# Introduction to Fundamental Mathematics

**Kazuaki TAIRA**

# Contents

# Contents

1. **The purpose of these lectures is to provide students with basic knowledge and skills of calculus and linear algebra.**
2. **To understand natural sciences, students should learn some fundamental mathematics. Therefore, students will study also various topics from Physics, Biology, Chemistry and Technology.**

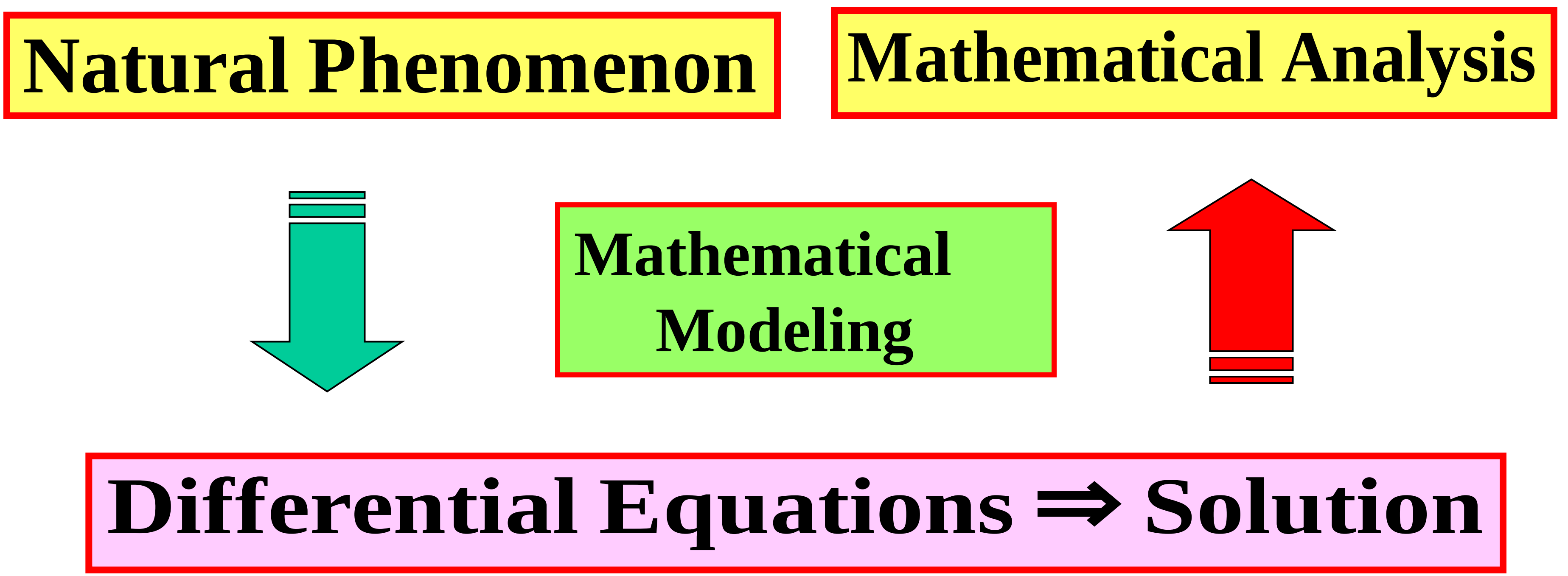
# Why do you study Mathematics ?

# The Role of Mathematics

# in

# Natural Sciences

# Mechanism of Mathematical Analysis



# Weather Forecast

# Mechanism of Weather Forecast

**Weather**

**Weather Forecast**

**Mathematical Modeling**

Navier - Stokes Equations $\underset{\text{Numerical Analysis}}{\Rightarrow}$ Approximation Solution

# Navier-Stokes Equations
# in
# Fluid Dynamics

$$\rho \frac{D\mathbf{V}}{Dt}$$

$$= -\nabla p + \rho \mathbf{B} + \mu \Delta \mathbf{V} + \frac{1}{3} \mu \nabla \cdot \mathrm{div}\, \mathbf{V}$$

**Inertia Force**

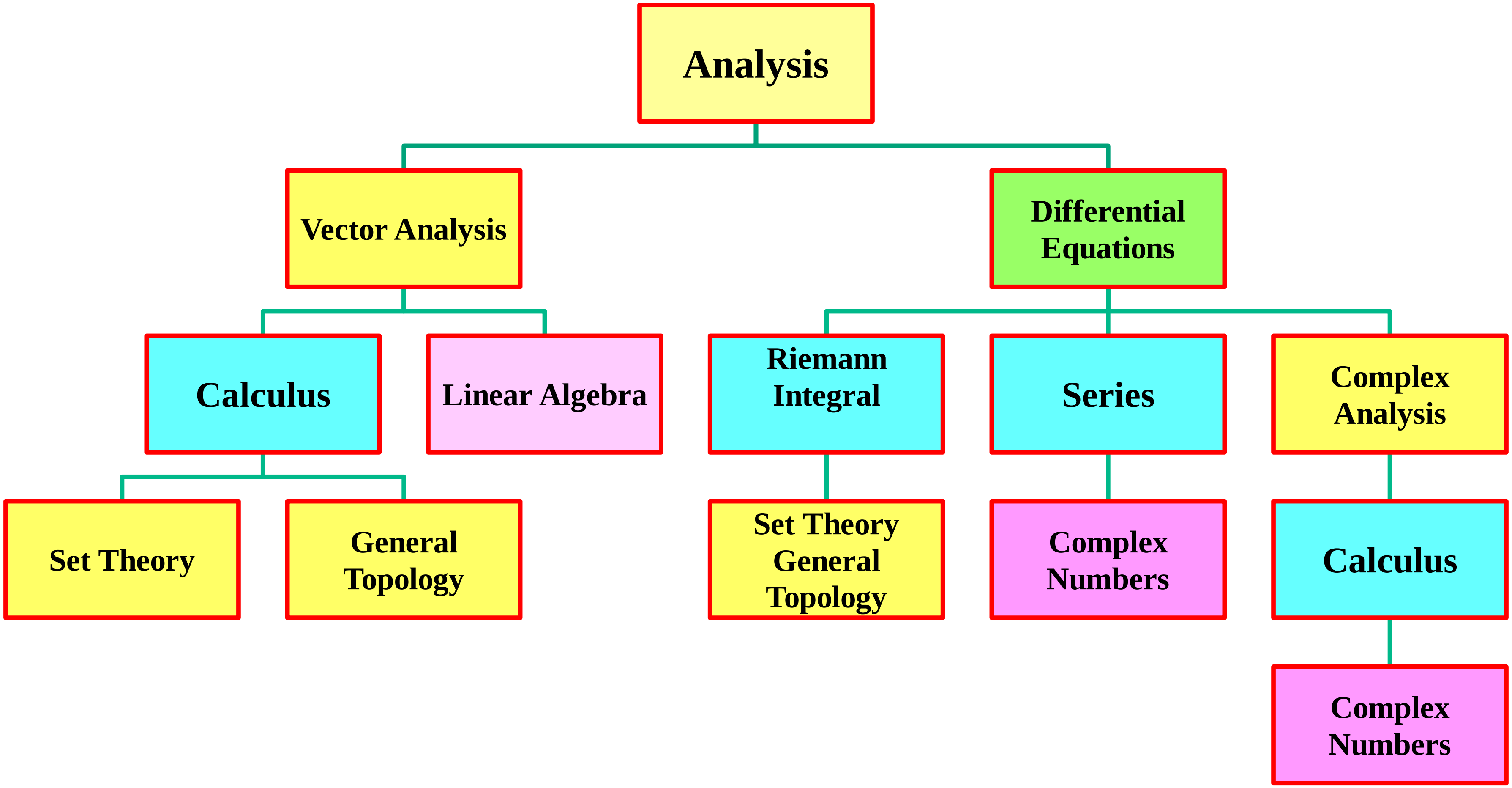
**= Pressure + Force + Viscosity + Stress**

# Bird's-Eye View

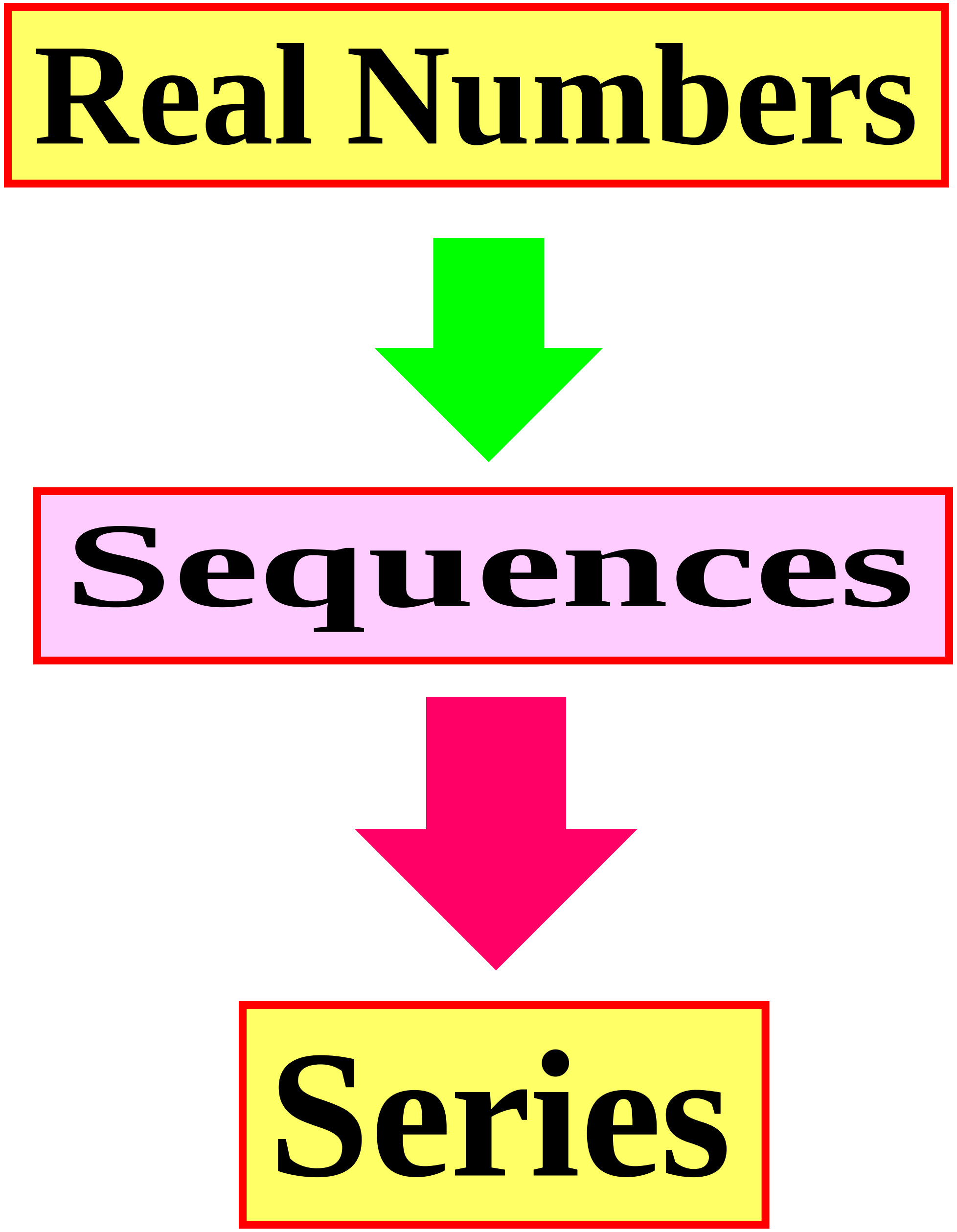
# of

# Fundamental Mathematics

# Bird's- Eye View



# Bird's-Eye View of Calculus


Real Numbers
Sequences
Series

**Sequences**

↓

**Differentiation**

↓

**Differential Equations**

**Series**

↓

**Integrals**

↓

**Vector Analysis**

# List of Mathematicians

# List (1)

- Archimedes（B. C. 287－B. C. 212）Greece
- Newton（1642－1727）England
- Leibniz（1646－1716）Germany
- Machin（1685－1751）England
- Fourier（1736－1813）France
- Lagrange（1736－1813）Italy, France
- Gauss（1777－1855）Germany
- Cauchy（1789－1857）France
- Abel（1802－1829）Norway

# List (2)

- Taylor（1685－1731）England
- Bolzano（1781－1848）Italy
- Hermite（1822－1901）France
- Maclaurin（1698－1746）Scotland
- Borel（1871－1956）France
- Dirichlet（1805－1859）Germany
- Weierstrass（1815－1897）Germany
- Dedekind（1831－1916）Germany

# List (3)

- Rolle（1652－1719） France
- Laplace（1749－1827）France
- Riemann（1826－1866）Germany
- Hilbert（1862－1943） Germany
- Hadamard（1865－1963） France
- Lebesgue（1875－1941） France
- Euler（1707－1783）Switzerland
- Poincare（1854－1912） France

# List (4)

- **Bernouille**（1667－1748）**Switzerland**
- **Bessel**（1784－1846）**Germany**
- **Cantor**（1845－1918) **Denmark/ Germany**
- **D'Alembert**（1717－1783）**France**
- **Darboux**（1842－1917） **France**
- **De Morgan**（1806－1871） **France**
- **Fubini**（1879－1943）**Italy**
- **de L'Hospital**（1661－1704） **France**

# List (5)

- Stokes（1819－1903）England
- Stirling（1962－1770）
- Simpson（1710－1761）England
- Schwarz（1843－1921）Germany
- Peano（1858－1932）Italy
- Napier（1550－1617）Scotland
- Jordan（1838－1922）France
- Landau（1887－1938）

# Mathematical Thoughts

# Mathematical Thoughts

(Ⅰ) Mathematical Reasoning
(Ⅱ) Mathematical Ideas
(Ⅲ) Mathematical Image

# Numerical Analysis

# Role of Numerical Analysis

| | | |
|---|---|---|
| **Mathematics** | **Analysis** | **Numerical Analysis** |
| **Physics** | **Theoretical Physics** | **Physical Experiments** |

# Mathematics versus Physics

# Bird's-Eye View

| Theme | Mathematics | Physics |
|---|---|---|
| Differential Equations | Ordinary Differential Equations | Newton's Equation of Motion |
| Infinite Series | Fourier Series | Eigenfunction Expansions （Principle of Superposition） |
| Vector Analysis | Calculus on Surfaces | Continuum Mechanics |

# Elasticity

# Importance of Elasticity

**A human body is an elastic material**

# Thoughts and Methods
# in
# Analysis

# Four Thoughts in Analysis

(Ⅰ) Discrete Case and Continuous Case

(Ⅱ) Principle of Superposition

(Ⅲ) Completeness

(Ⅳ) Numerical Analysis

# Discrete Case
# versus
# Continuous Case

# Vectors and Functions

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \left(\textbf{Finite - Dimensional Case}\right)$$

**Discrete Case**

**Continuous Case**

$$\int_a^b K(t, s) x(s) ds = y(t)$$

**(Infinite - Diemnsional Case)**

# Principle of Superposition

# Principle of Superposition

| Theme | Mathematics | Kinetics |
| --- | --- | --- |
| Infinite Series | Fourier Series | Eigenfunction Expansions |

# Principle of Superposition

$$Pu = f, \quad u = \sum_i u_i$$

**Decomposition into Fundamental Elements**

**Superposition of Solutions**

$$f = \sum_i f_i$$

**Find a solution**

$$Pu_i = f_i$$

# Fourier

- **Jean Baptiste Joseph Fourier (1768-1830)**

**French Mathematician and Physicist**

**La theorie analytique de la chaleur (1822)**

# Jean Baptiste Joseph Fourier

# Fourier's These

**Every function of period $2\pi$ can be approximated in terms of trigonometric functions.**

# Fourier Series Expansion (1)

$$f(x) = \sum_{j=0}^{\infty} f_j(x)$$

$$= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$

$$+ a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$+ a_j \cos jx + b_j \sin jx + \cdots$$

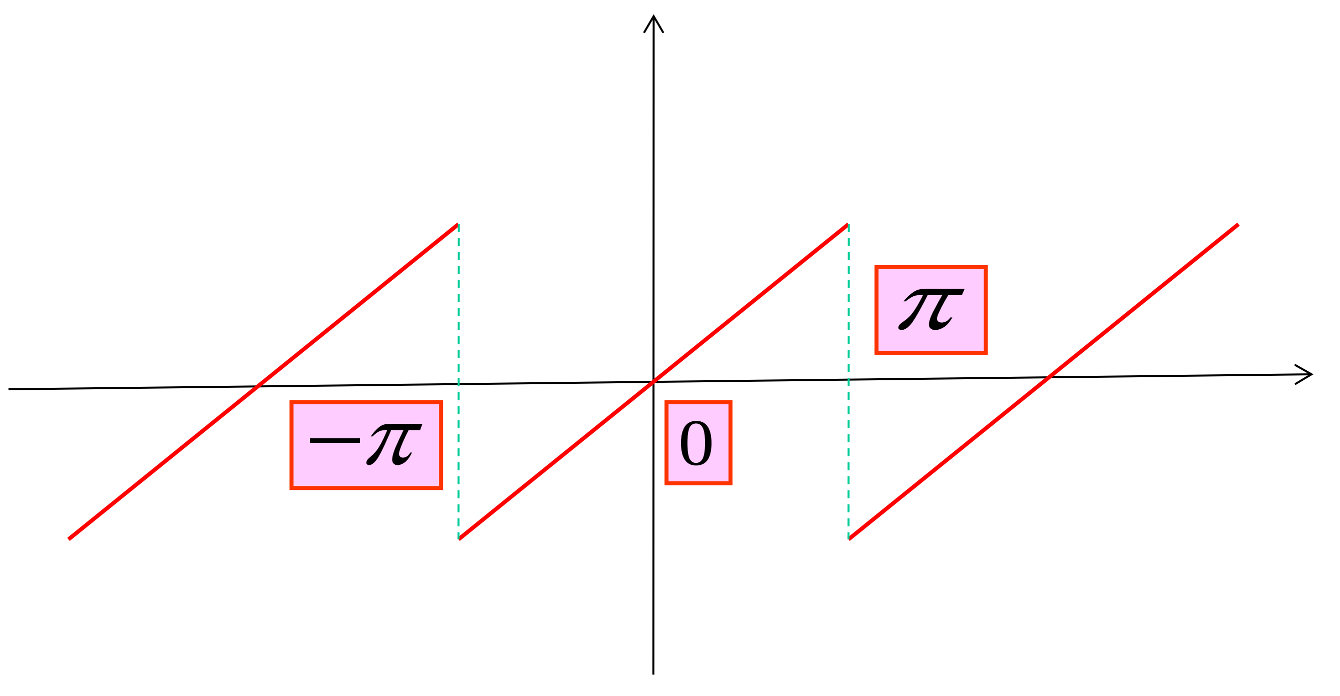
# Fourier Series Expansion (2)

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jx\,dx$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jx\,dx$$

# Example

$$f(x) = x, \qquad -\pi < x < \pi$$



# Fourier Coefficients

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos jx \, dx = 0$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin jx \, dx = \frac{2}{j}(-1)^{j+1}$$

$$(j \neq 0)$$

# Example of a Fourier Series

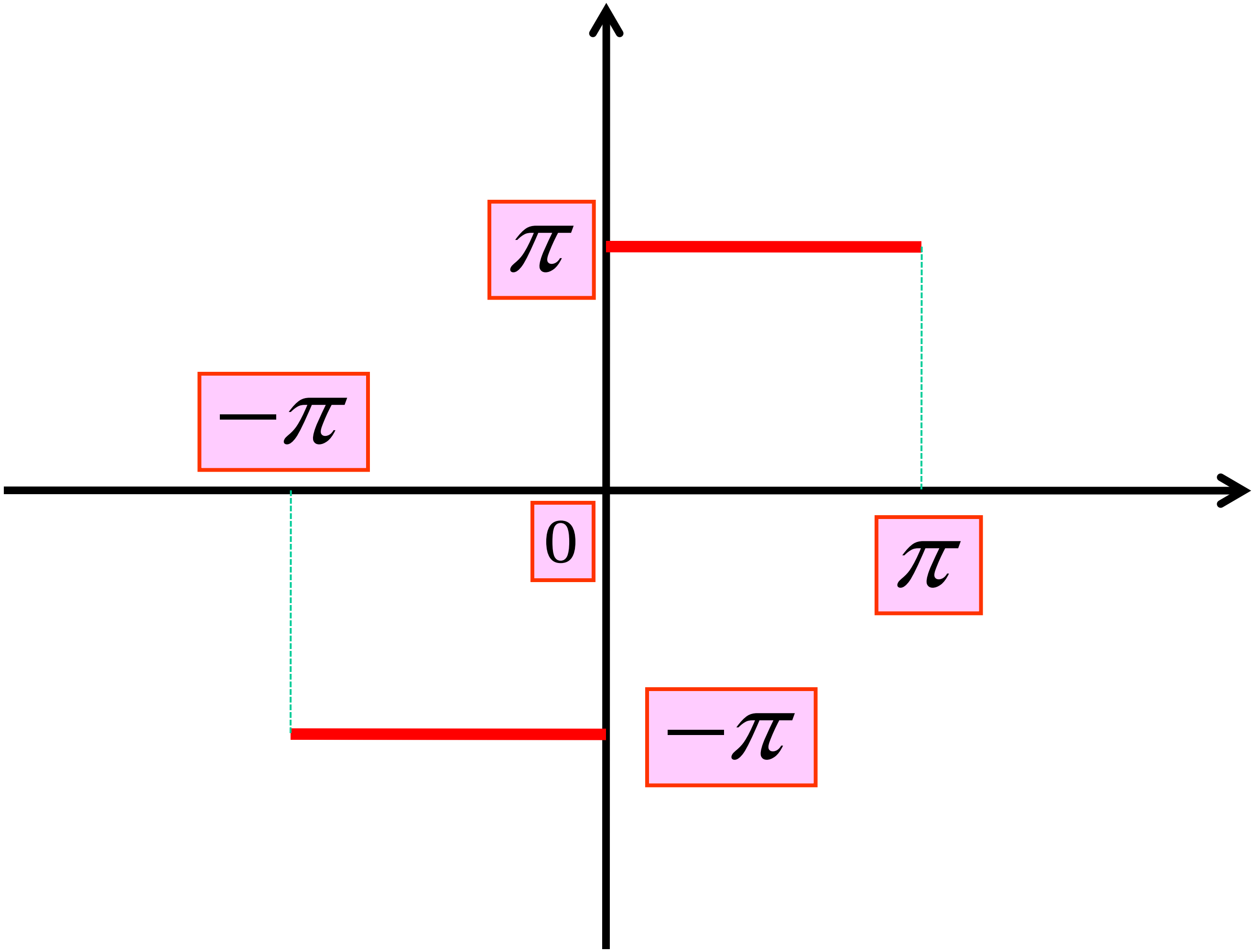
$$x = 2\sin x - 1\sin 2x + \cdots$$

$$+ \frac{2}{j}(-1)^{j+1}\sin jx + \cdots$$

$$(-\pi < x < \pi)$$

# Fourier Series of Step Functions

# Example of Step Functions



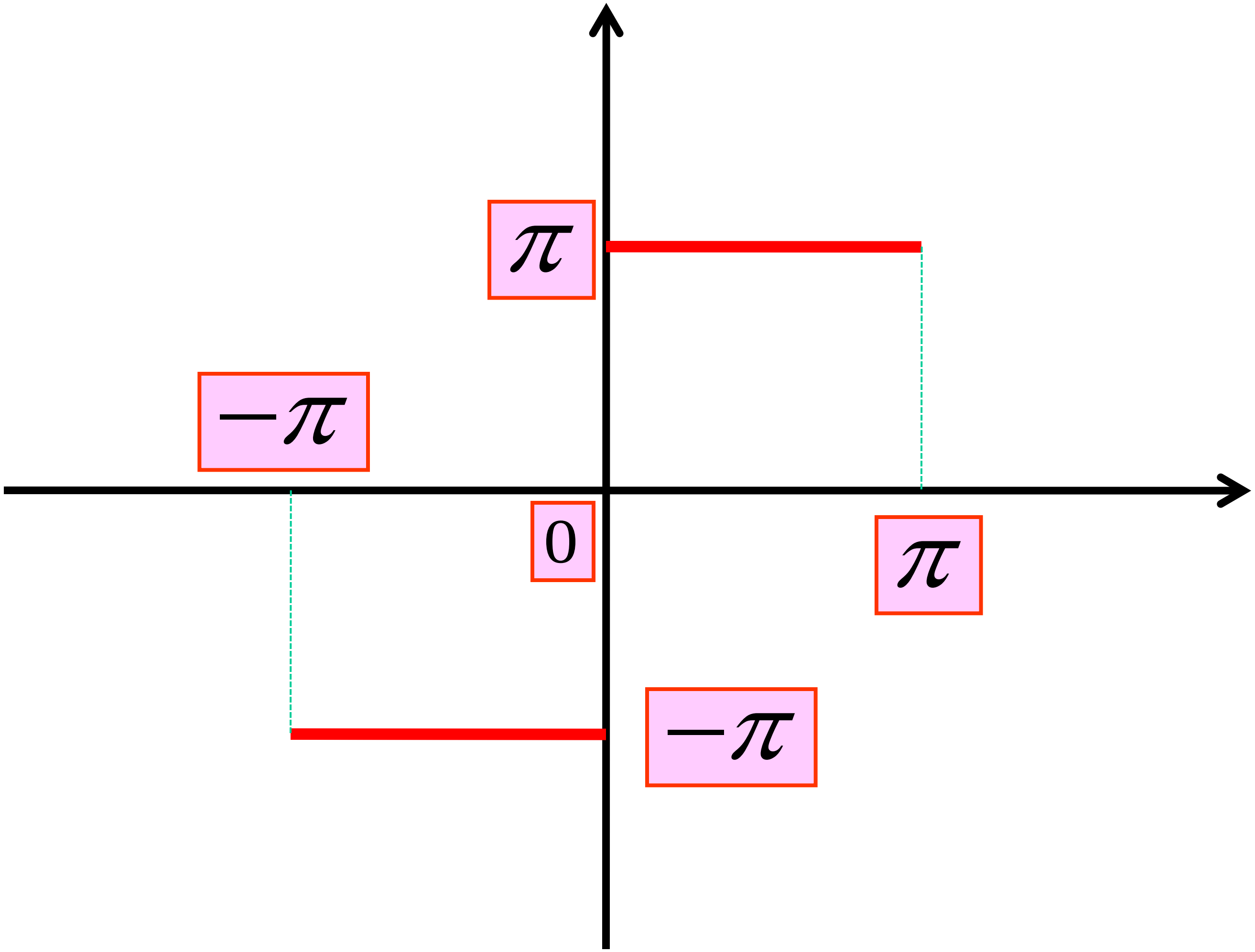
# Example of Fourier Series

$$\sum_{j=0}^{\infty} \frac{1}{2j-1} \sin(2j-1)x = \begin{cases} \dfrac{\pi}{4} & 0 < x < \pi \\ 0 & x = 0, \pi \\ -\dfrac{\pi}{4} & -\pi < x < 0 \end{cases}$$

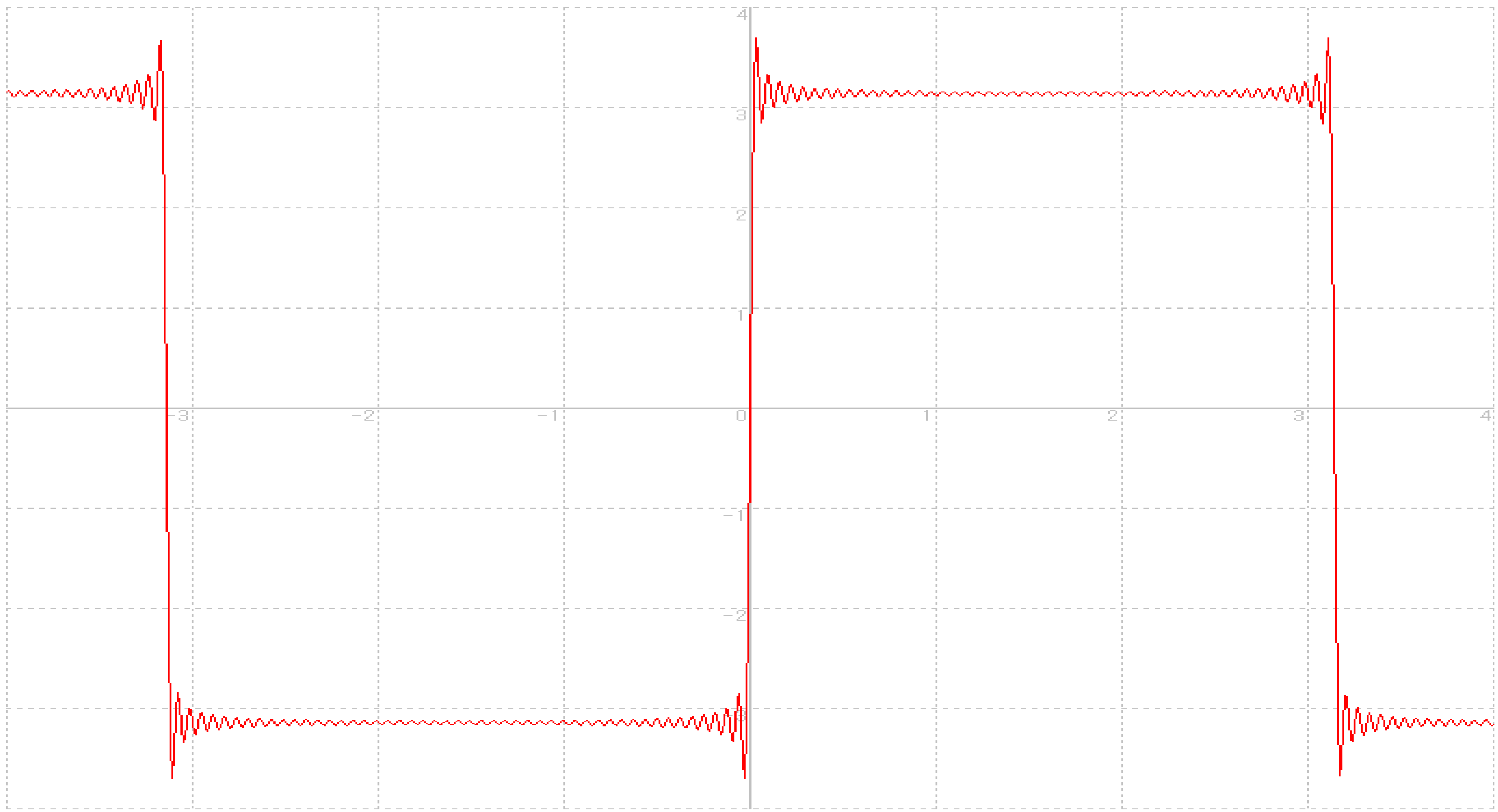
# Gibbs Phenomenon

# Numerical Computing with BASIC

# Example of Step Functions



# Example of Gibbs Phenomenon



# Weierstrass' Continuous Function

# Weierstrass's Function

$$f(x)=\sum_{k=0}^{\infty} a^k \cos\left(b^k x\right)$$

$$0<a<1, \quad ab\geq 1$$

# Numerical Computing with BASIC

# Example

$$f(x)=\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}\cos\left(3^{k}x\right)$$

$$a=\frac{1}{2},\,b=3\Rightarrow ab=\frac{3}{2}>1$$

$$s_0(x) = \cos x$$

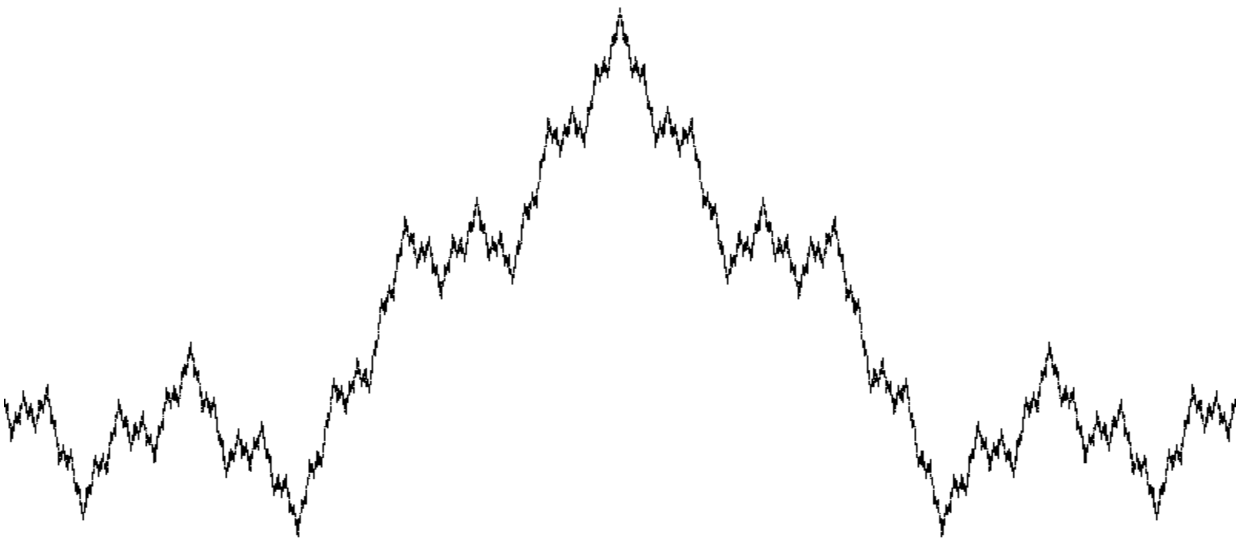
$$s_1(x) = \cos x + \frac{1}{2}\cos 3x$$

$$s_2(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x$$

$$s_3(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \frac{1}{8}\cos 27x$$

$$s_4(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \frac{1}{8}\cos 27x + \frac{1}{16}\cos 81x$$

# Weierstrass Function



# Heat Conduction (Fourier’s Work)

# Formulation of a Problem

**Steel bar of length $\pi$**

**Zero temperature on its ends**

**Initial temperature $f(x)$**

# Initial-Boundary Value Problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \quad 0<x<\pi,\ t>0$$

$u(0,t)=u(\pi,t)=0, \quad t>0$ （**Boundary Condition**）

$u(x,0)=f(x), \quad 0<x<\pi$ （**Initial Condition**）

# Fourier’s Method
# （Separation of Variables）

# Representation of a Solution (Heat Kernel)

$$u(x,t) = \int_0^{\pi} p(t,x,y) f(y) dy$$

$$p(t,x,y) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny$$

**(Heat Kernel)**

# Application to Series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

# Trace of a Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix}$$

$$\Rightarrow$$

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$ (**Sum of Eigenvalues**)

# Trace Formula (1)

$$\int_0^\pi p(t,x,x)\,dx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\left(\int_0^\pi \sin^2 nx\,dx\right)$$

$$= \sum_{n=1}^{\infty} e^{-n^2 t}$$

# Stationary Boundary Value Problem

$$v''(x) = g(x), \quad 0 < x < \pi$$

$$v(0) = v(\pi) = 0$$ （**Boundary Condition**）

# Representation of a Solution (Green's Function)

$$u(x,t) = \int_0^{\pi} G(x,y)g(y)dy$$

$G(x,y)$ **Green Function**

# Green's Function
# (Series Version)

$$G(x,y) = -\int_0^\infty p(t,x,y)\,dt$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^\infty e^{-n^2 t}\,dt\right)\sin nx \sin ny$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin nx \sin ny$$

# Trace Formula (2)

$$\int_0^{\pi} G(x,x)dx = -\int_0^{\infty}\int_0^{\pi} p(t,x,x)dxdt$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\int_0^{\pi}\sin^2 nx\,dx\right)$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n^2}$$ （**Sum of Eigenvalues**）

# Green's Function (Integral Kernel Version)

$$G(x,y) = \begin{cases} \left(\dfrac{y}{\pi} - 1\right)x & 0 \le x \le y \le \pi \\ \left(\dfrac{x}{\pi} - 1\right)y & 0 \le y \le x \le \pi \end{cases}$$

# Trace Formula (3)

$$\int_0^{\pi} G(x,x)\,dx$$

$$= \int_0^{\pi} \left( \frac{x^2}{\pi} - x \right) dx = -\frac{\pi^2}{6}$$

# Trace Formula (4)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^{\pi} G(x,x)\,dx = \frac{\pi^2}{6}$$

# Mathematical System of Numbers

| Set | Algebra | Analysis |
|---|---|---|
| Complex Numbers | $+ - \times \div$ | Complete |
| Real Numbers | $+ - \times \div$ | Complete |
| Rational Numbers | $+ - \times \div$ | |
| Integers | $+ - \times$ | |
| Natural Numbers | $+ \times$ | |

# Completeness

# Convergence of Sequences

# Definition of Convergence

$\{a_n\}$ **sequence of real numbers**

$\{a_n\}$ **converges to** $a$

$\overset{\text{def}}{\Leftrightarrow}$

$\forall \varepsilon > 0, \ \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$

# Cauchy’s Test

# Augustin Louis Cauchy

- **Augustin Louis Cauchy (1789-1857)**
  **French mathematician**

# Augustin Louis Cauchy

# Cauchy's Test

$\{a_n\}$ **converges**

$\Leftrightarrow$

$$\lim_{n,m\to\infty} |a_n - a_m| = 0$$

# Complex Numbers

# Gauss

- **Carl Friedrich Gauss (1777-1855)**
**German Mathematician and Physicist**

# Carl Friedrich Gauss

# Complex Number

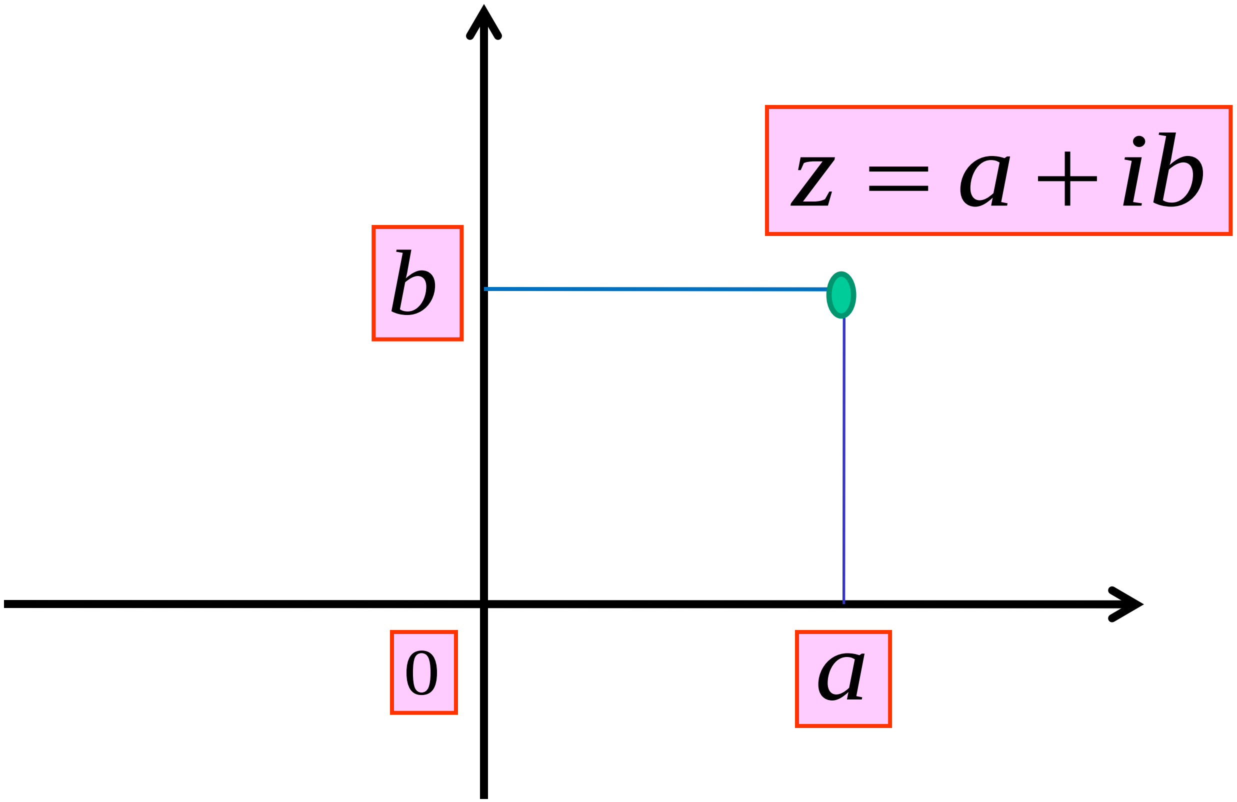
$$a + ib = c + id$$

$$\Leftrightarrow$$

$$a = c, b = d$$

$$i = \sqrt{-1}$$

# Complex Plane

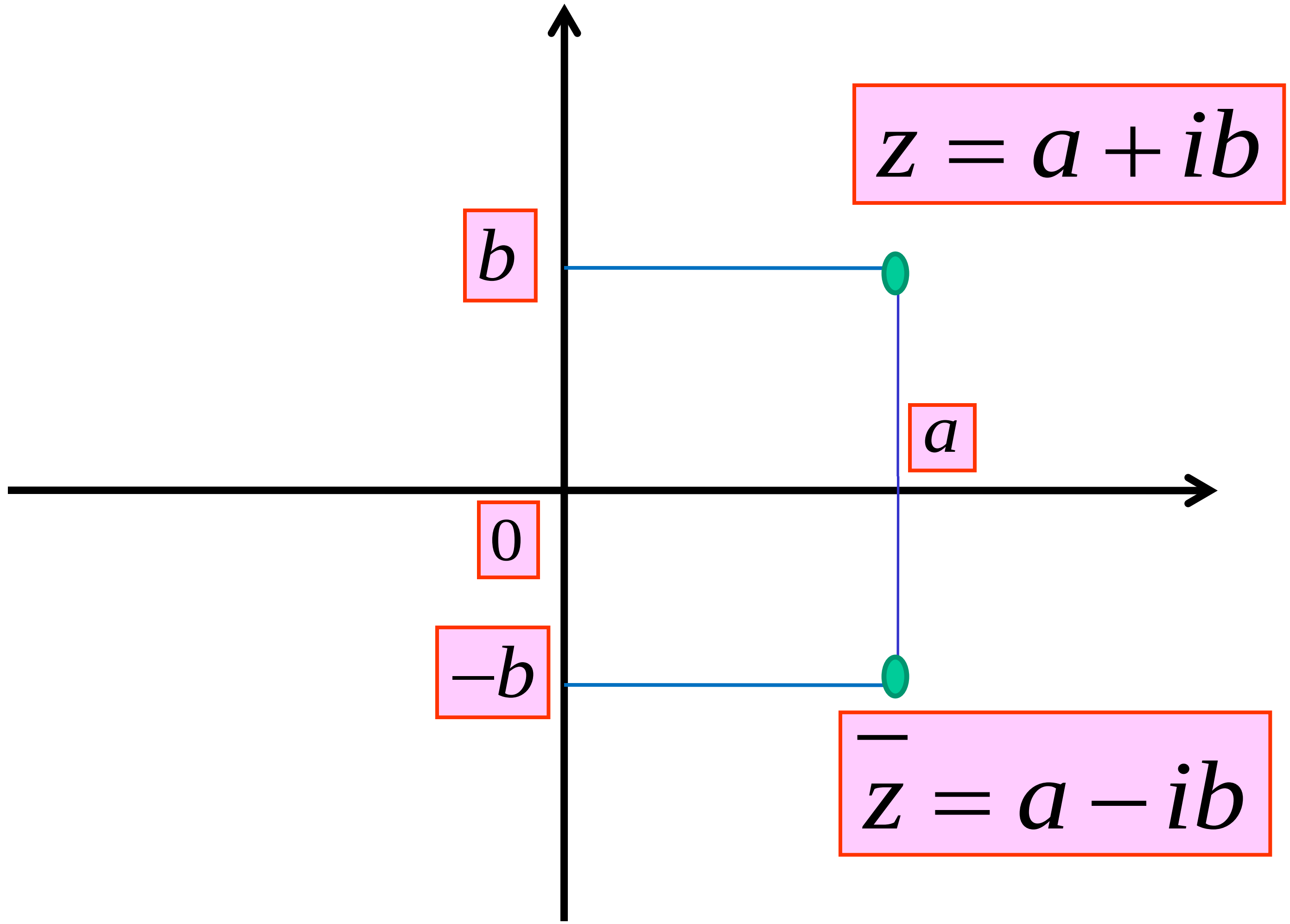


# Conjugate of a Complex Number

$$z = a + ib$$

$$\Rightarrow$$

$$\bar{z} = a + i(-b) = a - ib$$



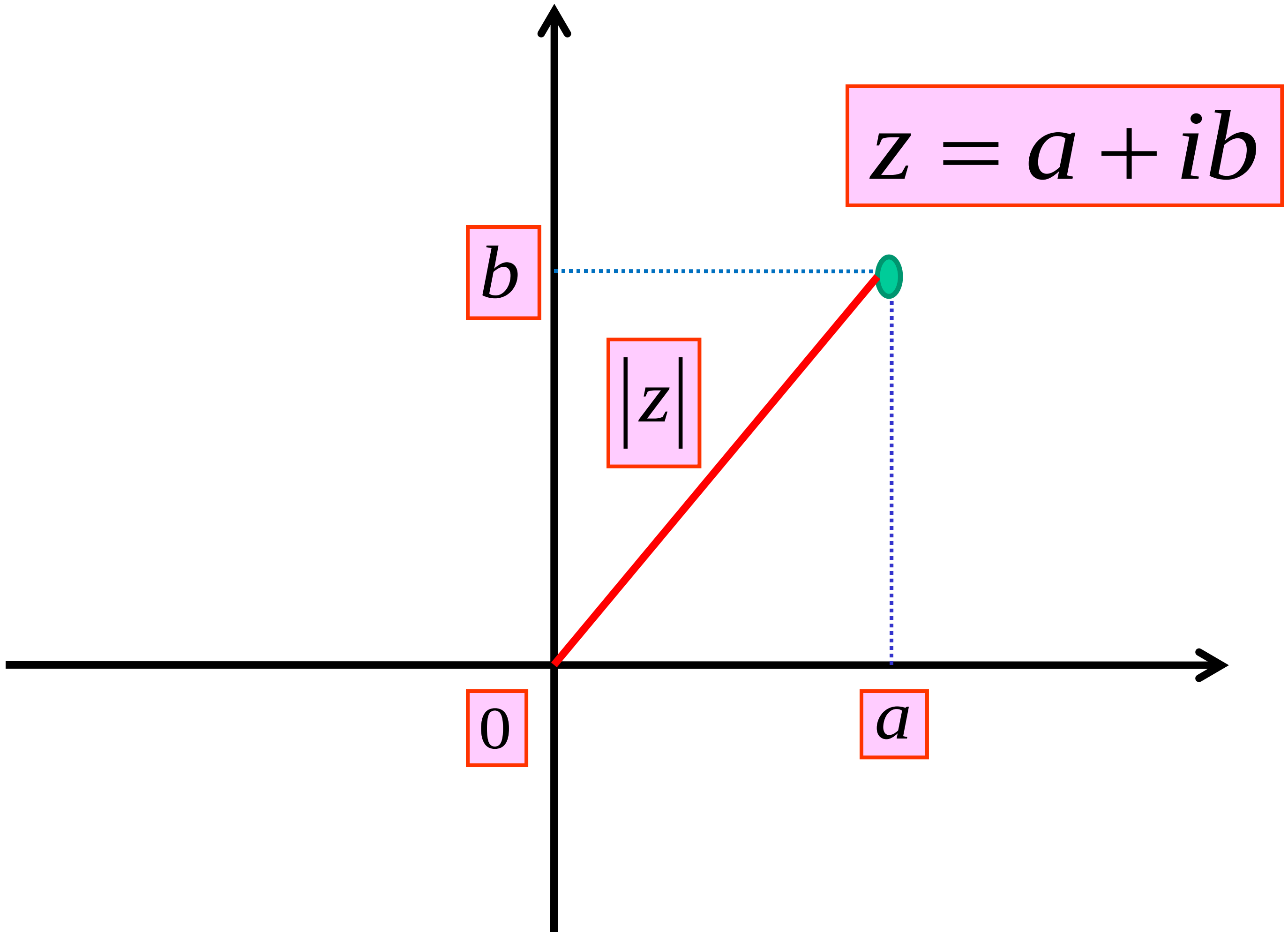
$z = a + ib$
$b$
$a$
$0$
$-b$
$\overline{z} = a - ib$

# Absolute Value of a Complex Number

$$z = a + ib$$

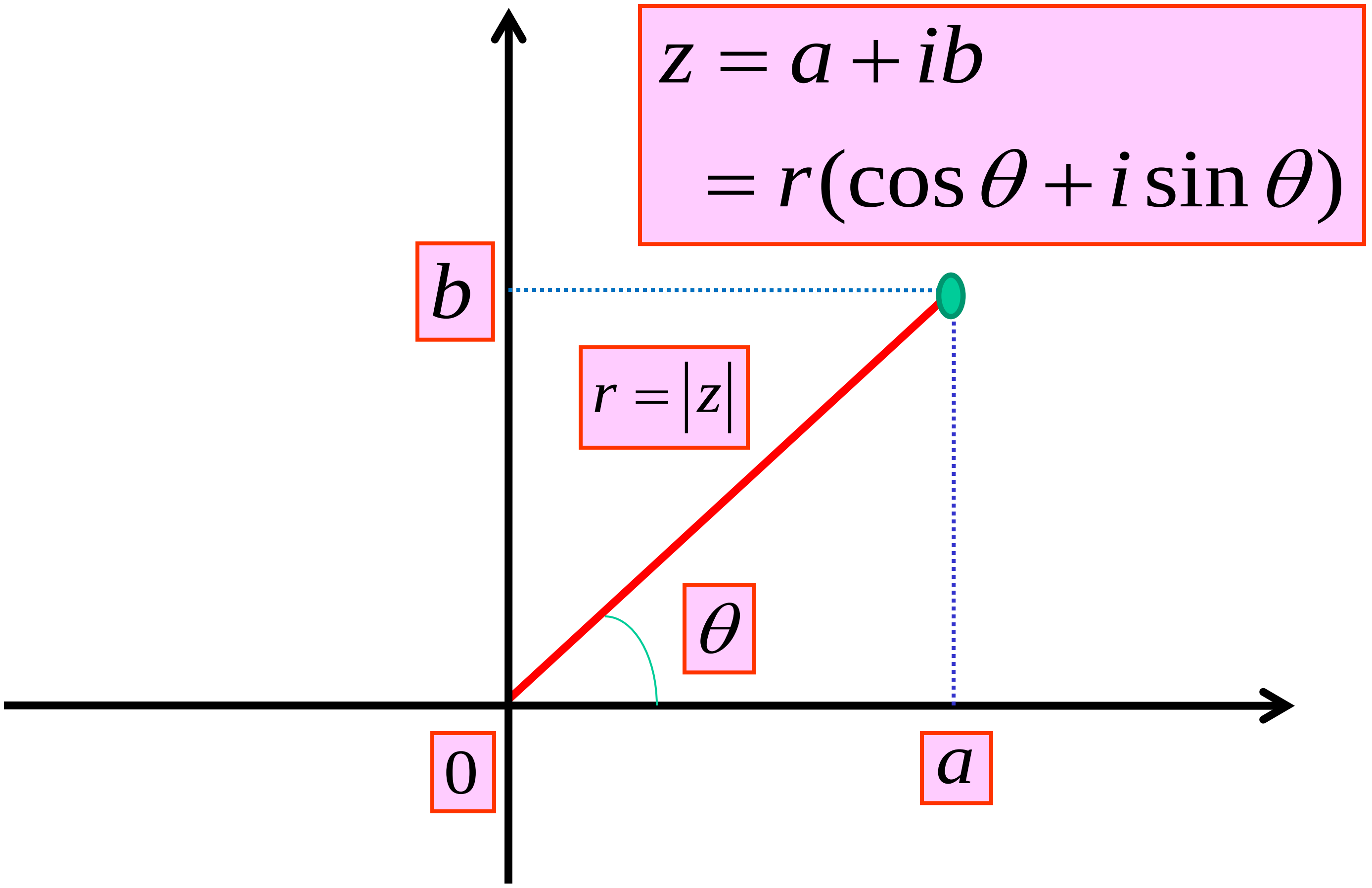
$$\Rightarrow$$

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$



z = a + ib
b
|z|
0
a

# Polar Coordinates of a Complex Number

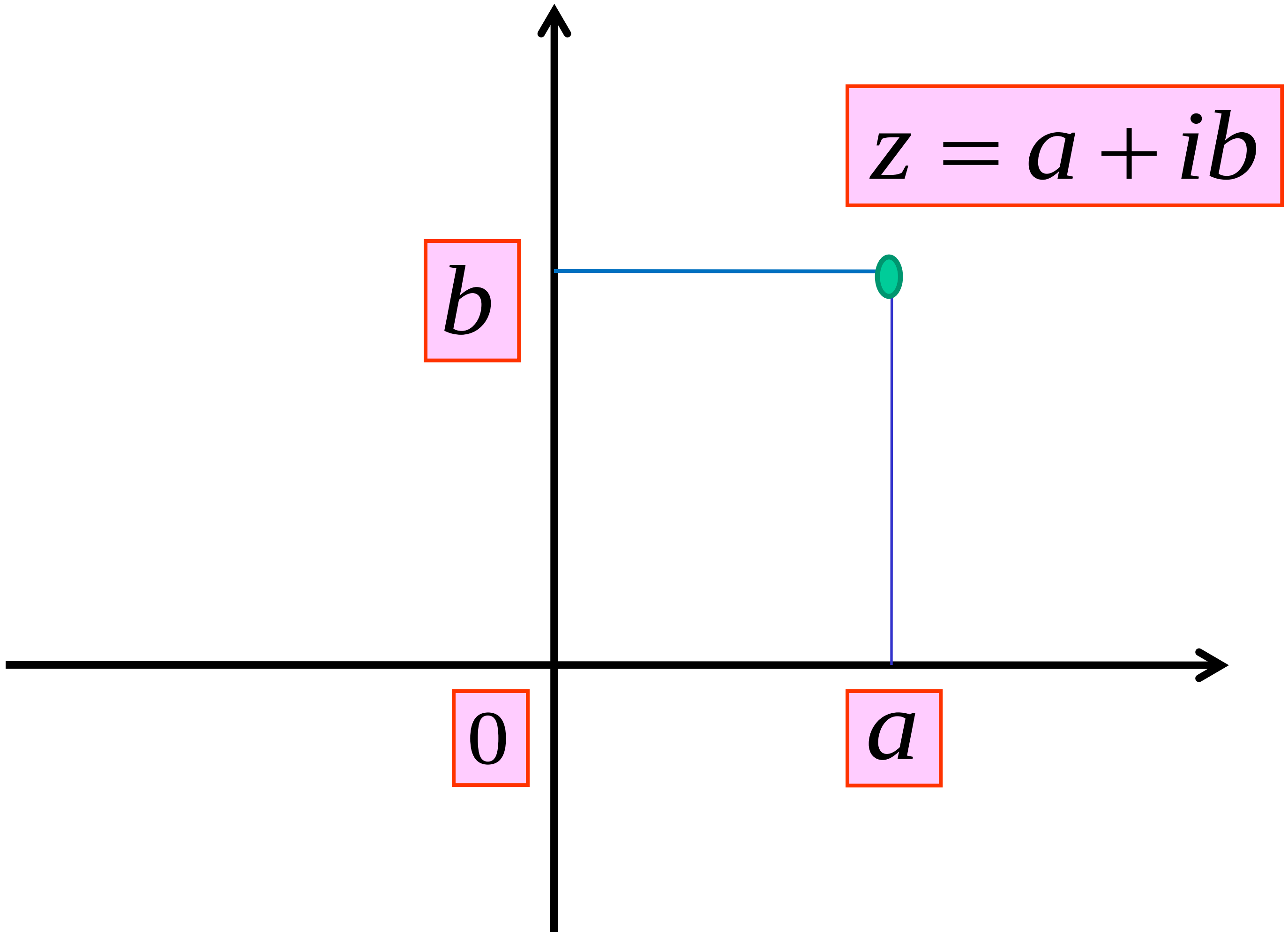


# Sum of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Rightarrow$$

$$z + w = (a + c) + i(b + d)$$



$z = a + ib$
b
0
a

# Difference of Complex Numbers

$$z = a + ib, \quad w = c + id$$

$$\Rightarrow$$

$$z - w = (a - c) + i(b - d)$$

# Product of Complex Numbers

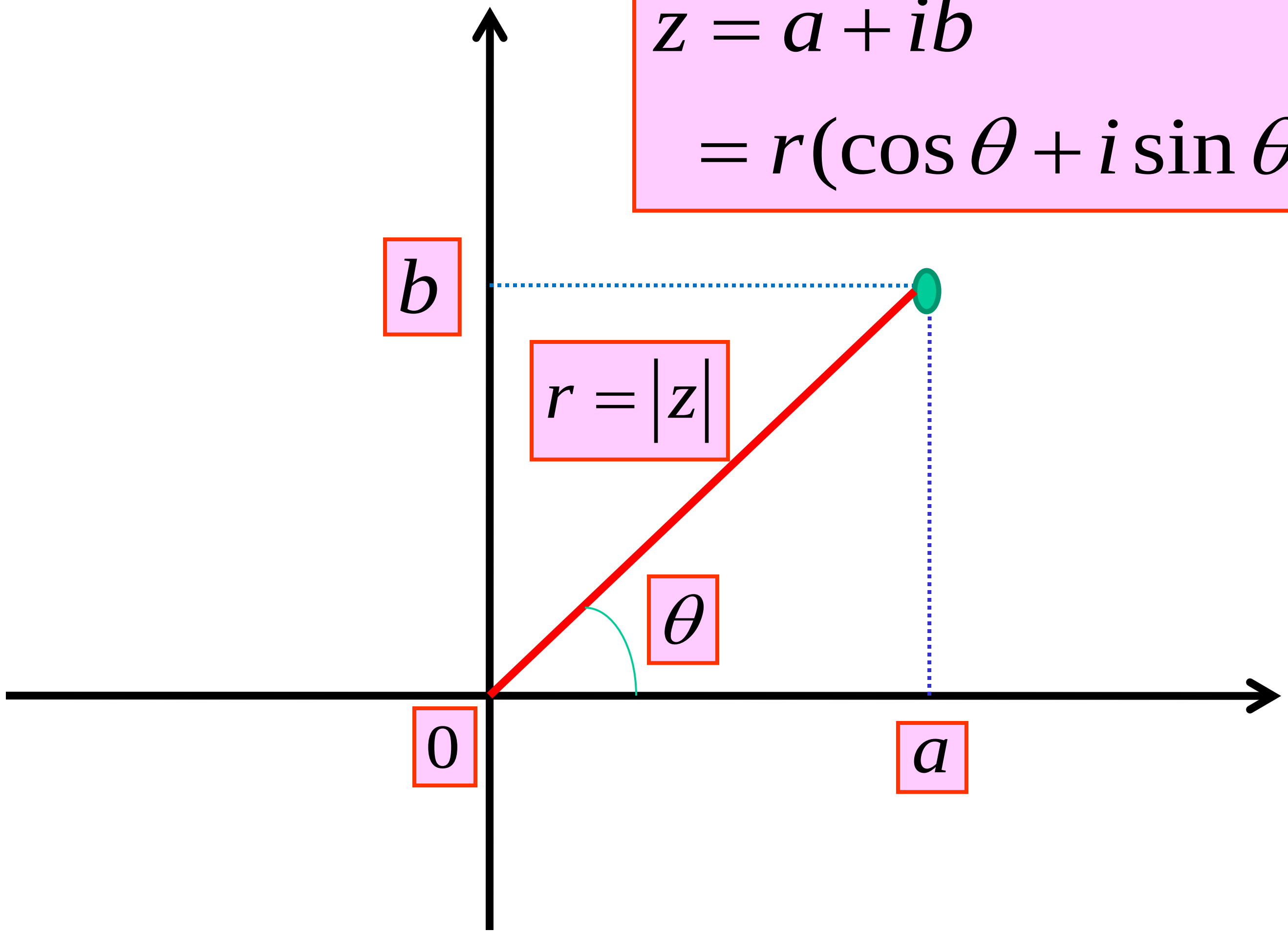
$$z = a + ib, \quad w = c + id$$

$$\Longrightarrow$$

$$zw = (ac - bd) + i(ad + bc)$$

$$i = \sqrt{-1} \Rightarrow i^2 = -1$$

$$z = a + ib$$
$$= r(\cos\theta + i\sin\theta)$$



# Product of Complex Numbers

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$w = s(\cos\omega + i\sin\omega) = se^{i\omega}$$

$$\Longrightarrow$$

$$zw = rs(\cos(\theta+\omega) + i\sin(\theta+\omega))$$

$$= rse^{i(\theta+\omega)}$$

# De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\forall n \in \mathbf{Z}$$

# Euler

- **Leonhard Euler (1707-1783)**

**Swiss Mathematician**

# Leonhard Euler

# Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

# Euler + De Moivre

$$
\begin{aligned}
(e^{i\theta})^n &= (\cos\theta + i\sin\theta)^n \\
&= \cos n\theta + i\sin n\theta \\
&= e^{in\theta} \qquad (\forall n \in \mathbf{Z})
\end{aligned}
$$

# Algebraic Equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

$$a_i \in \mathbf{C}$$

# Fundamental Theorem of Algebra (Gauss)

**Every algebraic equation**

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,\ a_0 \neq 0$$

**has $n$ roots in C counted with multiplicity.**

# Example (1)

$$ax + b = 0,\ a \neq 0$$

$$\Longrightarrow$$

$$x = -\frac{b}{a}$$

# Example (2)

$$ax^2 + bx + c = 0, \quad a \neq 0$$

$$\Rightarrow$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

# Imaginary Number

$$x^2 + 1 = 0$$

$$\Longrightarrow$$

$$x = \pm\sqrt{-1}$$

# Real Numbers

# Real Numbers and Decimal System

| Real Numbers | Decimal System | Classification |
|---|---|---|
| Natural Numbers | Positive Integers | Rational |
| Integers | Integers | Rational |
| Fractional Numbers | Finite Decimal | Rational |
| Fractional Numbers | Recurring Decimal | Rational |
| Non-Fractional Numbers | Non-Recurring Decimal | Irrational |

# Finite Decimal (1)

$$\frac{1}{4} = 0.25$$

$$\frac{118}{25} = 4.72$$

# Finite Decimal (2)

$$0.0625 = \frac{625}{10000}$$

$$= \frac{1}{16}$$

# Recurring Decimal (1)

$$\frac{83}{74} = 1.12162162162216\cdots$$

$$= 1.1\dot{2}\dot{1}\dot{6}$$

$$\frac{89}{13} = 6.846153846153\cdots$$

$$= 6.\dot{8}\dot{4}\dot{6}\dot{1}\dot{5}\dot{3}$$

# Recurring Decimal (2)

$$1.1\dot{2}1\dot{6} = 1.1216216216\cdots$$

$$= 1.1 + 0.0216 + 0.0000216 + \cdots$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4} + 216 \times \frac{1}{10^7} + \cdots$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4}\left(1 + \frac{1}{10^3} + \cdots\right)$$

$$= \frac{11}{10} + 216 \times \frac{1}{10^4} \times \frac{1}{1 - \frac{1}{10^3}}$$

$$= \frac{11205}{9990} = \frac{83}{74}$$

# Non-Recurring Decimal

$$\sqrt{2} = 1.41421356\cdots$$

$$e = 2.718281828845904\cdots$$

# The square root of a prime number is irrational (1)

Let $p$ be a prime number.

Assume that $\sqrt{p}$ is rational.

(*) $\sqrt{p} = \dfrac{n}{m}$

Here the right－hand side is irreducible.

# The square root of a prime number is irrational (2)

$(*) \Rightarrow$

$(**) \quad n^2 = pm^2$

$p$ **is a prime number**

$n^2$ **is a multiple of** $p \Leftrightarrow$

$n$ **is a multiple of** $p$

$n = pa$ **+** $(**) \Rightarrow$

$pm^2 = n^2 = p^2 a^2 \Rightarrow$

$m^2 = pa^2$

# The square root of a prime number is irrational (3)

$m^2 = pa^2$

implies that

$m$ is a multiple of $p$:

$m = pb$

$\Longrightarrow$

$$\sqrt{p} = \frac{n}{m} = \frac{pa}{pb} = \frac{a}{b}$$

(contradiction)

# Square Root of 2（1）

$$(1)\quad 1^2 < 2 < 2^2 \Rightarrow 1 < \sqrt{2} < 2$$

$$\sqrt{2} \in I_1 = [1, 2]$$

$$(2)\ (1.4)^2 = 1.96 < 2 < (1.5)^2 = 2.25$$

$$\Rightarrow 1.4 < \sqrt{2} < 1.5$$

$$\sqrt{2} \in I_2 = [1.4, 1.5]$$

$$(3)\ (1.41)^2 = 1.9881 < 2 < (1.42)^2 = 2.0164$$

$$\Rightarrow 1.41 < \sqrt{2} < 1.42$$

$$\sqrt{2} \in I_3 = [1.41, 1.42]$$

# Square Root of 2（2）

$$(n)\quad {a_n}^2 < 2 < {b_n}^2 \Longrightarrow a_n < \sqrt{2} < b_n$$

$$b_n - a_n = \frac{1}{10^n}$$

$$\sqrt{2} \in I_n = [a_n, b_n]$$

$$\Longrightarrow$$

$$\begin{cases} a_n \uparrow \alpha \\ b_n \downarrow \alpha \end{cases}$$

$$\alpha = \sqrt{2}$$

# Theory
# of
# Real Numbers

# Main Theme

**How do we characterize irrational numbers ?**

**What is the convergence of sequences ?**

# Four Fundamental Theorems in Real Numbers

**(Ⅰ) Dedekind Cut**
**(Ⅱ) Supremum and Infimum**
**(Ⅲ) Convergence of bounded monotone sequences**
**(Ⅳ) Cantor’s Nested-Interval Property**

# Dedekind Cut

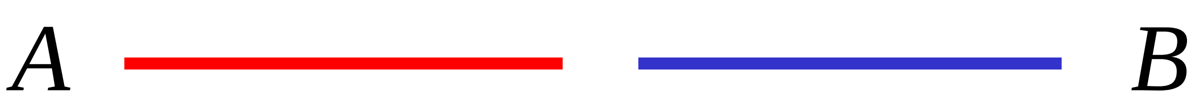
# Cut of Real Numbers

**A cut of real numbers :** $(A, B)$

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\mathbb{R} = A \cup B$$

$$\forall a \in A, \forall b \in B \Rightarrow a < b$$



# Examples

$$A = \{x \in \mathbf{R} : x \le 0\}, \quad B = \{x \in \mathbf{R} : 0 < x\}$$

$$A = \{x \in \mathbf{N} : 0 \le x \le 5\}, \quad B = \{x \in \mathbf{N} : 6 \le x\}$$

$$A = \{x \in \mathbf{N} : 0 \le x < \tfrac{1}{2}\}, \quad B = \{x \in \mathbf{N} : \tfrac{1}{2} < x\}$$

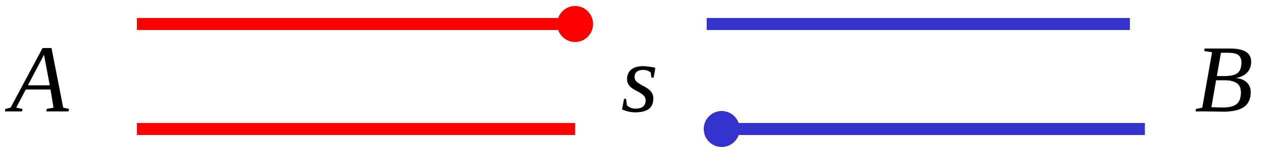
$$A = \{x \in \mathbf{Z} : -\infty < x < 0\}, \quad B = \{x \in \mathbf{Z} : 0 \le x < \infty\}$$

# Dedekind Cut

**A cut** $(A, B)$ **defines a number** $s$ **such that :**

**(1)** $s$ **is the maximum of** $A$**, but** $B$ **has no minimun.**

**(2)** $s$ **is the minimum of** $B$**, but** $A$ **has no maximun.**



# Supremum and Infimum

# Upper Bound and Lower Bound

**(1) The set $S$ is bounded from above**

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\exists M : a \leq M \quad \forall a \in S$$

**$M$ is called a upper bound**

**(2) The set $S$ is bounded from below**

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\exists L : L \leq a \quad \forall a \in S$$

**$L$ is called a lower bound**

# Example (1)

(1) $I = (-\infty, 1)$

$\Rightarrow$

$I$ **is bounded from above**

# Example (2)

$(2)\ J = (-1, \infty)$

$\Rightarrow$

$J$ **is bounded from below**

# Example (3)

$(3)\ K = [-1, 1]$

$\Longrightarrow$

$K$ **is bounded from below and from above**

# Supremum and Infimum (1)

**(I)** $a$ : **supremum of** $S$**:**

(I-1) $a$ **is a upper bound of** $S$.

(I-2) $a$ **is the least upper bound.**

**(II)** $b$ : **infimum of** $S$**:**

(II-1) $b$ **is a lower bound of** $S$.

(II-2) $b$ **is the greatest lower bound.**

# Supremum and Infimum (2)

**(I) Supremum of** $S: a = \sup S$

**(II) Infimum of** $S: b = \inf S$

**(I) Supremum = Generalization of Maximum**

**(II) Infimum = Generalization of Minimum**

# Example (1)

$$(1)\ I = (-1, 1)$$

$$\Rightarrow$$

$$\sup I = 1$$

$$\inf\ I = -1$$

# Example (2)

$$(2)\ J=(-1,1]$$

$$\Rightarrow$$

$$\sup J=\max J=1$$

$$\inf J=-1$$

# Example (3)

$$(3)\ K=[-1,1]$$

$$\Rightarrow$$

$$\sup K=\max K=1$$

$$\inf\ K=\min K=-1$$

# Example (4)

$$a_n = (-1)^n + \frac{1}{n}$$

$$a_1 = 0,\ a_3 = -\frac{2}{3},\ a_5 = -\frac{4}{5}$$

$$a_2 = \frac{3}{2},\ a_4 = \frac{5}{4},\ a_6 = \frac{7}{6}$$

$$\Longrightarrow$$

$$\begin{cases} \sup a_n = \max a_n = \frac{3}{2} \\ \inf a_n = -1 \end{cases}$$

# Weierstrass' Theorem

# Weierstrass（1815—1897）

# Existence of Supremum and Infimum

**(I) A set** $S$ **of bounded from above has the supremum** $a : \exists a = \sup S$

**(II) A set** $S$ **of bounded from below has the infimum** $b : \exists b = \inf S$

# Sequences

# Sequences versus Functions

| | Domain of Definition | Range |
|---|---|---|
| Sequence | Natural Numbers | Real Numbers |
| Functions | Real Numbers | Real Numbers |

# Definition

**The sequnce** $\{a_n\}$ **converges to** $a$

$$\overset{\text{def}}{\Longleftrightarrow}$$

$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$

**Notation :** $\lim_{n\to\infty} a_n = a$

# Fundamental Example

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

# Proof

$$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbf{N} \text{ such that}$$

$$\forall n \geq N \Rightarrow \frac{1}{n} < \varepsilon$$

# Archimedes' Principle

# Archimedes' Principle

$$\forall\ a, b > 0,\ \ \exists n \in \mathbf{N} \text{ such that}$$

$$\boxed{na > b}$$

$$\forall a, b > 0,\ \exists n \in \mathbf{N} \text{ such that}$$

$$\boxed{na > b}$$

$$\Updownarrow$$

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

# Examples (1)

$$(1)\ \lim_{n\to\infty}\frac{n+1}{n}=1$$

$$(2)\ \lim_{n\to\infty}\frac{n+1}{n^2}=0$$

$$(3)\ \lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)=0$$

# Example (2)

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \textbf{if} \quad 0 < a < 1 \\ 1 & \textbf{if} \quad a = 1 \\ +\infty & \textbf{if} \quad a > 1 \end{cases}$$

# Proof

$$a > 1 \Rightarrow a = 1 + h, \;\; h > 0$$

$$\Rightarrow a^n = (1 + h)^n$$

$$= 1 + nh + \frac{n(n-1)}{2} h^2 + \cdots$$

$$> 1 + nh$$

$$\Rightarrow \lim_{n \to \infty} a^n = +\infty$$

# Examples (3)

$$(1)\ \lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad \textbf{for} \quad a > 0$$

$$(2)\ \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

# Binomial Theorem

$$\forall a, b \in \mathbf{R}, n \in \mathbf{N}$$

$$(a+b)^n = \sum_{r=0}^{n} {}_nC_r \, a^{n-r} b^r$$

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

# Examples

$$(1)\ (a+b)^2 = a^2 + 2ab + b^2$$

$$(2)\ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(3)\ (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

# Bounded Sequences

# Fact

**A convergent sequence is bounded.**

# Bolzano-Weierstrass Theorem

# Bolzano（1781—1848）

# Weierstrass（1815—1897）

# Bolzano-Weierstrass Theorem

**Every bounded sequence has a convergent subsequence.**

# Example (1)

$$a_n = (-1)^n$$

$$\Rightarrow \begin{cases} a_{2k} = 1 \to 1 \\ a_{2k+1} = -1 \to -1 \end{cases}$$

# Example (2)

$$a_n = (-1)^n + \frac{1}{n}$$

$$\Rightarrow \begin{cases} a_{2k} = 1 + \dfrac{1}{2k} \to 1 \\ a_{2k+1} = -1 + \dfrac{1}{2k+1} \to -1 \end{cases}$$

# Bounded Monotone Sequence

# Fundamental Theorem

**Every bounded, monotone increasing sequence itself converges.**

$a_n \leq \exists M$ （**Bounded**）

$a_n \leq a_{n+1}$ （**Monotone increasing**）

# Example 1 (Golden Ratio)

$$a_1 = 1, \quad a_2 = \sqrt{1 + a_1}$$

$$a_n = \sqrt{1 + a_{n-1}} \quad (n \geq 3)$$

$$\Rightarrow$$

$$(1)\, 1 \leq a_n < 3$$

$$(2)\, 1 \leq a_n < a_{n+1}$$

$$(3)\, \lim_{n \to \infty} a_n = \frac{1 + \sqrt{5}}{2}$$

# Example 2 (Napier's Number)

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$e = {}^{\exists}\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

# Proof (1)

$0 < a_n < a_{n+1}$ **(Monotone increasing)**

$\because$

$$a_n = 1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1-\frac{1}{n}}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \cdots + \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})}{n!}$$

# Proof (2)

$0 < a_n < 3$ **(Boundedness)**

$\because$

$$a_n < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + 2\left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$< 3$$

# Geometric Series

$$\forall b, r \in \mathbf{R}, n \in \mathbf{N}$$

$$\sum_{k=1}^{n} br^{k-1}$$

$$= b + br + br^2 + \cdots + br^{n-2} + br^{n-1}$$

$$= \begin{cases} nb & \textbf{if} \quad r = 1 \\ \dfrac{b(1-r^n)}{1-r} & \textbf{if} \quad r \neq 1 \end{cases}$$

# Example 3 (Euler's Number)

$$b_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

$$\gamma = {}^{\exists}\lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right)$$

# Numerical Analysis

# Numerical Computing with BASIC

# Napier's Number

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= 2.718281828459045\cdots$$

# Napier's Number (Sequence Version)

$$e = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n$$

$$a_1 = 2$$
$$a_2 = 2.25$$
$$a_3 = 2.37037037037037$$
$$a_4 = 2.44140625$$
$$a_5 = 2.48832$$
$$a_6 = 2.52162637174211$$
$$a_7 = 2.54649969704071$$
$$a_8 = 2.56578451395034$$
$$a_9 = 2.58117479171319$$
$$a_{10} = 2.5937424601$$

# Napier's Number (Series Version)

$$e = \lim_{n \to \infty} A_n$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

$$A_1 = 1$$
$$A_2 = 2$$
$$A_3 = 2.5$$
$$A_4 = 2.666666666666665$$
$$A_5 = 2.708333333333333$$
$$A_6 = 2.716666666666663$$
$$A_7 = 2.718055555555554$$
$$A_8 = 2.718253968253968$$
$$A_9 = 2.718278769841270$$
$$A_{10} = 2.718281525573192$$

# Euler's Number

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$$

$$= 0.57721\cdots$$

$$b_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

$$b_n = 0.5776322736976\,98 \ (n = 1200)$$

$$b_n = 0.5774656440680\,48 \ (n = 2000)$$

$$b_n = 0.5772656640678\,27 \ (n = 10000)$$

# Square Root of 2

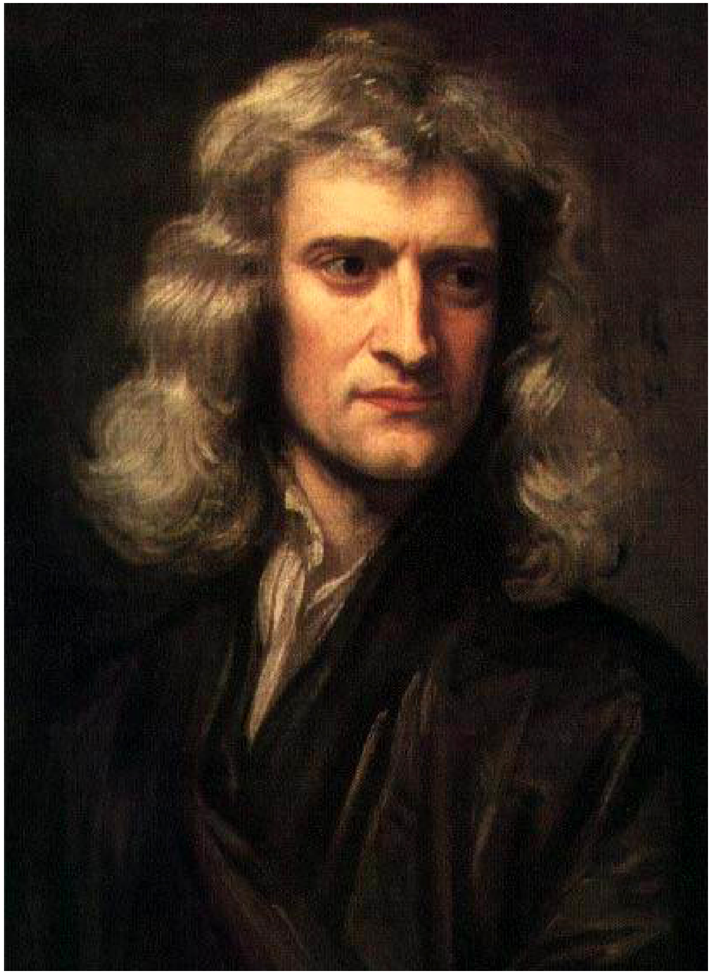
$$\sqrt{2} = 1.41421356\cdots$$

# Newton's Method versus Bisection Method

| Method | Newton's Method | Bisection Method |
| --- | --- | --- |
| Hypotheses | Differentiability<br>Monotonicity | Continuity |
| Merits<br>Demerits | Strong Hypotheses<br>Rapid Convergence | Weak Hypotheses<br>Slow Convergence |
| Background | Convergence of Monotone Sequences | Intermediate Value Theorem |

# Newton's Approximation Method

# Isaac Newton (1642-1727)



# Fundamental Theorem

**Every bounded, monotone increasing sequence itself converges.**

$a_n \leq \exists M$ （**Bounded**）

$a_n \leq a_{n+1}$ （**Monotone increasing**）

# Newton's Approximation Method

$$r > 0,\ a_0 > 0$$

$$a_{n+1} := \frac{1}{2}\left( a_n + \frac{r}{a_n} \right), \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow$$

$$a_n \downarrow \sqrt{r} \quad (n \to \infty)$$

# Example (Square root of 2)

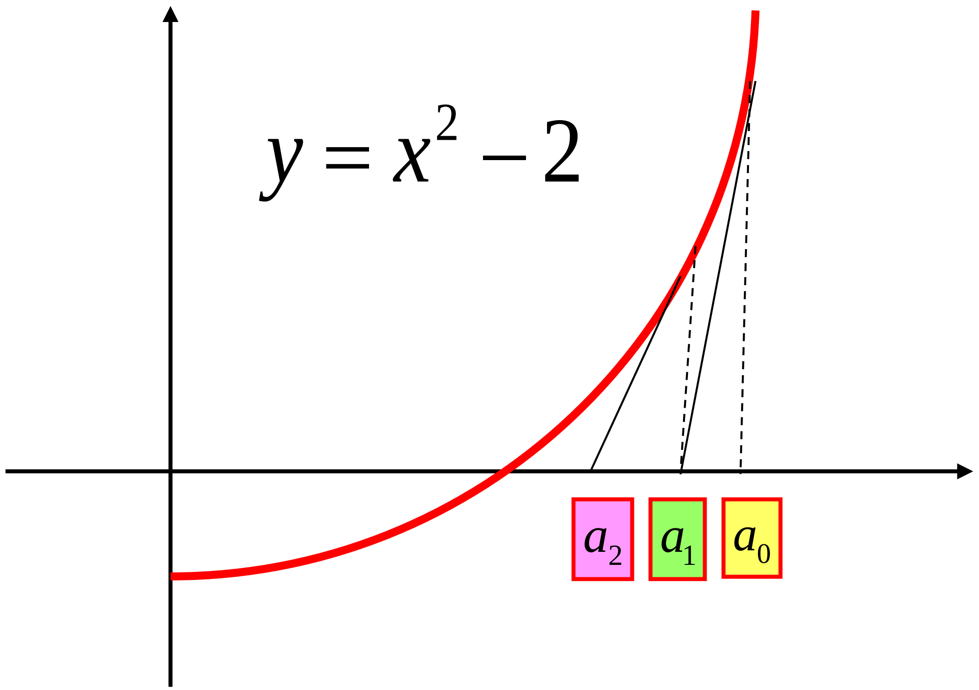
$$a_0 = 2, \qquad a_1 = \frac{3}{2}$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

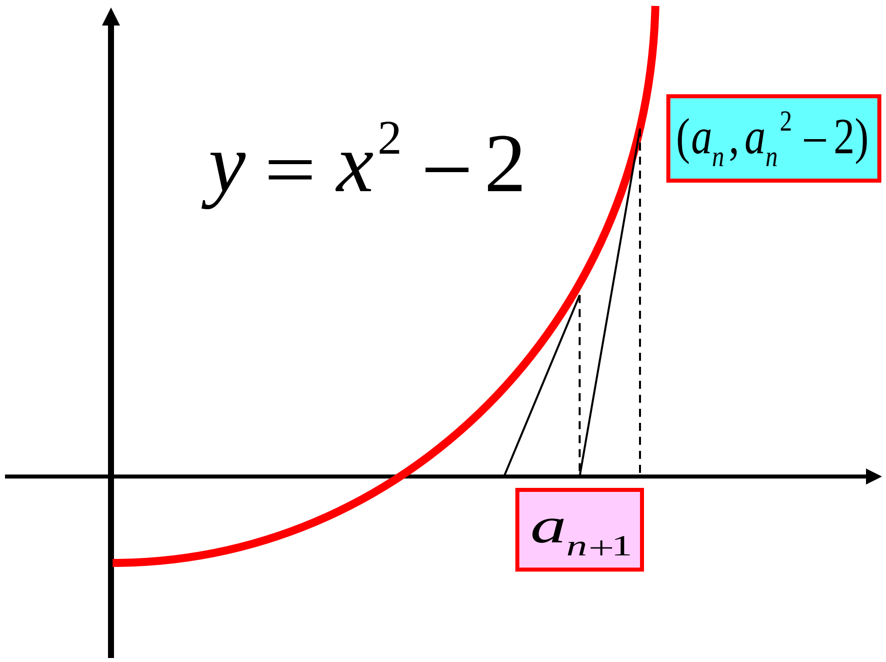
$$\Longrightarrow$$

$$\lim_{n\to\infty} = \sqrt{2}$$

# Newton's Method (1)



# Newton's Method (2)




**Tangent Line at** $(a_n, a_n^2 - 2)$ **:**

$$y = 2a_n(x - a_n) + a_n^2 - 2 = 2a_n x - a_n^2 - 2$$

# Numerical Computing
# with
# BASIC

$$a_1 = 1.5$$

$$a_2 = 1.41666666666666667$$

$$a_3 = 1.41421568627450999$$

$$a_4 = 1.41421356237468999$$

$$a_5 = 1.41421356237309510$$

$$a_6 = 1.41421356237309500$$

# Bisection Method

# Principle of Successive Subdivision

# Cantor (1845—1918)

# Cantor's Nested-Interval Property

$\{I_n\}$ **Sequence of closed intervals**

(1) $I_{n+1} \subset I_n$

(2) $|I_n| \to 0$

$\Longrightarrow$

$$\bigcap_{n=1}^{\infty} I_n = \{\textbf{One Point}\}$$

# Sequence Version

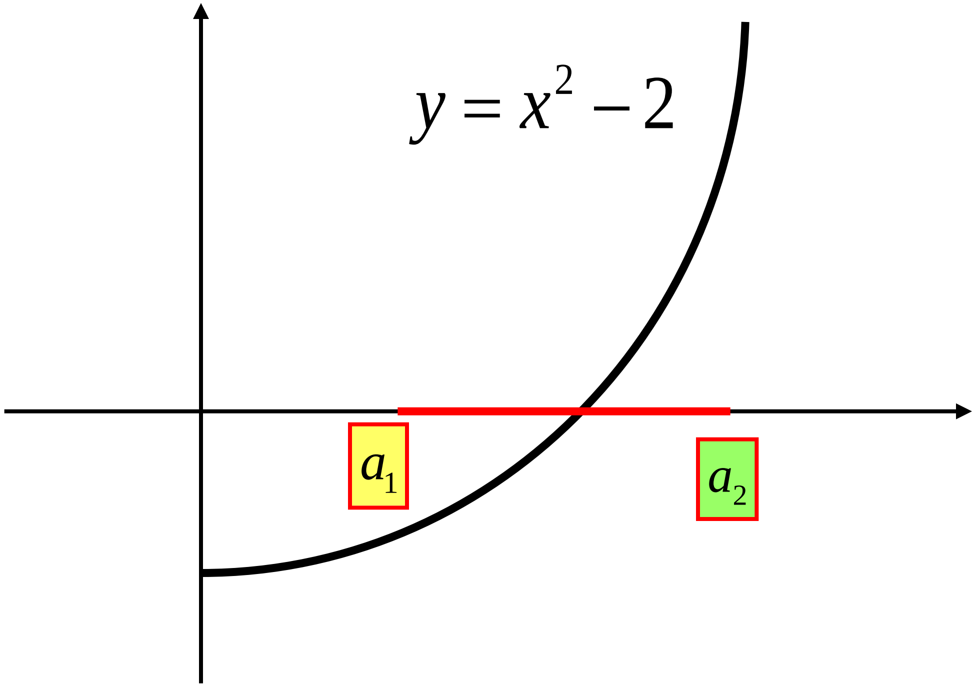
(1) $a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots \le b_{n+1} \le b_n \le b_2 \le b_1$

(2) $b_n - a_n \to 0$ as $n \to \infty$

$\Rightarrow$

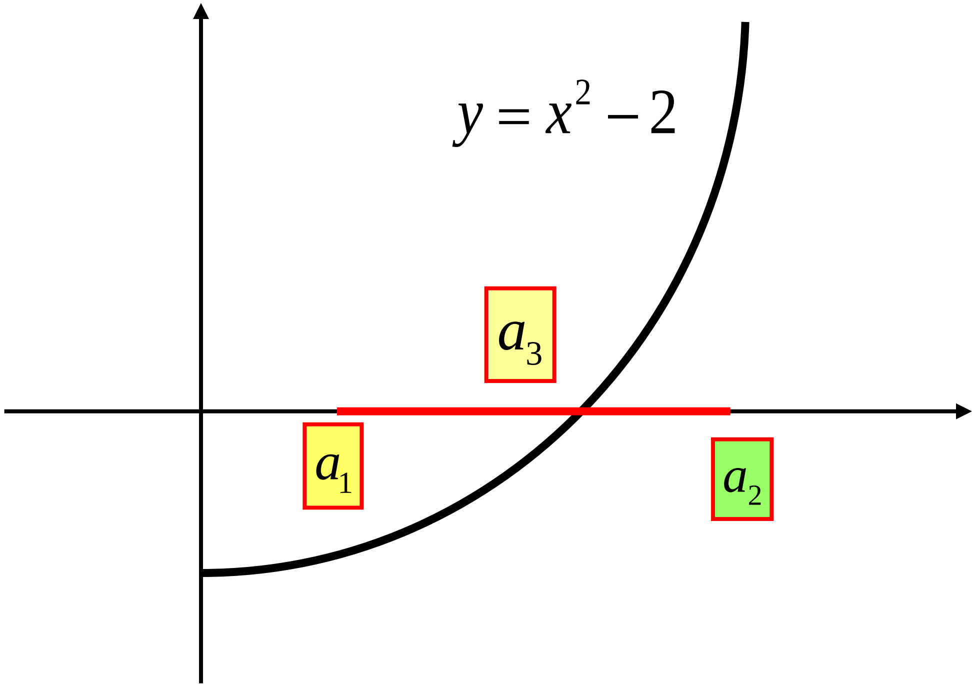
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

# Bisection Method（1）

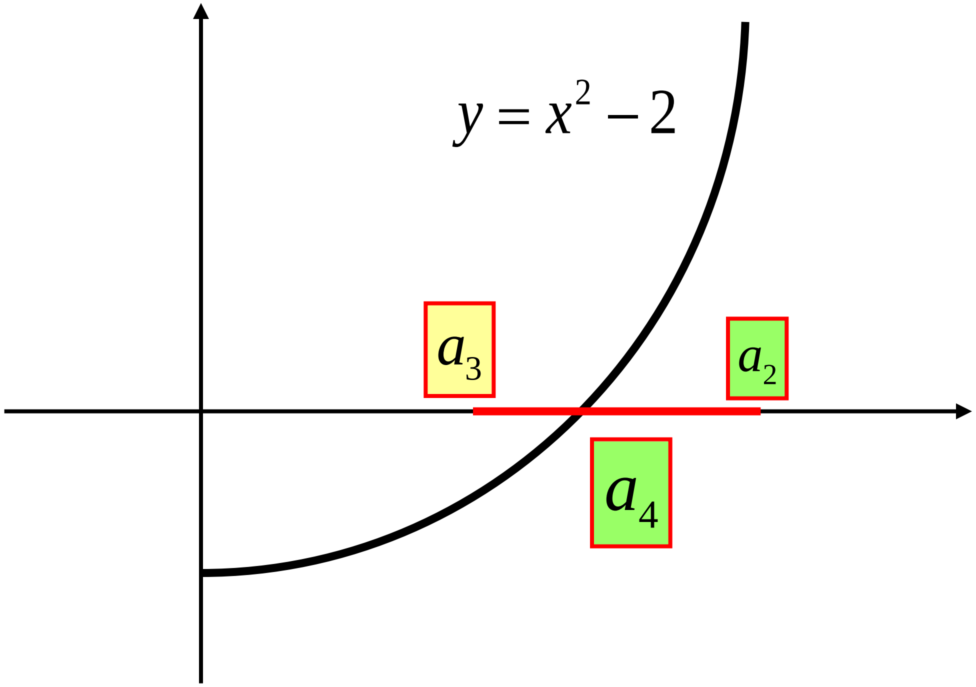



$\sqrt{2}$ : **Square Root of 2**

# Bisection Method（2）



# Bisection Method（3）



# Numerical Computing
# with
# BASIC

$$a_1 = 1.5$$
$$a_2 = 1.25$$
$$a_3 = 1.375$$
$$a_4 = 1.4375$$
$$a_5 = 1.40625$$
$$a_6 = 1.421875$$
$$a_7 = 1.4140625$$
$$a_8 = 1.41796875$$
$$a_9 = 1.416015625$$
$$a_{10} = 1.4150390625$$

# Number Pi

$$\pi = 3.14159265\cdots$$

$$\frac{\pi}{4} = 0.785398163397459\cdots$$

# Taylor Series Version

# Taylor Series

$$\tan^{-1} x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$+ (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \cdots$$

# Abel's Theorem

$A = \sum_{n=1}^{\infty} a_n$ **converges**

$\Rightarrow$

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \to A \text{ as } x \uparrow 1$$

# Leibniz's Series

$$\frac{\pi}{4} = \tan^{-1} 1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

# Numerical Computing
# with
# BASIC

$$a_1 = 1$$
$$a_2 = 0.666666666666667$$
$$a_3 = 0.866666666666667$$
$$a_4 = 0.723809523809524$$
$$a_5 = 0.834920634920634$$
$$a_6 = 0.744011544011544$$
$$a_7 = 0.820934620934621$$
$$a_8 = 0.754267954267954$$
$$a_9 = 0.813091483679719$$
$$a_{10} = 0.760459904732351$$

# Machin’s Series

# Machin's Series

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{5}\right)^{2n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{239}\right)^{2n+1}$$

# Proof (1)

$$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{2\tan A}{1 - \tan^2 A}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

# Proof (2)

$$\tan A = \frac{1}{5}$$

$$\Rightarrow$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{5}{12}$$

$$\tan 4A = \frac{2\tan 2A}{1 - \tan^2 2A} = \frac{120}{119}$$

# Proof (3)

$$\tan\left(4A - \frac{\pi}{4}\right) = \frac{\tan 4A - \tan\frac{\pi}{4}}{1 + \tan 4A \tan\frac{\pi}{4}}$$

$$= \frac{\tan 4A - 1}{1 + \tan 4A} = \frac{1}{239}$$

# Proof (4)

$$\tan^{-1}\left(\frac{1}{239}\right) = 4A - \frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}$$

$$\Rightarrow$$

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

# Taylor Series

# Example

$$\tan^{-1} x$$

$$= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$(-1 < x \leq 1)$$

# Numerical Computing
# with
# BASIC

$$A_1 = 0.7595815899581590$$
$$A_2 = 0.785149257331515$$
$$A_3 = 0.785405257331259$$
$$A_4 = 0.785397943045544$$
$$A_5 = 0.785398170601100$$
$$A_6 = 0.785398163153827$$
$$A_7 = 0.785398163405899$$
$$A_8 = 0.785398163397151$$
$$A_9 = 0.785398163397459$$
$$A_{10} = 0.785398163397448$$

# Series

# Series of Positive Terms

# Series of Positive Terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

$$a_n \geq 0$$

# Geometric Series

$$a + ar + ar^2 + \cdots + ar^n + \cdots$$

$$= \begin{cases} +\infty & \text{if } r = 1 \\ \dfrac{a}{1-r} & \text{if } 0 < r < 1 \end{cases}$$

# Proof

$$0 < r < 1$$

$$\Rightarrow S_n = a + ar + \cdots + ar^{n-1}$$

$$= \frac{a(1-r^n)}{1-r} \to \frac{a}{1-r}$$

$$\left(\because \lim_{n\to\infty} r^n = 0\right)$$

# Example (1)

$$1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$$

# Example (2)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$$

# Example (3)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \textbf{converges for } p > 1 \\ \textbf{diverges for } 0 < p \le 1 \end{cases}$$

# Cauchy's Root Test

$\sum_{n=1}^{\infty} a_n$ : **Series of positive terms**

$$\exists \lim_{n\to\infty} \sqrt[n]{a_n} = r$$

$$\Rightarrow$$

$$(1)\ \ r<1 \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$$

$$(2)\ \ r>1 \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$$

# D'Alembert's Test

$\sum_{n=1}^{\infty} a_n$ : **Series of positive terms**

$$\exists \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

$\Rightarrow$

$$(1)\quad r < 1 \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$$

$$(2)\quad r > 1 \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$$

# Numerical Computing with BASIC

# Square of Number Pi

$$\frac{\pi^2}{6} = 1.64493406684823\cdots\cdot$$

$$A_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

$$A_1 = 1$$

$$A_2 = 1.25$$

$$A_{50} = 1.625137273362152$$

$$A_{100} = 1.634983900184 9$$

$$A_{300} = 1.641606282897 63$$

$$A_{600} = 1.643268788298 87$$

$$A_{700} = 1.643506515341 94$$

$$A_{800} = 1.643684847772 75$$

$$A_{900} = 1.643823572792 52$$

$$A_{1000} = 1.643934566681 61$$

$$A_{1500} = 1.644267622354 4$$

# Alternating Series

# Alternating Series

$$\sum_{n=1}^{\infty}(-1)^{n-1} a_n = a_1 - a_2 + a_3 \cdots + a_{2k-1} - a_{2k} + \cdots$$

$$a_n > 0$$

# Leibniz's Theorem

(1) $a_n > a_{n+1}$ **(monotone decreasing)**

(2) $\lim_{n \to \infty} a_n = 0$

$\Rightarrow$

$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ **converges**

# Examples

$$(1)\quad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\log_e 2$$

$$(2)\quad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$$

# Continuity of Functions

# Definition of Continuity

**Let** $f(x)$ **be a function defined on an interval** $I$.

$f(x)$ **is continuous at** $a \in I$

$$\overset{\text{def}}{\Leftrightarrow}$$

$\forall \varepsilon > 0,\ \exists \delta = \delta(a, \varepsilon) > 0$ **such that**

$x, y \in I,\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

# Example (1)

$$f(x) = x^2$$

$$I = [0, \infty)$$

# Proof

$$(1)\ x > a:$$

$$\delta_1(a,\varepsilon) = \sqrt{a^2+\varepsilon} - a$$

$$(2)\ 0 \le x < a:$$

$$\delta_2(a,\varepsilon) = a - \sqrt{a^2-\varepsilon}$$

$$\Rightarrow$$

$$\delta(a,\varepsilon) = \min\{\delta_1(a,\varepsilon), \delta_2(a,\varepsilon)\}$$

# Example (2)

$$g(x) = \frac{1}{x}$$

$$I = (0, \infty)$$

# Proof

(1) $x > a:$

$$\delta_1(a,\varepsilon) = \frac{\varepsilon a^2}{1-\varepsilon a}$$

(2) $0 \le x < a:$

$$\delta_2(a,\varepsilon) = \frac{\varepsilon a^2}{1+\varepsilon a}$$

$\Rightarrow$

$$\delta(a,\varepsilon) = \min\{\delta_1(a,\varepsilon), \delta_2(a,\varepsilon)\}$$

$$= \frac{\varepsilon a^2}{1+\varepsilon a}$$

# Criterion of Continuity (Sequence Version)

$f(x)$ **is continuous** at $x = a$

$\Leftrightarrow$

$x_n \to a \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(a)$

# Example (1)

$$\sin x$$

$$\because \sin(a+h)$$

$$= \sin a \cos h + \cos a \sin h$$

$$\to \sin a \quad \textbf{as} \quad h \to 0$$

# Example (2)

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \textbf{if } x \neq 0, \\ 0 & \textbf{if } x = 0 \end{cases}$$

$$\because |f(h) - f(0)|$$

$$= \left| h \sin \frac{1}{h} \right|$$

$$\leq |h| \longrightarrow 0 \quad \textbf{as } h \longrightarrow 0$$

# Example of
# a Discontinuous Function

$$g(x) = \begin{cases} \sin\dfrac{1}{x} & \textbf{if } x \neq 0, \\ 0 & \textbf{if } x = 0 \end{cases}$$

# Proof

$$(1)\ x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \longrightarrow 0 \quad \textbf{as } n \longrightarrow \infty$$

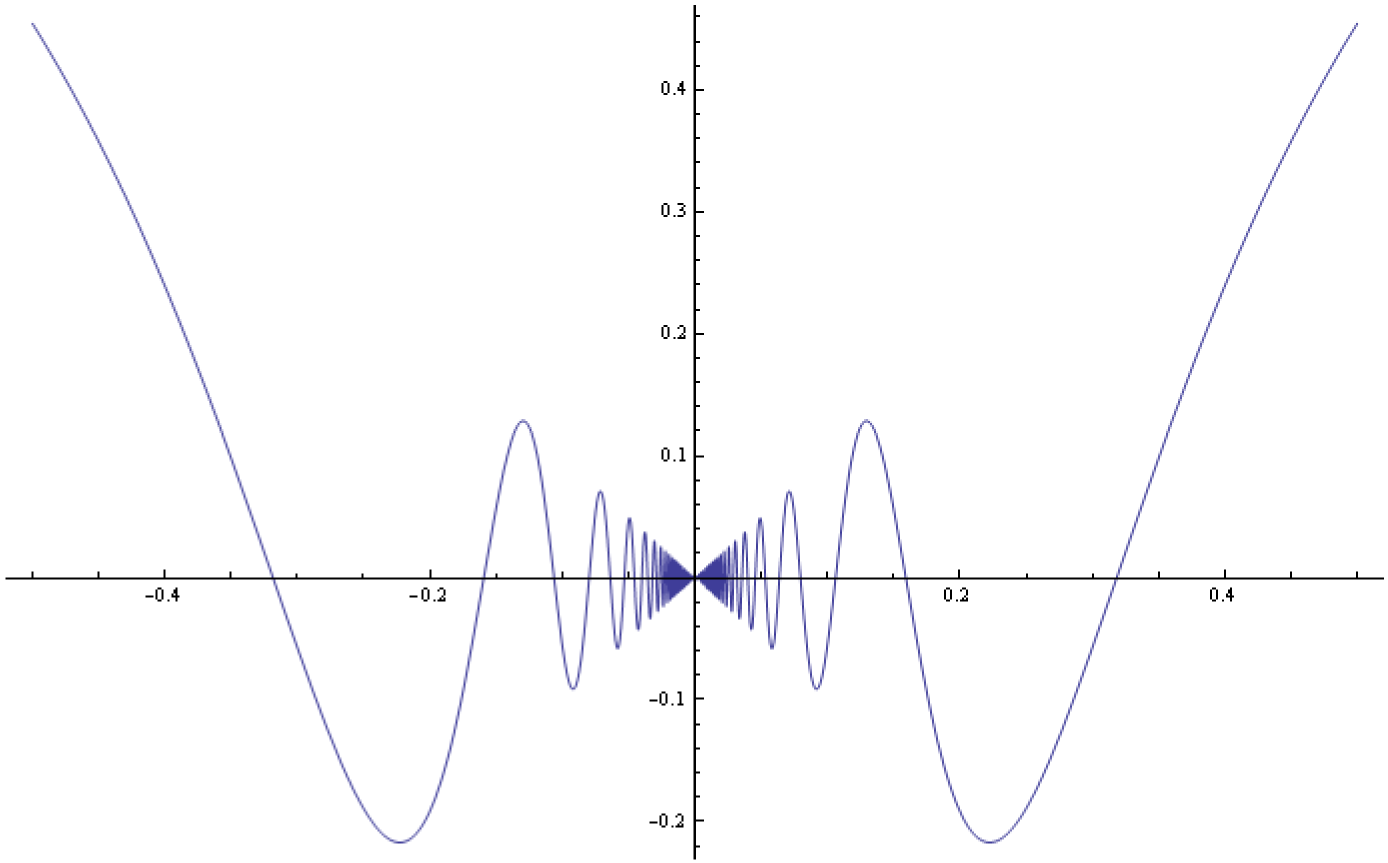
$$g(x_n) = \sin\frac{1}{x_n} = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$$

$$(2)\ y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi} \longrightarrow 0 \quad \textbf{as } n \longrightarrow \infty$$

$$g(y_n) = \sin\frac{1}{y_n} = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = -1$$

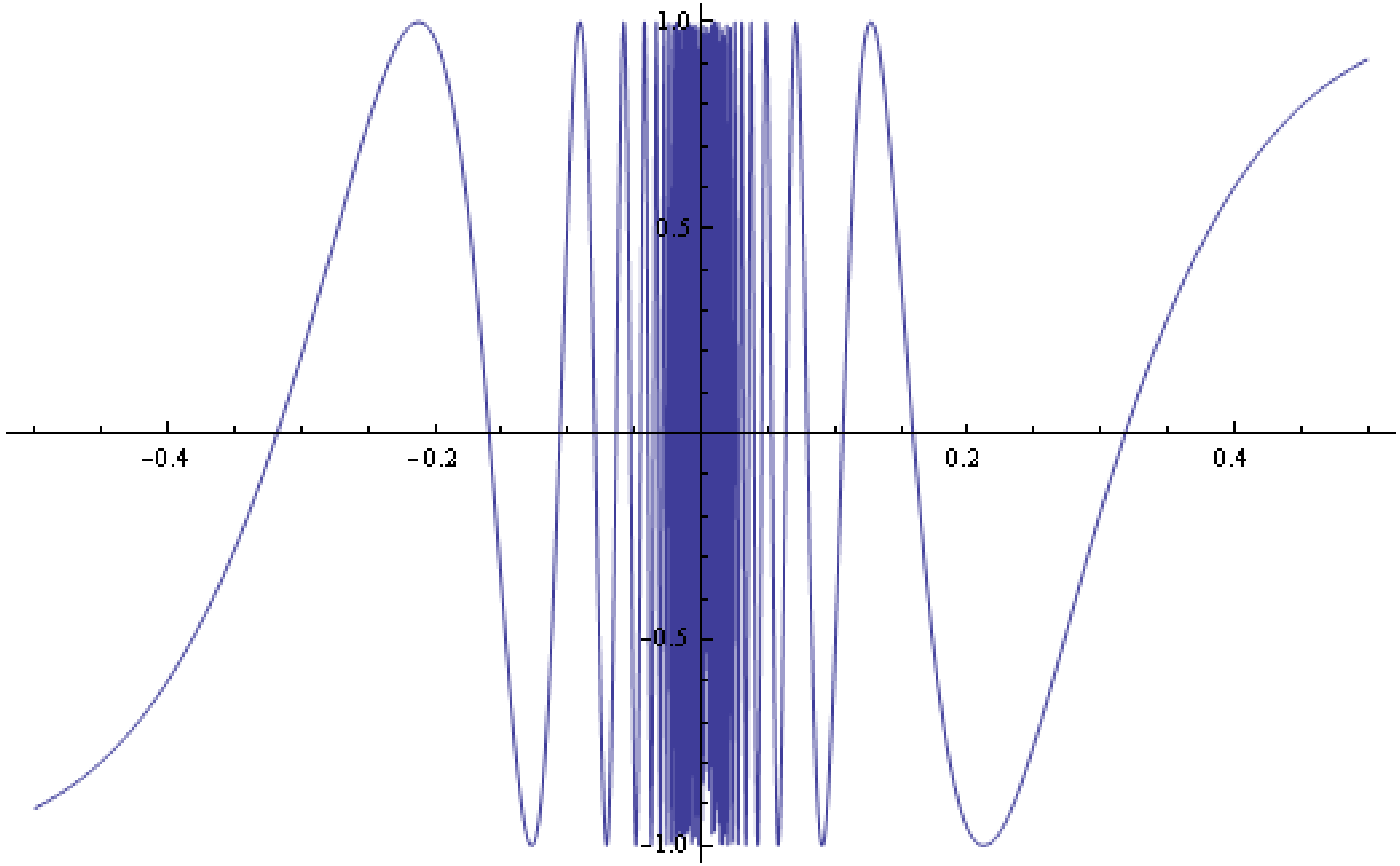
# Numerical Computing
# with
# BASIC

# Graph of $f(x)$



# Numerical Computing with BASIC

# Graph of $g(x)$



# Operations of Continuous Functions

$f(x), g(x)$ **are continuous**

$\Longrightarrow$

**(1)** $f(x) \pm g(x)$ **is continuous**

**(2)** $kf(x)$ **is continuous**

**(3)** $f(x)g(x)$ **is continuous**

**(4)** $\dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$ **is continuous**

# Continuity of Composite functions

$f(u)$ **is continuous**

$u = g(x)$ **is continuous**

$\Longrightarrow$

$f(g(x))$ **is continuous**

# Weierstrass' Theorem

# Maximum Value Theorem

$f(x)$ **is continuous on** $I = [a, b]$

$\Rightarrow$

$f(x)$ **takes its maximum**

# Proof (1)

$$\exists \alpha = \sup\{ f(x) \mid a \le x \le b \}$$

$$\Rightarrow$$

$$\forall n \in \mathbf{N},\ \ \exists x_n \in [a,b]$$

**such that**

$$\alpha - \frac{1}{n} < f(x_n) \le \alpha$$

# Proof (2)

$$\exists\{x_{n'}\}\subset\{x_n\},\ \exists c\in[a,b]$$

**such that**

$$x_{n'}\longrightarrow c$$

**(Bolzano - Weierstrass)**

# Proof (3)

$$(a)\ \alpha - \frac{1}{n'} < f(x_{n'}) \leq \alpha$$

$$(b)\ x_{n'} \to c$$

$$\Rightarrow$$

$$f(c) = \lim_{n' \to \infty} f(x_{n'}) = \alpha$$

# Minimum Value Theorem

$f(x)$ **is continuous on** $I = [a, b]$

$\Rightarrow$

$f(x)$ **takes its minimum**

# Proof (1)

$$\exists \beta = \inf \{ f(x) \mid a \le x \le b \}$$

$$\Rightarrow$$

$$\forall n \in \mathbf{N},\ \ \exists y_n \in [a,b]$$

**such that**

$$\beta \le f(y_n) < \beta + \frac{1}{n}$$

# Proof (2)

$$\exists \{ y_{n'} \} \subset \{ y_n \}, \exists d \in [a,b]$$

**such that**

$$y_{n'} \to d$$

**(Bolzano - Weierstrass)**

# Proof (3)

$$(a)\ \beta \le f(y_{n'}) < \beta + \frac{1}{n'}$$

$$(b)\ y_{n'} \longrightarrow d$$

$$\Longrightarrow$$

$$f(d) = \lim_{n' \to \infty} f(y_{n'}) = \beta$$

# Intermediate Value Theorem

$f(x)$ **is continuous on** $I = [a, b]$

$f(a) < 0, \; f(b) > 0$

$\Rightarrow$

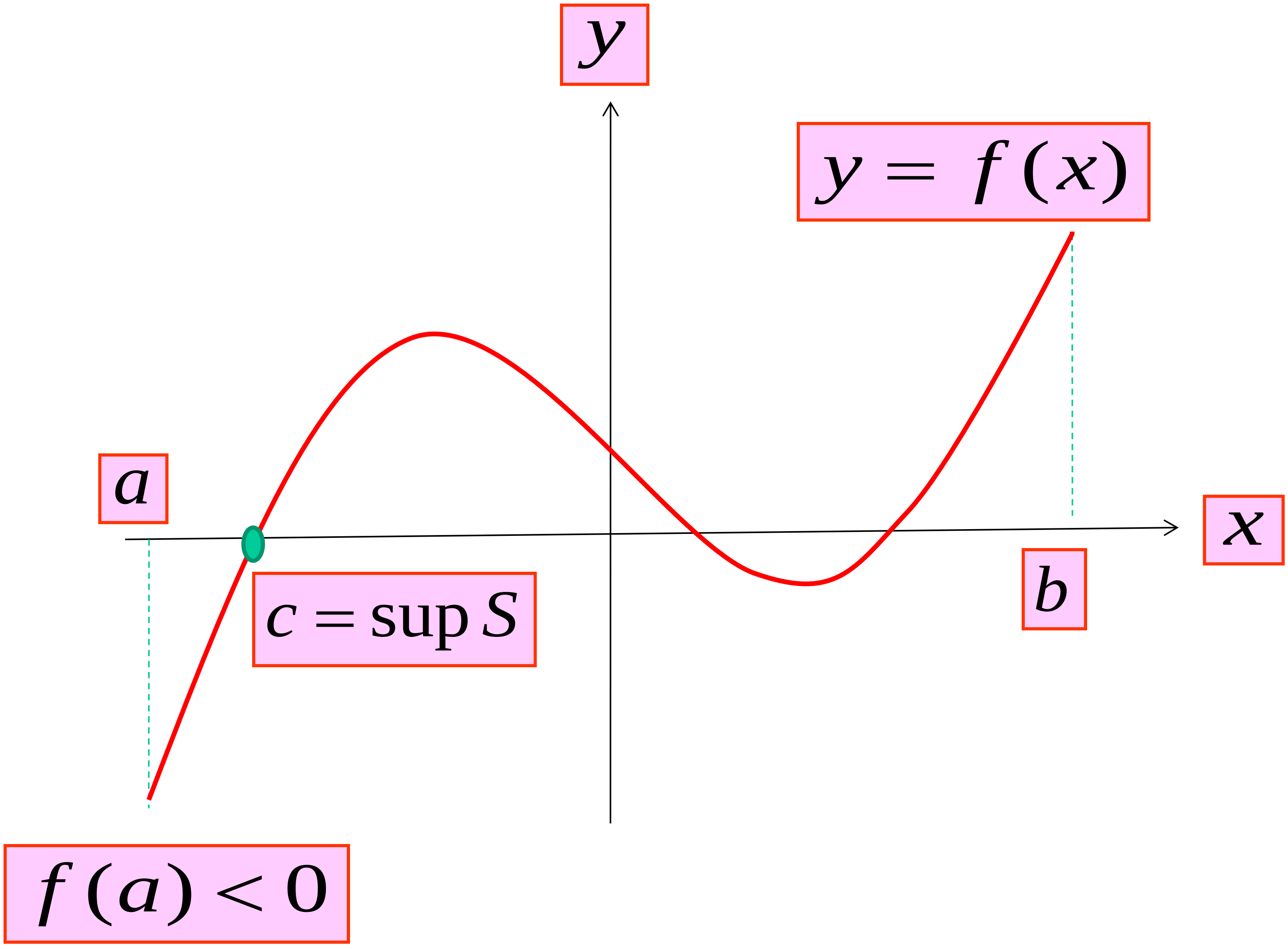
$a < \exists c < b$ **such that** $f(c) = 0$

# Proof (1)

$$S := \{ d \mid f(x) < 0,\ a \leq \forall x < d \}$$

$$\Rightarrow$$

$$\exists c = \sup S$$



y
y = f(x)
a
x
b
c = sup S
f(a) < 0

# Proof (2)

$$S = \{d \mid f(x) < 0,\ a \le \forall x < d\}$$

$$c = \sup S$$

$$\Rightarrow$$

$$f(c) = 0$$

# Corollary

$f(x)$ **is continuous on** $I=[a,b]$

$$\Rightarrow$$

$$\{ f(x) \mid a \le x \le b \} = [\alpha, \beta]$$

$$\alpha = \inf \{ f(x) \mid a \le x \le b \}$$

$$\beta = \sup \{ f(x) \mid a \le x \le b \}$$

# Fixed-Point Theorem

$f(x)$ **is continuous on** $I=[a,b]$

$f(I)\subseteq I$

$\Rightarrow$

$\exists c\in I$ **such that** $f(c)=c$

# Proof (1)

$$\varphi(x) := x - f(x)$$

$$\Rightarrow$$

$$\varphi(a) = a - f(a) \leq 0$$

$$(i)\ \varphi(a) = 0 \Rightarrow c = a$$

$$(ii)\ \varphi(a) < 0:$$

$$\varphi(b) = b - f(b) \geq 0$$

# Proof (2)

$(ii\text{-}1)\ \varphi(b) = 0 \Rightarrow c = b$

$(ii\text{-}2)\ \varphi(b) > 0 \quad (\varphi(a) < 0)$

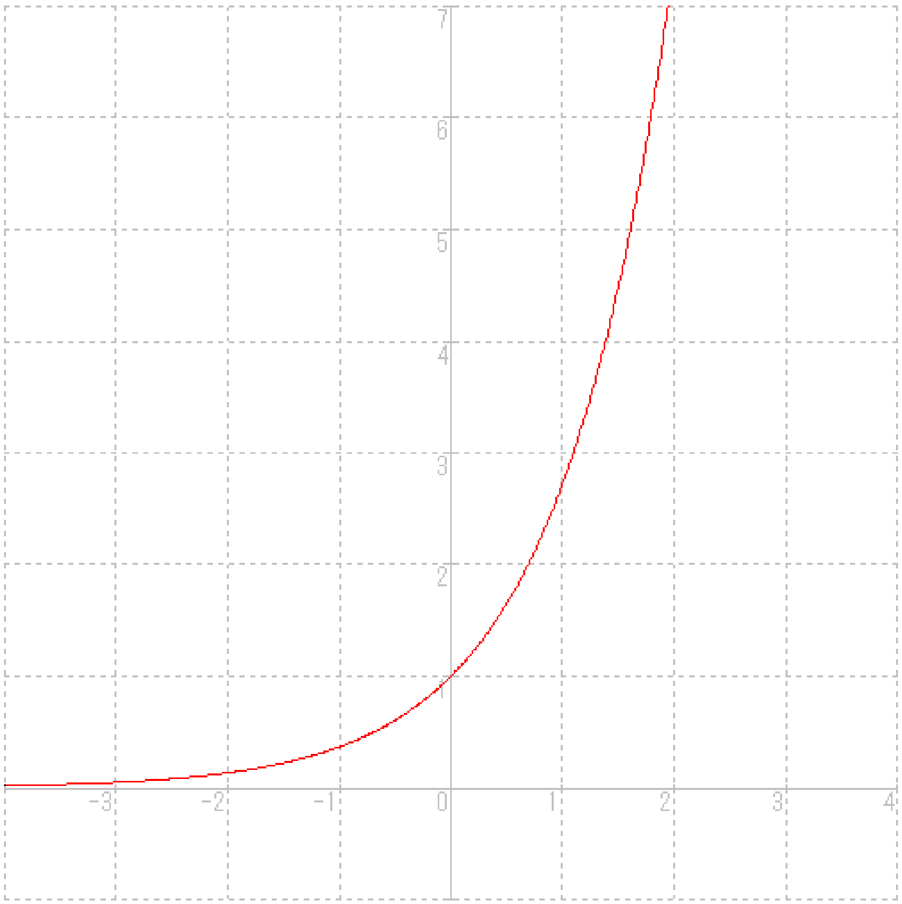
$\Rightarrow$

$a < \exists c < b$ **such that**

$\varphi(c) = c - f(c) = 0$

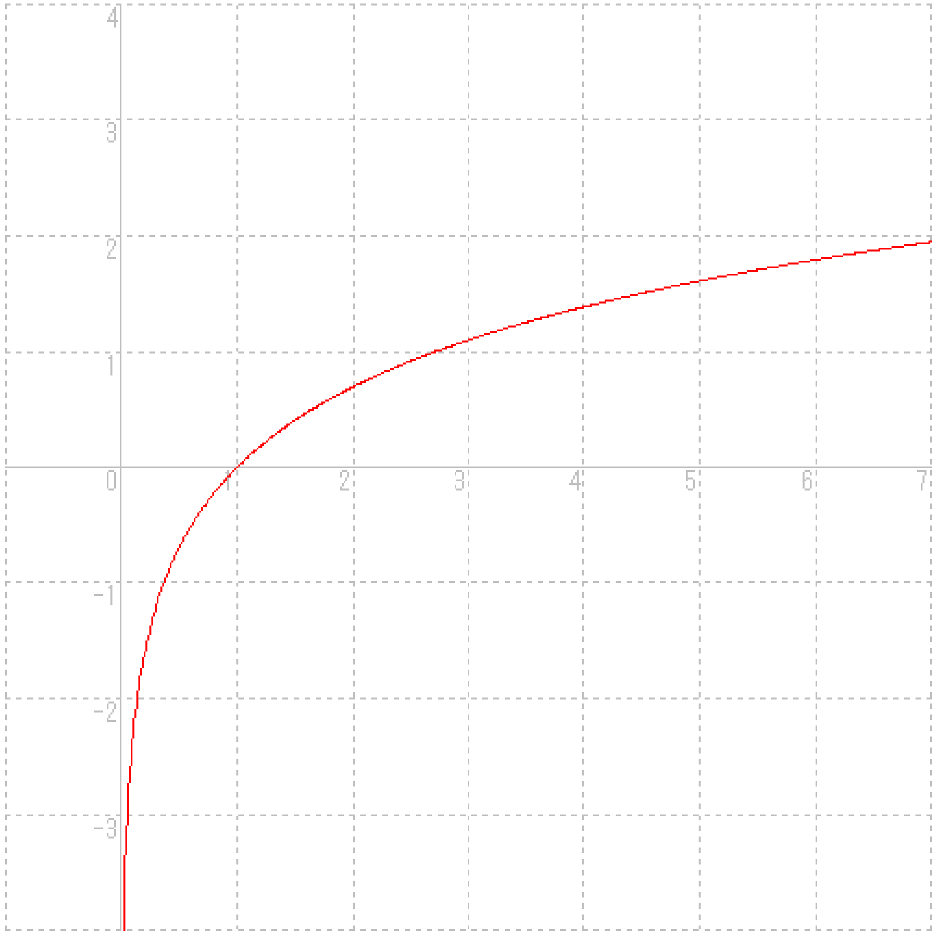
**(Intermediate Value Theorem)**

# Exponential Function



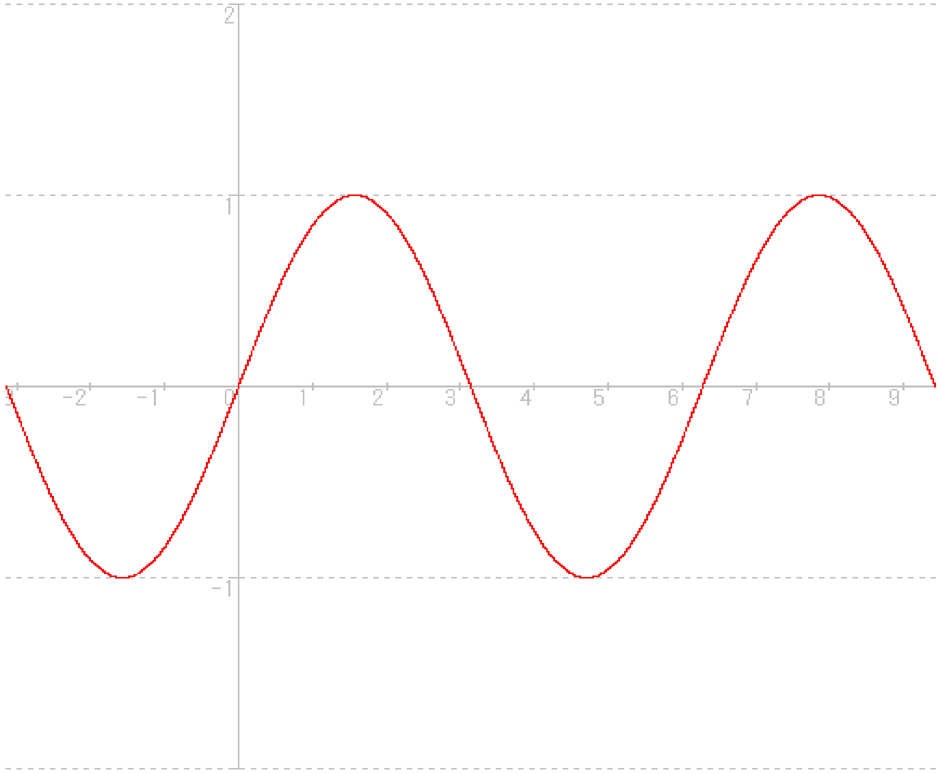

$$y = e^{x}, \quad -\infty < x < \infty$$

# Logarithm Function



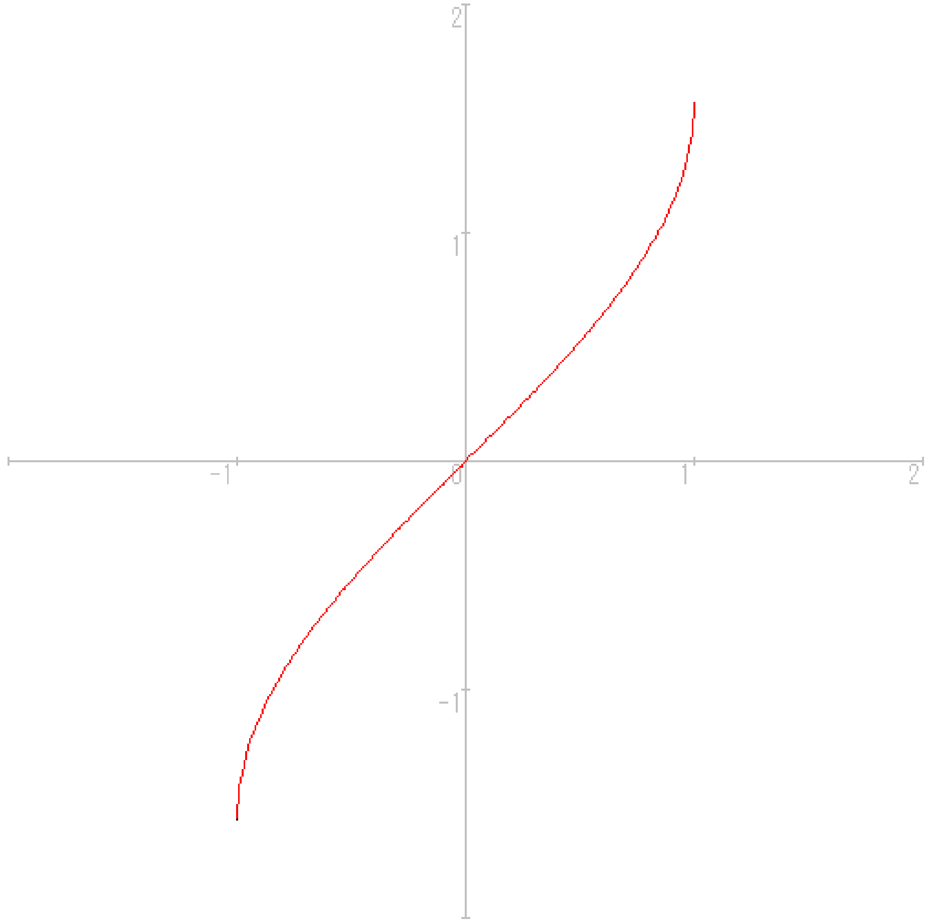

$$y = \log_e x, \ \ 0 < x < \infty$$

# Sine Function (1)



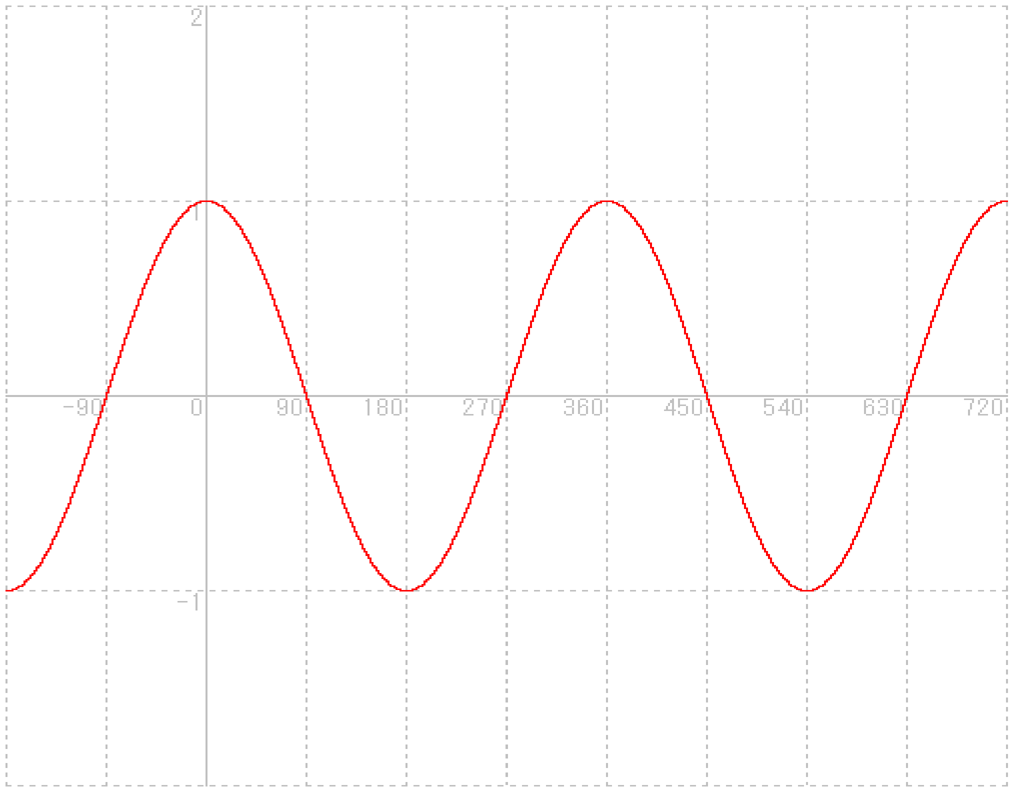

$$y = \sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

# Arcsine Function (1)



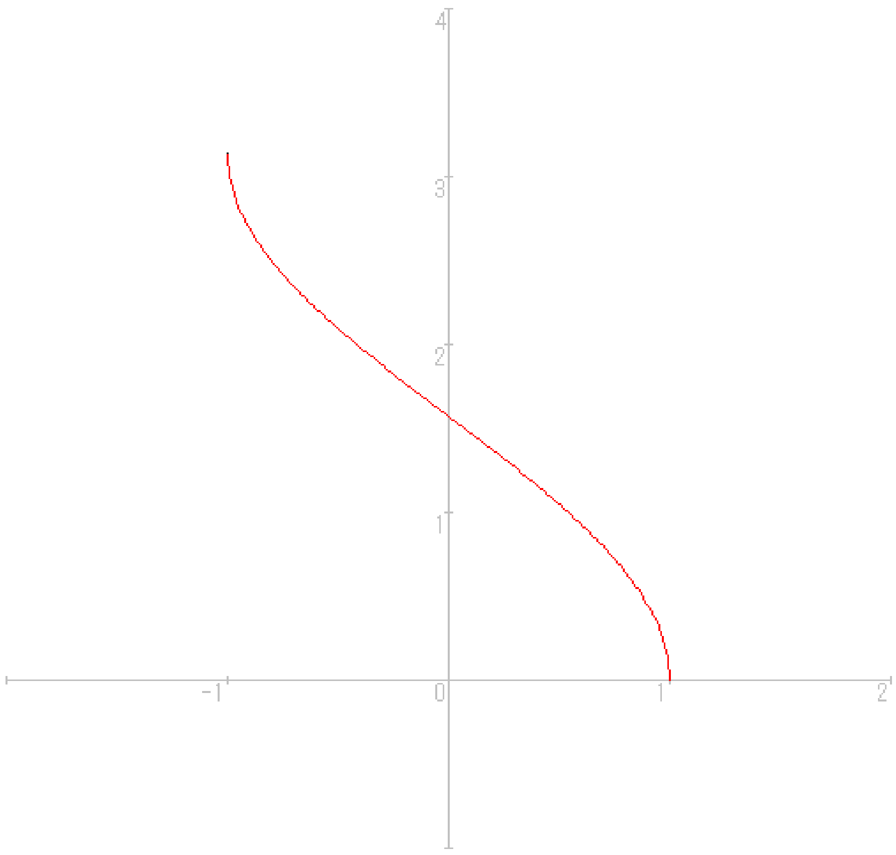

$$y = \sin^{-1} x, \ -1 \le x \le 1$$

# Cosine Function (2)



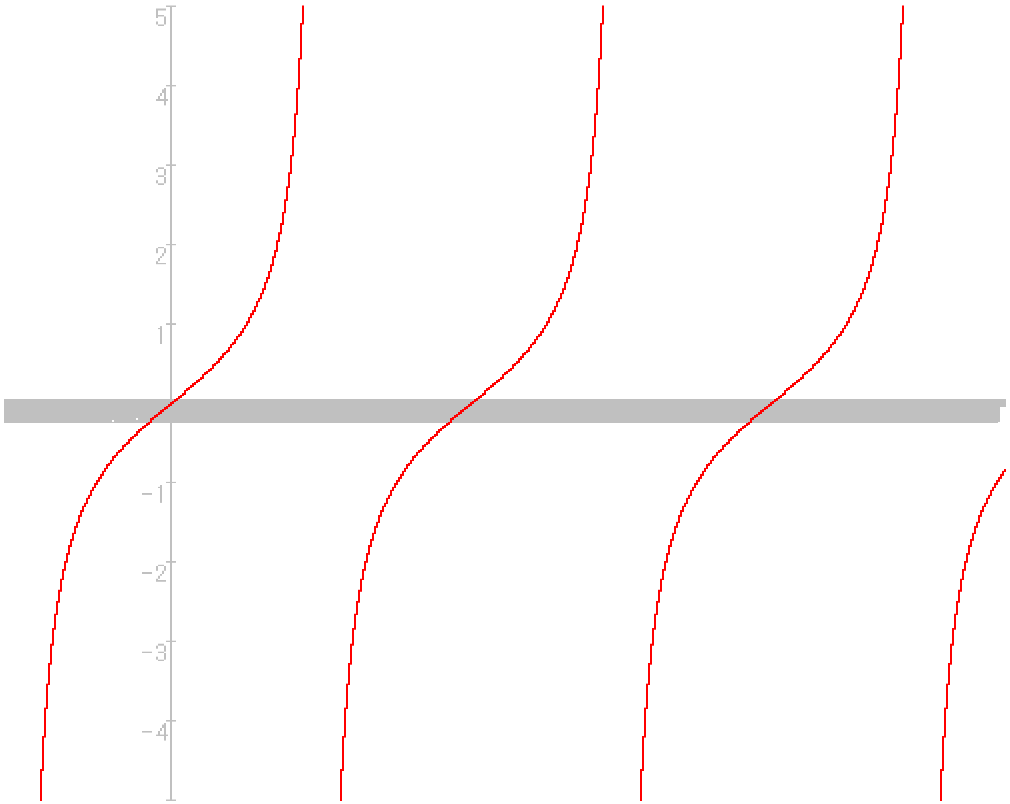

$$y = \cos x, \;\; 0 \le x \le \pi$$

# Arccosine (2)



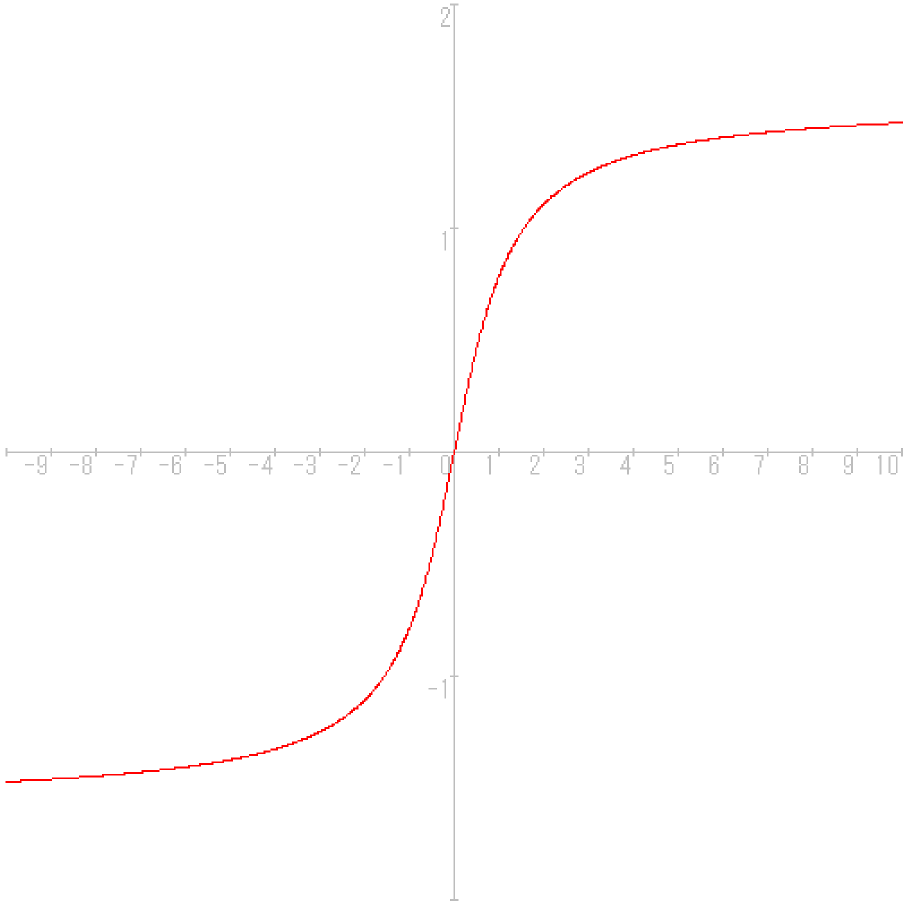

$$y = \cos^{-1} x, \ -1 \le x \le 1$$

# Tangent Function (3)




$$y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

# Arctangent Function (3)




$$y = \tan^{-1} x, \ -\infty < x < \infty$$

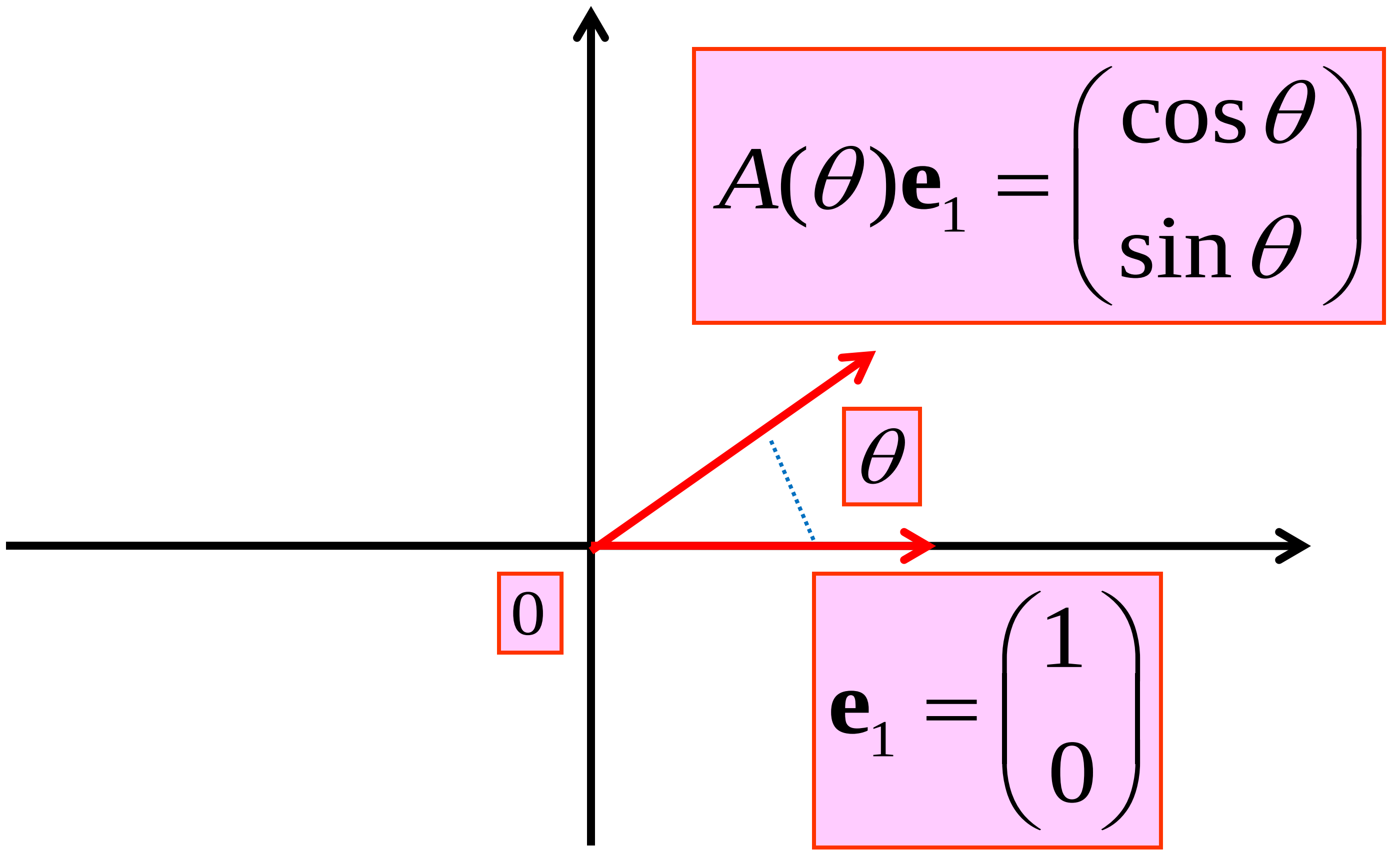
# Napier's Number (Continuous Version)

$$(1)\ e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}$$

$$(2)\ e = \lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}$$

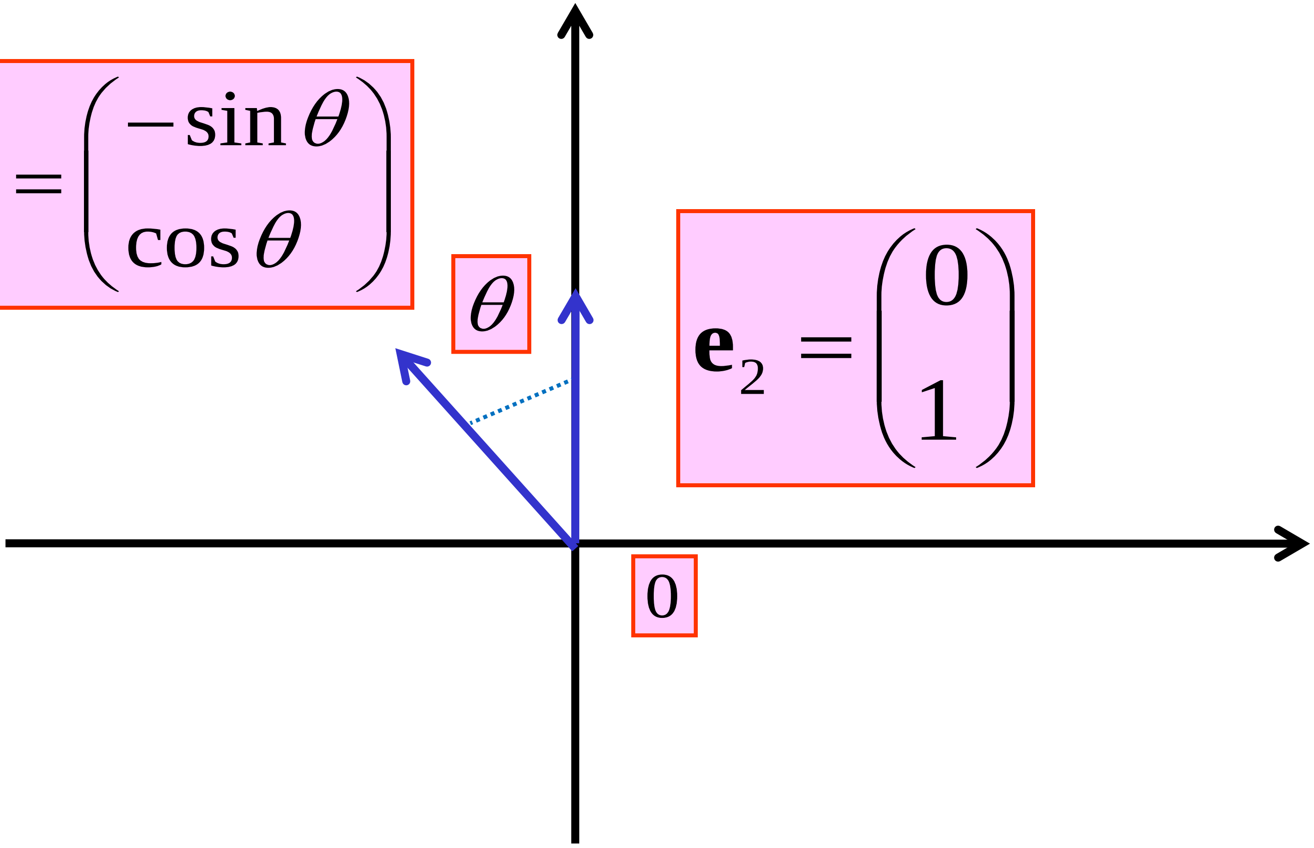
# Addition Theorem of Trigonometric Functions

# Rotation (1)


$A(\theta)\mathbf{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$
$\theta$
$0$
$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

# Rotation (2)

$$A(\theta)\mathbf{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$




$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

# Matrix of Rotation (1)

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

**Rotation of $\theta$**

# Matrix of Rotation (2)

$$A(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A(\theta)^{-1}$$

**Rotation of** $-\theta$

# Composition of Rotations (1)

$$A(\alpha)\mathbf{e}_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

# Composition of Rotations (2)

$$A(\beta)\left(A(\alpha)\mathbf{e}_1\right)$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha \end{pmatrix}$$

# Composition of Rotations (3)

$$A(\alpha)\mathbf{e}_2 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$$

# Composition of Rotations (4)

$$A(\beta)\left(A(\alpha)\mathbf{e}_2\right)$$

$$=\begin{pmatrix}\cos\beta & -\sin\beta \\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}-\sin\alpha \\ \cos\alpha\end{pmatrix}$$

$$=\begin{pmatrix}-\cos\beta\sin\alpha-\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha+\cos\beta\cos\alpha\end{pmatrix}$$

# Composition of Rotations (5)

$$A(\beta)A(\alpha)$$

$$=\begin{pmatrix}\cos\beta & -\sin\beta \\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}$$

$$=\begin{pmatrix}\cos\beta\cos\alpha-\sin\beta\sin\alpha & -\cos\beta\sin\alpha-\sin\beta\cos\alpha \\ \sin\beta\cos\alpha+\cos\beta\sin\alpha & -\sin\beta\sin\alpha+\cos\beta\cos\alpha\end{pmatrix}$$

# Composition of Rotations (6)

$$A(\beta)\left(A(\alpha)\mathbf{e}_1\right) = A(\beta)A(\alpha)\mathbf{e}_1$$

$$A(\beta)\left(A(\alpha)\mathbf{e}_2\right) = A(\beta)A(\alpha)\mathbf{e}_2$$

# Composition of Rotations (7)

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$A(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$\Rightarrow$$

$$A(\alpha)A(\beta) = A(\beta)A(\alpha) = A(\alpha+\beta)$$

# Composition of Rotations (8)

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= A(\alpha+\beta)$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

# Addition Theorem (1)

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

# Addition Theorem (2)

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

# Addition Theorem (3)

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

# Addition Theorem (4)

$$\sin A \ \sin B = \frac{1}{2}\left(\cos(A - B) - \cos(A + B)\right)$$

$$\cos A \cos B = \frac{1}{2}\left(\cos(A - B) + \cos(A + B)\right)$$

$$\sin A \cos B = \frac{1}{2}\left(\sin(A + B) + \sin(A - B)\right)$$

# Addition Theorem (5)

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\sin 2A = 2 \sin A \cos A$$

# Addition Theorem (6)

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$1+\tan^2 A=\frac{1}{\cos^2 A}$$

# Uniform Continuity of Functions

# Uniform Continuity

**Let** $f(x)$ **be a function defined on an interval** $I$.

$f(x)$ **is uniformly continuous on** $I$

$$\overset{\text{def}}{\Leftrightarrow}$$

$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0$ **such that**

$$\forall x, y \in I,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

# Fundamental Theorem

**Every continuous function defined on a bounded, closed interval is uniformly continuous.**

# Proof (1)

**Assume, to the contrary, that**

$$\exists \varepsilon_0 > 0,\ \forall n \in \mathbf{N}$$

$$\begin{cases} \exists x_n, y_n \in I,\ |x_n - y_n| < \dfrac{1}{n} \\ |f(x_n) - f(y_n)| \geq \varepsilon_0 \end{cases}$$

# Proof (2)

**Bolzano - Weierstrass**

$$\begin{cases} \exists x_{n'}, y_{n'} \in I, \ \ |x_{n'} - y_{n'}| < \dfrac{1}{n'} \\ x_{n'} \to \exists c, \ \ y_{n'} \to \exists d \end{cases}$$

$$\Rightarrow c = d$$

# Proof (3)

$$\begin{cases} x_{n'}, y_{n'} \in I \\ x_{n'} \to c, \quad y_{n'} \to c \end{cases}$$

$$\Rightarrow$$

$$0 = \lim_{n' \to \infty} \left| f(x_{n'}) - f(y_{n'}) \right| \geq \varepsilon_0 > 0$$

**(Contradiction)**

# Example (1)

$$f(x) = x^2$$

$$I = [0, K], \quad K > 0$$

# Proof

(1) $x > a:$

$$\delta_1(K,\varepsilon) = \sqrt{K^2+\varepsilon} - K$$

(2) $0 \le x < a:$

$$\delta_2(K,\varepsilon) = K - \sqrt{K^2-\varepsilon}$$

$\Rightarrow$

$$\delta(K,\varepsilon) = \min\{\delta_1(K,\varepsilon),\delta_2(K,\varepsilon)\}$$

# Example (2)

$$g(x) = \frac{1}{x}$$

$$I = [\alpha, \infty), \ \alpha > 0$$

# Proof

(1) $x > a$ :

$$\delta_1(\alpha, \varepsilon) = \frac{\varepsilon\alpha^2}{1 - \varepsilon\alpha}$$

(2) $0 \le x < a$ :

$$\delta_2(\alpha, \varepsilon) = \frac{\varepsilon\alpha^2}{1 + \varepsilon\alpha}$$

$\Rightarrow$

$$\delta(\alpha, \varepsilon) = \min\{\delta_1(\alpha, \varepsilon), \delta_2(\alpha, \varepsilon)\}$$

$$= \frac{\varepsilon\alpha^2}{1 + \varepsilon\alpha}$$

# Lipschitz Continuity

**Let** $f(x)$ **be a function defined on an interval** $I$.

$f(x)$ **is Lipschitz continuous on** $I$

$$\overset{\text{def}}{\Longleftrightarrow}$$

$\exists L > 0$ **such that**

$$\left| f(x) - f(y) \right| \le L \left| x - y \right|, \quad \forall x, y \in I$$

# Example

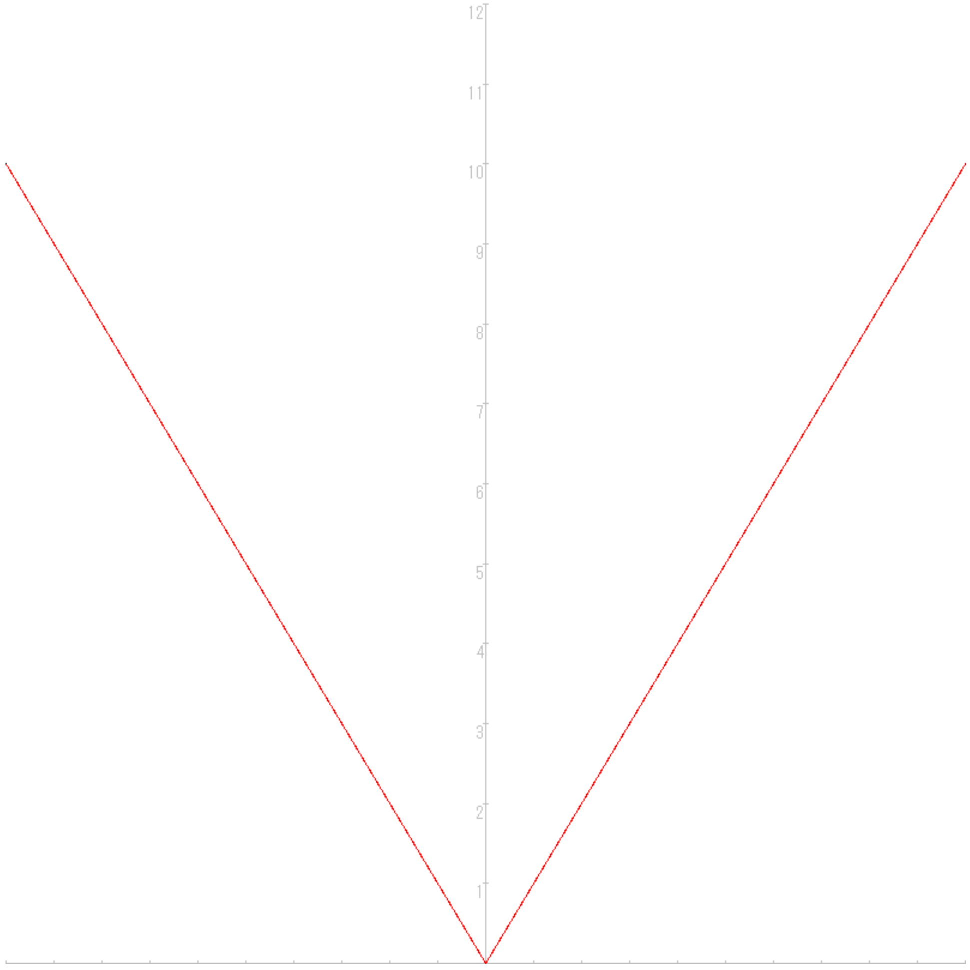
$$f(x) = |x|, \quad I = (-\infty, \infty)$$

$\because$

$$\big||x| - |y|\big| \leq |x - y|$$

# Numerical Computing with BASIC

# Piecewise Smooth Curve




$$y = |x|$$

# Differentiation

# Differentiability of Functions

# Definition of Differentiability

**Let** $f(x)$ **be a function defined on an open interval** $I$.

$f(x)$ **is differentiable at** $a \in I$

$$\overset{\text{def}}{\Longleftrightarrow}$$

$$\exists \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \alpha$$

**Notation :** $\alpha = f'(a) = \frac{df}{dx}(a)$

# Criterion for Differentiability

$$\begin{cases} f(x) = f(a) + \alpha(x-a) + R(x)(x-a) \\ \lim_{x\to a} R(x) = 0 \end{cases}$$

$$\begin{cases} f(a+h) = f(a) + \alpha h + R(a+h)h \\ \lim_{h\to 0} R(a+h) = 0 \end{cases}$$

# Geometrical Meaning

**Differentiability**

**Existence of Tangent Lines**

# Differentiability implies Continuity

$$\begin{cases} f(x) = f(a) + \alpha(x-a) + R(x)(x-a) \\ \lim_{x \to a} R(x) = 0 \end{cases}$$

$$\Rightarrow$$

$$\left| f(x) - f(a) \right| \le \left| \alpha + R(x) \right| \left| x - a \right|$$

$$\to 0 \ \ \textbf{as}\ x \to a$$

# Examples

# Examples (1)

$$(1)\left(x^{\alpha}\right)' = \alpha x^{\alpha-1}$$

$$(2)\left(e^{x}\right)' = e^{x}$$

$$(3)\left(a^{x}\right)' = a^{x}\log_{e} a \quad (a > 0)$$

$$(4)\left(\log_{e}|x|\right)' = \frac{1}{x}$$

# Examples (2)

$$(1)\ (\sin x)' = \cos x$$

$$(2)\ (\cos x)' = -\sin x$$

$$(3)\ (\tan x)' = \frac{1}{\cos^2 x}$$

$$(4)\ \left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(5)\ \left(\cos^{-1} x\right)' = -\frac{1}{\sqrt{1 - x^2}}$$

$$(6)\ \left(\tan^{-1} x\right)' = \frac{1}{1 + x^2}$$

# Example (1)

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}, \quad x > 0$$

$$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}$$

# Proof (1)

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

# Proof (2)

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}$$

# Example (2)

$$\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$$

# Proof (1)

$$(a)\ \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

$$(b)\ \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

$$(c)\lim_{h\to 0}(1+h)^{\frac{1}{h}}=e$$

# Proof (2)

$$\frac{\log_e(x+h)-\log_e x}{h}$$

$$=\log_e\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$=\log_e\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

# Proof (3)

$$\log_e \left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \log_e (1 + y)^{\frac{1}{xy}}$$

$$\left(y = \frac{h}{x}\right)$$

$$= \frac{1}{x} \log_e (1 + y)^{\frac{1}{y}}$$

# Proof (4)

$$\lim_{h \to 0} \frac{\log_e(x+h) - \log_e x}{h}$$

$$= \frac{1}{x} \lim_{y \to 0} \log_e (1+y)^{\frac{1}{y}}$$

$$= \frac{1}{x}$$

# Example (3)

$$\frac{d}{dx}\left(e^{x}\right)=e^{x}$$

# Proof

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

# Example (3)

$$\frac{d}{dx}(\sin x) = \cos x$$

# Proof (1)

$$\frac{1}{h}\left(\sin(x+h)-\sin x\right)$$

$$=\frac{2}{h}\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

$$=\frac{2}{h}\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)$$

# Proof (2)

$$\cos\left(x+\frac{h}{2}\right)$$

$$=\cos x\cos\frac{h}{2}-\sin x\sin\frac{h}{2}$$

$$\longrightarrow \cos x \quad \textbf{as } h \longrightarrow 0$$

# Proof (3)

$$\frac{2}{h}\sin\frac{h}{2} = \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$\rightarrow 1 \quad \textbf{as } h \rightarrow 0$$

# Proof (4)

$$\frac{1}{h}\left(\sin(x+h)-\sin x\right)$$

$$=\frac{2}{h}\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)$$

$$\longrightarrow \cos x \quad \textbf{as} \quad h \longrightarrow 0$$

# Example (4)

$$\frac{d}{dx}(\cos x) = -\sin x$$

# Proof (1)

$$\frac{1}{h}\left(\cos\left(x+h\right)-\cos x\right)$$

$$=-\frac{2}{h}\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

$$=-\frac{2}{h}\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)$$

# Proof (2)

$$\sin\left(x+\frac{h}{2}\right)$$

$$=\sin x\cos\frac{h}{2}+\cos x\sin\frac{h}{2}$$

$$\longrightarrow \sin x \quad \textbf{as} \;\; h\longrightarrow 0$$

# Proof (3)

$$\frac{2}{h}\sin\frac{h}{2} = \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$\longrightarrow 1 \quad \textbf{as} \ h \longrightarrow 0$$

# Proof (4)

$$\frac{1}{h}\left(\cos(x+h)-\cos x\right)$$

$$=-\frac{2}{h}\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)$$

$$\longrightarrow -\sin x \quad \textbf{as} \quad h\longrightarrow 0$$

# Operations of Differentiable Functions

$f(x), g(x)$ **are differentiable**

$\Longrightarrow$

**(1)** $f(x) \pm g(x)$ **are differentiable**

**(2)** $kf(x)$ **is differentiable**

**(3)** $f(x)g(x)$ **is differentiable**

**(4)** $\dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$ **is differentiable**

# Proof of (3-1)

$$f(a+h) = f(a) + f'(a)h + R(h)h$$

$$\lim_{h\to 0} R(h) = 0$$

$$g(a+h) = g(a) + g'(a)h + S(h)h$$

$$\lim_{h\to 0} S(h) = 0$$

# Proof of (3-2)

$$f(a+h)g(a+h)$$
$$=(f(a)+f'(a)h+R(h)h)$$
$$\times(g(a)+g'(a)h+S(h)h)$$
$$=f(a)g(a)+(f'(a)g(a)+f(a)g'(a))h$$
$$+(f(a)S(h)+g(a)R(h)+f'(a)g'(a)h)h$$

# Proof of (3-3)

$$f(a+h)g(a+h)$$

$$= f(a)g(a) + (f'(a)g(a) + f(a)g'(a))h$$

$$+(f(a)S(h) + g(a)R(h) + f'(a)g'(a)h)h$$

**Here :**

$$\lim_{h\to 0}(f(a)S(h) + g(a)R(h) + f'(a)g'(a)h) = 0$$

# Chain Rule

# Chain Rule

$y = f(u)$ **is differentiable at** $u = g(a)$

$u = g(x)$ **is differentiable at** $x = a$

$\Longrightarrow$

**The composite function** $f(g(x))$

**is differentiable at** $x = a$**:**

$$\frac{d}{dx} f(g(x))\Big|_{x=a} = \frac{dy}{du}(g(a)) \cdot \frac{du}{dx}(a)$$

# Proof (1)

$$g(a+h)=g(a)+g'(a)h+S(h)h$$

$$\lim_{h\to 0} S(h)=0$$

$$f(g(a)+k)=f(g(a))+f'(g(a))k+R(k)k$$

$$\lim_{k\to 0} R(k)=0$$

# Proof (2)

$$
\begin{aligned}
& f(g(a+h)) \\
&= f(g(a)+g'(a)h+S(h)h) \\
&= f(g(a))+f'(g(a))(g'(a)h+S(h)h) \\
&+R(g'(a)h+S(h)h)(g'(a)h+S(h)h)
\end{aligned}
$$

# Proof (3)

$$f(g(a+h))$$
$$= f(g(a)) + f'(g(a))g'(a)h$$
$$+ f'(g(a))S(h)h$$
$$+R(g'(a)h + S(h)h)(g'(a) + S(h))h$$

# Proof (4)

$$\lim_{h\to 0} R(g'(a)h + S(h)h)(g'(a) + S(h))$$

$$= \lim_{h\to 0} R(g'(a)h + S(h)h)\bullet g'(a)$$

$$= 0$$

# Proof (5)

$$f(g(a+h))$$

$$= f(g(a)) + f'(g(a))g'(a)h + T(h)h$$

$$\lim_{h \to 0} T(h) = 0$$

# Proof (6)

$$f(g(a+h))$$

$$= f(g(a)) + f'(g(a))g'(a)h + T(h)h$$

$$\Rightarrow$$

$$\frac{d}{dx} f(g(x))\Big|_{x=a} = f'(g(a))g'(a)$$

$$= \frac{dy}{du}(g(a)) \cdot \frac{du}{dx}(a)$$

# Example (1)

$$\frac{d}{dx}\left(\log_e\left(x+\sqrt{x^2+1}\right)\right)$$

$$=\frac{\left(x+\sqrt{x^2+1}\right)'}{x+\sqrt{x^2+1}}$$

$$=\frac{1}{\sqrt{x^2+1}}$$

# Example (2)

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \textbf{if } x \neq 0, \\ 0 & \textbf{if } x = 0 \end{cases}$$

# Remark

**The derivative**

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \textbf{if } x \neq 0, \\ 0 & \textbf{if } x = 0 \end{cases}$$

**is not continuous.**

# Mean Value Theorem

**Maximum Value Theorem**

**Minimum Value Theorem**

↓

**Rolle's Theorem**

↓

**Mean Value Theorem**

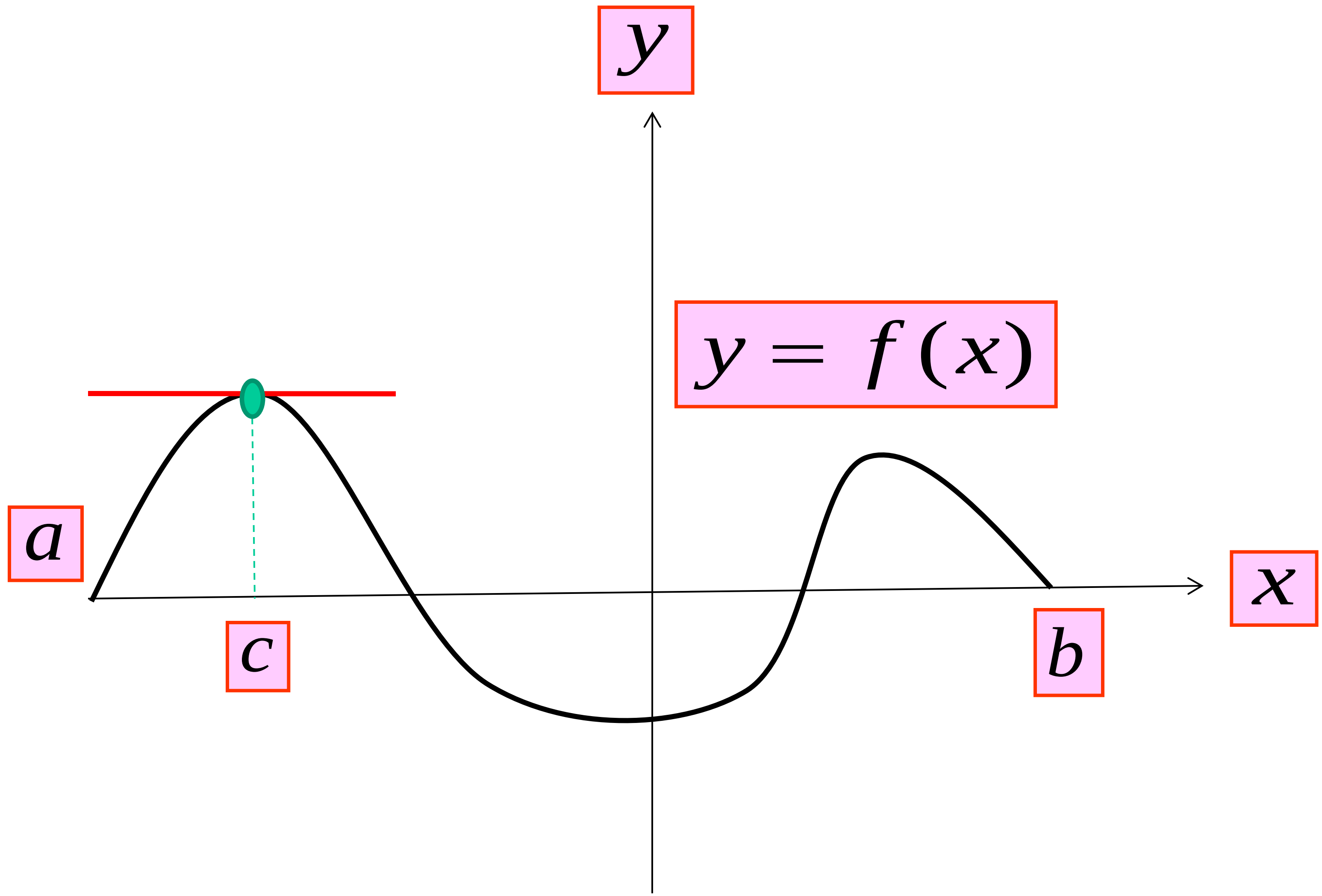
# Rolle's Theorem

$f(x)$ **is continuous on** $[a,b]$ **and is differentiable in** $(a,b)$

$f(a)=f(b)$

$\Rightarrow$

$a<\exists c<b$ **such that** $f'(c)=0$



y
y = f(x)
a
c
b
x

# Mean Value Theorem

$f(x)$ is continuous on $[a,b]$ and is differentiable in $(a,b)$

$\Rightarrow$

$a < \exists c < b$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

# Proof

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\Rightarrow$$

$$F(a) = F(b) = 0$$

$$\Rightarrow$$

$a < \exists c < b$ **such that**

$$F'(c) = 0$$

# Behavior of Functions

**Let** $f(x)$ **be continuous on** $[a,b]$ **and differentiable in** $(a,b)$**.**

**On** $(a,b)$

**(1)** $f'(x)=0 \Rightarrow f(x)$ **is constant**

**(2)** $f'(x)\geq 0 \Rightarrow f(x)$ **is monotone increasing**

**(3)** $f'(x)\leq 0 \Rightarrow f(x)$ **is monotone decreasing**

# Maximal and Minimal (1)

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$

$$+\frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

**(1)** $f'(a) \neq 0$: $f(x)$ **is not extremal**

**(2)** $f'(a) = 0$:

**(2-1)** $f''(a) > 0 \Rightarrow f(x)$ **is minimal**

**(2-2)** $f''(a) < 0 \Rightarrow f(x)$ **is maximal**

# Maximal and Minimal (2)

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$

$$+ \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$+ \frac{f''''(a)}{4!}(x-a)^4 + \cdots$$

**(3)** $f'(a) = f''(a) = 0$:

$f'''(a) \neq 0 \Rightarrow f(x)$ **is not extremal**

**(4)** $f'(a) = f''(a) = f'''(a) = 0$:

**(4-1)** $f''''(a) > 0 \Rightarrow f(x)$ **is minimal**

**(4-2)** $f''''(a) < 0 \Rightarrow f(x)$ **is maximal**

# Mean Value Theorem

**Cauchy's Mean Value Theorem**

**$\Longrightarrow$ de l'Hospital's Theorem**

**Taylor's Theorem $\Rightarrow$ Polynomial Approximation**

# Cauchy’s Mean Value Theorem

$f(x)$ is continuous on $[a,b]$ and
is differentiable in $(a,b)$
$g(x)$ is continuous on $[a,b]$ and
is differentiable in $(a,b)$ with $g'(x) \neq 0$
$\Rightarrow$
$a < \exists c < b$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

# de l'Hospital's Theorem

**Let** $f(x), g(x)$ **be continuous near** $a$ **and differentiable except for** $a$.

$f(a) = g(a) = 0, \;\; g'(x) \neq 0$

**Then :**

$$\exists \lim_{x \to a} \frac{f'(x)}{g'(x)} = \alpha \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \alpha$$

# Taylor Series

# Taylor's Theorem (Lagrange)

$f(x)$ is of class $C^{n-1}$ on $[a,b]$ and

is of class $C^n$ in $(a,b)$ where $n \geq 2$

$\Rightarrow$

$a < \exists c < b$ such that

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}$$

$$+\frac{f^{(n)}(c)}{n!}(b-a)^n$$

# Taylor's Theorem (Cauchy)

$f(x)$ is of class $C^{n-1}$ on $[a,b]$ and

is of class $C^n$ in $(a,b)$ where $n \geq 2$

$\Rightarrow$

$a < \exists c < b$ such that

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}$$

$$+\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1}(b-a)$$

# Taylor's Theorem

(1) $f(x)$ **is infinitely differentiable in** $(-R, R)$

(2) $\sup_{|x|<R} \left| f^{(n)}(x) \right| \leq \exists M_R, \quad n = 1, 2, \cdots$

$\Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad |x| < R$$

# Examples

# Example (1)

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots + \frac{x^{n}}{n!} + \cdots$$

# Remark (Napier's Number)

$$e = e^1$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

$$(x = 1)$$

# Example (2)

$$(\sin x)^{(n)} = \sin\left(x + \frac{n}{2}\pi\right)$$

$$\Rightarrow$$

$$(\sin x)^{(n)}\Big|_{x=0} = \sin\left(\frac{n}{2}\pi\right)$$

$$= \begin{cases} 0 & \textbf{for } n = 2k \\ (-1)^k & \textbf{for } n = 2k+1 \end{cases}$$

# Example (2)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

# Example (3)

$$(\cos x)^{(n)} = \cos\left(x + \frac{n}{2}\pi\right)$$

$$\Rightarrow$$

$$(\cos x)^{(n)}\Big|_{x=0} = \cos\left(\frac{n}{2}\pi\right)$$

$$= \begin{cases} (-1)^k \ \textbf{for} \ \ n = 2k \\ 0 \quad \textbf{for} \ n = 2k+1 \end{cases}$$

# Example (3)

$$\cos x$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$+ (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

# Example (4)

$$\log_e(1+x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$+ \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} + \cdots$$

$$(-1 < x \leq 1)$$

# Example (5)

$$\tan^{-1} x$$

$$= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$(-1 < x \leq 1)$$

# Abel's Theorem

$$A = \sum_{n=1}^{\infty} a_n$$ **converges**

$$\Rightarrow$$

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \to A \text{ as } x \uparrow 1$$

# Examples

$$(1)\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\log(1+1)=\log_e 2$$

$$(2)\ 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\tan^{-1}1=\frac{\pi}{4}$$

# Computational Approach

# Numerical Computing with BASIC

# Alternating Series Version

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
$$= 0.693147180559945$$

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

0.69314718054981 $(n = 10)$

0.6931471805599944 $(n = 100)$

0. 6931471805599944 $(n = 1000)$

# Taylor Series Version

# Taylor Series Version (1)

$$\log_e(1+x)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
$$+ \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} + \cdots$$
$$(-1 < x \leq 1)$$

# Taylor Series Version (2)

$$\log_e(1-x)$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$- \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} - \cdots$$

$$(-1 \leq x < 1)$$

# Taylor Series Version (3)

$$\log_e \frac{1+x}{1-x} = \log_e(1+x) - \log_e(1-x)$$

$$= 2\left( x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots \right)$$

$$(-1 < x < 1)$$

$$\log_e 2 = \log_e \frac{1+\frac{1}{3}}{1-\frac{1}{3}} \qquad \left(x = \frac{1}{3}\right)$$

$$= 2\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \cdots + \frac{1}{2n-1}\left(\frac{1}{3}\right)^{2n-1} + \cdots\right)$$

$$= 0.69314718055994 5\cdots$$

# Computational Approach

# Newton's Iteration Method

$$I = [a, b]$$

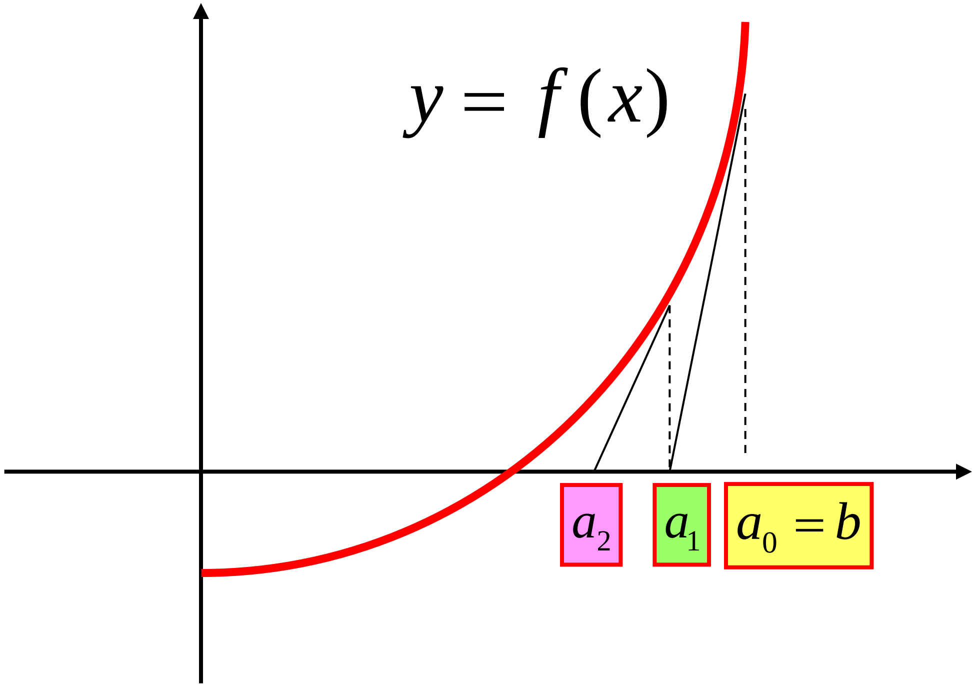
$$\begin{cases} f(a) < 0,\ f(b) > 0 \\ f''(x) > 0 \end{cases}$$

$$\Longrightarrow$$

$$\begin{cases} a_0 = b, \\ a_n = a_{n-1} - \dfrac{f(a_{n-1})}{f'(a_{n-1})}, \quad n = 1, 2, \dots \end{cases}$$

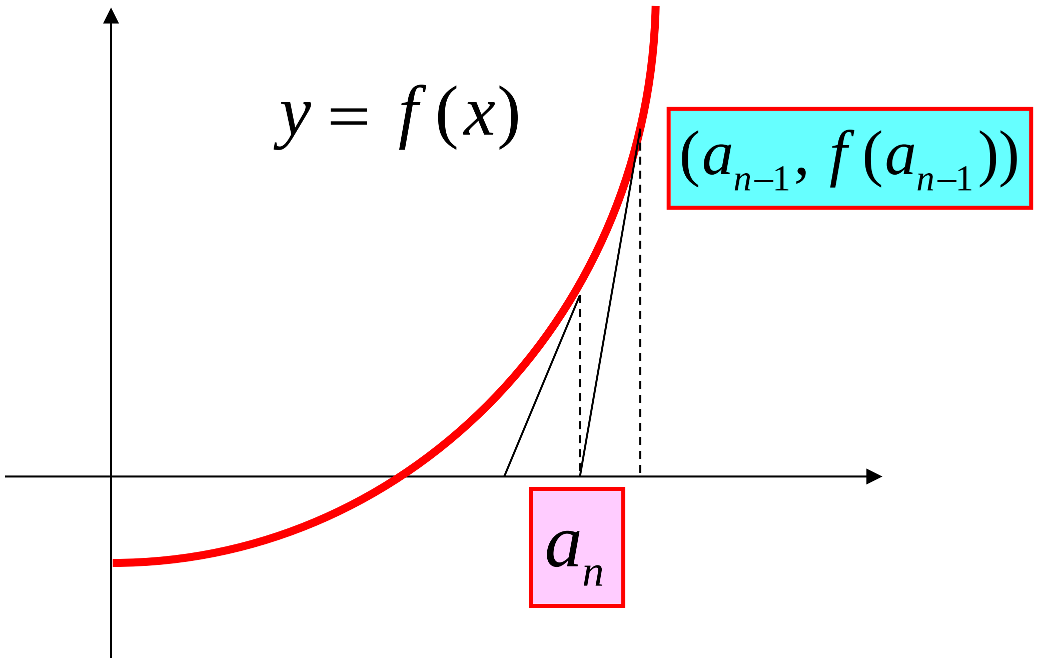
$\{a_n\}$ **converges to the solution** $\xi$ **of the equation** $f(x) = 0$

# Newton's Method (1)



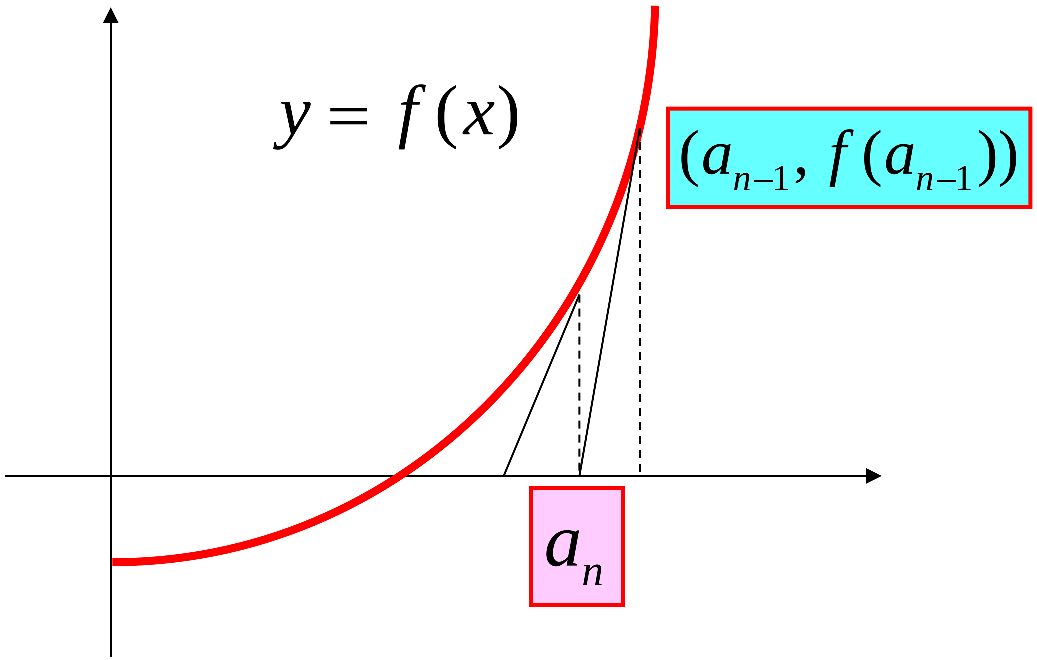

$$f(x) = 0$$

# Newton's Method (2)




$$y = f'(a_{n-1})(x - a_{n-1}) + f(a_{n-1})$$

# Newton's Method (3)

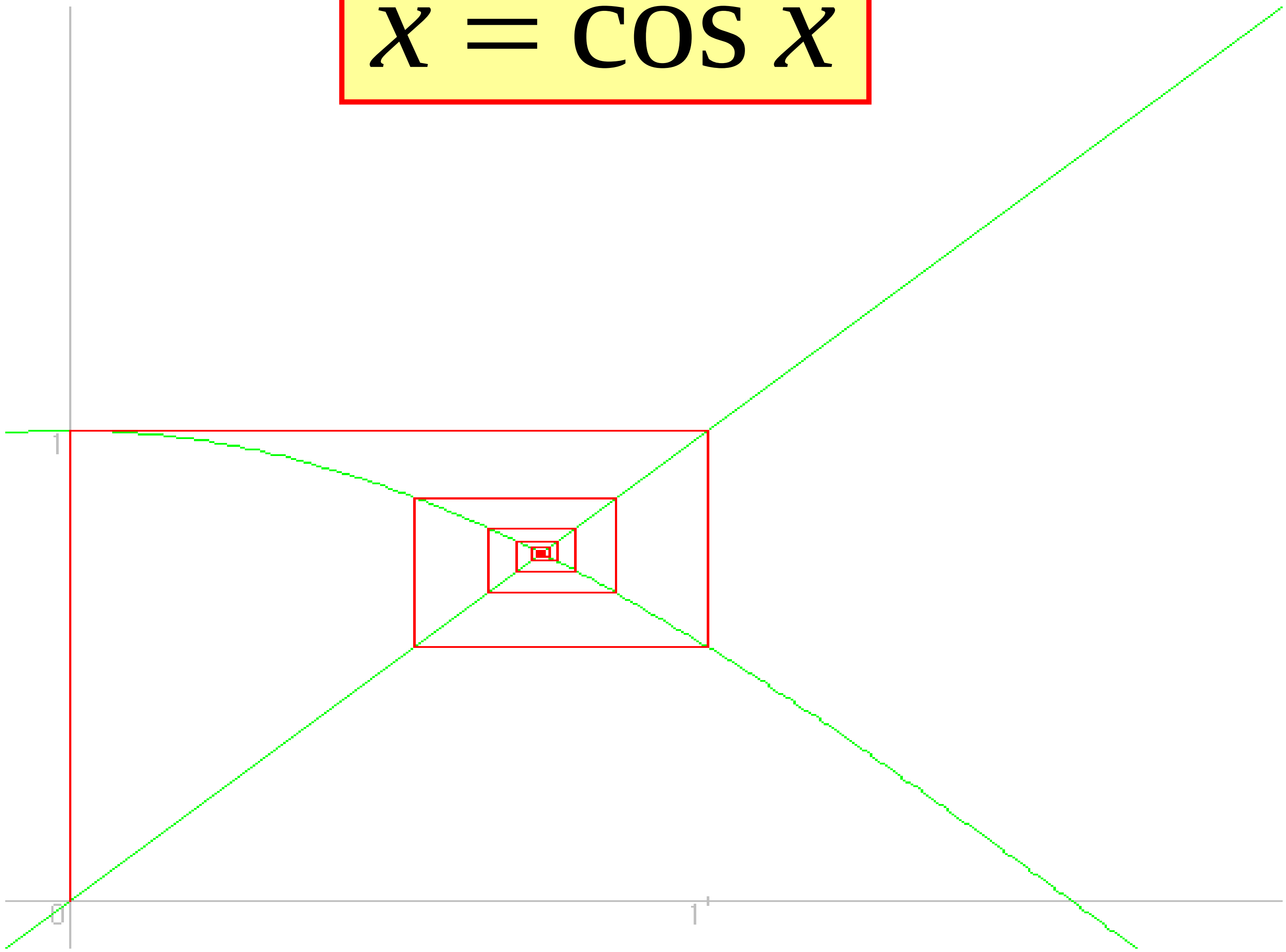



**Definition of $\{a_n\}$**

$$0 = f'(a_{n-1})(a_n - a_{n-1}) + f(a_{n-1})$$

# Numerical Computing with BASIC

$$x = \cos x$$



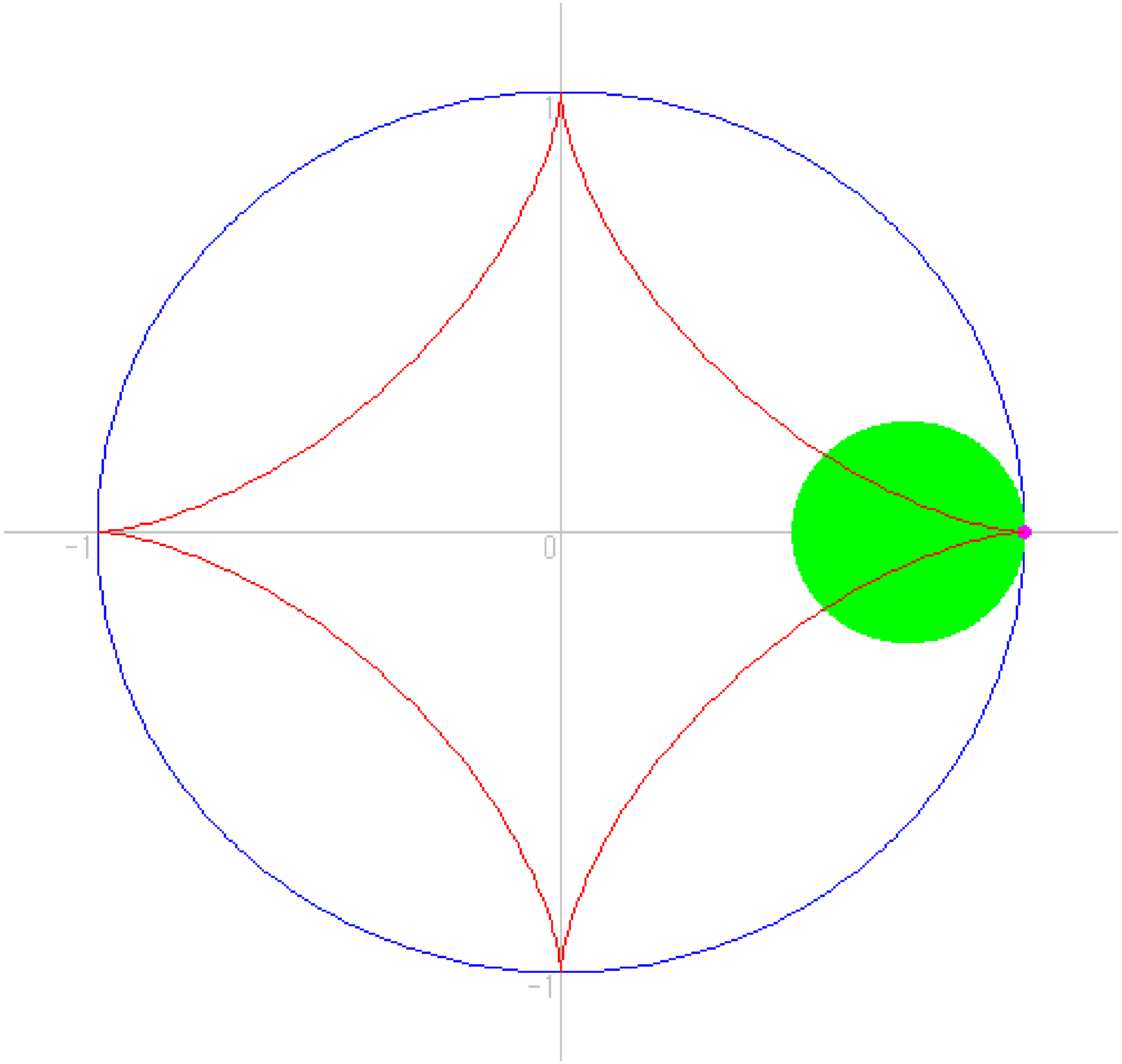
$$x = \cos x$$

$$x \doteq 0.739085133215166$$

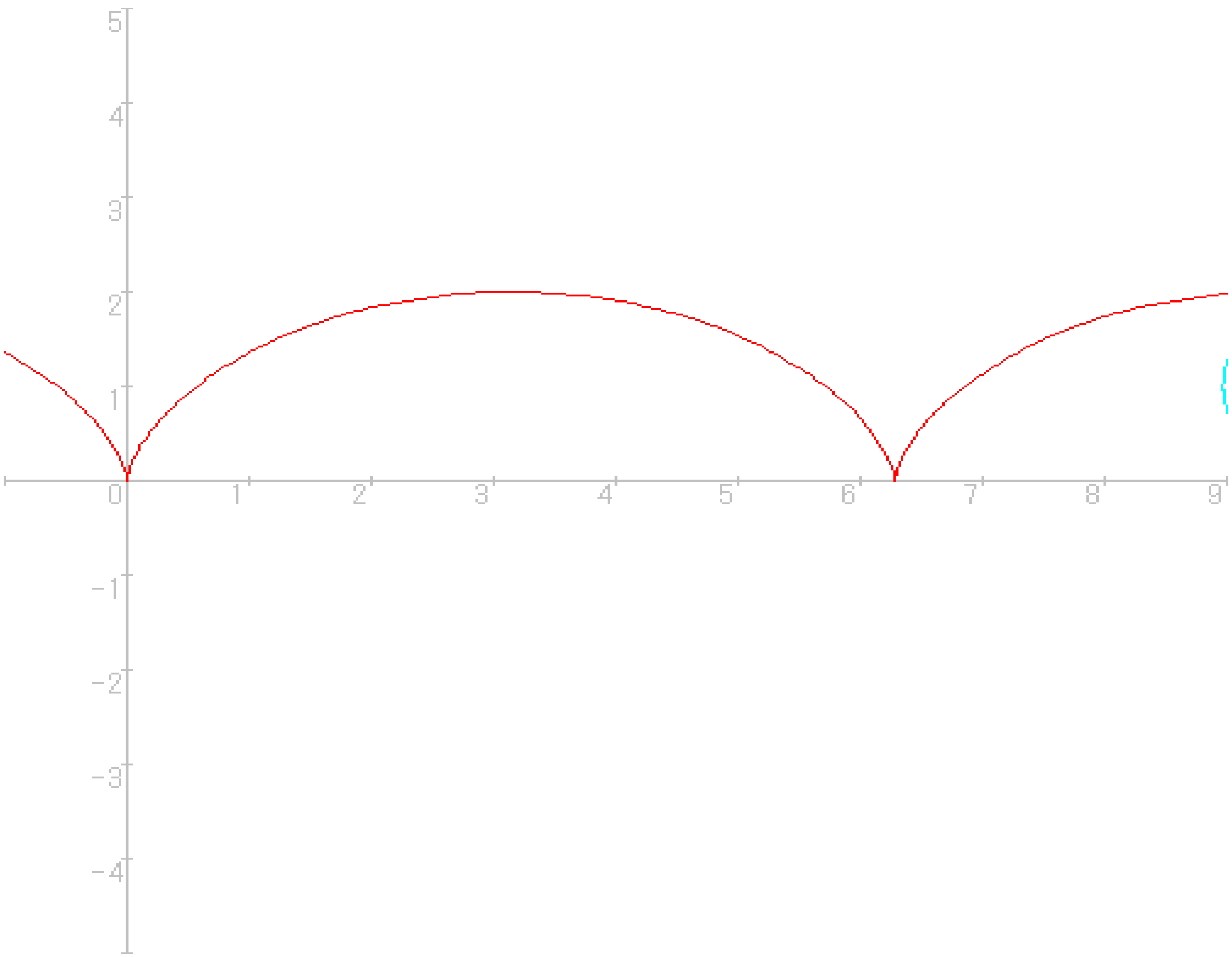
# Miscellaneous Curves

# Numerical Computing with BASIC

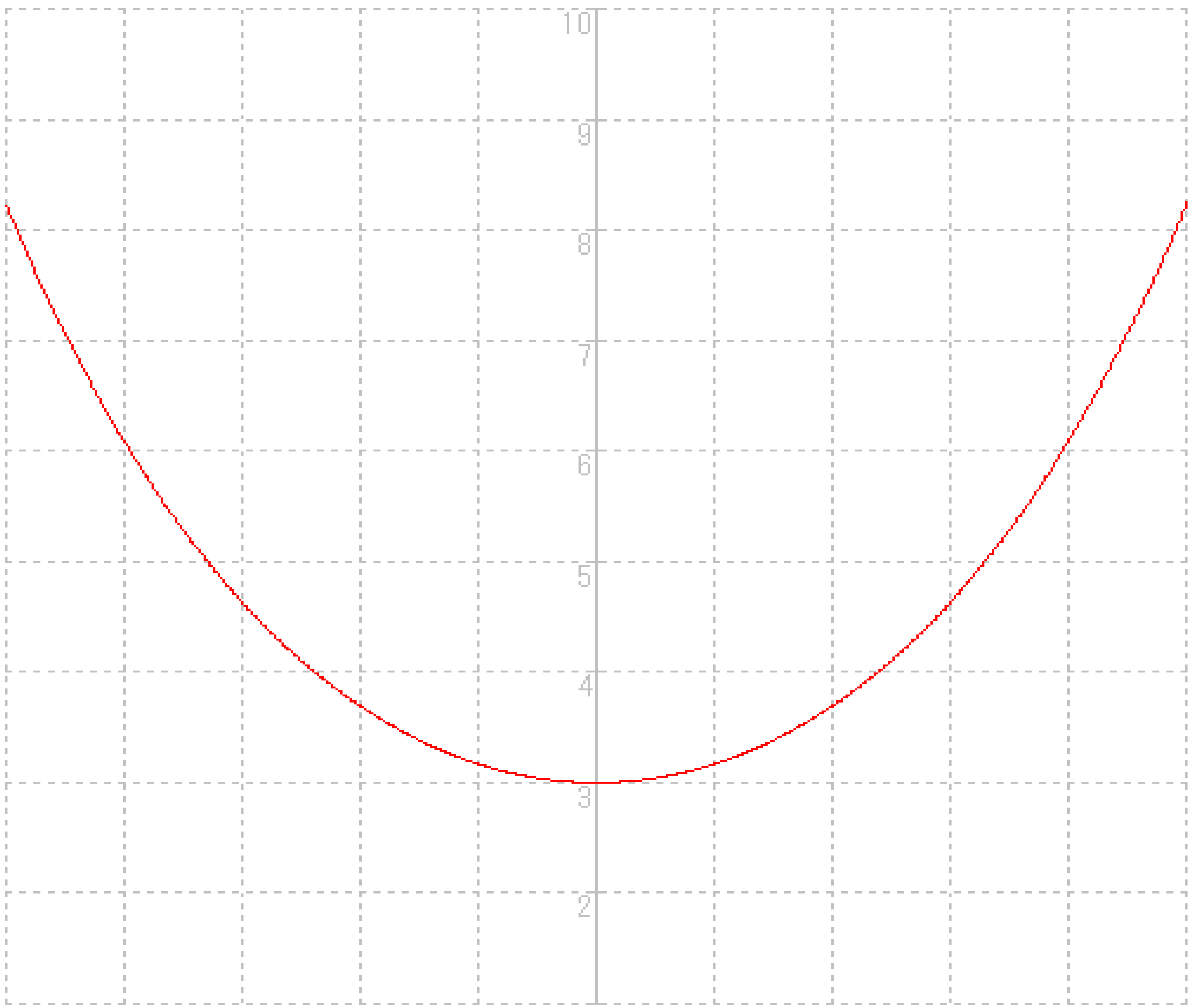
# Asteroid



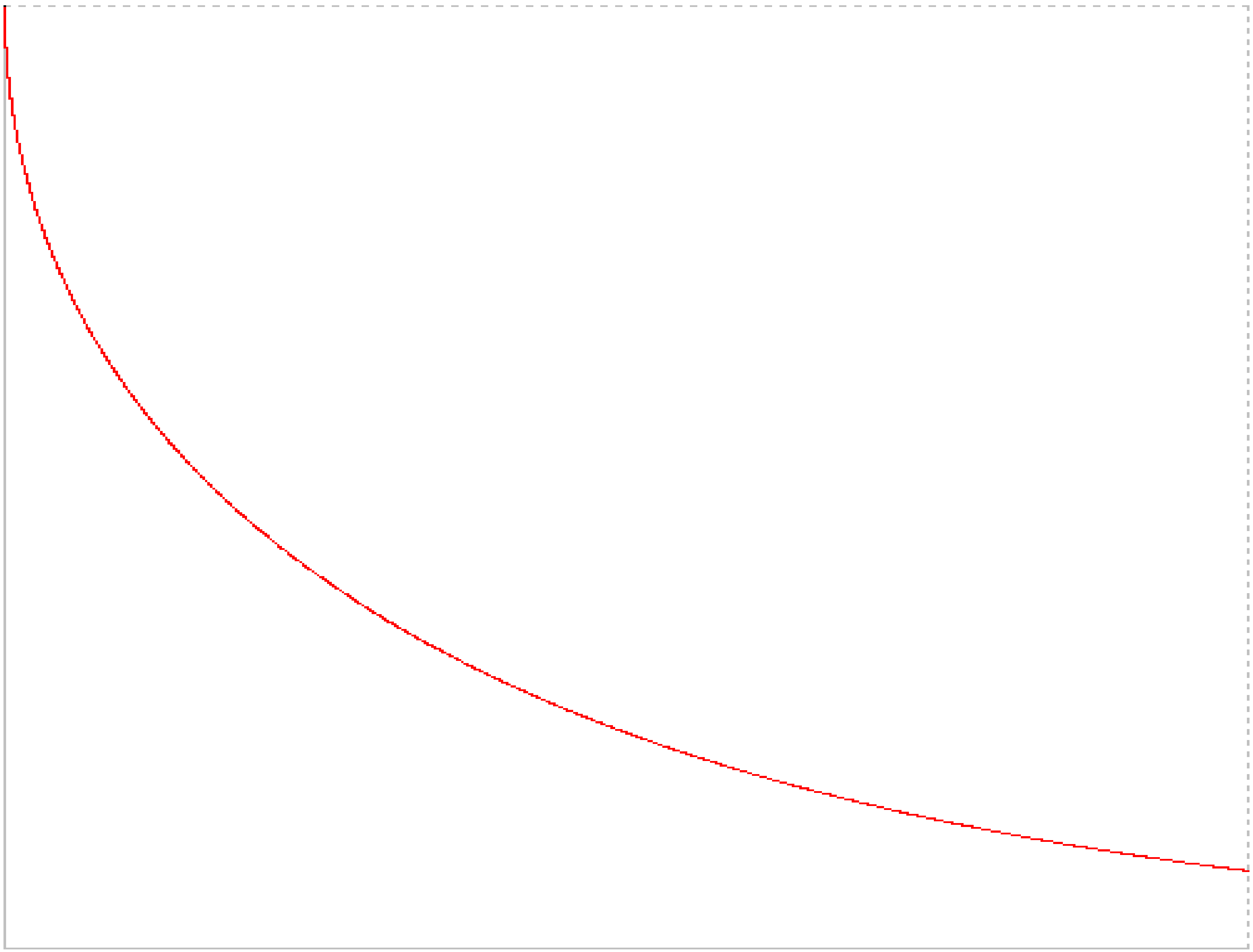
# Cycloid



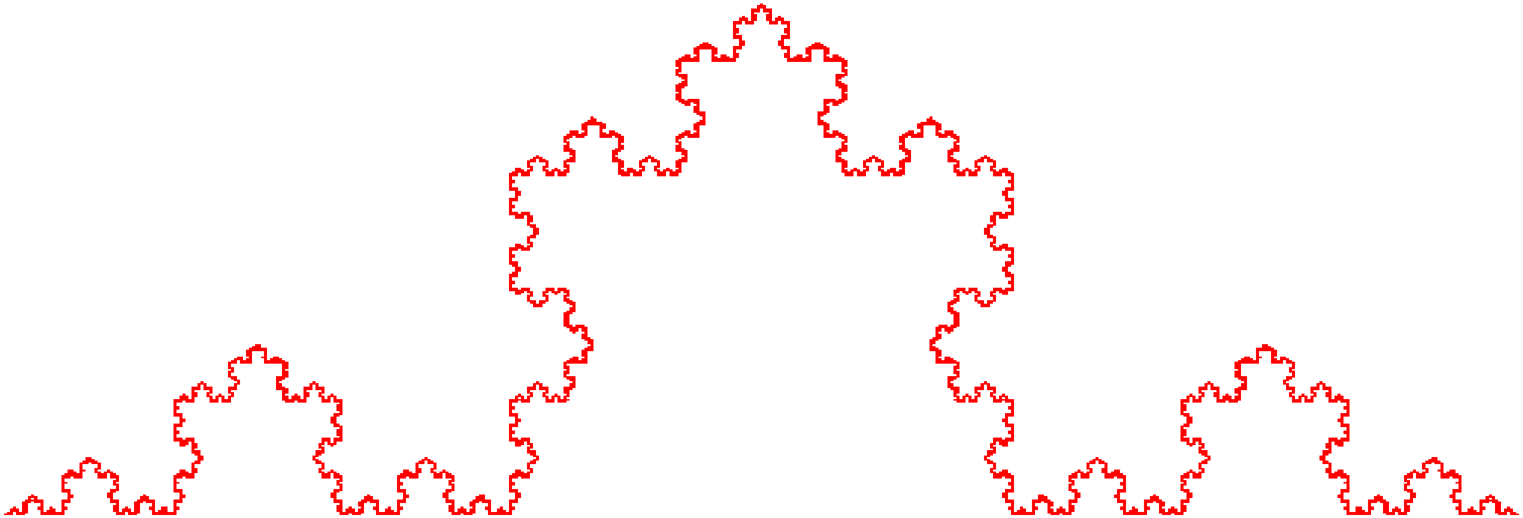
# Catenary



# Tractrix



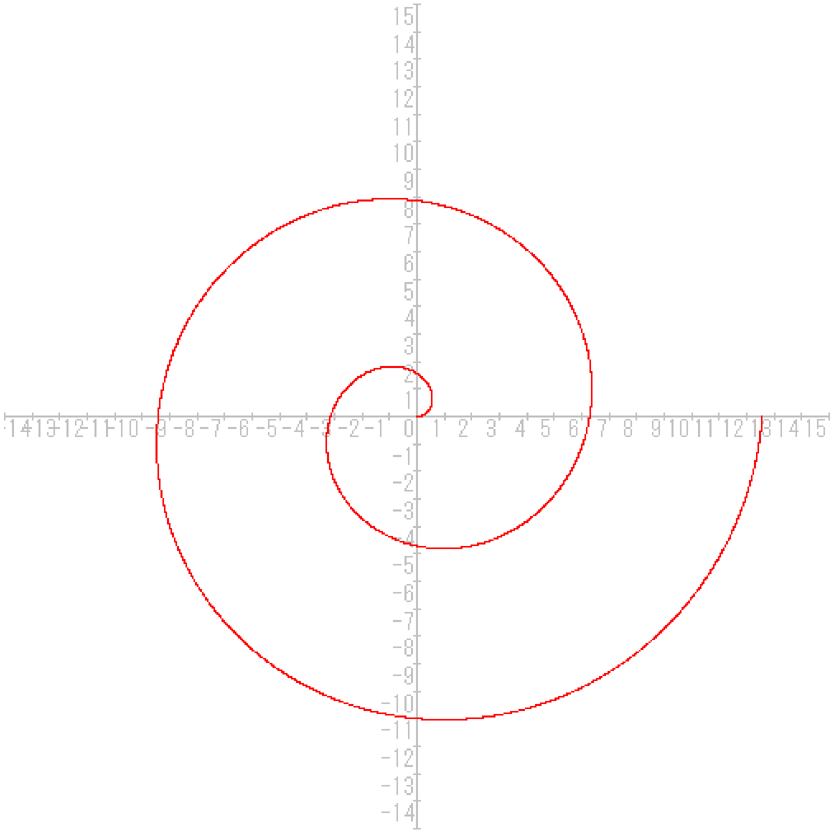
# von Koch's Curve



# Curves defined by Polar Coordinates

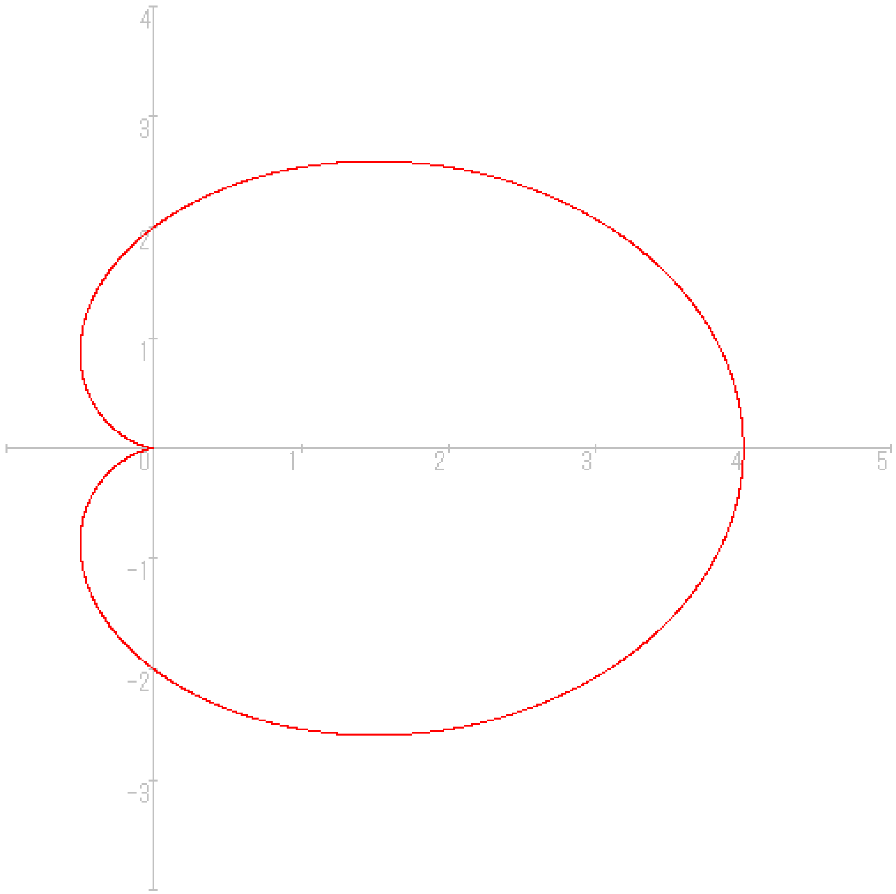
# Numerical Computing
# with
# BASIC

# Archimedes' Spiral



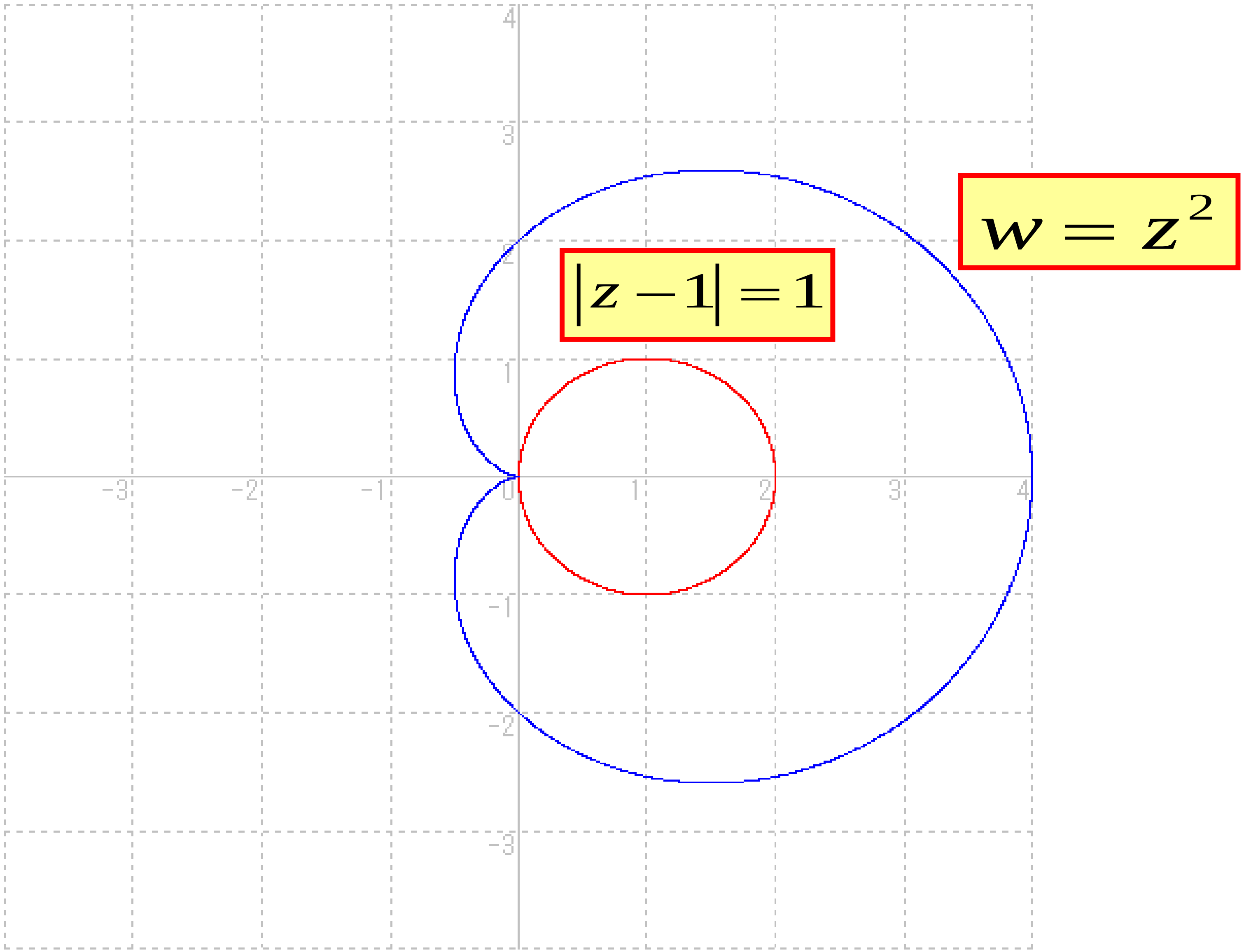
$$r = \theta$$

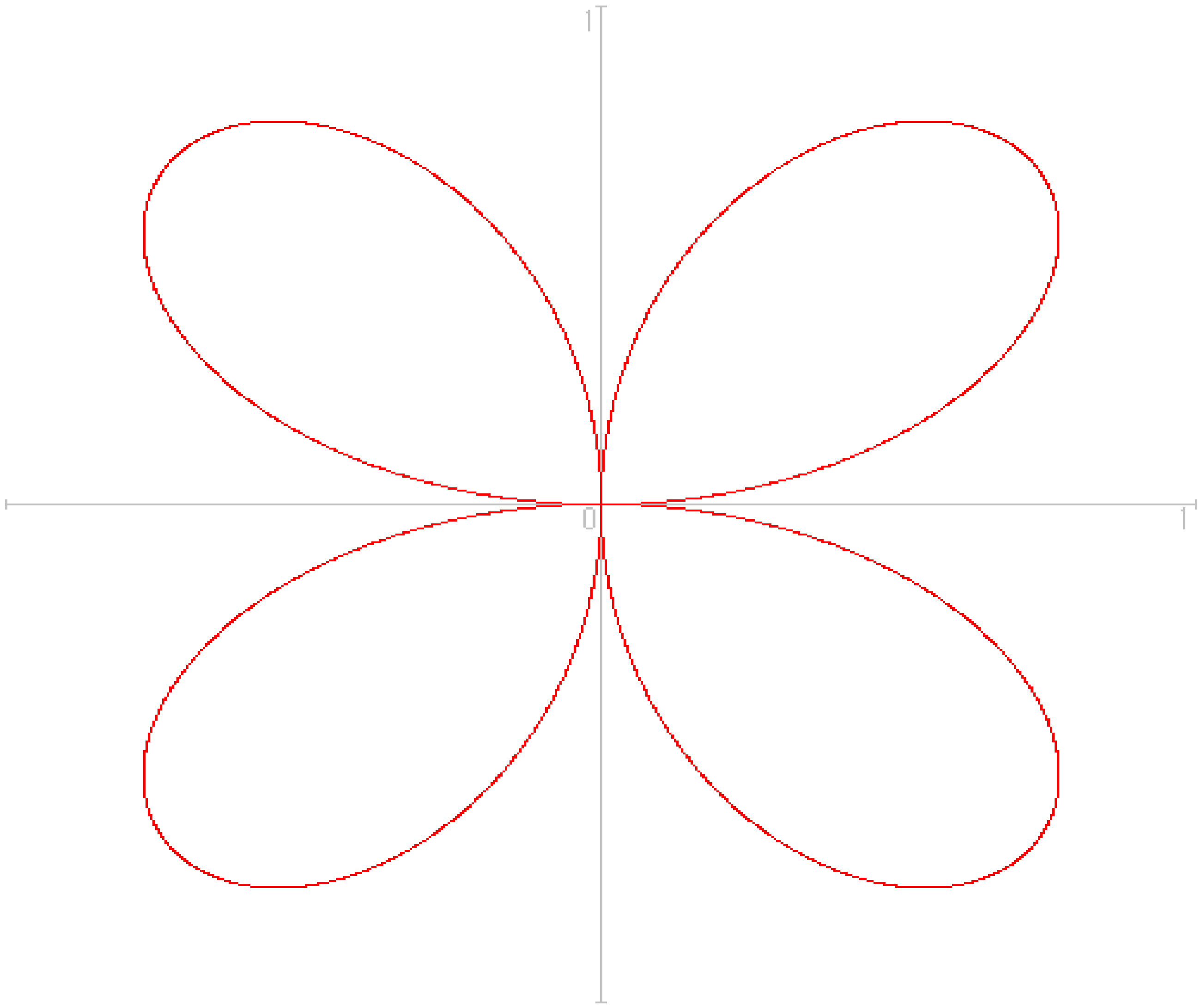
# Cardioid



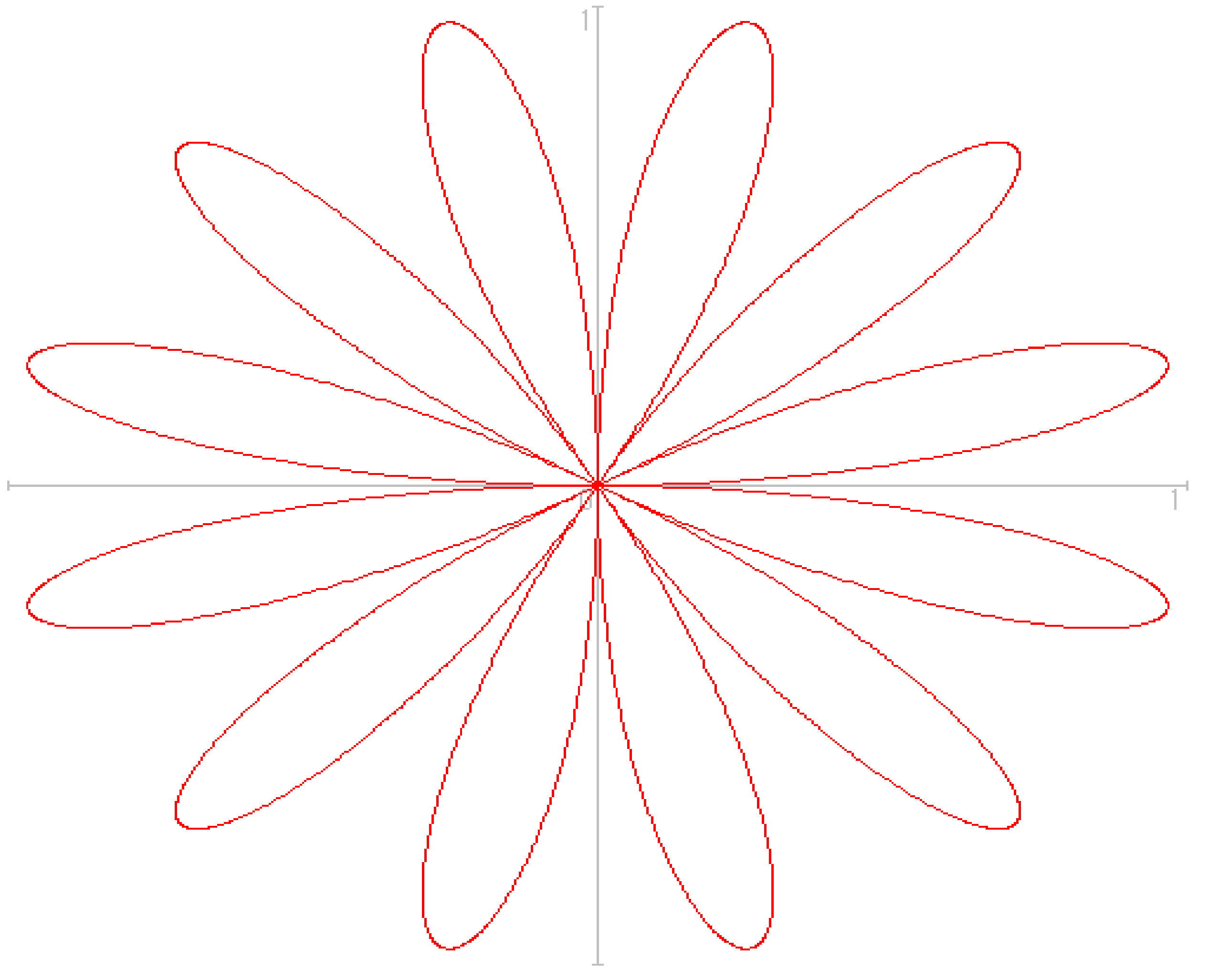

$$r = 2(1 + \cos\theta)$$

# Cardioid (Complex Version)






$$r = \sin 2\theta$$




$$r = \sin 6\theta$$

# Conic Section

$$r(\theta) = \frac{\ell}{1+\varepsilon\cos\theta}$$

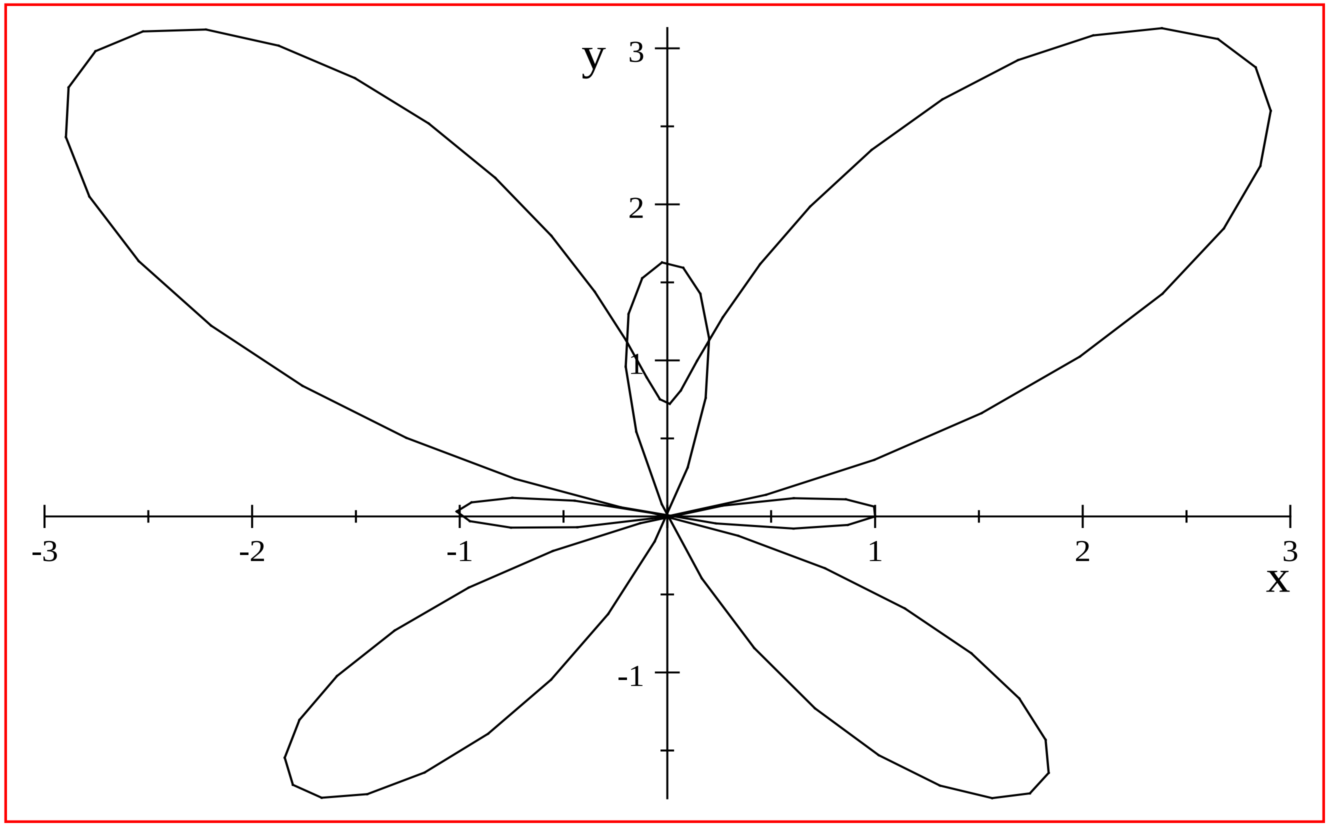
$$0<\varepsilon<1\,(\text{Ellipse}): \quad r+r' = \frac{2\ell}{1-\varepsilon^2}$$

$$\varepsilon>1\,(\text{Hyperbola}): \quad r'-r = \frac{2\ell}{\varepsilon^2-1}$$

$$\varepsilon=1\,(\text{Parabola}): \quad x = -\frac{1}{2\ell}y^2+\frac{\ell}{2}$$

# Numerical Computing with MuPAD

# Butterfly



# Compactness

| | |
|---|---|
| **Real Numbers** | **Bolzano-Weierstrass' Theorem (Sequences)** |
| **Calculus** | **Ascoli-Arzela's Theorem (Continuous Functions)** |

# Bolzano-Weierstrass Theorem

**Every bounded sequence has a convergent subsequence.**

# Ascoli-Arzela Theorem

**If a sequence of continuous functions is uniformly bounded and equicontinuous, then it has a subsequence which converges in the uniform topology.**

(1) $\exists M > 0: \ |f_n(x)| \le M$

(**Uniformly Bounded**)

(2) $\forall \varepsilon > 0, \ \exists \delta = \delta(\varepsilon) > 0:$

$|x - y| < \delta \Longrightarrow |f_n(x) - f_n(y)| < \varepsilon$

(**Equicontinuous**)

# Indefinite Integrals

# Antiderivative

A function $F(x)$ is called an **antiderivative** of $f(x)$ on an interval $I$ if it satisfies the condition

$$F'(x) = f(x), \ \forall x \in I$$

# Fundamental Theorem

If $F(x)$ is an antiderivative of $f(x)$ on an interval $I$, then the most general **antiderivative** of $f$ on $I$ is of the form

$F(x)+C,$

where $C$ is an arbitrary constant.

# Examples (1)

$$(1)\int x^{\alpha}dx=\frac{1}{\alpha+1}x^{\alpha+1},\quad \alpha\neq-1$$

$$(2)\int \frac{1}{x}dx=\log_e|x|\quad (\alpha=-1)$$

$$(3)\int a^{x}dx=\frac{a^{x}}{\log_e a},\quad a>0$$

$$\int e^{x}dx=e^{x}\quad (a=e)$$

# Examples (2)

$$(4)\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{x^2+1}dx = \tan^{-1}x \quad (a=1)$$

$$(5)\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log_e\left|\frac{x-a}{x+a}\right|, \quad a \neq 0$$

# Examples (3)

$$(6)\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0$$

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1} x \quad (a = 1)$$

$$(7)\int \log_e x\,dx = x\left(\log_e x - 1\right)$$

# Examples (4)

$$(8)\int \sqrt{a^2 - x^2}\, dx$$

$$= \frac{1}{2}\left( x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a} \right), \quad a > 0$$

$$(9)\int \frac{1}{\sqrt{x^2 + A}} dx = \log_e \left| x + \sqrt{x^2 + A} \right|, A > 0$$

# Examples (5)

$$(10)\int \sqrt{x^2 + A}\,dx$$

$$= \frac{1}{2}\left(x\sqrt{x^2 + A} + A\log_e\left|x + \sqrt{x^2 + A}\right|\right),\ A > 0$$

# Examples

$$(1) \int \sin x \, dx = -\cos x$$

$$(2) \int \cos x \, dx = \sin x$$

$$(3) \int \frac{1}{\cos^2 x} dx = \tan x$$

$$(4) \int \frac{1}{\sin^2 x} dx = -\cot x$$

$$(5) \int \tan x \, dx = -\log|\cos x|$$

$$(6) \int \cot x \, dx = \log|\sin x|$$

# Indefinite Integrals of Rational Functions

# Example

$$\frac{1}{x\left(x^2+1\right)^2}$$

$$=\frac{1}{x}-\frac{x}{x^2+1}-\frac{x}{(x^2+1)^2}$$

# Formula (5)

$$(1)\int \frac{1}{x-a}dx = \log_e |x-a|$$

$$(2)\int \frac{1}{(x-a)^l}dx$$

$$= -\frac{1}{(l-1)(x-a)^{l-1}}, \quad l \neq 1$$

# Formula (6)

$$\int \frac{x}{x^2 + a^2} dx$$

$$= \frac{1}{2} \log\left(x^2 + a^2\right), \quad a \neq 0$$

# Formula (7)

$$\int \frac{x}{\left(x^2+a^2\right)^m}\,dx$$

$$= \frac{1}{2(m-1)\left(x^2+a^2\right)^{m-1}},$$

$$a \neq 0, \quad m \neq 1$$

# Example 1-1

$$\int \frac{1}{x\left(x^2+1\right)^2}dx$$

$$= \int \frac{1}{x}dx - \int \frac{x}{x^2+1}dx$$

$$- \int \frac{x}{\left(x^2+1\right)^2}dx$$

# Example 1-2

$$\int \frac{1}{x\left(x^2+1\right)^2}dx$$

$$=\log_e \frac{|x|}{\sqrt{x^2+1}}+\frac{1}{2\left(x^2+1\right)}$$

# Formula (8)

$$I_m = \int \frac{1}{\left(x^2 + a^2\right)^m} dx,$$

$$a \neq 0, \quad m \geq 1$$

# Formula (9)

$$I_1 = \int \frac{1}{x^2 + a^2} dx$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a}$$

# Formula (10)

$$I_2 = \int \frac{1}{\left(x^2 + a^2\right)^2} dx$$

$$= \frac{1}{2a^2} \left( \frac{x}{2\left(x^2 + a^2\right)} + \frac{1}{a} \tan^{-1} \frac{x}{a} \right)$$

# Formula (11)

$$I_m = \int \frac{1}{\left(x^2 + a^2\right)^m} dx \qquad (m \geq 2)$$

$$= \frac{1}{a^2}\left(\frac{x}{2(m-1)\left(x^2 + a^2\right)^{m-1}} + \frac{2m-3}{2m-2} I_{m-1}\right)$$

# Definite Integrals

# Series and Integrals

$$\sum_{n=1}^{\infty} a_n$$

**Discrete Case**

**Continuous Case**

$$\int_a^b f(x)\,dx$$

# Riemann Integral

# Georg Friedrich Bernhard Riemann
# (1826-1866)

# Definition of Riemann Integral

**Let** $f(x)$ **be a bounded function defined on an interval** $I=[a,b]$

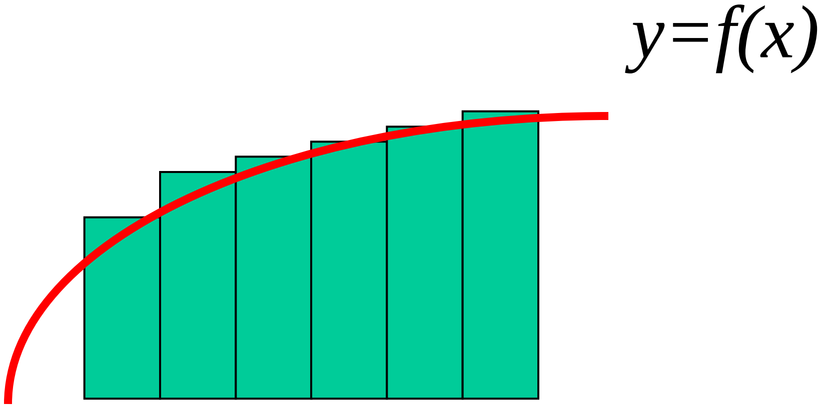
$\Delta: x_0 = a < x_1 < \cdots < x_n = b$ (**partition of** $I$)

$$|\Delta| = \max_{1\le i\le n} (x_i - x_{i-1})$$

$$S(\Delta, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),\ \ M_i = \sup_{x_{i-1}\le t\le x_i} f(t)$$

$$s(\Delta, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}),\ \ m_i = \inf_{x_{i-1}\le t\le x_i} f(t)$$

# Upper Integral

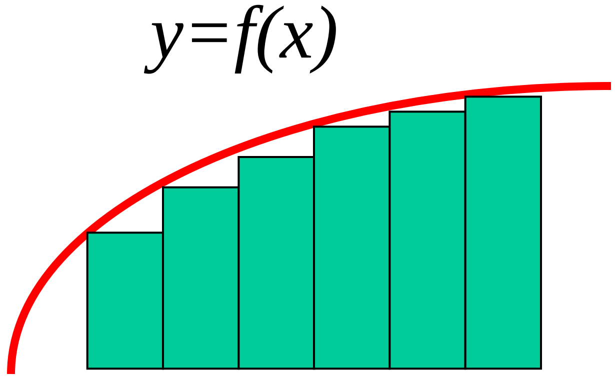



**Approximation area from outside**

# Upper Integral

$$\overline{\lim}\int_a^b f(x)\,dx = \inf_{\Delta} S(\Delta, f)$$

# Lower Integral




**Approximation area from inside**

# Lower Integral

$$\underline{\lim}\int_a^b f(x)\,dx = \sup_{\Delta} s(\Delta, f)$$

# Definition of Riemannian Integrability

$$\underline{\lim}\int_a^b f(x)\,dx = \overline{\lim}\int_a^b f(x)\,dx$$

# Examples

**(1) Continuous functions**

**(2) Monotone increasing (decreasing) functions**

**(3) Bounded variation functions**

# Darboux's Theorem

# Darboux's Theorem

$$(1)\ \overline{\lim}\int_a^b f(x)\,dx = \lim_{|\Delta|\to 0} S(\Delta, f)$$

$$(2)\ \underline{\lim}\int_a^b f(x)\,dx = \lim_{|\Delta|\to 0} s(\Delta, f)$$

# Mensuration by Parts

# Riemannian Sum

**Let** $f(x)$ **be a Riemann integrable function defined on an interval** $I=[a,b]$

$$\Delta : x_0 = a < x_1 < \cdots < x_n = b$$

$$\left|\Delta\right| = \max_{1\le i\le n} (x_i - x_{i-1})$$

$$\Rightarrow$$

$$\int_a^b f(x)dx = \lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \quad x_{i-1} \le \forall t \le x_i$$

# Example

**Let** $f(x)$ **be a continuous function defined on the interval** $I = [0,1]$

$\Rightarrow$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$$

# Example

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}}=\int_0^1\frac{1}{\sqrt{x^2+1}}dx$$

$$=\log_e\left|x+\sqrt{x^2+1}\right|_0^1$$

$$=\log_e\left(1+\sqrt{2}\right)$$

# Fundamental Theorem

# of

# Calculus

# Fundamental Theorem of Calculus
# Part 1

If $f(x)$ is continuous on $[a, b]$, then

$$g(x) = \int_a^x f(t)dt, \quad a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on $(a, b)$, and

$$g'(x) = f(x).$$

# Fundamental Theorem of Calculus Part 2

If $f(x)$ is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

# Notation

Because of the relation given by
the fundamental theorem of calculus
between antiderivatives and integrals,
the notation

$$\int f(x)dx$$

is traditionally used for an antiderivative of $f(x)$, and is called an **indefinite integral**.

# Strategy for Integration

# Formula for Integration by Parts

If $f(x)$ and $g(x)$ are

$C^1$ functions on $[a,b]$, then

$$\int_a^b f'(x)g(x)dx$$

$$=\left[f(x)g(x)\right]_{x=a}^{x=b}-\int_a^b f(x)g'(x)dx$$

# Substitution Rule for Definite Integrals

If $g(x)$ is a $C^1$ function on $[a,b]$ and

if $f(x)$ is continuous on the range of $g$,

then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

# Areas between Curves

If a region $D$ is bounded by the curves with equations

$$y = f(x), y = g(x)$$

and the lines

$$x = a, x = b,$$

where

$$f(x) \geq g(x), \quad a \leq x \leq b,$$

then the area $A$ of $D$ is

$$A = \int_a^b [f(x) - g(x)] dx$$

# Center of Mass

If a region $D$ is bounded by the curves with equations

$y = f(x),\ y = g(x)$

and the lines

$x = a,\ x = b,$

where

$f(x) \geq g(x), \quad a \leq x \leq b,$

then the center of mass the $D$ is located at

$$\overline{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]dx$$

$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]dx$$

# Volume of Revolution

If $S$ is the solid generated when the region bounded by the curves with equations

$$y = f(x),\ y = g(x) \quad (f(x) \geq g(x))$$

and the lines

$$x = a,\ x = b,$$

is rotated about the $x$-axis,

then the volume $V$ of $S$ is

$$V = \pi \int_a^b \left[ f(x)^2 - g(x)^2 \right] dx$$

# Improper Integrals

# Improper Integral of Type 1

(I) If the definite integral

$$\int_a^t f(x)dx$$

exists for every $t \geq a,$ then

$$\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx$$

provided the limit exists

as a finite number.

# Example 1

$$\int_1^\infty \frac{1}{x^\alpha}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^\alpha}dx$$

$$= \lim_{t\to\infty}\left(\frac{1}{\alpha-1} - \frac{1}{\alpha-1}\frac{1}{t^{\alpha-1}}\right)$$

$$= \frac{1}{\alpha-1}, \quad \alpha > 1$$

# Improper Integral of Type 1

(II) If the definite integral

$$\int_t^b f(x)dx$$

exists for every $t \leq b,$ then

$$\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$$

provided the limit exists as a finite number.

# Improper Integral of Type 1

(III) If the improper integrals

$$\int_a^{\infty} f(x)dx, \quad \int_{-\infty}^{a} f(x)dx$$

exist, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

# Improper Integral of Type 2

(I) If $f(x)$ is continuous on $[a,b)$
and is discontinuous at $b$, then

$$\int_a^b f(x)dx = \lim_{t \uparrow b} \int_a^t f(x)dx$$

provided the limit exists
as a finite number.

# Improper Integral of Type 2

(II) If $f(x)$ is continuous on $(a, b]$

and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \downarrow a} \int_t^b f(x)dx$$

provided the limit exists

as a finite number.

# Example 2

$$\int_0^1 \frac{1}{x^\alpha}dx = \lim_{t\to 0}\int_t^1 \frac{1}{x^\alpha}dx$$

$$= \lim_{t\to 0}\left(\frac{1}{1-\alpha} - \frac{1}{1-\alpha}t^{1-\alpha}\right)$$

$$= \frac{1}{1-\alpha}, \quad 0 < \alpha < 1$$

# Improper Integral of Type 2

(III) If $f(x)$ is discontinuous at $c \in (a, b)$,
and the improper integrals

$$\int_a^c f(x)dx, \quad \int_c^b f(x)dx$$

exist, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

# Application to Taylor Series

$$\tan^{-1} x$$

$$= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$(-1 < x \leq 1)$$

# Proof (1)

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{x^2 + 1}$$

$$\int_0^x \frac{1}{t^2 + 1} dt = \tan^{-1} x$$

# Proof (2)

$$\frac{1}{1+t^2} = \frac{1}{1-(-t^2)}$$

$$= 1 + (-t^2) + \cdots + (-1)^n t^{2n} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n t^{2n}$$ **(Geometric Series)**

$$(-1 < t < 1)$$

# Proof (3)

$$\tan^{-1} x$$

$$= \int_0^x \frac{1}{1+t^2} dt$$

$$= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} dt$$

$$= x - \frac{x^3}{3} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$(-1 < x < 1)$$

# Numerical Analysis

# Trapezoidal Rule

$$\int_a^b f(x)dx \doteq \frac{b-a}{n}$$

$$\times\frac{1}{2}\left\{ f(x_0)+2\Big( f(x_1)+\cdots+ f(x_{n-1})\Big)+ f(x_n)\right\}$$

# Simpson's Rule

$$\int_a^b f(x)dx \doteq \frac{b-a}{2n}$$

$$\times\frac{1}{3}\left\{ f(x_0)+2\left( f(x_2)+\cdots+ f(x_{2n-2})\right)+ f(x_{2n})\right\}$$

$$+\frac{b-a}{2n}\times\frac{1}{3}\left\{4\left( f(x_1)+\cdots+ f(x_{2n-1})\right)\right\}$$

# Computational Approach

# Example

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3} = 0.6666666666\cdots$$

# Numerical Computing
# with
# BASIC

# Trapezoidal Rule

```
REM 台形公式による近似積分
REM 関数 sqr(x) 積分区間 [0,1]
PRINT "台形公式により積分の近似計算をします"
PRINT "何等分しますか？"
INPUT PROMPT "n=": n
LET s=0
LET h = 1/n
FOR k = 0 TO n-1
   LET  x = k*h
   LET  y = (k+1)*h
   LET  s = s + (SQR(x) + SQR(y))*h/2
NEXT k
PRINT s
PRINT 0.6666666666666666
END
```

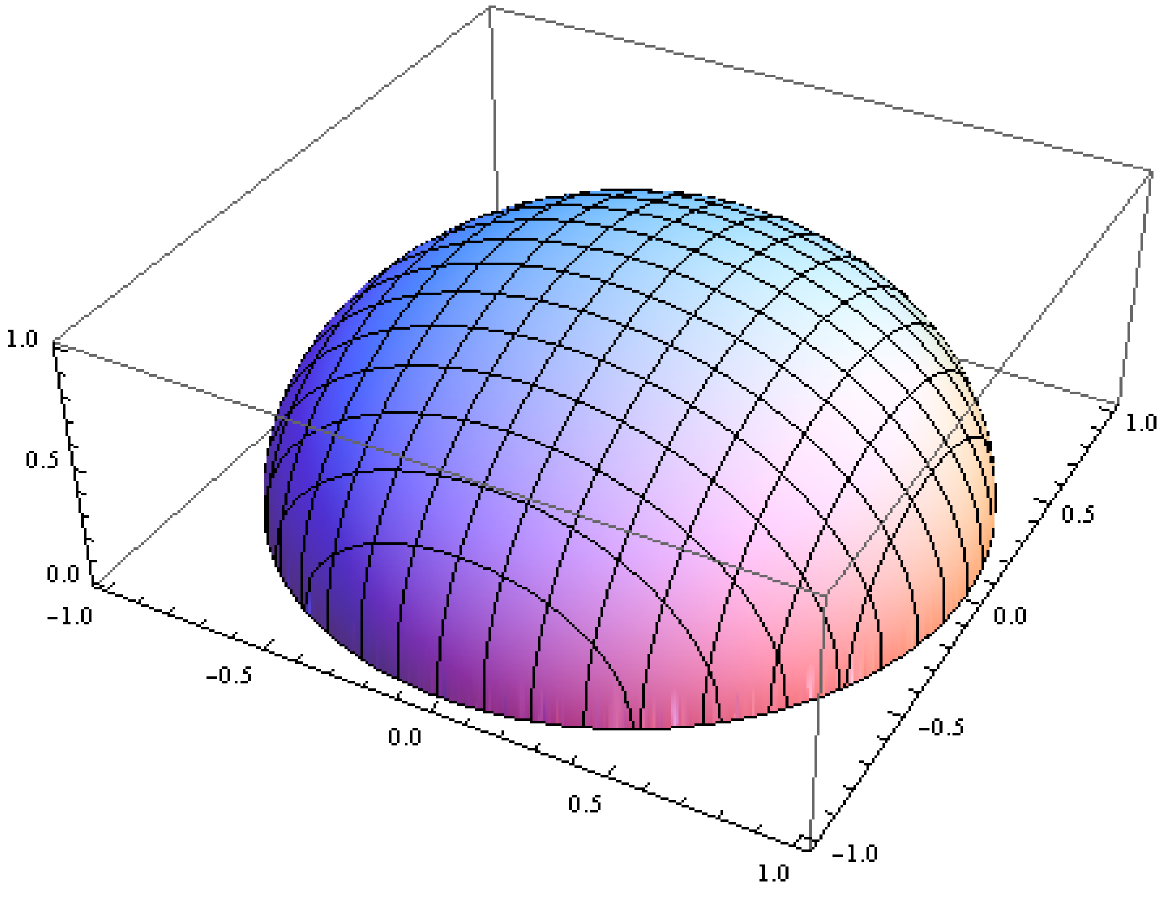
# Simpson's Rule

```
REM シンプソン公式による近似積分
REM 関数 sqrt(x) 積分区間 [0,1]
PRINT "シンプソン公式による近似面積"
PRINT "何等分しますか？"
INPUT PROMPT "n=": n
LET h = 1/(2*n)
FOR k = 0 TO n-1
   LET  x = 2*k*h
   LET  m = (2*k+1)*h
   LET  y = (2*k+2)*h
   LET  s = s + (SQR(x)+4*SQR(m)+SQR(y))*(h/3)
NEXT k
PRINT s
PRINT 0.66666666666666666666
END
```

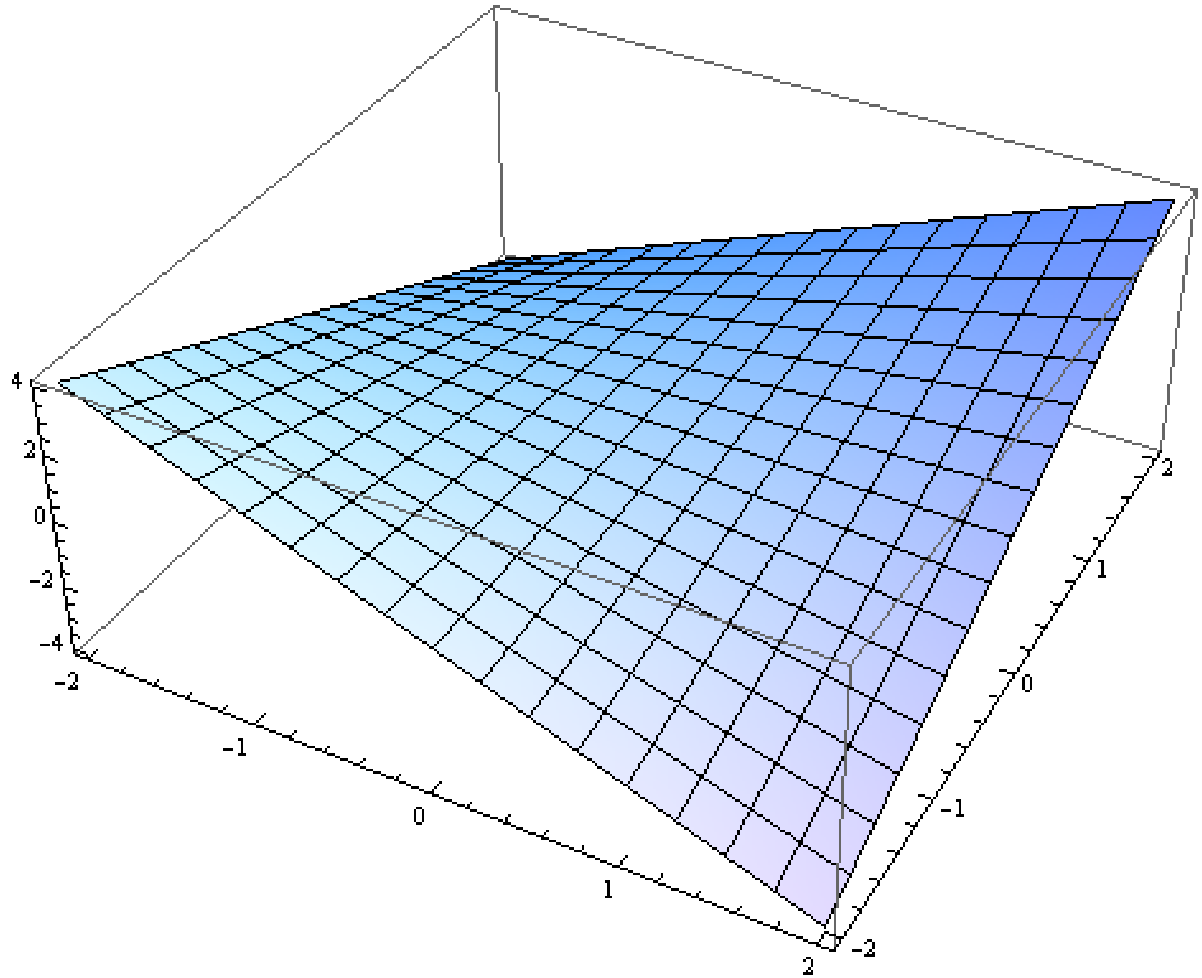
| Mesh | Trapezoidal Rule | Simpson’s rule |
|---|---|---|
| n=10 | .660509341706818 | .664099589757422 |
| n=100 | .666462947103147 | .666585482066722 |
| n=1000 | .666660134393675 | .666664099383542 |
| n=10000 | .666666459197103 | .666666585482054 |

# Calculus
# of
# Two Variables

$$z=\sqrt{1-x^2-y^2}$$



$$z = xy$$



# Continuity of Functions

# Continuity

Let $D$ be a domain in $\mathbf{R}^2$

A function $f(x,y)$ defined in $D$

is **continuous** at $(a,b)\in D$

$$\Leftrightarrow$$

$\forall\varepsilon>0,\ \exists\delta=\delta((a,b),\varepsilon)>0$ such that

$$|x-a|<\delta,\ |y-b|<\delta\Rightarrow|f(x,y)-f(a,b)|<\varepsilon$$

# Differentiation

# Partial Differentiability of Functions

# Partial Differentiability

Let $D$ be a domain in $\mathbf{R}^2$

A function $f(x,y)$ defined in $D$

is **partially differentiable** at $(a,b)\in D$

$$\Leftrightarrow$$

$$\exists \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}=\frac{\partial f}{\partial x}(a,b)$$

$$\exists \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}=\frac{\partial f}{\partial y}(a,b)$$

# Total Differentiability
# of
# Functions

# Total Differentiability

Let $D$ be a domain in $\mathbf{R}^2$

A function $f(x, y)$ defined in $D$

is **totally differentiable** at $(a, b) \in D$

$$\Leftrightarrow$$

$\exists \alpha \in \mathbf{R}, \exists \beta \in \mathbf{R}$ such that

$$f(a+h, b+k)$$

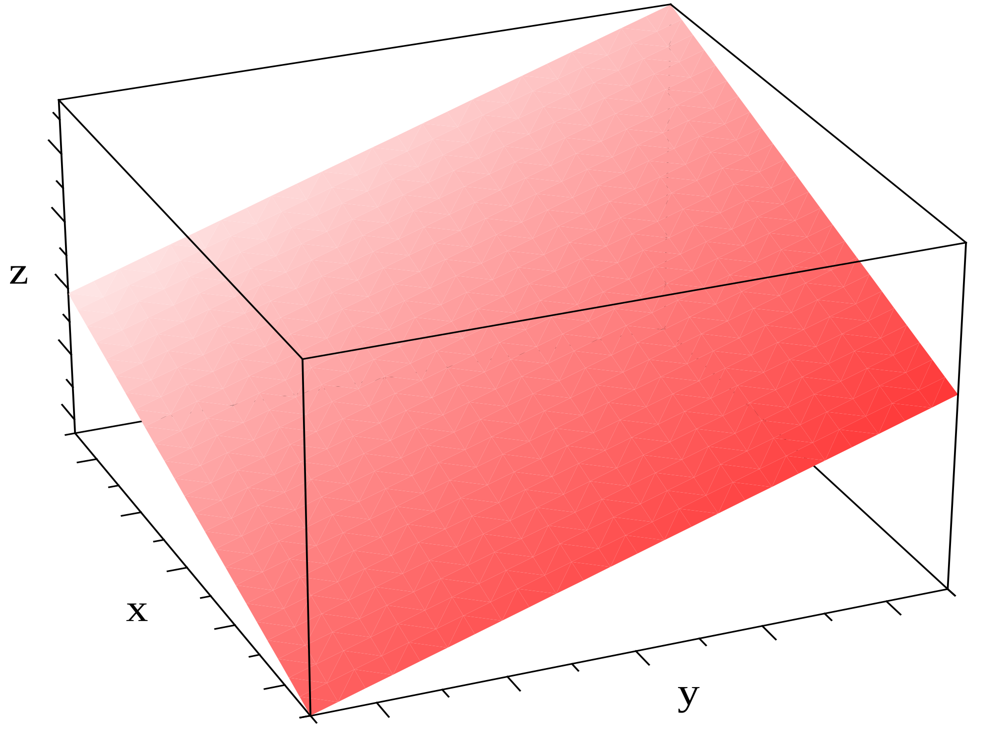
$$= f(a,b) + \alpha h + \beta k + o\left(\sqrt{h^2 + k^2}\right)$$

# Geometric Meaning
# of
# Total Differentiability

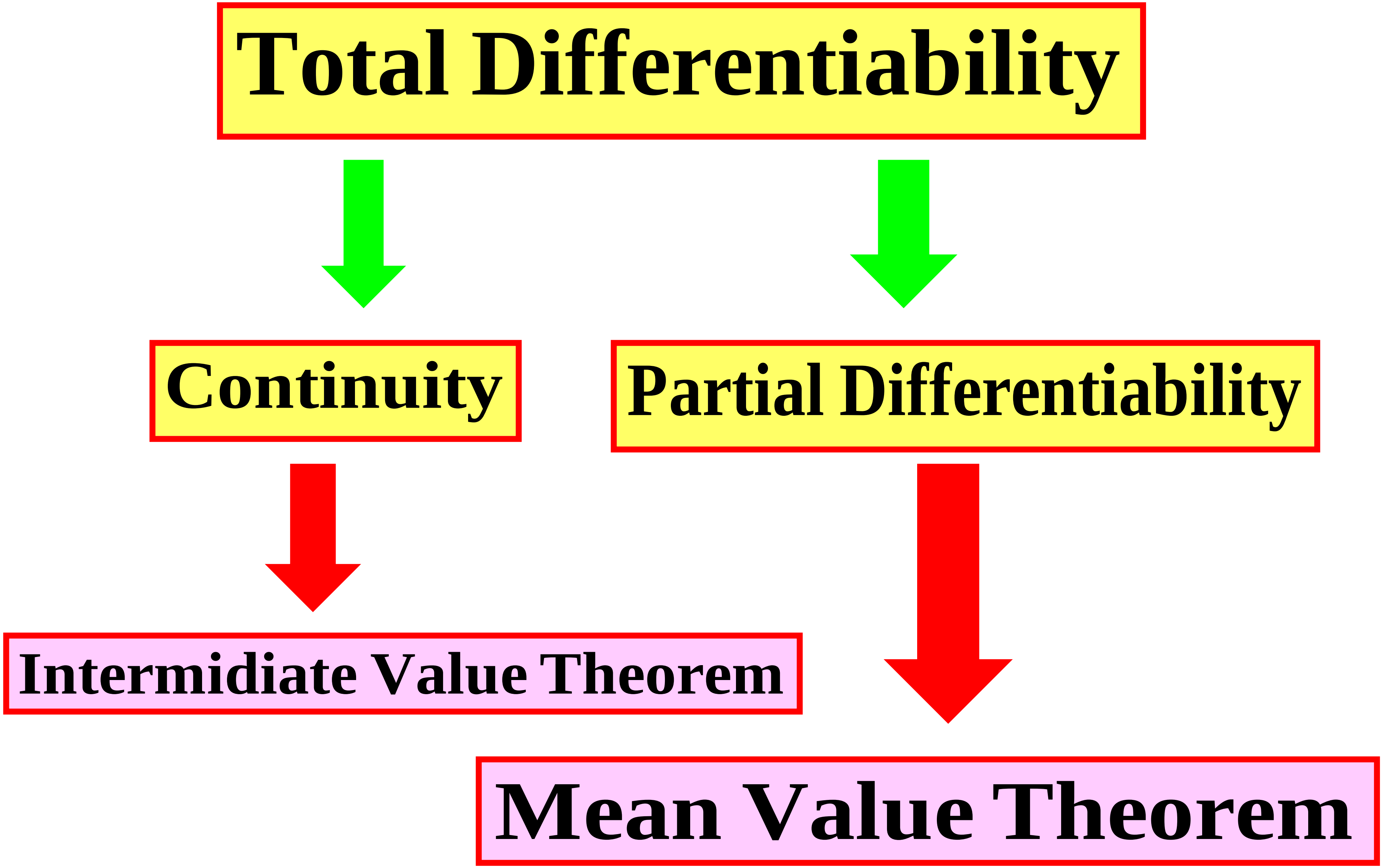
# Tangent Plane

$$\begin{pmatrix} x \\ y \\ f(a,b) + \alpha(x-a) + \beta(y-b) \end{pmatrix} = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} + (x-a)\begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} + (y-b)\begin{pmatrix} 0 \\ 1 \\ \beta \end{pmatrix}$$

# Equation of a Plane




$$-\alpha x - \beta y + z = d$$

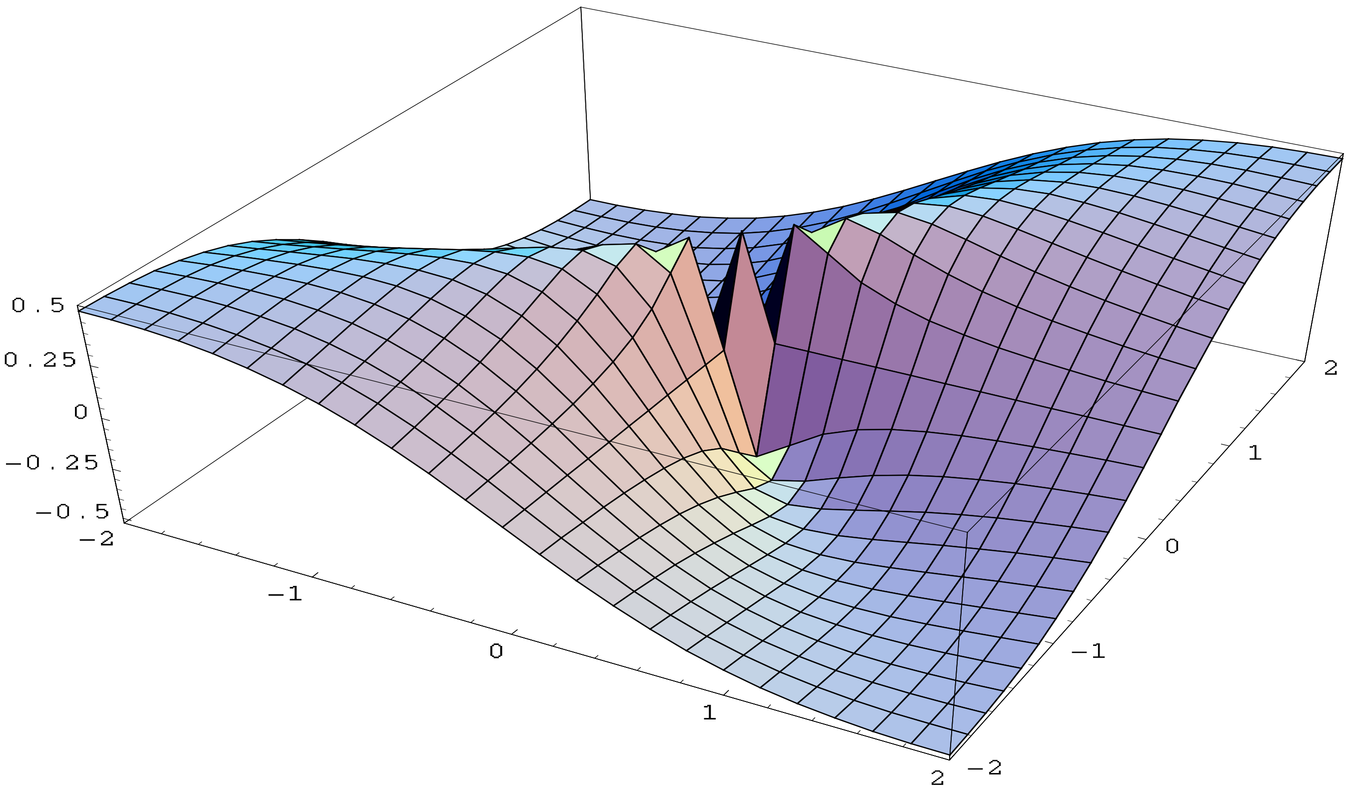

Total Differentiability
Continuity
Partial Differentiability
Intermidiate Value Theorem
Mean Value Theorem

# Example of Functions (Surfaces)

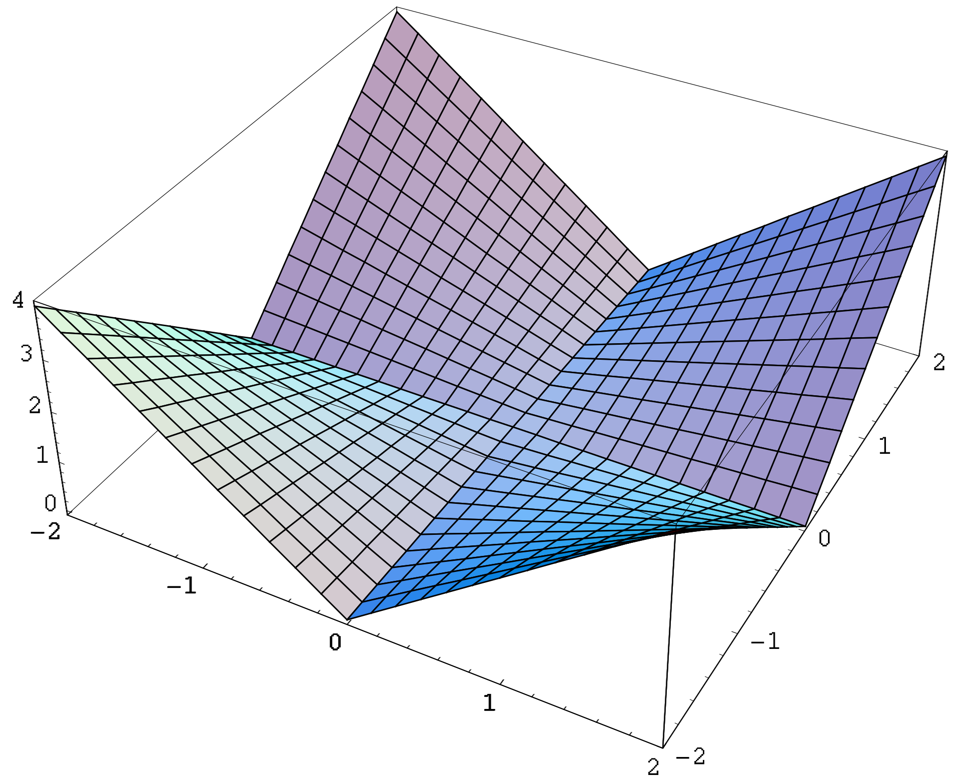
| Function | Continuity | Partial Differentiability | Total Differentiability |
| --- | --- | --- | --- |
| $\dfrac{xy}{x^2+y^2}$ | × | ○ | × |
| $\lvert xy \rvert$ | ○ | × | × |
| $xy\,e^{-\frac{x^2+y^2}{2}}$ | ○ | ○ | ○ |

# Numerical Computing with MATHEMATICA

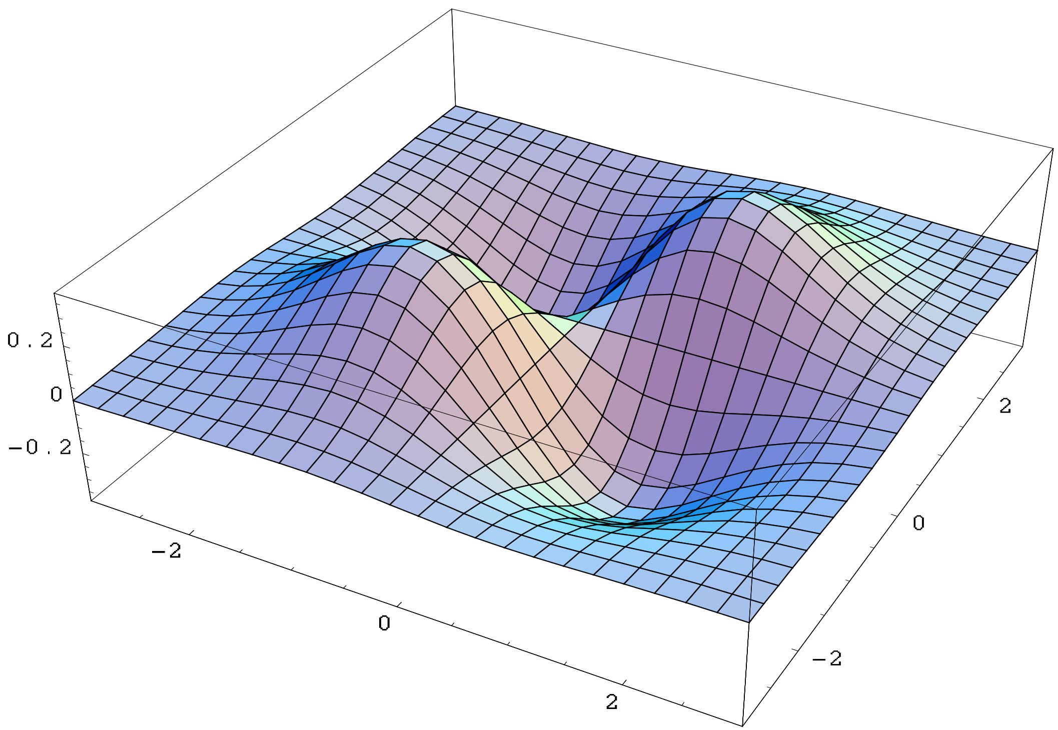
$$z = \frac{xy}{x^2 + y^2}$$



$$z = |xy|$$



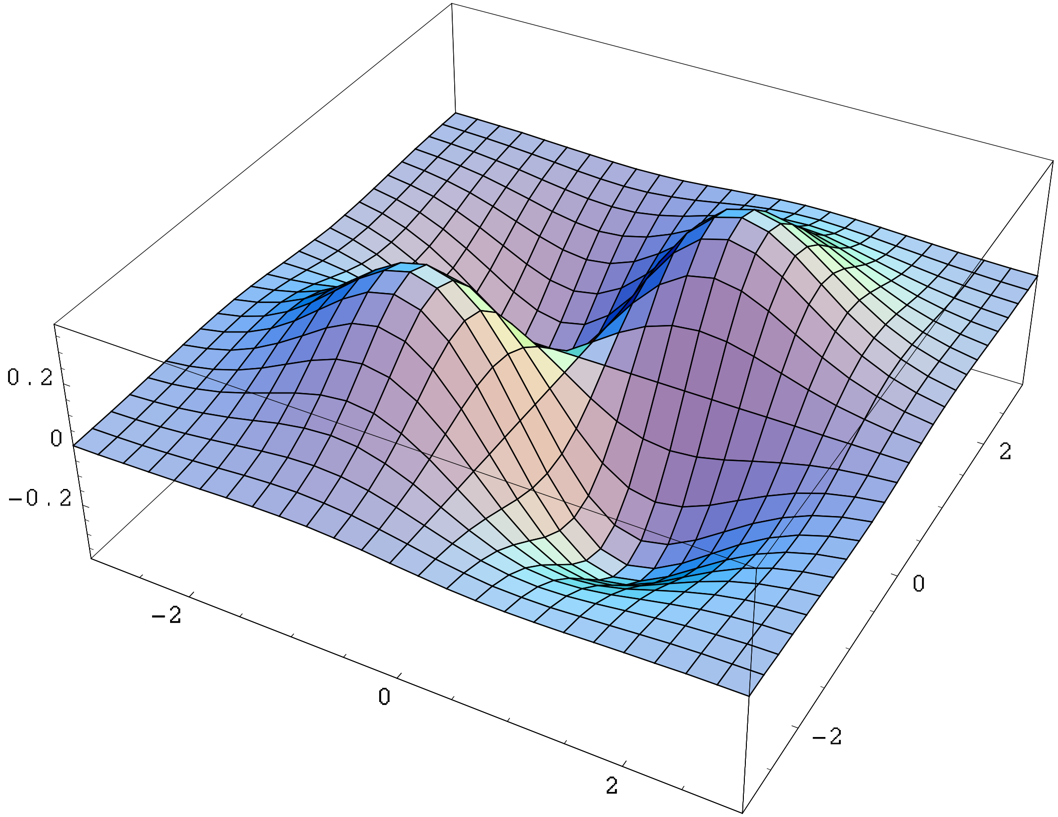
$$z = xy\,e^{-\frac{x^2+y^2}{2}}$$



# Extremes of Functions

| | | | | | |
|---|---|---|---|---|---|
| Point $(x, y)$ | $(0,0)$ | $(1,1)$ | $(-1,-1)$ | $(1,-1)$ | $(-1,1)$ |
| $xy\,e^{-\frac{x^2+y^2}{2}}$ | $0$ | $\frac{1}{e}$ | $\frac{1}{e}$ | $-\frac{1}{e}$ | $-\frac{1}{e}$ |
| **Behavior** | **Saddle Point** | **Maximal Value** | **Maximal Value** | **Minimal Value** | **Minimal Value** |

$$z = xy\, e^{-\frac{x^2+y^2}{2}}$$



# Canonical Forms
# of
# Polynomials of second-order

**Mean Value Theorem**

$\Downarrow$

**Taylor's Theorem**

$\Downarrow$

**Polynomial Approximation**

# Polynomial

$$z = f(x, y)$$
$$= ax^2 + 2bxy + cy^2$$

# Matrix Form

$$z = f(x, y)$$

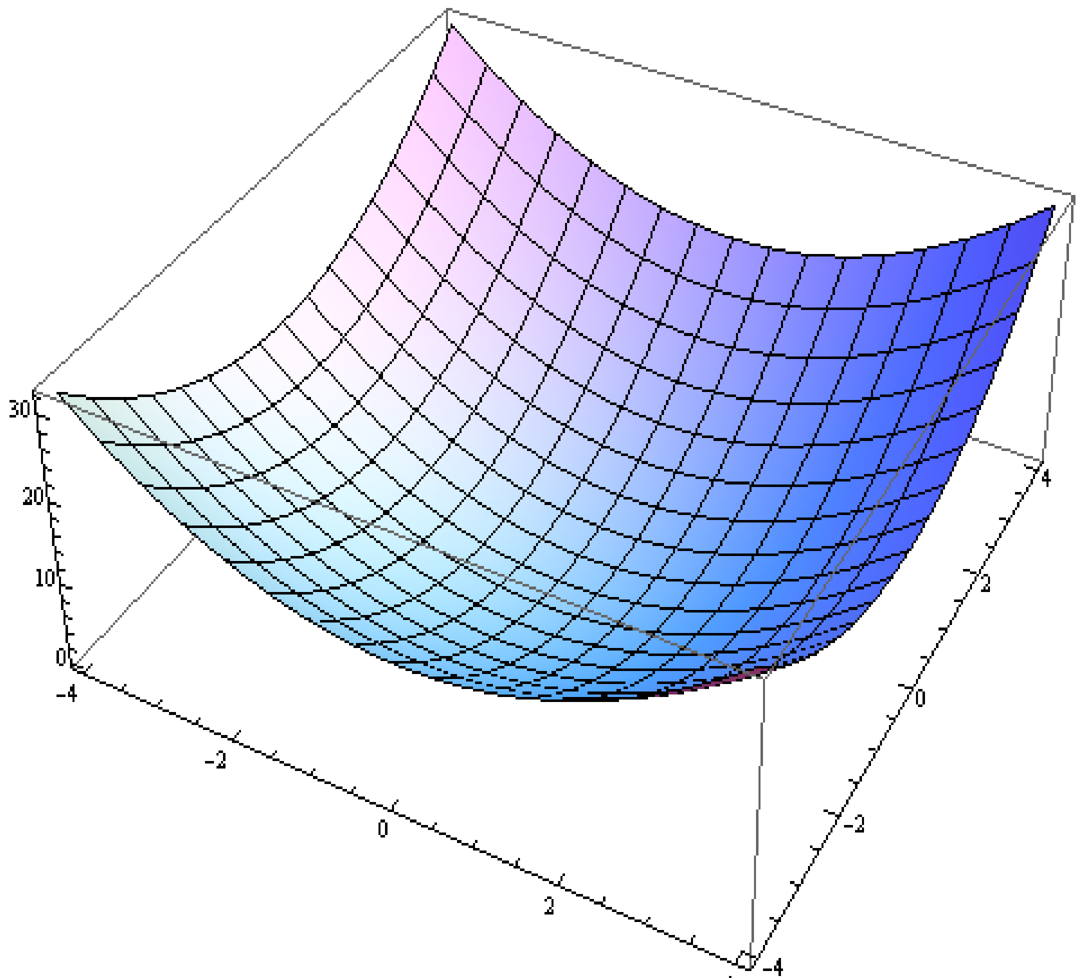
$$= ax^2 + 2bxy + cy^2$$

$$\Longrightarrow$$

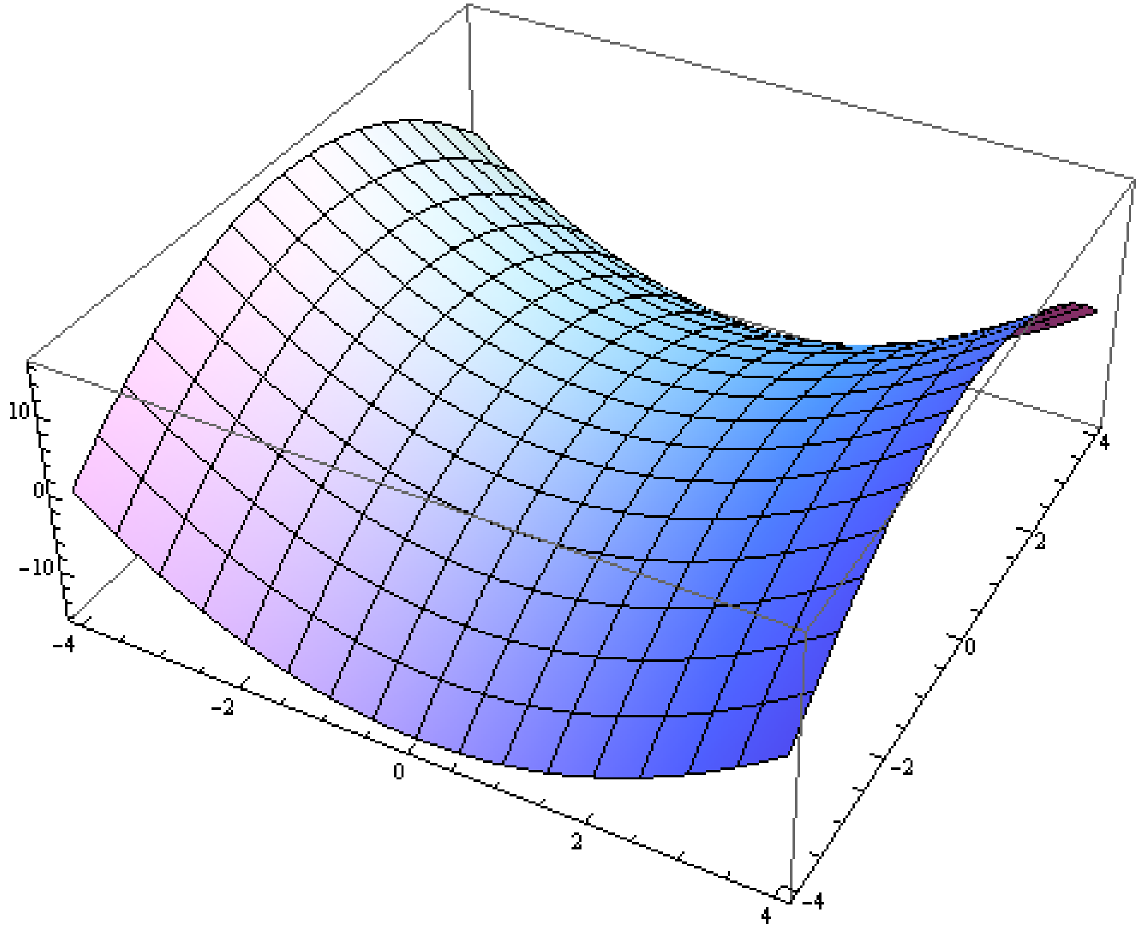
$$ax^2 + 2bxy + cy^2$$

$$= \left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

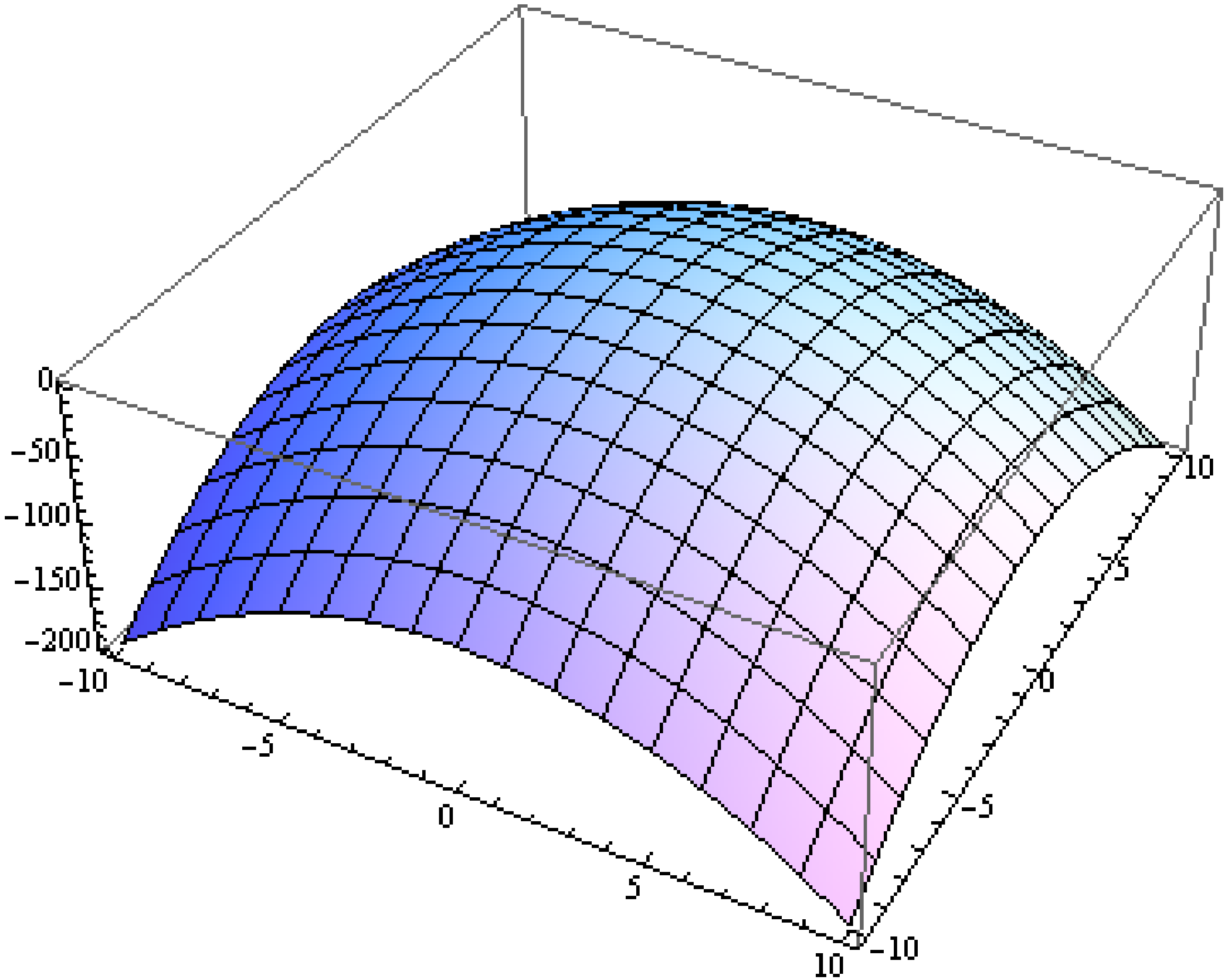
$z = x^2 + y^2$ **(minimal point)**



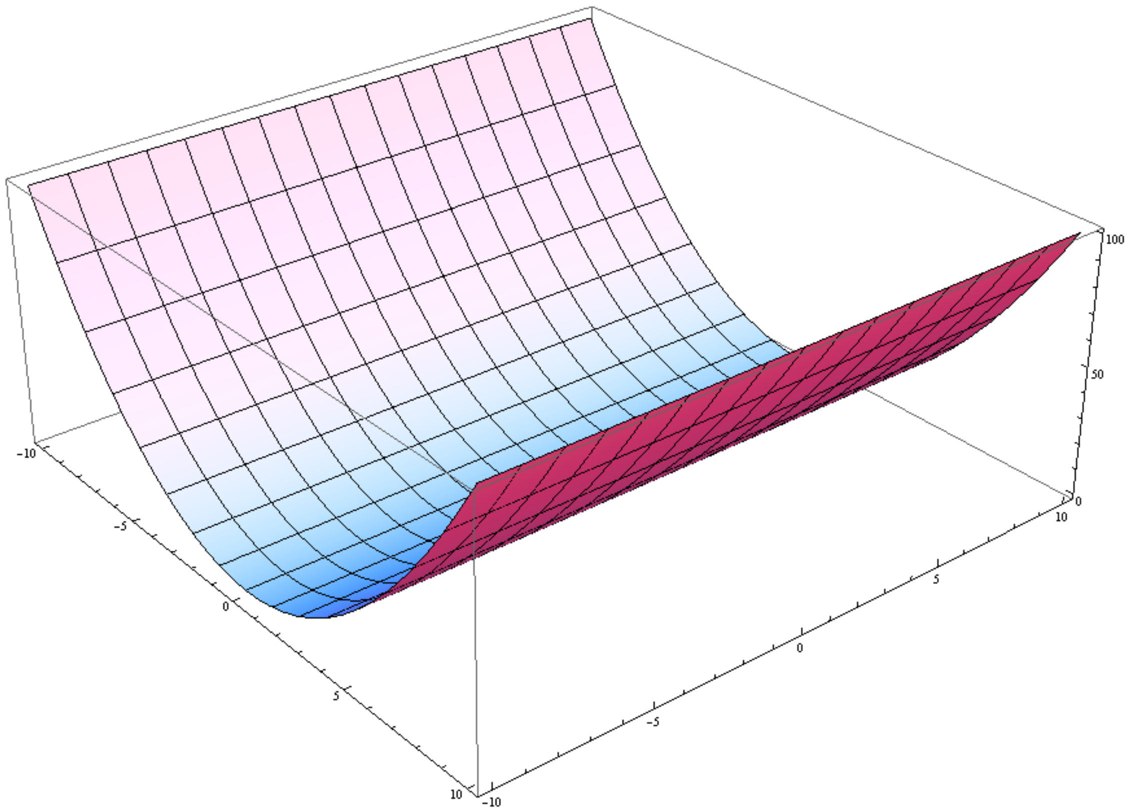
$z = x^2 - y^2$ **(saddle point)**



$z = -x^2 - y^2$ **（maximal point）**



$z = x^2$ （degenerate point）



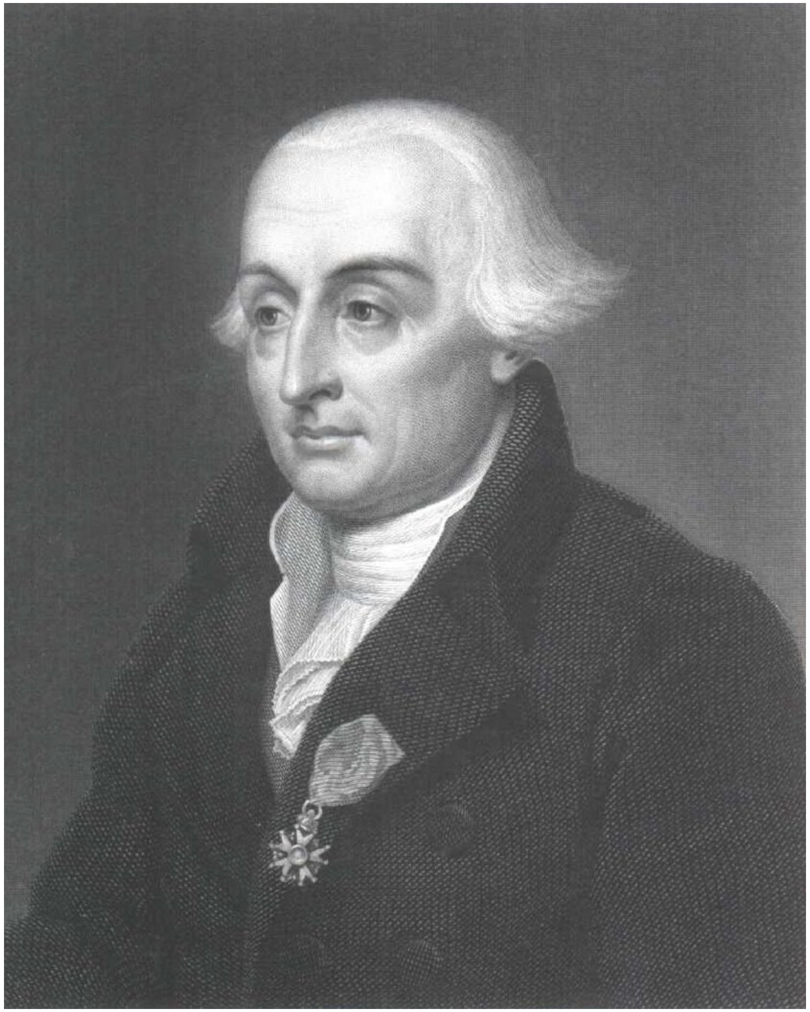
# Lagrange's Method
# of
# Undetermined Multipliers

# Lagrange

- Joseph Louis Lagrange (1736-1813)

**Italian and French Mathematician**

# Joseph Louis Lagrange



# Extreme Problem (1)

$f(x, y), \ \varphi(x, y): C^1$ **functions**

**Under the condition**

$\varphi(x, y) = 0$

**find the extremes of the function**

$f(x, y)$

# Idea (1)

**Introduce the function**

$$F(x, y, \lambda) = f(x, y) - \lambda \varphi(x, y)$$

$\lambda$ **: parameter**

# Idea (2)

**Assume that** $f(x,y)$ **takes an extreme at** $(a,b)$

**Then either (1) or (2) holds true :**

(1) $dF(x,y,\lambda)=0$ :

$$\begin{cases} f_x(a,b)-\lambda\varphi_x(a,b)=0 \\ f_y(a,b)-\lambda\varphi_y(a,b)=0 \\ \varphi(a,b)=0 \end{cases}$$

(2) $d\varphi(x,y)=0$ :

$$\varphi_x(a,b)=\varphi_y(a,b)=0$$

# Example 1

**Under the condition**

$$p_1 x + p_2 y = C$$

**find the maximum of the function**

$$z = a x^s y^t, \quad s + t = 1, s > 0, t > 0$$

# Example 2

**Under the condition**

$$4x + 5y - 6 = 0, \quad x > 0, \; y > 0$$

**find the maximum of the function**

$$z = x^{1/3} y^{2/3}$$

# Extremes Problem (2)

$f(x, y), \ g(x, y) : C^1$ **functions**

**Under the condition**

$g(x, y) \geq 0$

**find the extremes of the function**

$f(x, y)$

# Idea

**Introduce the function**

$$F(x, y, z, \lambda) = f(x, y) - \lambda\left(g(x, y) - z\right)$$

**Consider the extremes of the function**

$$F(x, y, z, \lambda)$$

**in the half space**

$$\mathbf{R}_+^3 = \left\{(x, y, z) \in \mathbf{R}^3 : z \geq 0\right\}$$

# Example

$$\Omega = \left\{ (x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

**Find the maximum and minimum of the function $x + y$ in $\Omega$.**

# Extreme Problem (3)

$f(x, y), \ g(x, y), h(x, y) : C^1$ **functions**

**Under the two conditions**

$g(x, y) \geq 0, h(x, y) \geq 0$

**find the extremes of the function**

$f(x, y)$

# Idea

**Introduce the function**

$$F(x, y, z, w, \lambda, \mu)$$

$$= f(x, y) - \lambda\left(g(x, y) - z\right) - \mu\left(h(x, y) - w\right)$$

**Find the extremes of the function**

$$F(x, y, z, w, \lambda, \mu)$$

**in the domain**

$$\left\{(x, y, z, w) \in \mathbf{R}^4 : z \geq 0, w \geq 0\right\}$$

# Example

$$\Omega = \left\{ (x, y) \in \mathbf{R}^2 : y \geq 0,\ y \leq 1 - x^2 \right\}$$

**Find the maximum and minimum of the function** $2x + y$ **in** $\Omega$**.**

# Differential Equations

# Bird's-Eye View

| Theme | Mathematics | Mechanics |
|---|---|---|
| Differential Equation | Second-order Differential Equation | Newtonian Equation of Motion |

# List of Mathematicians

# List of Mathematicians

- **Isaac Newton (1642-1727) England**
- **Leonhard Euler (1707-1783) Switzerland**
- **Jean-Baptiste Fourier (1768-1830) France**
- **Joseph Louis Lagrange (1736-1813) Italy, France**
- **Augustin Louis Cauchy (1789-1857) France**
- **Thomas Robert Malthus (1766-1834) England**
- **Pierre Francois Verhulst (1804-1849) Belgium**

# Method of Quadrature

# Linear Case

# First-Order Case

$$\frac{dx}{dt} + p(t)x = q(t)$$

# Variables Separable Form

# Homogeneous Case

$$\frac{dx}{dt} + p(t)x = 0$$

# General Solution

$$\frac{dx}{dt} + p(t)x = 0$$

$$\Longrightarrow$$

$$x(t) = Ce^{-\int_{t_0}^{t} p(s)ds}$$

$C$ : **Constant**

# Example 1

$$\frac{dx}{dt} - ax = 0 \quad (a \in \mathbf{R})$$

$$\Rightarrow$$

$$x(t) = Ce^{at}$$

# Structure Theorem (1)

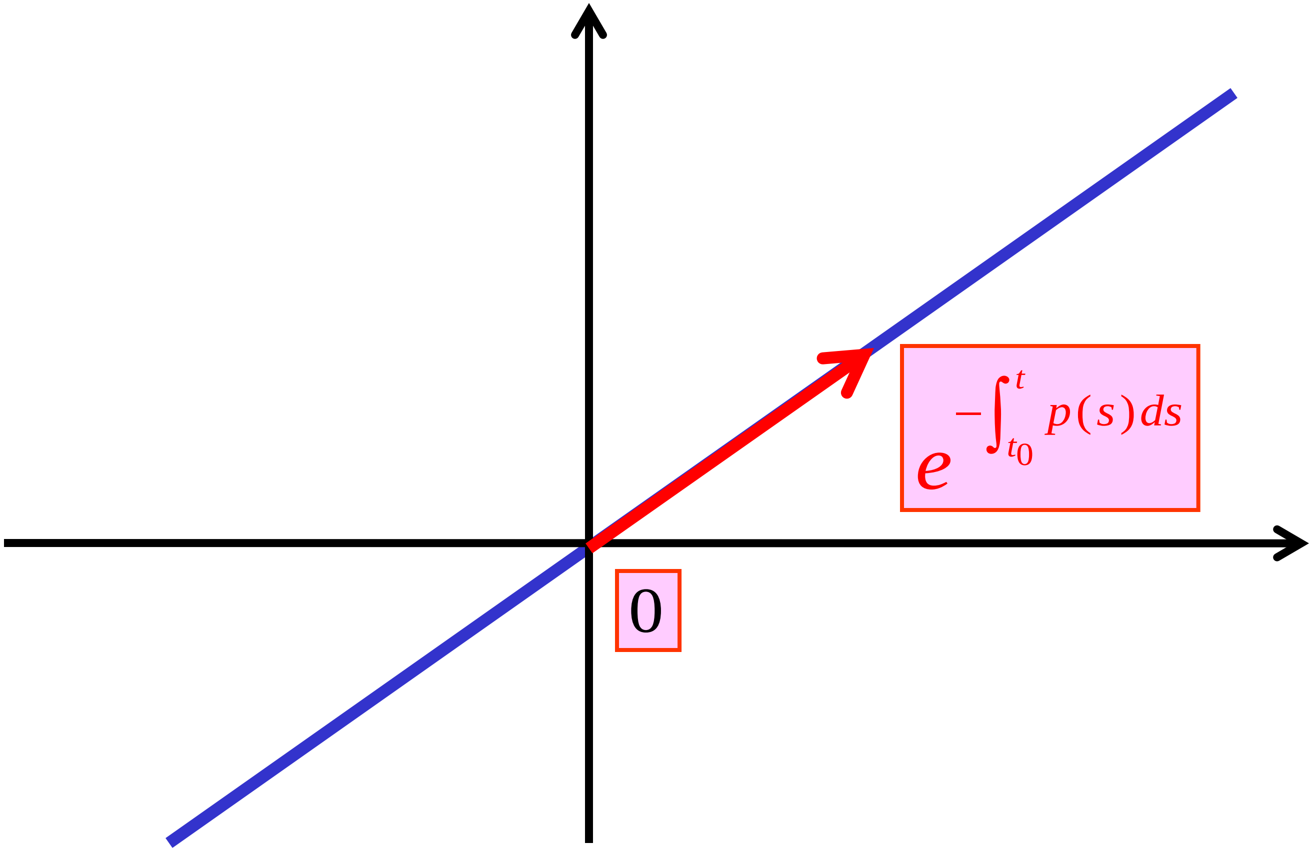
**The solution space of the homogeneous equation**

$$\frac{dx}{dt} + p(t)x = 0$$

**forms a one dimensional vector space spanned by a solution**

$$e^{-\int_{t_0}^{t} p(s)ds}$$

# Image (1)



# Proof (1)

$$\frac{dx}{dt} + p(t)x = 0$$

$$\Longrightarrow$$

$$\frac{dx}{x} = -p(t)dt$$

# Proof (2)

$$\frac{dx}{x} = -p(t)dt$$

$$\Longrightarrow$$

$$\int_{x_0}^{x} \frac{dy}{y} = -\int_{t_0}^{t} p(s)ds$$

$$\Longrightarrow$$

$$\log_e x = -\int_{t_0}^{t} p(s)ds + C_1$$

# Proof (3)

$$\log_e x(t) = -\int_{t_0}^{t} p(s)ds + C_1$$

$$\Longrightarrow$$

$$x(t) = Ce^{-\int_{t_0}^{t} p(s)ds}$$

# Non-Homogeneous Case

$$\frac{dx}{dt} + p(t)x = q(t)$$

# General Solution

$$\frac{dx}{dt} + p(t)x = q(t)$$

$$\Rightarrow$$

$$x(t) = \int_{t_0}^{t} q(s) \cdot e^{\int_{t_0}^{s} p(\tau)d\tau} ds \bullet e^{-\int_{t_0}^{t} p(s)ds}$$

$$+ Ce^{-\int_{t_0}^{t} p(s)ds}$$

$C$ : **Constant**

# Example 2

$$\frac{dx}{dt} - 2x = e^{3t}$$

$$\Rightarrow$$

$$x(t) = e^{3t} + Ce^{2t}$$

# Structure Theorem (2)

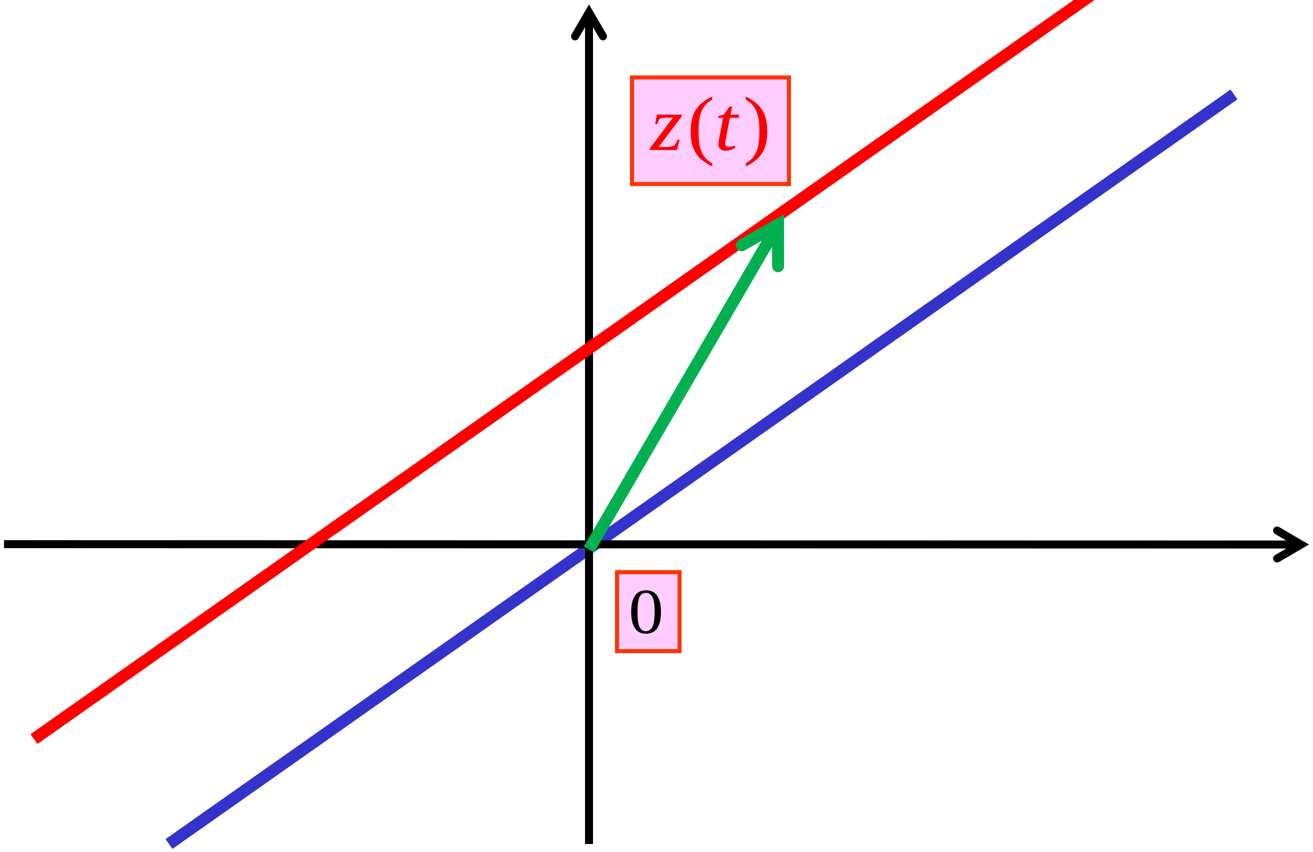
**The solution space of the non-homogeneous equation**

$$\frac{dx}{dt} + p(t)x = q(t)$$

**forms a one dimensional affine space translated by a particular solution**

$$z(t) = \int_{t_0}^{t} q(s) \cdot e^{\int_t^s p(\tau)d\tau} ds$$

# Image (2)



# Method

# of

# the Variation of Constants

# Proof (1)

$$x(t) = C(t) e^{-\int_{t_0}^{t} p(s)ds}$$

$$\Rightarrow$$

$$\frac{dx}{dt} + p(t)x = \frac{dC}{dt} e^{-\int_{t_0}^{t} p(s)ds}$$

# Proof (2)

$$\frac{dC}{dt} e^{-\int_{t_0}^{t} p(s)ds}$$

$$= \frac{dx}{dt} + p(t)x = q(t)$$

$$\Rightarrow$$

$$\boxed{\frac{dC}{dt} = q(t)e^{\int_{t_0}^{t} p(s)ds}}$$

# Proof (3)

$$C(t) = \int_{t_0}^{t} q(s) \cdot e^{\int_{t_0}^{s} p(\tau) d\tau} ds + C$$

$$\Rightarrow$$

$$x(t) = C(t)\, e^{-\int_{t_0}^{t} p(s) ds}$$

$$= \left( \int_{t_0}^{t} q(s) \cdot e^{\int_{t_0}^{s} p(\tau) d\tau} ds + C \right) e^{-\int_{t_0}^{t} p(s) ds}$$

# Non-Linear Case

# Example

$$\begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{1}{K}x(t)\right)x(t) \\ x(0) = x_0 \end{cases}$$

# Logistic Equation

$$\begin{cases} \dfrac{dx}{dt} = a(A - x)x \\ x(0) = C \end{cases}$$

# General Solution

$$x(t) = \frac{CA}{C + (A - C)e^{-aAt}}$$

$C$ : **Constant**

# Variables Separable Form

# Proof (1)

$$\frac{dx}{dt} = a(A - x)x$$

$$\Longrightarrow$$

$$adt = \frac{dx}{(A - x)x}$$

$$= \frac{1}{A}\left(\frac{dx}{x} + \frac{dx}{A - x}\right)$$

# Proof (2)

$$\boxed{\frac{1}{A}\left(\int\frac{dx}{x}-\int\frac{dx}{x-A}\right)=\int a\,dt}$$

$$\Rightarrow$$

$$\boxed{\frac{1}{A}\log_e\frac{x}{x-A}=at+C_1}$$

# Proof (3)

$$\frac{1}{A}\log_e \frac{x(t)}{x(t)-A} = at + C_1$$

$$\Rightarrow$$

$$x(t) = \frac{CA}{C + (A-C)e^{-aAt}}$$

$$x(0) = C$$

# General Case

# Initial-Value Problem

$$\frac{dx}{dt} = f(t, x(t))$$

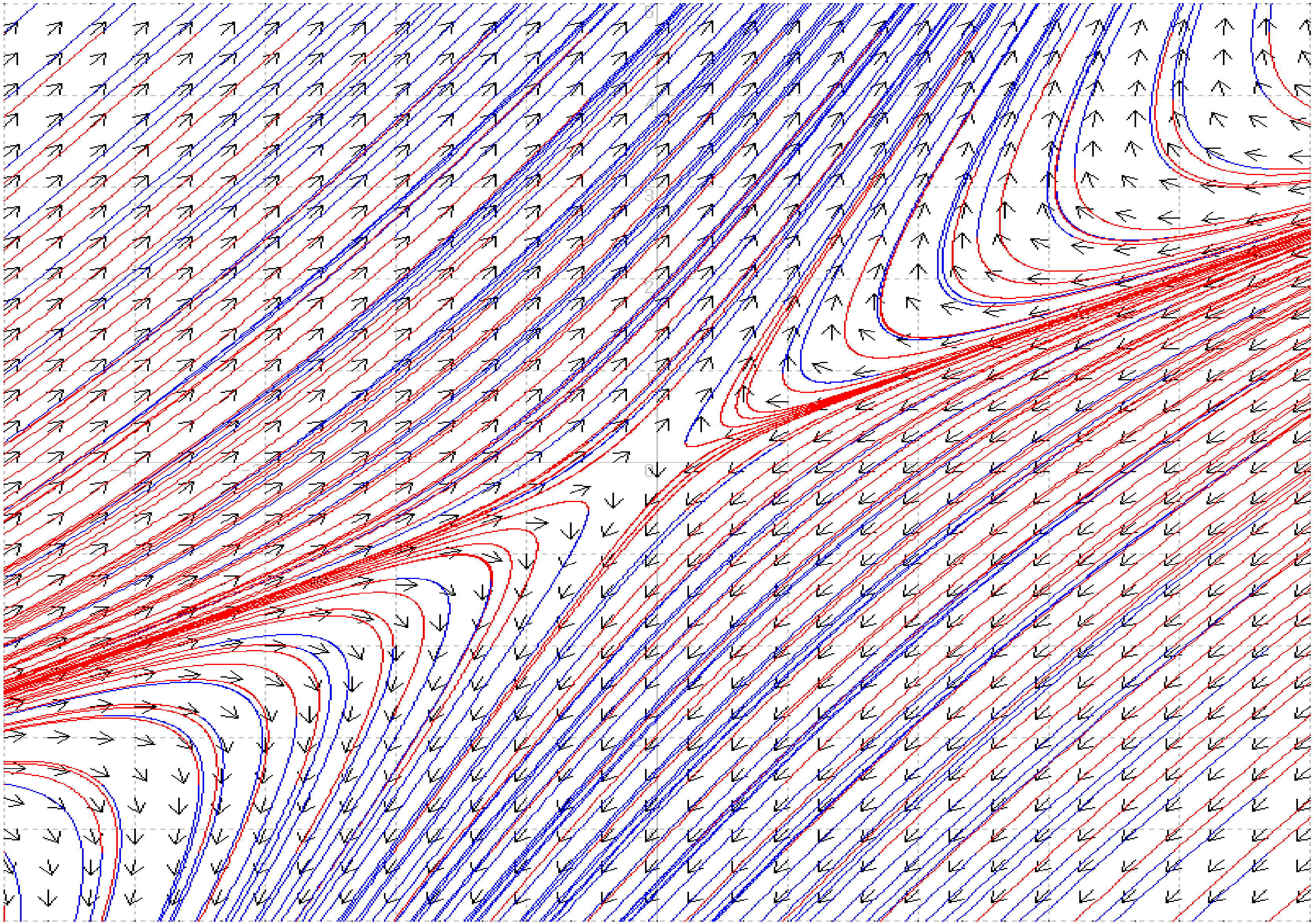
$$x(0) = x_0$$ **（Initial Condition）**

# Examples of $f(t, x)$

# Numerical Computing with BASIC

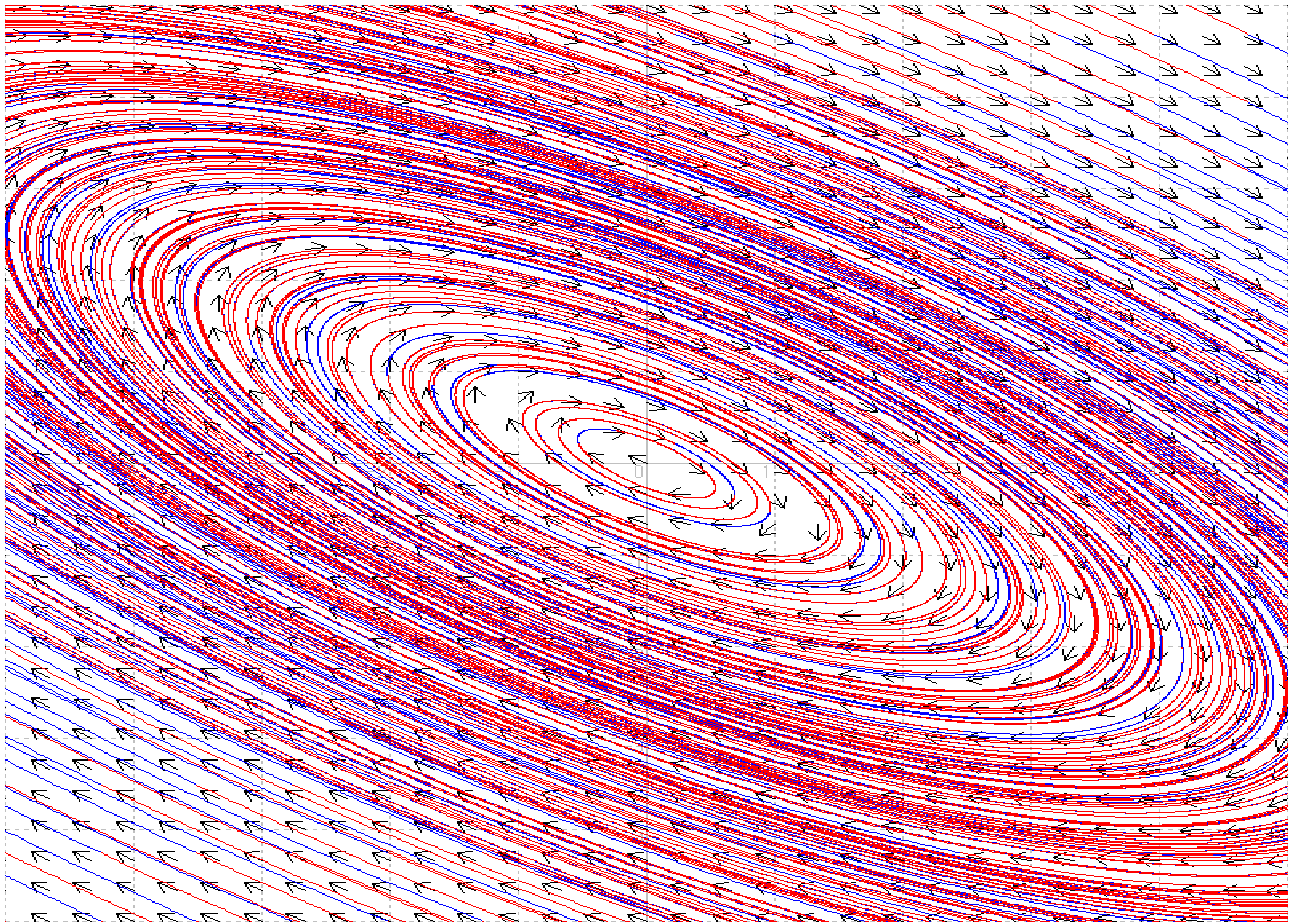
# Example (1)

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -2x_1 + 3x_2 \end{cases}$$



# Example (2)

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -x_1 - x_2 \end{cases}$$



# Picard's Method
# of
# Successive Approximation

# Reduction to an Integral Equation

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

**Differentiation** ↑ ↓ **Integration**

$$x(t) = x_0 + \int_0^t f(s, x(s))\,ds$$

# Initial-Value Problem

$$\frac{dx}{dt} = f(t, x(t))$$

$$x(0) = x_0$$ **（Initial Condition）**

# Algorithm
# for
# Successive Approximation

# Successive Approximation (1)

$$\begin{cases} \dfrac{dx_n}{dt} = f(t, x_{n-1}(t)) \\ x_n(0) = x_0 \end{cases}$$

# Successive Approximation (2)

$$x_1(t) = x_0 + \int_0^t f(s, x_0)ds$$

$$x_2(t) = x_0 + \int_0^t f(s, x_1(s))ds$$

.

.

$$x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))ds$$

($n$-th **Approximation**)

# Successive Approximation (3)

$x_n(t) \to \exists x(t)$ (**Uniform Convergence**)

$\Rightarrow$

$$x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))ds$$

$\Rightarrow \quad (n \to \infty)$

$$x(t) = x_0 + \int_0^t f(s, x(s))ds$$

# Example (1)

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) = 1 \end{cases}$$

**(Solution** : $x(t) = e^{at}$ )

# Successive Approximation (1)

$$x_1(t) = 1 + a\int_0^t 1\,ds = 1 + at$$

$$x_2(t) = 1 + a\int_0^t x_1(s)\,ds = 1 + at + \frac{(at)^2}{2!}$$

.

.

$$x_n(t) = 1 + a\int_0^t x_{n-1}(s)\,ds$$

$$= 1 + at + \cdots + \frac{(at)^n}{n!}$$

# Successive Approximation (2)

$$x_n(t) = 1 + at + \cdots + \frac{(at)^n}{n!}$$

$$\to e^{at} \quad (n \to \infty)$$

# Example (2)

$$\begin{cases} \dfrac{dx}{dt} = tx + \sqrt{t} \quad t > 0 \\ x(0) = 1 \end{cases}$$

# Successive Approximation (1)

$$x_1(t) = 1 + \int_0^t (s + \sqrt{s})ds = 1 + \frac{t^2}{2} + \frac{2}{3}t^{\frac{3}{2}}$$

$$x_2(t) = 1 + \int_0^t (sx_1(s) + \sqrt{s})ds$$

$$= 1 + \frac{t^2}{2} + \frac{2}{4 \cdot 2}t^4 + \frac{2}{3}t^{\frac{3}{2}} + \frac{2 \cdot 2}{7 \cdot 3}t^{\frac{7}{2}}$$

# Successive Approximation (2)

$$x_n(t) = 1 + \int_0^t \left( s x_{n-1}(s) + \sqrt{s} \right) ds$$

$$= 1 + \sum_{k=0}^{n-1} \frac{1}{2^k (1+k)!} t^{2(k+1)}$$

$$+ \sum_{k=0}^{n-1} \frac{2^{k+1}}{3(3+4)\cdots(3+4k)} t^{\frac{3}{2}+2k}$$

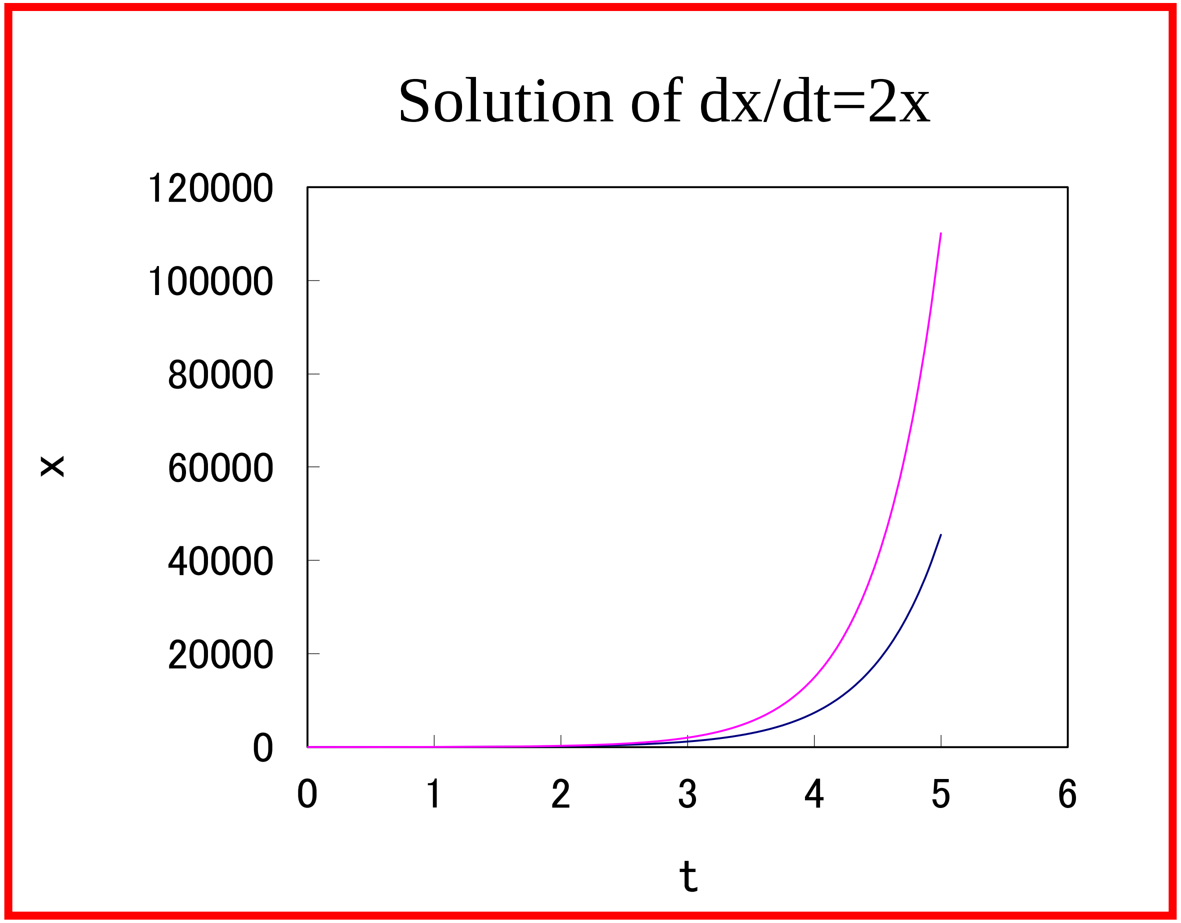
# Numerical Computing
# with
# Excel (VBA)

# Example

$$\begin{cases} \dfrac{dx}{dt} = 2x \\ x(0) = 5 \end{cases}$$

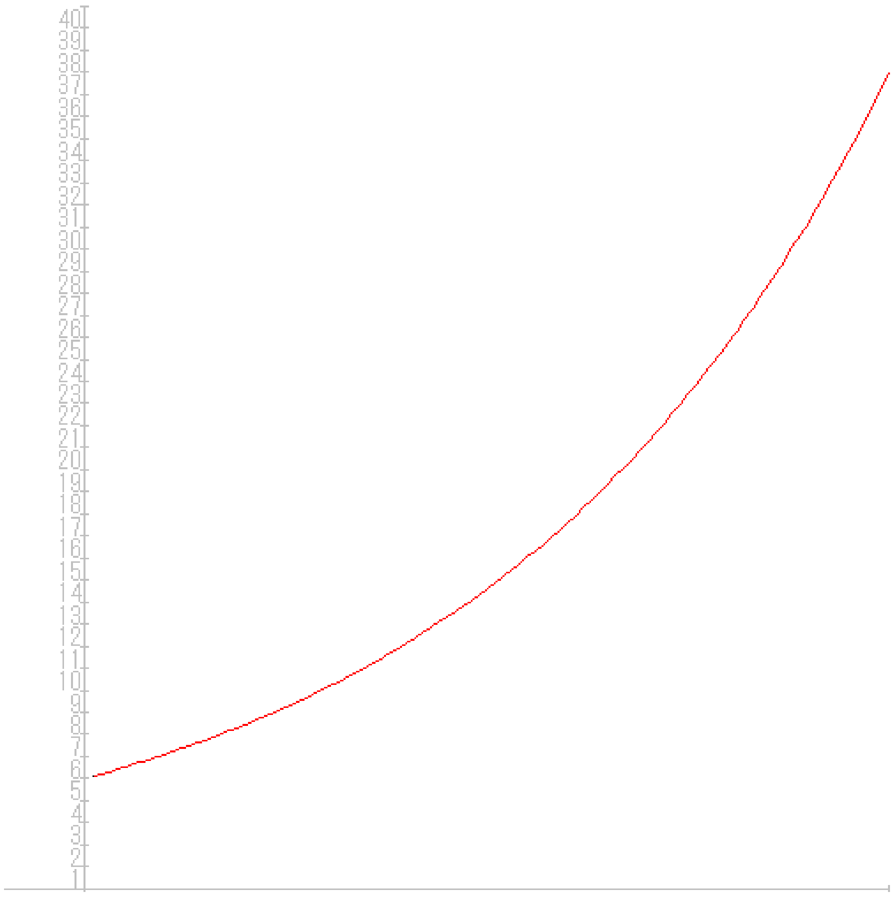
(**Solution** : $x(t) = 5e^{2t}$ )

# Euler's Method



# Numerical Computing with BASIC

# Runge-Kutta Method




$$x(t) = 5e^{2t}$$

# Example

$$\begin{cases} \dfrac{dx}{dt} = x^2 + 1 \\ x(0) = 0 \end{cases}$$

(**Solution** : $x(t) = \tan t$)

# Runge-Kutta Method (1)

```
REM ルンゲ・クッタ法による正弦関数の計算
OPTION ANGLE RADIANS
DEF F(t,x) = x^2+1
SET WINDOW 0,PI/2,0,10
DRAW grid
DRAW axes
!'tの初期値
LET  t = 0
!'xの初期値
LET  x = 0
!'tの１ステップの変化量
LET  h = 0.25
!'何回計算するか
LET  N = 2
FOR i = 0 TO N-1
   LET  k1 = F(t, x)
   LET  k2 = F(t + h, x + h * k1)
   LET  x = x + h * (k1 + k2) / 2
   PLOT LINES: t,x;
   SET LINE COLOR 4
NEXT i
PRINT x
END
```

# Runge-Kutta Method (2)

x=0.550499174772995

x-Tan(1/2)=-0.0632930624066609

# Successive Approximation
# and
# Fixed-Point Theorem

# Initial-Value Problem

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

**（Initial Condition）**

# Reduction to an Integral Equation

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

**Differentiation**

**Integration**

$$x(t) = x_0 + \int_0^t f(s, x(s))\,ds$$

# Solution and Fixed-Point

$$x(t) = x_0 + \int_0^t f(s, x(s))ds$$

$$Fx(t) := x_0 + \int_0^t f(s, x(s))ds$$

$$\Rightarrow$$

$$Fx = x$$ (**Fixed - point of** $F$)

# Banach's

# Fixed-Point Theorem

# Banach

- ◆Stefan Banach (1892-1945)

  Polish Mathematician

# Stefan Banach

# Banach's Fixed-Point Theorem

$(X, d)$ **Complete Metric Space**

$F : X \to X$ **Contraction Map:**

$$\begin{cases} 0 < \exists k < 1 \\ d(F(x), F(y)) \le k\, d(x, y), \forall x, y \in X \end{cases}$$

$\Rightarrow$

$\exists! z \in X$ **such that** $F(z) = z$

# Linear Case

# Second-Order Case

$$\begin{cases} u''(t) + 2bu'(t) + cu(t) = 0, \\ u(0) = u_0, \\ u'(0) = u_1 \end{cases}$$

# General Solutions

# General Solution（1）

$$D/4 = b^2 - c > 0$$

$$u(t) = e^{-bt}\left(Ae^{t\sqrt{b^2-c}} + Be^{-t\sqrt{b^2-c}}\right)$$

$A, B$ : **Constants**

# Example

$$\begin{cases} x''(t) - x(t) = t \\ x(0) = x'(0) = 0 \end{cases}$$

**Solution** : $x(t) = \frac{1}{2}(e^{t} - e^{-t}) - t$

# General Solution（2）

$$D/4 = b^2 - c < 0$$

$$u(t) = e^{-bt}\left( A\cos\sqrt{c-b^2}\,t + B\sin\sqrt{c-b^2}\,t \right)$$

$A, B$ : **Constants**

# General Solution（3）

$$D/4 = b^2 - c = 0$$

$$u(t) = e^{-bt}\left(At + B\right)$$

$A, B$ : **Constants**

# Linea Algebra
# and
# Differential Equations

# Exponential Matrix

# Main Idea

$$u''(t) + 2bu'(t) + cu(t) = 0$$

**Matrix Representation**

$$u''(t) + 2bu'(t) + cu(t) = 0$$

**Original Form**

$$\frac{dU(t)}{dt} = AU(t) \Rightarrow \textbf{Calculation of } e^{tA}$$

# Solution (1)

$$\begin{cases} u_1(t) = u(t), \\ u_2(t) = u'(t) \end{cases}$$

$$\begin{cases} u_1'(t) = u'(t) = u_2(t), \\ u_2'(t) = u''(t) = -2bu'(t) - cu(t) \\ \qquad\quad = -2bu_2(t) - cu_1(t) \end{cases}$$

# Solution (2)

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases}$$

# Solution (3)

$$U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\begin{cases} \dfrac{d}{dt}U(t) = AU(t), \\ U(0) = U_0 \end{cases}$$

# Solution (4)

$$U(t) = e^{tA} U_0$$

$$e^{tA} = I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots$$

(**Exponential Matrix**)

# Example
# of
# Exponential Matrices

# Simple Eigenvalue Case

# Calculation (1)

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c$$

# Calculation (2)

$$\textbf{Case}: D/4 = b^2 - c \neq 0$$

$$\begin{cases} \lambda_1 = -b + \sqrt{b^2 - c}, \\ \lambda_2 = -b - \sqrt{b^2 - c} \end{cases}$$

# Calculation (3)

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -b+\sqrt{b^2-c} & -b-\sqrt{b^2-c} \end{pmatrix}$$

$P^{-1}AP = \Lambda$ （**Diagonal**）

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -b+\sqrt{b^2-c} & 0 \\ 0 & -b-\sqrt{b^2-c} \end{pmatrix}$$

# Calculation (4)

$$P^{-1}e^{tA}P$$

$$= P^{-1}\left( I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right) P$$

$$= P^{-1}P + t\left(P^{-1}AP\right) + \frac{t^2}{2!}\left(P^{-1}AP\right)\left(P^{-1}AP\right) + \cdots +$$

$$+ \frac{t^n}{n!}\underbrace{\left(P^{-1}AP\right)\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}_{n-\text{times}} + \cdots$$

$$= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= e^{t\Lambda}$$

# Calculation (5)

$$e^{t\Lambda} = I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} \lambda_1{}^2 & 0 \\ 0 & \lambda_2{}^2 \end{pmatrix} + \cdots$$

$$+ \frac{t^n}{n!}\begin{pmatrix} \lambda_1{}^n & 0 \\ 0 & \lambda_2{}^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

# Calculation (6)

$$e^{tA} = P e^{t\Lambda} P^{-1}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & -e^{\lambda_1 t} + e^{\lambda_2 t} \\ \lambda_1 \lambda_2 \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right) & -\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

# Calculation (7)

**Case** : $D/4 = b^2 - c \neq 0$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & -e^{\lambda_1 t} + e^{\lambda_2 t} \\ \lambda_1 \lambda_2 \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right) & -\lambda_1 e^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

# Double Eigenvalue Case

# Jordan Canonical Form
# of
# Matrices

# Marie Ennemond Camille Jordan

# Jordan

- **Marie Ennemond Camille Jordan (1838-1922)**

**French Mathematician**

# Jordan's Canonical Form

$$P^{-1}AP = \Lambda \quad \textbf{(Jordan Form)}$$

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

# Calculation (1)

$$A = \begin{pmatrix} 0 & 1 \\ -c & -2b \end{pmatrix}$$

$$\left|\lambda I - A\right| = \begin{vmatrix} \lambda & -1 \\ c & \lambda + 2b \end{vmatrix} = \lambda^2 + 2b\lambda + c$$

# Calculation (2)

$$\textbf{Case}: D/4 = b^2 - c = 0$$

$$\lambda = -b \quad \textbf{(Double Root)}$$

$$P = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

# Calculation (3)

$$P^{-1}AP = \Lambda$$ **(Jordan Form)**

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -b & 1 \\ 0 & -b \end{pmatrix}$$

# Calculation (4)

$$
\begin{aligned}
&P^{-1}e^{tA}P \\
&= P^{-1}\left( I + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots \right) P \\
&= P^{-1}P + t\left(P^{-1}AP\right) + \frac{t^2}{2!}\left(P^{-1}AP\right)\left(P^{-1}AP\right) + \cdots + \\
&\qquad + \frac{t^n}{n!}\underbrace{\left(P^{-1}AP\right)\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}_{n-\text{times}} + \cdots \\
&= I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots \\
&= e^{t\Lambda}
\end{aligned}
$$

# Calculation (5)

$$e^{t\Lambda} = I + t\Lambda + \frac{(t\Lambda)^2}{2!} + \cdots + \frac{(t\Lambda)^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} + \cdots$$

$$+ \frac{t^n}{n!}\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

# Calculation (6)

$$e^{tA} = Pe^{t\Lambda}P^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} - \lambda te^{\lambda t} & te^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1)e^{\lambda t} \end{pmatrix}$$

# Calculation (7)

$$\textbf{Case:}\ D/4 = b^2 - c = 0$$

$$U(t) = e^{tA} U_0,$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} - \lambda t e^{\lambda t} & t e^{\lambda t} \\ -\lambda^2 + e^{\lambda t} & (\lambda t + 1) e^{\lambda t} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

# 2-dimensional Autonomous System

# Linear Case

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

# Matrix Form

$$U(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow$$

$$\frac{d}{dt}U(t) = AU(t)$$

# Stability of Solutions

# Computational Approach

# Numerical Computing with BASIC

# Example 1 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = y \end{cases}$$

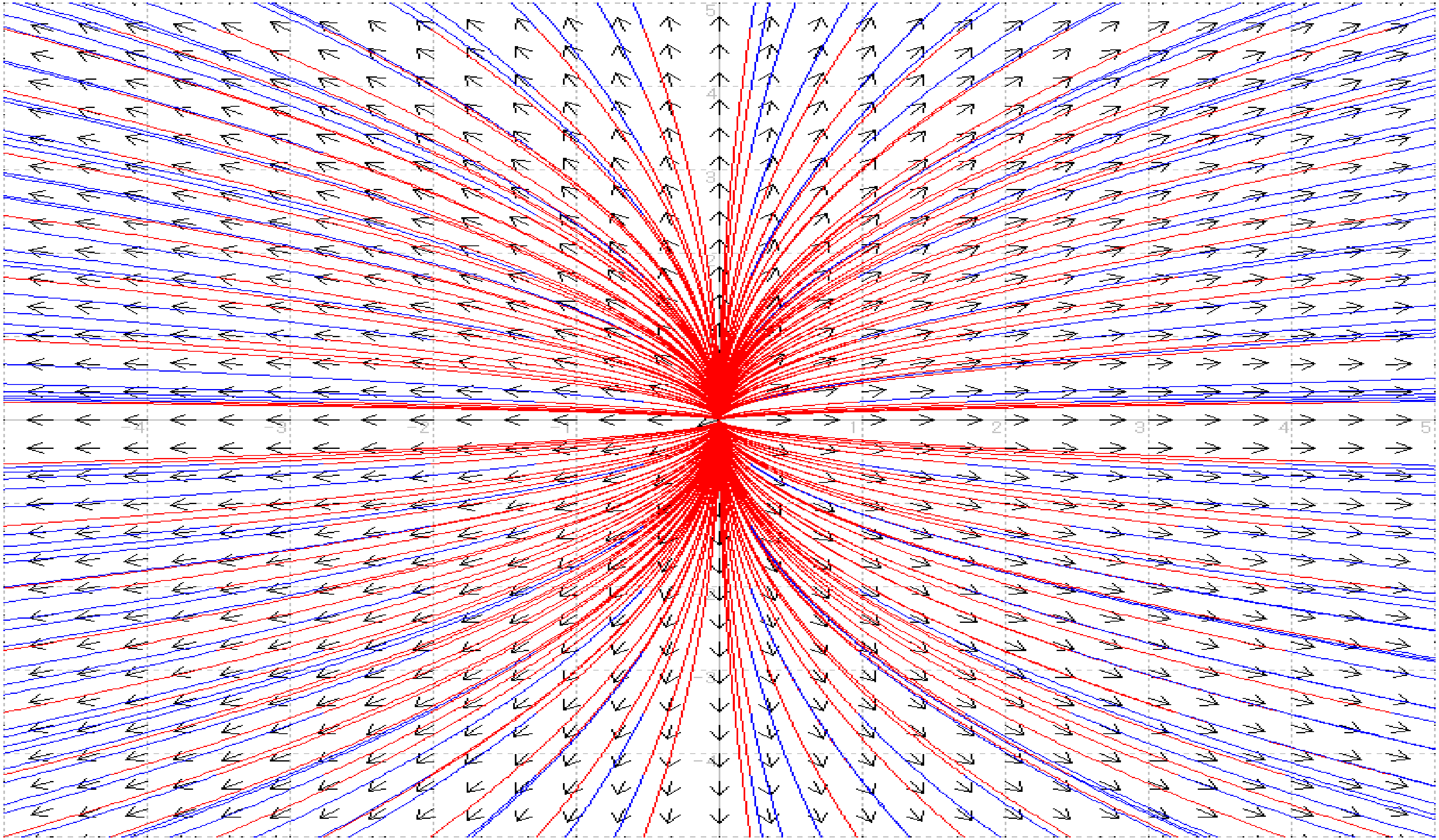
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

**Eigenvalues:** 2, 1

# Unstable Node



# Example 2 (Saddle Point)

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y \end{cases}$$

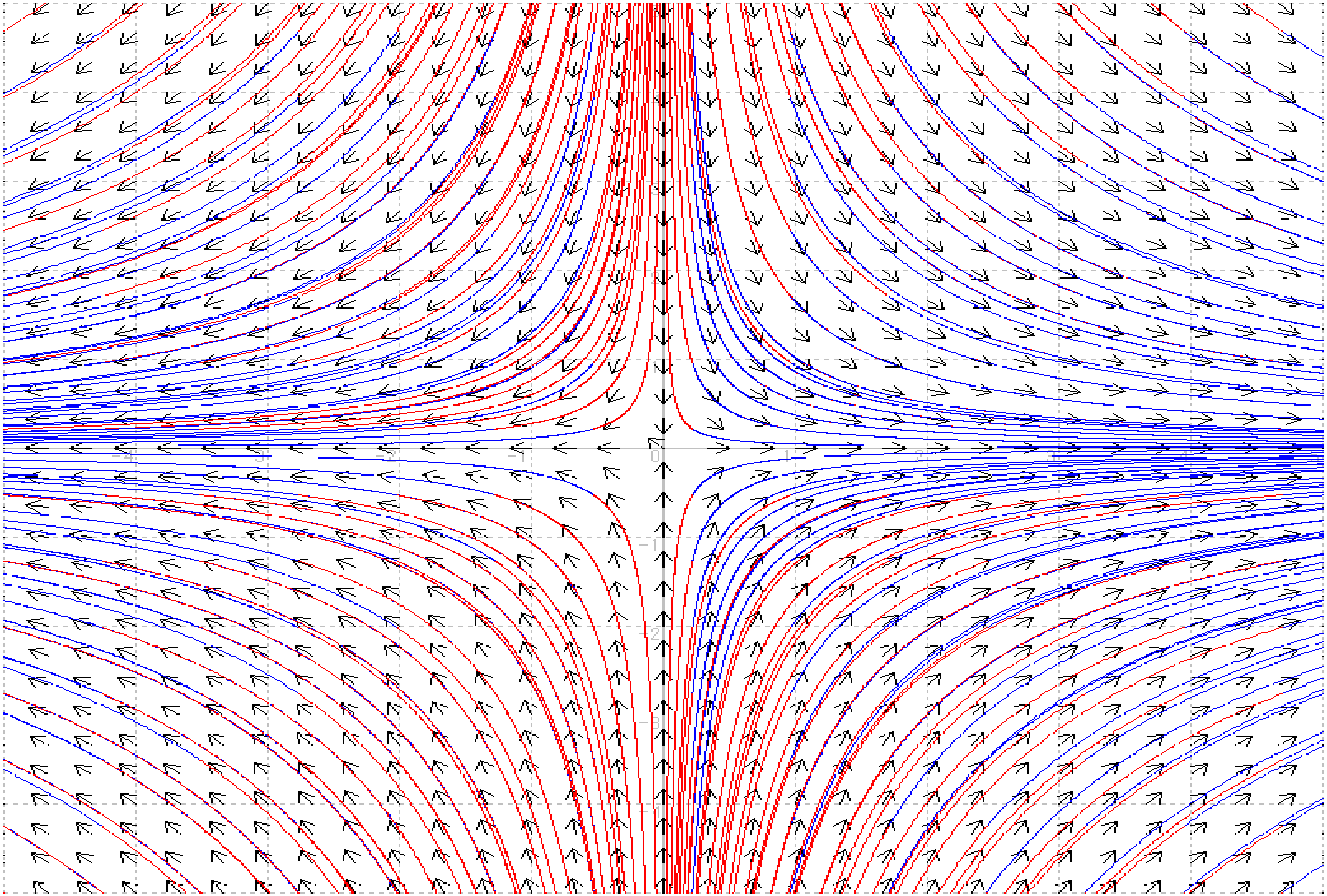
$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 1, \quad -1$$

# Saddle Point



# Example 3 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = 3x + 2y \end{cases}$$

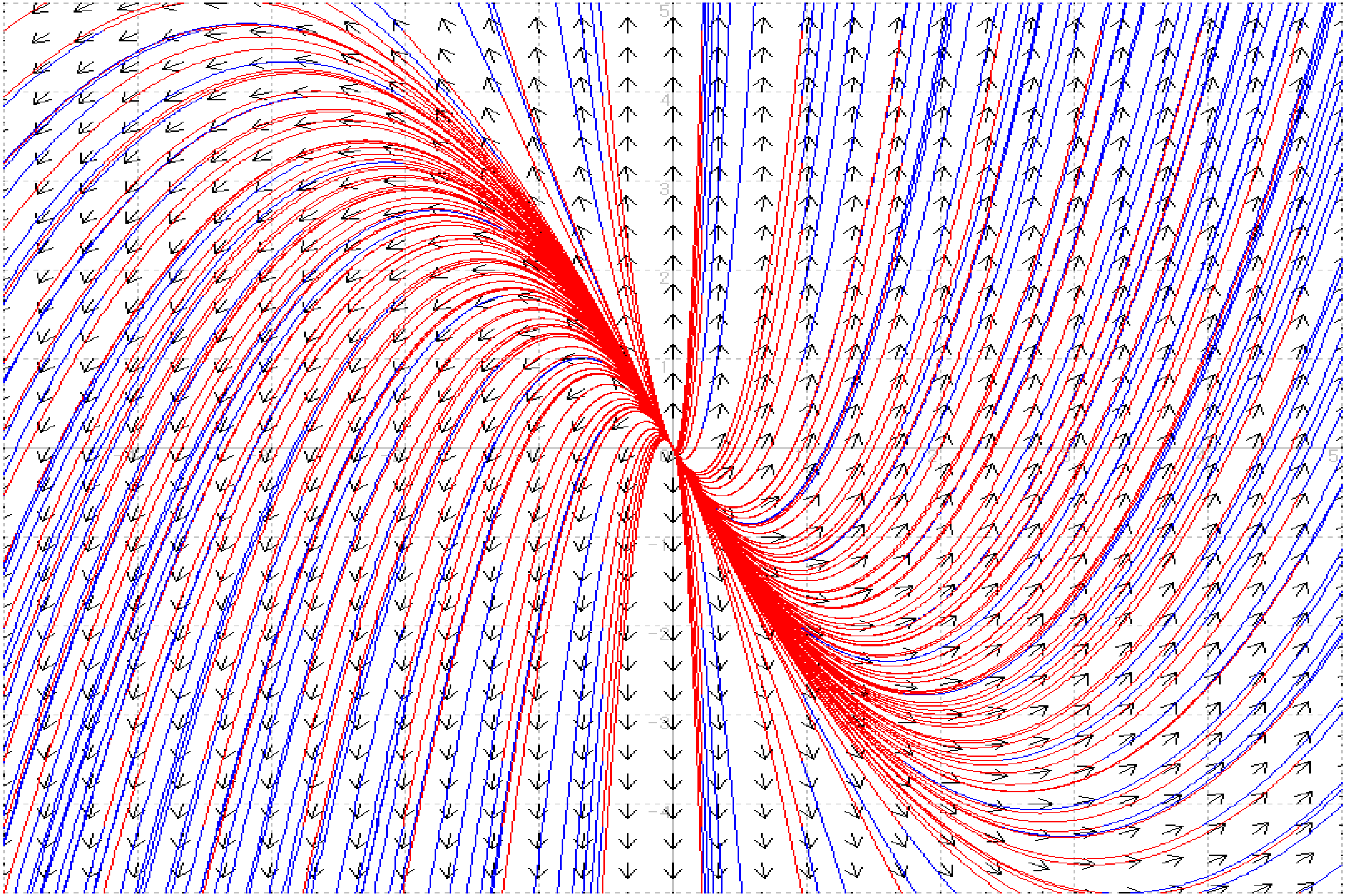
$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 1, \quad 2$$

# Unstable Node



# Example 4 (Stable Node)

$$\begin{cases} \dfrac{dx}{dt} = -2x - 1.5y \\ \dfrac{dy}{dt} = x - 5.5y \end{cases}$$

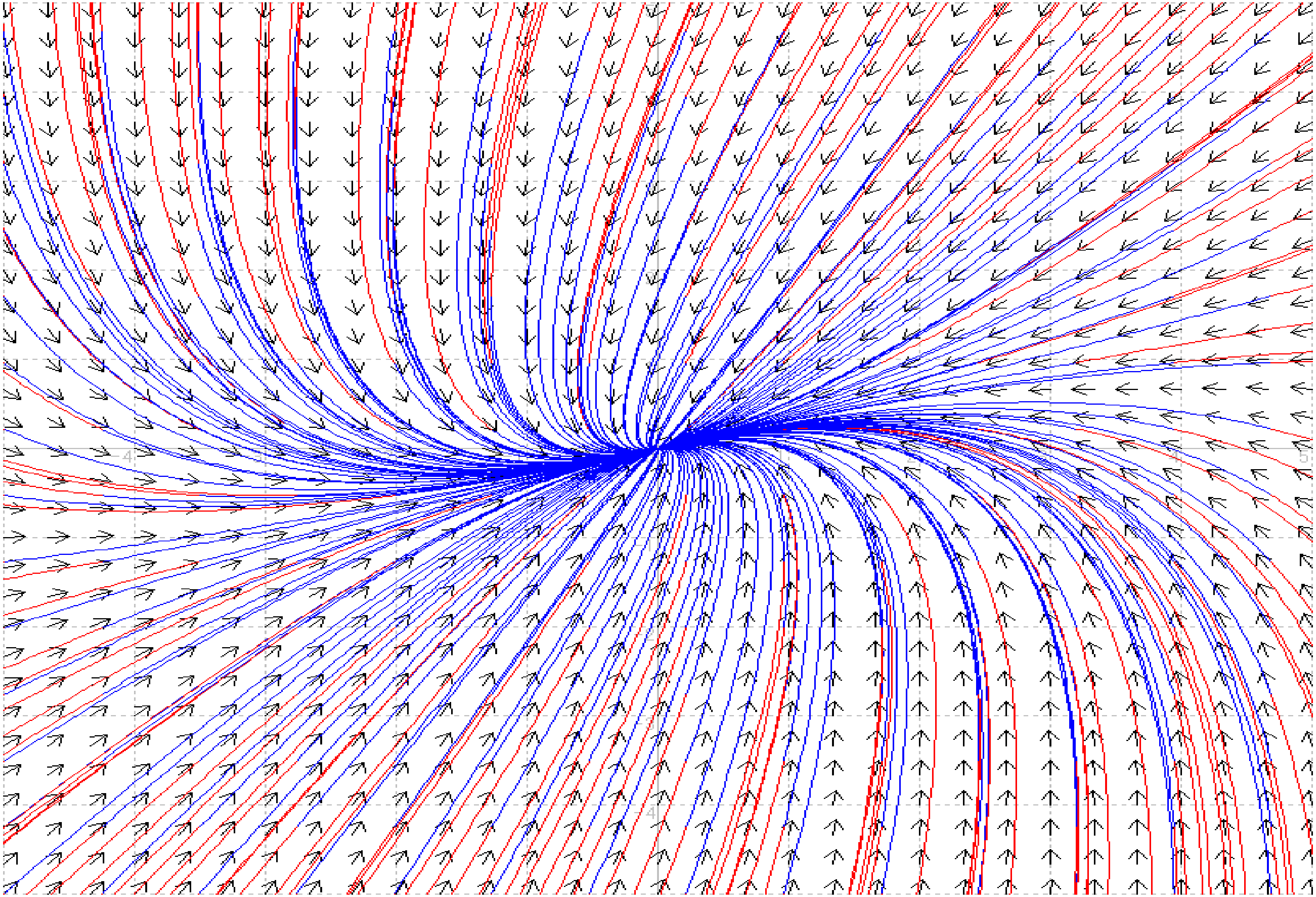
$$A = \begin{pmatrix} -2 & -1.5 \\ 1 & -5.5 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} -2 & -1.5 \\ 1 & -5.5 \end{pmatrix}$$

$$\textbf{Eigenvalues:} \; -2.5, \; -5$$

# Stable Node



# Example 5 (Saddle Point)

$$\begin{cases} \dfrac{dx}{dt} = -2x + 2y \\ \dfrac{dy}{dt} = -2x + 3y \end{cases}$$

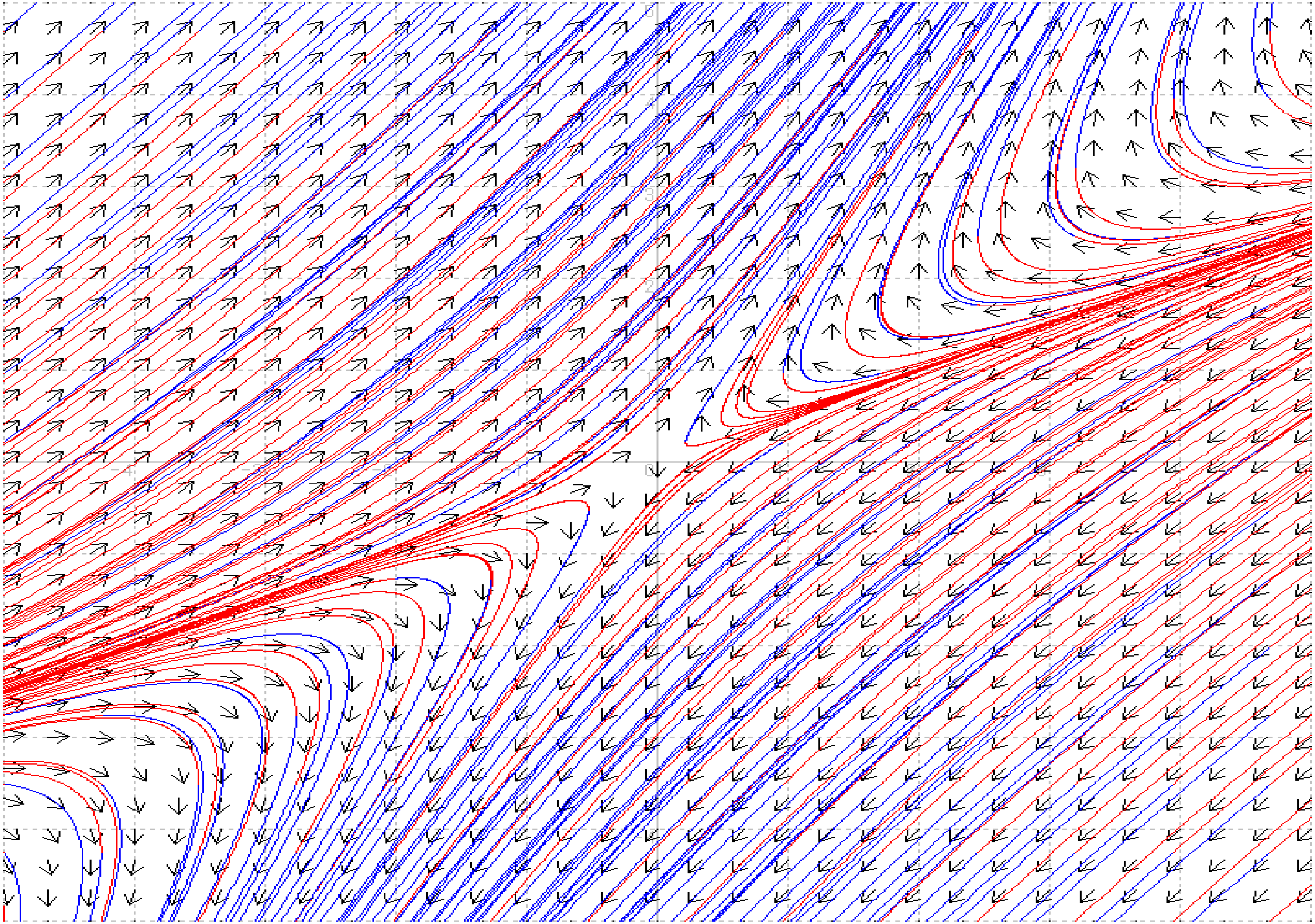
$$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 2, \ -1$$

# Saddle Point



# Example 6 (Unstable Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x + y \\ \dfrac{dy}{dt} = 2y \end{cases}$$

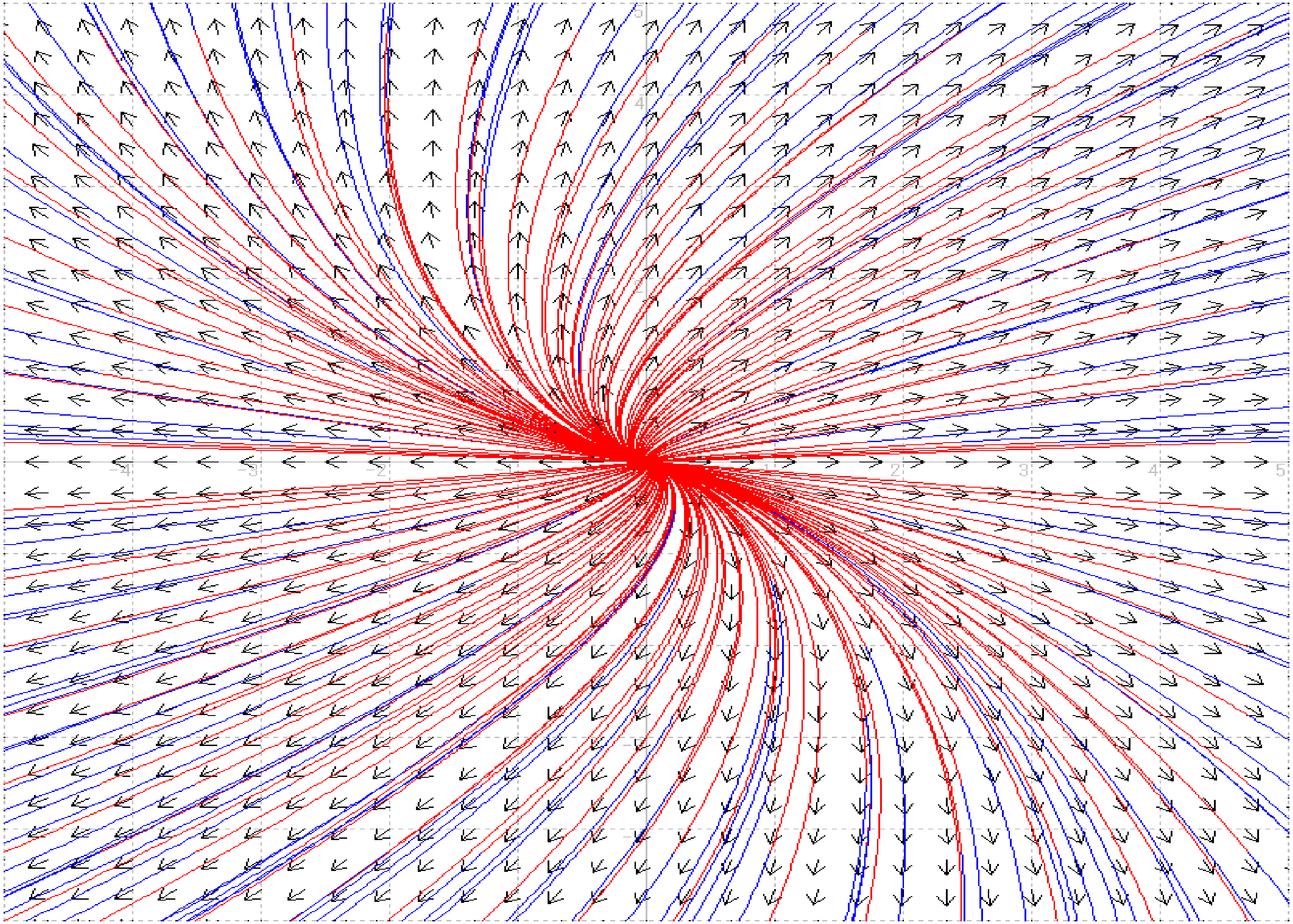
$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

**Eigenvalues :** 2, 2

# Unstable Node



# Example 7 (Center)

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = -x - y \end{cases}$$

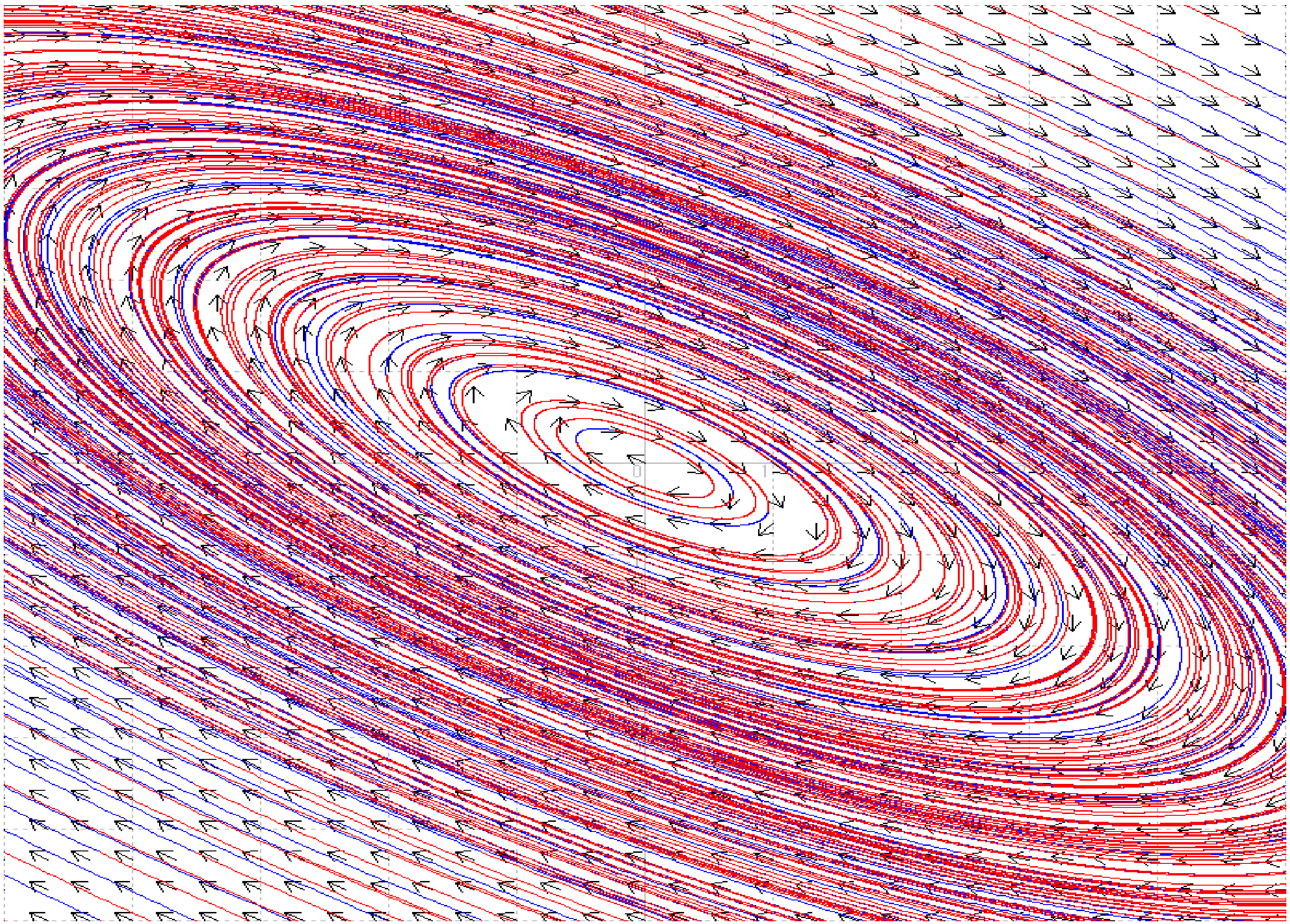
$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$\textbf{Eigenvalues:}\ \sqrt{-1},\ -\sqrt{-1}$$

# Center



# Example 8 (Unstable Focus)

$$\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = -2x + y \end{cases}$$

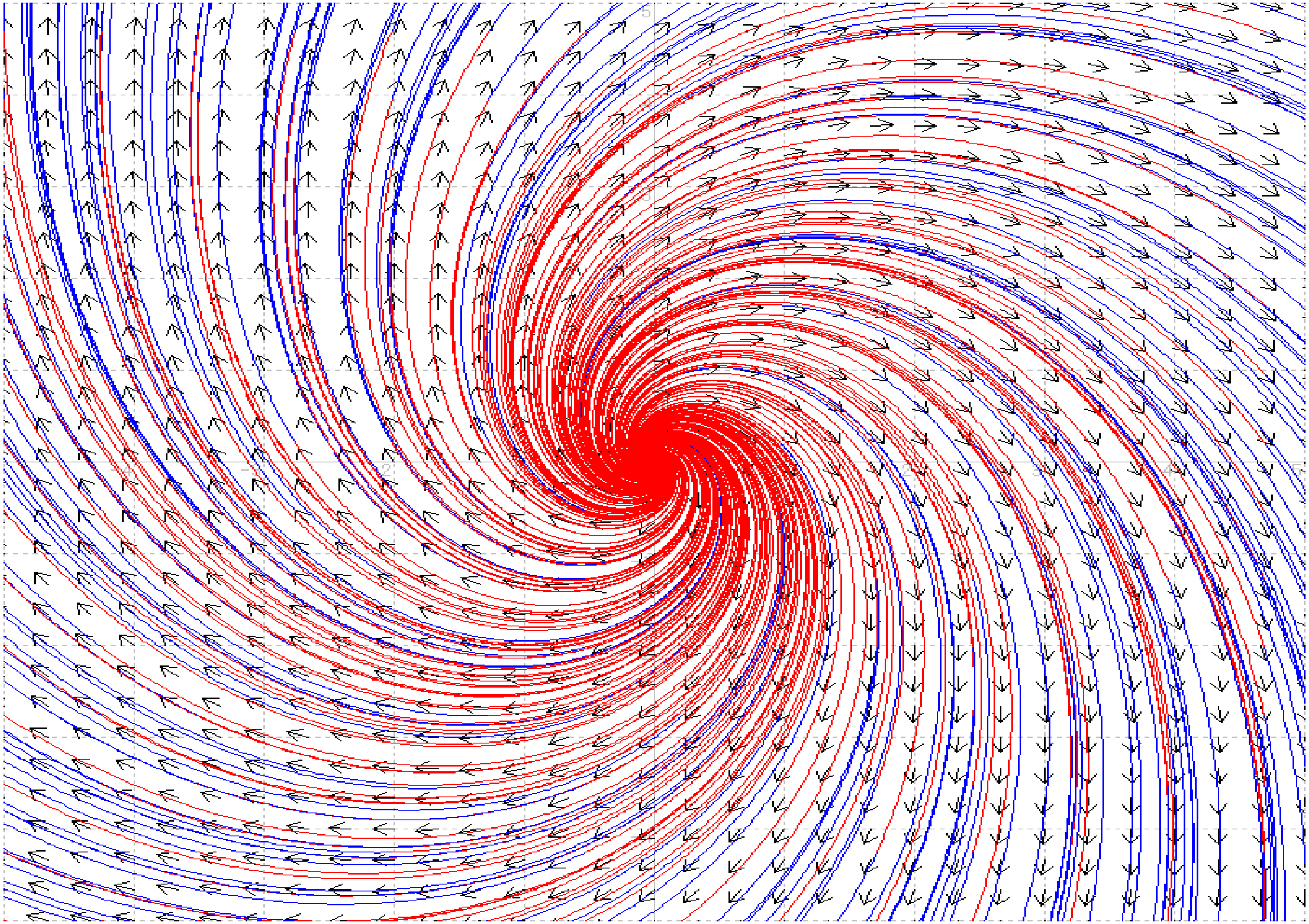
$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$\textbf{Eigenvalues:}\ 1+\sqrt{2}i,\ 1-\sqrt{2}i$$

# Unstable Node



# Example 9 (Degenerate Node)

$$\begin{cases} \dfrac{dx}{dt} = 2x + 2y \\ \dfrac{dy}{dt} = 3x + 3y \end{cases}$$

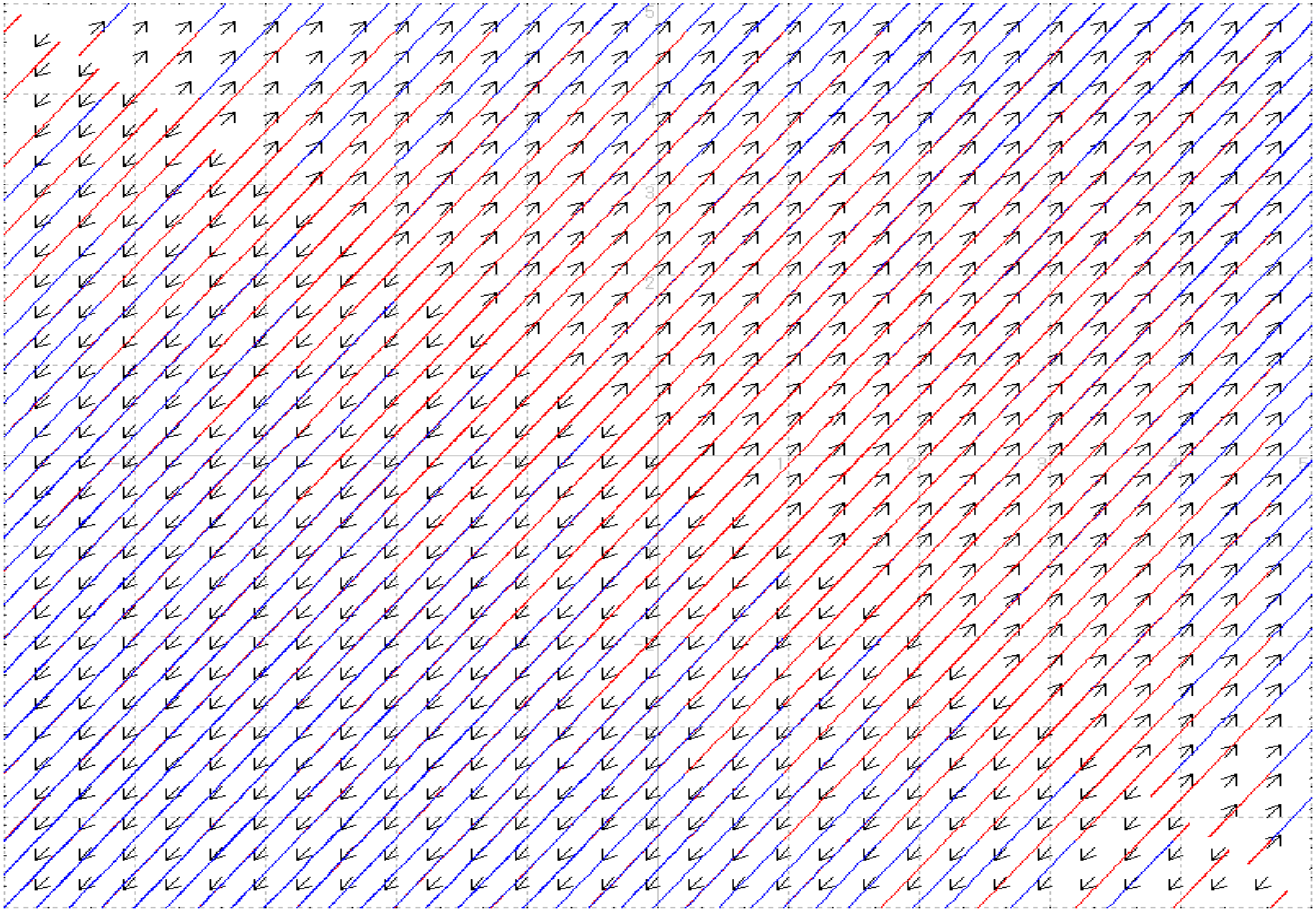
$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

# Signature of Eigenvalues

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

$$\textbf{Eigenvalues}: 0, \quad 5$$

# Degenerate Node



# Vector Analysis

# Bird's- Eye View

| Theme | Mathematics | Mechanics |
| --- | --- | --- |
| Vector Analysis | Calculus on Surfaces | Continuum Mechanics |

# Line Integrals

# Example (1)

$$\int_{x^2+y^2=1} x\,dx = \int_0^{2\pi} \cos\theta \left( d\cos\theta \right)$$

$$= -\frac{1}{2}\int_0^{2\pi} \sin 2\theta\, d\theta$$

$$= 0$$

# Example (2)

$$\int_{x^2+y^2=1} x\,dy = \int_0^{2\pi} \cos\theta \left(d\sin\theta\right)$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1+\cos 2\theta\right) d\theta$$

$$= \pi$$

# Example (3)

$$\int_{x^2+y^2=1} ydx + xdy = 0$$

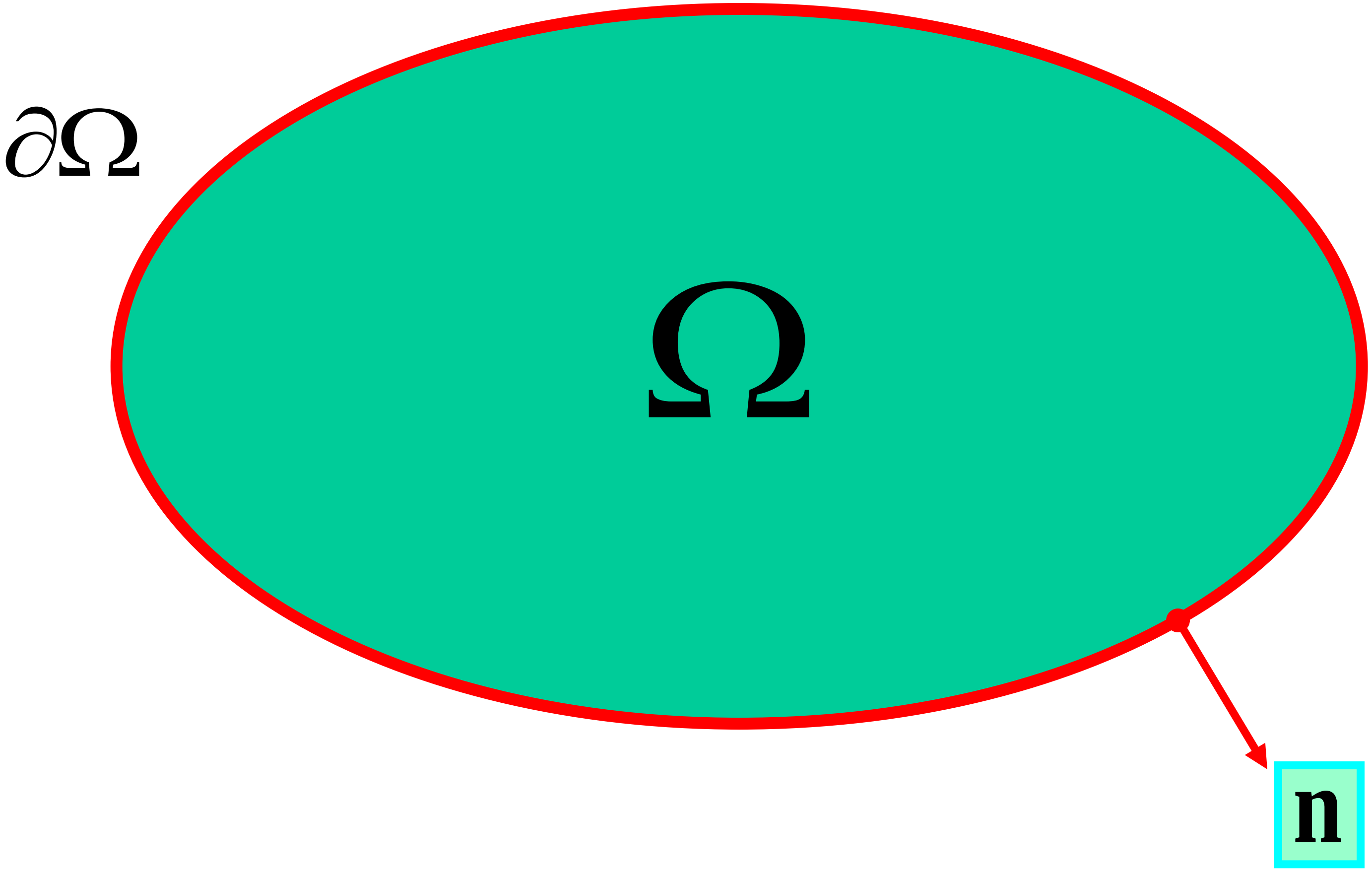
# Green's Theorem

# Green

◆**George Green (1793-1841)**

**British Mathematical Physicist**

# 2-dimensional Domain



# Green's Theorem (1)

$$\iint_{\Omega}\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)dxdy$$

$$=\int_{\partial\Omega} fdy+gdx$$

# Example (1)

$$|\Omega| = \iint_{\Omega} 1\,dxdy$$

$$= \frac{1}{2}\int_{\partial\Omega} xdy - ydx$$

# Green's Theorem (2)

$$\iint_{\Omega} \mathrm{div}\mathbf{F}\, dv = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, ds$$

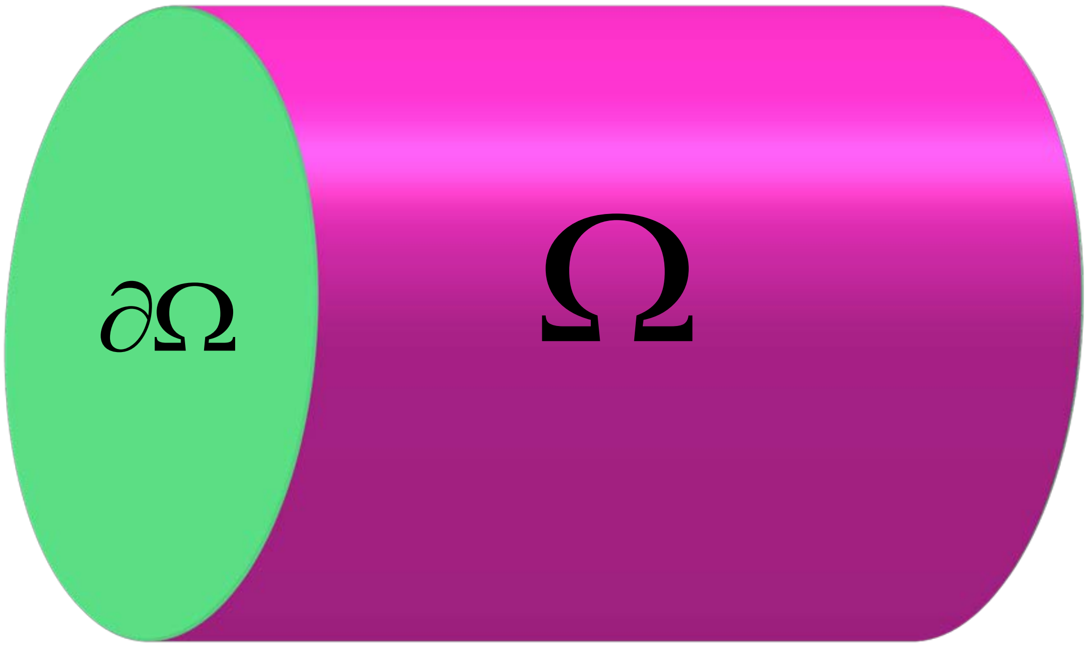
$$\mathbf{F} = (f, g)$$

# Example (2)

$$\iint_{\Omega} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dxdy = \int_{\partial\Omega} \frac{\partial f}{\partial \mathbf{n}} \, ds$$

# Gauss' Divergence Theorem

# 3-dimensional Domain



# Gauss' Divergence Theorem (1)

$$\iiint_{\Omega} \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dxdydz$$

$$= \iint_{\partial\Omega} fdydz + gdzdx + hdxdy$$

# Example (1)

$$\left|\Omega\right| = \iiint_{\Omega} 1\,dxdydz$$

$$= \frac{1}{3}\iint_{\partial\Omega} xdydz + ydzdx + zdxdy$$

# Example (2)

$$\int_{x^2+y^2+z^2=1} x^3 dydz + y^3 dzdx + z^3 dxdy$$

$$= \frac{12}{5}\pi$$

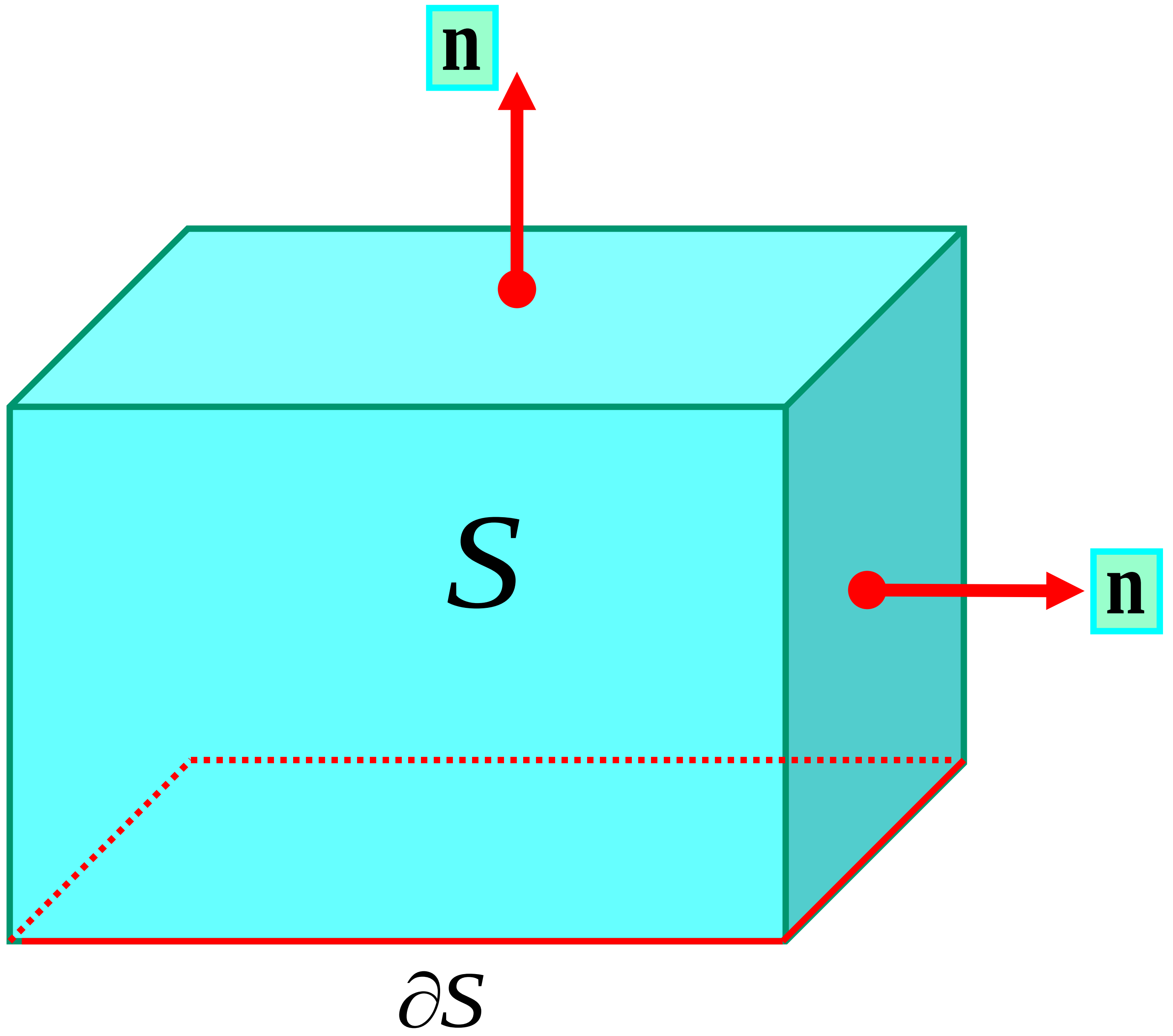
# Gauss' Divergence Theorem (2)

$$\iiint_D \mathrm{div}\mathbf{F}\, dV = \iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS$$

$$\mathbf{F} = (f, g, h)$$

# Stokes' Theorem

# George Gabriel Stokes
# (1819-1903)



n
n
S
∂S

# Stokes' Theorem (1)

$$\iint_S \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dydz + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dzdx + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy$$

$$= \int_{\partial S} fdx + gdy + hdz$$

# Example

$$\int_{(1,2-1)}^{(2,3,1)} y^2 z^3 dx + 2xyz^3 dy + 3xy^2 z^2 dz = 22$$

# Stokes' Theorem (2)

$$\iint_S \operatorname{rot} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s},$$

$$\mathbf{F} = (f, g, h)$$

# Differential Forms
# （Elie Cartan）

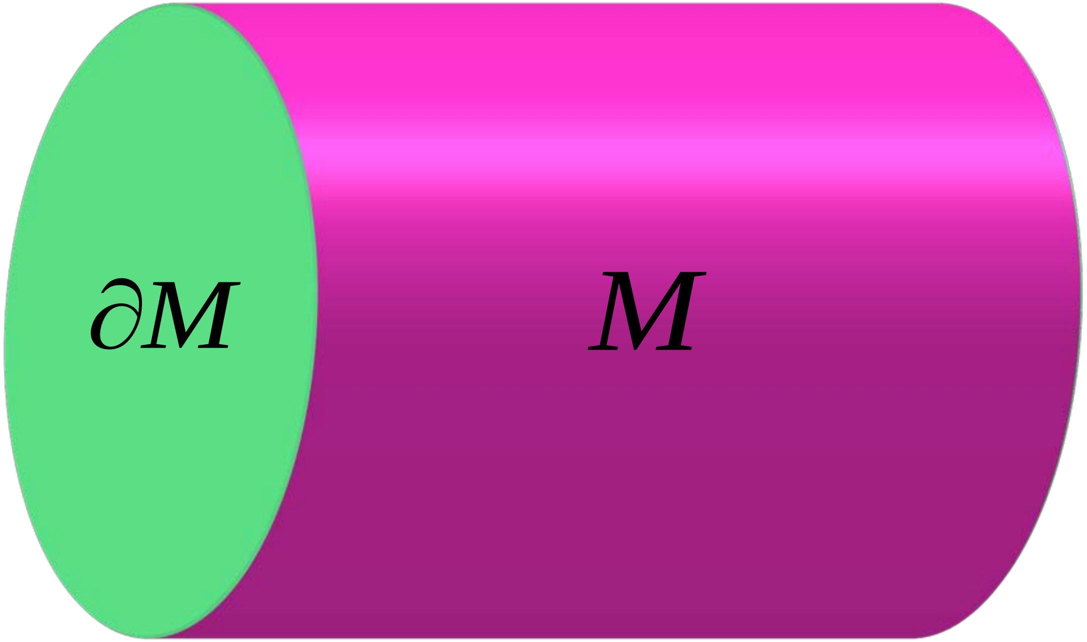
# Differential Forms and Figures

# Duality of Concepts

| Degree | Differential Forms | Figures |
|---|---|---|
| 0 | Functions | Points |
| 1 | $dx, dy, dz$ | Segments |
| 2 | $dxdy, dydz, dzdx$ | Rectangles |
| 3 | $dxdydz$ | Cubes |

# General Form of Stokes' Formula

# Manifold with Boundary



# Stokes' Formula

$$\int_M d\omega = \int_{\partial M} \omega$$

$$\langle M, d\omega \rangle = \langle \partial M, \omega \rangle$$

# Examples of Stokes' Formula

| Figure | Differential Form | Contents |
|---|---|---|
| Interval | Function | Fundamental Theorem of Calculus |
| 2-Domain | One Form | Green's Theorem |
| 3-Domain | Two Form | Gauss' Theorem |
| Surface | One Form | Stokes' Theorem |

# Exterior Derivation

# Gradient

$$\operatorname{grad} f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

# Rotation

$$\operatorname{rot}(f, g, h) = \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)$$

# Divergence

$$\operatorname{div}(f, g, h) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

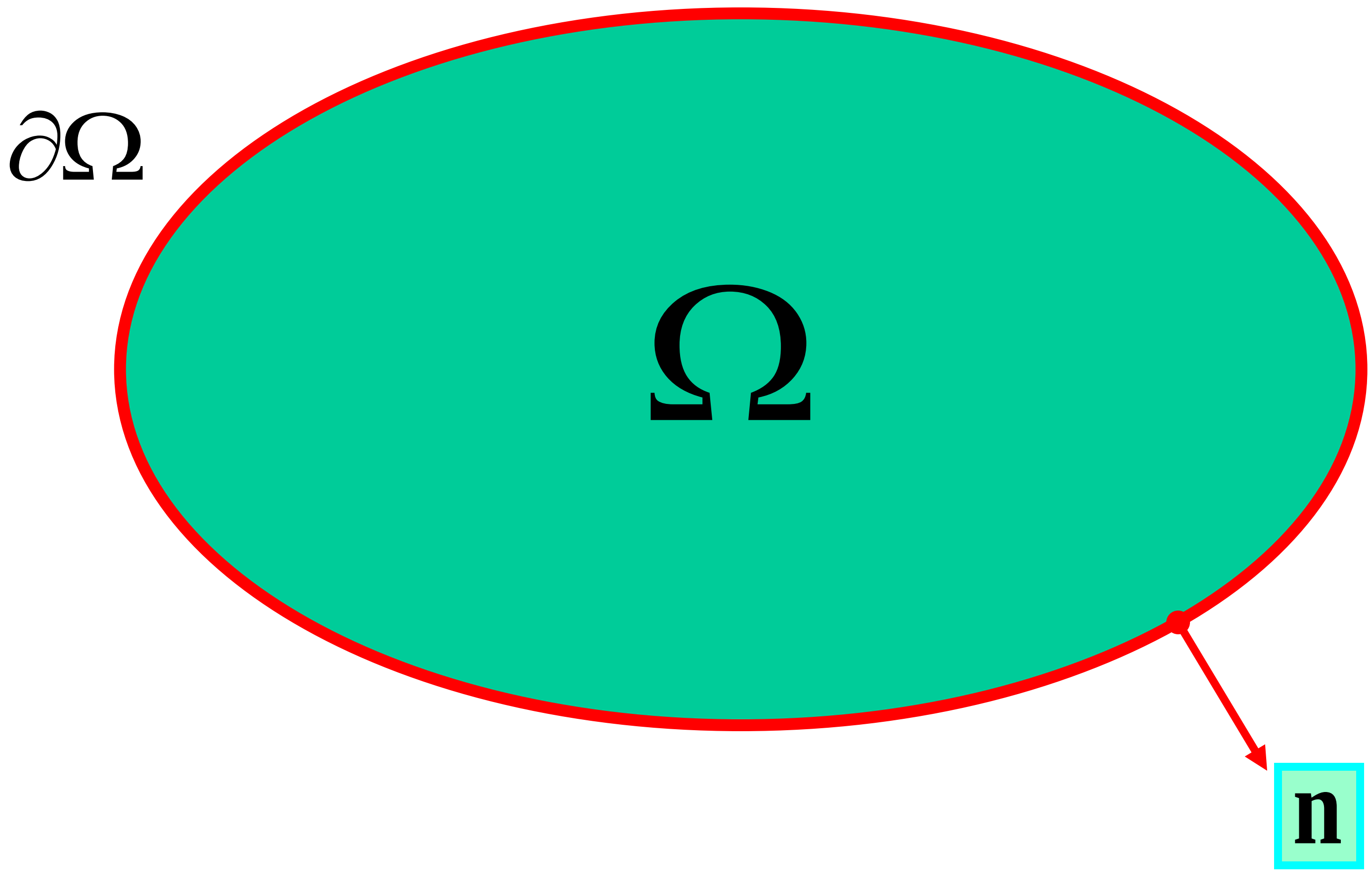
# Well-known Formulas

$$\mathrm{rot} \circ \mathrm{grad}\ f = 0$$

$$\mathrm{div} \circ \mathrm{rot}\ \mathbf{v} = 0$$

# Green’s Theorem

# 2-dimensional Domain



# Green's Theorem (1)

$$\iint_{\Omega}\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)dxdy$$

$$=\int_{\partial\Omega} fdy+gdx$$

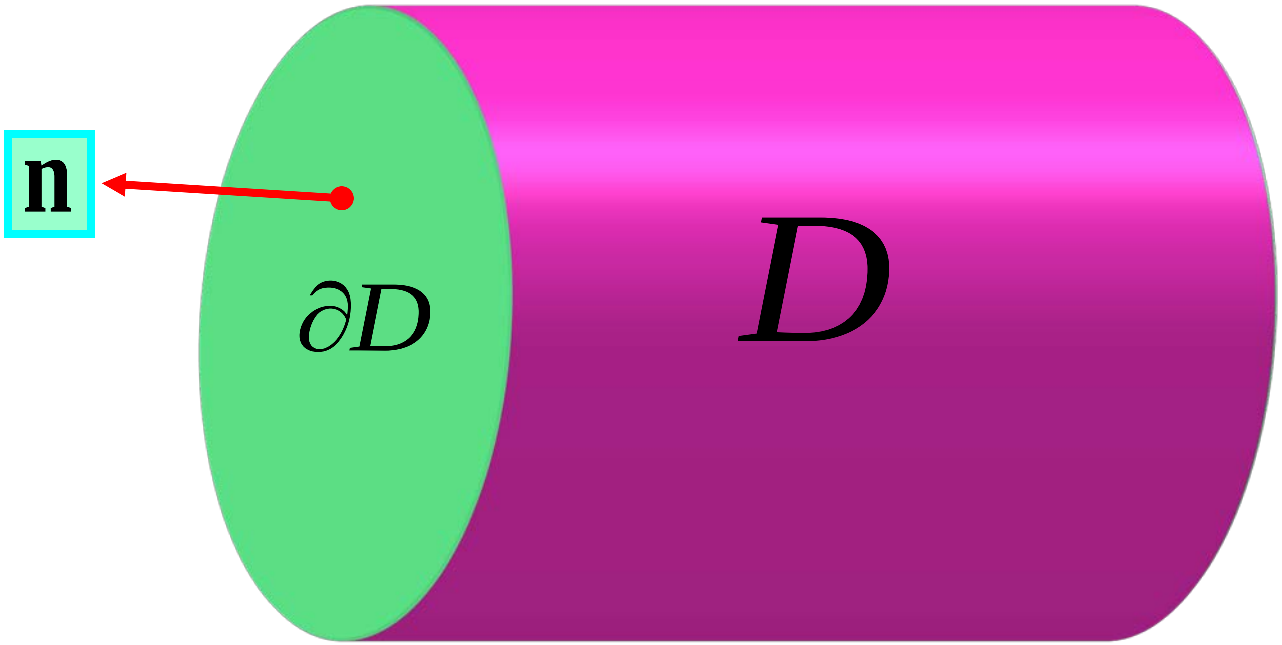
# Green's Theorem (1) via Differential Forms

$$\iint_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

$$\omega = fdy + gdx$$

# Gauss' Divergence Theorem

# 3-dimensional Domain



# Gauss' Divergence Theorem (1)

$$\iiint_D \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dxdydz$$

$$= \iint_{\partial D} fdydz + gdzdx + hdxdy$$

# Gauss' Divergence Theorem (1)

# via Differential Forms

$$\iint_D d\omega = \int_{\partial D} \omega$$

$$\omega = \; f dy dz + g dz dx + h dx dy$$

# Application to Electro-magnetism

# Gauss' Theorem (Magnetic Field)

$$\iint_{\partial D} \mathbf{B}(x) \cdot \mathbf{n}\, dS = 0$$

$\mathbf{B}(x) =$ **Magnetrostatics**

# Gauss' Theorem (Electric Field)

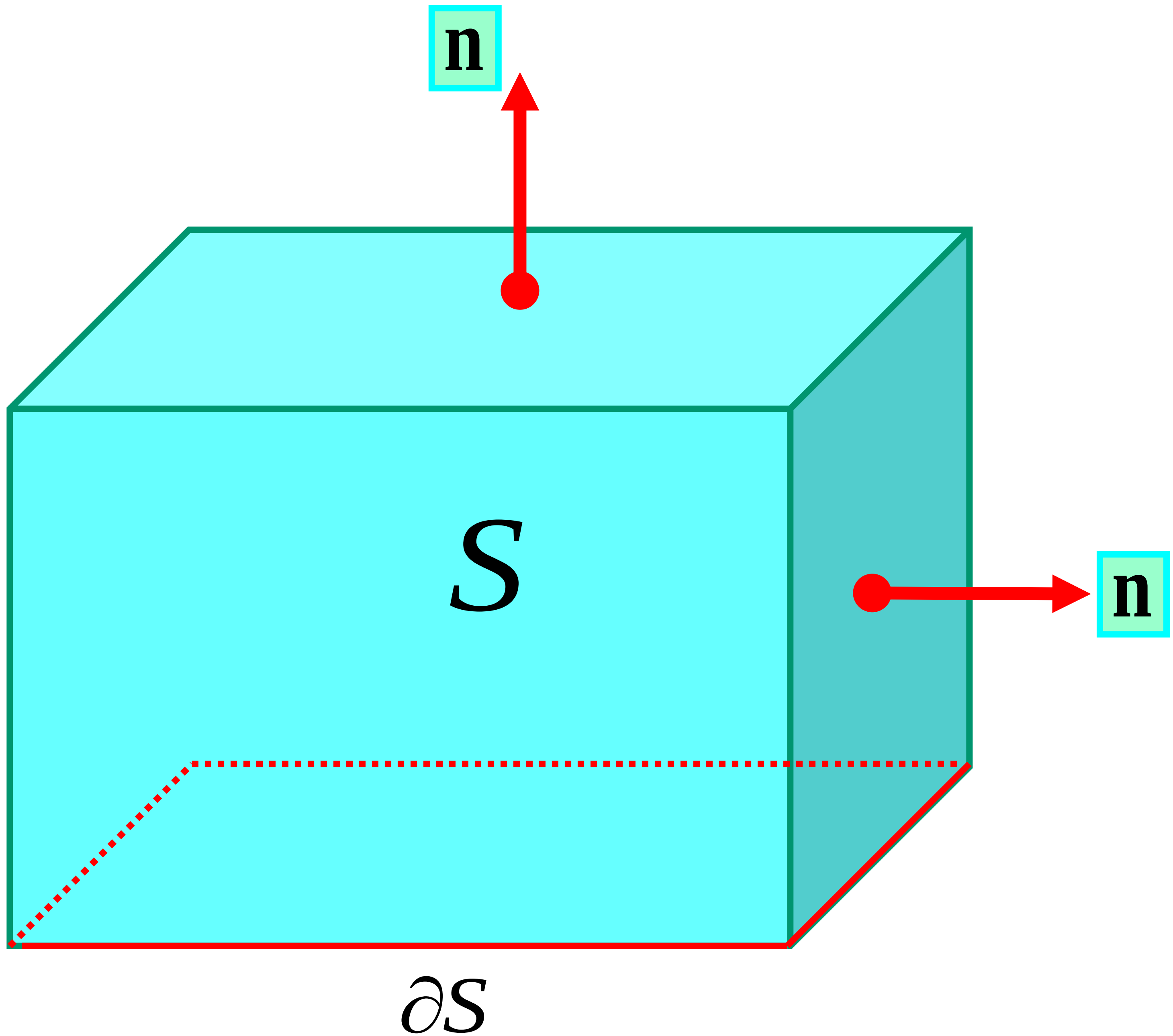
$$\iint_{\partial D} E(x) \cdot \mathbf{n}\, dS = \frac{1}{\varepsilon_0} \iiint_D \rho(x) dx$$

$E(x) =$ **Electrostatic Field**

$\rho(x) =$ **Electric Density**

$\varepsilon_0 =$ **Inductive Capacity in Free Space**

# Stokes' Theorem



n
n
S
∂S

# Stokes' Theorem (1)

$$\iint_S \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dydz + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dzdx + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy$$

$$= \int_{\partial S} fdx + gdy + hdz$$

# Stokes' Theorem (1) via Differential Forms

$$\iint_S d\omega = \int_{\partial S} \omega$$

$$\omega = f dx + g dy + h dz$$

# Application to Electro-magnetism

# Faraday's Law

# Faraday

◆Michael Faraday (1791-1867)
English Scientist

# Michael Faraday

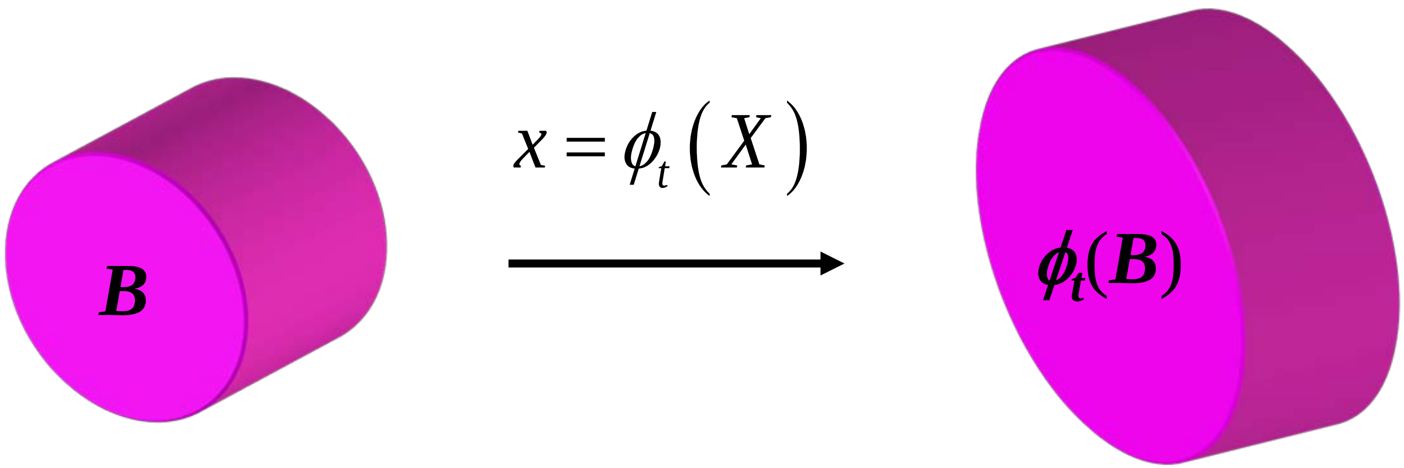
# Faraday's Law

$$-\frac{d}{dt}\left(\iint_{S} \mathbf{B}(x,t)\cdot\mathbf{n}\,dS\right) = \int_{\partial S} \mathbf{E}(x,t)\cdot d\mathbf{r}\,,$$

$$d\mathbf{r} = (dx, dy, dz)$$

# Mathematical Theory
# of
# Elasticity

# Motions and Configurations




**Reference configuration of a body**

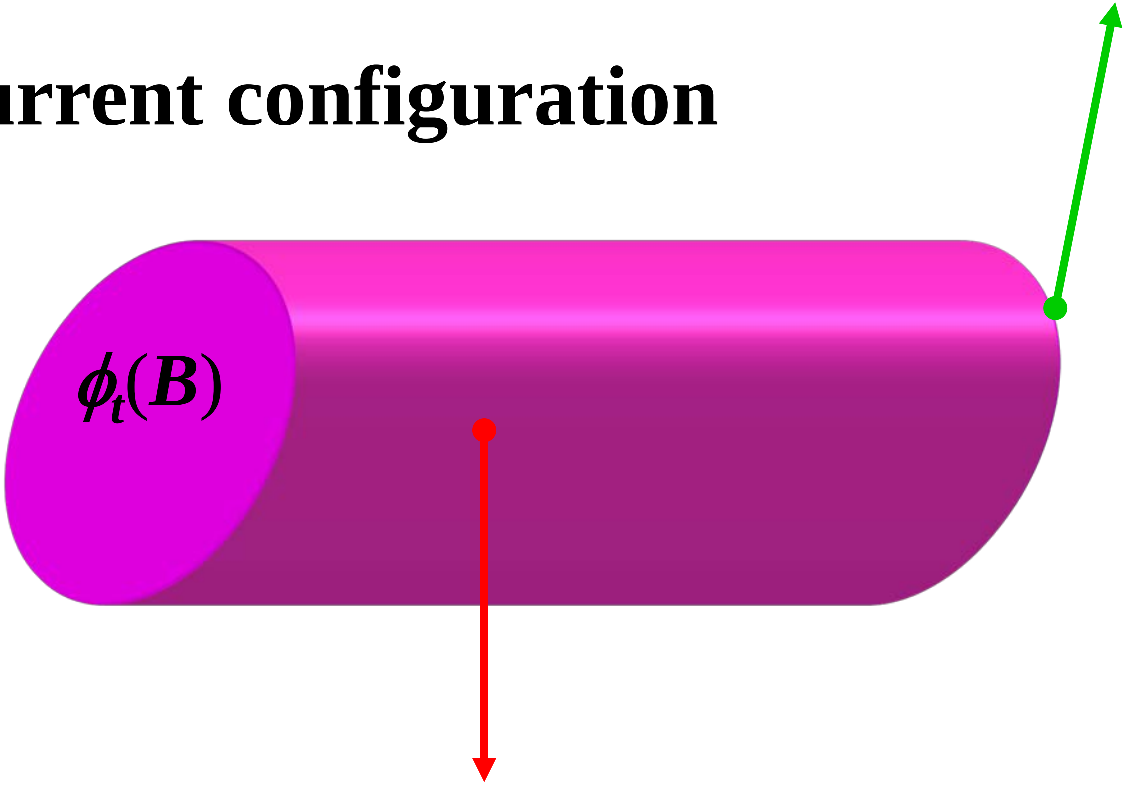
**Body after time *t***

# Two Descriptions in Elastodynamics

# Euler's Description

**Surface force** $\boldsymbol{\tau}(x,t)$

**Current configuration**

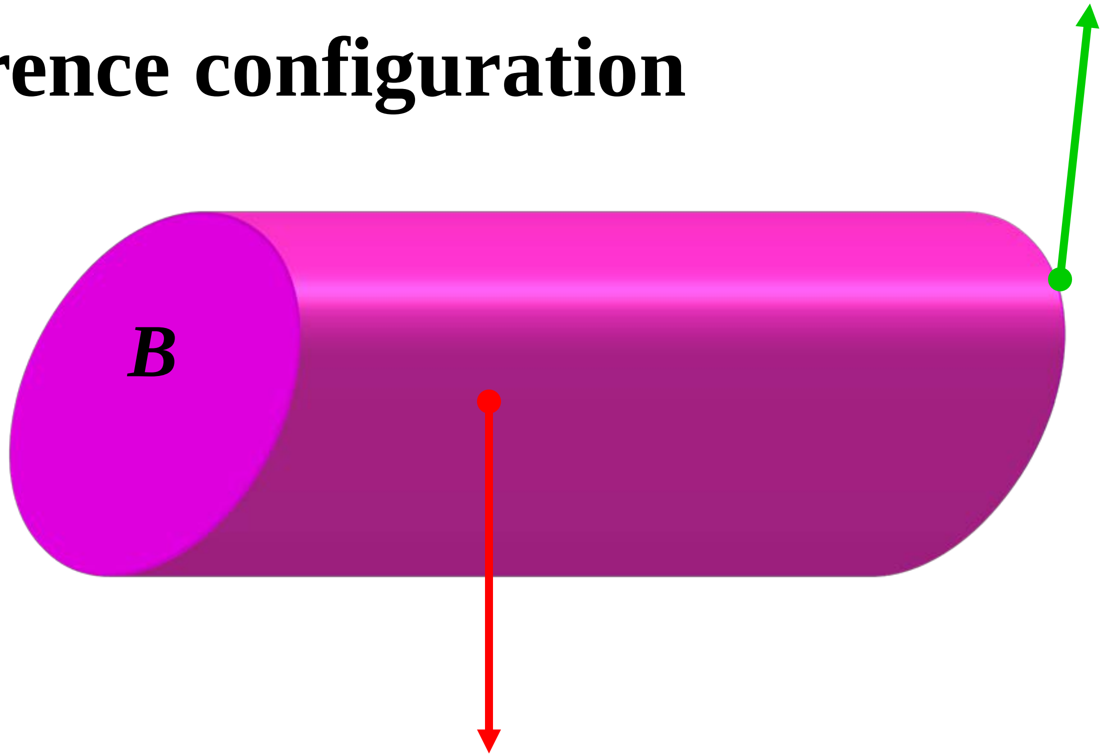



**Body force** $\mathbf{b}(x,t)$

# Lagrange's Description

**Surface force** $\boldsymbol{\tau}(X,t)$

**Reference configuration**




**Body force** $\mathbf{B}(X,t)$

# Continuum Mechanics (1)

| Description | Conservation Law of Mass | Balance Law of Momentum |
|---|---|---|
| Euler | $\dot{\rho} + \rho \, \mathrm{div}\, \mathbf{v} = 0$ | $\rho \, \dot{\mathbf{v}} = \mathrm{div}\, \boldsymbol{\sigma} + \rho \mathbf{b}$ |
| Lagrange | $\rho_0(X) = \rho(\phi_t(X), t) J(X, t)$ | $\rho_0 \dfrac{\partial \mathbf{V}}{\partial t} = \mathrm{Div}\, \mathbf{P} + \rho_0 \mathbf{B}$ |

# Continuum Mechanics (2)

| Description | Balance Law of Angular Momentum | Balance Law of Energy |
|---|---|---|
| Euler | $\boldsymbol{\sigma} = {}^{t}\boldsymbol{\sigma}$ | $\rho\dot{e} + \mathrm{div}\,\mathbf{q} = \mathrm{tr}(\boldsymbol{\sigma}\mathbf{d}) + \rho r$ |
| Lagrange | $\mathbf{S} = {}^{t}\mathbf{S}$ | $\rho_0 \dfrac{\partial E}{\partial t} + \mathrm{Div}\,\mathbf{Q} = \mathrm{tr}(\mathbf{SD}) + \rho_0 R$ |

# Probability and Calculus

# Weierstrass' Polynomial Approximation Theorem

# Weierstrass' Polynomial Approximation Theorem

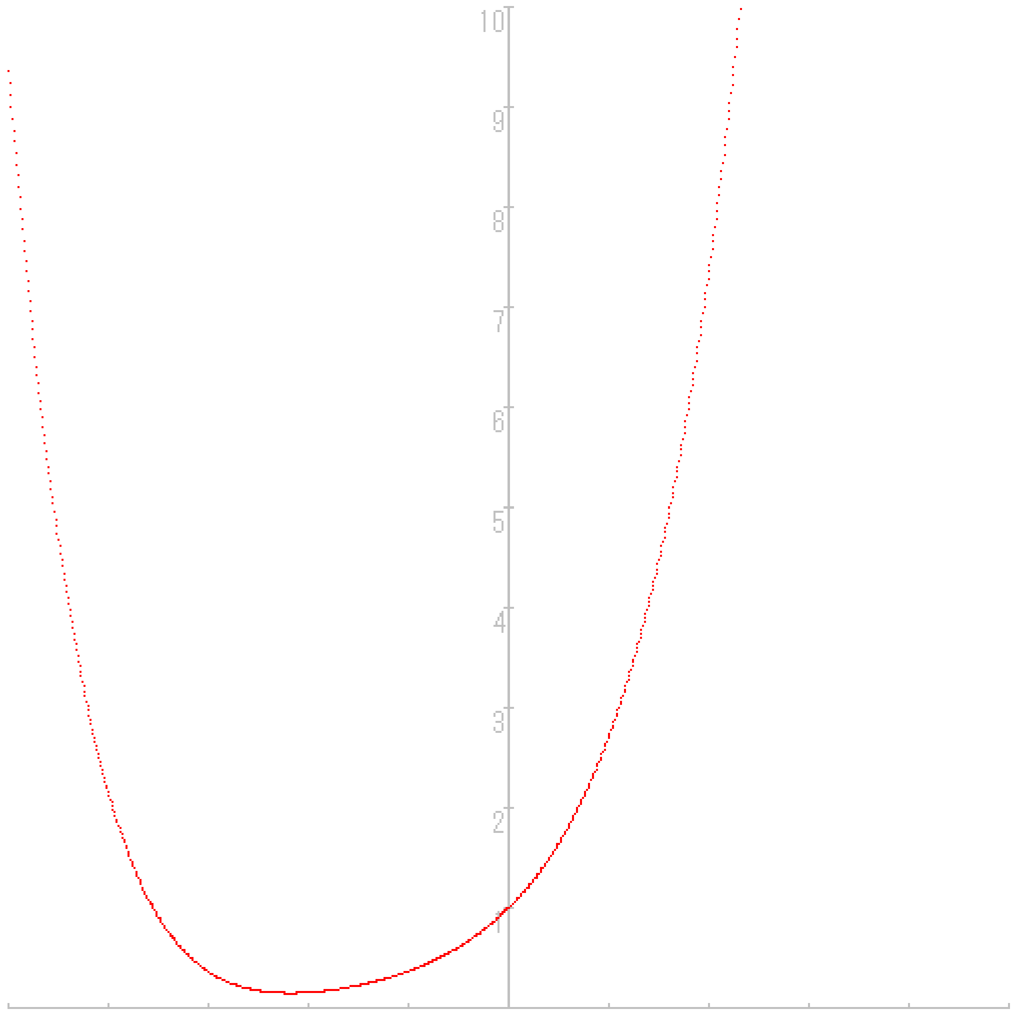
**Any continuous function defined on a bounded closed interval may be approximated uniformly by polynomials.**

# Examples of Taylor's Expansion

# Example 1

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

# Numerical Computing with BASIC

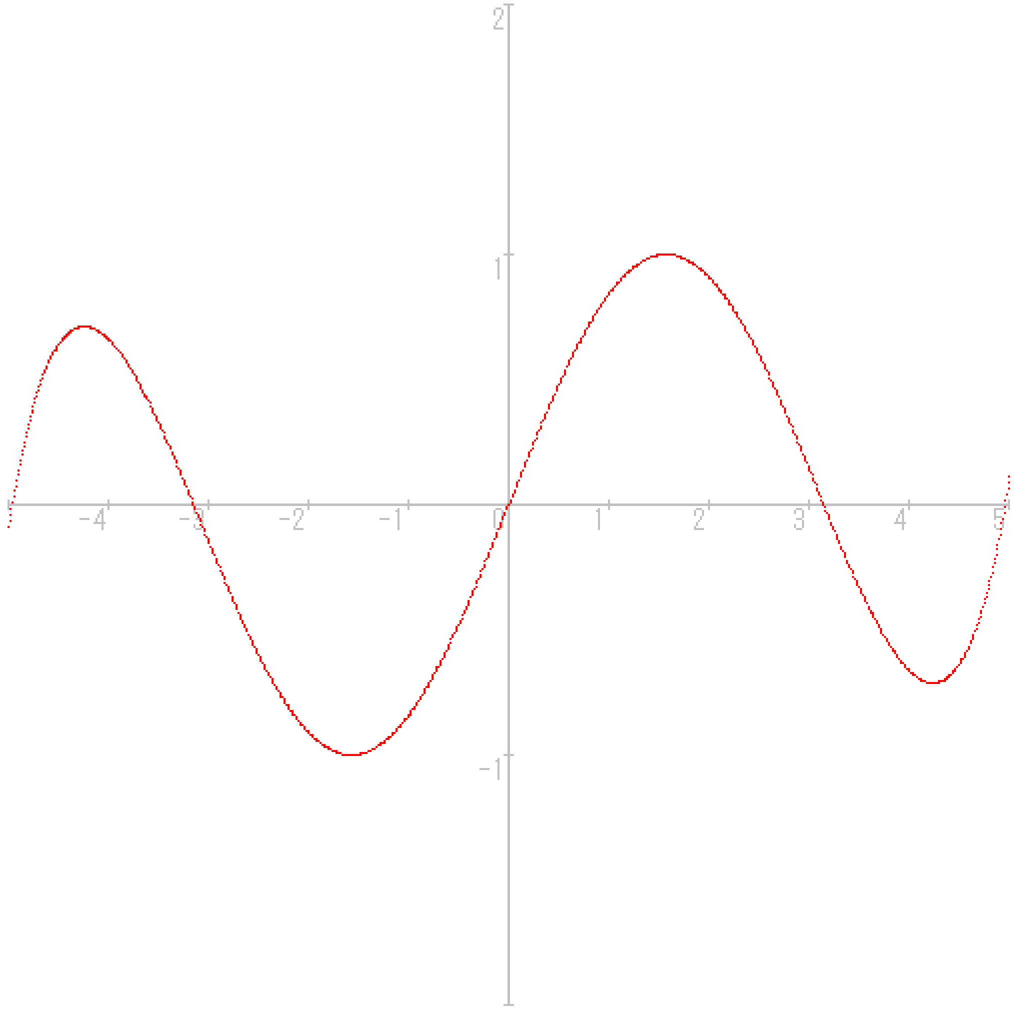


10
9
8
7
6
5
4
3
2
1

# Example 2

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

# Numerical Computing with BASIC



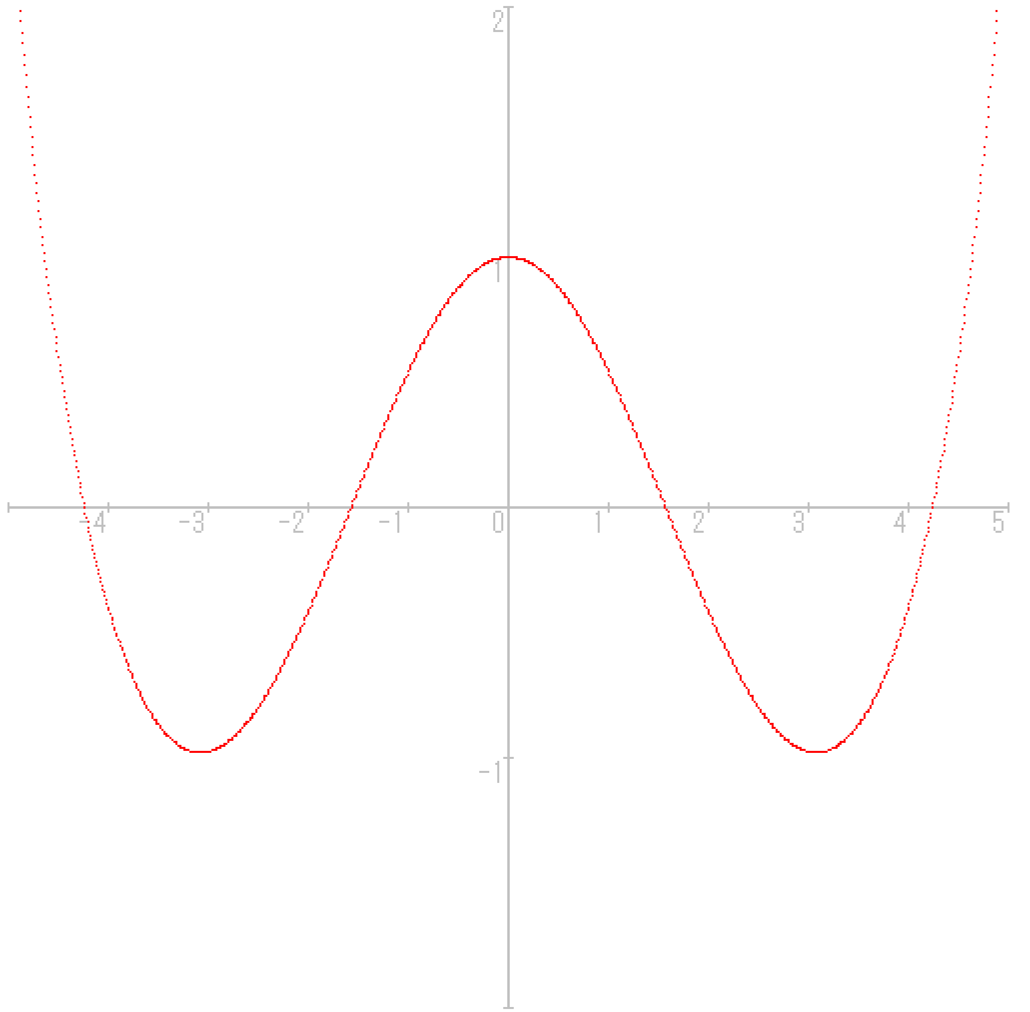
2
1
-1
-4
-3
-2
-1
0
1
2
3
4
5

# Example 3

$$\cos x$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

# Numerical Computing
# with
# BASIC


2
1
-1
-4
-3
-2
-1
0
1
2
3
4
5

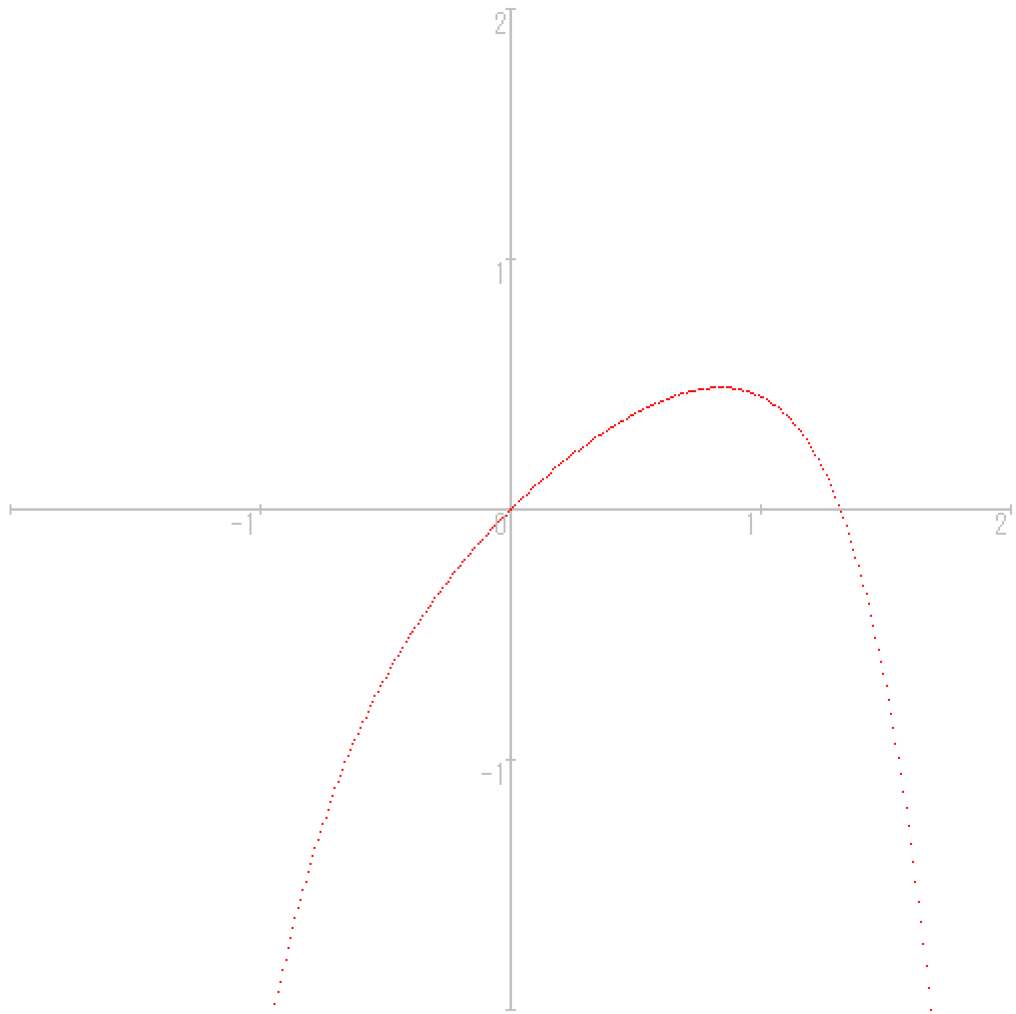
# Example 4

$$\log_e(1+x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(-1 < x \leq 1)$$

# Numerical Computing
# with
# BASIC



2
1
-1
0
1
2
-1

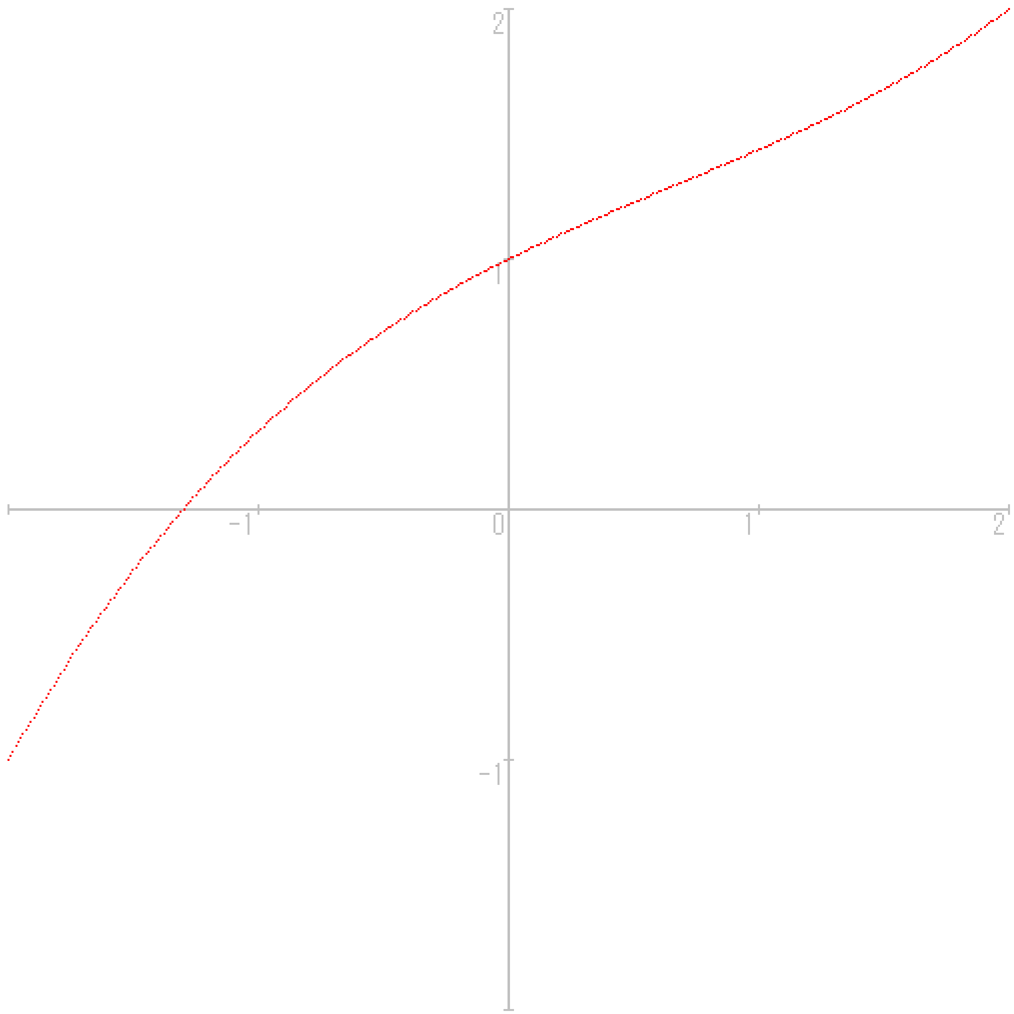
# Example 5

$$\sqrt{1+x}$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^{2}+\frac{1}{16}x^{3}-\cdots$$

$$(-1<x<1)$$

# Numerical Computing with BASIC



2
1
-1
0
1
2
-1

# Probabilistic Approach

# Purpose

**Any continuous function defined on the closed interval [0,1] may be approximated uniformly by Bernstein's polynomials.**

# Bernstein's Polynomial Approximation Theorem

# Bernstein's Polynomial Approximation Theorem

$$f(x) \in C[0,1]$$

$$f_n(p) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le p \le 1$$

($n-$ **th Bernstein's polynomial**)

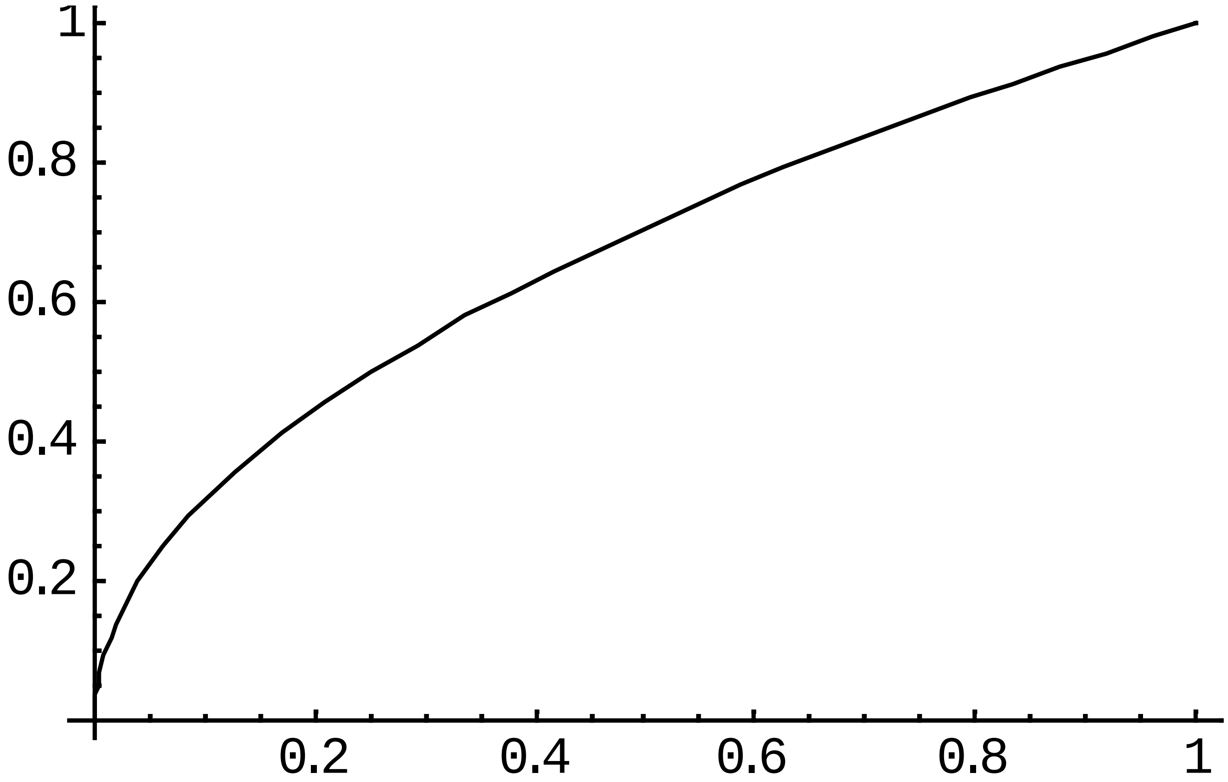
$\Rightarrow$

$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$$\forall n \ge N \Rightarrow \max_{0 \le p \le 1} \left| f(p) - f_n(p) \right| < \varepsilon$$

# Numerical Computing With MATHEMATICA

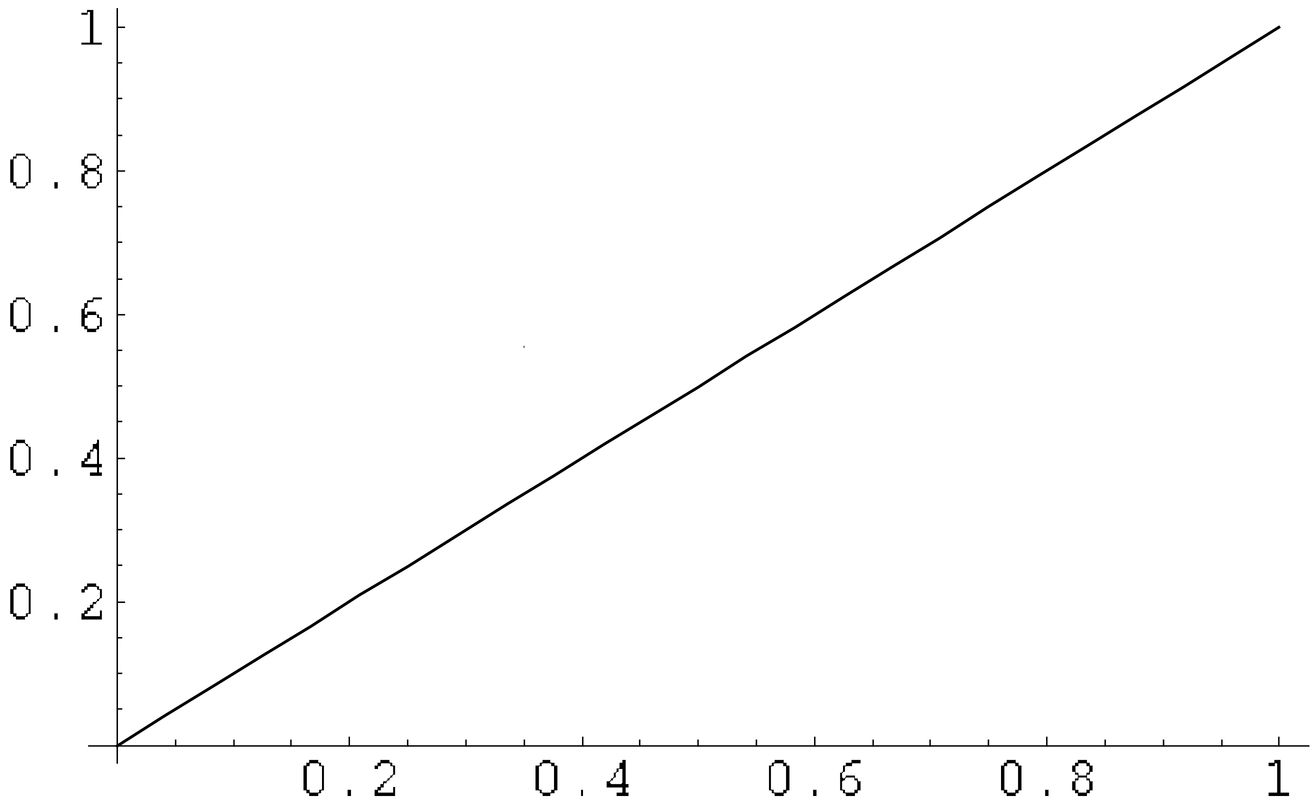
$$f(x) = \sqrt{x}$$



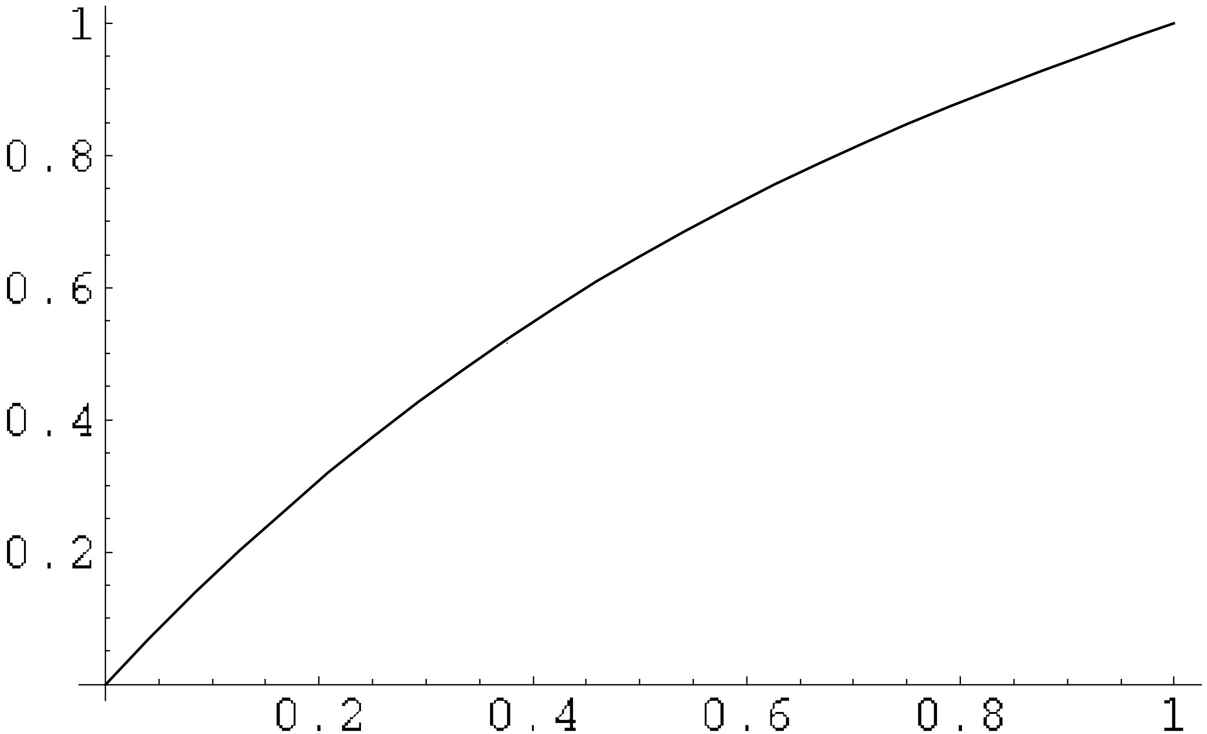
# Bernstein's Polynomial

$$f_n(x) := \sum_{k=0}^{n} \sqrt{\frac{k}{n}} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}, \quad 0 \le x \le 1$$

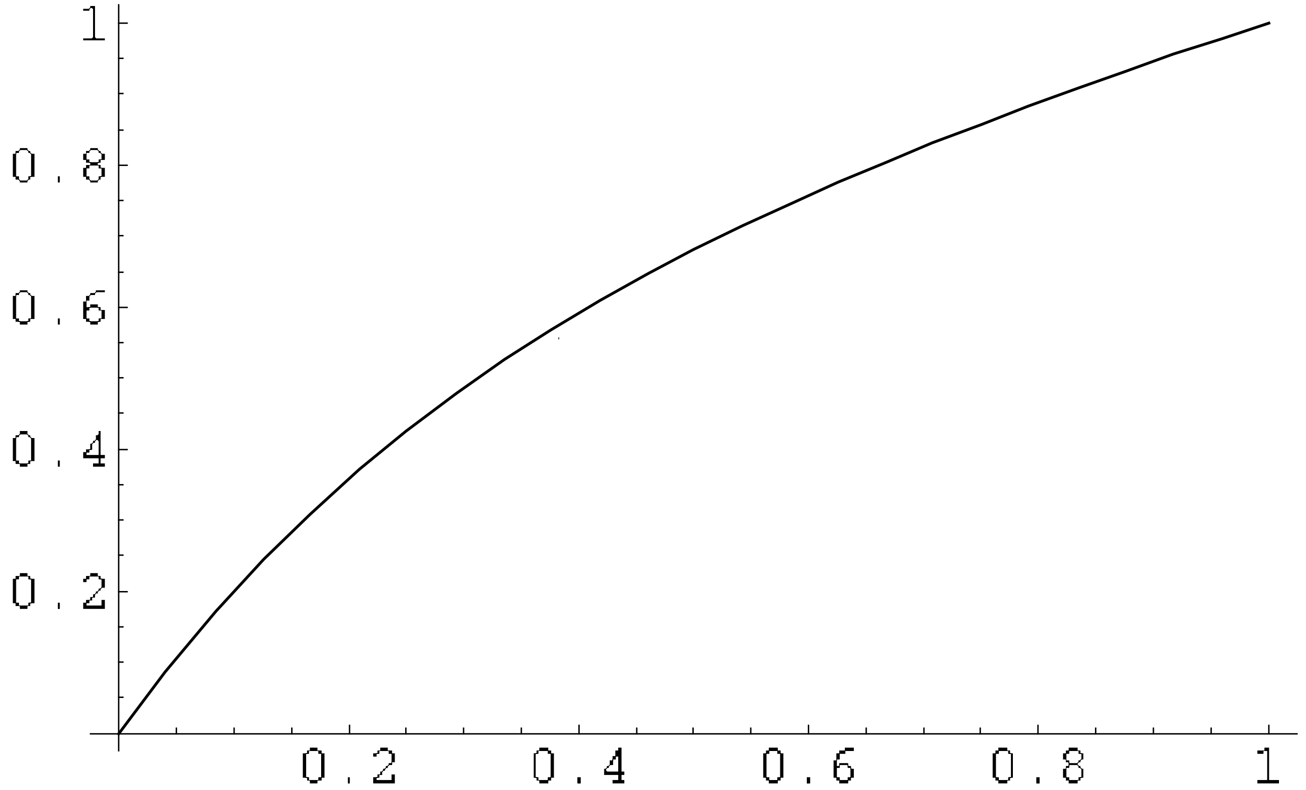
$f_1(x)$



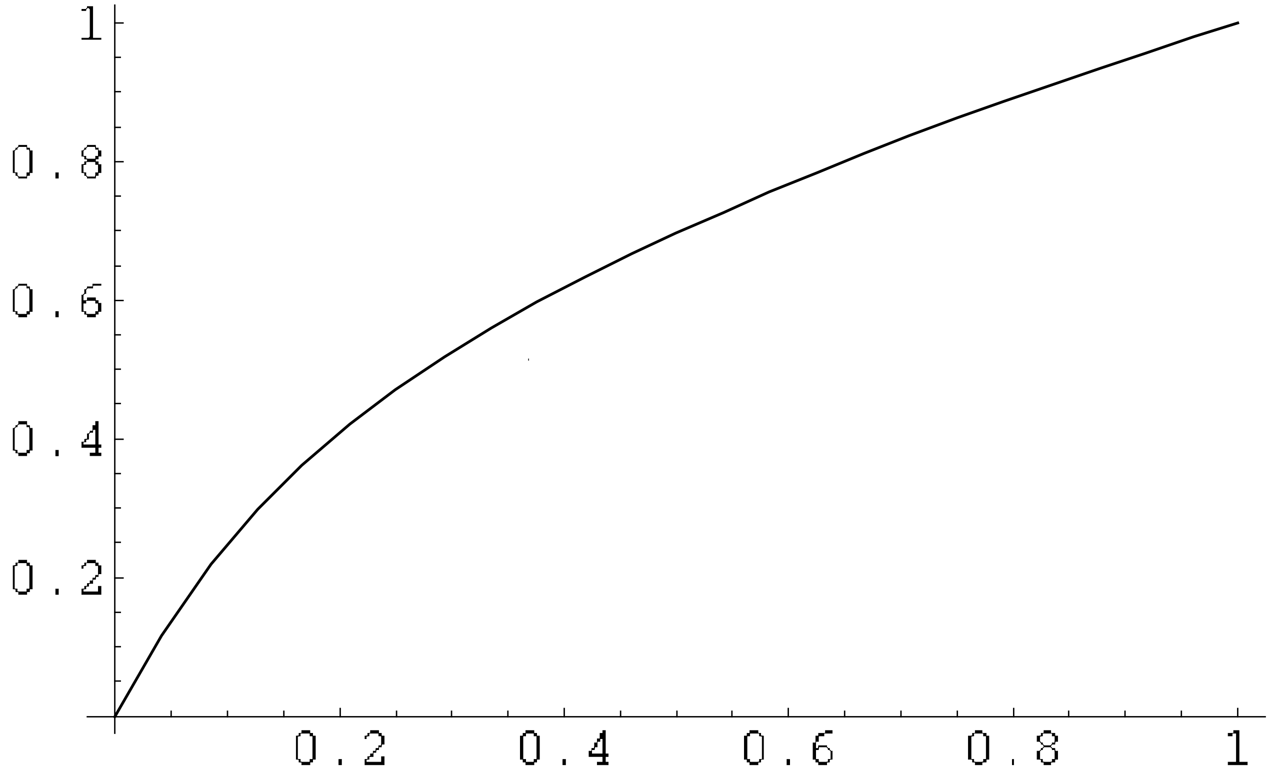
$f_3(x)$



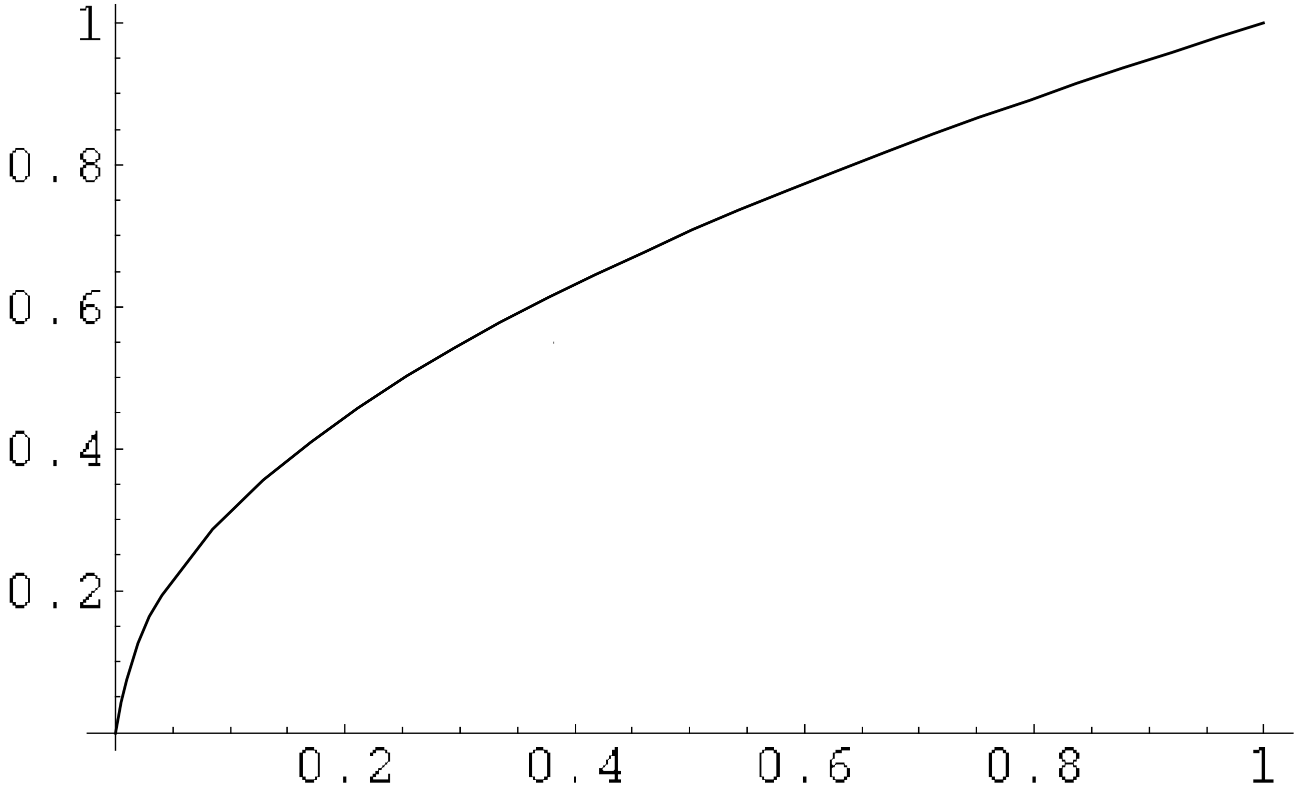
$f_5(x)$



$f_{10}(x)$



$f_{100}(x)$



# Probability

# Probability Space

$\Omega$ :**Sample Space**

$\mathrm{B}$ :**Completely Additive Class**

$\mathrm{P}$ :**Probability Measure**

$(\Omega, \mathrm{B}, \mathrm{P})$: **Probability Space**

$X_i$: **Random Variable (Measurable)**

$E(X_i) = \int_\Omega X_i(\omega) d\mathrm{P}$ : **Mean (Expectation)**

$V(X_i) = E\left((X_i - E(X_i))^2\right)$: **Variance**

# Independent Variables

$(\Omega, \mathrm{B}, \mathrm{P})$: **Probability Space**

$X_1, X_2$ **Independent**

$\underset{\text{def}}{\Longleftrightarrow}$

$\forall$**Borel Sets** $A_1, A_2$

$$\mathrm{P}(\{\omega \in \Omega \mid X_1(\omega) \in A_1, X_2(\omega) \in A_2\})$$

$$= \mathrm{P}(\{\omega \in \Omega \mid X_1(\omega) \in A_1\})\, \mathrm{P}(\{\omega \in \Omega \mid X_2(\omega) \in A_2\})$$

# Chebychev's Inequality

$$\begin{cases} (\Omega, \mathrm{B}, \mathrm{P}) \text{ } \textbf{Probability Space} \\ X : \textbf{Random Variable} \\ E(X^2) < \infty \end{cases}$$

$$\Rightarrow$$

$$\forall \varepsilon > 0$$

$$\mathrm{P}\left(\left\{\omega \in \Omega \mid \left|X(\omega) - E(X)\right| \geq \varepsilon\right\}\right) \leq \frac{V(X)}{\varepsilon^2}$$

# Coin Flipping

# Bernoulli Distribution

$$\begin{cases} \textbf{probability of a head 1}: & p \quad (0 \le p \le 1) \\ \textbf{probability of a tail 0}: & 1-p \end{cases}$$

$$X_n : \textbf{n-th throw}$$

$$\Rightarrow$$

$$\begin{cases} P(X_n = 1) = p \\ P(X_n = 0) = 1 - p \end{cases}$$

$$\begin{cases} E(X_n) = 1 \cdot \mathrm{P}(X_n = 1) + 0 \cdot \mathrm{P}(X_n = 0) = p \\ V(X_n) = E(X_n^2) - E(X_n)^2 = p(1-p) \end{cases}$$

# Bernstein's Polynomial Approximation Theorem

$f(x) \in C[0,1]$

$$f_n(p) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le p \le 1$$

($n-$ **th Bernstein's polynomial**)

$\Rightarrow$

$\forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbf{N}$ **such that**

$$\forall n \ge N \Rightarrow \max_{0 \le p \le 1} \left| f(p) - f_n(p) \right| < \varepsilon$$

# Mathematical Biology

# Population Models

- Two predator-prey type species residing in a common district.
- Two competing species residing in a common district.

# Predator-Prey Model (Lotka-Volterra)

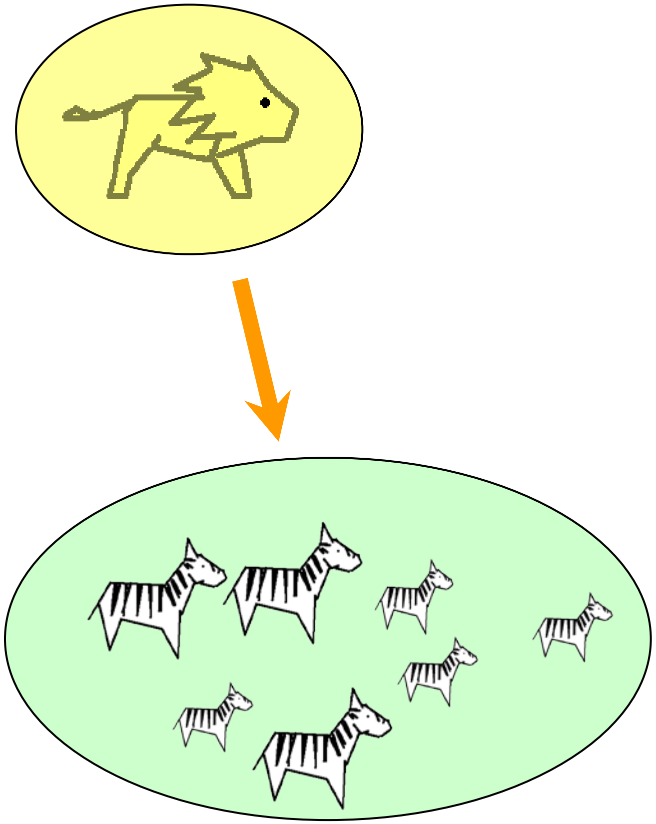
# Volterra

◆**Vito Volterra (1860-1940)**

**Italian Mathematician and Physicist**

# Vito Volterra

# Predator-Prey Model



Model

$$\begin{cases} \dfrac{dx}{dt} = (1 - y(t))\,x(t) \\ \dfrac{dy}{dt} = (-1 + x(t))\,y(t) \\ x(0) = 20 \\ y(0) = 2 \end{cases}$$

**Differential Equations**

# Predator-Prey Model

$$\begin{cases} \dfrac{dx}{dt} = r_1(1 - ay(t))x(t) \\ \dfrac{dy}{dt} = r_2(-1 + bx(t))y(t) \end{cases}$$

$x(t)$ : **density of the Prey**

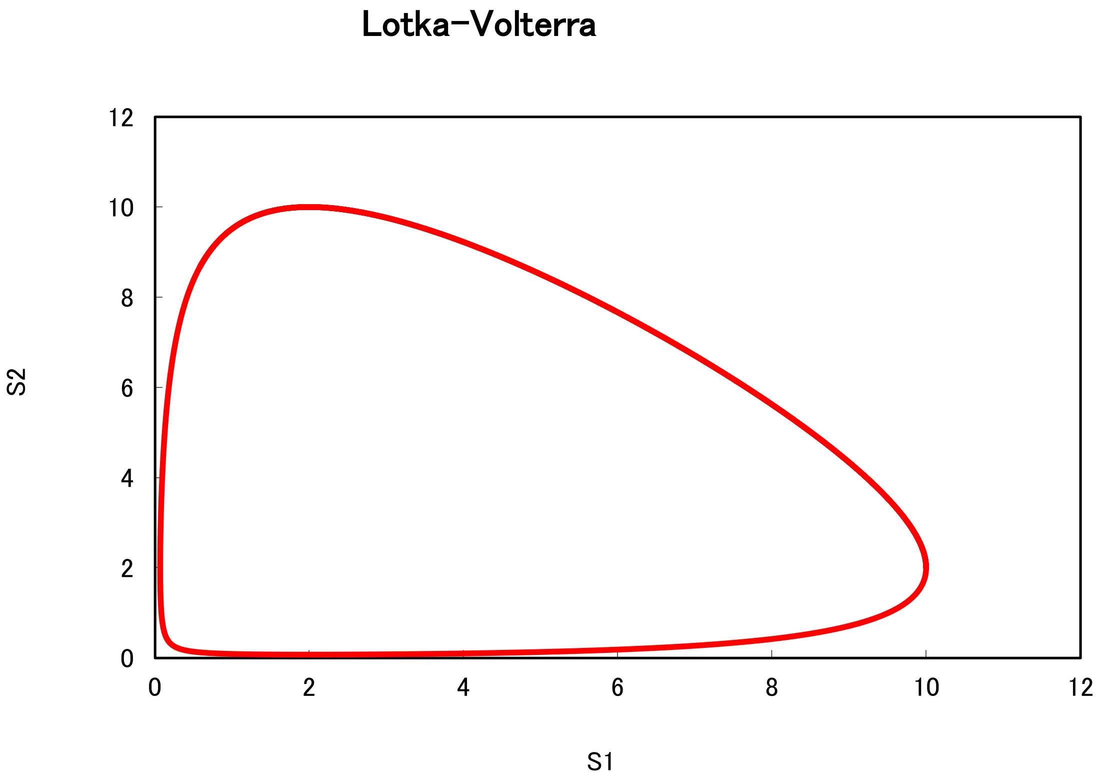
$y(t)$ : **density of the Predator**

# Numerical Computing
# with
# Excel (VBA)

# Lotka-Volterra's Model

$$\begin{cases} \dfrac{dx}{dt} = (2 - y(t))\,x(t) \\ \dfrac{dy}{dt} = (-2 + x(t))\,y(t) \\ x(0) = 10 \\ y(0) = 2 \end{cases}$$

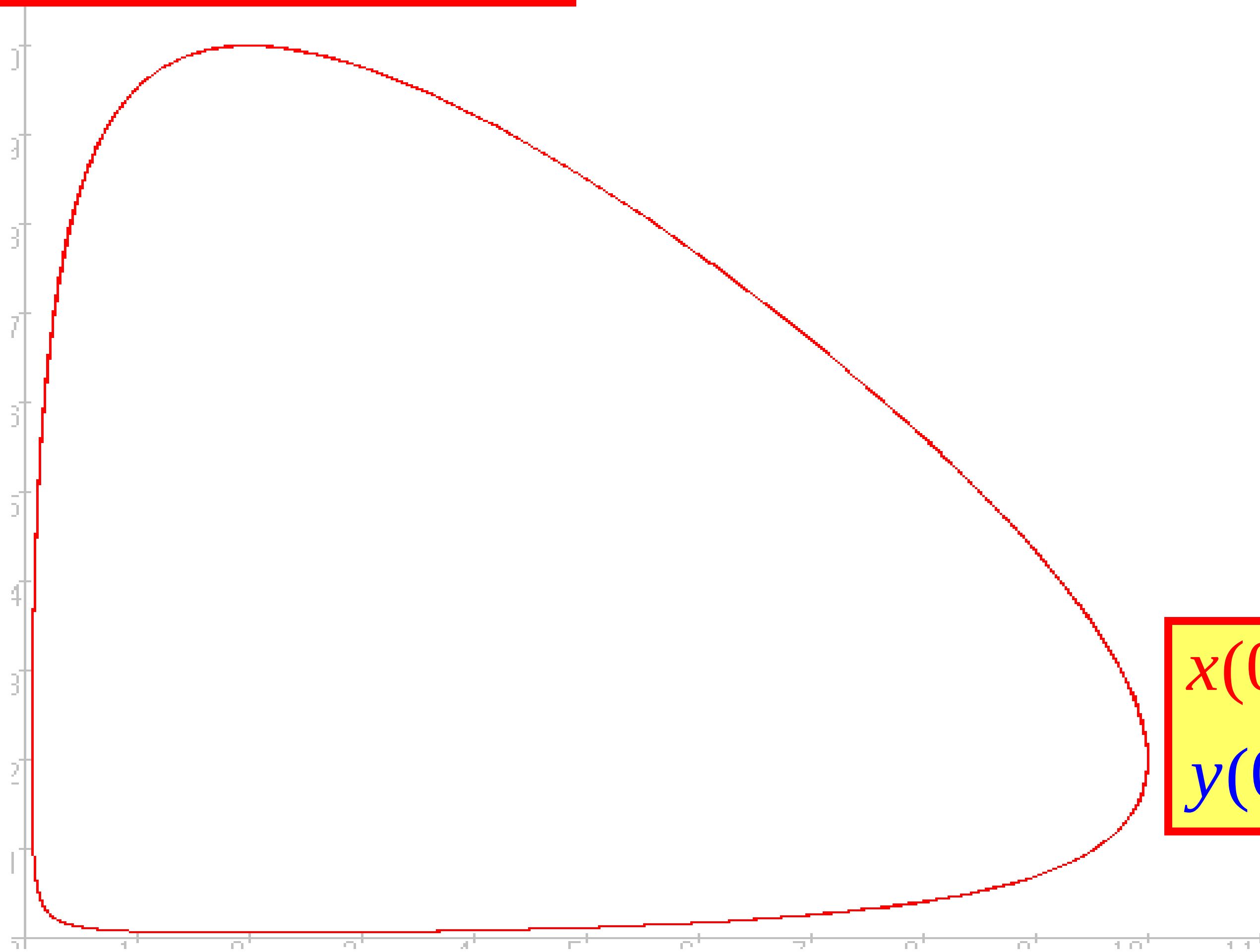
# Runge-Kutta Method



# Numerical Computing
# with
# BASIC

# Runge-Kutta Method

$y(t)$ : **density of the Predator**

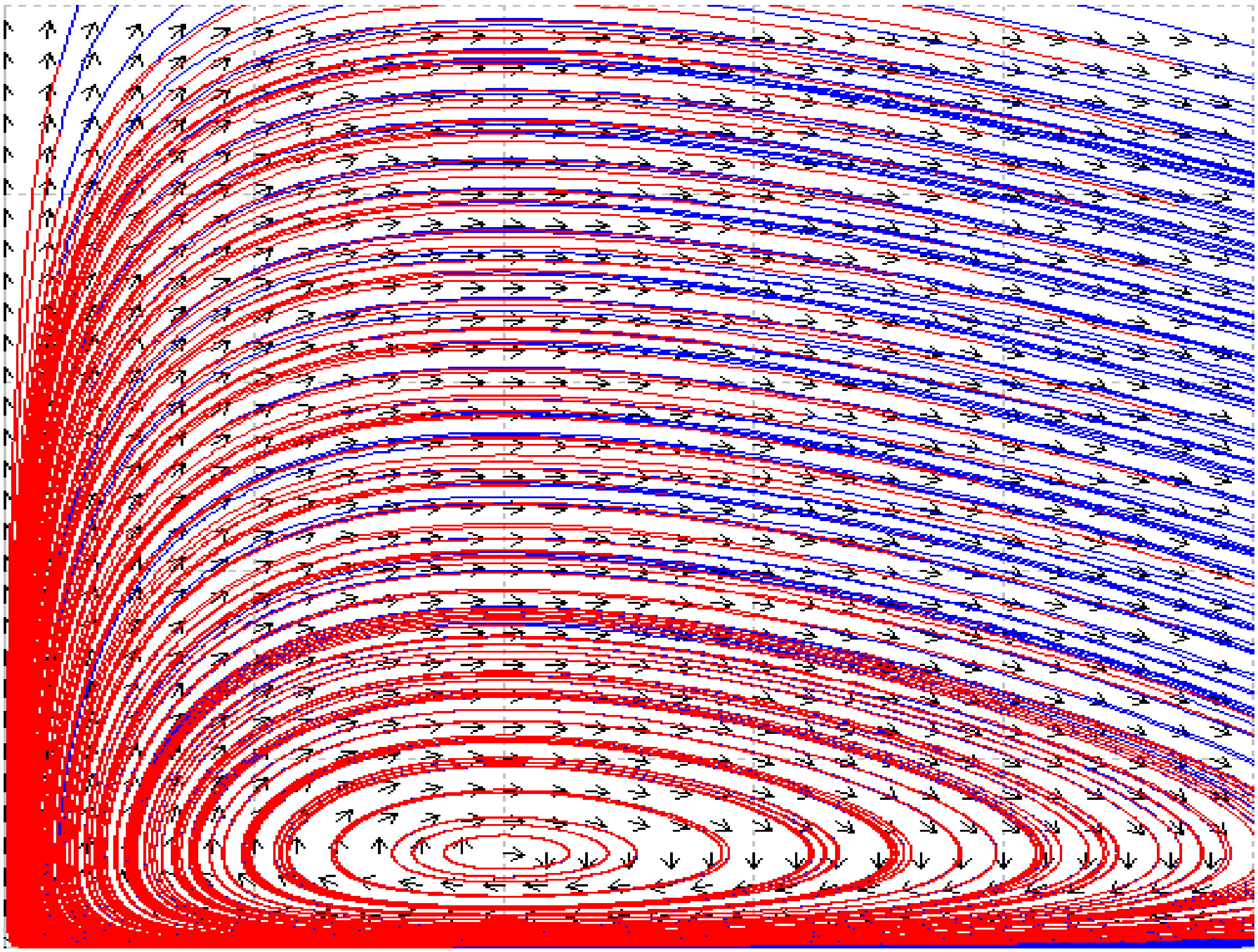


$x(0) = 10$

$y(0) = 2$

$x(t)$ : **density of the Prey**

# Trajectories depending on Initial Values



# Competitive Model (Volterra)

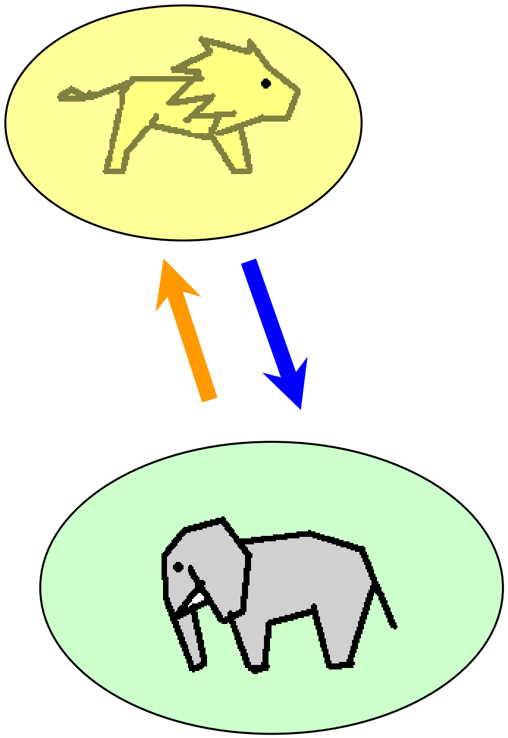
# Competitive Model

$$\begin{cases} \dfrac{dx}{dt} = r_1(1 - ax(t) - by(t))x(t) \\ \dfrac{dy}{dt} = r_2(1 - cy(t) - dx(t))y(t) \end{cases}$$

$x(t)$ : **density of species 1**

$y(t)$ : **density of species 2**

# Competitive Model



$$\begin{cases} \dfrac{dx}{dt} = r_1(1 - ax(t) - by(t))x(t) \\ \dfrac{dy}{dt} = r_2(1 - cy(t) - dx(t))y(t) \\ x(0) = \alpha \\ y(0) = \beta \end{cases}$$

**Model**

**Differential Equations**

# Stability of Solutions

# Numerical Computing
# with
# BASIC

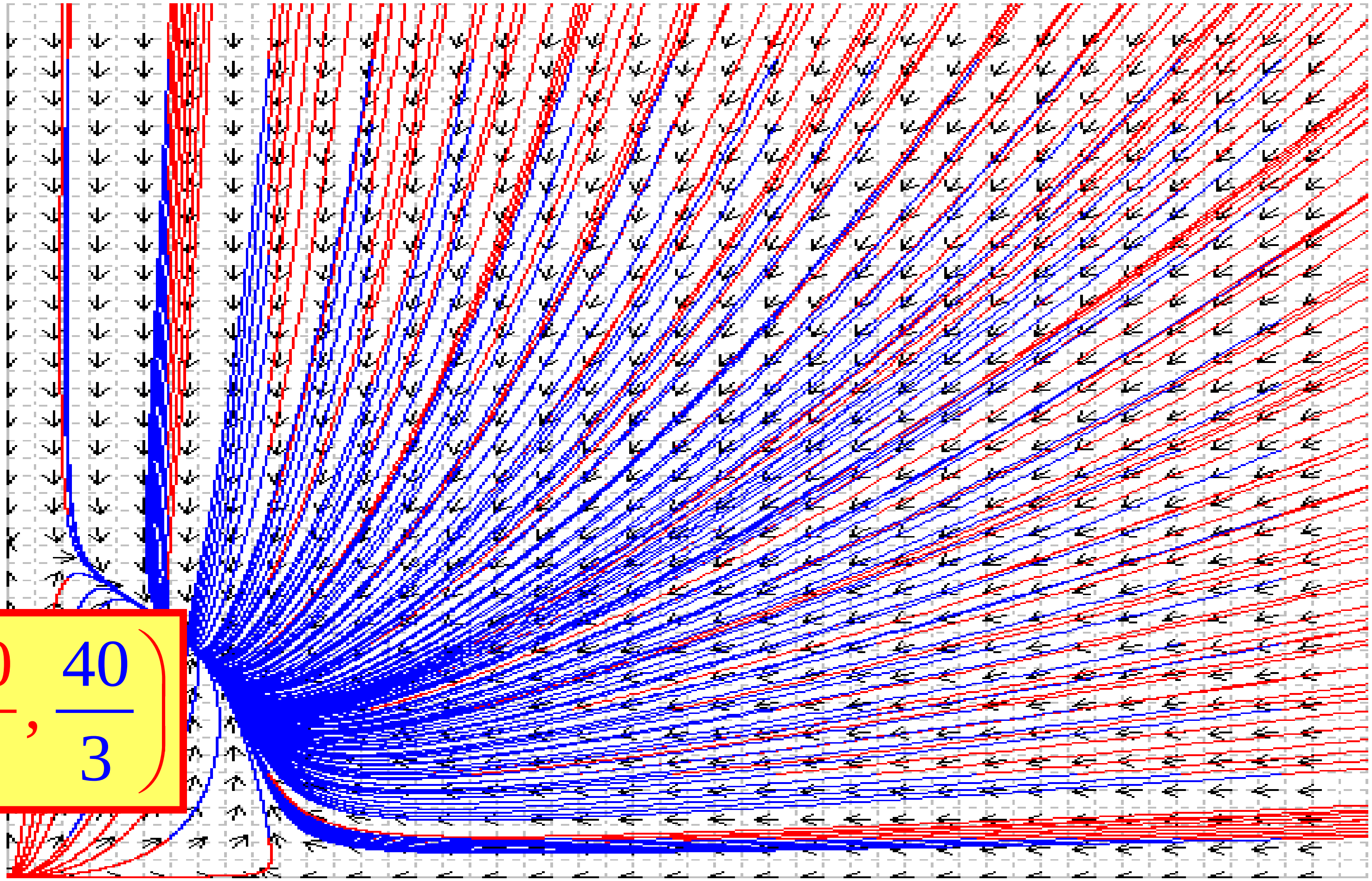
# Coexistence Model

$$\begin{cases} \dfrac{dx}{dt} = (1 - 0.1x(t) - 0.025y(t))x(t) \\ \dfrac{dy}{dt} = 2(1 - 0.05y(t) - 0.05x(t))y(t) \end{cases}$$

# Runge-Kutta Method

```
DEF F(X,Y)=(1-0.1*X-0.025*Y)*X
DEF G(X,y)=2*(1-0.05*Y-0.05*X)*Y
DEF FF(X,Y)=F(X,Y)/SQR(F(X,Y)^2+G(X,Y)^2)
DEF GG(X,Y)=G(X,Y)/SQR(F(X,Y)^2+G(X,Y)^2)
LET  h0=0.005
LET M=50
SET WINDOW 0,M,0,M
LET MS=M/30
DRAW Grid
FOR Y=-M TO M STEP MS
  FOR X=-M TO M STEP MS
    PLOT LINES
    PLOT LINES: X,Y;
    PLOT LINES : X+FF(X,Y)*MS/2,  Y+GG(X,Y)*MS/2
    PLOT LINES : X+(FF(X,Y)+GG(X,Y))*MS/5,  Y+(GG(X,Y)-FF(X,Y))*MS/5;
    PLOT LINES : X+FF(X,Y)*MS/2,  Y+GG(X,Y)*MS/2
    PLOT LINES : X+(FF(X,Y)-GG(X,Y))*MS/5,  Y+(GG(X,Y)+FF(X,Y))*MS/5;
    PLOT LINES : X+FF(X,Y)*MS/2,  Y+GG(X,Y)*MS/2
  NEXT X
NEXT Y
PAUSE
FOR Y00=-M TO M STEP MS*SQR(5)
  FOR X00=-M TO M STEP MS*SQR(5)
    FOR K=1 TO 2
      LET x0=X00
      LET Y0=Y00
      LET H=(-1)^(K-1)*H0
      SET COLOR 2^k
      PLOT LINES
      PLOT LINES: X0,Y0;
      LET COUNT=0
      DO WHILE ABS(X0)<2*M  AND ABS(Y0)<2*M  AND COUNT<1000
        LET COUNT=COUNT+1
        LET X=X0+F(X0,Y0)*H/2
        LET Y=Y0+G(X0,Y0)*H/2
        LET X1=X0+F(X,Y)*H
        LET Y1=Y0+G(X,Y)*H
        PLOT LINES: X1,Y1;
        LET X0=X1
        LET Y0=Y1
      LOOP
    NEXT K
  NEXT X00
NEXT Y00
END
```

# One Stable Point



$$\left(\frac{20}{3}, \frac{40}{3}\right)$$

# Stable Point

$$\boxed{\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0}$$

$$\Longrightarrow$$

$$\begin{cases} 1 - 0.1x_\infty - 0.025y_\infty = 0 \\ 1 - 0.05y_\infty - 0.05x_\infty = 0 \end{cases}$$

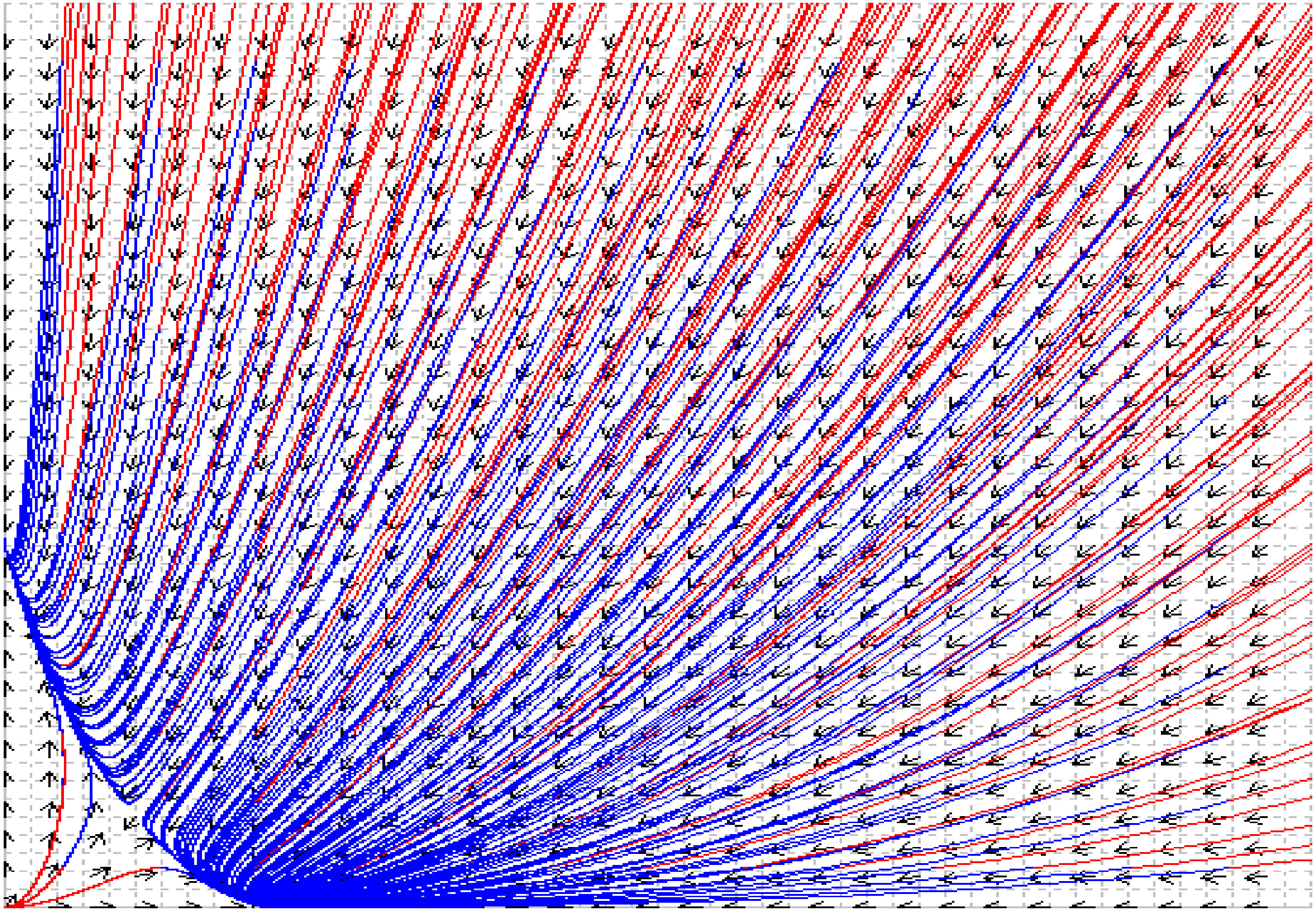
$$\Longrightarrow$$

$$\boxed{x_\infty = \frac{20}{3}, \quad y_\infty = \frac{40}{3}}$$

# Bistable Model

$$\begin{cases} \dfrac{dx}{dt} = (1 - 0.1x(t) - 0.1y(t))x(t) \\ \dfrac{dy}{dt} = 2(1 - 0.05y(t) - 0.15x(t))y(t) \end{cases}$$

# Two Stable Points



(0, 20)

(10, 0)

# Stable Point (1)

$$\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0$$

$$\Longrightarrow$$

$$\begin{cases} 1 - 0.1x_\infty - 0.1y_\infty = 0 \\ y_\infty = 0 \end{cases}$$

$$\Longrightarrow$$

$$x_\infty = 10, \quad y_\infty = 0$$

# Stable Point (2)

$$\boxed{\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0}$$

$$\Longrightarrow$$

$$\begin{cases} x_\infty = 0 \\ 1 - 0.05 y_\infty - 0.15 x_\infty = 0 \end{cases}$$

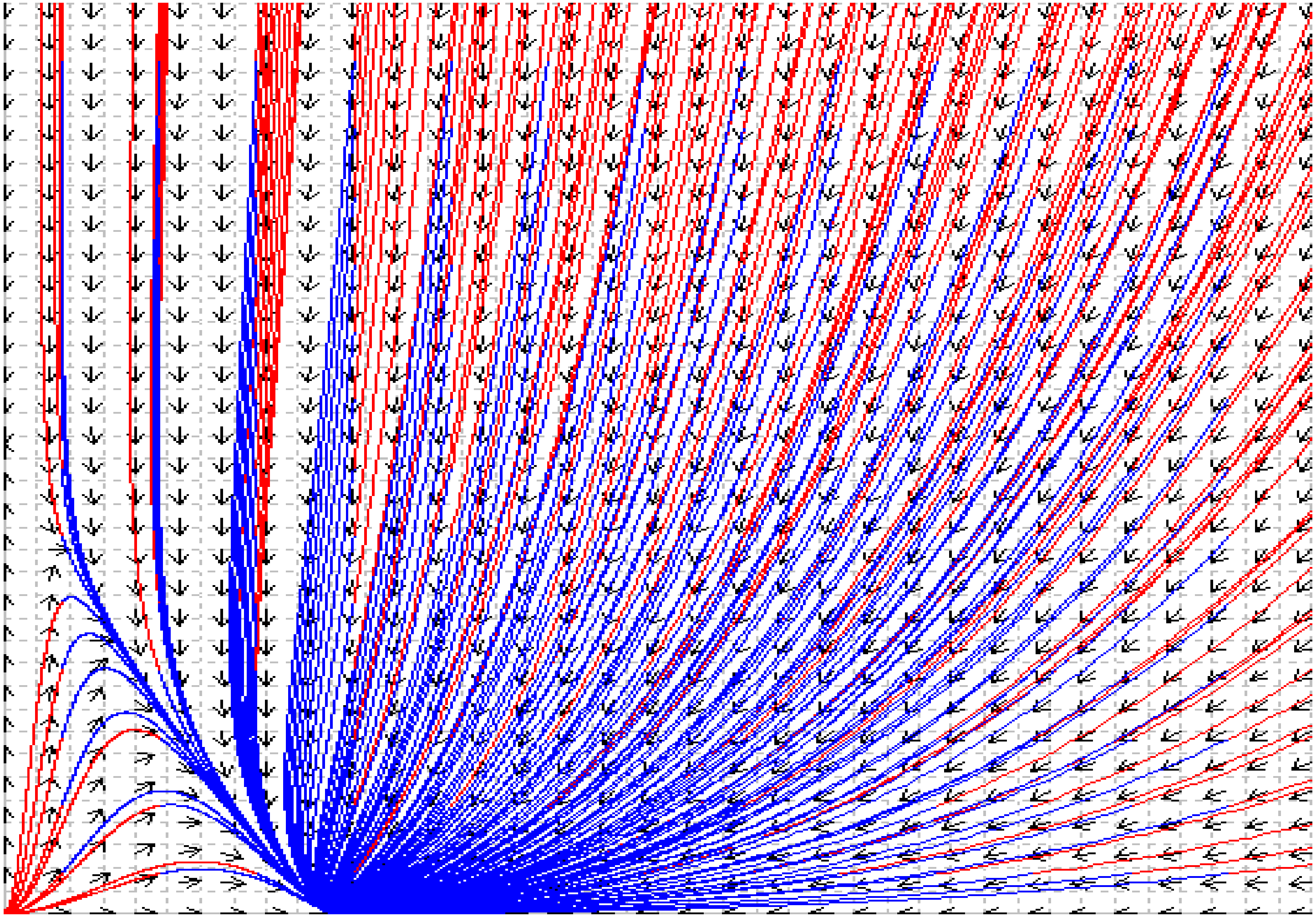
$$\Longrightarrow$$

$$\boxed{x_\infty = 0, \quad y_\infty = 20}$$

# Dominant Model (1)

$$\begin{cases} \dfrac{dx}{dt} = (1 - 0.1x(t) - 0.025y(t))x(t) \\ \dfrac{dy}{dt} = 2(1 - 0.05y(t) - 0.15x(t))y(t) \end{cases}$$

# Dominant Model (1)



$(10, 0)$

# Stable Point (1)

$$\boxed{\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0}$$

$$\Longrightarrow$$

$$\begin{cases} 1 - 0.1x_\infty - 0.025y_\infty = 0 \\ y_\infty = 0 \end{cases}$$

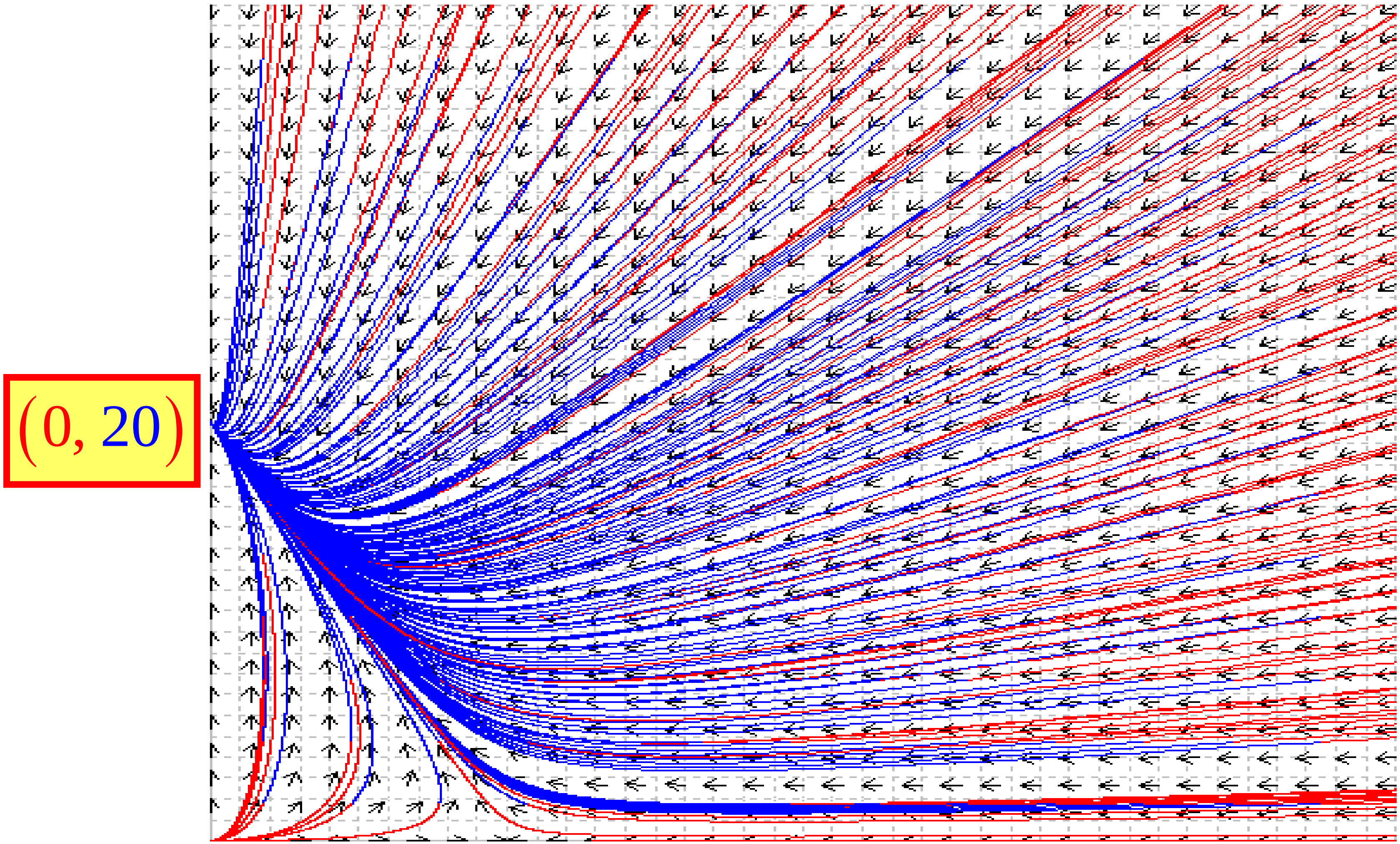
$$\Longrightarrow$$

$$\boxed{x_\infty = 10, \quad y_\infty = 0}$$

# Dominant Model (2)

$$\begin{cases} \dfrac{dx}{dt} = (1 - 0.1x(t) - 0.1y(t))x(t) \\ \dfrac{dy}{dt} = 2(1 - 0.05y(t) - 0.05x(t))y(t) \end{cases}$$

# Dominant Model (2)



# Stable Point (2)

$$\boxed{\frac{dx_\infty}{dt} = 0, \quad \frac{dy_\infty}{dt} = 0}$$

$$\Rightarrow$$

$$\begin{cases} x_\infty = 0 \\ 1 - 0.05y_\infty - 0.05x_\infty = 0 \end{cases}$$

$$\Rightarrow$$

$$\boxed{x_\infty = 0, \quad y_\infty = 20}$$

# Mathematical Study
# of
# Population Dynamics

# Catchphrase

| Theme | Real World |
| --- | --- |
| Population Dynamics | Problem of Population Growth |

# Malthus

- **Thomas Robert Malthus (1766-1834)**

**English Economist**

**An Essay on the Principle of Population (1798)**

# Thomas Robert Malthus

# Idea Credited to Malthus

- **A population will grow exponentially until limited by lack of available resources.**

# Malthus Model

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) = x_0 \quad \textbf{(Initial Condition)} \end{cases}$$

$x(t)$ : **Population Density**

$a$ : **Growth Rate**

# Computational Approach

# Numerical Computing with BASIC

# Example of Malthus

$$\begin{cases} \dfrac{dx}{dt} = 2x \\ x(0) = 5 \end{cases}$$

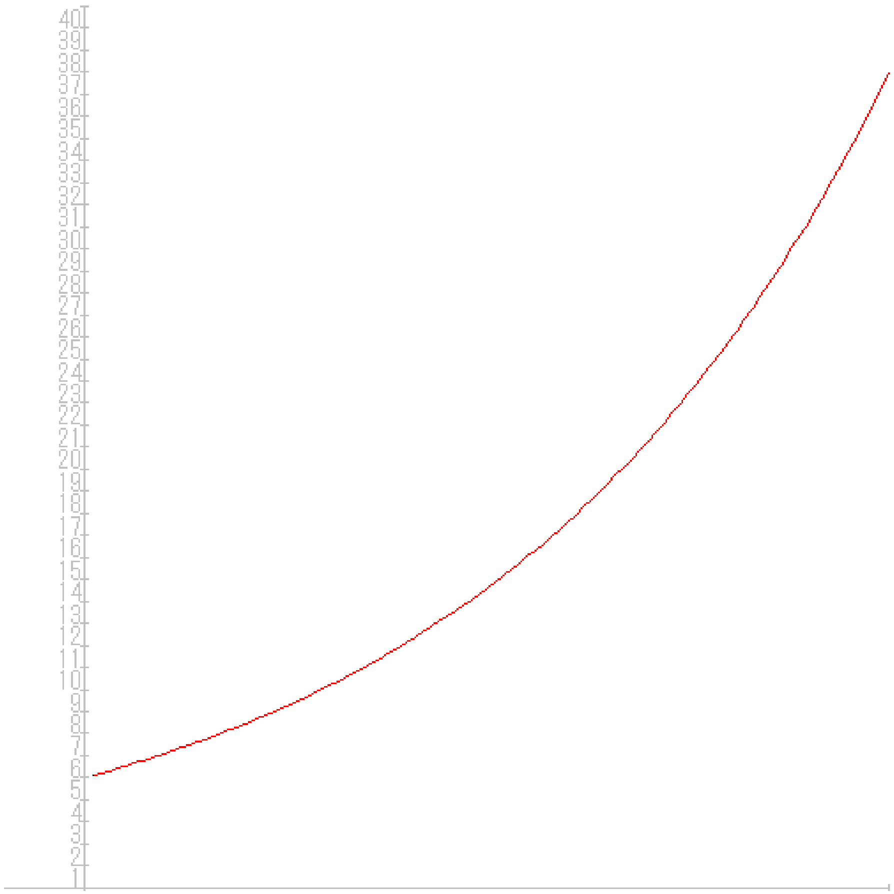
$$\Longrightarrow$$

$$\boxed{x(t) = 5e^{2t}}$$

# Runge-Kutta Method

```
DEF F(x, y)=2*y
SET WINDOW -0.1,3,-0.1,60
DRAW axes
LET  x = 0
LET  y = 5
LET  h = 0.01
LET  N = 10

FOR i = 0 TO N STEP 0.01
   LET  k1 = F(x, y)
   LET  k2 = F(x + h / 2, y + h * k1 / 2)
   LET  k3 = F(x + h / 2, y + h * k2 / 2)
   LET  k4 = F(x + h, y + h * k3)

   LET  x = x + h
   LET  y = y + h * (k1 + 2 * k2 + 2 * k3 + k4) / 6
   PLOT LINES: x,y;
   SET LINE COLOR "red"
   WAIT DELAY 0.01
NEXT i
END
```

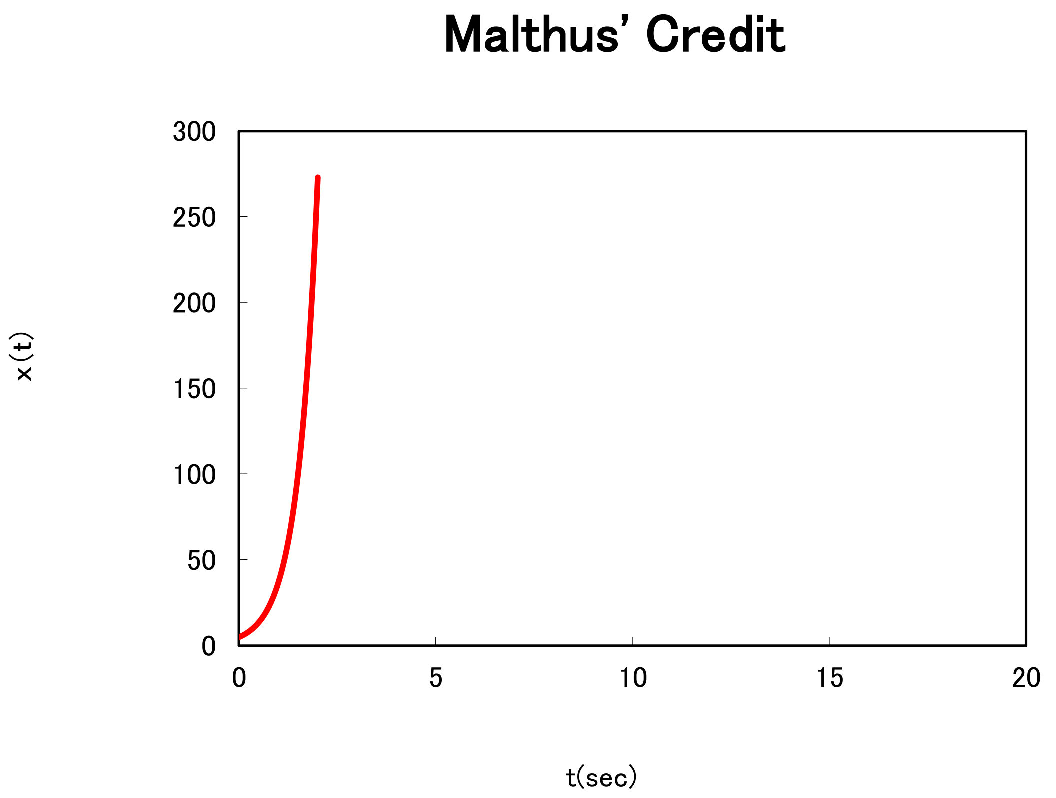
# Runge-Kutta Method



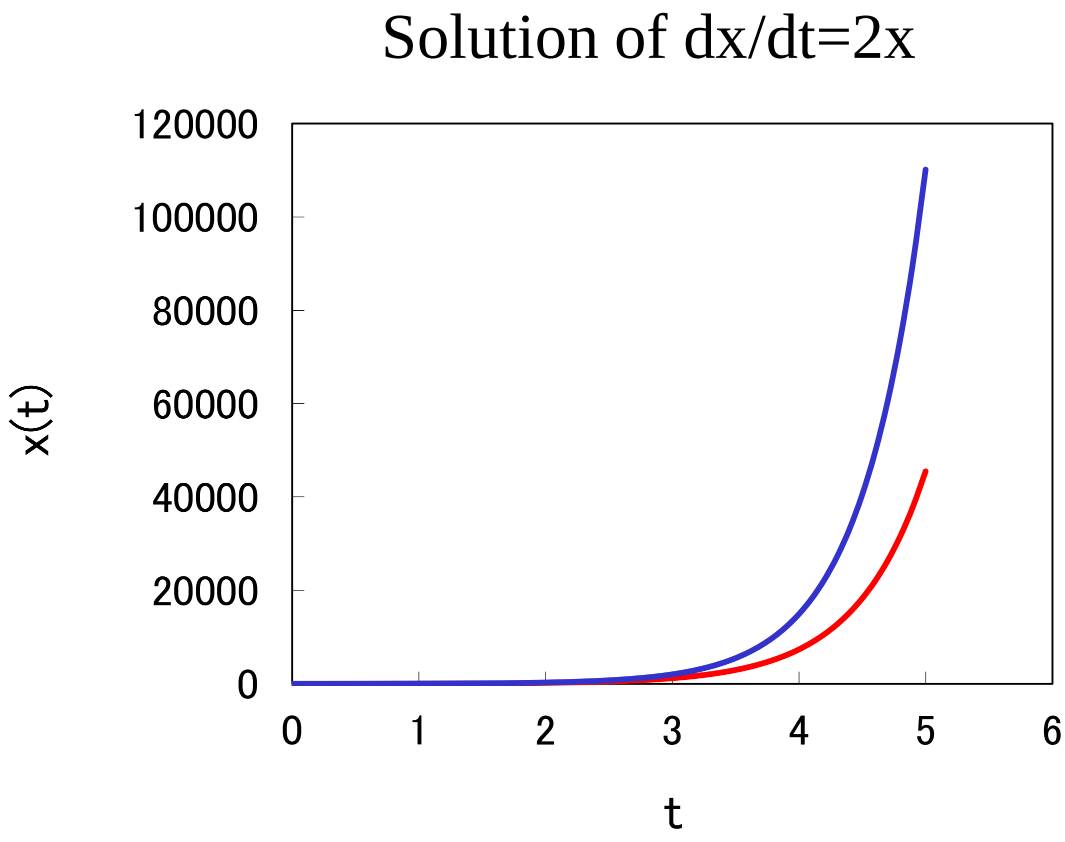

**A population will grow exponentially.**

# Numerical Computing with Excel (VBA)

# Runge-Kutta Method



# Euler's Method



# Verhulst

- **Pierre Francois Verhulst (1804-1849)**
  **Belgian Mathematical Biologist**
  **Notice sur la loi que la population poursuit dans son accroissement (1838)**

# Pierre Francois Verhulst

# Idea Credited to Verhulst

- **The growth rate of a population will depend on the effect of crowding within the population.**

# Logistic Model (1)

$$\begin{cases} \dfrac{dx}{dt} = x(t)(\varepsilon - \lambda x(t)) \\ x(0) = x_0 \quad \textbf{(Initial Condition)} \end{cases}$$

$x(t)$ : **Population Density**

$\varepsilon$ : **Intrinsic Growth Rate**

$\lambda$ : **Coefficient of Intraspecific Competition**

# Logistic Model (2)

$$\begin{cases} \dfrac{dx}{dt} = \varepsilon x(t)\left(1 - \dfrac{x(t)}{K}\right) \\ x(0) = x_0 \quad \textbf{(Initial Condition)} \end{cases}$$

$x(t)$ : **Population Density**

$K = \dfrac{\varepsilon}{\lambda}$ : **Carrying Capacity**

# Logistic Model (3)

$$\begin{cases} \dfrac{dx}{dt} = ax(t)(A - x(t)) \\ x(0) = x_0 \quad \text{(Initial Condition)} \end{cases}$$

$x(t)$ : **Population Density**

$a = \dfrac{\varepsilon}{K}$ : **Growth Rate**

$A = K$ : **Carrying Capacity of the Environment**

# Simplified Logistic Model (1)

$$\begin{cases} \dfrac{dx}{dt} = a\left(A - x(t)\right)x(t) \\ x(0) = x_0 \end{cases}$$

# Simplified Logistic Model (2)

$$x(t) = \frac{x_0 A}{x_0 + (A - x_0)e^{-aAt}}$$

$$\rightarrow \frac{x_0 A}{x_0} = A \quad (t \rightarrow +\infty)$$

# Biological Interpretation

- **For small populations, we get exponential growth with rate *aA*.**
- **As *x(t)* increases, the growth slows down and the population gradually reaches the carrying capacity of the environment.**

# Computational Approach

# Numerical Computing
# with
# BASIC

# Example (Large Initial Data)

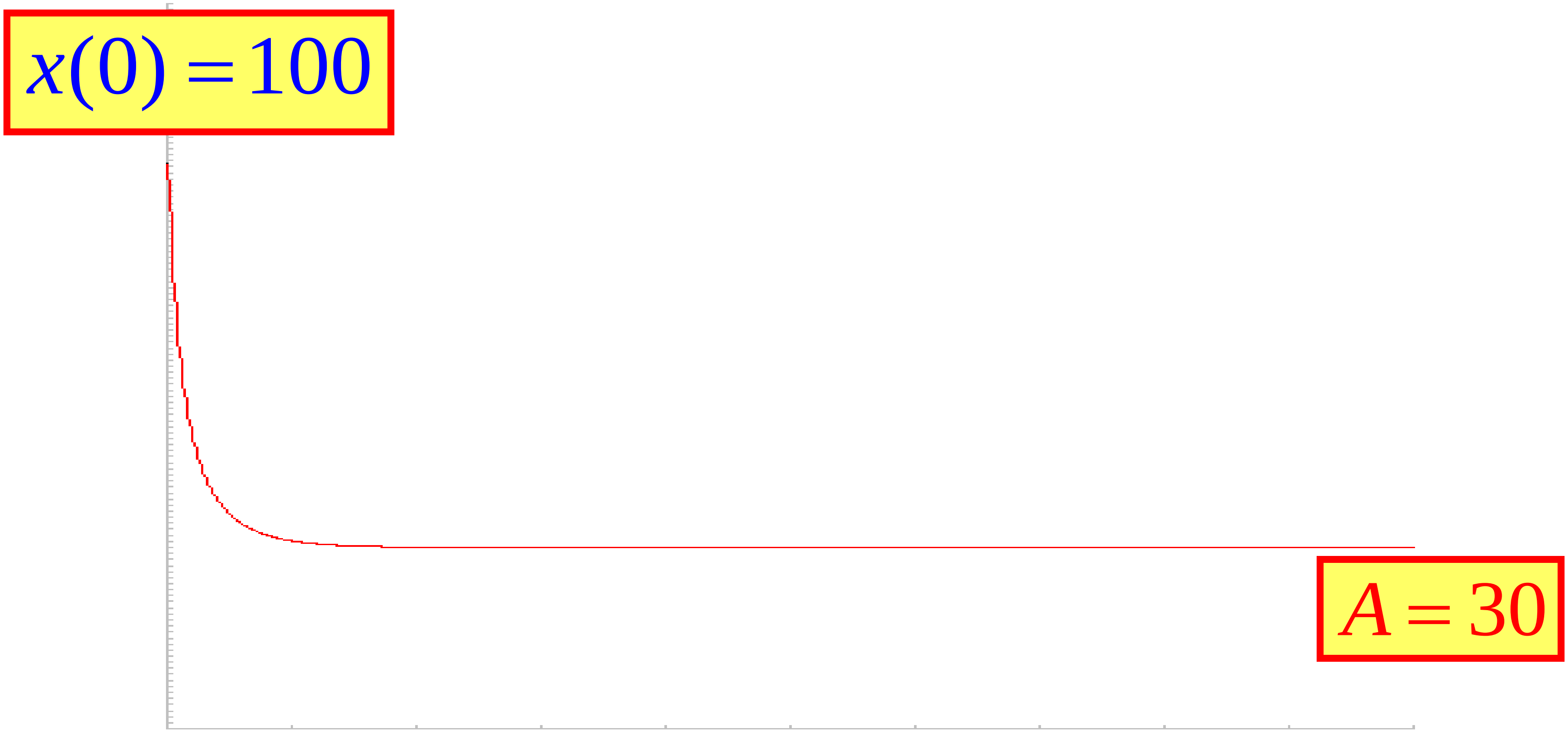
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{10}\big(30 - x(t)\big)\,x(t) \\ x(0) = 100 > 30 \end{cases}$$

$$a = \frac{1}{10}, \quad A = 30, \quad x_0 = 100$$

# Runge-Kutta Method

```
DEF F(t,x) = (3 - 0.1 * x) * x
SET WINDOW  0,10,0,40
DRAW axes

LET  t = 0
LET  x = 100

LET  h = 0.01
LET  N = 10

FOR i = 0 TO N STEP 0.01
   LET  k1 = F(t, x)
   LET  k2 = F(t + h / 2, x + h * k1 / 2)
   LET  k3 = F(t + h / 2, x + h * k2 / 2)
   LET  k4 = F(t + h, x + h * k3)

   LET  t = t + h
   LET  x = x + h * (k1 + 2 * k2 + 2 * k3 + k4) / 6
   PLOT LINES: t,x;
   SET LINE COLOR 4
   WAIT DELAY 0.01
NEXT i
END
```

# Runge-Kutta Method (Large Initial Data)

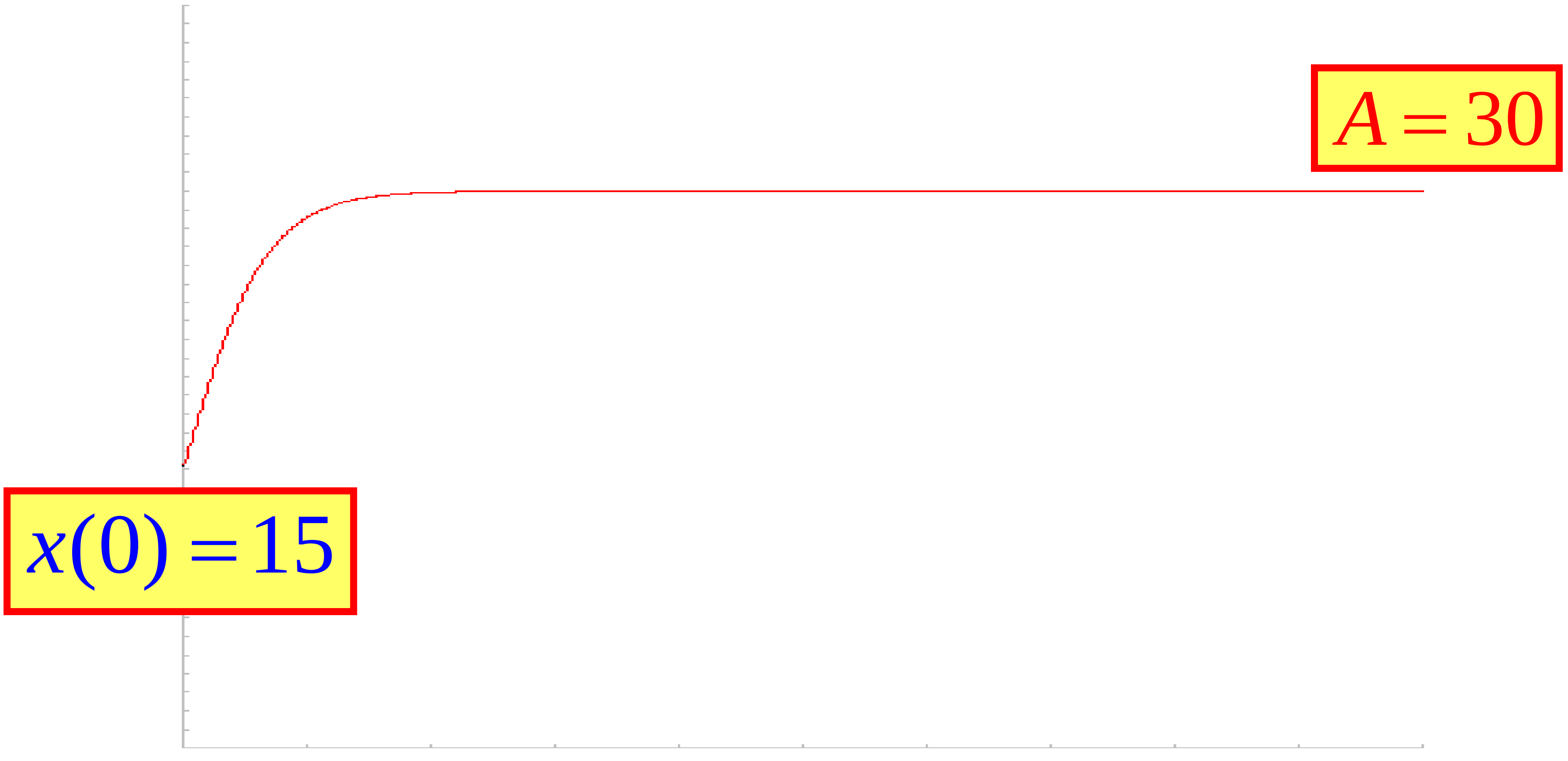


# Example (Small Initial Data)

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{10}\left(30 - x(t)\right)x(t) \\ x(0) = 15 < 30 \end{cases}$$

$$a = \frac{1}{10}, \quad A = 30, \quad x_0 = 15$$

# Runge-Kutta Method (Small Initial Data)



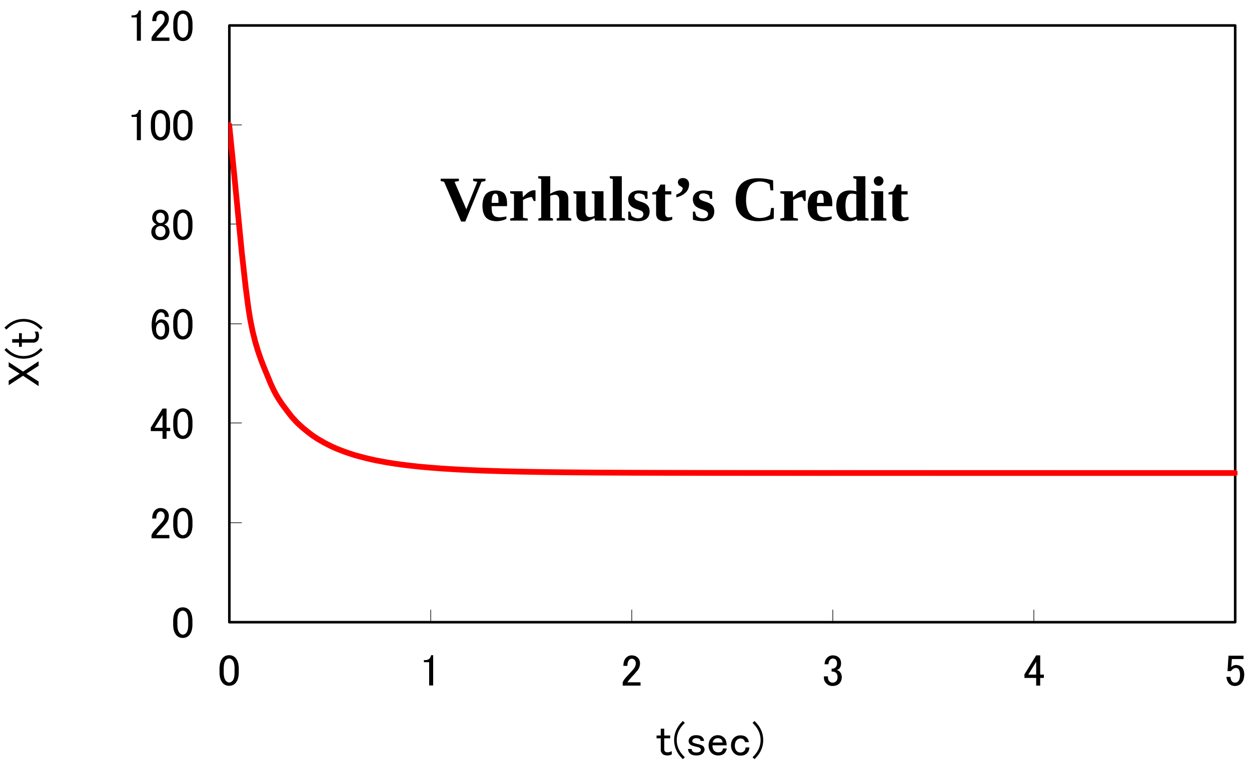
# Numerical Computing with Excel (VBA)

# Example (Large Initial Data)

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{10}\big(30 - x(t)\big)\,x(t) \\ x(0) = 100 > 30 \end{cases}$$

$$a = \frac{1}{10}, \quad A = 30, \quad x_0 = 100$$

# Runge-Kutta Method (Large Initial Data)

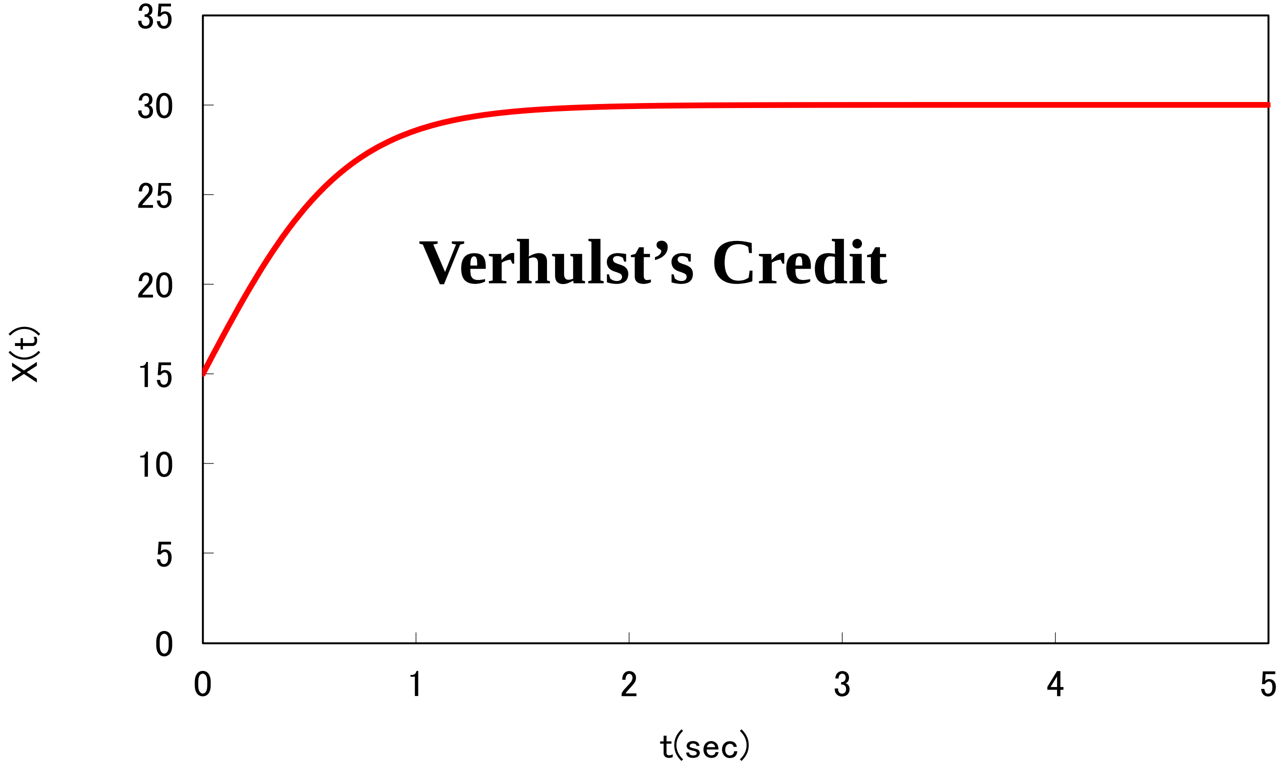


# Example (Small Initial Data)

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{10}\big(30 - x(t)\big)\,x(t) \\ x(0) = 15 < 30 \end{cases}$$

$$a = \frac{1}{10}, \quad A = 30, \quad x_0 = 15$$

# Runge-Kutta Method (Small Initial Data)



# END