

1/17(木) 第7回 数理学ⅢB

tangent vector (接ベクトル)

$M = \text{manifold space } x \in M$
 $t: \mathbb{R} \rightarrow M$ $t(0) = x$ 無限小の流れ
 時間

vector field (ベクトル場)

$\pi: M^D \xrightarrow{\text{projection}} M$ $\pi(t) = t(0)$

$\leftarrow X$ 切断 (section) $\pi \circ X = \text{id}_M$

$X \in (M^D)^M = M^{D \times M} = (M^M)^D$
 無限小の流れ
 議論における指数法則
 無限小変換

3つの観点からとらえることができる。

命題 $(d_1, d_2) \in D(2) = \{(d_1, d_2) \in D^2 \mid d_1 d_2 = 0\}$ を好む
 $X_{d_1} \circ X_{d_2} = X_{d_1+d_2}$
 quasi-colimit diagram

証明 $(d_1, d_2) \in D(2) \mapsto$
 $(1) \begin{matrix} d_1 = d & d_2 = 0 & X_d \circ X_0 = X_{d+0} \\ d_1 = 0 & d_2 = d & X_0 \circ X_d = X_{0+d} \end{matrix}$

Corollary $\forall (d, -d) \in D(2)$
 $(-d, d)$

$X_d \circ X_{-d} = X_0 \circ \text{id}_M$
 $X_{-d} \circ X_d = \text{id}_M$

命題 $X, Y: \text{vector fields on } M$

$(X+Y)_d = X_d \circ Y_d \circ l_{(X,Y)} = D(2) \rightarrow M^M$
 $= Y_d \circ X_d$

$(d_1, d_2) \in D(2) \mapsto X_{d_1} \circ Y_{d_2} \in M^M$
 $d_1 = d \quad d_2 = 0 \quad X_d$ たし算の場合
 $d_1 = 0 \quad d_2 = d \quad Y_d$ こたに d とおく

$\mathcal{L}(M)$ M 上のベクトル場の全体 \mathbb{R} 加群

Lie bracket (Lie かけ算)

$n \times n$ の行列の全体 $A+B$ αA
 $[A, B] = AB - BA$ 二重線型
 $[A, B] = -[B, A]$ antisymmetric
 $[A, A] = 0$

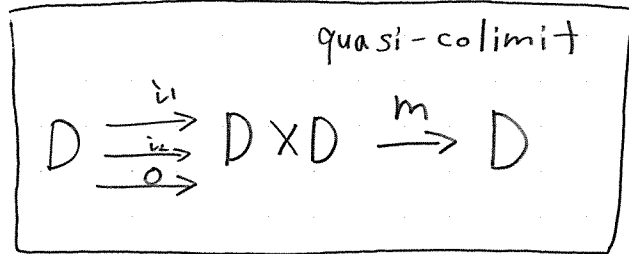
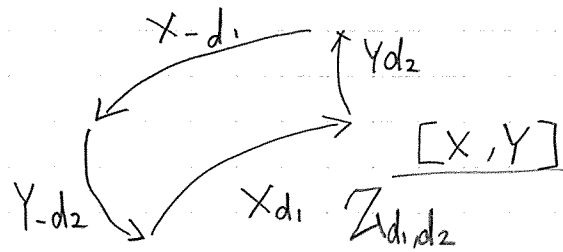
A, B, C Lie 代数 Jacobi の恒等式

$[[A, B], C] + [[B, C], A] + [[C, A], B]$
 $= (AB - BA)C - C(AB - BA) + (BC - CB)A - A(BC - CB)$
 $+ (CA - AC)B - B(CA - AC)$
 $= 0$

$$X_{-d} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} X_d \quad X_{-d} \circ X_d = id_M$$

$$[X, Y] \quad (d_1, d_2) \in D \times D \rightarrow M^M$$

$$Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$$



$$d_1 = d, d_2 = 0$$

$$\textcircled{Y_0} \circ X_{-d} \circ \textcircled{Y_0} \circ X_d = id_M$$

$$d_1 = 0, d_2 = d$$

id_M

anti symmetric

$$(X_d)^{-1} = X_{-d}$$

$$[X, Y] = -[Y, X]$$

$$[X, Y]_{d_1 d_2} = [X, Y]_{(-d_1) d_2}$$

$$= ([X, Y]_{(-d_1) d_2})^{-1}$$

$$= (Y_{-d_2} \circ X_{d_1} \circ Y_{d_2} \circ X_{-d_1})^{-1}$$

$$= (X_{-d_1})^{-1} \circ (Y_{d_2})^{-1} \circ (X_{d_1})^{-1} \circ (Y_{-d_2})^{-1}$$

$$= X_{d_1} \circ Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2}$$

$$= [Y, X]_{d_2 (-d_1)}$$

$$= (-[Y, X])_{d_1 d_2}$$

$[,] =$ 重線型

$$\varphi: D \rightarrow \mathbb{R}$$

$$\varphi(d) = \varphi(0) + ad$$

E \mathbb{R} -module

Euclidean

\mathbb{R}^n

$$\varphi: D \rightarrow E$$

$$\exists! a \in E$$

$$\varphi(d) = \varphi(0) + a d$$

$V: \mathbb{R}$ -module

$E =$ Euclidean \mathbb{R} -module

$x \in V$

$$f: V \rightarrow E$$

$$\forall a \in D \rightarrow f(x + a d) \in E$$

$$f(x + a d) = f(x) + \textcircled{?} d$$

\uparrow
 E

$$f'(x)(a) \quad f'(x): V \rightarrow E$$

線型

$$f'(x)(a_1 + a_2) = f'(x)(a_1) + f'(x)(a_2) \quad f: V \rightarrow ?$$

$$f(x + (a_1 + a_2)d) = f(x) + f'(x)(a_1 + a_2)d$$

$$f(x + a_1 d + a_2 d) = f(x + a_1 d) + f'(x + a_1 d)(a_2)d$$

$$= f(x) + f'(x)(a_1)d + \{f'(x) + f''(x)(a_1)d\}(a_2)d$$

$$= f(x) + f'(x)(a_1)d + f'(x)(a_2)d$$

$$\textcircled{\text{宿}} f(x)(da) = d f'(x)(a)$$

補題 $f: V \rightarrow E$

homogeneous $f(dx) = d f(x)$