec{\mathcal{V}}^2(\lambda)(\vec{g}_\pm, \vec{h}) &= \sum_{j=1}^\infty \Big(S_+(\zeta_{j,+}^0 \vec{u}_{j,+}^0) - \zeta_{j,+}^0 S_+(\vec{u}_{j,+}^0) + \gamma_{j1}^{0,+}\gamma_{j2}^{0,+}\lambda^{-1}(\mathrm{div}\,(\zeta_{j,+}^0\vec{u}_{j,+}^0) - \zeta_{j,+}^0 \mathrm{div}\,\vec{u}_{j,+}^0)I\Big)\vec{n}_j^0 \\ &\quad - \sum_{j=1}^\infty \Big(S_-(\zeta_{j,-}^0 \vec{u}_{j,-}^0) - \zeta_{j,-}^0 S_-(\vec{u}_{j,-}^0) + \gamma_{j1}^{0,-}\gamma_{j2}^{0,-}\lambda^{-1}(\mathrm{div}\,(\zeta_{j,-}^0\vec{u}_{j,-}^0) - \zeta_{j,-}^0 \mathrm{div}\,\vec{u}_{j,-}^0)I\Big)\vec{n}_j^0, \\ \vec{\mathcal{V}}^3(\lambda)(\vec{g}_\pm, \vec{h}_-) &= \sum_{j=1}^\infty \Big(S_-(\zeta_{j,-}^1 \vec{u}_{j,-}^1) - \zeta_{j,-}^1 S_-(\vec{u}_{j,-}^1) + \gamma_{j1}^{1,-}\gamma_{j2}^{1,-}\lambda^{-1}(\mathrm{div}\,(\zeta_{j,-}^1\vec{u}_{j,-}^1) - \zeta_{j,-}^1 \mathrm{div}\,\vec{u}_{j,-}^1)I\Big)\vec{n}_j^1. \end{aligned}$$

By (4.8), we have $\vec{\mathcal{V}}_\pm^1(\lambda)(\vec{g}_\pm, \vec{h}, \vec{h}_-) \in L_q(\Omega)^N$ and $\vec{\mathcal{V}}^2(\lambda)(\vec{g}_\pm, \vec{h}), \vec{\mathcal{V}}^3(\lambda)(\vec{g}_\pm, \vec{h}_-) \in W_q^1(\Omega)^N$, and

$$\|F_\lambda(\vec{\mathcal{V}}_\pm^1(\lambda)(\vec{g}_\pm, \vec{h}, \vec{h}_-), \vec{\mathcal{V}}^2(\lambda)(\vec{g}_\pm, \vec{h}), \vec{\mathcal{V}}^3(\lambda)(\vec{g}_\pm, \vec{h}_-))\|_{L_q(\Omega)} \le C\lambda_0^{-1/2}\|F_\lambda(\vec{g}_\pm, \vec{h}, \vec{h}_-)\|_{L_q(\Omega)} \tag{4.9}$$

for any $\lambda \in \Gamma_{\varepsilon,\lambda_0}$, where $F_\lambda(\vec{g}, \vec{h}, \vec{h}_-) = (\vec{g}_\pm, \lambda^{1/2}\vec{h}, \nabla\vec{h}, \lambda^{1/2}\vec{h}_-, \nabla\vec{h}_-)$. Setting

$$\begin{aligned} \vec{\mathcal{V}}_+(\lambda)(\vec{g}_+, \vec{h}, \vec{h}_-) &= (\vec{\mathcal{V}}_+^1(\vec{g}_+, \vec{h}, \vec{h}_-), \vec{\mathcal{V}}^2(\vec{g}_+, \vec{h})), \\ \vec{\mathcal{V}}_-(\lambda)(\vec{g}_-, \vec{h}, \vec{h}_-) &= (\vec{\mathcal{V}}_-^1(\vec{g}_-, \vec{h}, \vec{h}_-), \vec{\mathcal{V}}^2(\vec{g}_-, \vec{h}), \vec{\mathcal{V}}^3(\vec{g}_-, \vec{h}_-)) \end{aligned}$$

and choosing $\lambda_0 \ge 1$ so large that $C\lambda_0^{-1/2} \le 1/2$ in (4.9) we see that $(I - \vec{\mathcal{V}}_\pm)^{-1} \in \mathcal{L}(Y_q(\Omega_\pm))$ exists and $\vec{u}_\pm = \vec{\mathcal{U}}_\pm(I - \vec{\mathcal{V}}_\pm)^{-1}(\vec{g}_+, \vec{g}_-, \vec{h}, \vec{h}_-)$ is a solution to problem (2.6). The uniquness follows from the existence of solutions to the dual problem, so that we may omit its proof. Moreover the $\mathcal{R}$-boundedness of solution operator from the argument due to Enomoto, Below and Shibata [5] so that we may omit its proof. Therefore this completes the proof of Theorem 2.6.

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